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$\bar{X}$ Charts With Variable Sampling Intervals

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The usual practice in using a control chart to monitor a process is to take samples from the process with fixed sampling intervals. This article considers the properties of the $\bar{X}$ chart when the sampling interval between each pair of samples is not fixed but rather depends on what is observed in the first sample. The idea is that the time interval until the next sample should be short if a sample shows some indication of a change in the process and long if there is no indication of a change. The proposed variable sampling interval (VSI) $\bar{X}$ chart uses a short sampling interval if $\bar{X}$ is close to but not actually outside the control limits and a long sampling interval if $\bar{X}$ is close to target. If $\bar{X}$ is actually outside the control limits, then the chart signals in the same way as the standard fixed sampling interval (FSI) $\bar{X}$ chart. Properties such as the average time to signal and the average number of samples to signal are evaluated. Comparisons between the FSI and the VSI $\bar{X}$ charts indicate that the VSI chart is substantially more efficient.

KEY WORDS: Control chart; Process control; Quality control; Variable time delay.

1. INTRODUCTION

Shewhart control charts are widely used to display sample data from a process for purposes of determining whether a process is in control, for bringing an out-of-control process into control, and for monitoring a process to make sure that it stays in control. A control chart is maintained by taking samples from a process and plotting in time order on the chart some statistic computed from the samples. Control limits on the chart represent the limits within which the plotted points would fall with high probability if the process is operating in control. A point outside the control limits is taken as an indication that something, sometimes called an *assignable cause* of variation, has happened to change the process. When the chart signals that an assignable cause is present, then rectifying action is taken to remove the assignable cause and bring the process back into control.

The usual practice in maintaining a control chart is to take samples from the process at fixed-length sampling intervals, say every hour. This article investigates the modification of the standard practice in which the sampling interval or time between samples is not fixed but instead can vary depending on what is observed from the data. The idea of using a *variable sampling interval* (VSI) control chart is intuitively reasonable. If a sample point falls close to a control limit (but not actually outside where a signal is produced), then one would naturally wonder whether subsequent points may actually be outside the control limits. Since it is important to quickly detect changes in the process, the natural inclination in this situation would be to take another sample quickly, say in 10 minutes, rather than wait the usual hour for the next sample. In this way the suspicion about the point close to the control limit could be more quickly confirmed or denied. On the other hand, if the current point is close to the target so that there is no indication of trouble, it might be reasonable to wait longer than the usual hour for the next sample. Thus the proposed control procedure is to let the time until the next sample depend on what is observed in the current sample. The time will be short if there is some indication that there may be a problem and longer if there is no indication of a problem. If the indication of a problem is strong enough, then a signal will be produced at the current sample in the same way as in the usual *fixed sampling interval* (FSI) chart.

This article investigates the situation in which the distribution of observations from the process is normal and the objective is to control the process mean by using the sample means in an $\bar{X}$ chart. The sampling interval on the $\bar{X}$ chart has undoubtedly been varied on an ad hoc basis by control-chart users. For example, a short interval is sometimes used as a check following an adjustment; however, we were unable to find any development of the properties of a VSI $\bar{X}$ chart in the quality control literature. The problem of determining a sampling plan with variable time intervals between samples was investigated by Arnold (1970), Crigler (1973), Smeach and Jernigan (1977), and Crigler and Arnold (1979, 1986), but they did not explicitly consider the problem of controlling a process by using a control chart to signal when there appears to be a change in the process. Hui (1980) and Hui and Jensen (1980) extended the model of Arnold (1970) to allow for the control procedure to signal, but the properties of the procedure when the process is out of control were not explicitly considered. Theoretical aspects of the problem of variable sampling intervals in controlling a general parameter using control charts were investigated by Reynolds (1986) and Reynolds and Arnold (1986), who gave additional references to related problems and approaches. This article concentrates on details for the application of these ideas to the $\bar{X}$ chart.

2. DESCRIPTION OF THE VSI CHART

Consider the situation in which the distribution of observations from the process is normal with mean μ and known variance σ^2, where μ_0 denotes the target value for the mean. Assume that random samples of size n are taken at each sampling point, and let $\mathbf{X}_i = (X_{i1}, X_{i2}, \ldots, X_{in})$ represent the sample taken at the ith point. When the ith sample is taken, the sample mean $\bar{X}_i$ is computed and plotted on a control chart with centerline μ_0 and control limits $\mu_0 \pm \gamma\sigma/\sqrt{n}$, where γ is frequently taken to be 3. If $\bar{X}_i$ falls outside the control limits, then a signal is given. In practice it will frequently be necessary to estimate μ_0 and σ^2 from past data, but for simplicity it will be assumed here that μ_0 and σ^2 are known. The proposed VSI chart can be used with estimated values of μ_0 and σ^2, although the properties of the resulting chart will be different from the case in which μ_0 and σ^2 are known.

In the standard $\bar{X}$ chart, the length of the time interval between samples is fixed, but in the proposed VSI chart, the interval between samples $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ will depend on the value of $\bar{X}_i$. In this article we

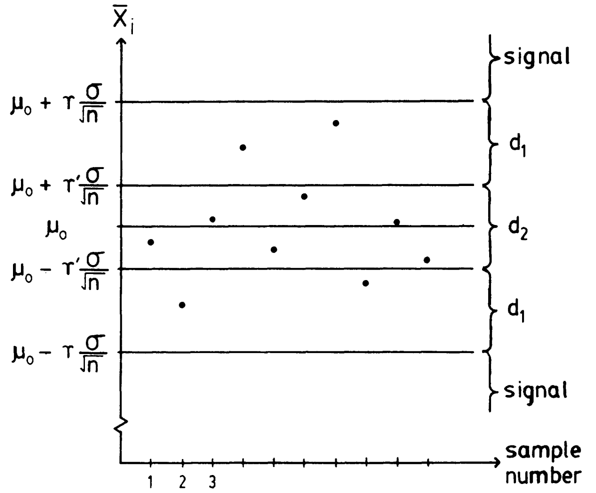

Figure 1. $\bar{X}$ Chart With Variable Sampling Intervals.

assume that the VSI chart uses a finite number of interval lengths $d_1, d_2, \ldots, d_\eta$, where $d_1 < d_2 < \cdots < d_\eta$, and these possible interval lengths must be chosen to satisfy $l_1 \le d_i \le l_2$. The minimum possible interval length $l_1 > 0$ might be determined by physical considerations such as the time required to take a sample. The maximum interval length l_2 might be determined by the maximum amount of time that process engineers are willing to allow the process to run without sampling. The choice of a sampling interval as a function of $\bar{X}_i$ can be represented by a sampling interval function $d(x)$, which specifies the sampling interval to be used when $\bar{X}_i = x$ is observed. Let the interval $(\mu_0 - \gamma\sigma/\sqrt{n},\ \mu_0 + \gamma\sigma/\sqrt{n})$ be partitioned into η regions $I_1, I_2, \ldots, I_\eta$ such that

$$d(x) = d_j, \qquad x \in I_j.$$

Thus the sampling interval used between $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ is $d(\bar{X}_i)$. More general sampling interval functions were considered by Reynolds (1986) and Reynolds and Arnold (1986).

