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IDENTIFYING THE TIME OF A STEP CHANGE WITH $\bar{X}$ CONTROL CHARTS

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Introduction

Control charts are used to monitor for changes in a process by distinguishing between common and special causes of variability (1). When a control chart signals that a special cause is present, process engineers must initiate a search for the special cause of the process disturbance. The search depends on the process engineers' expertise and knowledge of the process. Identifying which combination of the many process variables is responsible for a change in the process allows engineers to improve quality by preventing or avoiding changes in those variables which lead to poor quality, and by perpetuating those changes and optimizing those variables which can lead to better quality.

Knowing *when* a process has changed would simplify the search for the special cause. If the time of the change could be determined, process engineers would have a smaller search window within which to look for the special cause. Consequently, the special cause can be identified more quickly, and appropriate actions needed to improve quality can be implemented sooner.

In this article, we will consider the Shewhart $\overline{X}$ control chart, the most commonly used statistical process control chart in industry. The Shewhart $\overline{X}$ control chart plots individual subgroup means against upper and lower control limits. It is well known that the Shewhart $\overline{X}$ control chart can issue a signal of a change in a process mean a substantial amount of time *after* the change in the process mean actually occurred. Estimating the time of the process change with the time that the control chart issued a signal would lead to a badly biased and, therefore, possibly misleading estimate of the time of the process change. This bias is due to the potentially large delay in generating a signal from the control chart.

In this article, we consider the use of an estimator of the time of change of the process mean once the Shewhart $\overline{X}$ control chart issues a signal. The derivation of the estimator that we propose to use is due to Hinkley (2). Hinkley discusses the asymptotic properties of the estimator, which are different from the small-sample quality control situation we are considering here. Our method is applied when an $\overline{X}$ chart signals that a special cause is present. Our estimator provides process engineers with a useful estimate of the time of the process change. We investigate the performance of our estimator and give an example to illustrate its use.

In the next section, we consider a model for a step change in the location of a process. A step change for a process mean occurs when the mean suddenly changes its value and then does not change again until corrective action has been taken. Based on this step-change model, we introduce an estimator of the time of the process change. A numerical example is given to explain how the estimator can be used in practice. We then analyze the performance of the change point estimator using Monte Carlo simulation.

Process Step-Change Model

We will assume that the process is initially in control, with observations coming from a Normal distribution with a known mean of μ_0 and a known standard deviation of σ_0. However, after an unknown point in time τ (known as the process change point), the process location changes from μ_0 to $\mu_1 = \mu_0 + \delta\sigma_0/\sqrt{n}$, where n is the subgroup size and δ is the unknown magnitude of the change. We also assume that once this step change in the process location occurs, the process remains at the new level of μ_1 until the special cause has been identified and removed.

We will assume that $\overline{X}_T$ is the first subgroup average to exceed a control limit and that this signal is not a false alarm. Thus, $\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_\tau$ are the subgroup averages that came from the in-control process, whereas $\overline{X}_{\tau+1}, \overline{X}_{\tau+2}, \ldots, \overline{X}_T$ are from the changed process. We now introduce our estimator of the time of the process change with a hypothetical example that illustrates its use.

Illustrative Example

Consider a production process for forged piston rings. The specified outer diameter (OD) of the piston rings is 100 mm. The piston ring OD is the quality characteristic to be monitored. The in-control production process can be modeled as a Normal process whose mean is equal to 100. The process standard deviation is 5 and subgroups of size $n = 4$ are drawn. Thus, the upper control limit (UCL) of the Shewhart $\overline{X}$ control chart is 107.50 and the lower control limit (LCL) is 92.50.

The subgroup averages are shown in Table 1. We see that a total of 27 subgroup averages were obtained before one of them exceeded one of the control limits. It can be seen that $\overline{X}_{27} >$ UCL, and, thus, the Shewhart $\overline{X}$ control chart has signaled at time $T = 27$. At this point, our proposed change point estimator can be applied.

To apply our proposed method (see the Appendix for the derivation), we need to find the value of t in the range $0 \le t < T$ which maximizes $C_t = (T - t)(\overline{\overline{X}}_{T,t} - \mu_0)^2$. The reverse cumulative average, $\overline{\overline{X}}_{T,t} = 1/(T - t)\Sigma_{i=t+1}^{T}\overline{X}_i$, is the overall average of the $T - t$ most recent subgroups. The value of t which maximizes the C_t values is our estimator of the last subgroup from the in-control process.

