Chapter 8: Regression Models for Quantitative and Qualitative Predictors

STAT 3240

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UPEI

Learning Objectives for Section 8.1

After Section 8.1, you should be able to

- Understand the utility and disadvantages of polynomial regression
- Understand the need for centering
- Understand the danger of overfitting
- Compute and interpret parameters in a polynomial regression model

8.1: Polynonrial Regression Models

Polynomial regression models have two basic types of uses:

1. When the true curvilinear response function is indeed a polynomial function.
2. When the true curvilinear response function is unknown (or complex) but a polynomial function is a good approximation to the true function.

Show 20 entries Search:

	province	type_of_dwelling	income	marital_status	age_group
1	NL	single_detached	68000	never_married	30-34
2	NL	single_detached	48000	never_married	25-29
3	NL	single_detached	30000	married	35-39
4	NL	row_house	30000	never_married	30-34
5	NL	single_detached	35000	married	25-29

Showing 1 to 20 of 500 entries

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Suppose that we are not sure about the nature of the response function in the range of the factors studied, so we decide to fit the second-order polynomial regression model:

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_{11} x_{i1}^2 + \beta_{22} x_{i2}^2 + \beta_{12} x_{i1} x_{i2} + \varepsilon_i$$

We are particularly interested in whether interaction effects and curvature effects are required in the model for the range of X values considered.

We will center the predictors in order to avoid the computational inaccuracies that arise with highly collinear predictors:

- $income\_c_i = income_i - \overline{income}$
- $food\_expenditure\_c_i = food\_expenditure_i - \overline{food\_expenditure}$

```
with(spending_subset, data.frame(clothing= clothing_expenditure, income
```

```
##             clothing income  food income2 food2 income_food
## clothing       1.000  0.224 0.333   0.197 0.305       0.341
## income         0.224  1.000 0.083   0.975 0.054       0.732
## food           0.333  0.083 1.000   0.068 0.964       0.682
## income2        0.197  0.975 0.068   1.000 0.044       0.715
## food2          0.305  0.054 0.964   0.044 1.000       0.649
## income_food    0.341  0.732 0.682   0.715 0.649       1.000
```

```
spending_subset_c = with(spending_subset, data.frame(clothing= clothing_
spending_subset_c %>% cor() %>% round(3)
```

```
##               clothing income_c food_c income_c2 food_c2 incomec_foodc
## clothing         1.000    0.224  0.333     0.022   0.091        -0.101
## income_c         0.224    1.000  0.083     0.473  -0.052        -0.014
## food_c           0.333    0.083  1.000    -0.010   0.433        -0.090
## income_c2        0.022    0.473 -0.010     1.000  -0.006         0.136
```

An even better approach is to use **Orthogonal Polynomials**

This approach involves using new variables

- $z_1 = a_1 + m_{11}x$
- $z_2 = a_2 + m_{21}x + m_{22}x^2$
- $z_3 = a_3 + m_{31}x + m_{32}x^2 + m_{33}x^3$
- etc

Where these variables are orthogonal (i.e., where $z_j' z_k = 0$ for all j and k).

By using centered variables (i.e., $z_1 = -\bar{x} + x$, $z_2 = \bar{x}^2 - 2\bar{x}x + x^2$, etc.) we trade interpretability of the predictors for stability of the estimators. When using orthogonal polynomials, we go even farther with this idea.

```
spending = with(spending_subset, data.frame(clothing= clothing_expenditu
with(spending, data.frame(clothing, income.1 = poly(income, degree=2)[,1
```

```
##             clothing income.1 income.2 food.1 food.2 income_food
## clothing       1.000    0.224   -0.096  0.333 -0.059      -0.101
## income.1       0.224    1.000    0.000  0.083 -0.097      -0.014
## income.2      -0.096    0.000    1.000 -0.056  0.051       0.162
## food.1         0.333    0.083   -0.056  1.000  0.000      -0.090
## food.2        -0.059   -0.097    0.051  0.000  1.000       0.129
## income_food   -0.101   -0.014    0.162 -0.090  0.129       1.000
```

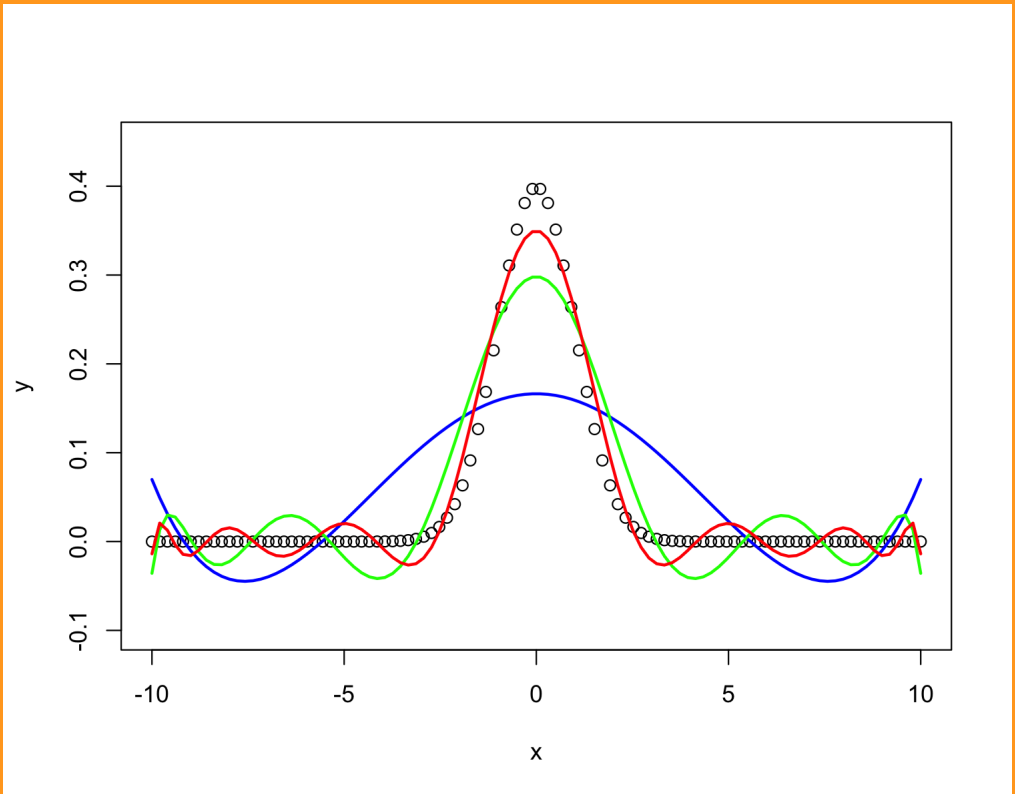
Note the change in the correlation between the two components of *income* and *food expenditure*.

Note that instead of using polynomial regresssion, often it will be an even better idea to use *Piecewise polynomials* (**splines**) that offer more flexibility by not assuming that the relationship of interest is the same across the whole range of X (similar to the idea of LOWESS curves).

Polynomial Regression

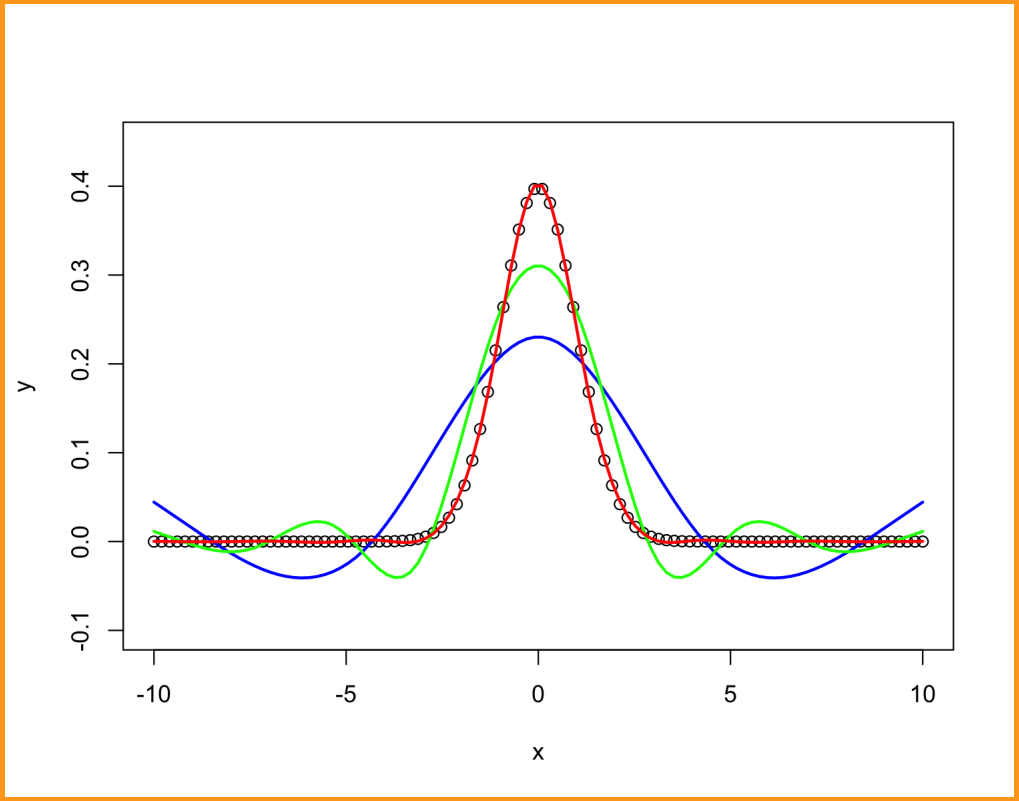
```
plot(x, y, ylim=c(-.1, .45))

lines(x, predict(lm(y~poly(x, 5))), type="l", col="blue", lwd=2)
lines(x, predict(lm(y~poly(x, 10))), type="l", col="green", lwd=2)
lines(x, predict(lm(y~poly(x, 15))), type="l", col="red", lwd=2)
```



(Restricted Cubic) Spline Regression

```
plot(x, y, ylim=c(-.1, .45))
require(rms)
lines(x, predict(lm(y~rcs(x, 5))), type="l", col="blue", lwd=2)
lines(x, predict(lm(y~rcs(x, 10))), type="l", col="green", lwd=2)
lines(x, predict(lm(y~rcs(x, 15))), type="l", col="red", lwd=2)
```



Keep this in mind, but we won't discuss Splines further in this course.

Fitting the Model

```
clothing_model = lm(clothing~income_c+food_c + income_c2+food_c2+incomec
msummary(clothing_model)
```

```
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    2.59e+03   1.00e+02   25.91  < 2e-16
## income_c       2.19e-02   4.48e-03    4.88  1.4e-06
## food_c         1.94e-01   2.81e-02    6.90  1.6e-11
## income_c2     -2.97e-07   1.83e-07   -1.62     0.11
## food_c2       -4.04e-06   5.84e-06   -0.69     0.49
## incomec_foodc -1.92e-06   1.44e-06   -1.33     0.18
##
## Residual standard error: 1550 on 494 degrees of freedom
## Multiple R-squared:  0.16,    Adjusted R-squared:  0.151
## F-statistic: 18.8 on 5 and 494 DF,  p-value: <2e-16
```

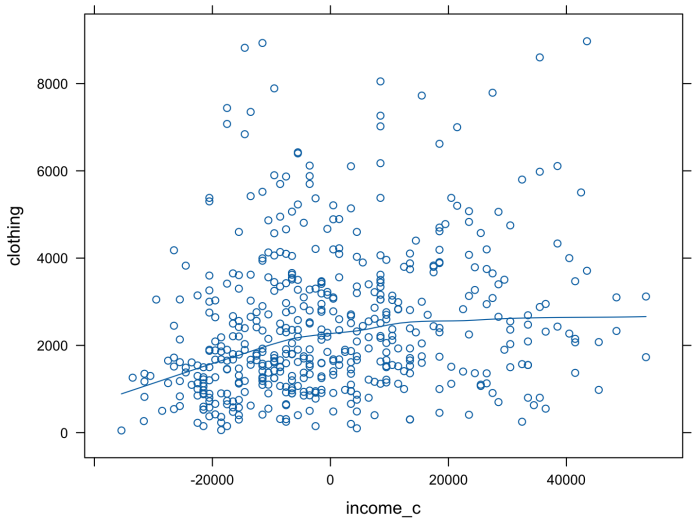
```
msummary(lm(clothing_expenditure~income + food_expenditure + I(income^2)
```

```
##                              Estimate Std. Error t value Pr(>|t|)
## (Intercept)                -8.81e+02   5.74e+02   -1.53   0.1257
## income                      5.96e-02   1.90e-02    3.13   0.0018
## food_expenditure            3.29e-01   1.05e-01    3.12   0.0019
## I(income^2)                -2.97e-07   1.83e-07   -1.62   0.1056
## I(food_expenditure^2)      -4.04e-06   5.84e-06   -0.69   0.4892
## income:food_expenditure -1.92e-06   1.44e-06   -1.33   0.1834
##
## Residual standard error: 1550 on 494 degrees of freedom
## Multiple R-squared:  0.16,    Adjusted R-squared:  0.151
## F-statistic: 18.8 on 5 and 494 DF,  p-value: <2e-16
```