An example of a VSI chart is shown in Figure 1. This chart uses two interval lengths, d_1 and d_2, with

$$I_1 = (\mu_0 - \gamma(\sigma/\sqrt{n}),\ \mu_0 - \gamma'(\sigma/\sqrt{n})] \cup [\mu_0 + \gamma'(\sigma/\sqrt{n}),\ \mu_0 + \gamma(\sigma/\sqrt{n}))$$

and

$$I_2 = (\mu_0 - \gamma'(\sigma/\sqrt{n}),\ \mu_0 + \gamma'(\sigma/\sqrt{n})),$$

where $0 < \gamma' < \gamma$. The way that the VSI chart works to improve the detection ability of the $\bar{X}$ chart can be explained with reference to Figure 1. Suppose that in Figure 1 $\gamma = 3$ and $\gamma' = 1$. Then when $\mu = \mu_0$, $\Pr(\bar{X} \in I_1) = .3146$ and $\Pr(\bar{X} \in I_2) = .6827$, so the longer interval will be used approximately twice as often as the shorter interval. Now suppose that μ shifts to $\mu_1 = \mu_0 + 2\sigma/\sqrt{n}$. In this case $\Pr(\bar{X} \in I_1) = .6840$ and $\Pr(\bar{X} \in I_2) = .1573$, so the shorter interval will be used much more often than the longer interval. By using the shorter interval more often when μ shifts, the frequency of sampling is increased and the time required to obtain a sample mean outside the control limits (to produce a signal) is substantially reduced.

The sample means in the chart in Figure 1 are plotted against the sample number. In practical applications of the chart, it would be necessary to record on the chart the times that the samples are taken, since the constant interval between the points on the chart disguises the fact that the actual time intervals between samples are not the same. For example, as the points are plotted, the interval between samples 1 and 2 is d_2, whereas the interval between samples 2 and 3 is d_1. An alternate way to construct a chart would involve plotting the sample means against time on the horizontal axis. Then the intervals between points on the chart would vary corresponding to the actual time intervals between samples. This alternate construction would be relatively easy to implement by hand if $d_2 = cd_1$, where c is a small positive integer. If d_2 is very much larger than d_1, then large blank spaces will appear on the chart whenever d_2 is used.

3. PROPERTIES OF THE VSI CHART

The properties of a control chart are determined by the length of time it takes the chart to produce a signal. If the process is in control, then this time should be long so that the rate of false alarms is low, but if the process mean shifts, then the time from the shift to the signal should be short so that detection is quick. The number of samples before a signal is usually called the run length in the quality-control literature, and the expected number of samples is called the *average run length* (ARL). With a fixed interval between samples, the ARL can be easily converted to the expected time to signal by multiplying by the fixed interval length, so the ARL can be thought of as the expected time to signal. In addition, the sampling rate will be constant regardless of the value of the process mean μ. With a VSI chart, however, the time to signal is not a constant multiple of the number of samples to signal, and the sampling rate depends on the value of μ. Thus, for the VSI chart, it is necessary to keep track of both the number of samples to signal and the time to signal. Since the traditional meaning of the ARL relates to both number of samples and time, it seems preferable to define new quantities for use with the VSI chart.

Define the *number of samples to signal* as the number of samples taken from the start of the process to the time the chart signals, and let the *average number of samples to signal* (ANSS) be the expected value of the number of samples to signal. Define the *time to signal* as the time from the start of the process to the time when the chart signals, and similarly let the *average time to signal* (ATS) be the expected value of the time to signal. Both the ANSS and the ATS would, of course, be functions of the process mean μ. When $\mu = \mu_0$ the ATS should be large, and when μ shifts from μ_0 the ATS should be small. To avoid large sampling costs the ANSS should not be excessively large compared with the ATS, particularly when $\mu = \mu_0$, since the process will presumably operate with $\mu = \mu_0$ most of the time.

As long as the control limits remain fixed, the VSI feature will have no effect on the probability that $\bar{X}$ falls outside the control limits. Thus the probability that $\bar{X}$ falls outside the control limits will be

$$q = \Pr(\bar{X} \le \mu_0 - \gamma\sigma/\sqrt{n} \text{ or } \bar{X} \ge \mu_0 + \gamma\sigma/\sqrt{n}),$$

regardless of the interval between samples. If N = number of samples to signal, then N has a geometric distribution with parameter q when the process does not change. Thus the ANSS is

$$E(N) = 1/q, \tag{3.1}$$

and the variance of N is $\text{var}(N) = (1 - q)/q^2$. If T = time to signal, and R_i = sampling interval used before the ith sample is taken, then $T = \sum_{i=1}^{N} R_i$. Note that the distribution of the R_i's as used here must be the conditional distribution of $d(\bar{X})$ given that there is no signal, since $d(\bar{X})$ was defined only for those values of $\bar{X}$ between the control limits. For simplicity we assume that the chart is started at time 0 and that R_1, the interval used before the first sample, is determined according to a random interval with the same distribution as the other R_i's, although no sample is taken at time 0. In practice it might be preferable to use the shortest interval d_1 at first when the process is just starting to give additional protection against problems that arise during start-up. This type of modification of the chart could be handled easily with a slight increase in the complexity of the resulting expressions. An adjusted ATS that does not require any assumption about how the first sampling interval is determined is developed in Section 5.

