

# Math models for Algorithmic trade 2020 Homework 2

Romà Domènech Masana A20450272

February 2020

## Contents

### 1 Concepts on Finance

#### 1.1 What is a risky asset?

**Asset:** It is a property owned by a person or company. It has a value and can meet debts. We can call these

$$S_i^t,$$

this represents the value of an asset that we call  $S$ , of a company or person  $i$  at a specific time  $t$ . An example would be the value of the amount of apples ( $S$ ) that a peasant  $i$  has today ( $t$ ) in his warehouse. We assume the assets  $\mathbf{S}$  to be a random vector  $\mathbf{S} : \Omega \rightarrow \mathbb{R}^d$ .

**Rate of return** (of 1 asset): Is the net gain on an asset over a period of time. It is expressed as a percentage with respect to the investment's initial cost. More precisely it is

$$R_i^t = \frac{S_i^t - S_i^{t-1}}{S_i^{t-1}}. \quad (1)$$

This is just telling us how the value of that amount of apples owned by the peasant, is changing over time. It could decrease naturally since the apples expire. Or rise, if the people suddenly feel hungry for apples and wanna buy them and consume them at any cost.

**Portfolio:** A portfolio  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)$  is the set of investments held by a person or company. Each investment is made for a specific product, and then it becomes an asset. A portfolio can also be understood then as a set of assets we hold.  $\xi_1$  is the asset for apples for instance. Then we can also call this asset of ours our “share” of apples from the peasant's farm. A share is an indivisible unit of capital of a company, owned by a shareholder or investor. To explain things step by step, farm  $i$  is divided into an amount of shares. We buy or possess  $\xi_i$  of them, then the product  $\xi_i S_i^t$  is telling us the asset or capital or value we have from that company. We can play with it, we can sell it, buy more, or do nothing.

Is share, asset and investment the same thing? What are the differences? Also, according to Wiki a share is “an indivisible unit of capital, expressing the ownership relationship between the company and the shareholder.” So is a share \$1? Is an asset a collection of shares? I got lost here.

**Rate of return** (of a portfolio): To put things simple. If we have only two shares  $\boldsymbol{\xi} = (\xi_1, \xi_2)$  in two different farms  $S_1, S_2$ , then just realize that our initial (at time  $t$ ) assets or investment is  $\xi_1 S_1^t + \xi_2 S_2^t$ . Then what we have at time  $t + 1$  is just  $\xi_1 S_1^{t+1} + \xi_2 S_2^{t+1}$ . So the rate of return for the set of assets  $\boldsymbol{\xi}$  is just:

$$\begin{aligned} R_{\boldsymbol{\xi}}^{t+1} &= \frac{\sum_{i=1}^d \xi_i S_i^{t+1} - \sum_{i=1}^d \xi_i S_i^t}{\sum_{i=1}^d \xi_i S_i^t} \\ &= \frac{\sum_{i=1}^d \xi_i (S_i^{t+1} - S_i^t)}{\sum_{i=1}^d \xi_i S_i^t} \\ &= \frac{\boldsymbol{\xi} \cdot (\mathbf{S}^{t+1} - \mathbf{S}^t)}{\boldsymbol{\xi} \cdot \mathbf{S}^t}, \end{aligned} \quad (2)$$

where  $d = 2$ .

Why is  $S_i$  multiplied by  $\xi_i$ ? In other words, why is a money quantity  $S_i$  (the asset price) multiplied by another money quantity  $\xi_i$ ? Don't we then obtain  $\$^{2??}$ ? Isn't this a contradiction?

$\xi_i$  is not a money quantity! It is only the ammount of shares of a company  $i$ . The product  $\xi_i S_i$  is a money quantity

It is reasonable to call our total money 1, and then normalize the portfolio  $\xi$  into a normalised portfolio  $\alpha$

**Weights** (of a portfolio  $\xi$  and set of assets  $S$ ) are the quantities  $\alpha = (\alpha_1, \dots, \alpha_d)$  where

$$\alpha_i = \frac{\xi_i S_i}{\xi \cdot S}. \quad (3)$$

We observe that

$$\alpha_1 + \dots + \alpha_d = 1. \quad (4)$$

We also observe that, given  $d$  rates of return

$$R^t = (R_1^t, \dots, R_d^t) = \left( \frac{S_1^t - S_1^{t-1}}{S_1^{t-1}}, \dots, \frac{S_d^t - S_d^{t-1}}{S_d^{t-1}} \right)$$

and the fact that:

$$\begin{aligned} \frac{\xi_i(S_i^{t+1} - S_i^t)}{\xi \cdot S} &= \frac{\xi_i S_i^t \frac{S_i^{t+1} - S_i^t}{S_i^t}}{\xi \cdot S} \\ &= \alpha_i^t R_i^{t+1}, \end{aligned} \quad (5)$$

?? becomes actually

$$\begin{aligned} R_\xi^{t+1} &= \sum_{i=1}^d \frac{\xi_i(S_i^{t+1} - S_i^t)}{\xi \cdot S^t} \\ &= \sum_{i=1}^d \alpha_i^t R_i^{t+1} \\ &= \alpha \cdot R^t =: R_\alpha^t. \end{aligned} \quad (6)$$

So now we can talk about a *portfolio* by  $\xi$  or by  $\alpha$  so that  $\alpha_1 + \dots + \alpha_d = 1$ .

## 1.2 Expectations and variance of returns

I think  $R_\xi$  is a time process and for each time step  $t$ , the rate of return  $R_\xi^t$  is a random variable, just as Brownian motion is!

Here we will compute expectations and variances over time. So we will change the notations and instead of writing  $R_\xi^t$  for describing the rate of return at time  $t$  of a portfolio  $\xi$ , we will simply write  $R_\xi$  for describing all the process of rate of returns over time. We will compute the expectations and variances of these time-spanning processes.

I understand now the rate of return as a time process

$$R_\xi = (R_\xi^{t=1}, R_\xi^{t=2}, \dots, R_\xi^{t=T}), \quad (7)$$

and the expected rate of return, in practice, as the average value over time:

$$\mathbb{E}R_\xi = \frac{R_\xi^{t=1} + R_\xi^{t=2} + \dots + R_\xi^{t=T}}{T} \quad (8)$$

In reality it could be a random process  $R_\xi^t$  having a certain mean and variance.

Let's call the expected rate of return of an asset  $S_i$  by

$$\mu_i := \mathbb{E}R_i,$$

then the

**Expected rate of return** (of a portfolio  $\xi$ ) is, using ?? and the fact that the weights  $\alpha$  are assumed **now to NOT depend on time**. Why do we not take the expectation of  $\alpha$ ? Is it not a random process??  $t$  (constant portfolio or weights over time), they are constant, and therefore we have that:

$$\begin{aligned} \mu_\xi = \mathbb{E}R_\xi &= \sum_{i=1}^d \alpha_i \mathbb{E}R_i \\ &= \sum_{i=1}^d \alpha_i \mu_i. \end{aligned} \quad (9)$$

Also, if we call

$$\sigma_{ij}^2 := \text{Cov}(R_i, R_j) = \mathbb{E}(R_i - \mu_i)(R_j - \mu_j),$$

the variance of the rate of return of an asset  $S_i$ , then the

**Variance of the rate return** (of a two portfolio  $\xi$ ) will be, in two dimensions,

$$\begin{aligned} \sigma_\xi^2 = \text{Var}(R_\xi) &= \mathbb{E}(R_\xi)^2 - \mu_\xi^2 \\ &= \mathbb{E}[\alpha_1 R_1 + \alpha_2 R_2]^2 - (\alpha_1 \mu_1 + \alpha_2 \mu_2)^2 \\ &= \mathbb{E}[\alpha_1^2 R_1^2 + \alpha_2^2 R_2^2 + 2\alpha_1 \alpha_2 R_1 R_2] - \alpha_1^2 \mu_1^2 - \alpha_2^2 \mu_2^2 - 2\alpha_1 \alpha_2 \mu_1 \mu_2 \\ &= \alpha_1^2 \mathbb{E}[R_1^2 - \mu_1^2] + \alpha_2^2 \mathbb{E}[R_2^2 - \mu_2^2] + 2\alpha_1 \alpha_2 [\mathbb{E}(R_1 R_2) - \mu_1 \mu_2] \\ &= \alpha_1^2 \mathbb{E}[R_1 - \mu_1]^2 + \alpha_2^2 \mathbb{E}[R_2 - \mu_2]^2 + 2\alpha_1 \alpha_2 \mathbb{E}(R_1 - \mu_1)(R_2 - \mu_2) \\ &= \sum_{i,j=1}^2 \alpha_i \alpha_j \sigma_{ij}^2. \end{aligned} \tag{10}$$

