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A Simple Method for Studying Run-Length Distributions of Exponentially Weighted Moving Average Charts

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A numerical procedure using integral equations is presented for the tabulation of moments of run lengths of exponentially weighted moving average (EWMA) charts. Both average run lengths (ARL's) and standard deviations of run lengths (SDRL's) are presented for the two-sided EWMA chart assuming normal observations, along with an example illustrating how to design such a chart. The procedure given extends easily to many nonnormal cases and to one-sided versions of the EWMA chart.

KEY WORDS: Average run length; Standard deviation of run length; Fredholm integral equation; Gaussian quadrature.

1. INTRODUCTION

One control charting procedure alternative to the standard Shewhart X-bar chart is the exponentially weighted moving average (EWMA) chart. For purposes of comparing an EWMA chart to another control scheme or designing an EWMA chart, knowledge of the run-length distribution associated with a particular EWMA chart is important.

Roberts (1959) compared the average run length (ARL) properties of control-chart tests based on such averages with tests based on ordinary moving averages and the standard control-chart test, assuming an iid normal process. He presented ARL curves based on simulation results and concluded that the EWMA tests cannot improve on the standard Shewhart X-bar control chart in detecting relatively large shifts in the process mean. For relatively small shifts, however, the EWMA provides greater sensitivity to the change.

Robinson and Ho (1978) presented a numerical procedure for the approximation of ARL's of an EWMA chart. They expressed the generation of successive EWMA's as an autoregressive [AR(1)] process. The pdf and cdf of the process were approximated using an Edgeworth expansion. The ARL's were found by summing an approximated survival function and truncating the sum at a finite stage.

The purpose of this article is to present a general methodology for studying EWMA procedures assuming an iid process. The approach uses integral equations for moments of run-length distributions associated with EWMA charts. Tables of ARL's and standard deviations of run lengths (SDRL's) for the normal case are included. I believe that ease of use and computational considerations favor the integral-equation approach over that given by Robinson and Ho (1978). In addition, the results that follow extend easily to cases in which the distribution of the statistic generating the EWMA is nonnormal.

2. AVERAGE RUN LENGTHS FOR AN EWMA CHART

Suppose we are plotting an EWMA of sample statistics y_t. The successive values plotted can be described by

$$Q_t = (1-\lambda)Q_{t-1} + \lambda y_t, \qquad 0 < \lambda \le 1,\ t = 1, 2, \ldots.$$

Here λ is a smoothing constant, Q_t is the value of the EWMA after observation t, where the subscript t represents the observation number as well as an index of a point in time. Hunter (1986) pointed out that the EWMA can be thought of as a compromise between the Shewhart X-bar and cumulative sum (CUSUM) charting procedures. For $\lambda = 1$, the

EWMA places all of its weight on the most recent observation, as does the X-bar chart. For λ close to 0, the most recent observation receives little weight, and the EWMA resembles the CUSUM. I wish to study the ARL properties of a procedure based on the Q_t's when the y_t's are assumed to be iid with pdf $f(\cdot)$. Without loss of generality, we will take the desired (target) mean value to be $\mu = 0$ and the known process variance to be $\sigma^2 = 1$. Using the Q_t's to monitor the process mean level, the process will be deemed "out of control" if Q_t is too large or too small. That is, at observation t, we will conclude that a shift in the mean has occurred if $|Q_t| > h$, for h a specified constant. Let $L(u)$ be the ARL, given that the EWMA starts with $Q_0 = u$. If the first observation y_1 is such that $|(1-\lambda)u + \lambda y_1|$ is greater than h, a signal is given. Otherwise, the run continues from $(1-\lambda)u + \lambda y_1$, with $L((1-\lambda)u + \lambda y_1)$ representing the additional expected run length. Thus

$$\begin{aligned} L(u) &= 1 \cdot \Pr(|(1-\lambda)u + \lambda y_1| > h) \\ &\quad + \int_{\{|(1-\lambda)u+\lambda y| \le h\}} (1 + L((1-\lambda)u + \lambda y)) f(y)\, dy \\ &= 1 + \frac{1}{\lambda} \int_{-h}^{h} L(y) f\left(\frac{y-(1-\lambda)u}{\lambda}\right) dy. \qquad (2.1) \end{aligned}$$

This integral equation for $L(\cdot)$ is a Fredholm integral equation of the second kind, and $L(\cdot)$ can be approximated numerically.

