سؤال ⑥

$$L(\mu,\sigma \mid x_1,x_2,\dots,x_n) = L(\mu,\sigma\mid x_1)\times\dots\times L(\mu,\sigma\mid x_n)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i-\mu)/2\sigma^2} \longrightarrow \text{likelihood}$$ از log تابع

استفاده می‌کنیم

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$$\longrightarrow \ln\left[L(\mu,\sigma \mid x_1,\dots,x_n)\right] = \frac{-n}{2}\ln(2\pi) - n\ln(\sigma) - \sum \frac{(x_i-\mu)^2}{2\sigma^2}$$

$$\Longrightarrow \begin{cases} \frac{\partial}{\partial\mu}\ln(L) = \frac{1}{\sigma^2}\left[(x_1+\dots+x_n)-n\mu\right] \overset{\text{set}}{=} 0 \\ \quad \hookrightarrow \hat{\mu} = \frac{(x_1+\dots+x_n)}{n} = \bar{X} \\ \frac{\partial}{\partial\sigma}\ln(L) = \frac{-n}{\sigma} + \frac{1}{\sigma^3}\left[(x_1-\mu)^2+\dots+(x_n-\mu)^2\right] \overset{\text{set}}{=} 0 \\ \quad \hookrightarrow \hat{\sigma} = \sqrt{\frac{(x_1-\mu)^2+\dots+(x_n-\mu)^2}{n}} \end{cases}$$

$$X_1,\dots,X_n \sim N(\mu,\sigma^2) \quad \begin{cases} H_0 : \mu > \mu_0 \ (\circ) \\ H_1 : \mu \leq \mu_0 \ (\circ) \end{cases}$$

$$\Lambda(x) = \frac{\sup\{L(\theta\mid x) : \theta\in\varphi_0\}}{\sup\{L(\theta\mid x) : \theta\in\varphi\}} = \frac{L_0}{L}, \quad L = \left(\frac{1}{2\pi\hat{\sigma}^2}\right)^{\frac{n}{2}} \exp\left[-\frac{\sum(x_i-\mu_0)^2}{2\hat{\sigma}^2}\right]$$

با توجه به قسمت قبل می‌دانیم $\hat{\mu} = \bar{x}$ و $\hat{\sigma}^2 = \frac{1}{n}\sum(x_i-\bar{x})^2$

$$\longrightarrow L = \left(\frac{1}{2\pi\frac{1}{n}\sum(x_i-\bar{x})^2}\right)^{\frac{n}{2}} \exp\left(\frac{-n}{2}\right) = \left(\frac{ne^{-1}}{2\pi\sum(x_i-\bar{x})^2}\right)^{\frac{n}{2}}$$

$$\longrightarrow L_0 = \left(\frac{ne^{-1}}{2\pi\sum(x_i-\mu_0)^2}\right)^{\frac{n}{2}} \Longrightarrow \Lambda(x) = \left[\frac{\sum(x_i-\bar{x})^2}{\sum(x_i-\mu_0)^2}\right]^{\frac{n}{2}}$$

$$\sum(x_i-\mu_0)^2 = \sum(x_i-\bar{x})^2 + n(\bar{x}-\mu_0)^2 \longleftarrow$$ با توجه به اینکه

$$\Lambda(x) = \left(\frac{1}{1+\frac{n(\bar{x}-\mu_0)^2}{\sum(x_i-\bar{x})^2}}\right)$$ صفحه بعد

$\longleftarrow$ برای likelihood Ratio داریم

$$\Lambda = \left(\frac{1}{1 + \frac{n(\bar{x}-\mu_0)^2}{(x_i-\bar{x})^2}} \right) \leqslant c^{\frac{2}{n}} \rightarrow \frac{n(\bar{x}-\mu_0)^2}{\frac{1}{n-1}\sum (x_i-\bar{x})^2} \geqslant (n-1)\left(c^{\frac{-2}{n-1}}\right) = k^*$$

→ (b) مربوط به قسمت

$$\Longrightarrow \frac{(\bar{x}-\mu_0)^2}{S^2/n} \geqslant k^* \rightarrow S^2 = \text{sample variance}$$

$$\rightarrow \frac{|\bar{x}-\mu_0|}{S/\sqrt{n}} \geqslant k^{**}$$

$$\boxed{\frac{\bar{x}-\mu_0}{s/\sqrt{n}} \rightarrow T \text{ statistic}}$$

باتوجه به نتایج بدست آمده خواهیم داشت:

اگر $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ آنگاه T از توزیع t پیروی می‌کند (با درجه آزادی $(n-1)$)

واگر بخواهیم α را fix کنیم خواهیم داشت:

$$\boxed{k^{**} = t_{\alpha/2, n-1}}$$

$$t_0 = \frac{\bar{X}-\mu_0}{S/\sqrt{n}}$$

ناحیه رد برای t_0 ⟵ $(-\infty, -t_\alpha] = (-\infty, t_{1-\alpha}]$

ناحیه پذیرش t_0 ⟵ $(-t_\alpha, \infty) = (t_{1-\alpha}, \infty)$

P_value ⟵ $P(t_{n-1} \leqslant t_0)$