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Exact Results for Shewhart Control Charts With Supplementary Runs Rules

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This article gives a simple and efficient method, using Markov chains, to obtain the exact run-length properties of Shewhart control charts with supplementary runs rules. Average run-length comparisons are made among the Shewhart $\bar{X}$ chart with supplementary runs rules, the basic Shewhart $\bar{X}$ chart, and the cumulative sum (CUSUM) chart.

KEY WORDS: Quality control; Markov chain; Average run length.

1. INTRODUCTION

Control charts are often used to monitor a parameter (or parameters) of the distribution of a quality characteristic of items as they are produced. In this article, we consider the Shewhart $\bar{X}$ control chart with supplementary runs rules for detecting a shift in the mean of a normal distribution. Several runs rules have been suggested by Page (1955), the Western Electric Company (1956), Roberts (1958), Bissell (1978), and Wheeler (1983), among others. These rules may be stated in the following form: An out-of-control signal is given if k of the last m standardized sample means fall in the interval (a, b), where $k \leq m$ and $a < b$.

Although runs rules have been used for many years, no practical approach has been developed to determine the exact properties of many of these charts. In this article, a Markov chain approach is used to evaluate the average run length (ARL) and the run-length distribution of the Shewhart control chart with supplementary runs rules. A Markov-chain approach has been considered by others—for example, Page (1955) and Bissell (1978)—but only for the simpler combinations of runs rules. The approach has been considered to be cumbersome or intractable when several runs rules are used simultaneously. It is shown here that determining the exact run-length properties is a tractable problem, and we provide a solution for the runs rules usually considered in the literature. The methods presented, however, are general and can be applied to other types of control charts such as the R chart.

In Section 3, the ARL performance of the Shewhart $\bar{X}$ chart with supplementary runs rules is compared to that of the basic $\bar{X}$ chart and the cumulative sum (CUSUM) chart. The construction of the Markov-chain representation used to obtain the ARL values is explained in the Appendix.

Roberts (1958) provided a method of approximating the ARL of two-sided $\bar{X}$ charts with supplementary runs rules. First he calculated the average number of samples required for each separate one-sided rule to signal. These values were then combined to approximate the ARL for each one-sided part of the chart. Finally, these two approximations, ARL_L and ARL_U, are used in the following equation to approximate the ARL of the two-sided chart:

$$\frac{1}{\mathrm{ARL}} \cong \frac{1}{\mathrm{ARL}_L} + \frac{1}{\mathrm{ARL}_U}. \tag{1.1}$$

Wheeler (1983) provided exact expressions for up to 10 run-length probabilities for some one-sided Shewhart $\bar{X}$ charts with supplementary runs rules. The recursive method presented in our Appendix can be used with one-sided charts or with the more commonly used two-sided charts, and it can be applied to calculate any number of run-length probabilities.

Montgomery (1985) and Duncan (1986) gave the following expression for the overall probability of a false alarm when t separate rules are used to indicate an out-of-control situation:

$$\alpha = 1 - \prod_{i=1}^{t} (1 - \alpha_i), \tag{1.2}$$

where α_i is the probability of a false alarm at each sample for the ith rule ($i = 1, 2, \ldots, t$). As Duncan (1986) pointed out, (1.2) is valid only if the rules are independent. It is clear, however, that the independence assumption is not valid with the usual runs rules. In addition, the value of α_i is not clearly defined for a runs rule, because an out-of-control signal involves several observations.

2. SUPPLEMENTARY RUNS RULES

Consider a random variable X that measures the quality of a product. We observe successively the independent sample means $\bar{X}_1, \bar{X}_2, \ldots,$ assuming $\bar{X}_i \sim N(\mu_i, \sigma^2/n)$, $i = 1, 2, \ldots$, where σ^2 is known and remains constant. For the case in which the observations are autocorrelated, see Alwan and Roberts (in press). Assuming that $\mu_i = \mu$ ($i = 1, 2, \ldots$), the condition $\mu = \mu_0$ is to be maintained. It is convenient to base decisions on the standardized sample means, $Z_i = (\bar{X}_i - \mu_0)/(\sigma/\sqrt{n})$, $i = 1, 2, \ldots$. The Shewhart control chart is based on the present sample mean with a signal being given at the first stage N such that $|Z_N| > c$, where usually $c = 3$. This procedure has the disadvantage of not being sensitive to small shifts in the mean. One way to improve the chart's sensitivity to small changes in the mean is to add more rules for signaling that the process is out-of-control. One type of rule that is often suggested is the runs rule.