3. EXPECTATIONS OF GENERAL FUNCTIONS OF RUN LENGTH

To study the run-length distribution of the EWMA chart in greater detail, it is possible to use an integral equation more general than (2.1). I proceed to derive this equation.

Let N stand for the run length associated with the EWMA procedure described earlier, and let $\Pr(n, u) = \Pr(N = n$ given that the EWMA starts at $Q_0 = u)$. For the run length N of an EWMA beginning at $Q_0 = u$ to equal n, the first observation y_1 must satisfy $|(1-\lambda)u + \lambda y_1| \le h$, and the EWMA continuing from $(1-\lambda)u + \lambda y_1$ must have further run length $n-1$ for every such y_1. Therefore, we have for $n \ge 2$ the equation

$$\begin{aligned} \Pr(n, u) &= \int_{\{|(1-\lambda)u+\lambda y| \le h\}} \Pr(n-1, (1-\lambda)u + \lambda y) f(y)\, dy \\ &= \frac{1}{\lambda} \int_{-h}^{h} \Pr(n-1, y) f\left(\frac{y-(1-\lambda)u}{\lambda}\right) dy, \qquad (3.1) \end{aligned}$$

where $f(\cdot)$ is the density of y_1.

Now for a function g, define

$$G_l(u) = \sum_{n=1}^{\infty} g(n+l) \Pr(n, u).$$

$G_l(u)$ is clearly the mean of $g(N + l)$ given a start at u. Then

$$\begin{aligned} G_0(u) &= \sum_{n=1}^{\infty} g(n) \Pr(n, u) \\ &= g(1) \Pr(1, u) + \sum_{n=2}^{\infty} g(n) \Pr(n, u). \end{aligned}$$

Using (3.1),

$$\begin{aligned} G_0(u) &= g(1) \cdot \Pr(1, u) \\ &\quad + \frac{1}{\lambda} \int_{-h}^{h} \sum_{n=2}^{\infty} g(n) \Pr(n-1, y) f\left(\frac{y-(1-\lambda)u}{\lambda}\right) dy \\ &= g(1) \cdot \Pr(1, u) + \frac{1}{\lambda} \int_{-h}^{h} G_1(y) f\left(\frac{y-(1-\lambda)u}{\lambda}\right) dy \\ &= g(1)\left(1 - F\left(\frac{h-(1-\lambda)u}{\lambda}\right) + F\left(\frac{-h-(1-\lambda)u}{\lambda}\right)\right) \\ &\quad + \frac{1}{\lambda} \int_{-h}^{h} G_1(y) f\left(\frac{y-(1-\lambda)u}{\lambda}\right) dy, \qquad (3.2) \end{aligned}$$

where F is the cdf of y_1. Equation (3.2) can be used to obtain integral equations for the mean of many useful functions of N. For example, taking $g(x) = x$ yields an expression for the mean run length, as in (2.1). Setting $g(x) = x^2$ produces an equation for the second moment of the run length. From (3.2) we have that

$$\begin{aligned} V(u) &= 1 + \frac{2}{\lambda} \int_{-h}^{h} L(y) f\left(\frac{y-(1-\lambda)u}{\lambda}\right) dy \\ &\quad + \frac{1}{\lambda} \int_{-h}^{h} V(y) f\left(\frac{y-(1-\lambda)u}{\lambda}\right) dy, \qquad (3.3) \end{aligned}$$

where $L(u)$ and $V(u)$ are the first and second moments, respectively, of the run length given a start at u. Once we have an approximation for $L(u)$ from (2.1), we can use the preceding expression to approximate $V(u)$.

Setting $g(x) = e^{tx}$ yields an equation for the moment-generating function (MGF) of N. Again, using (3.2), we have that

$$\begin{aligned} &e^{-t} \operatorname{MGF}(t, u) \\ &\quad = 1 - F\left(\frac{h-(1-\lambda)u}{\lambda}\right) + F\left(\frac{-h-(1-\lambda)u}{\lambda}\right) \\ &\qquad + \frac{1}{\lambda} \int_{-h}^{h} \operatorname{MGF}(t, y) f\left(\frac{y-(1-\lambda)u}{\lambda}\right) dy, \qquad (3.4) \end{aligned}$$

where $\operatorname{MGF}(t, u)$ is the moment-generating function of the run length given a start at u. So it is possible to study the run-length distribution in detail by applying (3.1) and (3.2) and standard numerical integration methods.