Actually this can be written as

$$\sigma_\xi^2 = (\alpha_1, \alpha_2) \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \tag{11}$$

In  $d$  dimensions the result will be the same, the proof is just more cumbersome, and we will obtain that

$$\sigma_\xi^2 = \text{Var}(R_\xi) = \sum_{i,j=1}^d \alpha_i \alpha_j \sigma_{ij}^2. \tag{12}$$

Or, in other words:

$$\sigma_\xi^2 = (\alpha_1, \dots, \alpha_d) \begin{pmatrix} \sigma_{11}^2 & \dots & \sigma_{1d}^2 \\ \vdots & \dots & \vdots \\ \sigma_{d1}^2 & \dots & \sigma_{dd}^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \tag{13}$$

### 1.2.1 Diversification of portfolio for independent assets

Assume that we have  $d$  independent assets  $S_1, \dots, S_d$  and define the “uniformly distributed” portfolio

$$\xi_i = \frac{1}{d} \frac{1}{S_i} \quad \forall i, \tag{14}$$

then observe first that

$$\alpha_i = \frac{\xi S_i}{\xi \cdot \mathbf{S}} = \frac{1/d}{\frac{1}{d} + \dots + \frac{1}{d}} = \frac{1/d}{1} = \frac{1}{d} \quad \forall i.$$

Then the variance of your portfolio is, using ??:

$$\begin{aligned} \sigma_\xi^2 &= \sum_{i,j=1}^d \alpha_i \alpha_j \sigma_{ij}^2 \\ &= \frac{1}{d^2} \sum_{i,j=1}^d \sigma_{ij}^2 \\ &= \frac{1}{d} \left[ \frac{1}{d} \sum_{i,j=1}^d \sigma_{ij}^2 \right] \\ &= \frac{1}{d} \left[ \frac{1}{d} \sum_{i=1}^d \sigma_{ii}^2 \right], \end{aligned} \tag{15}$$

where the last equality is due to the fact that

$$\sigma_{ij} = 0 \quad \text{if } i \neq j$$

since we assumed the  $d$  assets to be independent of each other.

Now, if the average variance of the rates of returns

$$\frac{1}{d} \sum_{i=1}^d \text{Var}(R_i) \leq M < \infty,$$

can be assumed to be bounded by a constant  $M$ , then

$$\sigma_\xi^2 \leq \frac{M}{d} \xrightarrow{d \rightarrow \infty} 0. \tag{16}$$

This literally means that the portfolio variance  $\sigma_\xi^2$  vanishes if we completely uniformly diversify our assets, and keep increasing the amount of assets of our portfolio. *This event occurs as long as the assets of our portfolio are independent between themselves! And given that their average variance (or at least the variance of each of them) is bounded by a constant.*

**Riskless portfolio:** It is a portfolio  $\xi$  such that

$$\sigma_\xi^2 = 0.$$

Then according to our sample variance ?? this would mean that the rate of returns

$$R_i^t = R_i^{t+1} \quad \forall t,$$

In other words, the rate of returns for each asset  $S_i$  are constant over time.

### 1.3 Example 1: Zero variance $\sigma_\alpha^2 = 0$ portfolio selection

Let  $d = 2$  and consider the vector of rate of return means

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \mathbb{E}R_1 \\ \mathbb{E}R_2 \end{pmatrix} = \begin{pmatrix} 0.10 \\ 0.14 \end{pmatrix}, \quad (17)$$

With covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11}^2 & \sigma_{11}\sigma_{22} \\ \sigma_{11}\sigma_{22} & \sigma_{22}^2 \end{pmatrix}, \quad (18)$$

where  $\sigma_{11}^2 = 0.08^2$  and  $\sigma_{22}^2 = 0.18^2$ .

The question is, can we find a portfolio  $\alpha = (\alpha_1, \alpha_2)$  such that

$$\sigma_\alpha^2 = 0?$$

By taking a look at ?? we just need for instance that the product

$$\begin{pmatrix} \sigma_{11}^2 & \sigma_{11}\sigma_{22} \\ \sigma_{11}\sigma_{22} & \sigma_{22}^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0. \quad (19)$$

But this is easy, since the matrix  $\Sigma$  can be rewritten as

$$\Sigma = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{22} \\ \sigma_{11} & \sigma_{22} \end{pmatrix}, \quad (20)$$

which clearly has determinant zero, and the choice

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = c \begin{pmatrix} -\sigma_{22} \\ \sigma_{11} \end{pmatrix} = c \begin{pmatrix} -0.18 \\ 0.08 \end{pmatrix}, \quad (21)$$

makes the product  $\Sigma\alpha = 0$ , for any constant  $c \in \mathbb{R}$ , which of course, makes the product

$$\sigma_\alpha^2 = \alpha' \Sigma \alpha = 0.$$

Just as we desired.

But recall that they are normalized, so we must have that  $\alpha_1 = 1 - \alpha_2$  which implies that  $c = -10$  and so

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 0.8 \end{pmatrix}. \quad (22)$$

If we do this we obtain that the *expected rate of return* for this portfolio is

$$\mathbb{E}R_\alpha = \alpha \cdot \mu = c(-\mu_1\sigma_{22} + \mu_2\sigma_{11}) = -c0.0068 = 0.068, \quad (23)$$

this means that the riskless rate is 6.8%.

**Exercise 1** A riskless return is attainable if and only if, the rate of return covariance matrix  $\Sigma$  is invertible. We clearly see that if the matrix  $\Sigma$  is not invertible, then there must exist a non-trivial vector

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \neq 0, \quad (24)$$

such that

$$\begin{pmatrix} \sigma_{11}^2 & \dots & \sigma_{1d}^2 \\ \vdots & \dots & \vdots \\ \sigma_{d1}^2 & \dots & \sigma_{dd}^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} = 0. \quad (25)$$

We just need to see the opposite, namely, that if  $\Sigma$  is NOT invertible, then a vector  $\alpha$  satisfying equation ?? must be 0. But just note that all entries of the matrix  $\Sigma$  are  $\sigma_{ij}^2 \geq 0$ , this implies that  $\Sigma$  is non-negative definite, and therefore

$$\alpha' \cdot \Sigma \cdot \alpha = 0 \rightarrow \alpha = 0.$$

## 1.4 Diversification to reduce unsystematic risk (correlated assets)

If we express again the Covariance of the return rates for a uniform diversification portfolio

$$\alpha = (1/d, \dots, 1/d),$$

we see that

$$\begin{aligned} \sigma_\alpha^2 &= \alpha' \begin{pmatrix} \sigma_{11}^2 & 0 & \dots & 0 \\ 0 & \sigma_{22}^2 & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & \sigma_{dd}^2 \end{pmatrix} \alpha + \alpha' \begin{pmatrix} 0 & \sigma_{12}^2 & \dots & \sigma_{1d}^2 \\ \sigma_{21}^2 & 0 & \dots & \sigma_{2d}^2 \\ \sigma_{31}^2 & \dots & \ddots & \sigma_{3d}^2 \\ \sigma_{d1}^2 & \sigma_{d2}^2 & \dots & 0 \end{pmatrix} \alpha \\ &= \frac{1}{d} \frac{1}{d} (\sigma_{11}^2 + \dots + \sigma_{dd}^2) + \frac{1}{d^2} \sum_{i \neq j} \sigma_{ij}^2 \\ &= \frac{1}{d} \bar{\sigma}_d^2 + \frac{d(d-1)}{d^2} \overline{\text{Cov}}_d \xrightarrow{d \rightarrow \infty} \overline{\text{Cov}}, \end{aligned} \quad (26)$$

if we assume (1) a bounded average variance:

$$\bar{\sigma}_d^2 = \frac{1}{d} (\sigma_{11}^2 + \dots + \sigma_{dd}^2) \leq M < \infty,$$

and (2) the Covariance limit to exist:

$$\overline{\text{Cov}}_d = \frac{1}{d(d-1)} \sum_{i \neq j} \sigma_{ij}^2 \xrightarrow{d \rightarrow \infty} \overline{\text{Cov}}. \quad (27)$$

We call this Covariance limit the **systematic risk**. Just realise that we obtained that:

$$\sigma_\alpha^2 \xrightarrow{d \rightarrow \infty} \overline{\text{Cov}}, \quad (28)$$

under assumptions (1) and (2). The **unsystematic risk** would be  $\sigma_\alpha^2$ .

## 2 Std. Dev-Mean space $(\sigma_\alpha, \mu_\alpha)$

Here we assume that **mean**  $\mu_\alpha$  and **variance**  $\sigma_\alpha^2$  of the rate of return of the portfolio  $R_\alpha$ , are the only two things of the portfolio  $\alpha$  that we care about.