The runs rule, which signals if k of the last m standardized sample means fall in the interval (a, b), $a < b$, will be denoted by $T(k, m, a, b)$. Thus the usual Shewhart $\bar{X}$ chart is denoted by $\{T(1, 1, -\infty, -3), T(1, 1, 3, \infty)\}$, and the $\bar{X}$ chart with supplementary runs rules is given by a larger collection of rules of the form $T(k, m, a, b)$.

Various combinations of the following rules will be considered in this article:

Rule 1: $C_1 = \{T(1, 1, -\infty, -3), T(1, 1, 3, \infty)\}$.
Rule 2: $C_2 = \{T(2, 3, -3, -2), T(2, 3, 2, 3)\}$.
Rule 3: $C_3 = \{T(4, 5, -3, -1), T(4, 5, 1, 3)\}$.
Rule 4: $C_4 = \{T(8, 8, -3, 0), T(8, 8, 0, 3)\}$.
Rule 5: $C_5 = \{T(2, 2, -3, -2), T(2, 2, 2, 3)\}$.
Rule 6: $C_6 = \{T(5, 5, -3, -1), T(5, 5, 1, 3)\}$.
Rule 7: $C_7 = \{T(1, 1, -\infty, -3.09), T(1, 1, 3.09, \infty)\}$.
Rule 8: $C_8 = \{T(2, 3, -3.09, -1.96), T(2, 3, 1.96, 3.09)\}$.
Rule 9: $C_9 = \{T(8, 8, -3.09, 0), T(8, 8, 0, 3.09)\}$.

For example, Rule 3 signals if four of the last five standardized sample means are between -3 and -1 or if four of the last five are between 1 and 3. These nine rules can be combined to form most of the control charts suggested in the literature. The last three rules use the probability limits common in British practice. A more general chart will be defined using the following notation:

$$C_{ij\cdots k} = C_i \cup C_j \cup \cdots \cup C_k.$$

One of the rules, 1 or 7, is used in all charts to be considered, corresponding to the usual case in practice.

The Appendix contains the details of the Markov-chain representation of the control charts. The procedure yields the minimal number of states required to represent the charts. The nonzero elements of the transition probability matrix are determined and used to construct recursive equations for obtaining the run-length probabilities. The fact that the run-length probabilities are geometric in their limiting form is used to obtain accurate approximations to the ARL and percentage points of the run-length distribution. Champ (1986) provided a more detailed explanation and further results on the Markov chain approach. The BASIC computer programs used to determine the states of the Markov-chain representation and to calculate the run-length properties are available from Champ.

3. ARL COMPARISONS

The ARL's of several Shewhart control charts with supplementary runs rules are shown in Table 1. The shifts in the mean are measured in units of the process standard deviation, that is, the shift is $d = \sqrt{n}|\mu - \mu_0|/\sigma$. Table 1 shows that the supplementary runs rules increase the sensitivity of the Shewhart chart to small shifts in the mean. The supplementary runs rules also reduce the ARL at the target value μ_0, however, and thus result in more false alarms. The in-control ARL for C_{1234} is only 91.75, and the use of shorter runs or additional stopping rules would reduce the ARL to an even smaller value. Any desired ARL at the target value could be obtained, however, by widening the control limits.

