Math models for Algorithmic trade 2020 Homework 2

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Contents

1 Concepts on Finance

1.1 What is a risky asset?

Asset: It is a property owned by a person or company. It has a value and can meet debts. We can call these

$$S_i^t,$$

this is represents the value of an asset that we call S, of a company or person i at a specific time t. An example would be the value of the amount of apples (S) that a peasant i has today (t) in his warehouse.
We assume the assets $\boldsymbol{S}$ to be a random vector $\boldsymbol{S} : \Omega \to \mathbb{R}^d$.

Rate of return (of 1 asset): Is the net gain on an asset over a period of time. It is expressed as a percentage with respect to the investment's initial cost. More precisely it is

$$R_i^t = \frac{S_i^t - S_i^{t-1}}{S_i^{t-1}}. \tag{1}$$

This is just telling us how the value of that amount of apples owned by the peasant, is changing over time. It could decrease naturally since the apples expire. Or rise, if the people suddenly feel hungry for apples and wanna buy them and consume them at any cost.

Portfolio: A portfolio $\boldsymbol{\xi} = (\xi_1, ..., \xi_d)$ is the set of investments held by a person or company. Each investment is made for a specific product, and then it becomes and asset. A portfolio can also be understood then as a set of assets we hold. ξ_1 is the asset for apples for instance. Then we can also call this asset of ours our "share" of apples from the peasant's farm. A share is an indivisible unit of capital of a company, owned by a shareholder or investor. To explain things step by step, farm i is divided into an mount of shares. We buy or possess ξ_i of them, then the product $\xi_i S_i^t$ is telling us the asset or capital or value we have from that company. We can play with it, we can sell it, buy more, or do nothing.

Is share, asset and investment the same thing? What are the differences? Also, according to Wiki a share is "an indivisible unit of capital, expressing the ownership relationship between the company and the shareholder." So is a share $1? Is an asset a collection of shares? I got lost here.

Rate of return (of a portfolio): To put things simple. If we have only two shares $\boldsymbol{\xi} = (\xi_1, \xi_2)$ in two different farms S_1, S_2, then just realize that our initial (at time t) assets or investment is $\xi_1 S_1^t + \xi_2 S_2^t$. Then what we have at time $t+1$ is just $\xi_1 S_1^{t+1} + \xi_2 S_2^{t+1}$. So the rate of return for the set of assets $\boldsymbol{\xi}$ is just:

$$\begin{aligned} R_\xi^{t+1} &= \frac{\sum_{i=1}^d \xi_i S_i^{t+1} - \sum_{i=1}^d \xi_i S_i^t}{\sum_{i=1}^d \xi_i S_i^t} \\ &= \frac{\sum_{i=1}^d \xi_i (S_i^{t+1} - S_i^t)}{\sum_{i=1}^d \xi_i S_i^t} \\ &= \frac{\boldsymbol{\xi} \cdot (\boldsymbol{S^{t+1}} - \boldsymbol{S^t})}{\boldsymbol{\xi} \cdot \boldsymbol{S^t}}, \end{aligned} \tag{2}$$

where $d = 2$.

Why is S_i multiplied by ξ_i? In other words, why is a money quantitiy S_i (the asset price) multiplied by another money quantity ξ_i? Don't we then obtain $\$^2$?? Isn't this a contradiction?

ξ_i is not a money quantity! It is only the ammount of shares of a company i. The product $\xi_i S_i$ is a money quantity
It is reasonable to call our total money 1, and then normalize the portfolio $\boldsymbol{\xi}$ into a normalised portfolio $\boldsymbol{\alpha}$

Weights (of a portfolio $\boldsymbol{\xi}$and set of assets $\boldsymbol{S}$) are the quantities $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d)$ where

$$\alpha_i = \frac{\xi_i S_i}{\boldsymbol{\xi} \cdot \boldsymbol{S}}. \tag{3}$$

We observe that

$$\alpha_1 + ... + \alpha_d = 1. \tag{4}$$

We also observe that, given d rates of return

$$\boldsymbol{R}^t = (R_1^t, ..., R_d^t) = \left(\frac{S_1^t - S_1^{t-1}}{S_1^{t-1}}, ..., \frac{S_d^t - S_d^{t-1}}{S_d^{t-1}} \right)$$

and the fact that:

$$\begin{array}{rcl} \frac{\xi_i (S_i^{t+1} - S_i^t)}{\boldsymbol{\xi} \cdot \boldsymbol{S}} & = & \frac{\xi_i S_i^t}{\boldsymbol{\xi} \cdot \boldsymbol{S}} \frac{S_i^{t+1} - S_i^t}{S_i^t} \\ & = & \alpha_i^t R_i^{t+1}, \end{array} \tag{5}$$

Equation 2 becomes actually

$$\begin{array}{rcl} R_\xi^{t+1} & = & \sum_{i=1}^d \frac{\xi_i (S_i^{t+1} - S_i^t)}{\boldsymbol{\xi} \cdot \boldsymbol{S}^t} \\ & = & \sum_{i=1}^d \alpha_i^t R_i^{t+1} \\ & = & \boldsymbol{\alpha} \cdot \boldsymbol{R}^t =: R_\alpha^t. \end{array} \tag{6}$$

So now we can talk about a *portfolio* by $\boldsymbol{\xi}$ or by $\boldsymbol{\alpha}$ so that $\alpha_1 + ... + \alpha_d = 1$.

1.2 Expectations and variance of returns

I think R_ξ is a time process and for each time step t, the rate of return R_ξ^t is a random variable, just as Brownian motion is!
Here we will compute expectations and variances over time. So we will change the notations and instead of writing R_ξ^t for describing the rate of return at time t of a portfolio ξ, we will simply write R_ξ for describing all the process of rate of returns over time. We will compute the expectations and variances of these time-spanning processes.

I understand now the rate of return as a time process

$$\boldsymbol{R}_\xi = (R_\xi^{t=1}, R_\xi^{t=2}, ..., R_\xi^{t=T}), \tag{7}$$

and the expected rate of return, in practice, as the average value over time:

$$\mathbb{E}\boldsymbol{R}_\xi = \frac{R_\xi^{t=1} + R_\xi^{t=2} + ... + R_\xi^{t=T}}{T} \tag{8}$$

In reality it could be a random process R_ξ^t having a certain mean and variance.

Let's call the expected rate of return of an asset S_i by

$$\mu_i := \mathbb{E}R_i,$$

then the

Expected rate of return (of a portfolio $\boldsymbol{\xi}$) is, using Equation 6 and the fact that the weights $\boldsymbol{\alpha}$ are assumed now to NOT depend on time. Why do we not take the expectation of α? Is it not a random process?? t (constant portfolio or weights over time), they are constant, and therefore we have that:

$$\begin{array}{rcl} \mu_\xi = \mathbb{E}R_\xi & = & \sum_{i=1}^d \alpha_i \mathbb{E}R_i \\ & = & \sum_{i=1}^d \alpha_i \mu_i. \end{array} \tag{9}$$

Also, if we call

$$\sigma_{ij}^2 := \mathrm{Cov}(R_i, R_j) = \mathbb{E}(R_i - \mu_i)(R_j - \mu_j),$$

the variance of the rate of return of an asset S_i, then the

Variance of the rate return (of a two portfolio $\boldsymbol{\xi}$) will be, in two dimensions,

$$\begin{array}{rcc} \sigma_\xi^2 = \mathrm{Var}(R_\xi) & = & \mathbb{E}(R_\xi)^2 - \mu_\xi^2 \\ & = & \mathbb{E}[\alpha_1 R_1 + \alpha_2 R_2]^2 - (\alpha_1\mu_1 + \alpha_2\mu_2)^2 \\ & = & \mathbb{E}[\alpha_1^2 R_1^2 + \alpha_2^2 R_2^2 + 2\alpha_1\alpha_2 R_1 R_2] - \alpha_1^2\mu_1^2 - \alpha_2^2\mu_2^2 - 2\alpha_1\alpha_2\mu_1\mu_2 \\ & = & \alpha_1^2\mathbb{E}[R_1^2 - \mu_1^2] + \alpha_2^2\mathbb{E}[R_2^2 - \mu_2^2] + 2\alpha_1\alpha_2[\mathbb{E}(R_1R_2) - \mu_1\mu_2] \\ & = & \alpha_1^2\mathbb{E}[R_1 - \mu_1]^2 + \alpha_2^2\mathbb{E}[R_2 - \mu_2]^2 + 2\alpha_1\alpha_2\mathbb{E}(R_1 - \mu_1)(R_2 - \mu_2) \\ & = & \sum_{i,j=1}^2 \alpha_i\alpha_j\sigma_{ij}^2. \end{array} \tag{10}$$

Actually this can be written as

$$\sigma_\xi^2 = (\alpha_1, \alpha_2)\begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \tag{11}$$

In d dimensions the result will be the same, the proof is just more cumbersome, and we will obtain that

$$\sigma_\xi^2 = \mathrm{Var}(R_\xi) \quad = \quad \sum_{i,j=1}^d \alpha_i\alpha_j\sigma_{ij}^2. \tag{12}$$

Or, in other words:

$$\sigma_\xi^2 = (\alpha_1, ..., \alpha_d)\begin{pmatrix} \sigma_{11}^2 & ... & \sigma_{1d}^2 \\ \vdots & ... & \vdots \\ \sigma_{d1}^2 & ... & \sigma_{dd}^2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \tag{13}$$

1.2.1 Diversification of portfolio for independent assets

Assume that we have d independent assets $S_1, ..., S_d$ and define the "uniformly distributed" portfolio

$$\xi_i = \tfrac{1}{d}\tfrac{1}{S_i} \qquad \forall i, \tag{14}$$

then observe first that

$$\alpha_i = \frac{\xi S_i}{\boldsymbol{\xi}\cdot\boldsymbol{S}} = \frac{1/d}{\boldsymbol{\xi}\cdot\boldsymbol{S}} = \frac{1/d}{\frac{1}{d} + ... + \frac{1}{d}} = \frac{1}{d} \quad \forall i.$$

Then the variance of your portfolio is, using Equation 12:

$$\begin{array}{rcl} \sigma_\xi^2 & = & \sum_{i,j=1}^d \alpha_i\alpha_j\sigma_{ij}^2 \\ & = & \frac{1}{d^2}\sum_{i,j=1}^d \sigma_{ij}^2 \\ & = & \frac{1}{d}\left[\frac{1}{d}\sum_{i,j=1}^d \sigma_{ij}^2\right] \\ & = & \frac{1}{d}\left[\frac{1}{d}\sum_{i=1}^d \sigma_{ii}^2\right], \end{array} \tag{15}$$

where the last equality is due to the fact that

$$\sigma_{ij} = 0 \qquad \text{if } i \neq j$$

since we assumed the d assets to be independent of each other.