If the process mean is constant, then $R_1, R_2, \ldots$ will be iid and, using Wald's identity, the ATS can be written as

$$E(T) = E(N)E(R_i). \tag{3.2}$$

If $p_j = \Pr(d(\bar{X}) = d_j) = \Pr(\bar{X} \in I_j)$, then

$$E(R_i) = \sum_{j=1}^{\eta} d_j \frac{p_j}{1 - q}, \tag{3.3}$$

where the probabilities are conditional on no signal, since $\sum_{i=1}^{\eta} p_i = 1 - q < 1$. Thus the ATS is

$$E(T) = \sum_{j=1}^{\eta} d_j p_j \Big/ q(1 - q). \tag{3.4}$$

Since the distribution of each R_i is the conditional distribution of $d(\bar{X})$ given that $\bar{X}$ is within the control limits, it follows that $R_1, R_2, \ldots$ are conditionally independent of N and the variance of T can be expressed as

$$\operatorname{var}(T) = E(N)\operatorname{var}(R_i) + \operatorname{var}(N)(E(R_i))^2$$
$$= \frac{\sum_{j=1}^{\eta} d_j^2 p_j}{q(1-q)} + \frac{(1-2q)(\sum_{j=1}^{\eta} d_j p_j)^2}{q^2(1-q)^2}. \quad (3.5)$$

The probabilities q and p_j used previously depend on the value of μ. When it is necessary to distinguish between μ_0 and some other value $\mu_1 (\neq \mu_0)$ that is of interest, the probabilities under μ_0 will be denoted by q_0 and p_{0j} and the probabilities under μ_1 by q_1 and p_{1j}.

In addition to the moments of the number of samples to signal and the time to signal, it may be useful to know the joint distribution of the numbers of times each sampling interval is used before a signal is given. Let N_j = the number of times sampling interval d_j occurs before a signal is given. Then, conditional on the value of N, the joint distribution of $(N_1, N_2, \ldots, N_\eta)$ is multinomial, and

$$\Pr(N_1 = n_1, \ldots, N_\eta = n_\eta)$$
$$= \Pr(N_1 = n_1, \ldots, N_\eta = n_\eta \mid N = n)\Pr(N = n)$$
$$= \frac{n!}{n_1! \cdots n_\eta!}\left(\frac{p_1}{1-q}\right)^{n_1} \cdots \left(\frac{p_\eta}{1-q}\right)^{n_\eta} q(1-q)^{n-1}$$
$$= \frac{q}{1-q}\frac{n!}{n_1! \cdots n_\eta!} p_1^{n_1} \cdots p_\eta^{n_\eta},$$

where $n_j \geq 0$, $j = 1, 2, \ldots, \eta$, and $n = \sum_{j=1}^{\eta} n_j$. From this the marginal distribution of N_j can be shown to be

$$\Pr(N_j = n_j) = \frac{q(1-q-p_j)}{(1-q)(q+p_j)}, \qquad n_j = 0$$
$$= \frac{q p_j^{n_j}}{(1-q)(q+p_j)^{n_j+1}}, \qquad n_j \geq 1.$$

Then $E(N_j) = p_j / q(1-q)$, and the expected number of times that sampling interval d_j is used is proportional to p_j, as would be expected.

In some situations it may be useful to know the actual distribution of the time to signal T. The distribution of T can be derived for certain special cases. Consider the case in which each possible sampling interval is an integer multiple of a constant; that is,

$$d_j = m_j h \quad (3.6)$$

for $j = 1, 2, \ldots, \eta$, where $h > 0$ and m_j is a positive integer. Then the only times when samples can be taken will be integer multiples of h. If $\pi(i) = \Pr$ (a sample is taken at time hi), for $i = 1, 2, \ldots$, then $\pi(i)$ satisfies

$$\pi(i) = \sum_{j=1}^{\eta} \pi(i - m_j)p_j \quad (3.7)$$

for $i = 1, 2, \ldots$, where in addition we define $\pi(i) = 0$ for $i < 0$ and $\pi(0) = 1/(1-q)$. When a sample is taken at a point, the probability of a signal at that point is q and thus

$$\Pr(T = hi) = \pi(i)q. \quad (3.8)$$

Then the ATS can be expressed as

$$E(T) = \sum_{i=1}^{\infty} hi\pi(i)q. \quad (3.9)$$

An explicit expression for $\pi(i)$ can be obtained for the special case when $m_1 = 1$ and $m_2 = m$. In this case

$$\pi(i) = \frac{1}{1-q}\sum_{r=0}^{u}\binom{i-(m-1)r}{r} p_1^{i-mr} p_2^r \quad (3.10)$$

for $i = 1, 2, \ldots$, where u is the integer part of i/m. The expression (3.10) can be obtained by induction, by a combinatorial argument, or by using the probability-generating function of T.

4. COMPARING CONTROL CHARTS

In evaluating the usefulness of the VSI feature, it seems natural to compare the performance of the VSI $\bar{X}$ chart to the same chart using fixed-sampling intervals. If both FSI and VSI charts have control limits at $\mu_0 \pm \gamma\sigma/\sqrt{n}$, then both will have the same value of q and the same ANSS function. This means that the interval used in the VSI chart will have no effect on the number of samples required to signal. It will be convenient to take the interval used by the FSI chart as the unit of time. For example, if the FSI chart takes samples every two hours, then a time unit will be taken as a two-hour period. In this case the numerical value of the ATS function will be the same as the numerical value of the ANSS function for the FSI chart. If the VSI chart is set up so that $E(R_i)$, the expected interval length, is equal to one time unit when $\mu = \mu_0$, then the two charts will have the same ATS functions when $\mu = \mu_0$. Then the VSI chart will be "matched" to the FSI chart in the sense that they both require the same number of samples to signal, and when $\mu = \mu_0$ they both have the same average sampling rate and false-alarm rate over time. Then the values of the ATS functions for the two charts can be compared for various values of μ_1 to determine which chart will do a better job of detecting a change in μ.