Table 2 contains ratios of the ARL values of basic Shewhart $\bar{X}$ charts to the corresponding ARL values in Table 1. The control limits of each $\bar{X}$ chart were chosen to two decimal places such that the ARL at the target value would be as close as possible to but not more than the ARL of the corresponding Shewhart control chart with supplementary runs rules. It is clear from Table 2 that the charts with supplementary runs rules are more effective in detecting small shifts in the mean. The charts including runs rules are

Table 1. Average Run Lengths for Shewhart Control Charts With Supplementary Runs Rules

Shift d	Control charts C_1	C_7	C_{12}	C_{78}	C_{15}	C_{13}	C_{14}	C_{79}	C_{16}	C_{123}	C_{156}	C_{124}	C_{789}	C_{134}	C_{1456}	C_{1234}
.0	370.40	499.62	225.44	239.75	278.03	166.05	152.73	170.41	349.38	132.89	266.82	122.05	126.17	105.78	133.21	91.75
.2	308.43	412.01	177.56	185.48	222.59	120.70	110.52	120.87	279.53	97.86	208.44	89.14	91.19	76.01	96.37	66.80
.4	200.08	262.19	104.46	106.15	134.17	63.88	59.76	63.80	165.48	52.93	119.47	48.71	49.19	40.95	51.94	36.61
.6	119.67	153.86	57.92	57.80	75.27	33.99	33.64	35.46	89.07	28.70	63.70	27.49	27.57	23.15	29.01	20.90
.8	71.55	90.41	33.12	32.75	42.96	19.78	21.07	22.09	48.40	16.93	34.96	17.14	17.14	14.62	17.94	13.25
1.0	43.89	54.55	20.01	19.70	25.61	12.66	14.58	15.26	27.74	10.95	20.43	11.73	11.71	10.19	12.19	9.22
1.2	27.82	34.03	12.81	12.62	16.06	8.84	10.90	11.42	17.05	7.68	12.83	8.61	8.59	7.66	8.90	6.89
1.4	18.25	21.97	8.69	8.58	10.60	6.62	8.60	9.05	11.28	5.76	8.65	6.63	6.62	6.08	6.84	5.41
1.6	12.38	14.68	6.21	6.16	7.36	5.24	7.03	7.44	7.98	4.54	6.22	5.27	5.27	5.01	5.42	4.41
1.8	8.69	10.15	4.66	4.64	5.36	4.33	5.85	6.24	5.97	3.73	4.71	4.27	4.27	4.24	4.39	3.68
2.0	6.30	7.25	3.65	3.65	4.07	3.68	4.89	5.25	4.67	3.14	3.72	3.50	3.52	3.65	3.61	3.13
2.2	4.72	5.36	2.96	2.98	3.22	3.18	4.08	4.41	3.78	2.70	3.04	2.91	2.94	3.17	3.01	2.70
2.4	3.65	4.08	2.48	2.51	2.64	2.78	3.38	3.67	3.14	2.35	2.55	2.47	2.50	2.77	2.54	2.35
2.6	2.90	3.20	2.13	2.17	2.22	2.43	2.81	3.05	2.64	2.07	2.19	2.13	2.16	2.43	2.19	2.07
2.8	2.38	2.59	1.87	1.91	1.93	2.14	2.35	2.54	2.26	1.85	1.91	1.87	1.91	2.14	1.91	1.85
3.0	2.00	2.15	1.68	1.71	1.70	1.89	1.99	2.14	1.95	1.67	1.70	1.68	1.71	1.89	1.70	1.67

less effective in detecting larger shifts, but both types of charts tend to detect large shifts quickly. The increased sensitivity obtained by using supplementary runs rules cannot be obtained by simply narrowing the control limits of the basic Shewhart chart.

Table 3 contains ratios of the ARL values of CUSUM control charts to the corresponding ARL values in Table 1. The CUSUM chart signals when $S_i > H$ or $T_i < -H$, where $H > 0$,

$$S_i = \max(0, S_{i-1} + Z_i - K),$$

$$T_i = \min(0, T_{i-1} + Z_i + K), \qquad i = 1, 2, \ldots,$$

and $S_0 = T_0 = 0$. The value $K = .5$ was used for the CUSUM charts to obtain the ratios in Table 3. Woodall (1986) provided additional information on the design of CUSUM control charts.