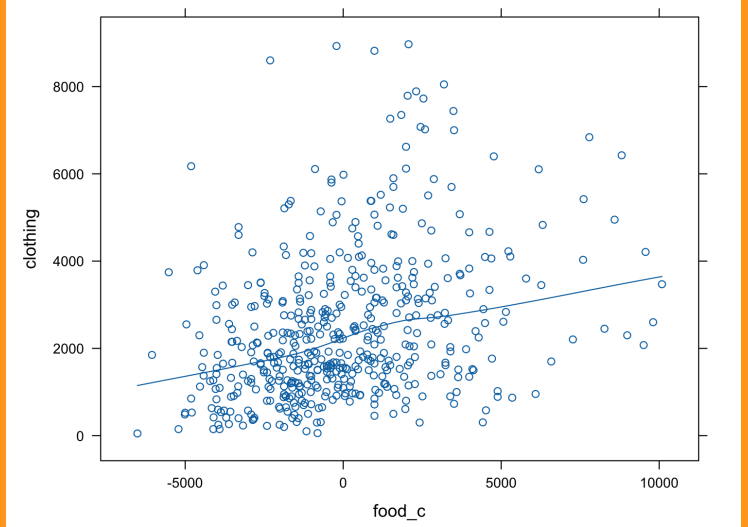
```
msummary(lm(clothing_expenditure~poly(income, degree=2) + poly(food_expe
```

```
##                                        Estimate Std. Error t value Pr(>|t|)
## (Intercept)                            3.01e+03   4.20e+02    7.16  2.9e-12
## poly(income, degree = 2)1              1.26e+04   4.23e+03    2.97   0.0031
## poly(income, degree = 2)2             -2.56e+03   1.58e+03   -1.62   0.1056
## poly(food_expenditure, degree = 2)1    1.66e+04   3.94e+03    4.21  3.0e-05
## poly(food_expenditure, degree = 2)2 -1.09e+03   1.58e+03   -0.69   0.4892
## income:food_expenditure               -1.92e-06   1.44e-06   -1.33   0.1834
##
## Residual standard error: 1550 on 494 degrees of freedom
## Multiple R-squared:  0.16,    Adjusted R-squared:  0.151
## F-statistic: 18.8 on 5 and 494 DF,  p-value: <2e-16
```

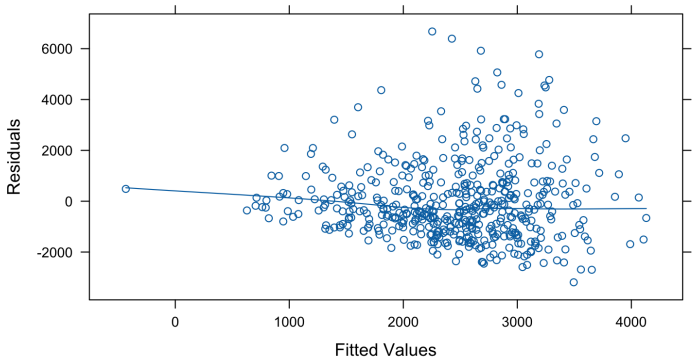
```
xyplot(clothing~income_c, data=sp
```



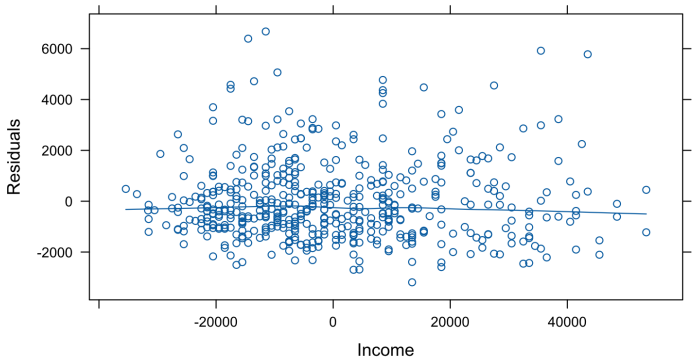

```
xyplot(clothing~food_c, data=spen
```



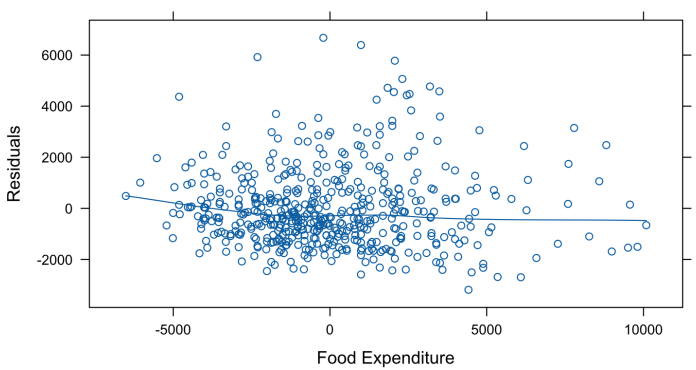
```
xyplot(resid(clothing_model)~pred
```



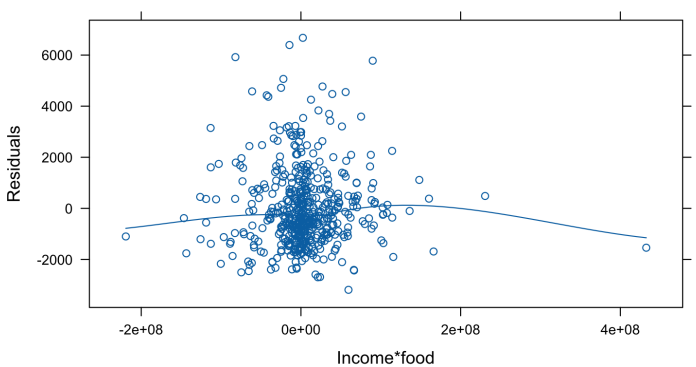

```
xyplot(resid(clothing_model)~spen
```



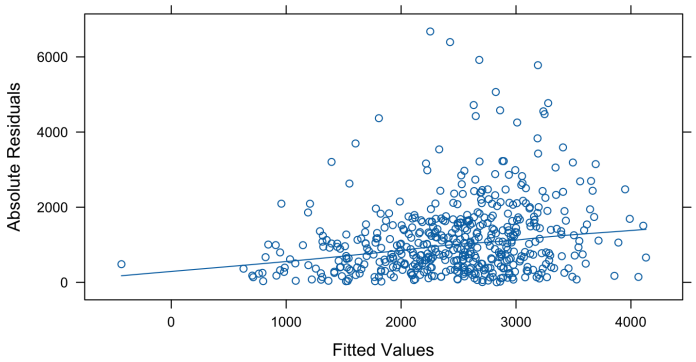

```
xyplot(resid(clothing_model)~spen
```



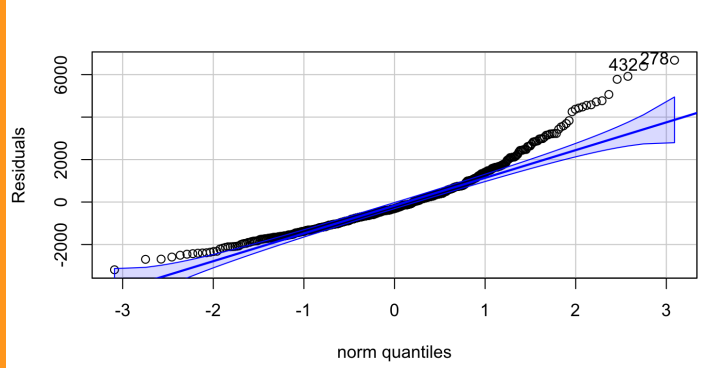

```
xyplot(resid(clothing_model)~spen
```



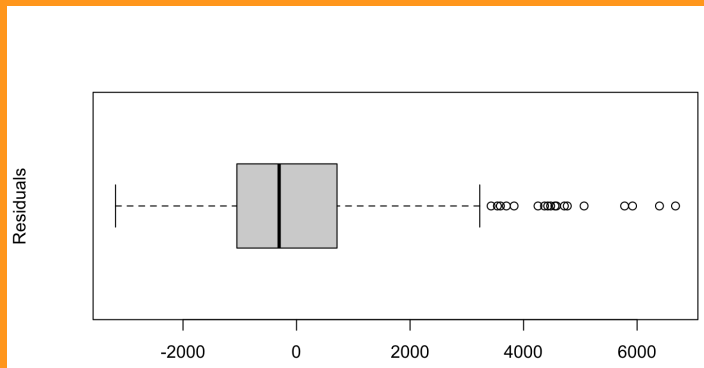
```
xyplot(abs(resid(clothing_model))
```



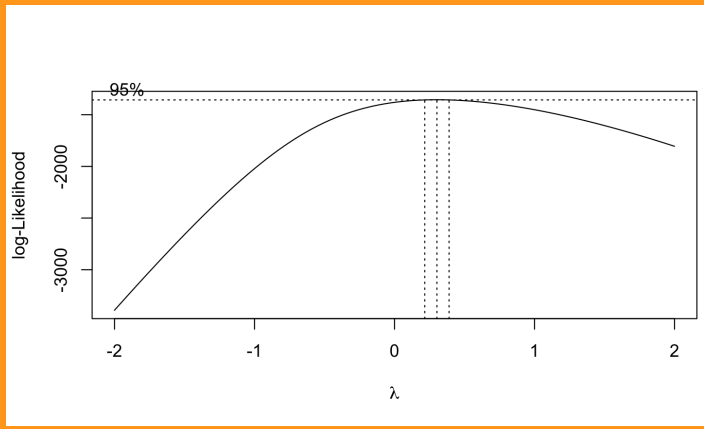

```
qqPlot(resid(clothing_model), yla
```




```
boxplot(resid(clothing_model), yl
```




```
MASS::boxcox(clothing_model)
```



Test of Fit

```
alr3::pureErrorAnova(clothing_model)
```

```
## Analysis of Variance Table
##
## Response: clothing
##                Df   Sum Sq  Mean Sq F value  Pr(>F)
## income_c        1 7.13e+07 7.13e+07   26.91 0.00057
## food_c          1 1.41e+08 1.41e+08   53.24 4.6e-05
## income_c2       1 8.65e+06 8.65e+06    3.27 0.10420
## food_c2         1 1.80e+06 1.80e+06    0.68 0.43111
## incomec_foodc   1 4.29e+06 4.29e+06    1.62 0.23514
## Residuals     494 1.19e+09 2.42e+06
##  Lack of fit  485 1.17e+09 2.41e+06    0.91 0.63789
##  Pure Error     9 2.38e+07 2.65e+06
```

Approximate Test of Fit

Show 20 entries Search:

	clothing	income_c	food_c	income_c2	food_
1	4200	26000	-3000	676000000	9000000	-7800
2	1930	6000	1000	36000000	1000000	600
3	2340	-12000	-2000	144000000	4000000	2400
4	2900	-12000	-1000	144000000	1000000	1200
5	1300	-7000	-2000	49000000	4000000	1400

Showing 1 to 20 of 500 entries

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```
clothing_model_1000 = lm(clothing~income_c+food_c + income_c2+food_c2+in
msummary(clothing_model_1000)
```

```
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    2.60e+03   1.00e+02   26.00  < 2e-16
## income_c       2.15e-02   4.40e-03    4.90  1.3e-06
## food_c         1.92e-01   2.77e-02    6.93  1.3e-11
## income_c2     -3.03e-07   1.83e-07   -1.66    0.098
## food_c2       -3.66e-06   5.60e-06   -0.65    0.513
## incomec_foodc -1.75e-06   1.42e-06   -1.23    0.220
##
## Residual standard error: 1550 on 494 degrees of freedom
## Multiple R-squared:  0.161,    Adjusted R-squared:  0.152
## F-statistic: 18.9 on 5 and 494 DF,  p-value: <2e-16
```