If two sampling intervals are used, the requirement that $E(R_i) = 1$ when $\mu = \mu_0$ means that

$$d_1 p_{01} + d_2 p_{02} = 1 - q_0. \quad (4.1)$$

Table 1. Values of the ATS for Matched FSI and VSI Charts for a Representative Number of Intervals

Shift $\sqrt{n}(\mu-\mu_0)/\sigma$	Fixed interval	Variable interval* $\eta = 2$	$\eta = 3$	$\eta = 5$	$\eta = 9$
0	370.40	370.40	370.40	370.40	370.40
.5	155.22	141.43	142.39	142.74	143.69
1.0	43.90	30.60	31.41	31.72	32.55
1.5	14.47	6.95	7.33	7.49	7.92
2.0	6.30	1.82	1.97	2.04	2.23
3.0	2.00	.27	.29	.30	.34
4.0	1.19	.13	.13	.13	.14

* Sampling intervals: Fixed, $d = 1$; $\eta = 2$, $d_i = (.1, 1.9)$; $\eta = 3$, $d_i = (.1, 1, 1.9)$; $\eta = 5$, $d_i = (.1, .5, 1, 1.5, 1.9)$; $\eta = 9$, $d_i = (.1, .3, .5, 1, 1.3, 1.5, 1.7, 1.9)$.

If γ is fixed, then q_0 is fixed, and (4.1) can be satisfied by specifying d_1 and d_2 and allowing these sampling intervals to determine p_{01} and p_{02} and then I_1 and I_2. Alternately I_1 and I_2 can be specified, and then these regions will determine p_{01} and p_{02}, and then d_1 and d_2. The approach taken here for investigating the behavior of the VSI chart will be to specify d_1 and d_2 and to let these sampling intervals determine the probabilities and the regions required to match the FSI chart. Then for given values of γ and $0 < d_1 < 1 < d_2$, p_{01} and p_{02} must satisfy

$$p_{01} = \frac{d_2 - 1}{d_2 - d_1}(1 - q_0), \qquad p_{02} = \frac{1 - d_1}{d_2 - d_1}(1 - q_0) \tag{4.2}$$

to satisfy (4.1), and (4.2) then determines γ', which determines I_1 and I_2.

The theoretical results of Reynolds (1986) and Reynolds and Arnold (1986) suggest that only two intervals need to be considered ($\eta = 2$) in the VSI control charts. In addition, these results suggest that d_1 and d_2 should be spaced far apart. These results are demonstrated numerically in Tables 1 and 2. Comparisons for $\eta = 2, 3, 5, 9$ are shown in Table 1 for the special case of sampling intervals symmetric about the fixed sampling interval of $d = 1$, and for equal interval probabilities p_{0i} ($i = 1, 2, \ldots, \eta$). All charts have $\gamma = 3$, and the shift in μ is expressed in units of the standard deviation of $\bar{X}$. The ATS values for $\mu > \mu_0$ are uniformly smaller for the VSI chart with $\eta = 2$ and gradually increasing as η increases. A similar pattern exists when comparing the standard deviations of the time to signal for the FSI and VSI charts.

In Table 2, comparisons are given for three symmetric and four asymmetric VSI charts, all using two sampling intervals matched to a FSI chart with $d = 1$ and $\gamma = 3$. As expected, the more widely spaced intervals yield smaller values of the ATS relative to the FSI chart. From an administrative point of view, it is fortunate that only two sampling intervals perform best.

Note that the proportionate reduction in the ATS achieved by the VSI technique is determined by the value of $E(R_i)$, since $E(N)$ in (3.2) is the same for both the FSI and VSI charts for all μ. The FSI chart had $E(R_i) = 1$ for all μ and a value of, say, $E(R_i) = .6$ for the VSI chart corresponds to an ATS for the VSI chart of 60% of the value of the ATS for the FSI chart.

The numerical comparisons given previously assume a shift in the mean at the beginning of the process. Although there are situations in which this is realistic, we believe that in most practical applications a process would shift at some random point in time between samples. A proposed model for this situation, along with comparisons of its effect on the VSI chart, is given in Section 5.

5. ADJUSTED TIME TO SIGNAL

When $\mu = \mu_1 \neq \mu_0$, the ATS is a measure of how long the chart will take to detect this deviation from target. The expressions for the ATS developed in the previous section give the ATS for any μ_1 under the

Table 2. Values of the ATS for Matched FSI and VSI Charts for Representative Interval Widths

Shift $\sqrt{n}(\mu-\mu_0)/\sigma$	Fixed interval $= 1$	Variable interval (d_1, d_2) Symmetric (.5, 1.5)	(.3, 1.7)	(.1, 1.9)	Asymmetric (.1, 1.1)	(.1, 1.3)	(.1, 1.5)	(.1, 4.0)
0	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
.5	155.22	147.56	144.49	141.43	149.11	145.03	143.17	139.53
1.0	43.90	36.51	33.56	30.60	37.30	33.60	32.03	29.15
1.5	14.97	10.51	8.73	6.95	10.36	8.38	7.61	6.31
2.0	6.30	3.81	2.62	1.82	3.30	2.39	2.08	1.59
3.0	2.00	1.04	.66	.27	.54	.35	.30	.25
4.0	1.19	.60	.36	.13	.19	.14	.13	.12
∞	1.00	.50	.30	.10	.10	.10	.10	.10

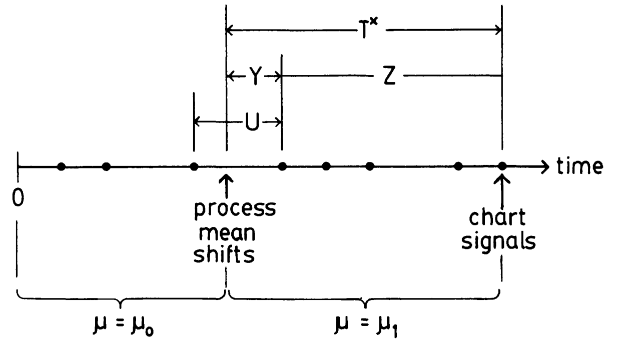

Figure 2. Time Required to Detect a Shift in the Process Mean When the Shift Occurs at a Random Time: •, Points Where Samples Are Taken.

simplifying assumption that this value of μ is the process mean from time 0 onward. But in practice the process may start out with $\mu = \mu_0$ and then shift from μ_0 to μ_1 at some random time in the future. In this case, the detection time that is of interest is the time from the process shift to the point at which the chart signals. For the type of chart considered here, the only difference between these two ways of determining the time to signal is that a shift occurring at a random time in the future may occur in the time interval between two samples.