Now, if the average variance of the rates of returns

$$\frac{1}{d}\sum_{i=1}^d \mathrm{Var}(R_i) \leq M < \infty,$$

can be assumed to be bounded by a constant M, then

$$\sigma_\xi^2 \leq \tfrac{M}{d} \xrightarrow{d\to\infty} 0. \tag{16}$$

This literally means that the portfolio variance σ_ξ^2 vanishes if we completely uniformly diversify our assets, and keep increasing the amount of assets of our portfolio. *This event occurs as long as the assets of our portfolio are independent between themselves! And given that their average variance (or at least the variance of each of them) is bounded by a constant.*

Riskless portfolio: It is a portfolio $\boldsymbol{\xi}$ such that

$$\sigma_{\xi}^2 = 0.$$

Then according to our sample variance Equation 8 this would mean that the rate of returns

$$R_i^t = R_i^{t+1} \quad \forall t,$$

In other words, the rate of returns for each asset S_i are constant over time.

1.3 Example 1: Zero variance $\sigma_\alpha^2 = 0$ portfolio selection

Let $d = 2$ and consider the vector of rate of return means

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \mathbb{E}R_1 \\ \mathbb{E}R_2 \end{pmatrix} = \begin{pmatrix} 0.10 \\ 0.14 \end{pmatrix}, \tag{17}$$

With covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11}^2 & \sigma_{11}\sigma_{22} \\ \sigma_{11}\sigma_{22} & \sigma_{22}^2 \end{pmatrix}, \tag{18}$$

where $\sigma_{11}^2 = 0.08^2$ and $\sigma_{22}^2 = 0.18^2$.

The question is, can we find a portfolio $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ such that

$$\sigma_\alpha^2 = 0?$$

By taking a look at Equation 11 we just need for instance that the product

$$\begin{pmatrix} \sigma_{11}^2 & \sigma_{11}\sigma_{22} \\ \sigma_{11}\sigma_{22} & \sigma_{22}^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0. \tag{19}$$

But this is easy, since the matrix Σ can be rewritten as

$$\Sigma = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{22} \\ \sigma_{11} & \sigma_{22} \end{pmatrix}, \tag{20}$$

which clearly has determinant zero, and the choice

$$\boxed{\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = c \begin{pmatrix} -\sigma_{22} \\ \sigma_{11} \end{pmatrix} = c \begin{pmatrix} -0.18 \\ 0.08 \end{pmatrix}}, \tag{21}$$

makes the product $\Sigma\boldsymbol{\alpha} = 0$, for any constant $c \in \mathbb{R}$, which of course, makes the product

$$\sigma_\alpha^2 = \boldsymbol{\alpha}'\Sigma\boldsymbol{\alpha} = 0.$$

Just as we desired.
But recall that they are normalized, so we must have that $\alpha_1 = 1 - \alpha_2$ which implies that $c = -10$ and so

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 0.8 \end{pmatrix}. \tag{22}$$

If we do this we obtain that the *expected rate of return* for this portfolio is

$$\mathbb{E}R_\alpha = \boldsymbol{\alpha} \cdot \boldsymbol{\mu} = c(-\mu_1\sigma_{22} + \mu_2\sigma_{11}) = -c0.0068 = 0.068, \tag{23}$$

this means that the riskless rate is 6.8%.

Exercise 1 A riskless return is attainable if and only if, the rate of return covariance matrix Σ is invertible.
We clearly see that if the matrix Σ is not invertible, then there must exist a non-trivial vector

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \neq 0, \tag{24}$$

such that

$$\begin{pmatrix} \sigma_{11}^2 & \dots & \sigma_{1d}^2 \\ \vdots & \dots & \vdots \\ \sigma_{d1}^2 & \dots & \sigma_{dd}^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} = 0. \tag{25}$$

We just need to see the opposite, namely, that if Σ is NOT invertible, then a vector $\boldsymbol{\alpha}$ satisfying equation Equation 25 must be 0. But just note that all entries of the matrix Σ are $\sigma_{ij}^2 \geq 0$, this implies that Σ is non-negative definite, and therefore

$$\boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha} = 0 \to \boldsymbol{\alpha} = 0.$$

1.4 Diversification to reduce unsystematic risk (correlated assets)

If we express again the Covariance of the return rates for a uniform diversification portfolio

$$\boldsymbol{\alpha} = (1/d, ..., 1/d),$$

we see that

$$\begin{aligned} \sigma_\alpha^2 &= \boldsymbol{\alpha}' \begin{pmatrix} \sigma_{11}^2 & 0 & \dots & 0 \\ 0 & \sigma_{22}^2 & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & \sigma_{dd}^2 \end{pmatrix} \boldsymbol{\alpha} + \boldsymbol{\alpha}' \begin{pmatrix} 0 & \sigma_{12}^2 & \dots & \sigma_{1d}^2 \\ \sigma_{21}^2 & 0 & \dots & \sigma_{2d}^2 \\ \sigma_{31}^2 & \dots & \ddots & \sigma_{3d}^2 \\ \sigma_{d1}^2 & \sigma_{d2}^2 & \dots & 0 \end{pmatrix} \boldsymbol{\alpha} \\ &= \tfrac{1}{d}\tfrac{1}{d}(\sigma_{11}^2 + \dots + \sigma_{dd}^2) + \tfrac{1}{d^2}\textstyle\sum_{i \neq j} \sigma_{ij}^2 \\ &= \tfrac{1}{d}\bar{\sigma}_d^2 + \tfrac{d(d-1)}{d^2}\overline{\mathrm{Cov}}_d \xrightarrow{d \to \infty} \overline{\mathrm{Cov}}, \end{aligned} \tag{26}$$

if we assume (1) a bounded average variance:

$$\bar{\sigma}_d^2 = \frac{1}{d}(\sigma_{11}^2 + \dots + \sigma_{dd}^2) \leq M < \infty,$$

and (2) the Covariance limit to exist:

$$\overline{\mathrm{Cov}}_d = \tfrac{1}{d(d-1)} \textstyle\sum_{i \neq j} \sigma_{ij}^2 \xrightarrow{d \to \infty} \overline{\mathrm{Cov}}. \tag{27}$$

We call this Covariance limit the **systematic risk**. Just realise that we obtained that:

$$\sigma_\alpha^2 \xrightarrow{d \to \infty} \overline{\mathrm{Cov}}, \tag{28}$$

under assumptions (1) and (2). The **unsystematic risk** would be σ_α^2.

2 Std. Dev-Mean space $(\sigma_\alpha, \mu_\alpha)$

Here we assume that **mean** μ_α and **variance** σ_α^2 of the rate of return of the portfolio R_α, are the only two things of the portfolio α that we care about.

In Example 1 at Equation 17, given a portfolio $\alpha = (\alpha_1, \alpha_2)$ we have that $\alpha_2 = 1 - \alpha_1$. So

$$\mu_\alpha = \mu_1 \alpha_1 + \mu_2 (1 - \alpha_1), \tag{29}$$

and

$$\begin{aligned} \sigma_\alpha^2 &= \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha} \\ &= \alpha_1^2 \sigma_{11}^2 + (1 - \alpha_1)^2 \sigma_{22}^2 + 2\alpha_1(1 - \alpha_1)\sigma_{12}^2. \end{aligned} \tag{30}$$

Note that if we restrict $\alpha \in [0, 1]$ then $\mu_\alpha \in [\mu_1, \mu_2]$, which makes sense. The question is, does $\sigma_\alpha \in [\sigma_1, \sigma_2]$ in this case? Or can σ_α fall outside of this interval?.