```
msummary(clothing_model)
```

```
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    2.59e+03   1.00e+02   25.91  < 2e-16
## income_c       2.19e-02   4.48e-03    4.88  1.4e-06
## food_c         1.94e-01   2.81e-02    6.90  1.6e-11
## income_c2     -2.97e-07   1.83e-07   -1.62     0.11
## food_c2       -4.04e-06   5.84e-06   -0.69     0.49
## incomec_foodc -1.92e-06   1.44e-06   -1.33     0.18
##
## Residual standard error: 1550 on 494 degrees of freedom
```

```
alr3::pureErrorAnova(clothing_model_1000)
```

```
## Analysis of Variance Table
##
## Response: clothing
##                Df   Sum Sq  Mean Sq F value  Pr(>F)
## income_c        1 7.12e+07 7.12e+07   30.85 1.1e-07
## food_c          1 1.43e+08 1.43e+08   62.04 4.4e-13
## income_c2       1 8.83e+06 8.83e+06    3.82   0.052
## food_c2         1 1.66e+06 1.66e+06    0.72   0.397
## incomec_foodc   1 3.64e+06 3.64e+06    1.58   0.211
## Residuals     494 1.19e+09 2.41e+06
##  Lack of fit  330 8.13e+08 2.46e+06    1.07   0.320
##  Pure Error   164 3.78e+08 2.31e+06
```

Transformed Outcome

```
spending_subset_c$clothing_transformed = spending_subset_c$clothing^(1/4
spending_subset_c %>% datatable()
```

Show 20 entries Search:

	clothing	income_c	food_c	income_c2	foo
1	4200	26482	-2871.85	701296324	8247522.42
2	1930	6482	538.1499999999996	42016324	289605.4224
3	2340	-11518	-1661.85	132664324	2761745.42
4	2900	-11518	-571.8500000000004	132664324	327012.4225
5	1300	-6518	-2331.85	42484324	5437524.42

Showing 1 to 20 of 500 entries

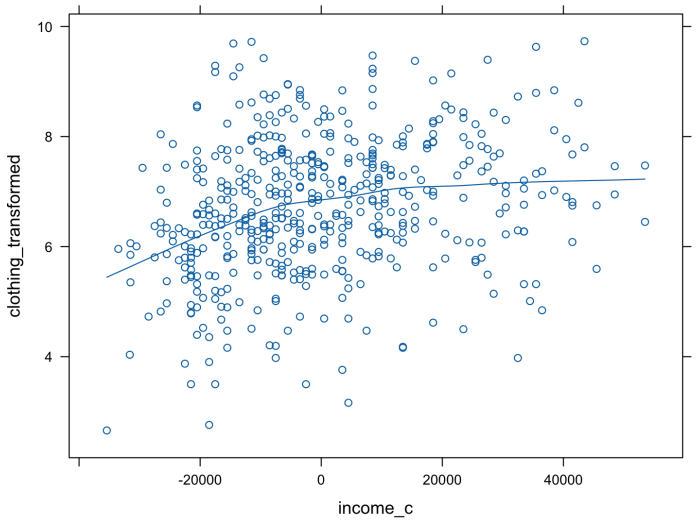
Previous 1 2 3 4 5 ... 25 Next

Fitting the Model

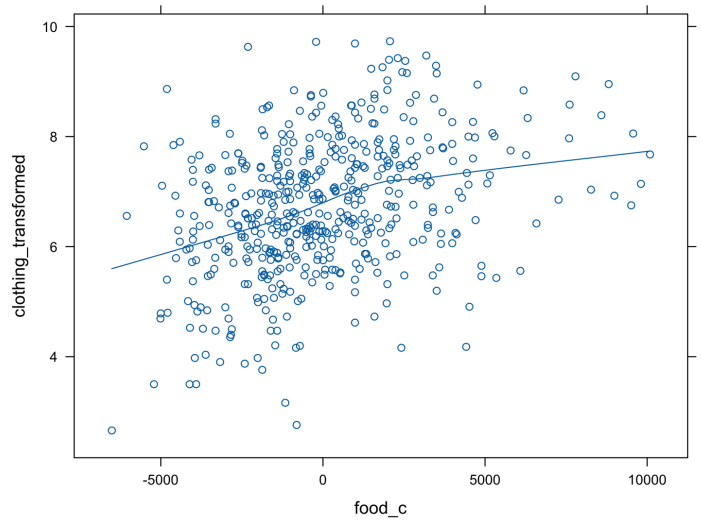
```
clothing_transformed_model = lm(clothing_transformed~income_c+food_c + 
msummary(clothing_transformed_model)
```

```
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    6.88e+00   7.18e-02   95.80  < 2e-16
## income_c       1.88e-05   3.22e-06    5.84  9.3e-09
## food_c         1.57e-04   2.02e-05    7.76  5.0e-14
## income_c2     -3.21e-10   1.32e-10   -2.44    0.015
## food_c2       -6.37e-09   4.20e-09   -1.52    0.130
## incomec_foodc -2.13e-09   1.04e-09   -2.06    0.040
##
## Residual standard error: 1.12 on 494 degrees of freedom
## Multiple R-squared:   0.2,    Adjusted R-squared:  0.192
## F-statistic: 24.7 on 5 and 494 DF,  p-value: <2e-16
```

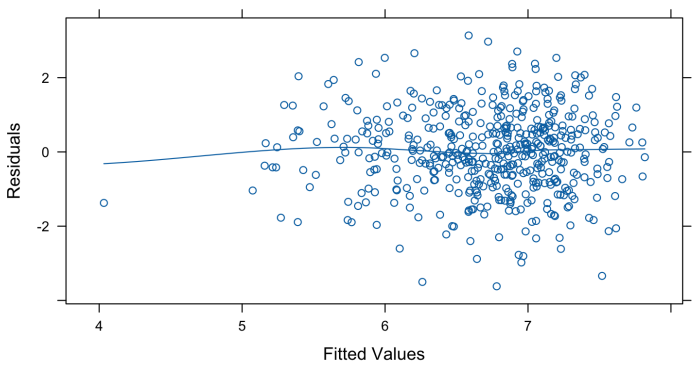
```
xyplot(clothing_transformed~incom
```



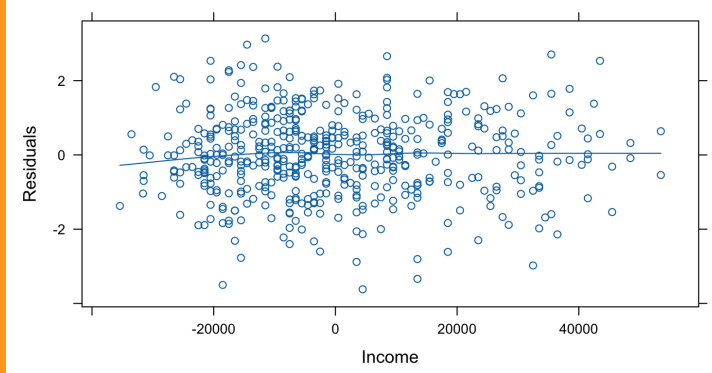

```
xyplot(clothing_transformed~food_
```



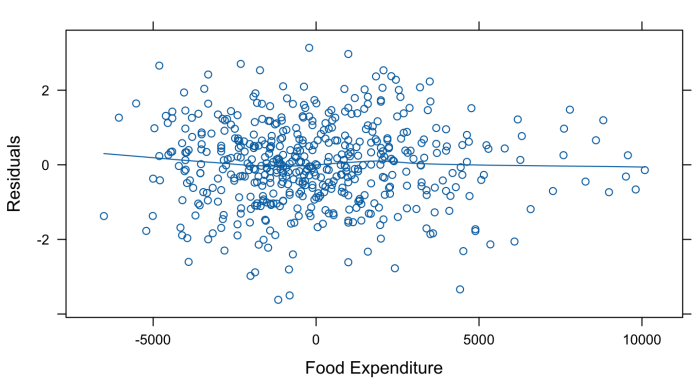
```
xyplot(resid(clothing_transformed
```



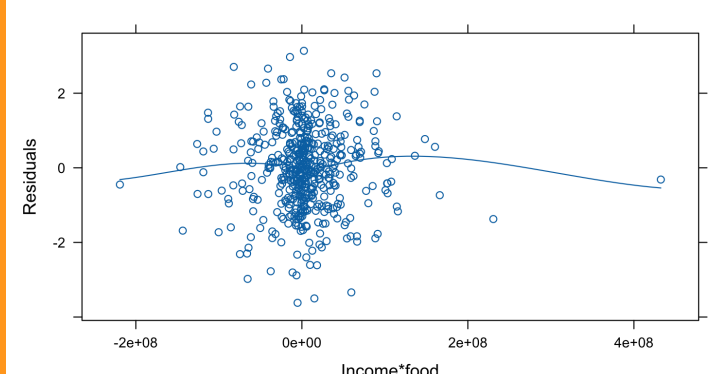

```
xyplot(resid(clothing_transformed
```



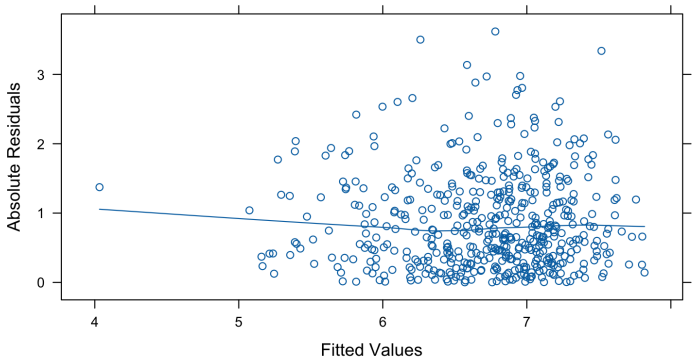

```
xyplot(resid(clothing_transformed
```



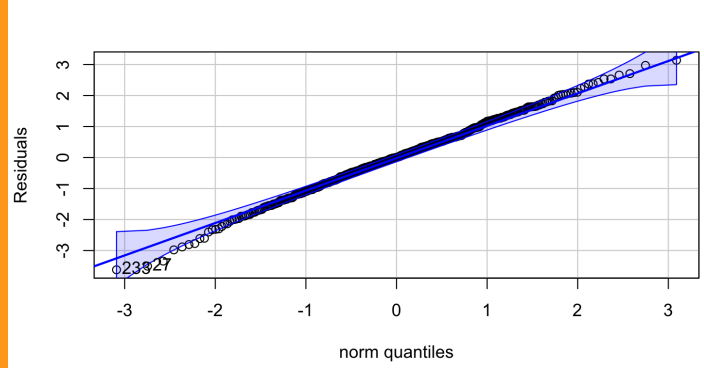

```
xyplot(resid(clothing_transformed
```



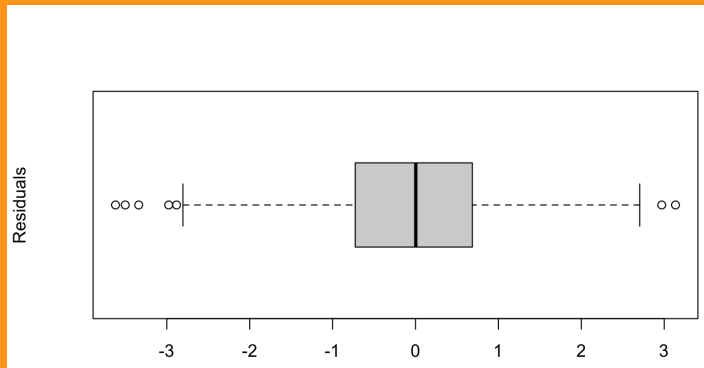
```
xyplot(abs(resid(clothing_transfo
```



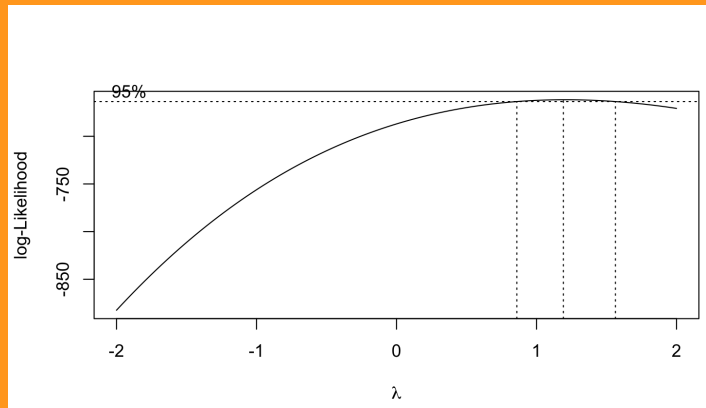