So, to apply our method, we first obtain the reverse cumulative averages for $t = 0, 1, 2, \ldots, T - 1$. In our example, the signal occurred at time $T = 27$. Thus, we need

Table 1. Subgroup Averages

subgroup i	$\overline{X}_i$
1	100.450
2	97.450
3	102.450
4	100.675
5	98.550
6	97.950
7	102.950
8	98.825
9	101.325
10	103.075
11	99.600
12	98.825
13	100.425
14	96.075
15	101.225
16	103.075
17	101.925
18	101.350
19	103.575
20	102.925
21	100.675
22	100.600
23	105.125
24	106.700
25	96.050
26	102.175
27	107.900

$$\overline{\overline{X}}_{27,t} = \frac{1}{27 - t} \sum_{i=t+1}^{27} \overline{X}_i$$

for t = 0, 1, 2, . . . , 26. Working with the most recent subgroups first, the reverse cumulative averages are

$$\overline{\overline{X}}_{27,26} = \frac{1}{1}(\overline{X}_{27}) = 107.90,$$

$$\overline{\overline{X}}_{27,25} = \frac{1}{2}(\overline{X}_{27} + \overline{X}_{26}) = \frac{1}{2}(107.90 + 102.18) = 105.04,$$

$$\overline{\overline{X}}_{27,25} = \frac{1}{3}(\overline{X}_{27} + \overline{X}_{26} + \overline{X}_{25})$$

$$= \frac{1}{3}(107.90 + 102.18 + 96.05) = 102.04,$$

and so on. All 27 of these reverse cumulative averages are shown in Table 2.

Once we have the $\overline{\overline{X}}_{T,t}$ values, we then compute the C_t values, where

$$C_t = (T - t)(\overline{\overline{X}}_{T,t} - \mu_0)^2.$$

Table 2. Example: Computations

subgroup i	$\overline{X}_i$	t	$\overline{\overline{X}}_{27,t}$	C_t
1	100.450	0	101.182	37.748
2	97.450	1	101.211	38.103
3	102.450	2	101.361	46.308
4	100.675	3	101.316	41.541
5	98.550	4	101.343	41.513
6	97.950	5	101.470	47.569
7	102.950	6	101.638	56.350
8	98.825	7	101.573	49.455
9	101.325	8	101.717	56.021
10	103.075	9	101.739	54.427
11	99.600	10	101.660	46.862
12	98.825	11	101.789	51.212
13	100.425	12	101.987	59.203
14	96.075	13	102.098	61.635
15	101.225	14	102.562	85.299
16	103.075	15	102.673	**85.734**
17	101.925	16	102.636	76.455
18	101.350	17	102.708	73.306
19	103.575	18	102.858	73.531
20	102.925	18	102.769	61.328
21	100.675	20	102.746	52.800
22	100.600	21	103.092	57.350
23	105.125	22	103.590	64.441
24	106.700	23	103.206	41.120
25	96.050	24	102.042	12.505
26	102.175	25	105.038	50.753
27	107.900	26	107.900	62.410

In our hypothetical example, because $\mu_0 = 100$ and $T = 27$, we compute

$$C_t = (27 - t)(\overline{\overline{X}}_{27,t} - 100)^2$$

for t = 0, 1, 2, . . . , 26. For example, working with the most recent reverse cumulative averages, we obtain

$$C_{26} = (1)(107.90 - 100)^2 = 62.41,$$

$$C_{25} = (2)(105.04 - 100)^2 = 50.75,$$

$$C_{24} = (3)(102.04 - 100)^2 = 12.51,$$

and so on. All 27 of these values are also given in Table 2. Calculations such as these can be performed easily by storing the historical data and using a spreadsheet for the computations once a signal is received from a control chart.

To obtain our proposed estimate of the time of the change in the process, we need to find the largest value of C_t. From Table 2, it can be seen that the largest C_t value is associated with subgroup 16. Thus, we estimate that subgroup 16 was the first subgroup obtained from the changed process and, consequently, that subgroup 15 was the last subgroup from the in-control process. Process engineers would then be instructed to examine their log books and records for a special cause that might have occurred between the formation of subgroups 15 and 16.

Upon examining their log books, the engineers discovered that the supplier of steel for the piston rings had been changed. The process engineers found that the daily production of piston rings with steel from the old supplier ended with subgroup number 15. Production of piston rings forged from steel from the new supplier began with formation of subgroup 16. Upon identifying this change in supplier as the probable special case, the process engineers initiated a comparative study of the physical properties of the steel from the new and old suppliers.