From Table 3, it is clear that CUSUM charts are more effective in detecting small shifts in the mean than Shewhart $\bar{X}$ charts with supplementary runs rules. The advantage of the CUSUM charts is lost for the larger shifts, but again both types of charts detect these large shifts quickly. As more runs rules are added to the Shewhart chart, the efficiency of the chart increases as compared to the CUSUM chart with a matching in-control ARL.

4. FAST INITIAL RESPONSE

A fast initial response (FIR) or head-start feature can easily be incorporated into a Shewhart control chart with supplementary runs rules. The FIR feature, introduced for the CUSUM charts by Lucas and Crosier (1982), gives out-of-control signals more quickly for initial out-of-control processes. The

Table 2. Ratios of Average Run Lengths: Shewhart Chart to the Corresponding Shewhart Chart With Supplementary Runs Rules (shown in parentheses)

Shift d	Shewhart control limit c 3.00 (C_1)	3.09 (C_7)	2.84 (C_{12})	2.86 (C_{78})	2.91 (C_{15})	2.74 (C_{13})	2.71 (C_{14})	2.75 (C_{79})	2.98 (C_{16})	2.67 (C_{123})	2.89 (C_{156})	2.64 (C_{124})	2.66 (C_{789})	2.59 (C_{134})	2.67 (C_{1456})	2.54 (C_{1234})
.0	1.00	1.00	.98	.98	1.00	.98	.97	.98	.99	.99	.97	.99	.98	.98	.99	.98
.2	1.00	1.00	1.06	1.08	1.04	1.15	1.15	1.19	1.04	1.16	1.05	1.17	1.18	1.19	1.18	1.18
.4	1.00	1.00	1.21	1.26	1.15	1.49	1.47	1.54	1.14	1.49	1.22	1.50	1.52	1.56	1.52	1.53
.6	1.00	1.00	1.34	1.42	1.25	1.77	1.66	1.74	1.27	1.76	1.40	1.71	1.75	1.80	1.74	1.77
.8	1.00	1.00	1.45	1.54	1.33	1.92	1.68	1.76	1.40	1.91	1.55	1.76	1.80	1.84	1.80	1.82
1.0	1.00	1.00	1.52	1.61	1.39	1.93	1.57	1.63	1.51	1.92	1.66	1.68	1.72	1.75	1.72	1.75
1.2	1.00	1.00	1.54	1.63	1.43	1.83	1.40	1.44	1.56	1.84	1.71	1.55	1.58	1.58	1.59	1.61
1.4	1.00	1.00	1.54	1.62	1.44	1.68	1.22	1.25	1.55	1.70	1.70	1.40	1.43	1.40	1.43	1.45
1.6	1.00	1.00	1.50	1.56	1.43	1.50	1.07	1.07	1.49	1.55	1.63	1.27	1.29	1.24	1.30	1.31
1.8	1.00	1.00	1.44	1.49	1.40	1.33	.94	.94	1.41	1.39	1.54	1.17	1.19	1.10	1.18	1.18
2.0	1.00	1.00	1.37	1.41	1.35	1.18	.86	.84	1.31	1.27	1.44	1.09	1.10	.99	1.10	1.08
2.2	1.00	1.00	1.29	1.32	1.30	1.07	.80	.78	1.21	1.16	1.34	1.04	1.04	.91	1.04	1.01
2.4	1.00	1.00	1.22	1.24	1.24	.98	.78	.75	1.13	1.08	1.25	1.00	1.00	.85	1.00	.96
2.6	1.00	1.00	1.16	1.16	1.19	.93	.78	.74	1.08	1.02	1.18	.97	.96	.81	.97	.92
2.8	1.00	1.00	1.11	1.10	1.13	.89	.80	.76	1.03	.98	1.13	.95	.94	.80	.95	.90
3.0	1.00	1.00	1.05	1.05	1.10	.88	.82	.78	1.01	.95	1.08	.93	.92	.80	.94	.89

Table 3. Ratios of Average Run Lengths: CUSUM Chart to the Corresponding Shewhart Chart With Supplementary Runs Rules (shown in parentheses)