```
qqPlot(resid(clothing_transformed
```




```
boxplot(resid(clothing_transforme
```




```
MASS::boxcox(clothing_transformed
```



Test of Fit

```
alr3::pureErrorAnova(clothing_transformed_model)
```

```
## Analysis of Variance Table
##
## Response: clothing_transformed
##                Df Sum Sq Mean Sq F value  Pr(>F)
## income_c        1     48    48.5   25.48 0.00069
## food_c          1     86    86.0   45.25 8.6e-05
## income_c2       1     10    10.3    5.43 0.04467
## food_c2         1      4     4.0    2.09 0.18257
## incomec_foodc   1      5     5.3    2.77 0.13025
## Residuals     494    616     1.2
##  Lack of fit  485    599     1.2    0.65 0.86928
##  Pure Error     9     17     1.9
```

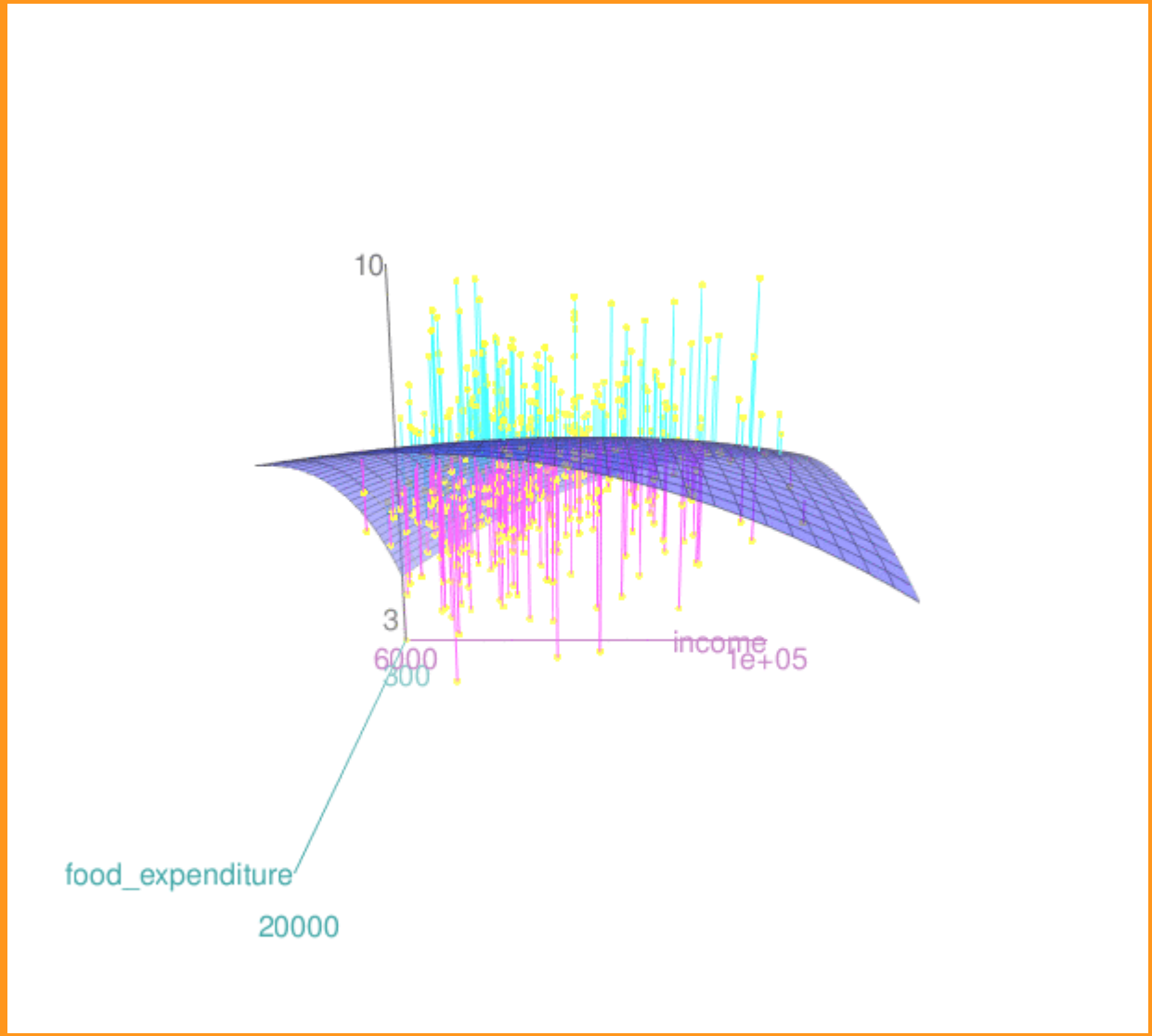
Approximate Test of Fit

```
clothing_transformed_model_1000 = lm(clothing_transformed~income_c+food_

alr3::pureErrorAnova(clothing_transformed_model_1000)
```

```
## Analysis of Variance Table
##
## Response: clothing_transformed
##                Df Sum Sq Mean Sq F value  Pr(>F)
## income_c        1     48    48.3   41.72 1.1e-09
## food_c          1     87    87.1   75.18 4.0e-15
## income_c2       1     10    10.4    8.95  0.0032
## food_c2         1      4     3.9    3.39  0.0673
## incomec_foodc   1      5     4.9    4.21  0.0418
## Residuals     494    615     1.2
##  Lack of fit  330    425     1.3    1.11  0.2222
##  Pure Error   164    190     1.2
```

```
car::scatter3d(I(clothing_expenditure^(1/4))~income+food_expenditure, da
```



Partial F Test:

Would a first order model be sufficient?

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_{11} x_{i1}^2 + \beta_{22} x_{i2}^2 + \beta_{12} x_{i1} x_{i2} + \varepsilon_i$$

- $Y = clothing\_expenditure^{1/4}$ for individual i
- x_{i1} is centered income for individual i
- x_{i2} is centered food expenditure for individual i

$H_0 : \beta_{11} = \beta_{22} = \beta_{12} = 0$ vs H_a : not all of those βs are 0

I.e., does the full model explain significantly more of the variation in Y than the following *reduced model*?

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$$

We can test this using

$$F^* = \frac{SSR(x_1^2, x_2^2, x_1 x_2 | x_1, x_2)}{3} \div MSE$$

Remember that

$$\begin{aligned} SSR(x_1^2, x_2^2, x_1x_2|x_1, x_2) =& SSR(x_1^2|x_1, x_2) \\ &+ SSR(x_2^2|x_1, x_2, x_1^2) \\ &+ SSR(x_1x_2|x_1, x_2, x_1^2, x_2^2) \end{aligned}$$

So we can find the test statistic by considering the extra sums of squares:

```
anova(clothing_transformed_model)
```

```
## Analysis of Variance Table
##
## Response: clothing_transformed
##                Df Sum Sq Mean Sq F value  Pr(>F)
## income_c        1     48    48.5   38.87 9.7e-10
## food_c          1     86    86.0   69.03 9.4e-16
## income_c2       1     10    10.3    8.29  0.0042
## food_c2         1      4     4.0    3.18  0.0750
## incomec_foodc   1      5     5.3    4.23  0.0402
## Residuals     494    616     1.2
```

Or we could get R to perform the general linear test:

```
clothing_transformed_model_reduced = lm(clothing_transformed~income_c+fo

anova(clothing_transformed_model_reduced, clothing_transformed_model)
```

```
## Analysis of Variance Table
##
## Model 1: clothing_transformed ~ income_c + food_c
## Model 2: clothing_transformed ~ income_c + food_c + income_c2 + food_c2 +
##     incomec_foodc
##   Res.Df RSS Df Sum of Sq    F Pr(>F)
## 1    497 635
## 2    494 616  3      19.6 5.23 0.0015
```

Note that RSS could be found in

```
anova(clothing_transformed_model_reduced)
```

```
## Analysis of Variance Table
##
## Response: clothing_transformed
##             Df Sum Sq Mean Sq F value  Pr(>F)
## income_c     1     48    48.5    37.9 1.5e-09
## food_c       1     86    86.0    67.3 2.0e-15
## Residuals  497    635     1.3
```

We could also consider the adequacy of other reduced models, e.g.

$$Y_i = \beta_0 + \beta_{11} x_{i1}^2 + \beta_{22} x_{i2}^2 + \beta_{12} x_{i1} x_{i2} + \varepsilon_i$$

which is equivalent to

$H_0 : \beta_1 = \beta_2 = 0$ vs H_a : not all of those βs are 0

```
clothing_transformed_model_reduced = lm(clothing_transformed~income_c2+

anova(clothing_transformed_model_reduced, clothing_transformed_model)
```

```
## Analysis of Variance Table
##
## Model 1: clothing_transformed ~ income_c2 + food_c2 + incomec_foodc
## Model 2: clothing_transformed ~ income_c + food_c + income_c2 + food_c2 +
##     incomec_foodc
##   Res.Df RSS Df Sum of Sq    F Pr(>F)
## 1    496 750
## 2    494 616  2       134 53.8 <2e-16
```

However, regardless of the outcome of this test, we usually take the **hierarchical approach to fitting**, which means, for example, that we keep x in the model whenever we use x^2, and we would keep x and x^2 in the model whenever we include x^3.

Warm-up Exercises

```
clothing_model = lm(clothing_expenditure~income + I(income^2) + I(income
msummary(clothing_model)
```

```
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)  1.77e+02   7.71e+02    0.23    0.819
## income       1.10e-01   5.63e-02    1.96    0.051
## I(income^2) -1.57e-06   1.25e-06   -1.26    0.210
## I(income^3)  7.93e-12   8.49e-12    0.93    0.351
##
## Residual standard error: 1640 on 496 degrees of freedom
## Multiple R-squared:  0.061,    Adjusted R-squared:  0.0553
## F-statistic: 10.7 on 3 and 496 DF,  p-value: 7.62e-07
```

```
clothing_model_centered = lm(clothing_expenditure~I(income-mean(income))
msummary(clothing_model_centered)
```