If process engineers had only examined their records corresponding to time $T = 27$ when the signal was given by the control chart, it is possible that they might have incorrectly identified a special cause or that no special cause may have been identified at all. Alternatively, process engineers might have started examining their log books at the time of the control chart signal (i.e., at time $T = 27$) and searched backward in time until a special cause was found. Our estimator of the time of the change suggests a more efficient method for searching for the special cause. Process engineers could initiate their search for the special cause at the time suggested by our estimator. From that point, they could expand their search by examining their log books corresponding to subgroups before and after our estimated change point.

Performance of the Estimator

We will now analyze the performance of our proposed change point estimator. We will consider the use of this estimator with the Shewhart $\overline{X}$ control chart. When the $\overline{X}$ control chart signals that a process change has occurred, the estimator is then applied to the data to estimate the time of the change.

A Monte Carlo simulation study was conducted to study the performance of our proposed change point estimator. Observations were randomly generated from a standard Normal distribution for subgroups 1, 2, . . . , 100. Then, starting with subgroup 101, observations were randomly generated from a Normal distribution with mean δ and standard deviation 1 until the Shewhart $\overline{X}$ control chart produced a signal. This procedure was repeated a total of 10,000 times for each of the values of δ that were studied, namely $\delta = 0.5, 1.0, 1.5, 2.0$, and 3.0. For each simulation run, the change point estimator was computed. The average of the estimates from the 10,000 simulation runs was computed along with its standard error.

In Table 3, we show the expected length of each simulation run for the different magnitudes of change in the process mean. This is the expected time at which the control chart signals a change in the process mean that occurred at time 100. We tabulate $\bar{\hat{\tau}}$, the average change point estimate from the simulation runs for various sizes of change in the process mean. We also give the standard error of the estimates.

As the actual change point for the simulations was at time 100, the average estimated time of the process change, $\bar{\hat{\tau}}$, should be close to 100. From Table 3, we see that for a process step change of standardized magnitude $\delta = 1$, the control chart issues a signal at time 143.86 on average. In this case, the average estimated time of the process change was 100.31, which is fairly close to the actual change point of 100. We also see that for a standardized process location change of size $\delta = 2$, the average time of the control chart signal is 106.30. The average estimated time of the change is 99.71. Also, Table 3 shows that for $\delta = 3$, the average estimated time of the change is 99.55. Thus, on average, our proposed maximum likelihood estimator (MLE) of the time of the process change is fairly close to the actual time of the change, regardless of the magnitude of the change.

The observed frequency with which our proposed estimator of the time of the step was within m subgroups of the actual time of the change, for $m = 0, 1, 2, \ldots, 10$ and 15, is shown in Table 4. This provides an indication of the precision of our proposed estimator. The proportion of the 10,000 runs where the estimated time of the change was within $\pm m$ of the actual change should be high and should increase as m increases.

From Table 4, we see that of the 10,000 simulation trials conducted for $\delta = 1$, 26% of those simulation trials identified the change point correctly. We also see that in 48% of the trials, the change point was estimated to be within ± 1 of the actual time of the process change. Also, in 61% of the trials, the estimate was within ± 2 subgroups; in 76% of the trials, it was within ± 4 subgroups; and in 90% of the trials, it was within ± 8 subgroups of the actual time of the process change.

The average run length (ARL) of a control chart is the expected number of subgroups required to detect a change in a process parameter. For a step change in the process mean of magnitude $\delta = 1$, it is easy to show that the ARL for a Shewhart $\overline{X}$ control chart is 43.86. Clearly, one would not want to estimate the time of the process change as the time of the signal from the Shewhart $\overline{X}$ control chart. Such an estimator would obviously be biased badly. If such an estimator of the time of the change were used, process engineers might not discover the special cause at all. Instead of examining process conditions, changes in process variables, and so on at the time of the actual process change, the process engineers would be looking at process log books, historical data, and records associated with a search window or time frame that was, on average, more than 40 subgroups *after* the process actually changed. Thus, the chances of discovering and identifying the special cause could be fairly small. Instead, if process engineers used our proposed change point estimator, they would have almost a 50% chance of identifying the time

Table 3. Simulation Results: Average Change Point Estimates and Associated Standard Error for Change Point of $\tau = 100$

	$\delta = 0.5$	$\delta = 1.0$	$\delta = 1.5$	$\delta = 2.0$	$\delta = 3.0$
Expected length of run, $E(T)$	261.29	143.86	114.97	106.30	102.00
$\bar{\hat{\tau}}$	103.77	100.31	99.87	99.71	99.55
$std.error\left(\bar{\hat{\tau}}\right)$	0.2319	0.0721	0.0413	0.0413	0.0442