In Example 1 at ??, given a portfolio  $\alpha = (\alpha_1, \alpha_2)$  we have that  $\alpha_2 = 1 - \alpha_1$ . So

$$\mu_\alpha = \mu_1 \alpha_1 + \mu_2 (1 - \alpha_1), \quad (29)$$

and

$$\begin{aligned} \sigma_\alpha^2 &= \alpha' \cdot \Sigma \cdot \alpha \\ &= \alpha_1^2 \sigma_{11}^2 + (1 - \alpha_1)^2 \sigma_{22}^2 + 2\alpha_1(1 - \alpha_1) \sigma_{12}^2. \end{aligned} \quad (30)$$

Note that if we restrict  $\alpha \in [0, 1]$  then  $\mu_\alpha \in [\mu_1, \mu_2]$ , which makes sense. The question is, does  $\sigma_\alpha \in [\sigma_1, \sigma_2]$  in this case? Or can  $\sigma_\alpha$  fall outside of this interval?.

## 2.1 $(\sigma_\alpha, \mu_\alpha)$ is a straight line if $\text{corr} = 1$

In the next example, it falls within the interval. We say then that the  $\sigma_\alpha$  is within a **straight interval**. (This is my definition of straight interval, I hope Prof. Nadtochiy doesn't disagree. )

So note that the mean of returns of the portfolio  $\mu_\alpha$  depends linearly on the portfolio, but not so does the variance of the rate of return of the portfolio  $\alpha$ ,  $\sigma_\alpha^2$ . Actually for the just cited example, we have that ?? is now

$$\begin{aligned}\sigma_\alpha^2 &= \alpha_1^2 \sigma_{11}^2 + (1 - \alpha_1)^2 \sigma_{22}^2 + 2\alpha_1(1 - \alpha_1)\sigma_{12}^2 \\ &= \alpha_1^2 (\sigma_{11} - \sigma_{22})^2 + \alpha_1 2[\sigma_{11}\sigma_{22} - \sigma_{22}^2] + \sigma_{22}^2 \\ &= (\alpha_1[\sigma_{11} - \sigma_{22}] + \sigma_{22})^2\end{aligned}\tag{31}$$

We can now take the square root of  $\sigma_\alpha^2$ . This gives us the standard deviation of the portfolio (we only consider the positive definition of it). Recall that  $\sigma_{11}^2 \leq \sigma_{22}^2$

$$\sigma_\alpha = +|\alpha_1(\sigma_{11} - \sigma_{22}) + \sigma_{22}| = \begin{cases} \alpha_1(\sigma_{11} - \sigma_{22}) + \sigma_{22} & \text{if } \alpha_1 \leq \frac{\sigma_{22}}{\sigma_{22} - \sigma_{11}} \\ \alpha_1(\sigma_{22} - \sigma_{11}) - \sigma_{22} & \text{if } \alpha_1 \geq \frac{\sigma_{22}}{\sigma_{22} - \sigma_{11}} \end{cases},\tag{32}$$

Which is a linear function on  $\alpha_1$  This is not a straight line! and gives the interval

$$\sigma_\alpha \in [\sigma_{11}, \sigma_{22}] \text{ for } \alpha_1 \in [0, 1].$$

Thus the plot of

$$\{(\sigma_\alpha, \mu_\alpha) : \alpha = (\alpha_1, 1 - \alpha_1), \alpha_1 \in \mathbb{R}\}$$

should give a  $90^\circ$  clockwise rotated absolute value function like the one in ?? on the Std dev.-Mean plane, using ?? and ??.

$$\begin{aligned}\sigma_{11}^2 &= 0.08^2 \\ \sigma_{22}^2 &= 0.18^2 \\ \sigma_{12}^2 &= \sigma_{11}\sigma_{22} = 0.00144 \\ \mu_1 &= 0.10 \\ \mu_2 &= 0.14,\end{aligned}\tag{33}$$

So if we plug in all these values for the mean and variance of rate of returns  $(\sigma_\alpha^2, \mu_\alpha)$  of the portfolio  $\alpha$ , we obtain the following plot

**Example 2.1** Consider the previous example, where the values are given by ??, but with

$$\sigma_{12} = 0,$$

then with  $\alpha \in [0, 1]$  we obtain the Mean-Variance plot of ??.

We observe that in this case we have

$$\sigma_\alpha \notin [\sigma_{11}, \sigma_{22}].\tag{34}$$

Why would we restrict  $\alpha_1 \in [0, 1]$ ???? - Question for Prof. Nadtochiy

**Note 2.1** Observe that in the case of a 2D portfolio  $\alpha = (\alpha_1, \alpha_2)$  since  $\alpha_1 + \alpha_2 = 1$ , we have that this draws a 1-dimensional line on the Mean-Variance space. It would draw a 2D area if the portfolio has dimension 3 or higher.

**Theorem 2.1** (Lemma 2 from the notes of Prof. Nadtochiy)

(a) If two assets  $S_1, S_2$  are perfectly correlated,

$$\sigma_{12}^2 = \pm \sigma_{11}\sigma_{22},$$

then

$$\sigma_\alpha^2 \in [\sigma_{11}^2, \sigma_{22}^2] \text{ if } \alpha \in [0, 1].$$

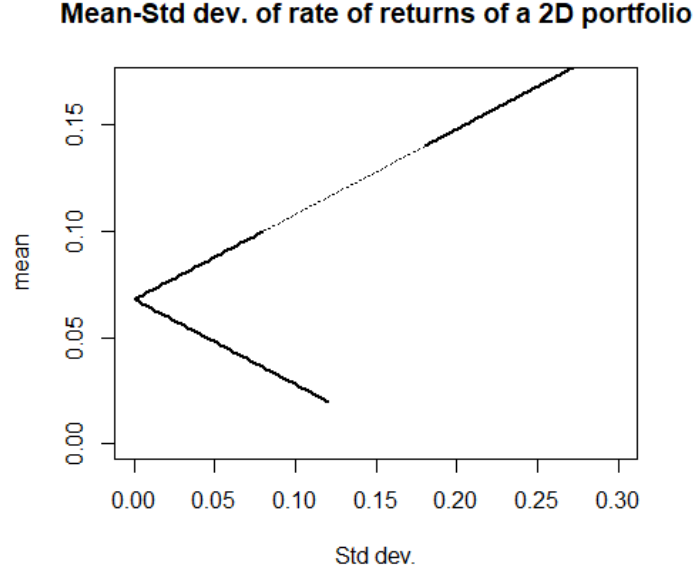


Figure 1: Standard deviation-Mean plot from the rate of returns of a 2 dimensional portfolio  $\alpha$  as specified in example 1 at ???. Here  $\alpha_1 \in [-1, 3]$ . But the region where  $\alpha_1 \in [0, 1]$  is highlighted with a dotted line.

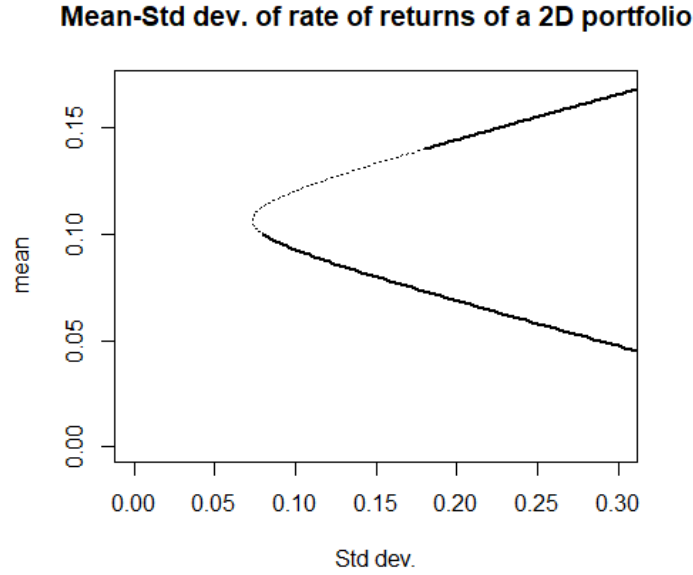


Figure 2: Standard deviation-Mean plot from the rate of returns of a 2 dimensional portfolio  $\alpha_1$  as specified in example 1 at ??, but with  $\sigma_{12} = 0$ . The region where  $\alpha_1 \in [0, 1]$  is highlighted with a dotted line.