To develop a model that adjusts the time to signal to allow for a shift to occur between samples, let T^* = adjusted time to signal (time from the process shift until a signal), U = length of the interval in which the shift occurs, Y = time from the process shift until the next sample, Z = time from the next sample after the process shift until a signal, and N = number of samples after the shift until a signal. The relationship between T^*, U, Y, and Z is shown in Figure 2. From Figure 2 it is clear that $T^* = Y + Z$, and that Z has the same distribution as $\sum_{i=1}^{N-1} R_i$, where each R_i has the conditional distribution of $d(\bar{X})$, given no signal, when the process mean has shifted. The ATS that has been adjusted for the shift occurring between samples is

$$\begin{aligned} E(T^*) &= E(Y) + E(Z) \\ &= E(Y) + (E(N) - 1)E(R_i). \qquad (5.1) \end{aligned}$$

The distribution of N is geometric with parameter q as before, and an expression for $E(R_i)$ is given by (3.3). Before proceeding with the development of a model for determining the distribution of Y, it may be useful to discuss the differences between the time to signal defined by T and the adjusted time to signal defined by T^*. The time T^* includes the time Y from the shift to the next sample, where $0 \le Y \le U$, and the distribution of U is determined when the process mean is at μ_0, since the sample immediately before the shift was taken when $\mu = \mu_0$. In contrast, the distribution of T is the same as the distribution of $R + Z$, where R has the same distribution as $d(\bar{X})$ conditional on no signal when $\mu = \mu_1$. This follows from the fact that when the process starts at time 0 with $\mu = \mu_1$ the distribution of the first interval is assumed to be the same as the distribution of future intervals determined when $\mu = \mu_1$. Thus the difference between T and T^* is the difference between R and Y.

The ATS determined by $E(T^*)$ is particularly relevant when a VSI chart has a large value of d_η, and the magnitude of the shift $(\mu_1 - \mu_0)$ is large. For example, suppose that a VSI chart using intervals of 10 minutes and two hours is being compared with a FSI chart using an interval of one hour. The intuitive objection to the VSI chart would be that a large shift in μ could occur early in a two-hour sampling interval and then the process would operate for a relatively long time at the wrong level before the next sample is taken. With the FSI chart, on the other hand, the maximum possible time until the next sample is only one hour. The use of $E(T^*)$ accounts for the possibility of the shift occurring in a long sampling interval when a VSI chart is being used. The difference between $E(T)$ and $E(T^*)$ will be important only when the shift in μ is large, since small or moderate shifts will require a relatively large number of samples for detection, and in this case Y is a relatively small component of T^*. When two FSI charts each using the same interval are being compared, the use of $E(T^*)$ may not be necessary, since both charts have the same probability of a shift between samples.

To actually evaluate $E(T^*)$, a model must be developed so that $E(Y)$ can be determined. The distribution of Y depends on the time that the shift in μ occurs. One approach to developing a model would involve using a "prior" distribution for the time of this shift and then deriving the distribution of Y. Since this approach appears to be quite difficult, an alterate approach is to assume that when the shift falls in a particular sampling interval the position within the interval is uniformly distributed over the interval. In addition, we assume that the probability of the shift falling in an interval of length d_i is proportional to the product of this length and the probability p_{0j} of the occurrence of the interval when the process is in control. This means that

$$f_{Y|U}(y \mid u) = 1/u, \qquad 0 \le y \le u,$$

and

$$\Pr(U = d_j) = d_j p_{0j} \Big/ \sum_{j=1}^{\eta} d_j p_{0j}, \qquad j = 1, 2, \ldots, \eta.$$

Table 3. Values of the Adjusted ATS for Matched FSI and VSI Charts

Shift $\sqrt{n}(\mu-\mu_0)/\sigma$	Fixed interval = 1.0	Variable interval (d_1, d_2): Symmetric (.5, 1.5)	Symmetric (.3, 1.7)	Symmetric (.1, 1.9)	Asymmetric (.1, 1.1)	Asymmetric (.1, 1.3)	Asymmetric (.1, 1.5)	Asymmetric (.1, 4.0)
0	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
.5	154.72	147.23	144.31	141.42	148.69	144.73	142.98	140.48
1.0	43.40	36.30	33.54	30.81	36.99	33.47	32.02	30.34
1.5	14.47	10.44	8.89	7.39	10.21	8.45	7.83	7.74
2.0	5.80	3.83	3.12	2.44	3.33	2.65	2.47	3.19
3.0	1.50	1.15	1.07	1.04	.82	.81	.88	1.97
4.0	.69	.72	.80	.93	.58	.66	.75	1.87
∞	.50	.63	.75	.91	.55	.64	.73	1.85

Then

$$f_Y(y) = \sum_{j=1}^{\eta} f_{Y|U}(y \mid d_j)\Pr(U = d_j)$$

$$= \sum_{\{j:d_j \geq y\}} p_{0j} \Big/ \sum_{j=1}^{\eta} d_j p_{0j}, \qquad 0 \leq y \leq d_\eta,$$

and

$$E(Y) = \int_0^{d_\eta} y f_Y(y)\, dy = \sum_{j=1}^{\eta} d_j^2 p_{0j} \Big/ \left(2 \sum_{j=1}^{\eta} d_j p_{0j}\right).$$

The adjusted ATS when the shift is to $\mu = \mu_1$ is then

$$E(T^*) = \sum_{j=1}^{\eta} d_j^2 p_{0j} \Big/ 2 \sum_{j=1}^{\eta} d_j p_{0j} + \frac{1}{q_1} \sum_{j=1}^{\eta} d_j p_{1j}.$$

The variance of the adjusted time to signal is

$$\operatorname{var}(T^*) = \operatorname{var}(Y) + E(N-1)\operatorname{var}(R_i) + \operatorname{var}(N-1)(E(R_i))^2$$

$$= \frac{\sum_{j=1}^{\eta} d_j^3 p_{0j}}{3 \sum_{j=1}^{\eta} d_j p_{0j}} - \frac{(\sum_{j=1}^{\eta} d_j^2 p_{0j})^2}{4(\sum_{j=1}^{\eta} d_j p_{0j})^2} + \sum_{j=1}^{\eta} d_j^2 p_{1j} \Big/ q_1 + \left(\sum_{j=1}^{\eta} d_j p_{1j}\right)^2 \Big/ q_1^2.$$

Limited simulation studies using several prior distributions for the time of the shift indicated that the assumptions that Y is uniform and that the distribution of U is proportional to the product of length and probability of an interval are quite reasonable. The only case in which these assumptions may not be reasonable seems to occur when the mean of the prior distribution is very small and the shift occurs very soon after the chart is started.