2.1 $(\sigma_\alpha, \mu_\alpha)$ is a straight line if corr = 1

In the next example, it falls within the interval. We say then that the σ_α is within a **straight interval**. (This is my definition of straight interval, I hope Prof. Nadtochiy doesn't disagree.)
So note that the mean of returns of the portfolio μ_α depends linearly on the portfolio, but not so does the variance of the rate of return of the portfolio α, σ_α^2. Actually for the just cited example, we have that Equation 30 is now

$$\begin{aligned} \sigma_\alpha^2 &= \alpha_1^2\sigma_{11}^2 + (1-\alpha_1)^2\sigma_{22}^2 + 2\alpha_1(1-\alpha_1)\sigma_{12}^2 \\ &= \alpha_1^2(\sigma_{11}-\sigma_{22})^2 + \alpha_1 2[\sigma_{11}\sigma_{22} - \sigma_{22}^2] + \sigma_{22}^2 \\ &= (\alpha_1[\sigma_{11}-\sigma_{22}] + \sigma_{22})^2 \end{aligned} \tag{31}$$

We can now take the square root of σ_α^2. This gives us the standard deviation of the portfolio (we only consider the positive definition of it). Recall that $\sigma_{11}^2 \leq \sigma_{22}^2$

$$\sigma_\alpha = +|\alpha_1(\sigma_{11}-\sigma_{22}) + \sigma_{22}| = \begin{cases} \alpha_1(\sigma_{11}-\sigma_{22}) + \sigma_{22} & \text{if } \alpha_1 \leq \frac{\sigma_{22}}{\sigma_{22}-\sigma_{11}} \\ \alpha_1(\sigma_{22}-\sigma_{11}) - \sigma_{22} & \text{if } \alpha_1 \geq \frac{\sigma_{22}}{\sigma_{22}-\sigma_{11}} \end{cases}, \tag{32}$$

Which is a linear function on α_1 This is not a straight line! and gives the interval

$$\sigma_\alpha \in [\sigma_{11}, \sigma_{22}] \text{ for } \alpha_1 \in [0,1].$$

Thus the plot of

$$\{(\sigma_\alpha, \mu_\alpha) : \boldsymbol{\alpha} = (\alpha_1, 1-\alpha_1),\ \alpha_1 \in \mathbb{R}\}$$

should give a 90° clockwise rotated absolute value function like the one in Figure 1 on the Std dev.-Mean plane, using Equation 29 and Equation 32.

$$\begin{aligned} \sigma_{11}^2 &= 0.08^2 \\ \sigma_{22}^2 &= 0.18^2 \\ \sigma_{12}^2 = \sigma_{11}\sigma_{22} &= 0.00144 \\ \mu_1 &= 0.10 \\ \mu_2 &= 0.14, \end{aligned} \tag{33}$$

So if we plug in all these values for the mean and variance of rate of returns $(\sigma_\alpha^2, \mu_\alpha)$ of the portfolio α, we obtain the following plot

Example 2.1 Consider the previous example, where the values are given by Equation 33, but with

$$\sigma_{12} = 0,$$

then with $\alpha \in [0,1]$ we obtain the Mean-Variance plot of Figure 2.

We observe that in this case we have

$$\sigma_\alpha \notin [\sigma_{11}, \sigma_{22}]. \tag{34}$$

Why would we restrict $\alpha_1 \in [0,1]$???? - Question for Prof. Nadtochiy

Note 2.1 Observe that in the case of a 2D portfolio $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ since $\alpha_1 + \alpha_2 = 1$, we have that this draws a 1-dimensional line on the Mean-Variance space. It would draw a 2D area if the portfolio has dimension 3 or higher.

Theorem 2.1 (Lemma 2 from the notes of Prof. Nadtochiy)

(a) If two assets S_1, S_2 are perfectly correlated,

$$\sigma_{12}^2 = \pm\sigma_{11}\sigma_{22},$$

then

$$\sigma_\alpha^2 \in [\sigma_{11}^2, \sigma_{22}^2] \text{ if } \alpha \in [0,1].$$

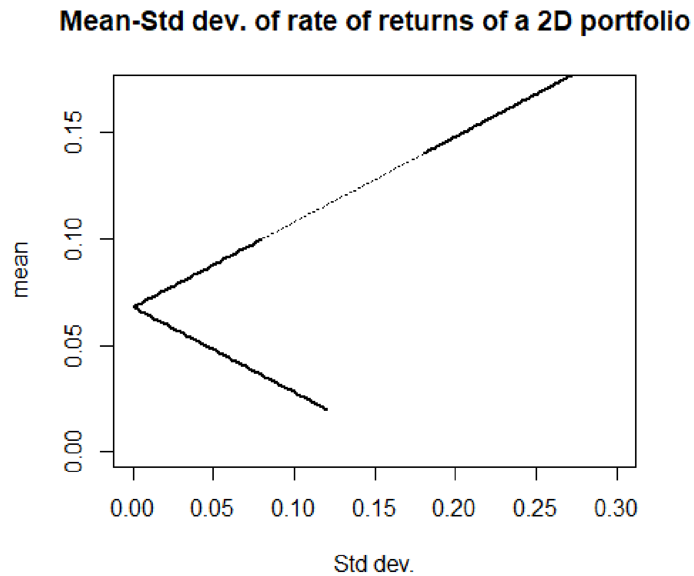

Figure 1: Standard deviation-Mean plot from the rate of returns of a 2 dimensional portfolio α as specified in example 1 at Equation 17. Here $\alpha_1 \in [-1, 3]$. But the region where $\alpha_1 \in [0, 1]$ is highlighted with a doted line.

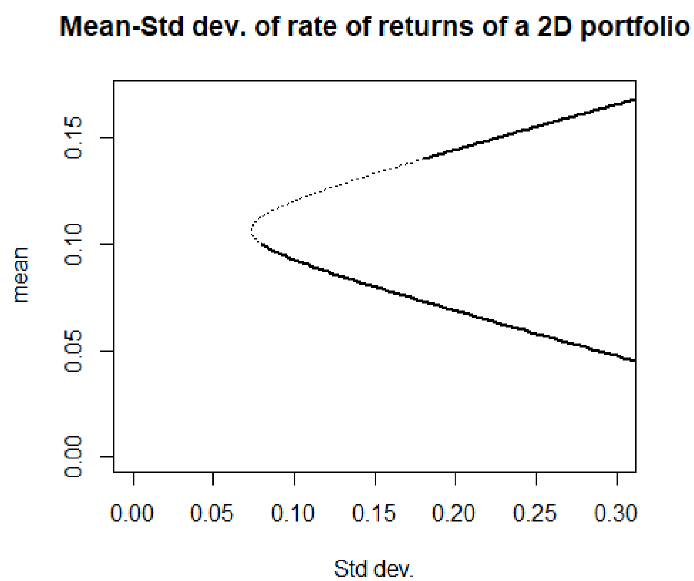

Figure 2: Standard deviation-Mean plot from the rate of returns of a 2 dimensional portfolio α_1 as specified in example 1 at Equation 17, but with $\sigma_{12} = 0$. The region where $\alpha_1 \in [0, 1]$ is highlighted with a doted line.

(b) If one of the assets (let's assume it is S_1) is riskless, or

$$\sigma_{11}^2 = 0,$$

then

$$\sigma_\alpha^2 \in [0, \sigma_{22}^2] \text{ if } \alpha \in [0, 1].$$

Proof 2.1 (a) Just take a look at Equation 32.

(b) Just notice that

$$0 = \sigma_{11}^2 = \mathbb{E}(R_1 - \mu_1)^2 \Rightarrow R_1 \overset{a.s.}{=} \mu_1. \tag{35}$$

This directly implies that

$$\sigma_{12}^2 = \mathrm{Cov}(R_1, R_2) = \mathbb{E}(R_1 - \mu)(R_2 - \mu_2) = 0,$$

so actually we have that

$$\sigma_{12}^2 = 0 = \sigma_{11}\sigma_{22},$$

and we can just apply case (a).

□

3 Efficient Frontier

Consider $d \geq 2$, and the corresponding set of d assets $\{S_1, ..., S_d\}$, together with its rates or return $\{R_1, ..., R_d\}$, and a portfolio $\boldsymbol{\alpha}$ such that $\alpha_1 + ... + \alpha_d = 1$. Then let's call **Feasible region** the Std dev-Mean space

$$Fea = \{(\sigma_\alpha, \mu_\alpha) : \mu_\alpha \in \mathbb{R}, \sigma_\alpha^2 \geq 0, \alpha_1 + ... + \alpha_d = 1\}$$

where

$$\begin{gathered} \mu_\alpha = \boldsymbol{\mu} \cdot \boldsymbol{\alpha} = (\mathbb{E}R_1, ..., \mathbb{E}R_d) \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} = \alpha_1 \mathbb{E}R_1 + ... + \alpha_d \mathbb{E}R_d, \\ \sigma_\alpha = \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha}, \end{gathered} \tag{36}$$

and where

$$\Sigma = \begin{pmatrix} \sigma_{11}^2 & \dots & \sigma_{1d}^2 \\ \vdots & \vdots & \vdots \\ \sigma_{d1}^2 & \dots & \sigma_{dd}^2 \end{pmatrix}, \tag{37}$$

is the covariance matrix of the rates of return. *Feasible* refers in Linear Optimization to the fact that

For $\mu_\alpha \in \mathbb{R}$, denote by

$$f(\mu_\alpha) := \min_{\mu_\alpha} \sigma_\alpha^2 = \min_\alpha \sigma_\alpha^2 \tag{38}$$

the smallest standard deviation attainable of the portfolio α, given the mean of the rate of return of the portfolio α, μ_α. Then **the upper part** of the curve

$$\{(f(\mu_\alpha), \mu_\alpha) : \alpha_1 + ... + \alpha_d = 1\}$$

is the **efficient frontier.**

3.1 Degenerate markets

Example 3.1 Consider the case where two assets S_1, S_2 are riskless, that is

$$\sigma_{11}^2 = \sigma_{22}^2 = 0.$$

Then Prof. claims that the entire line

$$\{(0, \mu) : \forall \mu \in \mathbb{R}\} \subseteq Fea.$$

That's true, consider

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0, ..., 0),$$

where $\alpha_1, \alpha_2 \neq 0$. Then

$$\begin{aligned} \mu_\alpha = \boldsymbol{\mu} \cdot \boldsymbol{\alpha} &= \alpha_1\mu_1 + \alpha_2\mu_2 \\ &= \alpha_1\mu_1 + (1 - \alpha_1)\mu_2 \\ &= \alpha_1(\mu_1 - \mu_2) + \mu_2 \in \mathbb{R} \text{ if } \alpha_1 \in \mathbb{R}. \end{aligned} \tag{39}$$

□

We can then claim that the **market is degenerate**, since the definition of a **degenerate asset** S_1 is that it has zero volatility, or, in other words, that

$$\sigma_{11}^2 = 0.$$

But now observe that ALL rates of returns $R_\alpha = \boldsymbol{R} \cdot \boldsymbol{\alpha}$ from this example have exactly 0 volatility! (for any portfolio $\boldsymbol{\alpha}$). This is because

$$\begin{array}{rcc} \sigma_\alpha & = & \boldsymbol{\alpha} \cdot \Sigma \cdot \boldsymbol{\alpha} \\ & = & (\alpha_1, 1-\alpha_1) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ 1-\alpha_1 \end{pmatrix} = 0. \end{array} \tag{40}$$

Recall that $\sigma_{12}^2 = 0$ when $\sigma_{11}^2 = 0$ as we explained short after item 35.
Thus, it makes sense to call this market or feasible set of portfolios degenerate. (all of them have 0 risk! It's financial heaven on Earth!). Moreover, if $\mu_1 \leq \mu_2$, by taking a portfolio α_1 very negative, we can make our portfolio mean of returns μ_α very big (as detailed in Equation 39), which is unrealistic.