(b) If one of the assets (let's assume it is  $S_1$ ) is riskless, or

$$\sigma_{11}^2 = 0,$$

then

$$\sigma_{\alpha}^2 \in [0, \sigma_{22}^2] \text{ if } \alpha \in [0, 1].$$

**Proof 2.1** (a) Just take a look at ??.

(b) Just notice that

$$0 = \sigma_{11}^2 = \mathbb{E}(R_1 - \mu_1)^2 \Rightarrow R_1 \stackrel{a.s.}{=} \mu_1. \quad (35)$$

This directly implies that

$$\sigma_{12}^2 = \text{Cov}(R_1, R_2) = \mathbb{E}(R_1 - \mu_1)(R_2 - \mu_2) = 0,$$

so actually we have that

$$\sigma_{12}^2 = 0 = \sigma_{11}\sigma_{22},$$

and we can just apply case (a).

□

### 3 Efficient Frontier

Consider  $d \geq 2$ , and the corresponding set of  $d$  assets  $\{S_1, \dots, S_d\}$ , together with its rates or return  $\{R_1, \dots, R_d\}$ , and a portfolio  $\alpha$  such that  $\alpha_1 + \dots + \alpha_d = 1$ . Then let's call **Feasible region** the Std dev-Mean space

$$Fea = \{(\sigma_\alpha, \mu_\alpha) : \mu_\alpha \in \mathbb{R}, \sigma_\alpha^2 \geq 0, \alpha_1 + \dots + \alpha_d = 1\}$$

where

$$\begin{aligned} \mu_\alpha &= \boldsymbol{\mu} \cdot \boldsymbol{\alpha} = (\mathbb{E}R_1, \dots, \mathbb{E}R_d) \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} = \alpha_1 \mathbb{E}R_1 + \dots + \alpha_d \mathbb{E}R_d, \\ \sigma_\alpha &= \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha}, \end{aligned} \quad (36)$$

and where

$$\Sigma = \begin{pmatrix} \sigma_{11}^2 & \dots & \sigma_{1d}^2 \\ \vdots & \ddots & \vdots \\ \sigma_{d1}^2 & \dots & \sigma_{dd}^2 \end{pmatrix}, \quad (37)$$

is the covariance matrix of the rates of return. *Feasible* refers in Linear Optimization to the fact that

For  $\mu_\alpha \in \mathbb{R}$ , denote by

$$f(\mu_\alpha) := \min_{\mu_\alpha} \sigma_\alpha^2 = \min_{\alpha} \sigma_\alpha^2 \quad (38)$$

the smallest standard deviation attainable of the portfolio  $\alpha$ , given the mean of the rate of return of the portfolio  $\alpha$ ,  $\mu_\alpha$ . Then **the upper part** of the curve

$$\{(f(\mu_\alpha), \mu_\alpha) : \alpha_1 + \dots + \alpha_d = 1\}$$

is the **efficient frontier**.

#### 3.1 Degenerate markets

**Example 3.1** Consider the case where two assets  $S_1, S_2$  are riskless, that is

$$\sigma_{11}^2 = \sigma_{22}^2 = 0.$$

Then Prof. claims that the entire line

$$\{(0, \mu) : \forall \mu \in \mathbb{R}\} \subseteq Fea.$$

That's true, consider

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0, \dots, 0),$$

where  $\alpha_1, \alpha_2 \neq 0$ . Then

$$\begin{aligned} \mu_\alpha &= \boldsymbol{\mu} \cdot \boldsymbol{\alpha} &= \alpha_1 \mu_1 + \alpha_2 \mu_2 \\ &= \alpha_1 \mu_1 + (1 - \alpha_1) \mu_2 \\ &= \alpha_1 (\mu_1 - \mu_2) + \mu_2 \in \mathbb{R} \text{ if } \alpha_1 \in \mathbb{R}. \end{aligned} \quad (39)$$

□



We can then claim that the **market is degenerate**, since the definition of a **degenerate asset**  $S_1$  is that it has zero volatility, or, in other words, that

$$\sigma_{11}^2 = 0.$$

But now observe that ALL rates of returns  $R_\alpha = \mathbf{R} \cdot \boldsymbol{\alpha}$  from this example have exactly 0 volatility! (for any portfolio  $\boldsymbol{\alpha}$ ). This is because

$$\begin{aligned} \sigma_\alpha &= \boldsymbol{\alpha} \cdot \Sigma \cdot \boldsymbol{\alpha} \\ &= (\alpha_1, 1 - \alpha_1) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ 1 - \alpha_1 \end{pmatrix} = 0. \end{aligned} \quad (40)$$

Recall that  $\sigma_{12}^2 = 0$  when  $\sigma_{11}^2 = 0$  as we explained short after ??.

Thus, it makes sense to call this market or feasible set of portfolios degenerate. (all of them have 0 risk! It's financial heaven on Earth!). Moreover, if  $\mu_1 \leq \mu_2$ , by taking a portfolio  $\alpha_1$  very negative, we can make our portfolio mean of returns  $\mu_\alpha$  very big (as detailed in ??), which is unrealistic.

Just a note, is **short sell** to sell assets, and therefore allow one of the  $\alpha_i < 0$  ? For some  $1 \leq i \leq d$ .

The conclusion from ?? is that it doesn't make sense to accept two riskless assets. Therefore we will only accept a **unique riskless asset** in the market. Otherwise the whole portfolio game stops being funD.

## 4 $\sigma_\alpha^2$ minimization using the Lagrangian

We want to minimize

$$\sigma_\alpha^2 = \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha},$$

such that

$$\sum_1^d \alpha_i = 1.$$

But like this problem is equivalent to minimizing

$$\frac{1}{2} \boldsymbol{\alpha}' \cdot \Sigma \cdot \boldsymbol{\alpha} - \lambda \left( \sum_1^d \alpha_i - 1 \right), \quad (41)$$

wich can be rewritten as

$$\frac{1}{2} \sum_{i,j=1}^d \alpha_i \alpha_j \sigma_{ij}^2 - \lambda \left( \sum_1^d \alpha_i - 1 \right). \quad (42)$$

Now, considering the variables

$$(\alpha_1, \dots, \alpha_d, \lambda),$$

we set first that

$$L = \frac{1}{2} \sigma_\alpha^2 - \lambda \left( \sum_1^d \alpha_i - 1 \right),$$

and then we take the derivative with respect to them and obtain that,

$$\begin{aligned} 0 &= \frac{\partial}{\partial \alpha_i} \sigma_\alpha^2 = \sum_{j=1}^d \alpha_j \sigma_{ij}^2 - \lambda, \quad i = 1, \dots, d. \\ 0 &= \frac{\partial}{\partial \lambda} \sigma_\alpha^2 = \sum_{j=1}^d \alpha_j - 1. \end{aligned} \quad (43)$$

Note that for case  $d = 2, i = 1$  the first equation of the above would be

$$\frac{\partial}{\partial \alpha_1} \left[ \frac{1}{2} (\alpha_1^2 \sigma_{11}^2 + 2\alpha_1 \alpha_2 \sigma_{12}^2 + \alpha_2^2 \sigma_{22}^2) - \lambda (\alpha_1 + \alpha_2 - 1) \right] = \alpha_1 \sigma_{11}^2 + \alpha_2 \sigma_{12}^2 - \lambda. \quad (44)$$

The above system of equations can be rewritten as

$$\begin{pmatrix} \sum & -1 \\ & \vdots \\ & -1 \\ 1 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (45)$$

$$\begin{aligned} &\Updownarrow \\ &Ax = b. \end{aligned}$$

**Note 4.1** The Professors says here that  $A$  is  $d + 1 \times d + 1$  and that  $x, b \in \mathbb{R}^{d+1}$  and that

$$A \text{ invertible} \Leftrightarrow \Sigma \text{ invertible} \text{ ???}$$

Judging by the form of  $A$  at ?? what is  $A^{-1}$ ?

I assumed the following matrix multiplication

$$Id_{d+1,d+1} = A \cdot A^{-1} = \begin{pmatrix} \Sigma & -1 \\ 1 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} B & \begin{pmatrix} c_1 \\ \vdots \\ c_d \\ p \end{pmatrix} \\ q_1 & \dots & q_d & p \end{pmatrix} = \begin{pmatrix} \Sigma \cdot B - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \times (q_1, \dots, q_d) & \Sigma \cdot \mathbf{c} + p \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} \\ (1, \dots, 1) \cdot B & c_1 + \dots + c_d \end{pmatrix} \quad (46)$$

But if this must be equal to the identity, then

$$\Sigma \cdot B + (1, \dots, 1) \times \begin{pmatrix} q_1 \\ \vdots \\ q_d \end{pmatrix} = Id_{d \times d}, \quad (47)$$

$$\Sigma \cdot \mathbf{c} + p \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} = 0. \quad (48)$$

$$c_1 + \dots + c_d = 1,$$

$$(1, \dots, 1) \cdot B = 0. \quad (49)$$

But the last equality is satisfied only if  $B = 0$ , which means that  $A^{-1}$  is not invertible, which is a contradiction!! How to solve this?