Note that the argument used to justify (5.1) can also be used to derive an unadjusted ATS for the case in which R_1, the interval used before the first sample, does not have the same distribution as R_2, R_3, ... For example, if d_1 is always used before the first sample, then the unadjusted ATS would be $d_1 + (E(N) - 1)E(R_2)$. This unadjusted ATS would be a measure of the ability of this VSI chart to detect shifts in μ that are present at the time that the process starts.

6. COMPARISONS BASED ON THE ADJUSTED ATS

Values of the adjusted ATS for several shifts in μ are given in Table 3 for the same FSI and VSI charts as in Table 2. Except for large shifts, the adjusted and unadjusted ATS values are very close together. The ATS for the VSI charts is substantially lower than the ATS of the FSI chart for moderate shifts in μ $(1.0 \leq \sqrt{n}(\mu - \mu_0)/\sigma \leq 2.0)$. For very large shifts, the adjusted ATS is higher for the VSI charts than for the FSI chart. For very large shifts, the shift is usually detected after only one sample, so $E(Z) \simeq 0$ and the adjusted ATS is essentially $E(Y)$. If a VSI chart has a large value for d_2, then the shift may occur in one of the intervals using d_2 and $E(T^*)$ will be large. From Table 3 it appears that a VSI chart with $(d_1, d_2) = (.1, 1.5)$ or $(.1, 1.9)$ gives good performance over a wide range of shifts in μ.

Additional information about good choices for d_1 and d_2 can be obtained by finding the values of d_1 and d_2 that minimize the adjusted ATS at a specified shift, subject to the constraint that the chart be matched to a special FSI chart when $\mu = \mu_0$. Analytically minimizing the adjusted ATS is difficult, so a numerical procedure was used for the case in which the FSI chart had $\gamma = 3$ and a fixed sampling interval of 1.0, and d_1 and d_2 are constrained to fall in the intervals $.1 \leq d_1 \leq .9$ and $1.1 \leq d_2 \leq 10.0$. The lower bound of $l_1 = .1$ for d_1 seems to be realistic, since it would usually be impractical to take a sample sooner than 1/10 of the fixed sampling interval. For all shifts considered, the optimal solution always has $d_1 = .1$. The optimal values of d_2 for $\sqrt{n}(\mu_1 - \mu_0)/\sigma = .1$, .5, 1.0, 1.5, 2.0, 3.0, 4.0 are, respectively, $d_2 = 9.77$, 9.97,

Table 4. Values of the Standard Deviation of the Adjusted Time to Signal for Matched FSI and VSI Charts

Shift $\sqrt{n}(\mu-\mu_0)/\sigma$	Fixed interval = 1.0	Variable interval (d_1, d_2) (.5, 1.5)	(.3, 1.7)	(.1, 1.9)	(.1, 1.1)	(.1, 1.3)	(.1, 1.5)	(.1, 4.0)
0	369.89	369.95	370.04	370.17	369.97	370.05	370.10	370.76
.5	154.72	147.21	144.29	141.41	148.69	144.72	142.97	140.45
1.0	43.39	36.23	33.46	30.76	36.98	33.45	31.99	30.21
1.5	14.46	10.28	8.71	7.26	10.18	8.39	7.74	7.40
2.0	5.79	3.60	2.82	2.18	3.25	2.51	2.29	2.58
3.0	1.44	.87	.72	.65	.63	.54	.56	1.27
4.0	.55	.50	.54	.57	.34	.39	.45	1.23
∞	.29	.44	.52	.57	.32	.38	.44	1.23

6.45, 2.78, 1.67, 1.14, 1.10. These results suggest that d_1 should be taken as small as is practical, whereas d_2 should be large if interest is in small shifts and small if interest is in large shifts. Although for some processes it may be possible to specify the magnitude of the shift that is likely to occur, it will usually be necessary to guard against a range of possible shifts. For this reason it may not be wise to use extremely large values of d_2, since large values of d_2 do not give as much protection against large shifts. Thus for most applications it seems appropriate to recommend $d_1 = l_1 = .1$ and d_2 in the range $1 < d_2 < 2$.

In addition to the effect of the variable-sampling interval on the mean of the time to signal distribution, there is the question of the effect on other characteristics of the distribution, such as the standard deviation. Table 4 gives the standard deviations of T^* for the same FSI and VSI charts as in Table 3. Unless the shift in μ is large, the time to signal distribution is highly skewed to the right and the value of the standard deviation is close to the value of the mean. When $\mu = \mu_0$, the VSI chart has a slightly larger standard deviation than the FSI chart. Table 5 gives the coefficients of variation for the charts in Table 3. For small shifts the coefficients of variation are very close to 1, and it is only for very large shifts that the FSI chart has a smaller standard deviation and a smaller coefficient of variation than the VSI charts.

7. PROPERTIES WHEN THE PROCESS MEAN DRIFTS

The properties of the VSI control chart have been evaluated for the case in which the process mean is constant. This means that the occurrence of an assignable cause results in a shift in the process mean that remains constant until the shift is detected and the assignable cause is eliminated. Another type of change in the process mean that can occur in applications is a gradual drift of the mean away from the target value. This drift can continue until it is detected and eliminated. In some applications, the drift may be predictable to some degree, such as with tool wear in machining operations. In the case of tool wear, the problem may be to monitor the process to decide on the optimal time to replace the tool. In other applications, the drift may be unpredictable in onset, direction, and magnitude, and it is primarily for this situation that the current VSI chart is being considered. The VSI idea could be adapted to the tool-wear problem in which samples would be taken more frequently when wear approaches a critical point.