Just a note, is **short sell** to sell assets, and therefore allow one of the $\alpha_i < 0$? For some $1 \leq i \leq d$.

The conclusion from Example 3.1 is that it doesn't make sense to accept two riskless assets. Therefore we will only accept a **unique riskless asset** in the market. Otherwise the whole portfolio game stops being funD.

4 σ_α^2 minimization using the Lagrangian

We want to minimize

$$\sigma_\alpha^2 = \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha},$$

such that

$$\sum_1^d \alpha_i = 1.$$

But like this problem is equivalent to minimizing

$$\tfrac{1}{2}\boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha} - \lambda \left(\textstyle\sum_1^d \alpha_i - 1 \right), \tag{41}$$

wich can be rewritten as

$$\tfrac{1}{2} \textstyle\sum_{i,j=1}^d \alpha_i \alpha_j \sigma_{ij}^2 - \lambda \left(\sum_1^d \alpha_i - 1 \right). \tag{42}$$

Now, considering the variables

$$(\alpha_1, ..., \alpha_d, \lambda),$$

we set first that

$$L = \frac{1}{2}\sigma_\alpha^2 - \lambda (\sum_1^d \alpha_i - 1),$$

and then we take the derivative with respect to them and obtain that,

$$\begin{array}{rcc} 0 = \frac{\partial}{\partial \alpha_i}\sigma_\alpha^2 & = & \sum_{j=1}^d \alpha_j \sigma_{ij}^2 - \lambda, \quad i = 1, ..., d. \\ 0 = \frac{\partial}{\partial \lambda}\sigma_\alpha^2 & = & \sum_{j=1}^d \alpha_j - 1. \end{array} \tag{43}$$

Note that for case $d = 2, i = 1$ the first equation of the above would be

$$\begin{array}{c} \frac{\partial}{\partial \alpha_1} \left[\frac{1}{2}(\alpha_1^2 \sigma_{11}^2 + 2\alpha_1\alpha_2\sigma_{12}^2 + \alpha_2^2\sigma_{22}^2) - \lambda(\alpha_1 + \alpha_2 - 1) \right] = \\ \alpha_1 \sigma_{11} + \alpha_2 \sigma_{12}^2 - \lambda. \end{array} \tag{44}$$

The above system of equations can be rewritten as

$$\begin{pmatrix} & & & -1 \\ & \Sigma & & \vdots \\ & & & -1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{45}$$

$$\Updownarrow$$

$$Ax = b.$$

Note 4.1 The Professors says here that A is $d+1xd+1$ and that $x, b \in \mathbb{R}^{d+1}$ and that

$$A \text{ invertible } \Leftrightarrow \Sigma \text{ invertible ????}$$

Judging by the form of A at Equation 45 what is A^{-1}?

Use the Sherman-Morrisson formula in the following, which says that

Theorem 4.1 (Sherman-Morrisson) Given a square invertible matrix $A \in \mathbb{R}^{dxd}$ and two vectors $u, v \in R^d$, then

$$A + vu' \text{ is invertible } \leftrightarrow 1 + u'Av \neq 0. \tag{46}$$

I assumed the following matrix mutliplication

$$Id_{d+1,d+1} = A \cdot A^{-1} =$$

$$\begin{pmatrix} & & & -1 \\ & \Sigma & & \vdots \\ & & & -1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} & & & c_1 \\ & B & & \vdots \\ & & & c_d \\ q_1 & \cdots & q_d & p \end{pmatrix} = \begin{pmatrix} \Sigma \cdot B - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \times (q_1, .., q_d) & \Sigma \cdot \boldsymbol{c} + p \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} \\ (1, \quad \cdots, \quad 1) \cdot B & c_1 + ... + c_d \end{pmatrix} \tag{47}$$

But if this must be equal to the identity, then

$$\Sigma \cdot B + (1, ..., 1) \times \begin{pmatrix} q_1 \\ \vdots \\ q_d \end{pmatrix} = Id_{dxd}, \tag{48}$$

$$\Sigma \cdot \boldsymbol{c} + p \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} = 0. \tag{49}$$

$$c_1 + ... + c_d = 1,$$

$$(1, ..., 1) \cdot B = 0. \tag{50}$$

But the last equality is satisfied only if $B = 0$, which means that A^{-1} is not invertible, which is a contradiction!! How to solve this?

Note 4.2 *Prof. says that since the* ***objective*** *is convex, and the constraints are linear, setting*

$$\frac{\partial}{\partial x_i} L = 0,$$

is a sufficient condition for optimality? Why?

This is lemma 3 and I still don't get it.

Example 4.1 Let $d = 2$ and

$$\Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 2 \end{pmatrix}, \boldsymbol{\mu} = \begin{pmatrix} 0.2 \\ 0.4 \end{pmatrix}, \tag{51}$$

this, leads the optimal point of

$$\begin{gathered} \lambda = 7/8 \\ \alpha_1 = 3/4 \\ \alpha_2 = 1/4. \end{gathered} \tag{52}$$

For the lagrangian of equation Equation 43. This means a portfolio

$$\boldsymbol{\alpha} = (3/4, 1/4)$$

. We get a portfolio *expected rate of return* μ_α of

$$\mu_\alpha = \boldsymbol{\alpha} \cdot \boldsymbol{\mu} = 0.25, \tag{53}$$

with variance

$$\begin{gathered} \sigma_\alpha^2 = \boldsymbol{\alpha}' \cdot \Sigma \boldsymbol{\alpha} = 28/32 = 7/8 = 0.875, \\ \downarrow \\ \sigma_\alpha = 0.9354. \end{gathered} \tag{54}$$

Now, what does this mean? Does this mean that for every dollar inverted in our portfolio $\boldsymbol{\alpha}$ we will get \$0.25 dollars back after the *return time*? And how do we interpret the *standard deviation of* $\sigma_\alpha = 0.935$? Does this mean that the expected return is

$$\mu_\alpha = 0.25 \pm 0.935? \tag{55}$$

5 Computing the efficient frontier

It's like minimizing the portfolio variance σ_α^2, but subject to a fixed $\mu_\alpha = y$. So, the problem would be now:

$$\begin{gathered} \min_\alpha \sigma_\alpha^2 = \min_\alpha \boldsymbol{\alpha}' \Sigma \boldsymbol{\alpha}, \\ \boldsymbol{\alpha} \cdot \boldsymbol{\mu} = y, \\ \sum_{i=1}^d \alpha_i = 1. \end{gathered} \tag{56}$$

For solving this problem we can write down the lagrangian:

$$L = \tfrac{1}{2} \sum_{i,j=1}^d \alpha_i \alpha_j \sigma_{ij}^2 - \lambda_1 \left(\sum_{i=1}^d \alpha_i \mu_i - y \right) - \lambda_2 \left(\sum_{i=1}^d \alpha_i - 1 \right). \tag{57}$$

Now if we take the derivatives of this we obtain that

$$\begin{gathered} 0 = \tfrac{\partial}{\partial \alpha_i} L = \sum_{j=1}^d \alpha_i \sigma_{ij} - \lambda_1 \mu_i - \lambda_2 \\ 0 = \tfrac{\partial}{\partial \lambda_1} L = \boldsymbol{\alpha} \cdot \boldsymbol{\mu} - y \\ 0 = \tfrac{\partial}{\partial \lambda_2} L = \sum_{i=1}^d \alpha_i - 1, \end{gathered} \tag{58}$$

For this case I get the matrix formulation:

$$\begin{gathered} \begin{pmatrix} & & & -\mu_1 & -1 \\ & \Sigma & & \vdots & \vdots \\ & & & -\mu_d & -1 \\ \mu_1 & \cdots & \mu_d & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y \\ 1 \end{pmatrix} \\ \Updownarrow \\ Ax = b. \end{gathered} \tag{59}$$

Note 5.1 We have that

$$A \text{ invertible } \Leftarrow \sum \text{ invertible}$$

But I don't know why!

Note 5.2 And this problem is degenerate for the case $d = 2$, since then we have to solve that

$$\alpha_1 \mu_1 + (1 - \alpha_1) \mu_2 = y, \tag{60}$$

which has solution

$$\alpha_1 = \tfrac{y - \mu_2}{\mu_1 - \mu_2}, \quad \alpha_2 = 1 - \alpha_1, \tag{61}$$

and this would lead to a

$$\sigma_\alpha^2 = ay^2 + by + c, \tag{62}$$

where a, b, c are constants that depend on $\boldsymbol{\mu}, \Sigma$ only. Therefore for any given mean of returns y, there will be only one possible variance σ_α available. No need for minimization. This makes perfect sense, since in 2D we are actually running on a 1D line. So for every given mean of returns there is exactly only one variance possible. And viceversa!