**Note 4.2** Prof. says that since the **objective** is convex, and the constraints are linear, setting

$$\frac{\partial}{\partial x_i} L = 0,$$

is a sufficient condition for optimality? Why?

This is lemma 3 and I still don't get it.

**Example 4.1** Let  $d = 2$  and

$$\Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 2 \end{pmatrix}, \mu = \begin{pmatrix} 0.2 \\ 0.4 \end{pmatrix}, \quad (50)$$

this, leads the optimal point of

$$\begin{aligned} \lambda &= 7/8 \\ \alpha_1 &= 3/4 \\ \alpha_2 &= 1/4. \end{aligned} \quad (51)$$

For the lagrangian of equation ?? . This means a portfolio

$$\alpha = (3/4, 1/4)$$

. We get a portfolio *expected rate of return*  $\mu_\alpha$  of

$$\mu_\alpha = \alpha \cdot \mu = 0.25, \quad (52)$$

with variance

$$\begin{aligned}\sigma_\alpha^2 &= \boldsymbol{\alpha}' \cdot \Sigma \boldsymbol{\alpha} = 28/32 = 7/8 = 0.875, \\ &\quad \downarrow \\ \sigma_\alpha &= 0.9354.\end{aligned}\tag{53}$$

Now, what does this mean? Does this mean that for every dollar invested in our portfolio  $\boldsymbol{\alpha}$  we will get \$0.25 dollars back after the *return time*? And how do we interpret the *standard deviation* of  $\sigma_\alpha = 0.935$ ? Does this mean that the expected return is

$$\mu_\alpha = 0.25 \pm 0.935?\tag{54}$$

## 5 Computing the efficient frontier

It's like minimizing the portfolio variance  $\sigma_\alpha^2$ , but subject to a fixed  $\mu_\alpha = y$ . So, the problem would be now:

$$\begin{aligned}\min_\alpha \sigma_\alpha^2 &= \min_\alpha \boldsymbol{\alpha}' \Sigma \boldsymbol{\alpha}, \\ \boldsymbol{\alpha} \cdot \boldsymbol{\mu} &= y, \\ \sum_{i=1}^d \alpha_i &= 1.\end{aligned}\tag{55}$$

For solving this problem we can write down the lagrangian:

$$L = \frac{1}{2} \sum_{i,j=1}^d \alpha_i \alpha_j \sigma_{ij}^2 - \lambda_1 \left( \sum_{i=1}^d \alpha_i \mu_i - y \right) - \lambda_2 \left( \sum_{i=1}^d \alpha_i - 1 \right).\tag{56}$$

Now if we take the derivatives of this we obtain that

$$\begin{aligned}0 &= \frac{\partial}{\partial \alpha_i} L = \sum_{j=1}^d \alpha_j \sigma_{ij}^2 - \lambda_1 \mu_i - \lambda_2 \\ 0 &= \frac{\partial}{\partial \lambda_1} L = \boldsymbol{\alpha} \cdot \boldsymbol{\mu} - y \\ 0 &= \frac{\partial}{\partial \lambda_2} L = \sum_{i=1}^d \alpha_i - 1,\end{aligned}\tag{57}$$

For this case I get the matrix formulation:

$$\begin{pmatrix} & & & -\mu_1 & -1 \\ & \Sigma & & \vdots & \vdots \\ & & & -\mu_d & -1 \\ \mu_1 & \cdots & \mu_d & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y \\ 1 \end{pmatrix}\tag{58}$$

$$\begin{aligned} &\Updownarrow \\ &Ax = b.\end{aligned}$$

**Note 5.1** We have that

$$A \text{ invertible} \Leftrightarrow \sum \text{ invertible}$$

But I don't know why!

**Note 5.2** And this problem is degenerate for the case  $d = 2$ , since then we have to solve that

$$\alpha_1 \mu_1 + (1 - \alpha_1) \mu_2 = y,\tag{59}$$

which has solution

$$\alpha_1 = \frac{y - \mu_2}{\mu_1 - \mu_2}, \quad \alpha_2 = 1 - \alpha_1,\tag{60}$$

and this would lead to a

$$\sigma_\alpha^2 = ay^2 + by + c,\tag{61}$$

where  $a, b, c$  are constants that depend on  $\boldsymbol{\mu}, \Sigma$  only. Therefore for any given mean of returns  $y$ , there will be only one possible variance  $\sigma_\alpha$  available. No need for minimization. This makes perfect sense, since in 2D we are actually running on a 1D line. So for every given mean of returns there is exactly only one variance possible. And viceversa!

**Example 5.1** In case  $\sigma_{12}^2 = 0$ , then

$$\sigma_\alpha^2 = \left( \frac{y - \mu_2}{\mu_1 - \mu_2} \right)^2 \sigma_{11}^2 + \left( \frac{\mu_1 - y}{\mu_1 - \mu_2} \right)^2 \sigma_{22}^2$$

We can try to find the minimal variance portfolio  $\alpha$  by differentiating this formula and setting the derivative equal to 0, and isolate the value for  $y$ .

## 6 Optimal portfolio for means minus variance

An investor wants to maximize **mean** and minimize **variance**. One way is to put the first minus the second and maximize the problem, i.e. maximize:

$$\alpha \cdot \mu - \gamma \alpha' \cdot \sum \alpha \quad (62)$$

so that

$$\sum_{i=1}^d \alpha_i = 1.$$

where  $\gamma$  is the reciprocal **risk tolerance**.

For solving this, use as before a lagrangian of the form

$$L = \alpha \cdot \mu - \gamma \frac{1}{2} \alpha' \cdot \sum \alpha - \lambda \left( \sum_{i=1}^d \alpha_i - 1 \right) \quad (63)$$

### 6.1 Adding robustness in mean returns

*It is difficult to find a reliable approximation of the mean returns  $\mu = (\mu_1, \dots, \mu_d)$ .* Thus, we may consider the robust (or **worst case**) mean, and maximize the objective.

so we want to compute

$$\begin{aligned} \max_\alpha \min_\mu [\alpha \cdot \mu] - \gamma \alpha' \cdot \sum \alpha \\ \Downarrow \\ \max_\alpha \min_\mu \sum_{i=1}^d \alpha_i \mu_i - \gamma \alpha_i \alpha_j \sigma_{ij}^2 \end{aligned} \quad (64)$$

where

$$\sum_{i=1}^d \alpha_i = 1, \quad \mu_i^0 \leq \mu_i \leq \mu_i^1, \quad i = 1, \dots, d \quad (65)$$

for some lower and upper bounds  $\mu_i^0, \mu_i^1$ . Denote

$$\hat{\mu}_i = \frac{\mu_i^0 + \mu_i^1}{2} \quad \epsilon_i = \frac{\mu_i^1 - \mu_i^0}{2}. \quad (66)$$

Then we have that

$$\min \mu_i = \hat{\mu}_i \alpha_i - \epsilon |\alpha_i|.$$

then since both  $|\alpha_i|$  and  $\alpha_i \alpha_j$  are concave functions, the objective itself is concave. Then it can be solved by standard numerical methods such as gradient descent or second-order methods.

#### 6.1.1 Implement it

Using python,

```
scipy.optimize.minimize()
scipy.optimize.LinearConstraint()
```

## 7 Two-fund or one-fund theorem

Consider the two dimensional case  $d = 2$  and that one of the assets is riskless, i.e.  $\sigma_{11}^2 = 0$ , this, by ?? implies as well that

$$\sigma_{12}^2 = 0.$$

Now note that a riskless asset  $S_1$  has no variance, therefore

$$\mu_1 = \mathbb{E}R_1 = R_1.$$

We then have that

$$\mu_\alpha = \alpha_1 R_1 + (1 - \alpha_1)\mu_2, \quad \sigma_\alpha = \sqrt{(1 - \alpha_1)^2 \sigma_{22}^2} = |1 - \alpha_1| \sigma_{22}, \quad (67)$$

where  $\alpha_1 \in \mathbb{R}$ .