```
##                                Estimate Std. Error t value Pr(>|t|)
## (Intercept)                    2.61e+03   9.87e+01   26.49   <2e-16
## I(income - mean(income))       2.07e-02   7.37e-03    2.81   0.0052
## I((income - mean(income))^2) -5.87e-07   2.63e-07   -2.23   0.0259
## I((income - mean(income))^3)  7.93e-12   8.49e-12    0.93   0.3509
##
```

Recap: Section 8.1

After Section 8.1, you should be able to

- Understand the utility and disadvantages of polynomial regression
- Understand the need for centering
- Understand the danger of overfitting
- Compute and interpret parameters in a polynomial regression model

Learning Objectives for Section 8.2

After Section 8.2, you should be able to

- Understand the utility and disadvantages of interactions in regression
- Compute and interpret parameters in regression models with interactions
- Compute and interpret parameters in curvilinear regression models with interactions

Interpretation of Interaction

Suppose

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i1} X_{i2} + \varepsilon_i.$$

Then

$$E[Y|X_1, X_2] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2.$$

We can use this to understand the meaning of the regression coefficients:

$$E[Y|0,0] = \beta_0 + \beta_1(0) + \beta_2(0) + \beta_3(0)(0) = \beta_0$$

$$\begin{aligned} E[Y|X_1+1,0] - E[Y|X_1,0] =& \beta_0 + \beta_1(X_1+1) + \beta_2(0) + \beta_3(X_1+1)(0) \\ & - (\beta_0 + \beta_1(X_1) + \beta_2(0) + \beta_3(X_1)(0)) \\ =& \beta_1 \end{aligned}$$

$$\begin{aligned} E[Y|X_1+1,X_2] - E[Y|X_1,X_2] =& \beta_0 + \beta_1(X_1+1) + \beta_2 X_2 + \beta_3(X_1+1)X_2 \\ & - (\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2) \\ =& \beta_1 + \beta_3 X_2 \end{aligned}$$

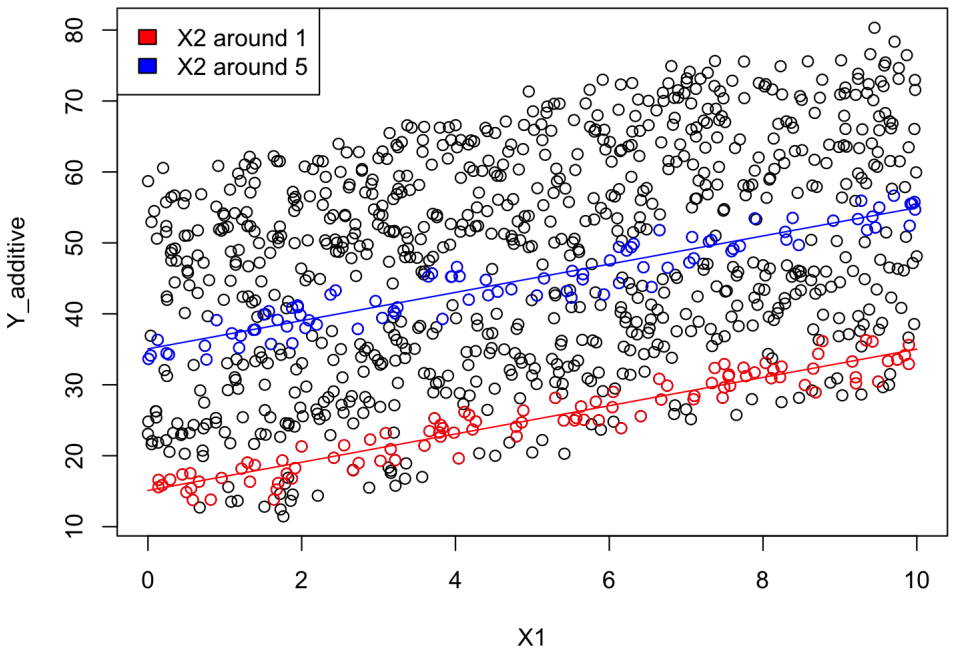
$$E[Y|X_1,X_2+1] - E[Y|X_1,X_2] = \beta_0 + \beta_1 X_1 + \beta_2(X_2+1) + \beta_3 X_1(X_2+1)$$

We could also see the effect of a regression coefficient by taking a partial derivative of the response surface:

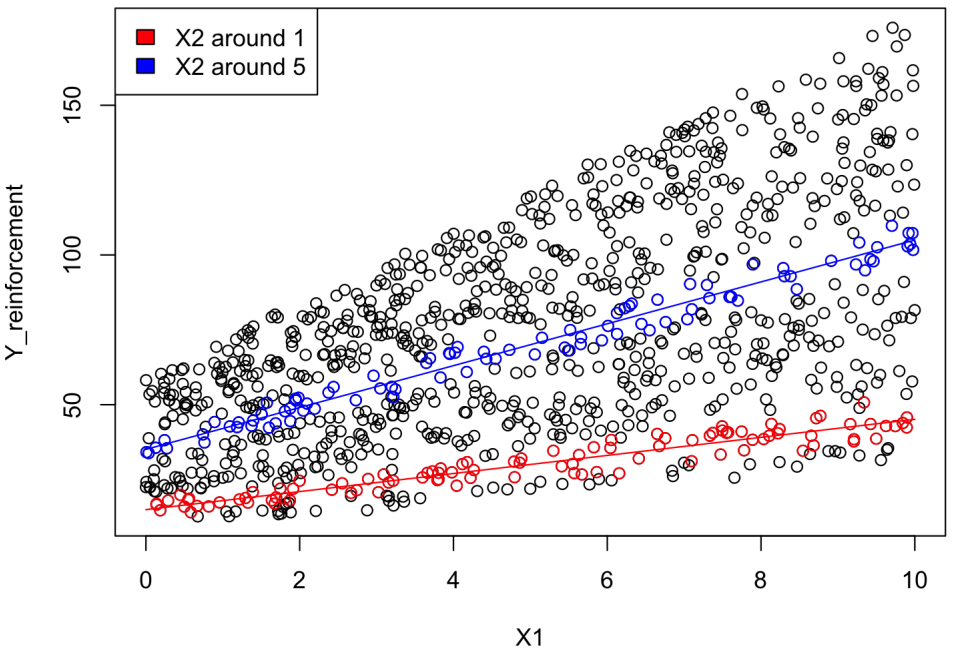
$$\begin{aligned}\frac{\partial E[Y|X_1, X_2]}{\partial X_2} =& \frac{\partial}{\partial X_2}(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2) \\ =& \beta_2 + \beta_3 X_1\end{aligned}$$

```
n=1000
X1 = runif(n, min=0, max=10)
X2 = runif(n, min=0, max=10)
Y_additive = rnorm(n, mean= 10 + 2*X1 + 5*X2 + 0 *X1*X2, sd=1)
Y_reinforcement = rnorm(n, mean= 10 + 2*X1 + 5*X2 + 1 *X1*X2, sd=1)
Y_interference  = rnorm(n, mean= 10 + 2*X1 + 5*X2 + -1 *X1*X2, sd=1)
```

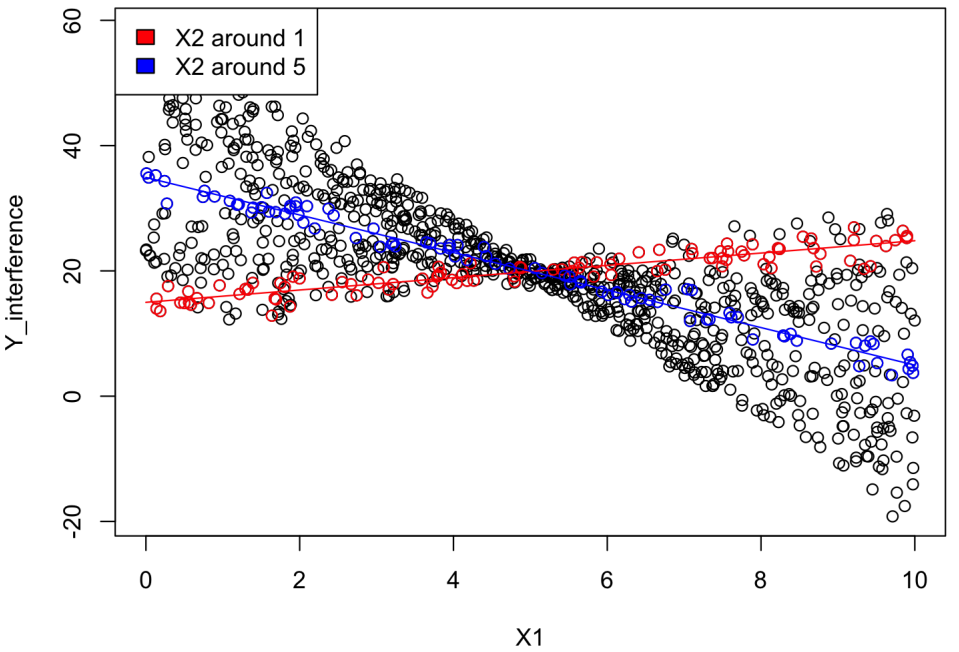
```r
additive_model = lm(Y_additive~X1*X2)
plot(X1, Y_additive)
points(X1[round(X2)==1], Y_additive[round(X2)==1], col="red")
points(X1[round(X2)==5], Y_additive[round(X2)==5], col="blue")
newX1 = seq(from=0, to=10, length.out=100)
lines(newX1, predict(additive_model, newdata=data.frame(X1 = newX1, X2=1
lines(newX1, predict(additive_model, newdata=data.frame(X1 = newX1, X2=5
legend("topleft", fill  =c("red", "blue"), legend=c("X2 around 1", "X2 a
```



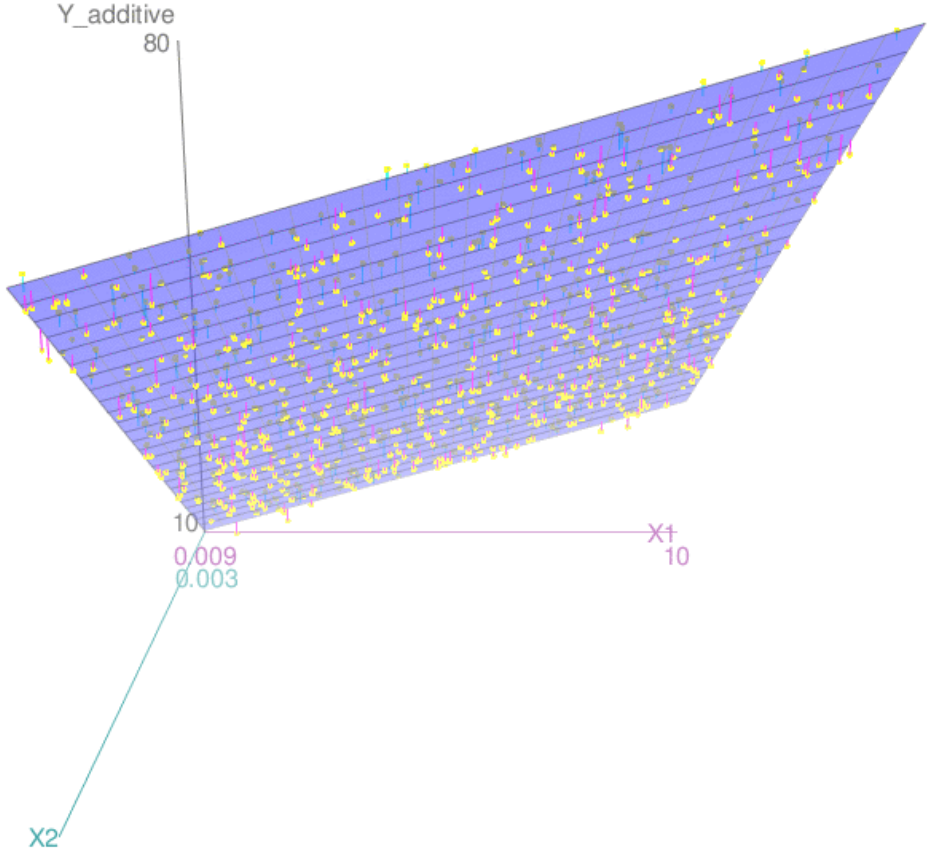
```
reinforcement_interaction_model = lm(Y_reinforcement~X1*X2)
plot(X1, Y_reinforcement)
points(X1[round(X2)==1], Y_reinforcement[round(X2)==1], col="red")
points(X1[round(X2)==5], Y_reinforcement[round(X2)==5], col="blue")
newX1 = seq(from=0, to=10, length.out=100)
lines(newX1, predict(reinforcement_interaction_model, newdata=data.frame
lines(newX1, predict(reinforcement_interaction_model, newdata=data.frame
legend("topleft", fill  =c("red", "blue"), legend=c("X2 around 1", "X2 a
```



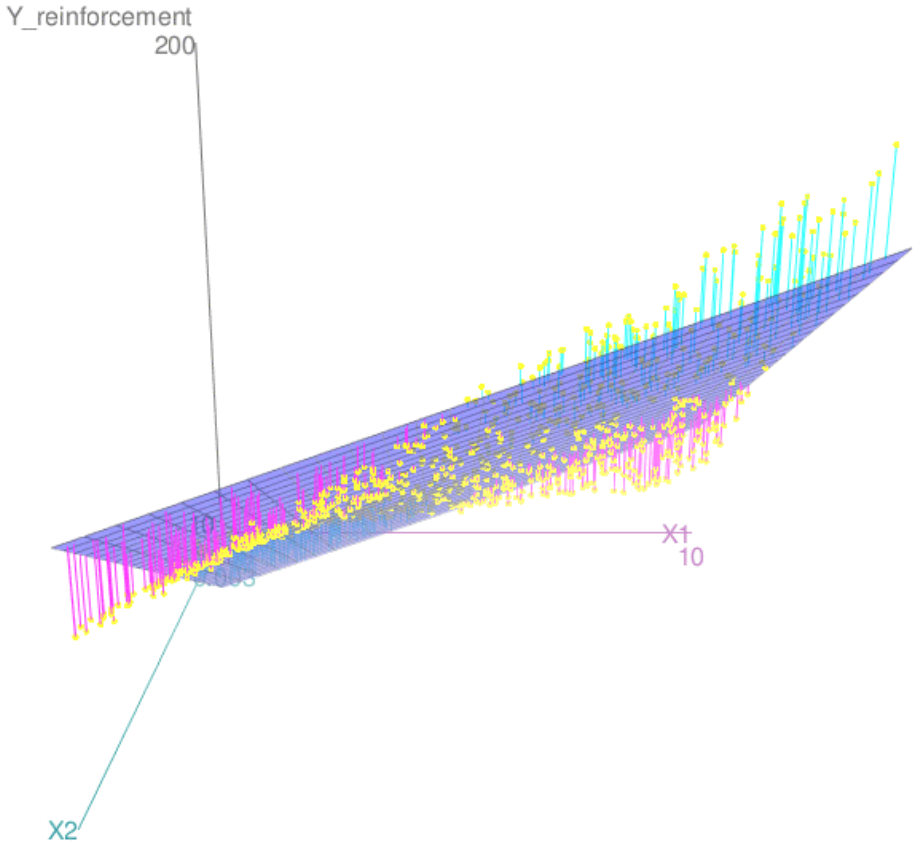
```
interference_interaction_model = lm(Y_interference~X1*X2)
plot(X1, Y_interference)
points(X1[round(X2)==1], Y_interference[round(X2)==1], col="red")
points(X1[round(X2)==5], Y_interference[round(X2)==5], col="blue")
newX1 = seq(from=0, to=10, length.out=100)
lines(newX1, predict(interference_interaction_model, newdata=data.frame
lines(newX1, predict(interference_interaction_model, newdata=data.frame
legend("topleft", fill  =c("red", "blue"), legend=c("X2 around 1", "X2 a
```



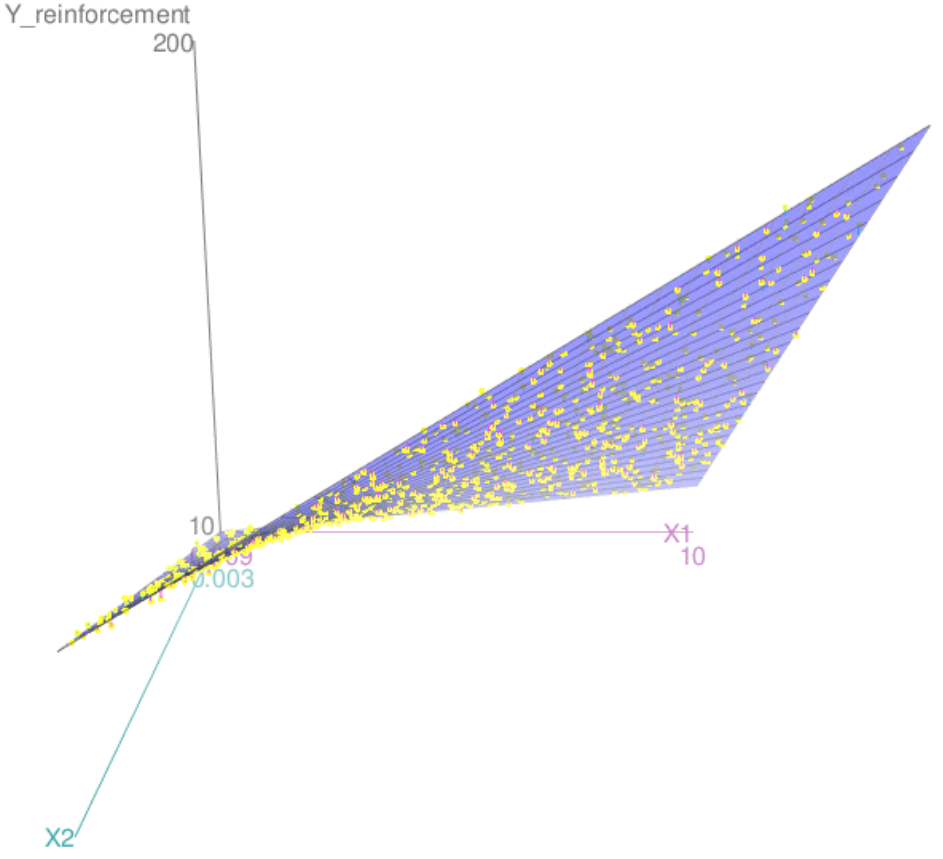
```
car::scatter3d(Y_additive~X1*X2, fit="linear")
```



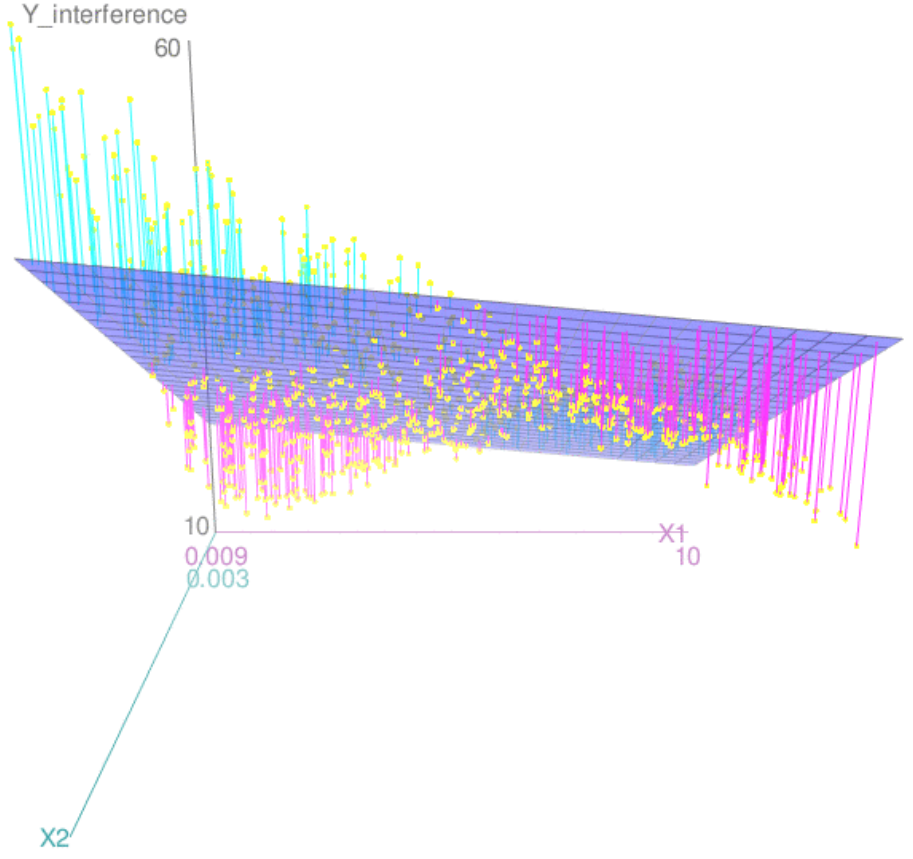
```
car::scatter3d(Y_reinforcement~X1*X2, fit="linear")
```



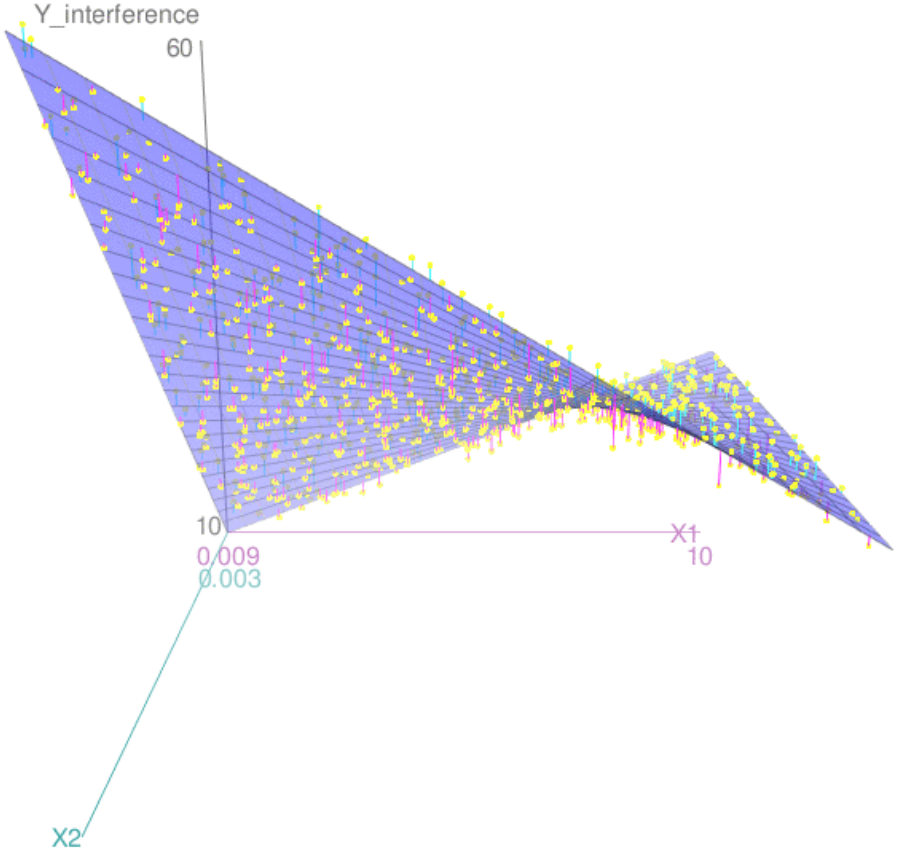
```
car::scatter3d(Y_reinforcement~X1*X2, fit="quad")
```