4. NORMAL DISTRIBUTION RESULTS

Values of $L(0)$, the ARL for the EWMA control scheme starting at 0, were approximated numerically and appear in Table 1. For large t, under an iid model, the variance of Q_t, the EWMA at observation t, is approximately

$$\sigma_Q^2 \equiv \mathrm{var}(Q_t) = (\lambda/(2-\lambda))(\sigma^2/n),$$

where the y_t's are sample means generating the EWMA and are distributed as $N(\mu, \sigma^2/n)$ random variables. Control limits for the EWMA chart are specified in terms of σ_Q, taken to be $\pm L\sigma_Q$. Thus a particular $L(0)$ value in Table 1 depends on the true mean μ, the weighting constant λ, and the constant L. The true mean is expressed in units of $\sigma/\sqrt{n}$, the standard deviation of the sample mean.

The solutions to the integral equations (2.1) and (3.3) were obtained by replacing the equations with a system of linear algebraic equations and solving them numerically using Gaussian quadrature. See Baker (1977) for more details. This approach was used to approximate the first and second moments of the run length given a start at 0. For a given (μ, λ, L) combination, the approximated ARL value converged as the number of quadrature points increased from 6 to 24. Increasing the number of points above 24 did not change the approximation appreciably over the range of parameter values considered.

The values appearing in Table 1 are consistent with the simulation results given by Roberts (1959) and the numerical approximations obtained in an entirely different way by Robinson and Ho (1978). Within the practical range of ARL's tabled, the results given here are comparable to those given by Robinson and Ho (1978). Differences that exist are partly because the values tabled here are $L(0)$ values, ARL's of EWMA procedures starting at 0. Robinson and Ho (1978) considered only an EWMA with a random start, that is, they tabled $E\{L(u)\}$, where u is chosen at random from the controlled steady-state distribution of an AR(1) process. For the in-control case ($\mu = 0$), $L(u)$ is maximized at $u = 0$, so it is to be expected that the in-control ARL's given here are larger than those given by Robinson and Ho (1978).

Values of $S(0) = (V(0) - L^2(0))^{1/2}$, the standard deviation of run length (SDRL) for the EWMA scheme starting at 0, were approximated numerically using Equations (2.1) and (3.3), they appear in Table 2. Notice that for large ARL values, the corresponding SDRL value is approximately equal to the ARL. For the $\lambda = 1$ case, of course, this must be true, since the ARL is p^{-1} and the SDRL is $(1-p)^{1/2}p^{-1}$, where p is the probability of a single y_t falling outside the "in control" region. In fact, numerical evaluation of the moment-generating function (3.4) about $t = 0$ suggests that more generally the run-length distribution for the in-control large ARL cases is approximately geometric with parameter $p = 1/L(0)$.

5. GENERAL CONSIDERATIONS IN THE DESIGN OF EWMA CHARTS

Various criteria for designing a quality control chart for monitoring a process mean have been suggested. Page (1961) recommended designing CUSUM charts to have specified ARL values at $\mu = 0$, the target mean, and at $\mu = \mu_1$, where μ_1 is the smallest shift in the mean considered important enough to be detected quickly. Robinson and Ho (1978) made the same recommendation for constructing an EWMA chart and outlined a three-step design procedure. Table 1 presented here can be used in a similar way to construct EWMA charts with specified ARL's at $\mu = 0$ and at $\mu = \mu_1$, some deviation from target.

Woodall (1985) proposed a strategy for designing a CUSUM chart that applies equally well to the design of EWMA charts. His approach was to first specify a region of acceptable values for the process mean. A control chart was then designed to have specified ARL values at two particular shifts in the underlying process mean. Table 1 presented here can be used to construct EWMA charts by specifying the ARL at $\mu = 0$ and choosing the chart based on its ARL profile over the out-of-control region of greatest interest.