Example 5.1 In case $\sigma_{12}^2 = 0$, then

$$\sigma_\alpha^2 = \left(\frac{y-\mu_2}{\mu_1-\mu_2}\right)^2 \sigma_{11}^2 + \left(\frac{\mu_1 - y}{\mu_1-\mu_2}\right)^2 \sigma_{22}^2$$

We can try to find the minimal variance portfolio α by differentiating this formula and setting the derivative equal to 0, and isolate the value for y.

6 Optimal portfolio for means minus variance

An investor wants to maximize **mean** and minimize **variance**. On way is to put the first minus the second and maximize the problem, i.e. maximize:

$$\boldsymbol{\alpha} \cdot \boldsymbol{\mu} - \gamma \boldsymbol{\alpha}' \cdot \sum \boldsymbol{\alpha} \tag{63}$$

so that

$$\sum_{i=1}^{d} \alpha_i = 1.$$

where γ is the reciprocal **risk tolerance**.
For solving this, use as before a lagrangian of the form

$$L = \boldsymbol{\alpha} \cdot \boldsymbol{\mu} - \gamma \tfrac{1}{2} \boldsymbol{\alpha}' \cdot \sum \boldsymbol{\alpha} - \lambda \left(\textstyle\sum_{i=1}^{d} \alpha_i - 1 \right) \tag{64}$$

6.1 Adding robustness in mean returns

It is difficult to find a reliable approximation of the mean returns $\boldsymbol{\mu} = (\mu_1, ..., \mu_d)$. Thus, we may consider the robust (or **worst case**) mean, and maximize the objective.

so we want to compute

$$\begin{gathered} \max_\alpha \min_\mu \left[\boldsymbol{\alpha} \cdot \boldsymbol{\mu}\right] - \gamma \boldsymbol{\alpha}' \cdot \sum \boldsymbol{\alpha} \\ \Updownarrow \\ \max_\alpha \min_\mu \textstyle\sum_{i=1}^{d} \alpha_i \mu_i - \gamma \alpha_i \alpha_j \sigma_{ij}^2 \end{gathered} \tag{65}$$

where

$$\textstyle\sum_{i=1}^{d} \alpha_i = 1, \quad \mu_i^0 \leq \mu_i \leq \mu_i^1, \quad i = 1, ..., d \tag{66}$$

for some lower and upper bounds μ_i^0, μ_i^1. Denote

$$\hat{\mu_i} = \tfrac{\mu_i^0 + \mu_i^1}{2} \quad \epsilon_i = \tfrac{\mu_i^1 - \mu_i^0}{2}. \tag{67}$$

Then we have that

$$\min \mu_i = \hat{\mu}_i \alpha_i - \epsilon |\alpha_i|.$$

then since both $|\alpha_i|$ and $\alpha_i \alpha_j$ are concave functions, the objective itself is concave. Then it can be solved by standard numerical methods such as gradient descent or second-order methods.

6.1.1 Implement it

Using python,

```
scipy.optimize.minimize()
scipy.optimize.LinearConstraint()
```

7 Two-fund or one-fund theorem

Consider the two dimensional case $d = 2$ and that one of the assets is riskless, i.e. $\sigma_{11}^2 = 0$, this, by Theorem 2.1 implies as well that

$$\sigma_{12}^2 = 0.$$

Now note that a riskless asset S_1 has no variance, therefore

$$\mu_1 = \mathbb{E}R_1 = R_1.$$

We then have that

$$\mu_\alpha = \alpha_1 R_1 + (1-\alpha_1)\mu_2, \quad \sigma_\alpha = \sqrt{(1-\alpha_2)^2\sigma_{22}^2} = |1-\alpha_2|\sigma_{22}, \tag{68}$$

where $\alpha_1 \in \mathbb{R}$.

Then $(\mu_\alpha, \sigma_\alpha)$ forms a line in $\mathbb{R}^2$ for $\alpha_1 \in \mathbb{R}$ and a half-line for $\alpha_1 \leq 1$, which depends on (μ_2, σ_{22}) The idea is the following:

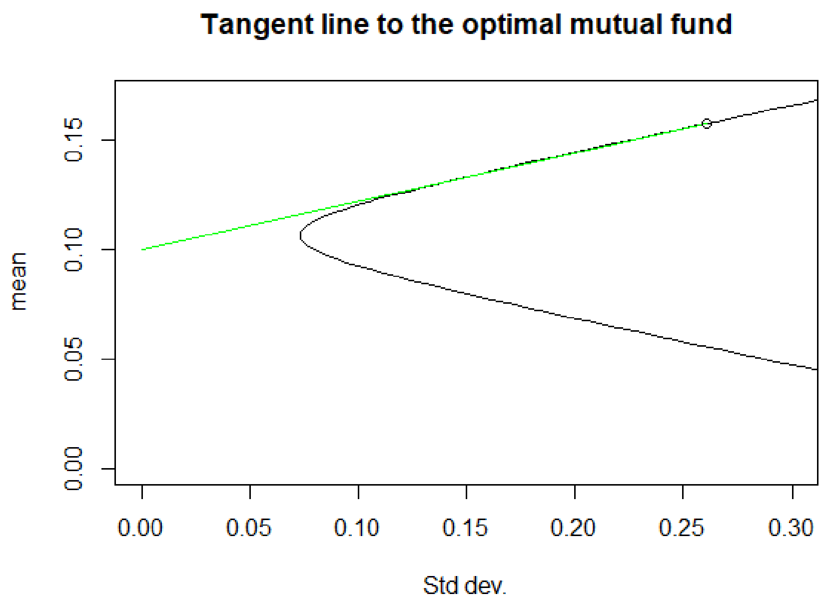

Figure 3: The tangent line to the Optimal Mutual fund. The optimal mutual fund is represented by the dot at which the green line is tangent to.

Note 7.1 If we have a riskless asset

$$\sigma_{00}^2 = 0$$

and the remaining risky assets cannot generate a riskless portfolio return $\sigma_\alpha^2 = 0$, then the matrix $\sum$ is invertible.

Note 7.2 Consider all (σ, μ) generated by portfolios of risky assets only. Then, under the above assumption of invertibility, at most one of these points lies on the efficient frontier of the **overall market**. We can call this point the **optimal mutual fund.** (the overall market *includes the riskless asset*).

7.1 Finding the optimal mutual fund

There is only one optimal mutual fund, if we assume that the submatrix

$$(\sigma_{ij})_{ij}$$

is invertible.
This point must lie on the efficient frontier of the *overall market*. Given a portfolio mean of returns $\mu_\alpha = y$, then we wanna find

$$\min_\alpha \boldsymbol{\alpha}' \cdot \sum \boldsymbol{\alpha} = \quad \min_\alpha \textstyle\sum_{i=0}^d \alpha_i \alpha_j \sigma_{ij}^2 \tag{69}$$

subject to

$$\textstyle\sum_{i=0}^d \alpha_i \mu_i = y, \quad \sum_{i=0}^d \alpha_i = 1. \tag{70}$$

Then using the same lagrangian as we did in Equation 57, but starting our sum at $j = 0$, we get

$$L = \tfrac{1}{2} \textstyle\sum_{i,j=0}^d \alpha_i \alpha_j \sigma_{ij}^2 - \lambda_1 \left(\sum_{i=0}^d \alpha_i \mu_i - y \right) - \lambda_2 \left(\sum_{i=0}^d \alpha_i - 1 \right). \tag{71}$$

Then, as we did in Equation 59, taking the derivatives

$$\tfrac{\partial}{\partial \alpha_i}, \quad \tfrac{\partial}{\partial \lambda_1}, \quad \tfrac{\partial}{\partial \lambda_2}, \tag{72}$$

we get the following matrix problem:

$$\begin{pmatrix} & & & & -\mu_0 & -1 \\ & & & & -\mu_1 & -1 \\ & \sum & & & \vdots & \vdots \\ & & & & -\mu_d & -1 \\ \mu_0 & \mu_1 & \cdots & \mu_d & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_d \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ y \\ 1 \end{pmatrix} \tag{73}$$

$$\Updownarrow$$
$$Ax = b.$$

Note that the matrix A has dimension $d + 3 \times d + 3$ here.

$$\textstyle\sum_{j=0}^d \sigma_{ij} \alpha_j - \lambda_1 \mu_i - \lambda_2 = 0, \tag{74}$$

Now. We should observe that the first asset S_0 is riskless, this means that $\sigma_{00}^2 = 0 \to \sigma_{i0}^2 = \sigma_{0i}^2 = 0$ for all i, as I proved before. This means that the Risk matrix $\sum$ is actually

$$\sum = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \sum_{11} & \\ 0 & & \end{pmatrix}, \tag{75}$$

where

$$\Sigma_{11}$$

is the risk matrix of the risky assets $\{\sigma_{ij}^2\}_{i,j \geq 1}$. Then the problem looks now like

$$\begin{pmatrix} 0 & \cdots & 0 & & -\mu_0 & -1 \\ \vdots & \sum_{11} & & & -\mu_1 & -1 \\ 0 & & & & \vdots & \vdots \\ & & & & -\mu_d & -1 \\ \mu_0 & \mu_1 & \cdots & \mu_d & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_d \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ y \\ 1 \end{pmatrix} \tag{76}$$

$$\Updownarrow$$
$$Ax = b.$$

The <u>question is</u>: the same as before! But now use the equation coming from the first row, in order to isolate λ_2, and assume we don't find the optimal mutual fund by inverting in the riskless asset (i.e., $\alpha_0 = 0$), then this system becomes:

$$\begin{pmatrix} & \Sigma_{11} & & \begin{matrix}-\mu_1 \\ \vdots \\ -\mu_d\end{matrix} & \begin{matrix}-1 \\ \vdots \\ -1\end{matrix} \\ \mu_1 & \cdots & \mu_d & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \\ \lambda_1 \\ -\mu_0\lambda_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y \\ 1 \end{pmatrix} \tag{77}$$

$$\Updownarrow$$
$$Ax = b.$$

This is exactly the same matrix formulation that we had in Equation 59, but now, we only have $d+1$ variables, instead of $d+2$. So if the covariance matrix of risky assets is invertible, this system may not find a solution depending on the values of the riskless return μ_0. Is this what the professor said in class? The system is now overdetermined.