```
car::scatter3d(Y_interference~X1*X2, fit="linear")
```



```
car::scatter3d(Y_interference~X1*X2, fit="quad")
```



Notes on Implementation of Interaction Regression Models

1. When interaction terms are added to a regression model, high multicollinearities may be introduced.
    - A partial remedy to improve computational accuracy is to center the predictor variables.
2. When the number of predictor variables in the regression model is large, there are a lot of potential interaction terms.
    - For example, if there were 8 predictors, there would be $\binom{8}{2} = 28$ 2-way interactions; we would need a large data set in order to be able to estimate all of these effects!
    - If we have a lot of interactions in our model, we are at risk of *overfitting*.
    - It is best to determine the interactions that are potentially important *a priori*.

```
clothing_model = lm(clothing_expenditure~ (recreation_expenditure + food
msummary(clothing_model)
```

```
##                                            Estimate Std. Error t value Pr(>|t
## (Intercept)                                9.38e+01   4.66e+02    0.20   0.84
## recreation_expenditure                     3.71e-02   1.33e-01    0.28   0.78
## food_expenditure                           1.57e-01   6.94e-02    2.26   0.02
## income                                     2.95e-02   1.06e-02    2.79   0.00
## recreation_expenditure:food_expenditure    2.60e-05   1.26e-05    2.06   0.04
## recreation_expenditure:income              1.03e-08   1.96e-06    0.01   0.99
## food_expenditure:income                   -2.10e-06   1.48e-06   -1.42   0.15
##
## Residual standard error: 1490 on 493 degrees of freedom
## Multiple R-squared:  0.229,    Adjusted R-squared:  0.219
## F-statistic: 24.4 on 6 and 493 DF,  p-value: <2e-16
```

Extra sum of squares
$(SSR(x_1), SSR(x_2|x_1), SSR(x_3|x_1, x_2), SSR(x_1x_2|x_1, x_2, x_3), \dots)$

```
anova(clothing_model)
```

```
## Analysis of Variance Table
##
## Response: clothing_expenditure
##                                              Df   Sum Sq  Mean Sq F value  Pr(>
## recreation_expenditure                        1 1.96e+08 1.96e+08   88.22 < 2e-
## food_expenditure                              1 8.64e+07 8.64e+07   38.89 9.6e-
## income                                        1 3.01e+07 3.01e+07   13.56 0.000
## recreation_expenditure:food_expenditure       1 7.33e+06 7.33e+06    3.30 0.069
## recreation_expenditure:income                 1 5.92e+05 5.92e+05    0.27 0.606
## food_expenditure:income                       1 4.51e+06 4.51e+06    2.03 0.154
## Residuals                                   493 1.10e+09 2.22e+06
```

Test $H_0 : \beta_4 = \beta_5 = \beta_6 = 0$ using

$$F^* = \frac{SSR(x_1x_2, x_1x_3, x_2x_3|x_1, x_2, x_3)}{3} \div MSE$$

```
clothing_model_reduced = lm(clothing_expenditure~ recreation_expenditure
anova(clothing_model_reduced, clothing_model)
```

```
## Analysis of Variance Table
##
## Model 1: clothing_expenditure ~ recreation_expenditure + food_expenditure +
##     income
## Model 2: clothing_expenditure ~ (recreation_expenditure + food_expenditure
##     income)^2
##   Res.Df      RSS Df Sum of Sq    F Pr(>F)
## 1    496 1.11e+09
## 2    493 1.10e+09  3  12431950 1.87   0.13
```

```
clothing_model = lm(clothing_expenditure~ income+  recreation_expenditur
msummary(clothing_model)
```

```
##                                  Estimate Std. Error t value Pr(>|t|)
## (Intercept)                      6.02e+01   4.23e+02    0.14   0.8869
## income                           3.86e-02   1.77e-02    2.18   0.0299
## recreation_expenditure           6.72e-01   1.37e-01    4.91  1.2e-06
## I(income^2)                     -2.16e-07   1.82e-07   -1.18   0.2366
## I(recreation_expenditure^2)     -4.10e-05   1.35e-05   -3.04   0.0025
## income:recreation_expenditure -1.65e-06   1.91e-06   -0.87   0.3867
##
## Residual standard error: 1530 on 494 degrees of freedom
## Multiple R-squared:  0.183,    Adjusted R-squared:  0.175
## F-statistic: 22.1 on 5 and 494 DF,  p-value: <2e-16
```

Extra sum of squares $(SSR(x_1), SSR(x_2|x_1), SSR(x_3|x_2, x_1), \dots)$