Reported design and analyses of EWMA control schemes have generally been based on ARL considerations, such as the preceding. More than ARL's may be needed, however. In many cases, the EWMA designer may be concerned with the probability of early false out-of-control signals for a given EWMA control scheme. For example, from Table 1, an EWMA chart with parameters $(\lambda, L) = (.50, 2.75)$ has an in-control ARL of approximately 184.56. An application of Equation (3.1) to the in-control case shows that the EWMA scheme signals within the first 10 observations with probability of approximately .05. For larger ARL cases, the fact that the run-length distribution is approximately geometric can be used to calculate such probabilities. Setting the in-control ARL at a desired level may not ensure that the probability of an early false signal is acceptable. This property could, in practice, be cause for concern. I recommend, once ARL considerations have led to a particular EWMA scheme, that the designer include an analysis of the probability of early false signals considered unacceptable in practice.

Table 1. Average Run Lengths for Two-Sided Exponentially Weighted Moving Average Charts

	λ						λ					
$\sqrt{n}\mu/\sigma$	1.0	.75	.50	.25	.10	.05	1.0	.75	.50	.25	.10	.05
			$L = 2.0$						$L = 2.25$			
.00	21.98	22.88	26.45	38.56	73.28	127.53	40.90	42.25	47.78	67.46	125.10	215.39
.25	19.13	18.86	20.12	24.83	34.49	43.94	34.53	33.07	33.48	37.86	47.50	56.78
.50	13.70	12.34	11.89	12.74	15.53	18.97	23.23	19.78	17.62	17.03	19.12	22.52
.75	9.21	7.86	7.29	7.62	9.36	11.64	14.67	11.64	9.93	9.49	11.02	13.44
1.00	6.25	5.26	4.91	5.24	6.62	8.38	9.41	7.31	6.30	6.27	7.63	9.55
1.25	4.40	3.76	3.95	3.59	5.13	6.56	6.29	4.95	4.42	4.62	5.84	7.42
1.50	3.24	2.84	2.80	3.19	4.20	5.41	4.41	3.58	3.34	3.66	4.74	6.08
1.75	2.49	2.26	2.29	2.68	3.57	4.62	3.24	2.75	2.68	3.04	4.01	5.17
2.00	2.00	1.88	1.95	2.32	3.12	4.04	2.49	2.21	2.23	2.61	3.48	4.51
2.25	1.67	1.61	1.70	2.06	2.78	3.61	2.00	1.85	1.92	2.30	3.09	4.02
2.50	1.45	1.42	1.51	1.85	2.52	3.26	1.67	1.60	1.69	2.07	2.79	3.63
2.75	1.29	1.29	1.37	1.69	2.32	2.99	1.45	1.42	1.51	1.88	2.55	3.32
3.00	1.19	1.19	1.26	1.55	2.16	2.76	1.29	1.29	1.38	1.73	2.36	3.06
3.25	1.12	1.13	1.18	1.43	2.03	2.56	1.19	1.19	1.27	1.59	2.21	2.85
3.50	1.07	1.08	1.12	1.32	1.93	2.39	1.12	1.13	1.19	1.48	2.09	2.66
3.75	1.04	1.05	1.08	1.24	1.83	2.26	1.07	1.08	1.13	1.37	1.99	2.49
4.00	1.02	1.03	1.05	1.17	1.73	2.15	1.04	1.05	1.08	1.28	1.91	2.34
			$L = 2.5$						$L = 2.75$			
.00	80.52	82.49	91.17	124.18	223.35	379.40	167.80	170.64	184.56	242.20	420.78	702.19
.25	65.77	61.07	58.33	59.66	66.59	73.98	132.28	119.08	107.13	98.32	96.17	98.23
.50	41.49	33.26	27.16	23.28	23.63	26.64	78.12	59.04	43.97	32.89	29.50	31.50
.75	24.61	18.05	13.96	11.96	12.95	15.41	43.51	29.47	20.44	15.34	15.20	17.56
1.00	14.92	10.57	8.27	7.52	8.75	10.79	24.91	16.03	11.19	9.11	9.99	12.11
1.25	9.46	6.75	5.52	5.39	6.60	8.31	14.96	9.58	7.04	6.29	7.42	9.25
1.50	6.30	4.65	4.03	4.18	5.31	6.78	9.46	6.24	4.92	4.78	5.92	7.51
1.75	4.41	3.43	3.14	3.43	4.46	5.75	6.30	4.39	3.72	3.86	4.94	6.34
2.00	3.24	2.67	2.57	2.92	3.86	5.00	4.41	3.28	2.98	3.25	4.26	5.49
2.25	2.49	2.17	2.18	2.56	3.42	4.43	3.24	2.59	2.49	2.82	3.75	4.86
2.50	2.00	1.83	1.90	2.29	3.07	4.00	2.49	2.13	2.14	2.51	3.37	4.37
2.75	1.67	1.59	1.69	2.08	2.80	3.64	2.00	1.81	1.88	2.27	3.06	3.98
3.00	1.45	1.41	1.52	1.91	2.57	3.36	1.67	1.58	1.68	2.09	2.81	3.66
3.25	1.29	1.29	1.39	1.77	2.39	3.12	1.45	1.41	1.53	1.93	2.60	3.39
3.50	1.19	1.19	1.28	1.64	2.24	2.92	1.29	1.29	1.40	1.80	2.42	3.17
3.75	1.12	1.13	1.20	1.52	2.13	2.74	1.19	1.20	1.29	1.69	2.27	2.99
4.00	1.07	1.08	1.13	1.42	2.04	2.58	1.12	1.13	1.21	1.57	2.16	2.82