He says that Set $\lambda_1 = 1$, the the system of Equation 77 has a unique solution (so far so good). Now, if this solution $\boldsymbol{\alpha}$ is such that

1. $$\sum_{i=1}^{d} \alpha_i \neq 0,$$

2. $$sign\left(\sum_{i=1}^{d} \alpha_i\right) = sign\left(\sum_{i=1}^{d} \alpha_i(\mu_i - \mu_1)\right),$$

then we can multiply $\boldsymbol{\alpha}$ by a cosntant so that the resulting vector solves the desired system Equation 74?? and produces the optimal mutual fund??

8 Dickey Fuller test

The null hypotheses of the Dickey Fuller test is that X_t **is a random walk.** The alternative is that it is a stationary process.

Augmented DIckey fuller test: We test now a richer hypotheses.

$$\begin{aligned} X_t = &\quad \textstyle\sum_{i=1}^{p} \phi_i X_{t-i} + \epsilon_t - AR(p) \\ \phi = &\quad 1 - \textstyle\sum_{i=1}^{p} \phi_i z^i \end{aligned} \tag{78}$$

He doesn't want to write down the statistic for this test. I t is complicated. Python can chose the p for this sum.

H_0 Here exists a root of ϕ inside the unit circle. Not stat
H_1 not stationary.

In python:

```
my_DF = stattools.adfuller(my_AR1,1,'c',None)
#1 is binding or python tests up to 1, i.e. choose ``'p' up to 1, if I choose ``c'' forces order 1.
print(my_DF[0])#test statistic value
print(my_DF[1])#pvalue
print(my_DF[2])#orer of AR tested
# he obtains
-4.50217
0.0001948 # this means that process is indeed stationary
1
```

Let's now trick python- Sergey says-
It is too difficult to test for stationaryti. So we impose AR assumption. Now I put MA which is not AR, let's see what happens, we obtain:

```
my_DF = stattools.adfuller(my_MA2)
print(my_DF[0])#test statistic value
print(my_DF[1])#pvalue
print(my_DF[2])#orer of AR tested
# he obtains
-6.162288
7.1259e-08
21# you have to keep track of your entire history, it is not useful, Nadthochiy says.
```

Now let's see what happens with a random walk.

$$\phi = 1$$

```
my_DF = stattools.adfuller(rw)
print(my_DF[0])#test statistic value
print(my_DF[1])#pvalue
print(my_DF[2])#orer of AR tested
# he obtains
-1.0518
0.73390# my confidence for rejecting H0 is only 20\%. Cannot reject H0
2
```

This is what we expected because this is not a stationary series.

Now let's apply it to real data.

```
my_DF = stattools.adfuller(price)
print(my_DF[0])#test statistic value
print(my_DF[1])#pvalue
print(my_DF[2])#orer of AR tested
# he obtains
```

Gives not stationarity. As expected. Now let's try it on log returns.

```
my_DF = stattools.adfuller(l_ret,1)
print(my_DF[0])#test statistic value
print(my_DF[1])#pvalue
print(my_DF[2])#orer of AR tested
# he obtains
-33
0.0
0
```

But order 0 AR(0) is white noise. So this log returns are completely stationary. But we cannot trade based on the log returns. Because a return can go up, but the price can keep going down for that while. So we actually would make no money.

We look for autocorrelation function or autocovariance function, and if we find a white noise then we are done. But **what is the distribution of the white nosie ϵ_t?**

8.1 distribution of the white nosie ϵ_t

$$\{\epsilon_t\} - WN$$

Assumption:

$$\epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

Ways to test this:

1. Histogram: In each bin we put the of ϵ_t in the bin, and report this number as the height of the bin. In python use

   ```
   plt.hist(price,30)
   plt.hist(price,30,(-0.07,0-07))#to specify x limits
   ```

   We want to apply a histogram on *white noise*, something that is iid. Otherwise it has no meaning. We want to analyse it as the

   $$\Pr(\epsilon_t \in bin).$$

   The histogram should remind us, after normalization, of the density function. We look for “heavy tails”. The heavier the tails, the higher the risk of loosing money or ganining money.

2. Q-Q plots. Quantile-Quantile function plots. let’s call q the quantile function of $\mathcal{N}(0, 1)$, and q^N the sample quantile function. (normalized). We plot these quantiles, they should form a diagonal line. He draws a growing ladder. Plots the sample CDF (a staircase) and inverts numerically.
   We expect, if the data is $\mathcal{N}(0, 1)$ to form a Q-Q diagonal plot

   ```
   iid_norm = iid/np.std(iid)
   l_ret_norm = (l_ret - np.mean(l_ret))/np.std(l_ret)

   gofplots.qqplot(iid_norm)#by default python will compare to std normal
   gofplots.qqplot(l_ret_norm)#now we plot real financial data
   ```

   We observe heavy tails when we do the Q-Q plot for real market data.

8.2 Cointegration

Now we get to the heart of the chapter.
Definition: 2 time series X, Y such that a linear combination of them is stationary, then we say they are **cointegrated**.

P_i price process of asset i.
P_j price of process of asset j.

If they are cointegrated, $aP_i + bP_j$ are *tradable*!. You can make money.
Typically $a, b \neq 0$ and

$$ab < 0.$$

There are reasons for this to happen. a can be something like a 1000. If we trade with expensive stocks the the **trading errors will be large**. If stocks are cheap, a gets to be big, and rounding errors will be small.

So we can divide by

$$P_j - C_{ij} Pi \text{ stationary} : C_{ij} > 0, \tag{79}$$

and this will be stationary.
question: why is it natural to assume that such C_{ij} exists?
Answer: **Relative Misspricing**. Ass soon as P_j is higher than P_i then people will start selling j and buying i and viceverse, so it will oscillate around 0. this is why it is stationary. You won’t trust statistics. *You only trust both statistics and economics analysis. You only accept cointegrated companies if they trade on similar assets.* Even if statistics say so, you would never consider that Shell is cointegrated with Education First. Oil and Education are very independent.

$$\begin{aligned} P_i^t &= a_i + \beta_i R^t + S_i^t \\ P_j^t &= a_j + \beta_j R^t + S_j^t \end{aligned} \tag{80}$$

R^t "industry index".
S_i, S_j independent stationary, mean zero.

$$\begin{aligned} C_{ij} = & \frac{\beta_j}{\beta_i} \Rightarrow Z_{ij}^t = P_i - C_{ij} P_j^t \\ = & \quad a_i - C_{ij} a_j + S_i^t - C_{ij} S_i^t. \end{aligned} \tag{81}$$

These two assets i, j are correlated. But I can kill the common component R^t. Correlated assets that are a random walk, can still be a random walk, and therefore useless for making money. If two assets are cointegrated, we can trade with a linear combination of them. is stationary.

This is related to CAPM model.
Some of the python code:

```
my_arima = arima_model.ARMA(price,[1,0],price2)
arma_results = my_arima.fit()
print(arma_results.arparams)
print(arma_results.)
```

He also runs a dikey fuller test on this. He gets 0.123 as p-value, 88% of rejecting null hypotheses.

9 27.02.2020

1. Find candidate C_{ij}
2. Check if
$$Z_{ij} = P_i^t - C_{ij} P_j^t = \hat{a} + \hat{S}^t$$
is stationary. whre
$$C_{ij} = \beta_i / \beta_j$$

OLS

The problem in making
$$\Delta P_i^t = C_{ij} \Delta P_j^t + \Delta Z_{ij}^t$$
it a linear regression is not possible. Because we need the errors
$$\Delta Z_{ij}^t$$
to be independent of time

Book for Cointegration and Time Series: book of Carmona. It also contains information of Q-Q plots.

The second option is using the **Kalman filter** using

```
statsmodels.tsa.arima_model.ARMA()
```

to fit
$$P_i - C_{ij} P_j^t = \epsilon^t + \phi P_i^{t-1} \tag{82}$$
The exogenous coefficient would be C_{ij}

He wants to reject with a small p value, he doesn't want to trade with that pai. A pair is **EIG.?**

He says that other methods are not significantly better than the OLS regression. Even though the theoretical framework of the OLS regression, Prof. claims, is not met.

ϕ close to 1, means that your model
$$X_t = \phi X_{t-1} + \epsilon^t$$

will do long excursions up and down. A $\sigma = 0.5$. Recall that

$$X := Z_{ij} = P_i - C_{ij} P_j$$

. "Buying a pair" means buying units on P_i and *short selling* units of P_j.