```
anova(clothing_model)
```

```
## Analysis of Variance Table
##
## Response: clothing_expenditure
##                                Df   Sum Sq  Mean Sq F value  Pr(>F)
## income                          1 7.13e+07 7.13e+07   30.34 5.9e-08
## recreation_expenditure          1 1.58e+08 1.58e+08   67.47 1.9e-15
## I(income^2)                     1 5.85e+06 5.85e+06    2.49  0.1150
## I(recreation_expenditure^2)     1 2.24e+07 2.24e+07    9.53  0.0021
## income:recreation_expenditure   1 1.76e+06 1.76e+06    0.75  0.3867
## Residuals                     494 1.16e+09 2.35e+06
```

```
clothing_model_reduced = lm(clothing_expenditure~ income*recreation_expe
msummary(clothing_model_reduced)
```

```
##                                  Estimate Std. Error t value Pr(>|t|)
## (Intercept)                      7.41e+02   2.92e+02    2.54   0.0113
## income                           2.21e-02   6.87e-03    3.21   0.0014
## recreation_expenditure           3.77e-01   8.66e-02    4.36  1.6e-05
## income:recreation_expenditure -2.42e-06   1.89e-06   -1.28   0.2003
##
## Residual standard error: 1550 on 496 degrees of freedom
## Multiple R-squared:  0.165,    Adjusted R-squared:  0.159
## F-statistic: 32.6 on 3 and 496 DF,  p-value: <2e-16
```

```
anova(clothing_model_reduced, clothing_model)
```

```
## Analysis of Variance Table
##
## Model 1: clothing_expenditure ~ income * recreation_expenditure
## Model 2: clothing_expenditure ~ income + recreation_expenditure + I(income^
##     I(recreation_expenditure^2) + income:recreation_expenditure
##   Res.Df      RSS Df Sum of Sq    F Pr(>F)
## 1    496 1.19e+09
## 2    494 1.16e+09  2  26069841 5.55 0.0041
```

Warm-up Exercises

```
clothing_model_centered_interaction = lm(clothing_expenditure~sex*(I(cen
msummary(clothing_model_centered_interaction)
```

```
##                                Estimate Std. Error t value Pr(>|t|)
## (Intercept)                    2.78e+03   1.45e+02   19.22   <2e-16
## sexmale                       -3.42e+02   1.92e+02   -1.78    0.075
## I(center(income))              2.94e-02   6.37e-03    4.62    5e-06
## I(center(income)^2)           -4.24e-07   2.85e-07   -1.49    0.138
## sexmale:I(center(income))     -2.61e-03   9.62e-03   -0.27    0.786
## sexmale:I(center(income)^2) -3.55e-08   3.91e-07   -0.09    0.928
##
## Residual standard error: 1640 on 494 degrees of freedom
## Multiple R-squared:  0.0701,    Adjusted R-squared:  0.0607
## F-statistic: 7.45 on 5 and 494 DF,  p-value: 9.48e-07
```

Recap: Section 8.2

After Section 8.2, you should be able to

- Understand the utility and disadvantages of interactions in regression
- Compute and interpret parameters in regression models with interactions
- Compute and interpret parameters in curvilinear regression models with interactions

Learning Objectives for Sections 8.3-8.4

After Sections 8.3-8.4, you should be able to

- Implement and interpret regression using indicator (dummy) variables

Qualitative Predictor with Two Classes

There are many ways of quantitatively identifying the classes of a qualitative variable. We shall use indicator variables that take on the values 0 and 1. These indicator variables are easy to use and are widely employed, but they are by no means the only way to quantify a qualitative variable.

For the insurance innovation example, where the qualitative predictor variable has two classes ("stock company" or "mutual company"), we might define two indicator variables X_2 and X_3 as follows:

$$X_2 = \begin{cases} 1 & \text{if stock company} \\ 0 & \text{otherwise} \end{cases}$$

$$X_3 = \begin{cases} 1 & \text{if mutual company} \\ 0 & \text{otherwise} \end{cases}$$

We might think that a first-order model would then be

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \varepsilon_i$$

However, suppose that we have $n = 4$ observations, the first two being stock firms, and the second two being mutual firms. The design matrix would then be

$$\mathbb{X} = \begin{bmatrix} 1 & X_{11} & 1 & 0 \\ 1 & X_{21} & 1 & 0 \\ 1 & X_{31} & 0 & 1 \\ 1 & X_{41} & 0 & 1 \end{bmatrix}$$

where the first column is equal to the sum of X_2 and X_3, so the columns are linearly dependent! Thus

$$\mathbb{X}'\mathbb{X} = \begin{bmatrix} 4 & \sum_{i=1}^{4} X_{i1} & 2 & 2 \\ \sum_{i=1}^{4} X_{i1} & \sum_{i=1}^{4} X_{i1}^2 & \sum_{i=1}^{2} X_{i1} & \sum_{i=3}^{4} X_{i1} \\ 2 & \sum_{i=1}^{2} X_{i1} & 2 & 0 \\ 2 & \sum_{i=3}^{4} X_{i1} & 0 & 2 \end{bmatrix}$$

has linearly dependent columns and, therefore, does not have an inverse. That means that we could not possibly find $b = (\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Y}$

We can also see that this coding would not work by attempting to decipher what the parameters would represent:

For stock companies, the response function would be

$$E[Y] = \beta_0 + \beta_1 X_{i1} + \beta_2(1) + \beta_3(0) = (\beta_0 + \beta_2) + \beta_1 X_{i1}$$

For the other type of company (mutual companies), the response function would be

$$E[Y] = \beta_0 + \beta_1 X_{i1} + \beta_2(0) + \beta_3(1) = (\beta_0 + \beta_3) + \beta_1 X_{i1}$$

We are using 3 parameters $(\beta_0, \beta_2, \beta_3)$ to capture 2 things (the y-intercept for stock companies and the y-intercept for mutual companies). This *over-parameterization* is what makes it impossible to uniquely estimate the parameters.

A simple way out of this difficulty is to drop one of the indicator variables.

In our example, we might drop X_3.

Dropping one indicator variable is not the only way out of the difficulty, but it leads to simple interpretations of the parameters.

In general, therefore, we shall follow the principle:

- A qualitative variable with c classes will be represented by $c - 1$ indicator variables, each taking on the values 0 and 1.

Indicator variables are frequently also called *dummy* variables or *binary* variables.

Interpretation of Regression Coefficients

We are now considering the regression model

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i$$

So, for stock companies, the response function would be

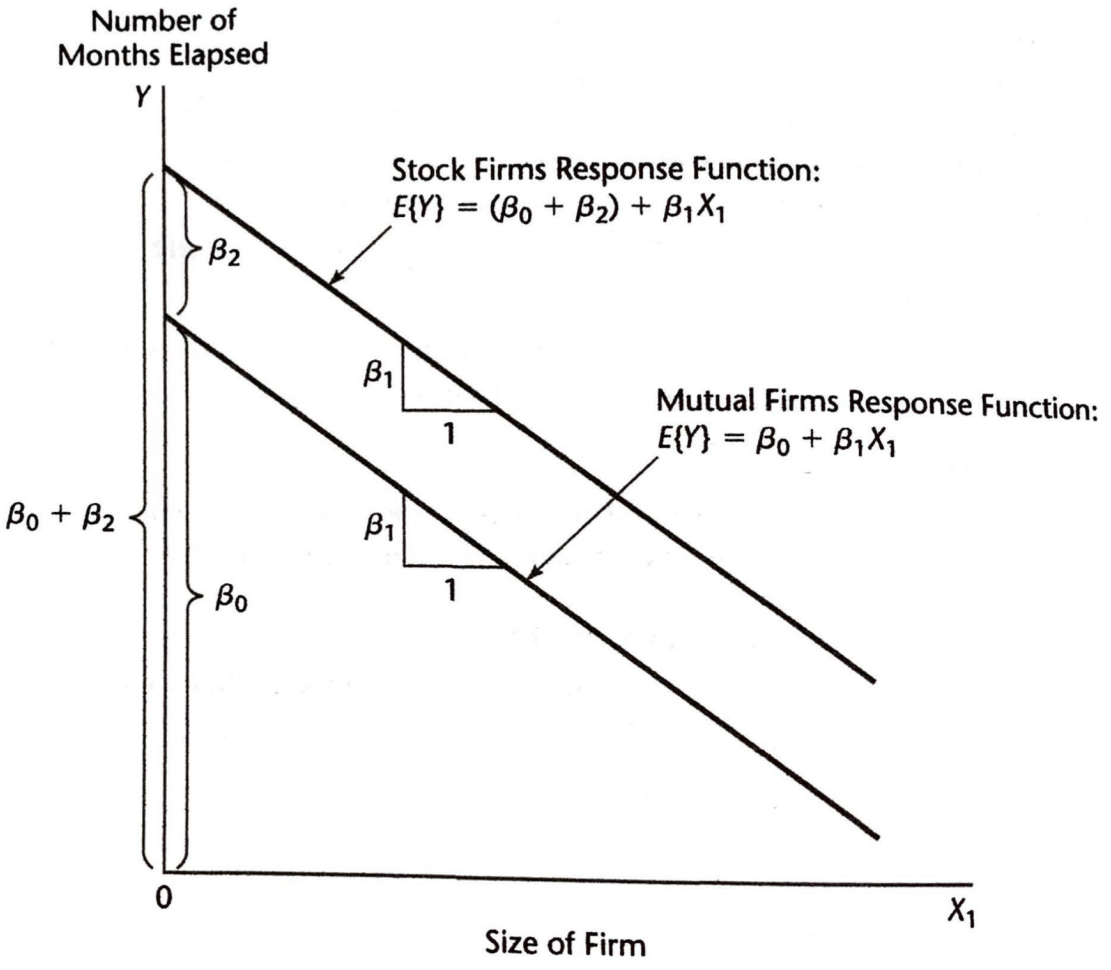
$$E[Y] = \beta_0 + \beta_1 X_{i1} + \beta_2(1) = (\beta_0 + \beta_2) + \beta_1 X_{i1}$$

For the other type of company (mutual companies), the response function would be

$$E[Y] = \beta_0 + \beta_1 X_{i1} + \beta_2(0) = \beta_0 + \beta_1 X_{i1}$$

So we can uniquely interpret β_0, β_1, and β_2.

FIGURE 8.11
Illustration of Meaning of Regression Coefficients for Regression Model (8.33) with Indicator Variable X_2—Insurance Innovation Example.



Note that we could have alternately chosen to drop β_0 from our over-parameterized regression model (i.e., used regression through the origin):

$$Y_i = \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \varepsilon_i$$

So, for stock companies, the response function would be

$$E[Y] = \beta_1 X_{i1} + \beta_2(1) + \beta_3(0) = \beta_2 + \beta_1 X_{i1}$$

For the other type of company (mutual companies), the response function would be

$$E[Y] = \beta_1 X_{i1} + \beta_2(0) + \beta_3(1) = \beta_3 + \beta_1 X_{i1}$$

We could still uniquely interpret our parameters, but the interpretations would be different!

Consider how one would test whether or not the two regression lines are identical.

Instead of testing $H_0 : \beta_2 = 0$ (which could be done using extra sums of squares), we would need to test $H_0 : \beta_2 = \beta_3$ (which we could accomplish by fitting a reduced model where $\beta_2 = \beta_3$ and using the general linear test approach to test the adequacy of that reduced model).

We could have also considered the regression model

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \varepsilon_i$$

where, instead of the usual dummy variable coding, we used

$$X_2 = \begin{cases} 1 & \text{if stock company} \\ -1 & \text{otherwise} \end{cases}$$

Then, for stock companies, the response function would be

$$E[Y] = \beta_0 + \beta_1 X_{i1} + \beta_2(1) = (\beta_0 + \beta_2) + \beta_1 X_{i1}$$

For the other type of company (mutual companies), the response function would be

$$E[Y] = \beta_0 + \beta_1 X_{i1} + \beta_2(-1) = (\beta_0 - \beta_2) + \beta_1 X_{i1}$$

Again, we could uniquely interpret our parameters, but the interpretations would again be different!

Qualitative Predictor with More than Two Classes

Show 20 entries Search:

	province	type_of_dwelling	income	marital_status	age_group
1	NL	single_detached	68000	never_married	30-34
2	NL	single_detached	48000	never_married	25-29
3	NL	single_detached	30000	married	35-39
4	NL	row_house	30000	never_married	30-34

Showing 1 to 20 of 3,347 entries

Previous 1 2 3 4 5 ... 168 Next