6. EXAMPLE

Suppose a two-sided EWMA chart is to be designed for controlling a process mean such that the chart will yield an ARL of 370 when the process is in control. Choice of λ and L will depend on the magnitude of shift, which should be detected quickly. Table 3 gives a numerical comparison of several EWMA charts, including in-control and out-of-control ARL's.

In Table 3, each of the (λ, L) combinations (.05, 2.50), (.12, 2.75), and (1.0, 3.00) yields an in-control ARL of approximately 370. These combinations can be found either by interpolation in Table 1 or, more accurately, by a numerical search using (2.1) and starting values from Table 1.

Clearly, if small shifts are of primary concern, the best combination in Table 3 is $(\lambda, L) = (.05, 2.50)$. If moderate shifts are of greatest concern, $(\lambda, L) = (.12, 2.75)$ is preferable, and if only large shifts are of concern, $(\lambda, L) = (1.0, 3.00)$ yields the smallest ARL's over the region of interest. Recall that $(1 - \lambda)$ is the weight given to the past history and λ is the weight given to the most recent observation. So for detecting increasingly large shifts, λ increases and more weight is given to the current data. Note that the usual Shewhart X-bar chart corresponds to $\lambda = 1$.

Although many (λ, L) combinations yield an in-control ARL of 370, Table 3 illustrates how comparisons can be made. In practice, Table 1 can be used to make rough comparisons of (λ, L) combinations for λ between .05 and 1.0 and L between 2 and 4. Table 2

Table 1 (continued)