	CUSUM decision interval H															
Shift d	4.78 (C_1)	5.07 (C_7)	4.29 (C_{12})	4.35 (C_{78})	4.50 (C_{15})	4.00 (C_{13})	3.91 (C_{14})	4.02 (C_{79})	4.72 (C_{16})	3.78 (C_{123})	4.46 (C_{156})	3.70 (C_{124})	3.73 (C_{789})	3.56 (C_{134})	3.78 (C_{1456})	3.42 (C_{1234})
.0	1.01	1.00	1.00	1.00	1.01	1.01	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.00	1.00	1.00
.2	.53	.49	.65	.65	.61	.77	.79	.78	.56	.81	.63	.83	.83	.88	.82	.89
.4	.27	.24	.42	.43	.36	.60	.62	.61	.32	.66	.40	.68	.69	.81	.67	.79
.6	.21	.17	.37	.38	.30	.57	.56	.55	.27	.63	.35	.64	.64	.72	.62	.76
.8	.20	.17	.39	.40	.32	.60	.55	.54	.29	.66	.38	.64	.65	.72	.63	.76
1.0	.23	.19	.45	.46	.37	.66	.56	.55	.35	.73	.46	.66	.67	.74	.65	.78
1.2	.27	.23	.53	.55	.44	.73	.58	.56	.44	.79	.55	.70	.70	.76	.69	.81
1.4	.33	.29	.64	.65	.54	.79	.59	.58	.53	.86	.66	.73	.74	.77	.72	.84
1.6	.41	.36	.75	.76	.66	.83	.61	.59	.63	.92	.77	.78	.78	.79	.77	.87
1.8	.51	.45	.86	.87	.78	.87	.63	.61	.73	.97	.88	.83	.84	.81	.82	.91
2.0	.61	.56	.97	.98	.90	.91	.67	.64	.82	1.02	.98	.90	.90	.84	.89	.95
2.2	.73	.68	1.07	1.08	1.02	.94	.72	.68	.90	1.06	1.08	.97	.97	.87	.95	.99
2.4	.86	.81	1.16	1.16	1.13	.98	.80	.75	.99	1.11	1.16	1.04	1.04	.91	1.03	1.03
2.6	.99	.94	1.24	1.24	1.24	1.04	.88	.83	1.08	1.17	1.25	1.12	1.11	.95	1.11	1.09
2.8	1.12	1.08	1.32	1.30	1.32	1.09	.98	.93	1.17	1.22	1.32	1.19	1.17	1.01	1.18	1.14
3.0	1.25	1.22	1.41	1.39	1.44	1.18	1.11	1.08	1.29	1.29	1.40	1.27	1.26	1.10	1.27	1.24

simple modification required in the Markov-chain approach by use of the FIR is discussed in the Appendix.

As an example, consider the chart C_{123} with an FIR rule that gives an out-of-control signal if the first two standardized means are both in the interval (1, 3), if both are in the interval $(-3, -1)$, or if $|Z_1| > 2$. The ARL values for this chart are shown in Table 4.

The FIR feature is useful if it can provide quicker detection of shifts from the target value with a relatively small decrease in the in-control ARL. In general, for the charts and head starts that we considered, the FIR feature did not result in benefits as substantial as those of the CUSUM FIR features. The reason may be that the effect of the head start is lost with runs rules unless a signal occurs very quickly, for example, within two observations for the C_{123} head start in our example.

5. CONCLUSIONS

We used a Markov-chain approach to model supplementary runs rules used with Shewhart control charts. Exact run-length probabilities can be calculated for any set of supplementary runs rules. We presented a method for approximating the ARL of control charts with supplementary runs rules and tabulated ARL's for the more commonly used charts. It has been shown that supplementary runs rules cause the Shewhart chart to be more sensitive to small shifts in the mean, but not as sensitive as the CUSUM chart.