Then  $(\mu_\alpha, \sigma_\alpha)$  forms a line in  $\mathbb{R}^2$  for  $\alpha_1 \in \mathbb{R}$  and a half-line for  $\alpha_1 \leq 1$ , which depends on  $(\mu_2, \sigma_{22})$ . The idea is the following:

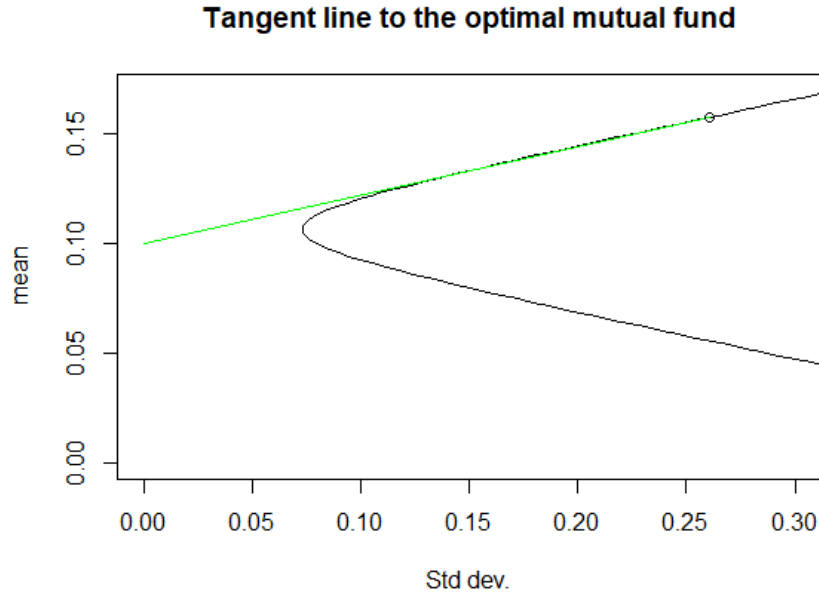


Figure 3: The tangent line to the Optimal Mutual fund. The optimal mutual fund is represented by the dot at which the green line is tangent to.

**Note 7.1** If we have a riskless asset

$$\sigma_{00}^2 = 0$$

and the remaining risky assets cannot generate a riskless portfolio return  $\sigma_\alpha^2 = 0$ , then the matrix  $\Sigma$  is invertible.

**Note 7.2** Consider all  $(\sigma, \mu)$  generated by portfolios of risky assets only. Then, under the above assumption of invertibility, at most one of these points lies on the efficient frontier of the **overall market**. We can call this point the **optimal mutual fund**. (the overall market *includes the riskless asset*).

### 7.1 Finding the optimal mutual fund

There is only one optimal mutual fund, if we assume that the submatrix

$$(\sigma_{ij})_{ij}$$

is invertible.

This point must lie on the efficient frontier of the *overall market*. Given a portfolio mean of returns  $\mu_\alpha = y$ , then we wanna find

$$\min_{\alpha} \alpha' \cdot \sum \alpha = \min_{\alpha} \sum_{i=0}^d \alpha_i \alpha_j \sigma_{ij}^2 \quad (68)$$

subject to

$$\sum_{i=0}^d \alpha_i \mu_i = y, \quad \sum_{i=0}^d \alpha_i = 1. \quad (69)$$

Then using the same lagrangian as we did in ?? we get, from  $i = 0$  until  $d$ .

$$\sum_{j=0}^d \sigma_{ij} \alpha_j - \lambda_1 \mu_i - \lambda_2 = 0, \quad (70)$$

Now. Recall that we are trying to find the *optimal mutual fund*. Then by taking a look at ?? we can see that this tangent point can be generated by the set of *risky assets*. We can add this bit of information to our problem, by not inverting in the riskless asset, i.e. by doing  $\alpha_0 = 0$  and trying to find a portfolio of risky assets only that gives us the optimal mutual fund. We get that, in the case  $i = 0$ ,

$$\lambda_2 = -\lambda_1 \mu_0, \quad (71)$$

from ?? and the fact (that I proved before) that  $\sigma_{00}^2 = 0 \rightarrow \sigma_{i0}^2 = 0$  for all  $i$ . Then

$$\begin{aligned} \sum_{j=1}^d \sigma_{ij}^2 \alpha_j &= \lambda_1 (\mu_i - \mu_0), \quad i = 1, \dots, d, \\ \sum_{i=1}^d &= 1. \end{aligned} \quad (72)$$

Just write this like

$$\sum \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} = \lambda_1 \begin{pmatrix} \mu_1 - \mu_0 \\ \vdots \\ \mu_d - \mu_0 \end{pmatrix} \quad (73)$$

## 8 Dickey Fuller test

The null hypotheses of the Dickey Fuller test is that  $X_t$  is a **random walk**. The alternative is that it is a stationary process.

Augmented Dickey fuller test: We test now a richer hypotheses.

$$\begin{aligned} X_t &= \sum_{i=1}^p \phi_i X_{t-i} + \epsilon_t - AR(p) \\ \phi &= 1 - \sum_{i=1}^p \phi_i z^i \end{aligned} \quad (74)$$

He doesn't want to write down the statistic for this test. It is complicated. Python can chose the  $p$  for this sum.

$H_0$  Here exists a root of  $\phi$  inside the unit circle. Not stat

$H_1$  not stationary.

In python:

```
my_DF = stattools.adfuller(my_AR1,1,'c',None)
#1 is binding or python tests up to 1, i.e. choose ``p' up to 1, if I choose ``c'' forces order 1.
print(my_DF[0])#test statistic value
print(my_DF[1])#pvalue
print(my_DF[2])#order of AR tested
# he obtains
-4.50217
0.0001948 # this means that process is indeed stationary
1
```

Let's now trick python- Sergey says-

It is too difficult to test for stationarity. So we impose AR assumption. Now I put MA which is not AR, let's see what happens, we obtain:

```
my_DF = stattools.adfuller(my_MA2)
print(my_DF[0])#test statistic value
print(my_DF[1])#pvalue
print(my_DF[2])#order of AR tested
# he obtains
-6.162288
7.1259e-08
21# you have to keep track of your entire history, it is not useful, Nadthochiy says.
```

Now let's see what happens with a random walk.

$$\phi = 1$$

```
my_DF = stattools.adfuller(rw)
print(my_DF[0])#test statistic value
print(my_DF[1])#pvalue
print(my_DF[2])#order of AR tested
# he obtains
-1.0518
0.73390# my confidence for rejecting H0 is only 20\%. Cannot reject H0
2
```

This is what we expected because this is not a stationary series.

Now let's apply it to real data.

```
my_DF = stattools.adfuller(price)
print(my_DF[0])#test statistic value
print(my_DF[1])#pvalue
print(my_DF[2])#order of AR tested
# he obtains
```

Gives not stationarity. As expected. Now let's try it on log returns.

```
my_DF = stattools.adfuller(l_ret,1)
print(my_DF[0])#test statistic value
print(my_DF[1])#pvalue
print(my_DF[2])#order of AR tested
# he obtains
-33
0.0
0
```

But order 0 AR(0) is white noise. So this log returns are completely stationary. But we cannot trade based on the log returns. Because a return can go up, but the price can keep going down for that while. So we actually would make no money.

We look for autocorrelation function or autocovariance function, and if we find a white noise then we are done. But **what is the distribution of the white noise  $\epsilon_t$ ?**

## 8.1 distribution of the white noise $\epsilon_t$

$$\{\epsilon_t\} - WN$$

Assumption:

$$\epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

Ways to test this:

1. **Histogram:** In each bin we put the of  $\epsilon_t$  in the bin, and report this number as the height of the bin.  
In python use

```
plt.hist(price,30)
plt.hist(price,30,(-0.07,0-07))#to specify x limits
```

We want to apply a histogram on *white noise*, something that is iid. Otherwise it has no meaning. We want to analyse it as the

$$\Pr(\epsilon_t \in bin).$$

The histogram should remind us, after normalization, of the density function. We look for “heavy tails”. The heavier the tails, the higher the risk of loosing money or gaining money.

2. **Q-Q plots.** Quantile-Quantile function plots. let’s call  $q$  the quantile function of  $\mathcal{N}(0, 1)$ , and  $q^N$  the sample quantile function. (normalized). We plot these quantiles, they should form a diagonal line. He draws a growing ladder. Plots the sample CDF (a staircase) and inverts numerically. We expect, if the data is  $\mathcal{N}(0, 1)$  to form a Q-Q diagonal plot

```
iid_norm = iid/np.std(iid)
l_ret_norm = (l_ret - np.mean(l_ret))/np.std(l_ret)

gofplots.qqplot(iid_norm)#by default python will compare to std normal
gofplots.qqplot(l_ret_norm)#now we plot real financial data
```

We observe heavy tails when we do the Q-Q plot for real market data.

## 8.2 Cointegration

Now we get to the heart of the chapter.

**Definition:** 2 time series  $X, Y$  such that a linear combination of them is stationary, then we say they are **cointegrated**.

$P_i$  price process of asset  $i$ .

$P_j$  price of process of asset  $j$ .

If they are cointegrated,  $aP_i + bP_j$  are *tradable!*. You can make money.