	λ						λ					
$\sqrt{n}\mu/\sigma$	1.0	.75	.50	.25	.10	.05	1.0	.75	.50	.25	.10	.05
			$L = 3.0$						$L = 3.5$			
.00	370.40	374.50	397.56	502.90	842.15	1,379.35	2,149.34	2,157.99	2,227.34	2,640.16	4,106.29	6,464.60
.25	281.15	245.76	208.54	171.09	144.74	134.86	1,502.76	1,245.90	951.18	624.78	385.49	281.09
.50	155.22	110.95	75.35	48.45	37.41	37.37	723.81	468.68	267.36	123.43	64.72	54.56
.75	81.22	50.92	31.46	20.16	17.90	19.95	334.40	182.12	88.70	38.68	25.33	25.63
1.00	43.89	25.64	15.74	11.15	11.38	13.51	160.95	78.05	35.97	17.71	14.79	16.60
1.25	24.96	14.26	9.21	7.39	8.32	10.23	81.80	37.15	17.64	10.48	10.37	12.32
1.50	14.97	8.72	6.11	5.47	6.57	8.26	43.96	19.63	10.19	7.25	8.00	9.84
1.75	9.47	5.80	4.45	4.34	5.45	6.94	24.96	11.46	6.70	5.52	6.54	8.21
2.00	6.30	4.15	3.47	3.62	4.67	6.00	14.97	7.33	4.86	4.47	5.55	7.06
2.25	4.41	3.16	2.84	3.11	4.10	5.30	9.47	5.08	3.78	3.77	4.83	6.21
2.50	3.24	2.52	2.41	2.75	3.67	4.76	6.30	3.76	3.10	3.28	4.29	5.56
2.75	2.49	2.09	2.10	2.47	3.32	4.32	4.41	2.94	2.63	2.91	3.87	5.03
3.00	2.00	1.79	1.87	2.26	3.05	3.97	3.24	2.40	2.30	2.63	3.54	4.60
3.25	1.67	1.57	1.69	2.09	2.82	3.67	2.49	2.03	2.05	2.41	3.26	4.25
3.50	1.45	1.41	1.53	1.95	2.62	3.42	2.00	1.76	1.85	2.23	3.03	3.95
3.75	1.29	1.29	1.41	1.84	2.45	3.22	1.67	1.56	1.69	2.10	2.84	3.69
4.00	1.19	1.20	1.31	1.73	2.30	3.04	1.45	1.40	1.55	1.99	2.66	3.47
			$L = 4.0$									
.00	15,787.2	15,806.3	16,051.3	18,069.9	26,240.4	39,725						
.25	10,090.2	7,984.40	5,576.65	2,998.63	1,330.23	1,058.61						
.50	4,236.81	2,520.47	1,233.62	406.11	130.08	82.07						
.75	1,729.98	833.01	324.04	90.06	37.73	33.42						
1.00	740.64	303.84	104.22	31.65	19.44	19.99						
1.25	335.59	122.96	41.15	15.81	12.90	14.45						
1.50	161.04	55.31	19.74	9.90	9.67	11.44						
1.75	81.80	27.67	11.23	7.11	7.77	9.52						
2.00	43.96	15.36	7.31	5.55	6.51	8.17						
2.25	24.96	9.40	5.27	4.57	5.63	7.16						
2.50	14.97	6.27	4.08	3.90	4.96	6.39						
2.75	9.47	4.51	3.34	3.42	4.45	5.78						
3.00	6.30	3.45	2.84	3.05	4.05	5.28						
3.25	4.41	2.76	2.48	2.77	3.72	4.86						
3.50	3.24	2.30	2.22	2.54	3.44	4.49						
3.75	2.49	1.97	2.01	2.36	3.22	4.18						
4.00	2.00	1.73	1.84	2.21	3.03	3.92						

can be used to approximate and compare second moments for each combination.

Once the choice of (λ, L) has been made, control limits are set at (assuming target 0) $\pm L\sigma_Q$. In general, if μ_0 is the target (nominal) mean, the control limits are set at $\mu_0 \pm L\sigma_Q$.

7. CONCLUSION

For the normal case, Robinson and Ho (1978) presented an iterative numerical technique for approximation of ARL's. Their approach assumes a random start and employs successive Edgeworth expansions for approximating pdf's and cdf's of an AR(1) process. An approximate method using the approximated cdf's yields the ARL values.

The integral-equation approach presented here gives an exact expression for moments of the run length associated with an EWMA scheme. A quadrature rule, requiring far less numerical work than the approach outlined by Robinson and Ho (1978), is then used to approximate the desired moments.

The integral-equation approach also extends easily to distributions that are nonnormal, an important feature that allows use of the approach when studying control procedures for process parameters other than a process mean. Future work will include investigation of using EWMA's to monitor a process variance. It is not obvious that the numerical methods of Robinson and Ho (1978) extend easily to the nonnormal case.

Table 2. Standard Deviation of Run Lengths for Two-Sided Exponentially Weighted Moving Average Charts