We integrate all stocks we have. We choose our favourite method to do Idk what, then the run a Dickey fuller test to see if they are cointegrated. But when do we have to sell and buy? Which are the values λ_1, λ_2 at which we do this?
We always wanna trade when the price is below the mean. We sell high and we buy low. The question is *How to do it?*

9.1 Cointegrated log-prices

We will say that *cumulative returns* are cointegrated. The log of the prices behaves as before. Not the Price itself. This model is much more realistic in reality and in theory. This is nothing else than CAPM. In CAPM we had to assume that S_j^t where white noise. Here we don't have to assume that.

$$\begin{aligned} \log P_i &= a_i + \beta_i R^t + S_i^t \\ \log P_j &= a_j + \beta_j R^t + S_j^t \end{aligned} \tag{83}$$

If we find a cointegrated pair, its logarithm is even more cointegrated.
We obtain

$$\log P_i - C_{ij} \log P_j^t \tag{84}$$

how can we make money out of this? using $t << 1$,

$$\log p_i^t / p_i^0 \sim= \frac{p_i^t}{p_i^0} - 1,$$

so

$$\frac{p_i^t}{p_i^0} - 1 + \log P_i^0 - C_{ij}(\frac{p_i^t}{p_i^0} - + \log P_k^0) = \frac{1}{p_i^0} p_i^t - \frac{c_{ij}}{p_i^0} P_j^t + ctt,$$

To buy log-cointegrated pair, invest +\$ 1 in i-th asset, and -\$ C_{ij} in j-th asset. **All this framework only works for $t << 1$. For a short amount of time.**

9.2 How to optimally choose the y-axis lebels λ_1, λ_2

When to buy and sell our assets?

$$\begin{aligned} P_i - C_{ij} P_j^t &= \epsilon^t + \phi P_i^{t-1} \\ X^t &= (1-p) X^{t-1} + \epsilon^t, \phi < 1, \end{aligned} \tag{85}$$

$\sigma^2 = \mathrm{Var}(\epsilon^t), \phi = 1 - p, p > 0$

$$\hat{\sigma}^2 = \mathrm{Var}\, X_t = \frac{\sigma^2}{1 - \phi^2}$$

Some people choose

$$\begin{aligned} \lambda_1 &= -\tfrac{3}{2}\hat{\sigma} \\ \lambda_2 &= -\tfrac{1}{2}\hat{\sigma} \end{aligned} \tag{86}$$

I want:

$$profit = l_2 - l_1,$$

to be large. But a principle in the market, is that you cannot make a lot of money without risk. Where does the risk come from?

-> mean reversion time - we want it small. We don't want to wait for 100 years to have a selling opportunity. This gives intermediate losses. That can be fatal for a company.

->The main risk of this process is that 2 assets stop being cointegrated.

Actually we care about profits per time.How long will it take to get to λ_2 ? We assume $\sigma = 0$

$$\begin{gathered} X^t = \phi X^{t-1} + \epsilon^t \\ \downarrow \\ \Delta X^t = -p X^{t-1} \\ \downarrow \\ \frac{dX^t}{dt} = -\rho X^t \\ X^0 = \lambda_1 \end{gathered} \tag{87}$$

we get the following solution:

$$\begin{gathered} X^t = \lambda_1 e^{-\rho t} \\ T = \log\left(\frac{|\lambda_1|}{|\lambda_2|}\right) / \log(1-p) \end{gathered} \tag{88}$$

$$\mathbb{E}\tau = \frac{\sqrt{\pi}}{\hat{\sigma}\sqrt{\rho}} \int_{\lambda_1}^{\lambda_2} e^{\frac{y^2\rho}{\hat{\sigma}^2}(1+\text{erf}\left(\frac{y\sqrt{p}}{\hat{\sigma}}\right))} dy \tag{89}$$

then given $x^* = \lambda_1$ maximize

$$\lambda_2 : \max_{\lambda_2 > \lambda_1 + c} \left[\frac{\lambda_2 - \lambda_1 - c}{\rho}(\tau - \mathbb{E}\tau)\right] \tag{90}$$

We want the expectation $\mathbb{E}\tau$ to be small, where c is the cost, T the maximum holding time and $P = P_i^0$ the initial price, so we normalize it.

10 About homework 3

He says tat

$$\begin{gathered} P_i - C_{ij} P_j = Z_{ij} \\ P_j - \frac{1}{c_{ij}} P_i = \frac{1}{c_{ij}} Z_{ij} \end{gathered} \tag{91}$$

wher $i < j$. Asset i is on the long side of the pair. The asset with the smallest index is on the long side of the pair.

11 Class of 03.03.2020

11.1 Optimal exit from a pair

He draws again a kind of random walk process and he asigns two y-values, $\lambda_1 < \lambda_2$. He says we buy for price λ_1 and sells when the asset gets price λ_2. The cointegrated prices can change. The **model risk** cannot be modelled within the model. There is no canonical choice of a simple objectice to maximize. *Sergey's choice is:* the one of Equation 90. Together with the condition that

$$\begin{gathered} P = P_i^0 \\ T - \text{ max. holding time} \\ c \sim 4P\lambda \end{gathered} \tag{92}$$

The situation is as follows He chooses the input parameters for the model

$$\begin{aligned} Z_{ij}^t &= (1-\rho) Z_{ij}^{t-1} + \epsilon^t, & \rho &> 0 \\ \sigma^2 &= \text{Var}(\epsilon^+), & \hat{\sigma}^2 &= \text{Var}(Z_{ij}^t) \end{aligned} \tag{93}$$

The expected time is 20 days. Now he will adress two more questions

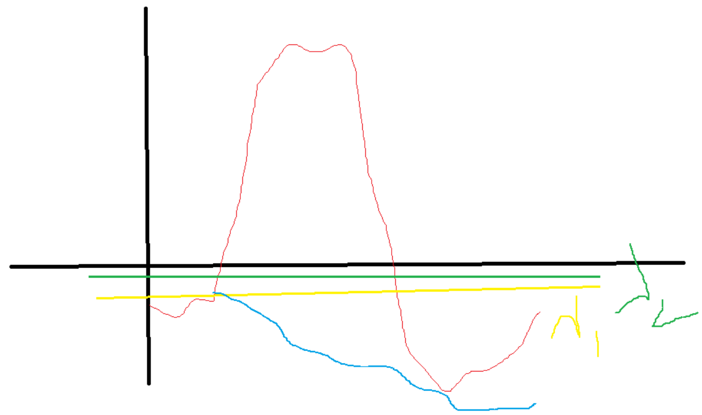

Figure 4: Framework of when we should exit or sell. The risky case is that since we buy for λ_1 price, the assets follows the blue path and we get losses.

11.2 What do do if a pair de-integrates

In general this is THE question. We can do the **stop-loss**: Liquidate pos. in the pair if

$$\text{holding time} > \bar{T}.$$

Or if

$$\bar{\lambda}$$

is $\hat{\sigma}$ units away from the mean. In these two situations we can say that the pair- de integrated.

Regime-switching (hidden Markov). Models - detection of switching.

12 How to trade multiple pairs

Now the question of optimality is done. Now we focus on a new problem. Question: How to allocate capital among cointegrated pairs?

1. Invest \$ $\bar{C}$ in the asset on the long side of each pair. (long or short).
2. We claim that some pairs are better than others. We will call V the value of

$$V = \max_{\lambda_2 > \lambda_1 + c} \left[\frac{\lambda_2 - \lambda_1 - c}{\rho} (\tau - \mathbb{E}\tau) \right]. \tag{94}$$

   We want to invest more in those assets that have large V. Invest \$ $\bar{C}V$ in the long side of the pair.

12.1 Implementation

Using a running window. Assume that the amount of days is $M = 2570$. We will use $N = 100$ days for callibration. We estimate everything using this 100 days sample. This is our estimation window. Now we find all cointegrated pairs $i < j, p \leq p^*(= 5\%)$ and we find out which are the associated

$$\{C_{ij}\}_{ij}.$$

For each cointegrated pair, we compute λ_2^*, V and we determine the investment size.

Then for the second part we have to check whether

1. It has reached λ_2^* (close it)
2. it is in *stop-loss* category (close it)

Compute PnL At each $t = N+1, ..., M$, I+II $\Rightarrow$ number of shares of each basic asset $i = 1, ..., d$ to hold at time t:

$$\begin{array}{c} \psi^t = (\psi_1^t, ..., \psi_d^t) \\ \Downarrow \\ \$L^t - \text{ total } \$ \\ \$S^t - \text{ total } \$ \text{ short} \end{array} \tag{95}$$

Absolute PnL (in \$) $\bar{P}$

$$\begin{Bmatrix} \bar{P}^t = \bar{P}^{t-1} + \psi^{t-1}(T^t - P^{t-1}) - |\psi^{t-1} - \psi^{t-2}|P^{t-1}\lambda, & t = N+1, ..., M \\ \psi^{N+1} = 0, & \bar{P}^N = 0 \end{Bmatrix} \tag{96}$$

12.2 Multiple cointegrated log-prices

$$\begin{array}{c} Z_{ij} = \log P_i - C_{ij} \log P_j \\ F^t = Z_{ij}^t, \end{array} \tag{97}$$

F^t is a strong predictive factor for the portfolio. Portfolio: For each \$ 1 we invest in i-th asset we sell $\$C_{ij}$ units from asset j.

Now each pair is an asset to trade with. Before, in the previous subsection, we bought the pair and hold it for weeks, until it returned to have a positive price. Now we are interested in selling after 1 day only. We will be doing mean-variance optimization, or CAPM.

$$\frac{P_i^{t+1} - P_i^t}{P_i^t} - C_{ij} \frac{P_j^{t+1} - P_j^t}{P_j^t} \sim Z_{ij}^{t+1} - Z_{ij}^t, \tag{98}$$

this is the absolute return of the pair.