When there is a drift in the process mean, the

Table 5. Values of the Coefficient of Variation of Adjusted Time to Signal for Matched FSI and VSI Charts

Shift $\sqrt{n}(\mu-\mu_0)/\sigma$	Fixed interval = 1.0	Variable interval (d_1, d_2) (.5, 1.5)	(.3, 1.7)	(.1, 1.9)	(.1, 1.1)	(.1, 1.3)	(.1, 1.5)	(.1, 4.0)
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.5	1.0000	.9999	.9999	.9999	.9999	.9999	.9998	.9978
1.0	1.0000	.9981	.9976	.9984	.9992	.9977	.9959	.9573
1.5	.9996	.9847	.9798	.9823	.9931	.9779	.9599	.7621
2.0	.9975	.9399	.9035	.8923	.9562	.8790	.8147	.6084
3.0	.9623	.7565	.6728	.6255	.6823	.5964	.5842	.6328
4.0	.8053	.6945	.6750	.6198	.5824	.5903	.6015	.6560
∞	.5774	.7024	.7011	.6298	.5887	.6019	.6133	.6622

Table 6. Values of the ATS and ANSS for Matched FSI and VSI Charts When the Process Mean Drifts

Drift θ	Fixed interval = 1.0, ATS = ANSS	Variable interval (.1, 1.1) ATS	(.1, 1.1) ANSS	(.1, 1.5) ATS	(.1, 1.5) ANSS	(.1, 1.9) ATS	(.1, 1.9) ANSS
.000	370.40	370.40	370.40	370.40	370.40	370.40	370.40
.005	134.11	130.75	136.60	128.08	138.64	127.39	139.17
.010	89.56	86.41	92.45	84.03	94.74	83.44	95.34
.025	49.37	46.80	52.35	45.06	54.58	44.66	55.14
.050	30.45	28.41	33.24	27.16	35.21	26.90	35.68
.100	18.43	16.88	20.90	16.05	22.50	15.90	22.86
.250	9.31	8.32	11.28	7.88	12.36	7.84	12.53
.500	5.52	4.84	7.09	4.61	7.80	4.61	7.81
1.0000	3.28	2.84	4.49	2.77	4.96	2.68	4.78

probabilities $p_1, p_2, \ldots, p_\eta, q$ change over time. The properties of the VSI chart can be evaluated for the special case (3.6) in which each sampling interval is an integer multiple of a constant $h > 0$. In this case, denote the sampling interval and signal probabilities at time hi by $q(i)$ and $p_j(i)$, respectively. Then (3.7) becomes

$$\pi(i) = \sum_{j=1}^{\eta} \pi(i - m_j)p_j(i - m_j), \tag{7.1}$$

and the ATS is

$$E(T) = \sum_{i=0}^{\infty} hi\pi(i)q(i). \tag{7.2}$$

In addition, the ANSS will be

$$E(N) = \sum_{i=0}^{\infty} \pi(i). \tag{7.3}$$

The ATS was evaluated numerically for the case in which there are two sampling intervals ($m_1 = 1$ and $m_2 = m$) and the process mean increases linearly over time. The amount of the drift at time hi is $\sqrt{n}\theta hi/\sigma$, where θ is the drift per unit time in units of the standard deviation of $\bar{X}$. To evaluate (7.1) and (7.2) numerically, it is necessary at each i to store only $\pi(i - m)$, $\pi(i - m + 1)$, $\ldots$, $\pi(i - 1)$ to compute $\pi(i)$. The sums in (7.2) and (7.3) can be truncated when $q(i)$ is close to 1. Some results for the FSI and VSI charts are given in Table 6. As in the case of a shift in the mean to a constant value, the VSI chart detects the drift faster than the FSI chart, although the detection is not dramatically faster. In contrast to the shift case, the VSI chart takes more samples to detect the drift. An apparent explanation is that for a slow drift it takes the process mean a relatively long time to get to a level at which detection is likely. For both the FSI and VSI charts, there is a relatively long period when detection is unlikely. When the process mean finally rises to a reasonably high level, the VSI chart will signal quickly, but the long initial period when the process mean is low tends to mask the difference between the detection times for the two charts. The fact that the VSI chart requires more samples for detection can be explained by the fact that the VSI chart detects the drift sooner when the process mean is lower, and with a lower mean more samples are required for detection.

Although the performance of the VSI chart in Table 6 is not as dramatic as in the case of a shift, when the drift does get to the point where the process mean is far from target, the VSI chart will detect this situation much faster than the FSI chart. More samples will be required for detection, but faster detection will usually be worth the extra sampling cost. The process will presumably operate with the mean at the target most of the time, and in this case the FSI and VSI charts have the same average sampling rate.

The ATS values computed previously for the case of a drift in the process mean assume that the drift starts at time 0. An adjusted ATS was not used because of the difficulty in computing an adjusted ATS. If the drift starts between samples, then the position of the process mean at the next sample depends on where the drift started in the interval. This means that the distribution of Z (in the notation of Sec. 5) depends on the value of Y. As long as the drift is not too fast, the ATS will be relatively large, and there will be little difference between adjusted and unadjusted values.

8. THE $\bar{X}$ CHART WITH RUNS RULES

This article has concentrated on the application of the VSI technique to the Shewhart $\bar{X}$ chart. The VSI technique can also be used with other types of charts, such as $\bar{X}$ charts with runs rules and cumulative sum (CUSUM) charts. When runs rules are used with $\bar{X}$ charts, the decision to signal depends on past samples in addition to the current sample. With the CUSUM chart, the decision to signal depends on a CUSUM of past sample means. When the VSI feature is added to such charts, the sampling-interval function could be allowed to depend on past samples. Although the principle is relatively straightforward, the actual calculation of properties and determination of optimal procedures is much more complicated. One simple extension will be investigated briefly in this section to show that the VSI technique can significantly improve the performance of charts other than the standard $\bar{X}$ chart.

Consider a control chart in which the decision to signal after the ith sample is taken can depend on previous samples in addition to current samples. For example, suppose that warning limits are added to the $\bar{X}$ chart at $\mu_0 \pm \xi\sigma/\sqrt{n}$, where the control limits (or action limits) are at $\mu_0 \pm \gamma\sigma/\sqrt{n}$ and $\xi < \gamma$. In many cases γ will be 3 and ξ will be between 1 and 2. As an example of a chart with runs rules using the warning limits, consider the chart that signals if

r' out of r sample means fall between the warning and action limits on one side of μ_0

or

the current mean falls outside the action limits. (8.1)

Suppose that the VSI modification is added to a chart of this type in which the sampling-interval function depends only on the current sample. A situation in which the sampling interval function depends on past samples was investigated by Cui and Reynolds (1988).

When the VSI feature with two sampling intervals d_1 and d_2 is used with the runs rules in applications, it would be convenient to choose $\gamma' = \xi$ so that the warning limits are the same as the limits that determine the sampling intervals d_1 and d_2. This would keep the chart simple and enable the user to determine quickly whether to signal and what sampling interval to use if the decision is not to signal. In this case it will usually be preferable to choose $\xi(= \gamma')$ to be less than 2 (e.g., $\xi = 1$) so that the probability of using d_1 is not excessively small.