Table 4. Average Run Lengths for C_{123} With FIR

d	ARL	d	ARL
.0	122.17	1.6	3.07
.2	89.28	1.8	2.43
.4	47.23	2.0	2.00
.6	24.74	2.2	1.71
.8	13.98	2.4	1.50
1.0	8.60	2.6	1.36
1.2	5.73	2.8	1.25
1.4	4.08	3.0	1.18

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APPENDIX: THE MARKOV-CHAIN REPRESENTATION

The possible values of the standardized sample mean can be partitioned depending on the particular runs rules used. If we denote by $r + 1$ the number of distinct values in the set $\{a_i, b_i\ (i = 1, \ldots, t)\}$ and by $A_0, A_1, \ldots, A_r$, their ordered values, then the region R_j is defined to be the interval (A_{j-1}, A_j) and p_j is defined to be the probability that an observation is in R_j $(j = 1, 2, \ldots, r)$. For example, the regions corresponding to C_{13} are $R_1 = (-\infty, -3)$, $R_2 = (-3, -1)$, $R_3 = (-1, 1)$, $R_4 = (1, 3)$, and $R_5 = (3, \infty)$. A realization of the chart is completely determined by the regions containing the successive values of the standardized sample means.

The states of the Markov chain indicate the status of the chart with respect to each runs rule. There is one absorbing state that corresponds to the out-of-control signal. Two absorbing states could be used to indicate which side of the chart produces the signal, but we do not consider this generalization.

For the rule $T(k_i, m_i, a_i, b_i)$, $m_i > 1$, define the vectors

$$\mathbf{W}_i' = (W_{i,1}, \ldots, W_{i,m_i-1})$$

and

$$\mathbf{X}_i' = (X_{i,1}, \ldots, X_{i,m_i-1}),$$

where

$$W_{i,j} = 1 \quad \text{if the } j\text{th previous observation was in } (a_i, b_i)$$
$$= 0 \quad \text{otherwise}$$

and

$$X_{i,j} = W_{i,j} \quad \text{if } \sum_{h=1}^{j} (1 - W_{i,h}) < m_i - k_i + 1$$
$$= 0 \quad \text{otherwise.}$$

The vector $\mathbf{X}_i'$ indicates by 1s only those observations falling in (a_i, b_i) that may contribute to an out-of-control signal. Thus a transient state of a chart using t rules can be represented by $\mathbf{S}' = (\mathbf{X}_1', \ldots, \mathbf{X}_t')$, where the subvector $\mathbf{X}_i'$ is defined as previously for the rule $T(k_i, m_i, a_i, b_i)$, $i = 1, 2, \ldots, t$, For any rule with $m = 1$, it is not necessary to remember any information about previous observations. Moreover, to reduce the length of the vector $\mathbf{S}'$, it is often convenient to replace the subvector associated with any rule $T(m, m, a, b)$, $m > 1$, with a counter l, where l can take the values $0, 1, \ldots, m - 1$. The number l is the number of consecutive points in the interval (a, b) since the last point not in (a, b).

Table A.1 contains the 30 states required in the Markov-chain representation of the chart C_{13}. The first four coordinates of each vector representing a particular state correspond to the rule $T(4, 5, -3, -1)$ and the next four correspond to the rule $T(4, 5, 1, 3)$. For example, state 21, denoted by (01001010), indicates that the past three observations fell into the intervals (1, 3), (−3, −1), and (1, 3), successively. If there was an observation preceding these three, it was such that it cannot contribute to an out-of-control signal by either rule.

The transition matrix P is defined as $P = [P_{i,j}]$, where $P_{i,j}$ is the probability of moving from state i to state j in one transition. The states are numbered from 1 to s, where state 1 is the initial state and state s is the absorbing state.

For each rule $T(k_i, m_i, a_i, b_i)$, let

$$X_{i,0} = 1 \quad \text{if present observation is in } (a_i, b_i)$$
$$= 0 \quad \text{otherwise.}$$

A transition to the absorbing state occurs if for some i

$$\sum_{j=0}^{m_i-1} X_{i,j} = k_i,$$

otherwise there is a transition to another transient state. This transient state can be represented by the vector $\mathbf{V}' = (\mathbf{y}_1, \ldots, \mathbf{y}_t)$, where $\mathbf{y}_i' = (y_{i,j}, \ldots, y_{i,m_i-1})$, $i = 1, \ldots, t$, and

$$y_{i,j} = X_{i,j-1} \quad \text{if } \sum_{h=0}^{j-1} (1 - X_{i,h}) < m_i - k_i + 1$$
$$= 0 \quad \text{otherwise}$$

$(j = 1, \ldots, m_i - 1; i = 1, \ldots, t)$.