Table 4. Simulation Results for Different Magnitudes of Process Change Based on 10,000 Trials

	$\delta = 0.5$	$\delta = 1.0$	$\delta = 1.5$	$\delta = 2.0$	$\delta = 3.0$
$\hat{P}(\hat{\tau} = \tau)$	0.08	0.26	0.45	0.61	0.82
$\hat{P}(\lvert\hat{\tau} - \tau\rvert \le 1)$	0.19	0.48	0.70	0.84	0.94
$\hat{P}(\lvert\hat{\tau} - \tau\rvert \le 2)$	0.27	0.61	0.81	0.92	0.97
$\hat{P}(\lvert\hat{\tau} - \tau\rvert \le 3)$	0.33	0.70	0.87	0.96	0.98
$\hat{P}(\lvert\hat{\tau} - \tau\rvert \le 4)$	0.38	0.76	0.91	0.98	0.99
$\hat{P}(\lvert\hat{\tau} - \tau\rvert \le 5)$	0.44	0.81	0.93	0.99	
$\hat{P}(\lvert\hat{\tau} - \tau\rvert \le 6)$	0.47	0.85	0.95		
$\hat{P}(\lvert\hat{\tau} - \tau\rvert \le 7)$	0.51	0.88	0.97		
$\hat{P}(\lvert\hat{\tau} - \tau\rvert \le 8)$	0.54	0.90			
$\hat{P}(\lvert\hat{\tau} - \tau\rvert \le 9)$	0.57	0.92			
$\hat{P}(\lvert\hat{\tau} - \tau\rvert \le 10)$	0.60	0.94			
$\hat{P}(\lvert\hat{\tau} - \tau\rvert \le 15)$	0.70	0.96			

of the process change within one subgroup of the actual change point and have approximately a 75% (90%) chance of identifying the change point within four (eight) subgroups of the actual change point. Thus, using our proposed change point estimator would increase the chances of identifying correctly the underlying special cause.

For a process step change of magnitude $\delta = 2$, the average of the estimated 10,000 change point estimates was 99.71. Our proposed change point estimator identified correctly the change point in 61% of the trials. Our proposed estimate was within one subgroup of the actual change point in 84% of the trials, and within two subgroups of the actual change point in 92% of the trials.

For a process location change of magnitude $\delta = 2$, the ARL for a Shewhart $\bar{X}$ control chart is 6.30. For a change of this magnitude, using the time of the control chart signal to estimate the change point would again result in a biased estimate of the time of the process change.

We also see from Table 4 that for process step changes of magnitude $\delta = 3$, our proposed estimator exhibited a good performance in identifying the time of change. For step changes of this magnitude, our proposed estimator correctly identified the time of the change in 82% of the trials and was within one (two) subgroups of the time of the actual process change in 94% (97%) of the trials.

For a location step change in a normal process mean of magnitude $\delta = 3$, the ARL for the Shewhart $\bar{X}$ control chart is 2. In this case, using the time of the control chart signal as an estimate of the first subgroup from the changed process would not produce a badly biased estimate. However, such an estimator will only identify correctly the time of the process change with a probability of 0.50. From Table 4, it can be seen that for a process change of magnitude $\delta = 3$, the estimated probability of correctly identifying the time of the process change using our proposed estimator is 0.84.

In Table 5, we compare the probabilities of identifying correctly the time of the change in the process mean when the time of the signal from the Shewhart $\bar{X}$ control chart is used and for the case when our proposed estimator is used. This comparison is done for different standardized magnitudes of the step change in the process mean.

From Table 5, it can be seen that for a step change of magnitude $\delta = 1$, the probability of identifying correctly

Table 5. Probabilities of Estimating Correctly the Time of a Step Change Using Two Different Methods

	Shewhart $\bar{X}$ control chart	Our proposed estimator
$\delta = 0.5$	0.01	0.08
$\delta = 1.0$	0.02	0.26
$\delta = 1.5$	0.07	0.45
$\delta = 2.0$	0.16	0.61
$\delta = 3.0$	0.50	0.82

the time of the change using the signal from the Shewhart $\bar{X}$ control chart is 0.02. As mentioned earlier, the Shewhart $\bar{X}$ control chart signals an average of 43.96 subgroups after the change in the process mean occurs. Thus, for a change of magnitude $\delta = 1$, it is unlikely that the control chart would issue a signal on the first subgroup after the change. On the other hand, using our proposed estimator, the estimated probability of correctly identifying the time of the change is about 0.26.