	λ						λ					
$\sqrt{n}\mu/\sigma$	1.0	.75	.50	.25	.10	.05	1.0	.75	.50	.25	.10	.05
			$L = 2.0$						$L = 2.25$			
.00	21.47	22.26	25.43	36.45	68.45	118.45	40.40	41.59	46.61	64.98	119.26	204.26
.25	18.62	18.18	18.97	22.44	29.00	33.97	34.03	32.34	32.14	34.94	40.61	44.27
.50	13.19	11.58	10.62	10.28	10.62	11.10	22.73	18.94	16.11	14.06	13.12	13.01
.75	8.69	7.05	6.00	5.35	5.26	5.48	14.16	10.75	8.41	6.78	6.13	6.16
1.00	5.73	4.45	3.66	3.20	3.16	3.34	8.90	6.40	4.85	3.86	3.57	3.69
1.25	3.87	2.95	2.42	2.13	2.15	2.30	5.77	4.05	3.06	2.48	2.38	2.51
1.50	2.69	2.05	1.71	1.53	1.58	1.71	3.88	2.72	2.08	1.74	1.73	1.85
1.75	1.93	1.49	1.28	1.17	1.22	1.33	2.69	1.91	1.51	1.30	1.33	1.44
2.00	1.41	1.13	1.00	.94	.99	1.09	1.93	1.41	1.16	1.03	1.07	1.16
2.25	1.06	.87	.81	.79	.82	.91	1.41	1.07	.93	.84	.89	.97
2.50	.80	.69	.67	.69	.70	.78	1.06	.84	.77	.72	.76	.83
2.75	.62	.55	.57	.62	.60	.70	.80	.67	.65	.64	.66	.72
3.00	.47	.44	.48	.57	.52	.63	.62	.54	.56	.59	.57	.65
3.25	.36	.35	.40	.53	.47	.58	.47	.44	.48	.56	.49	.60
3.50	.28	.28	.33	.48	.44	.52	.36	.35	.41	.53	.43	.57
3.75	.21	.22	.27	.43	.45	.46	.28	.28	.34	.49	.39	.53
4.00	.15	.16	.21	.37	.47	.38	.21	.22	.28	.45	.39	.49
			$L = 2.5$						$L = 2.75$			
.00	80.02	81.79	89.85	121.33	216.45	366.08	167.30	169.90	183.10	238.97	412.83	696.42
.25	65.27	60.27	56.77	56.19	58.19	58.69	131.78	118.22	105.37	94.30	86.19	79.91
.50	40.99	32.34	25.42	19.76	16.46	15.34	77.61	58.03	42.00	28.80	21.05	18.25
.75	24.10	17.06	12.21	8.77	7.20	6.93	43.01	28.39	18.46	11.63	8.53	7.81
1.00	14.41	9.57	6.60	4.72	4.05	4.05	24.40	14.93	9.29	5.87	4.60	4.45
1.25	8.94	5.76	3.96	2.92	2.64	2.72	14.45	8.49	5.26	3.47	2.93	2.95
1.50	5.78	3.69	2.59	1.99	1.89	1.99	8.95	5.18	3.28	2.30	2.06	2.14
1.75	3.88	2.51	1.81	1.46	1.44	1.54	5.78	3.37	2.21	1.65	1.56	1.64
2.00	2.70	1.79	1.35	1.13	1.15	1.24	3.88	2.32	1.59	1.25	1.23	1.32
2.25	1.93	1.33	1.06	.91	.95	1.03	2.70	1.67	1.21	1.00	1.01	1.09
2.50	1.41	1.03	.86	.76	.81	.88	1.93	1.26	.97	.82	.86	.93
2.75	1.06	.81	.73	.66	.71	.76	1.41	.98	.81	.69	.75	.81
3.00	.80	.66	.63	.60	.63	.67	1.06	.79	.69	.61	.67	.71
3.25	.62	.53	.55	.56	.55	.60	.80	.64	.61	.55	.60	.63
3.50	.47	.44	.48	.54	.48	.56	.62	.53	.54	.53	.54	.57
3.75	.36	.35	.41	.53	.41	.54	.47	.43	.48	.52	.47	.52
4.00	.28	.28	.35	.50	.35	.53	.36	.35	.42	.52	.40	.51
			$L = 3.0$						$L = 3.5$			
.00	369.90	373.71	395.86	499.32	833.19	1,623.03	2,148.84	2,157.11	2,225.51	2,635.96	4,128.46	6,379.19
.25	280.64	244.84	206.61	166.53	133.15	113.09	1,502.26	1,244.85	948.93	620.27	370.78	323.77
.50	154.72	109.87	73.18	43.78	27.59	21.99	723.31	467.46	264.81	117.62	51.87	33.74
.75	80.71	49.76	29.25	15.92	10.21	8.82	333.90	180.80	86.06	33.28	15.36	11.49
1.00	43.39	24.44	13.60	7.45	5.25	4.89	160.45	76.67	33.38	12.98	6.99	5.95
1.25	24.45	13.07	7.22	4.19	3.25	3.19	81.29	35.76	15.18	6.40	4.05	3.75
1.50	14.46	7.56	4.27	2.67	2.25	2.29	43.45	18.25	7.90	3.74	2.70	2.63
1.75	8.95	4.68	2.76	1.87	1.68	1.75	24.46	10.12	4.60	2.46	1.96	1.99
2.00	5.78	3.09	1.92	1.40	1.32	1.40	14.46	6.04	2.94	1.76	1.51	1.57
2.25	3.88	2.15	1.41	1.10	1.08	1.15	8.95	3.86	2.02	1.34	1.22	1.27
2.50	2.70	1.57	1.10	.89	.90	.98	5.78	2.61	1.48	1.06	1.01	1.06
2.75	1.93	1.19	.90	.74	.78	.85	3.88	1.86	1.14	.88	.86	.91
3.00	1.41	.94	.76	.63	.69	.75	2.70	1.39	.92	.74	.75	.81
3.25	1.06	.76	.66	.56	.63	.67	1.93	1.08	.77	.63	.66	.73
3.50	.80	.63	.60	.51	.58	.60	1.41	.87	.68	.54	.60	.67
3.75	.62	.52	.54	.49	.54	.54	1.06	.72	.61	.47	.57	.62
4.00	.47	.43	.48	.50	.48	.49	.80	.60	.57	.43	.55	.56