The set of required transient states can be determined iteratively. For a given initial state, one can determine the state resulting from each of the regions possibly containing the first sample point. This process is repeated for each new transient state until no new states are generated. Any states that have identical rows in the transition matrix can be combined into one state to further reduce the number of states required. The one-step transitions for C_{13} are given in Table A.1. For example, if state 17, (10100100), is

Table A. 1. States and One-Step Transitions for Control Chart C_{13}

Present state		Number of next state				
No.	Representation	R_1	R_2	R_3	R_4	R_5
1	(00000000)	30	2	1	11	30
2	(10000000)	30	4	3	12	30
3	(01000000)	30	5	1	11	30
4	(11000000)	30	7	6	13	30
5	(10100000)	30	8	3	12	30
6	(01100000)	30	9	1	11	30
7	(11100000)	30	30	10	14	30
8	(11010000)	30	30	6	13	30
9	(10110000)	30	30	3	12	30
10	(01110000)	30	30	1	11	30
11	(00001000)	30	16	15	19	30
12	(01001000)	30	17	15	19	30
13	(01101000)	30	18	15	19	30
14	(01111000)	30	30	15	19	30
15	(00000100)	30	2	1	20	30
16	(10000100)	30	4	3	21	30
17	(10100100)	30	8	3	21	30
18	(10110100)	30	30	3	21	30
19	(00001100)	30	23	22	24	30
20	(00001010)	30	16	15	25	30
21	(01001010)	30	17	15	25	30
22	(00000110)	30	2	1	26	30
23	(10000110)	30	4	3	27	30
24	(00001110)	30	29	28	30	30
25	(00001101)	30	23	22	30	30
26	(00001011)	30	16	15	30	30
27	(01001011)	30	17	15	30	30
28	(00000111)	30	2	1	30	30
29	(10000111)	30	4	3	30	30
30	Absorbing	30	30	30	30	30

Table A. 2. Properties of Markov Chain Transition Matrix

Control chart	No. of states	Percentage of nonzero elements
C_1, C_7	2	75.0
C_{15}	4	68.8
C_{12}, C_{78}	8	32.8
C_{16}	10	35.0
C_{156}	16	31.6
C_{14}, C_{79}	16	17.2
C_{13}	30	11.7
C_{124}, C_{789}	44	9.3
C_{1456}	64	9.8
C_{123}	72	6.5
C_{134}	110	4.1
C_{1234}	216	2.7

occupied and the next observation is in $R_2 = (-3, -1)$, then the resulting state is state 8, represented by (11010000). As seen from Table A.2, when the transition matrix is large, it is also sparse.

The initial state is represented by the zero vector unless the FIR feature is used. With the FIR feature the initial state will usually be an ephemeral state not ordinarily required in the Markov-chain representation. For example, if an FIR feature for C_{13} gives a signal if the first two points are both in (1, 3) or if both are in $(-3, -1)$, then the initial state is (01100110).

To calculate the run-length probabilities, define the $(s-1) \times 1$ vector $\mathbf{L}_h$ by

$$\mathbf{L}_h' = [\Pr(N_1 = h), \ldots, \Pr(N_{s-1} = h)],$$

where N_i is the run length of the chart with initial state i. Brook and Evans (1972) showed that these vectors can be calculated recursively using

$$\mathbf{L}_1 = (I - Q) \quad \mathbf{1}$$

$$\mathbf{L}_h = Q \quad \mathbf{L}_{h-1}, \qquad h = 2, 3, \ldots,$$

where $\mathbf{1}$ is a column vector of 1s and Q is the $(s-1) \times (s-1)$ matrix obtained from the transition matrix P by deleting its last row and column.