We now do something like

$$\begin{array}{c} \Delta Z_{ij}^{t+1} = Z_{ij}^{t+1} - Z_{ij}^t = -\rho Z_{ij}^t + \epsilon_{ij}^{t+1}, \\ \epsilon_{ij} \sim WN, \end{array} \tag{99}$$

Now we will take a mean and covariance so that we can apply the techniques of Chapter 1. Note that $\mathbb{E}_t$ refers to the conditional epectation up to time t. It would be $\mathcal{F}_t$.

$$\begin{array}{rl} \mu_{ij} = \mathbb{E}_t(Z_{ij}^{t+1} - Z_{ij}^t) = & -\rho Z_{ij}^t + \mathbb{E}_t \epsilon_{ij}^{t+1}, \\ -\rho Z_{ij}^t + 0.\epsilon_{ij} \sim WN, & \end{array} \tag{100}$$

and

$$\Sigma_{(ij)(i'j')} = \mathrm{Cov}_t(\Delta Z_{ij}^{t+1}, \Delta Z_{i'j'}^{t+1}) = \mathrm{Cov}(\epsilon_{ij}^{t+1}, \epsilon_{i'j'}^{t+1}) \sim \text{ sample covariance} \tag{101}$$

Now assume that we have $(i < j)$ cointegrated paris,

α_{ij} amount to invest in the i-th asset (+ or -)

The objective is now

$$\max_\alpha \left[\sum_{ij} \alpha_{ij}\mu_{ij} - \sum_{l=1}^d \lambda \left| \sum_{ij} \kappa_{ij}^l \alpha_{ij} - a_l^{t-1} \frac{p_l^t}{p_l^{t-1}} \right| - \gamma \sqrt{\sum_{(ij)(i'j')} \sigma_{(ij)(i'j')}^2 \alpha_{ij}\alpha_{i'j'}} \right] \tag{102}$$

so that

$$
\begin{aligned}
\textstyle\sum_{ij} |\alpha_{ij}| = \$\hat{C} > 0 \\
\lambda \text{ is the proportion of T-cost} \\
\kappa^l_{ij} = \begin{cases} 1, & \text{if } l = i \\ -C_{ij}, & \text{if } l = i, \\ 0, & \text{otherwise} \end{cases} \\
a^t_l = \text{ \$ invested in asset } l \text{ at time } t
\end{aligned}
\tag{103}
$$

and

$$
\begin{aligned}
\xi^t_l &= \frac{\sum_{ij} \kappa^l_{ij} \alpha^*_{ij}}{P^t_l}, \quad l = 1, ..., d \\
a^t_l &= \xi^t_l P^t_l
\end{aligned}
\tag{104}
$$

You don't truly believe in the model. You just believe in a trend, that it will end up going back to 0.

This is the end of pairs trading.

13 Fundamentals of Market micro-structure

What is this technology and what did it change in the last 20 years. Until the mid 1950's, all that was there was the *Old exchange*. There used to be the **Buyers** and the **Sellers**. To related the two, a middle man was introduced, called the **market-maker**, specialist or dealer, broker. So buyers and sellers would ONLY interact using this *specialist*.

Specialists quote prices. For the business to be profitable to the broker,

$$B < A$$

bid and ask.

$$A - B$$

is the spread or profit of the specialist. Electronic exchanges (NASDAQ) For the common stocks, however, for common non-stock assets, the maket is still old-fashioned.

What is the difference and why did we move from one to the other one? The idea of the electrionic exchange is that everyone can be a market maker! There is no longer need for the middle man. The strategy of **market making** is to be the middle man, and try to buy for less than you sell and make money out of this comission.

The *biggest risk* of the market maker is to have too much of some asset.

13.1 Limit order book (LOB)

In an electronic exchange, participants can submit two times of ?

1. market order quantity (MO) and direction (buy,sell). This is the old-fashioned way of buying and selling.
2. limit order (LO): here we have price, quantity and direction. We say how much we want to buy and at how much. But our order may meet no buyer/seller and is stored in a book until your offer/demand is matched.

14 Exercise 1 - finding the efficient frontier

(a) We produce the estimated vector of mean returns and the estimated covariance matrix. We save them in ".csv" files.

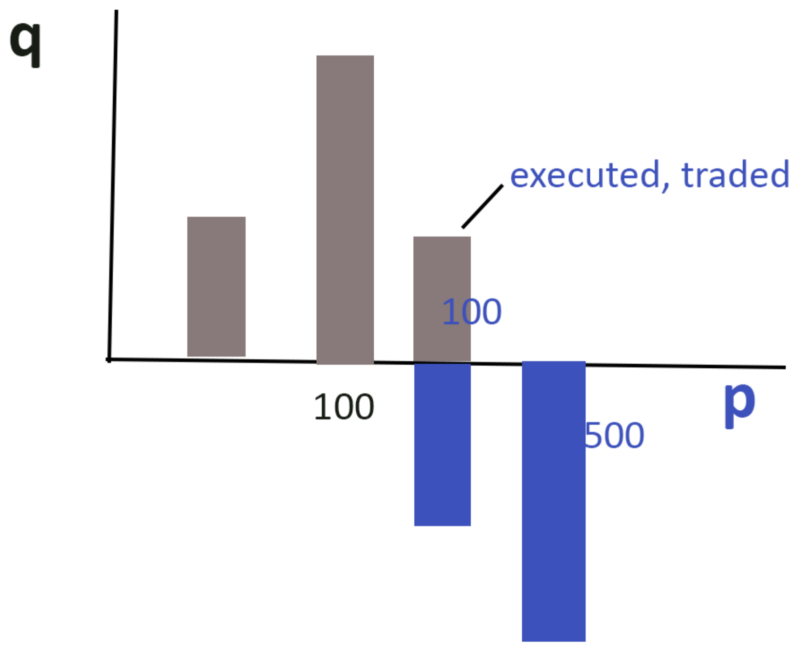

Figure 5: Limited book, when an offer is matched the order is exectued

So we have that the return of the stock of a company i is

$$R_i^t = \frac{P_i^t - P_i^{t-1}}{P_i^{t-1}},$$

where P_i^t is the closing price on date t, and $t-1$ is the last closing price before the closing price at time t was reported. In our data set

```
In [4]: print(stock_data)

Attributes    Adj Close                                                    \
Symbols            AAPL        MSFT        TSM       INTC       CSCO
Date
2009-01-02    11.253528   15.635055   5.567235  10.736197  13.128983
2009-01-05    11.728474   15.781175   5.458869  10.531362  13.245105
2009-01-06    11.535025   15.965750   5.689144  10.856274  13.771499
2009-01-07    11.285772   15.004425   5.221822  10.199384  13.407664
2009-01-08    11.495339   15.473547   5.106684  10.277082  13.577968
...                 ...         ...        ...        ...        ...
2019-01-28   153.621765  103.548927  35.749737  45.349712  44.510239
2019-01-29   152.029510  101.440102  34.689846  45.184669  44.714546
2019-01-30   162.418411  104.829979  35.692440  46.155552  45.444221
2019-01-31   163.587997  102.908394  35.921604  45.747780  46.008507
2019-02-01   163.666641  101.282433  35.873859  47.310894  46.057152
```

Figure 6: Viewing a part of our stock data set

We have one closing price P_i^t per day, during the almost 11 years we are studying which run from January 1st 2009 until December 31st 2019.

I did the following

```
import pandas as pd
import numpy as np
```

```
import csv
import matplotlib.pyplot as plt
import math
from pandas_datareader import data
import scipy.optimize
from scipy import stats

tickers_file = 'TechTickers.csv'
tickers = [];
f = open(tickers_file,"r",encoding='utf-8-sig')
for line in csv.reader(f):
    tickers.append(str(line[0]))
f.close
print(tickers)

#downoad the prices and volumes for the previously read list of tickers for the first month
#of the earliest year in the proposed time period
start_date = '2009-01-01'
end_date = '2019-01-31'
stock_data = data.get_data_yahoo(tickers, start_date, end_date)

#create a list of tickers whose adjusted closing prices in the first month of the first year
#do not have any missing values
stockArray = []
for ticker in tickers:
    stockArray.append(list(stock_data['Adj Close'][ticker]))
stockArray=np.array(stockArray)
tickers_liq = []
for i in range(len(tickers)):
    temp=0
    for j in range(len(stockArray[i,:])):
        if math.isnan(stockArray[i,j]):
            temp=1
    if (temp==0):
        tickers_liq.append(tickers[i])
print(len(tickers_liq))
print(tickers_liq)

#save the list of tickers without any initial missing values
tickers = list(tickers_liq)
np.savetxt("TechTickers_liq.csv",np.array(tickers),fmt='%1s',delimiter=',')

#trying commands:
np.size(stock_data.values[1,:])
ret=np.zeros(( np.size(stock_data.values[:,1])-1 , np.size(stock_data.values[1,:] ) ))
print(range(0,np.size(stock_data.values[1,:])))

adjclose = stock_data['Adj Close']
np.shape(adjclose)
#np.shape(stock_data)
#print(adjclose)

#1(a) np.size(stock_data.values[1,:])
adjclose = stock_data['Adj Close']
ret=np.zeros(( np.size(adjclose.values[:,1])-1 , np.size(adjclose.values[1,:] ) ))
for i in range(0,np.size(adjclose.values[1,:] )):
```

```
    pto = adjclose.values[0:np.size(adjclose.values[:,1])-1,i]
    ptf = adjclose.values[1:np.size(adjclose.values[:,1]),i]
    ret[:,i] = (ptf-pto)/pto
ret2 = adjclose.pct_change()
#print(ret2)
means = ret2.mean(axis=0)
covs = ret2.cov()
print(means)
print(covs)

#saving the files:
meanss=np.array(means)
np.savetxt("means1a.csv",np.transpose(meanss),delimiter=',')
covss=np.array(covs)
np.savetxt("covs1a.csv",np.transpose(covss),delimiter=',')
```

(b) Compute the weights of the minimal-variance portfolio and save them in '.csv' file.

(c) Compute the weights of the optimal mean-variance portfolio (i.e., maximizing a linear combination of mean and variance) with the risk tolerance $1/\gamma = 1$. Save the weights in a '.csv' file and plot them on a graph. Output the mean and variance of the resulting portfolio.