Table 2 (continued)

$\sqrt{n}\mu/\sigma$	λ = 1.0	.75	.50	.25	.10	.05
			$L = 4.0$			
.00	15,786.7	15,805.3	16,049.3	18,065.2	26,171	39,200
.25	10,089.7	7,983.24	5,574.16	2,992.37	1,369.6	841.3
.50	4,236.31	2,519.13	1,230.79	399.32	113.94	60.88
.75	1,729.48	831.55	321.06	83.50	25.09	15.67
1.00	740.14	302.31	101.21	25.77	9.70	7.56
1.25	335.09	121.39	38.23	10.73	5.14	4.57
1.50	160.54	53.74	16.99	5.56	3.26	3.10
1.75	81.30	26.12	8.68	3.36	2.30	2.28
2.00	43.45	13.85	4.98	2.27	1.73	1.78
2.25	24.46	7.95	3.14	1.65	1.38	1.43
2.50	14.46	4.90	2.14	1.28	1.13	1.15
2.75	8.95	3.21	1.55	1.03	.96	.94
3.00	5.78	2.23	1.18	.86	.83	.80
3.25	3.88	1.63	.94	.74	.73	.74
3.50	2.70	1.24	.78	.64	.64	.71
3.75	1.93	.99	.68	.55	.57	.68
4.00	1.41	.81	.61	.48	.53	.64

The approach taken here for studying the EWMA chart parallels the approach taken by Page (1954), Goel and Wu (1971), Lucas (1976), and others for studying the CUSUM chart. A one-sided version of the EWMA chart with restarts can easily be studied using this same approach. And run-length properties for one-sided EWMA's assuming a head start, an idea suggested by Lucas and Crosier (1982) for CUSUM's, can also easily be determined using the approach given here.

Table 3. Values of λ and L for Two-Sided EWMA Control Charts With ARL 370 When the Process Is in Control and Values of ARL When the Process Is Out of Control

$\sqrt{n}\mu/\sigma$	$L = 2.50$, $\lambda = .05$	$L = 2.75$, $\lambda = .12$	$L = 3.00$, $\lambda = 1.0$
.00	370	370	370
.25	74.0*	96.2	281
.50	26.6*	29.6	155
.75	15.4	14.9*	81.2
1.00	10.8	9.6*	43.9
1.25	8.3	7.1*	25.0
1.50	6.8	5.6*	15.0
1.75	5.7	4.7*	9.5
2.00	5.0	4.0*	6.3
2.25	4.4	3.5*	4.4
2.50	4.0	3.2*	3.2
2.75	3.6	2.9	2.5*
3.00	3.4	2.6	2.0*
3.25	3.1	2.4	1.7*
3.50	2.9	2.3	1.4*
3.75	2.7	2.2	1.3*
4.00	2.6	2.1	1.2*

*Minimum in the row.

A computer program to evaluate ARL's using (2.1) was given by Crowder (1987).

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