For large step changes in the process mean, the chances of identifying correctly the time of the change increase for both estimators. However, for our proposed estimator the chances of identifying correctly the time of the change are substantially higher. Thus, it can be seen that regardless of the magnitude of the change in the process mean, our proposed estimator produces better and more useful estimates of the time of the process change.

Conclusions

Control charts are used to detect whether or not a process has changed. When a control chart does signal that a process has changed, engineers must initiate a search for the special cause. However, given a signal from a control chart, process engineers generally do not know what caused the process to change or when the process changed. Knowing the time of the process change would simplify the search for the special cause. If the process engineers knew when the process changed, the search would simply reduce to discovering what aspect of the process changed at that time. Thus, process engineers would increase their chances of identifying correctly the special cause quickly. This would allow them to take the appropriate actions immediately to improve quality.

In this article, we have proposed an estimator that is useful for identifying the change point of a step change in a Normal process mean. We have included a hypothetical example to demonstrate the use of our estimator. The example shows that the procedure is simple to implement and can be performed easily using a spreadsheet.

We have discussed the performance of our change point estimator when it is used with the Shewhart $\bar{X}$ control chart. The results show that our estimator provides a useful and much better alternative to using the time of the signal from the control chart. The analysis of the performance of our estimator with the Shewhart $\bar{X}$ control chart shows that our proposed change point estimator performs well even in the cases of small changes in the process mean that have large average run lengths.

Appendix: Derivation of the Change Point Estimator

In this appendix, we consider the derivation of the maximum likelihood estimator (MLE) of τ, the process location change point. Maximum likelihood estimation techniques are discussed in Ref. 3, for example. We will denote the MLE of the change point τ as $\hat{\tau}$. Given the subgroup averages $\bar{x}_1, \ldots, \bar{x}_T$, the MLE of τ is the value of τ that maximizes the likelihood function or, equivalently, its logarithm. The logarithm of the likelihood function (apart from a constant) is

$$\log L(\tau, \mu_1, |\bar{\mathbf{x}}) = -\frac{n}{2\sigma_0^2}\left(\sum_{i=1}^{\tau} (\bar{x}_i - \mu_0)^2 + \sum_{i=\tau+1}^{T} (\bar{x}_i - \mu_1)^2\right)$$

$$= -\frac{n}{2\sigma_0^2}\left(\sum_{i=1}^{T} \bar{x}_i^2 - 2\mu_0\sum_{i=1}^{\tau} \bar{x}_i - \tau\mu_0^2 \right.$$

$$\left. -2\mu_1 \sum_{i=\tau+1}^{T} \bar{x}_i + (T-\tau)\mu_1^2\right). \quad (1)$$

We note that there are two unknowns in the log-likelihood function: τ and μ_1. If the change point τ were known, the MLE of μ_1 would be $\hat{\mu}_1 = \bar{\bar{X}}_{T,\tau} = (T-\tau)^{-1} \Sigma_{i=\tau+1}^{T} \bar{X}_i$, the average of the $T - \tau$ most recent subgroup averages. Substituting this back into Eq. (1), we get

$$\log L(\tau, |\bar{\mathbf{x}}) = -\frac{n}{2\sigma_0^2}\left(\sum_{i=1}^{T} \bar{x}_i^2 - 2\mu_0\sum_{i=1}^{\tau} \bar{x}_i + \tau\mu_0^2 - (T-\tau)\bar{\bar{x}}_{T,\tau}^2\right). \quad (2)$$

It can be shown that this is equivalent to

$$\log L(\tau, |\bar{\mathbf{x}}) = -\frac{n}{2\sigma_0^2}\left(\sum_{i=1}^{T} \bar{x}_i^2 - 2\mu_0\sum_{i=1}^{T} \bar{x}_i + T\mu_0^2 - (T-\tau)(\bar{\bar{x}}_{T,\tau} - \mu_0)^2\right). \quad (3)$$

It then follows that the value of τ that maximizes the log-likelihood function is

$$\hat{\tau} = \arg\max_t \left\{(T-t)(\bar{\bar{x}}_{T,t} - \mu_0)^2\right\} = \arg\max_t \{C_t\}, \quad (4)$$

where $C_t = (T-t)(\bar{\bar{x}}_{T,t} - \mu_0)^2$; that is, $\hat{\tau}$ is the value of t in the range $0 \le t < T$ which maximizes $C_t = (T-t)(\bar{\bar{x}}_{T,t} - \mu_0)^2$.

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