Typically  $a, b \neq 0$  and

$$ab < 0.$$

There are reasons for this to happen.  $a$  can be something like a 1000. If we trade with expensive stocks the the **trading errors will be large**. If stocks are cheap,  $a$  gets to be big, and rounding errors will be small.

So we can divide by

$$P_j - C_{ij}P_i \text{ stationary : } C_{ij} > 0, \quad (75)$$

and this will be stationary.

**question:** why is it natural to assume that such  $C_{ij}$  exists?

**Answer: Relative Misspricing.** Ass soon as  $P_j$  is higher than  $P_i$  then people will start selling  $j$  and buying  $i$  and viceverse, so it will oscillate around 0. this is why it is stationary. You won’t trust statistics. *You only trust both statistics and economics analysis. You only accept cointegrated companies if they trade on similar assets.* Even if statistics say so, you would never consider that Shell is cointegrated with Education First. Oil and Education are very independent.

$$\begin{aligned} P_i^t &= a_i + \beta_i R^t + S_i^t \\ P_j^t &= a_j + \beta_j R^t + S_j^t \end{aligned} \quad (76)$$



$R^t$  “industry index”.

$S_i, S_j$  independent stationary, mean zero.

$$\begin{aligned} C_{ij} &= \frac{\beta_j}{\beta_i} \Rightarrow Z_{ij}^t = P_i - C_{ij}P_j^t \\ &= a_i - C_{ij}a_j + S_i^t - C_{ij}S_j^t. \end{aligned} \quad (77)$$

These two assets  $i, j$  are correlated. But I can kill the common component  $R^t$ . Correlated assets that are a random walk, can still be a random walk, and therefore useless for making money. If two assets are cointegrated, we can trade with a linear combination of them. is stationary.

This is related to CAPM model.

Some of the python code:

```
my_arma = arima_model.ARMA(price,[1,0],price2)
arma_results = my_arma.fit()
print(arma_results.arparams)
print(arma_results.)
```

He also runs a dikey fuller test on this. He gets 0.123 as p-value, 88% of rejecting null hypotheses.

## 9 27.02.2020

1. Find candidate  $C_{ij}$
2. Check if

$$Z_{ij} = P_i^t - C_{ij}P_j^t = \hat{a} + \hat{S}^t$$

is stationary. whre

$$C_{ij} = \beta_i/\beta_j$$

OLS

The problem in making

$$\Delta P_i^t = C_{ij}\Delta P_j^t + \Delta Z_{ij}^t$$

it a linear regression is not possible. Because we need the errors

$$\Delta Z_{ij}^t$$

to be independent of time

*Book for Cointegration and Time Series: book of Carmona. It also contains information of Q-Q plots.*

The second option is using the **Kalman filter** using

```
statsmodels.tsa.arima_model.ARMA()
```

to fit

$$P_i - C_{ij}P_j^t = \epsilon^t + \phi P_i^{t-1} \quad (78)$$

The exogenous coefficient would be  $C_{ij}$

He wants to reject with a small p value, he doesn't want to trade with that pai. A pair is **EIG.?**

He says that other methods are not significantly better than the OLS regression. Even though the theoretical framework of the OLS regression, Prof. claims, is not met.

$\phi$  close to 1, means that your model

$$X_t = \phi X_{t-1} + \epsilon^t$$

will do long excursions up and down. A  $\sigma = 0.5$ . Recall that

$$X := Z_{ij} = P_i - C_{ij}P_j$$

. “Buying a pair” means buying units on  $P_i$  and *short selling* units of  $P_j$ .

We integrate all stocks we have. We choose our favourite method to do Idk what, then the run a Dickey fuller test to see if they are cointegrated. But when do we have to sell and buy? Which are the values  $\lambda_1, \lambda_2$  at which we do this?

We always wanna trade when the price is below the mean. We sell high and we buy low. The question is *How to do it?*

## 9.1 Cointegrated log-prices

We will say that *cumulative returns* are cointegrated. The log of the prices behaves as before. Not the Price itself. This model is much more realistic in reality and in theory. This is nothing else than CAPM. In CAPM we had to assume that  $S_j^t$  where white noise. Here we don't have to assume that.

$$\begin{aligned}\log P_i &= a_i + \beta_i R^t + S_i^t \\ \log P_j &= a_j + \beta_j R^t + S_j^t\end{aligned}\tag{79}$$

If we find a cointegrated pair, its logarithm is even more cointegrated.

We obtain

$$\log P_i - C_{ij} \log P_j\tag{80}$$

how can we make money out of this? using  $t \ll 1$ ,

$$\log p_i^t / p_i^0 \simeq \frac{p_i^t}{p_i^0} - 1,$$

so

$$\frac{p_i^t}{p_i^0} - 1 + \log P_i^0 - C_{ij} \left( \frac{p_i^t}{p_i^0} - 1 + \log P_j^0 \right) = \frac{1}{p_i^0} p_i^t - \frac{c_{ij}}{p_i^0} P_j^t + ctt,$$

To buy log-cointegrated pair, invest +\$ 1 in  $i$ -th asset, and -\$  $C_{ij}$  in  $j$ -th asset. **All this framework only works for  $t \ll 1$ . For a short amount of time.**

## 9.2 How to optimally choose the y-axis lebls $\lambda_1, \lambda_2$

When to buy and sell our assets?

$$\begin{aligned}P_i - C_{ij}P_j^t &= \epsilon^t + \phi P_i^{t-1} \\ X^t &= (1 - p)X^{t-1} + \epsilon^t, \phi < 1,\end{aligned}\tag{81}$$

$$\sigma^2 = \text{Var}(\epsilon^t), \phi = 1 - p, p > 0$$

$$\hat{\sigma}^2 = \text{Var } X_t = \frac{\sigma^2}{1 - \phi^2}$$

Some people choose

$$\begin{aligned}\lambda_1 &= -\frac{3}{2}\hat{\sigma} \\ \lambda_2 &= -\frac{1}{2}\hat{\sigma}\end{aligned}\tag{82}$$

I want:

$$profit = l_2 - l_1,$$

to be large. But a principle in the market, is that you cannot make a lot of money without risk. Where does the risk come from?

-> mean reversion time - we want it small. We don't want to wait for 100 years to have a selling opportunity. This gives intermediate losses. That can be fatal for a company.

->The main risk of this process is that 2 assets stop being cointegrated.

Actually we care about profits per time. How long will it take to get to  $\lambda_2$  ? We assume  $\sigma = 0$

$$\begin{aligned} X^t &= \phi X^{t-1} + \epsilon^t \\ &\downarrow \\ \Delta X^t &= -p X^{t-1} \\ &\downarrow \\ \frac{dX^t}{dt} &= -\rho X^t \\ X^0 &= \lambda_1 \end{aligned} \tag{83}$$

we get the following solution:

$$\begin{aligned} X^t &= \lambda_1 e^{-\rho t} \\ T &= \log \left( \frac{|\lambda_1|}{|\lambda_2|} \right) / \log(1 - p) \end{aligned} \tag{84}$$

$$\mathbb{E}\tau = \frac{\sqrt{\pi}}{\hat{\sigma}\sqrt{\rho}} \int_{\lambda_1}^{\lambda_2} e^{\frac{y^2 \rho}{\hat{\sigma}^2} (1 + \operatorname{erf}(\frac{y\sqrt{\rho}}{\hat{\sigma}}))} dy \tag{85}$$

then given  $x^* = \lambda_1$  maximize

$$\lambda_2 : \max_{\lambda_2 > \lambda_1 + c} \left[ \frac{\lambda_2 - \lambda_1 - c}{\rho} (\tau - \mathbb{E}\tau) \right] \tag{86}$$

We want the expectation  $\mathbb{E}\tau$  to be small, where  $c$  is the cost,  $T$  the maximum holding time and  $P = P_i^0$  the initial price, so we normalize it.

## 10 About homework 3

He says tat

$$\begin{aligned} P_i - C_{ij} P_j &= Z_{ij} \\ P_j - \frac{1}{c_{ij}} P_i &= \frac{1}{c_{ij}} Z_{ij} \end{aligned} \tag{87}$$

where  $i < j$ . Asset  $i$  is on the long side of the pair. The asset with the smallest index is on the long side of the pair.