This method of calculating the run-length probabilities coupled with the sparseness of the transition matrix gives simple recursive formulas for the run-length probabilities. For example, for the control chart C_{12}, the following recursive formulas are used to calculate the run-length probabilities:

$$\begin{aligned}
\Pr(N_1 = h) &= p_3 \Pr(N_1 = h - 1) \\
&\quad + p_2 \Pr(N_2 = h - 1) \\
&\quad + p_4 \Pr(N_4 = h - 1), \\
\Pr(N_2 = h) &= p_3 \Pr(N_3 = h - 1) \\
&\quad + p_4 \Pr(N_5 = h - 1), \\
\Pr(N_3 = h) &= p_3 \Pr(N_1 = h - 1) \\
&\quad + p_4 \Pr(N_4 = h - 1), \\
\Pr(N_4 = h) &= p_3 \Pr(N_6 = h - 1) \\
&\quad + p_2 \Pr(N_7 = h - 1), \\
\Pr(N_5 = h) &= p_3 \Pr(N_6 = h - 1), \\
\Pr(N_6 = h) &= p_3 \Pr(N_1 = h - 1) \\
&\quad + p_2 \Pr(N_2 = h - 1), \\
\Pr(N_7 = h) &= p_3 \Pr(N_3 = h - 1),
\end{aligned}$$

for $h = 2, 3, 4, \ldots$, where $\Pr(N_1 = 1) = 1 - p_2 - p_3 - p_4$, $\Pr(N_2 = 1) = 1 - p_3 - p_4$, $\Pr(N_3 = 1) = 1 - p_3 - p_4$, $\Pr(N_4 = 1) = 1 - p_2 - p_3$, $\Pr(N_5 = 1) = 1 - p_3$, $\Pr(N_6 = 1) = 1 - p_2 - p_3$, and $\Pr(N_7 = 1) = 1 - p_3$. Similar results hold for calculating the run-length probabilities of the other charts considered in this article, although more equations are usually required.

Brook and Evans (1972) showed that the ARL for initial state i can be calculated by adding the elements in the ith row of $(I - Q)^{-1}$. Woodall and Reynolds (1983) gave a method for obtaining the ARL for any case in which the run-length probabilities become geometric in their limiting form. This approach is more efficient in our case, because Q is usually a very large, sparse matrix. Woodall and Reynolds (1983) showed that

$$E(N) \cong \sum_{h=1}^{n^*} h \Pr(N = h) + \hat{\lambda} \Pr(N = n^*) \times [n^*/(1 - \hat{\lambda}) + 1/(1 - \hat{\lambda})^2],$$

where

$$\hat{\lambda} = \left[1 - \sum_{h=1}^{n^*} \Pr(N = h)\right] \Big/ \left[1 - \sum_{h=1}^{n^*-1} \Pr(N = h)\right]$$

and N is the run length. To illustrate the efficiency of this approach, $n^* = 21$ was the largest value required to calculate the ARL's in Table 1.

The percentage point N_α $(0 < \alpha < 1)$, defined to be the smallest integer such that $\Pr(N \le N_\alpha) \ge \alpha$, can be determined easily when $N_\alpha \le n^*$. As shown by Woodall and Reynolds (1983), if $N_\alpha > n^*$ then

$$N_\alpha \cong n^* - 1 + \frac{\ln[(1 - \hat{\lambda})(\sum_{h=1}^{n^*} \Pr(N = h) - \alpha)/\Pr(N = n^*) + \hat{\lambda}]}{\ln(\hat{\lambda})}.$$

If only the ARL is needed for a two-sided chart, then our Markov-chain approach can be used to calculate the exact ARL's of the two one-sided component charts and then (1.1) can be used to obtain an accurate approximation. The one-sided charts require fewer states in their Markov chain representation. For example, a one-sided version of C_{1234} requires only 91 states, whereas the two-sided C_{1234} requires 216 states. It should be noted that Equation (1.1) will be exact if all runs rules used have $k = m$. Under this condition, a signal from one of the one-sided charts implies that the other one-sided chart will be in its initial state and the approach used by Lucas and Crosier (1982) can be used to establish equality.

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