When runs rules are added to the $\bar{X}$ chart, the ANSS is no longer given by (3.1), since the distribution of N will not be geometric. Thus in this case a different method for evaluating $E(N)$ must be used. Methods for evaluating the ARL of charts using runs rules were developed by Page (1962), Weindling, Littauer, and de Oliveira (1970), and Champ and Woodall (1987). Since the ARL of a FSI chart is numerically equal to the ANSS, these methods of computing the ARL can be applied to the case of the VSI chart. The Markov chain approach used by Champ and Woodall (1987) was used to compute the ANSS. The ATS was computed by using the Markov chain to determine the expected number of times that each sampling interval is used before a signal is given.

Table 7. Values of the ATS for Matched FSI and VSI Charts With Runs Rules

Shift $\sqrt{n}(\mu-\mu_0)/\sigma$	Rule 1 Fixed	Rule 1 Variable	Rule 2 Fixed	Rule 2 Variable
0	225.87	225.87	349.39	349.39
.5	77.74	74.97	121.80	113.02
1.0	20.00	17.48	27.74	20.82
1.5	7.30	5.54	9.41	5.11
2.0	3.65	2.33	4.68	1.68
3.0	1.67	.67	1.95	.31
4.0	1.17	.28	1.19	.13
∞	1.00	.10	1.00	.10

A comparison of the $\bar{X}$ chart using runs rules both with and without the VSI feature can be done in the same way as the comparison for the charts without runs rules. If both charts use the same rule to decide when to signal, then the ANSS functions will be equal for all μ. If the fixed interval is chosen as 1 and the sampling interval function of the VSI chart is chosen so that the ATS is equal to the ANSS when $\mu = \mu_0$, then the two charts will be matched.

Values of the unadjusted ATS were computed for the FSI and VSI $\bar{X}$ chart using the runs rules of the form of (8.1) with $\gamma = 3$ for two cases. In the first case, called Rule 1, $d_1 = .1$, $d_2 = 1.078$, $\xi = \gamma' = 2$, $r' = 2$, and $r = 3$; in the second case, called Rule 2, $d_1 = .1$, $d_2 = 1.415$, $\xi = \gamma' = 1$, and $r' = r = 5$. The values of d_2 for Rules 1 and 2 were selected so that the ATS values for the FSI and VSI charts matched when $\mu = \mu_0$. In practical applications, it would be preferable to pick d_2 to be a convenient value, since it would be unnecessary to match a FSI chart.

The results given in Table 7 show that the VSI feature significantly improves the two $\bar{X}$ charts with runs rules. A more extensive investigation of this topic will be required before recommendations can be given on the best choice of runs rules for various situations.

9. CONCLUSIONS AND EXTENSIONS

The results in this article have demonstrated that the VSI feature of the $\bar{X}$ chart can substantially reduce the time required by the chart to detect small and moderate shifts in the process mean. Thus this feature should be particularly useful for situations in which the process has been brought into reasonably good control, and any shifts that occur in the process mean are likely to be relatively small. A nice feature of the VSI chart is that it is sufficient to use only two sampling intervals so that the complexity of the chart is kept to a reasonable level. With the chart set up as in Figure 1, it would be easy for the chart user to determine the time to wait until the next sample by simply glancing at the position of the previous point.

This article has considered the application of the VSI feature to the $\bar{X}$ chart, but a number of other extensions would seem to be reasonable. Reynolds (1986) and Reynolds and Arnold (1986) considered the problem of controlling a general process parameter, so the theory that has been developed can easily be applied to charts such as R charts and p charts. With the charts considered so far, the sampling interval between $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ depends only on $\mathbf{X}_i$. As mentioned in Section 8, a natural extension would allow the sampling interval to depend on all past samples. In particular, with a CUSUM chart the sampling interval could depend on the current value of the CUSUM used as the control statistic in the chart. Amin, Reynolds, and Arnold (1987) considered the extension of the VSI feature to CUSUM charts, and Reynolds (1988) showed that using two sampling intervals is optimal for charts that can be modeled as a Markov chain.

In most cases in which an $\bar{X}$ chart or a CUSUM chart is being used to control the process mean, some other kind of chart such as an R chart will be used to control the process variance. The VSI feature could not be used independently in each chart, since the sampling intervals for the next sample specified by each chart might be different. If two charts are to use the VSI feature simultaneously, some mechanism must be used to integrate information about the process mean and process variance to decide when to take the next sample. The same kind of problem would arise when several variables are being measured on each item and individual univariate control charts are used for each variable. One natural way around the problem in this case is to use a multivariate control chart and base the decision about the sampling interval on the value of the multivariate control statistic. Chengalur, Arnold, and Reynolds (1987) considered the problem of using the VSI feature when multiple parameters are being monitored.

One disadvantage of the VSI feature is that the cost and effort required to administer the chart may be greater than for a corresponding FSI chart. In addition, the sampling intervals specified by the chart may not correspond to the natural periods in the process, such as work shifts for plant personnel. These disadvantages may not be particularly important in many applications, but for situations in which they are important, a modification of the VSI idea could be used. This modification would keep a fixed sampling interval, say every hour, but allow additional sampling within the one-hour period if there is an indication of a problem with the process. Thus samples would always be taken at the hourly interval points, but an extra sample could be taken after, say, 10 minutes if necessary. If this extra sample gives a value of $\bar{X}$ close to target, then the next sample would be in 50 minutes in order to return to the standard one-hour spacing. If the extra sample is not close to target, then other samples could be taken at 10-minute intervals with the provision that sampling will eventually return to the one-hour spacing. The sampling cost and false-alarm rate would be slightly higher as compared with a chart that takes samples only at the one-hour spacing, but the greatly increased ability to detect small shifts in the mean might be worth the extra cost. Alternately, the sample size and control limits could be modified to give the same in control performance as the standard FSI. Although the spirit of such modifications is the same as for the other VSI charts discussed in this article, a different model must be used to evaluate the properties. A different model is necessary because the restriction that samples be taken on the one-hour spacings implies that the time until the next sample depends on the current position within the one-hour interval as well as on the value of $\bar{X}$.

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