## 11 Class of 03.03.2020

### 11.1 Optimal exit from a pair

He draws again a kind of random walk process and he assigns two y-values,  $\lambda_1 < \lambda_2$ . He says we buy for price  $\lambda_1$  and sells when the asset gets price  $\lambda_2$ . The cointegrated prices can change. The **model risk** cannot be modelled within the model. There is no canonical choice of a simple objective to maximize. *Sergey's choice is*: the one of ???. Together with the condition that

$$\begin{aligned} P &= P_i^0 \\ T &= \text{max. holding time} \\ c &\sim 4P\lambda \end{aligned} \tag{88}$$

The situation is as follows He chooses the input parameters for the model

$$\begin{aligned} Z_{ij}^t &= (1 - \rho) Z_{ij}^{t-1} + \epsilon^t, & \rho > 0 \\ \sigma^2 &= \operatorname{Var}(\epsilon^+), & \hat{\sigma}^2 = \operatorname{Var}(Z_{ij}^t) \end{aligned} \tag{89}$$

The expected time is 20 days. Now he will adress two more questions

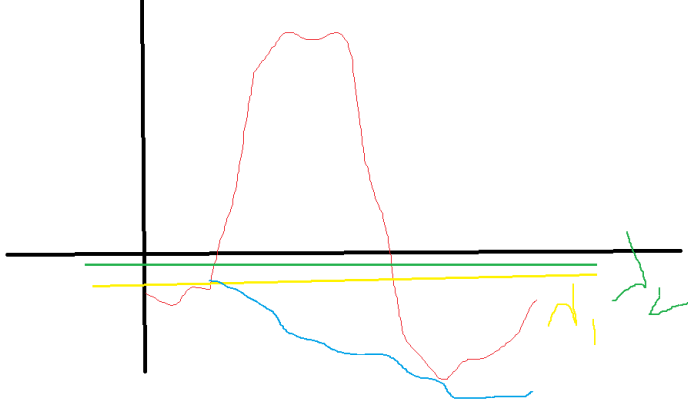


Figure 4: Framework of when we should exit or sell. The risky case is that since we buy for  $\lambda_1$  price, the assets follows the blue path and we get losses.

## 11.2 What do do if a pair de-integrates

In general this is THE question. We can do the **stop-loss**: Liquidate pos. in the pair if

$$\text{holding time} > \bar{T}.$$

Or if

$$\bar{\lambda}$$

is  $\hat{\sigma}$  units away from the mean. In these two situations we can say that the pair- de integrated.

Regime-switching (hidden Markov). Models - detection of switching.

## 12 How to trade multiple pairs

Now the question of optimality is done. Now we focus on a new problem. Question: How to allocate capital among cointegrated pairs?

1. Invest \$  $\bar{C}$  in the asset on the long side of each pair. (long or short).
2. We claim that some pairs are better than others. We will call  $V$  the value of

$$V = \max_{\lambda_2 > \lambda_1 + c} \left[ \frac{\lambda_2 - \lambda_1 - c}{\rho} (\tau - \mathbb{E}\tau) \right]. \quad (90)$$

We want to invest more in those assets that have large  $V$ . Invest \$  $\bar{C}V$  in the long side of the pair.

### 12.1 Implementation

Using a running window. Assume that the amount of days is  $M = 2570$ . We will use  $N = 100$  days for callibration. We estimate everything using this 100 days sample. This is our estimation window. Now we find all cointegrated pairs  $i < j, p \leq p^* (= 5\%)$  and we find out which are the associated

$$\{C_{ij}\}_{ij}.$$

For each cointegrated pair, we compute  $\lambda_2^*$ ,  $V$  and we determine the investment size.

Then for the second part we have to check whether

1. It has reached  $\lambda_2^*$  (close it)
2. it is in *stop-loss* category (close it)

Compute PnL At each  $t = N + 1, \dots, M$ , I+II  $\Rightarrow$  number of shares of each basic asset  $i = 1, \dots, d$  to hold at time  $t$ :

$$\begin{aligned} \psi^t &= (\psi_1^t, \dots, \psi_d^t) \\ &\Downarrow \\ \$L^t &- \text{total \$} \\ \$S^t &- \text{total \$ short} \end{aligned} \quad (91)$$

Absolute PnL (in \$)  $\bar{P}$

$$\left\{ \begin{array}{l} \bar{P}^t = \bar{P}^{t-1} + \psi^{t-1}(T^t - P^{t-1}) - |\psi^{t-1} - \psi^{t-2}|P^{t-1}\lambda, \quad t = N + 1, \dots, M \\ \psi^{N+1} = 0, \quad \bar{P}^N = 0 \end{array} \right\} \quad (92)$$

### 13 Exercise 1 - finding the efficient frontier

- (a) We produce the estimated vector of mean returns and the estimated covariance matrix. We save them in “.csv” files.

So we have that the return of the stock of a company  $i$  is

$$R_i^t = \frac{P_i^t - P_i^{t-1}}{P_i^{t-1}},$$

where  $P_i^t$  is the closing price on date  $t$ , and  $t - 1$  is the last closing price before the closing price at time  $t$  was reported. In our data set

```

In [4]: print(stock_data)

```

Attributes	Adj Close					
Symbols	AAPL	MSFT	TSM	INTC	CSCO	\
Date						
2009-01-02	11.253528	15.635055	5.567235	10.736197	13.128983	
2009-01-05	11.728474	15.781175	5.458869	10.531362	13.245105	
2009-01-06	11.535025	15.965750	5.689144	10.856274	13.771499	
2009-01-07	11.285772	15.004425	5.221822	10.199384	13.407664	
2009-01-08	11.495339	15.473547	5.106684	10.277082	13.577968	
...	...	...	...	...	...	
2019-01-28	153.621765	103.548927	35.749737	45.349712	44.510239	
2019-01-29	152.029510	101.440102	34.689846	45.184669	44.714546	
2019-01-30	162.418411	104.829979	35.692440	46.155552	45.444221	
2019-01-31	163.587997	102.908394	35.921604	45.747780	46.008507	
2019-02-01	163.666641	101.282433	35.873859	47.310894	46.057152	

Figure 5: Viewing a part of our stock data set

We have one closing price  $P_i^t$  per day, during the almost 11 years we are studying which run from January 1st 2009 until December 31st 2019.

I did the following

```

import pandas as pd
import numpy as np
import csv
import matplotlib.pyplot as plt

```

```

import math
from pandas_datareader import data
import scipy.optimize
from scipy import stats

tickers_file = 'TechTickers.csv'
tickers = [];
f = open(tickers_file,"r",encoding='utf-8-sig')
for line in csv.reader(f):
    tickers.append(str(line[0]))
f.close
print(tickers)

#download the prices and volumes for the previously read list of tickers for the first month
#of the earliest year in the proposed time period
start_date = '2009-01-01'
end_date = '2019-01-31'
stock_data = data.get_data_yahoo(tickers, start_date, end_date)

#create a list of tickers whose adjusted closing prices in the first month of the first year
#do not have any missing values
stockArray = []
for ticker in tickers:
    stockArray.append(list(stock_data['Adj Close'][ticker]))
stockArray=np.array(stockArray)
tickers_liq = []
for i in range(len(tickers)):
    temp=0
    for j in range(len(stockArray[i,:])):
        if math.isnan(stockArray[i,j]):
            temp=1
    if (temp==0):
        tickers_liq.append(tickers[i])
print(len(tickers_liq))
print(tickers_liq)

#save the list of tickers without any initial missing values
tickers = list(tickers_liq)
np.savetxt("TechTickers_liq.csv",np.array(tickers),fmt='%1s',delimiter=',')

#trying commands:
np.size(stock_data.values[1,:])
ret=np.zeros(( np.size(stock_data.values[:,1])-1 , np.size(stock_data.values[1,:]) ))
print(range(0,np.size(stock_data.values[1,:])))

adjclose = stock_data['Adj Close']
np.shape(adjclose)
#np.shape(stock_data)
#print(adjclose)

#1(a) np.size(stock_data.values[1,:])
adjclose = stock_data['Adj Close']
ret=np.zeros(( np.size(adjclose.values[:,1])-1 , np.size(adjclose.values[1,:]) ))
for i in range(0,np.size(adjclose.values[1,:]) ):
    pto = adjclose.values[0:np.size(adjclose.values[:,1])-1,i]
    ptf = adjclose.values[1:np.size(adjclose.values[:,1]),i]

```

```

    ret[:,i] = (ptf-pt0)/pt0
ret2 = adjclose.pct_change()
#print(ret2)
means = ret2.mean(axis=0)
covs = ret2.cov()
print(means)
print(covs)

#saving the files:
meanss=np.array(means)
np.savetxt("means1a.csv",np.transpose(meanss),delimiter=',')
covss=np.array(covs)
np.savetxt("covs1a.csv",np.transpose(covss),delimiter=',')

```

- (b) Compute the weights of the minimal-variance portfolio and save them in 'csv' file.
- (c) Compute the weights of the optimal mean-variance portfolio (i.e., maximizing a linear combination of mean and variance) with the risk tolerance  $1/\gamma = 1$ . Save the weights in a 'csv' file and plot them on a graph. Output the mean and variance of the resulting portfolio.