```
table(spending_subset$type_of_dwelling)
```

```
##
##           other single_detached   semi_detached       row_house         apart
##              96            1894             142             229
##          duplex
##             158
```

Let

$$X_1 = income$$

$$X_2 = \begin{cases} 1 & \text{if single\_detached} \\ 0 & \text{otherwise} \end{cases}$$

$$X_3 = \begin{cases} 1 & \text{if semi\_detached} \\ 0 & \text{otherwise} \end{cases}$$

$$X_4 = \begin{cases} 1 & \text{if row\_house} \\ 0 & \text{otherwise} \end{cases}$$

$$X_5 = \begin{cases} 1 & \text{if apartment} \\ 0 & \text{otherwise} \end{cases}$$

$$X_6 = \begin{cases} 1 & \text{if duplex} \\ 0 & \text{otherwise} \end{cases}$$

If we use the first-order regression model

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i4} + \beta_5 X_{i5} + \beta_6 X_{i6} + \varepsilon_i$$

then the response function is

$$E[Y] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_5 + \beta_6 X_6$$

So for someone who lives in a single detached house,

$$E[Y] = (\beta_0 + \beta_2) + \beta_1 X_1$$

So for someone who lives in a semi-detached house,

$$E[Y] = (\beta_0 + \beta_3) + \beta_1 X_1$$

etc.

$X_1 = income; X_2 = I(single\_detached); X_3 = I(semi\_detached);$
$X_4 = I(row\_house); X_5 = I(apartment); X_6 = I(duplex).$

$$E[Y] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_5 + \beta_6 X_6$$

- **What does β_0 represent?**
- **How much higher is the response function for those in single detached houses compared to those whose housing was listed as "other"?**
- **How much higher is the response function for those in single detached houses compared to those in semi-detached houses?**
- **How do we test whether the response function for those in single detached houses differs from that of those whose housing was listed as "other"?**
- **How do we test whether those in single detached and semi-detached houses are different in terms of mean clothing expenditure?**

```
msummary(lm(clothing_expenditure~income+type_of_dwelling, data=spending_
```

```
##                                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)                     1.14e+03   1.75e+02    6.53  7.5e-11
## income                          1.45e-02   1.46e-03    9.90  < 2e-16
## type_of_dwellingsingle_detached 5.61e+02   1.71e+02    3.29   0.0010
## type_of_dwellingsemi_detached   3.86e+02   2.15e+02    1.79   0.0734
## type_of_dwellingrow_house       5.51e+02   1.98e+02    2.78   0.0055
## type_of_dwellingapartment       3.39e+02   1.75e+02    1.93   0.0534
## type_of_dwellingduplex          4.30e+02   2.11e+02    2.04   0.0415
##
## Residual standard error: 1630 on 3340 degrees of freedom
## Multiple R-squared:  0.0379,    Adjusted R-squared:  0.0362
## F-statistic: 21.9 on 6 and 3340 DF,  p-value: <2e-16
```

```
spending_subset$type_of_dwelling = factor(spending_subset$type_of_dwell
msummary(lm(clothing_expenditure~income+type_of_dwelling, data=spending_
```

```
##                                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)                     1.70e+03   7.63e+01   22.32   <2e-16
## income                          1.45e-02   1.46e-03    9.90   <2e-16
## type_of_dwellingsemi_detached -1.75e+02   1.42e+02   -1.24   0.2153
## type_of_dwellingrow_house     -1.06e+01   1.14e+02   -0.09   0.9259
## type_of_dwellingapartment     -2.22e+02   6.82e+01   -3.25   0.0011
## type_of_dwellingduplex        -1.31e+02   1.35e+02   -0.98   0.3294
## type_of_dwellingother         -5.61e+02   1.71e+02   -3.29   0.0010
```

Indicator Variables versus Allocated Codes

Now suppose that we use the following coding

Let

$$X_1 = income$$

$$X_2 = \begin{cases} 1 & \text{if single\_detached} \\ 2 & \text{if semi\_detached} \\ 3 & \text{if row\_house} \\ 4 & \text{if apartment} \\ 5 & \text{if duplex} \\ 6 & \text{otherwise} \end{cases}$$

Then the response function corresponding to the first order regression model is

$$E[Y] = \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

So for someone who lives in a single detached house,

$$E[Y] = (\beta_0 + \beta_2) + \beta_1 X_1$$

For someone who lives in a semi-detached house,

$$E[Y] = (\beta_0 + 2\beta_2) + \beta_1 X_1$$

For someone who lives in a row house,

$$E[Y] = (\beta_0 + 3\beta_2) + \beta_1 X_1$$

For someone who lives in an apartment,

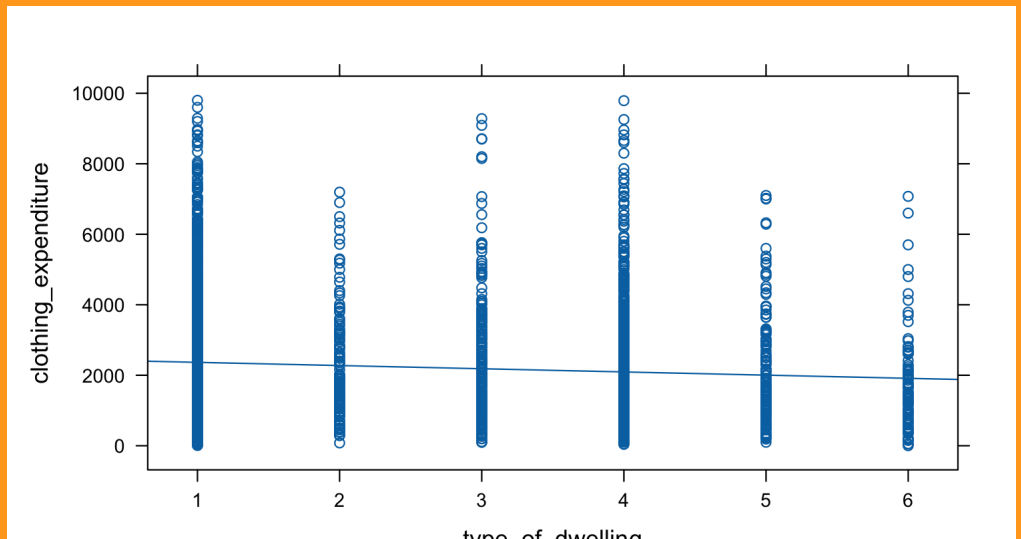
$$E[Y] = (\beta_0 + 4\beta_2) + \beta_1 X_1$$

etc.

```
spending_subset$type_of_dwelling = as.numeric(spending_subset$type_of_dw
msummary(lm(clothing_expenditure~income+type_of_dwelling, data=spending_
```

```
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)       1.77e+03   8.46e+01   20.90  < 2e-16
## income            1.46e-02   1.46e-03   10.04  < 2e-16
## type_of_dwelling -7.01e+01   1.81e+01   -3.87  0.00011
##
## Residual standard error: 1630 on 3344 degrees of freedom
## Multiple R-squared:  0.0364,    Adjusted R-squared:  0.0359
## F-statistic: 63.2 on 2 and 3344 DF,  p-value: <2e-16
```

```
xyplot(clothing_expenditure~type_of_dwelling, data=spending_subset, type
```



Indicator Variables versus Quantitative Variables

```
table(spending_subset$age_group)
```

```
##
##   <25 25-29 30-34 35-39 40-44 45-49 50-54 55-59 60-64 65-69 70-74 75-79 80-
##   201   366   422   378   421   482   426   348   228    61     9     4
```

```
msummary(lm(clothing_expenditure~income+age_group, data=spending_subset)
```

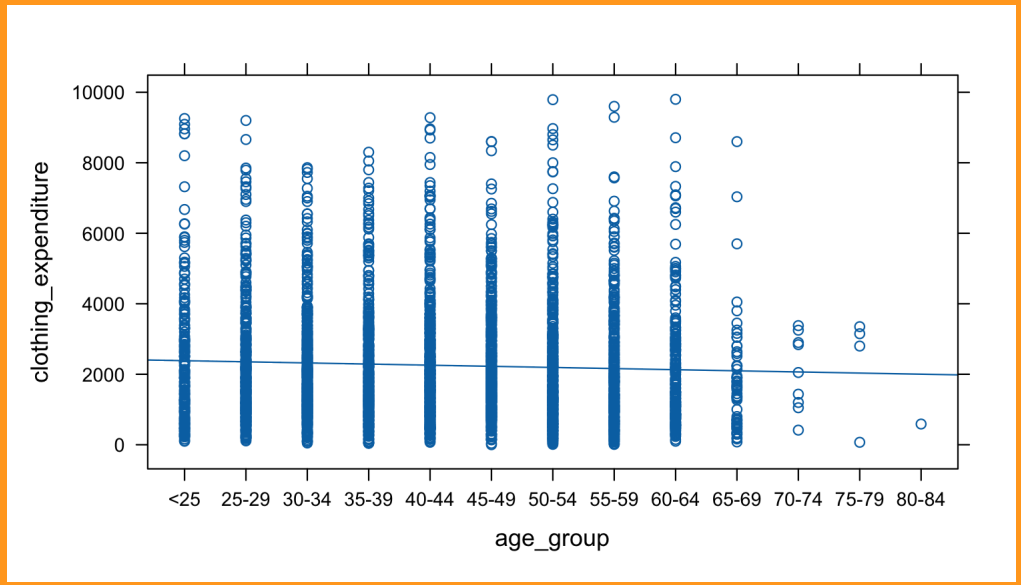
```
##                  Estimate Std. Error t value Pr(>|t|)
## (Intercept)     1.86e+03   1.24e+02   14.99  < 2e-16
## income          1.62e-02   1.47e-03   10.99  < 2e-16
## age_group25-29 -2.40e+02   1.44e+02   -1.67  0.09518
## age_group30-34 -3.24e+02   1.41e+02   -2.30  0.02153
## age_group35-39 -2.88e+02   1.43e+02   -2.01  0.04477
## age_group40-44 -1.61e+02   1.41e+02   -1.14  0.25331
## age_group45-49 -3.09e+02   1.38e+02   -2.24  0.02535
## age_group50-54 -3.53e+02   1.41e+02   -2.51  0.01218
## age_group55-59 -4.98e+02   1.46e+02   -3.41  0.00065
## age_group60-64 -5.19e+02   1.59e+02   -3.27  0.00109
## age_group65-69 -8.28e+02   2.39e+02   -3.47  0.00054
## age_group70-74 -5.28e+02   5.55e+02   -0.95  0.34085
## age_group75-79 -2.58e+02   8.22e+02   -0.31  0.75363
## age_group80-84 -1.77e+03   1.63e+03   -1.09  0.27792
##
```

```
table(as.numeric(spending_subset$age_group))
```

```
##
##   1   2   3   4   5   6   7   8   9  10  11  12  13
## 201 366 422 378 421 482 426 348 228  61   9   4   1
```

```
msummary(lm(clothing_expenditure~income+as.numeric(age_group), data=sper
```

```
##                          Estimate Std. Error t value Pr(>|t|)
## (Intercept)              1.80e+03   8.75e+01   20.52  < 2e-16
## income                   1.60e-02   1.46e-03   10.95  < 2e-16
## as.numeric(age_group) -4.70e+01   1.17e+01   -4.02  6.1e-05
##
## Residual standard error: 1630 on 3344 degrees of freedom
## Multiple R-squared:  0.0368,    Adjusted R-squared:  0.0362
## F-statistic: 63.8 on 2 and 3344 DF,  p-value: <2e-16
```


clothing_expenditure
10000
8000
6000
4000
2000
0
<25
25-29
30-34
35-39
40-44
45-49
50-54
55-59
60-64
65-69
70-74
75-79
80-84
age_group

```
anova(lm(clothing_expenditure~income+as.numeric(age_group), data=spendi
```

```
## Analysis of Variance Table
##
## Model 1: clothing_expenditure ~ income + as.numeric(age_group)
## Model 2: clothing_expenditure ~ income + age_group
##   Res.Df      RSS Df Sum of Sq    F Pr(>F)
## 1   3344 8.85e+09
## 2   3333 8.82e+09 11   2.9e+07 0.99   0.45
```

Other Codings for Indicator Variables

```
spending_subset$sex.1 = ifelse(spending_subset$sex=="male", 1, -1)
spending_subset %>% datatable()
```

Show 20 entries Search:

	province	type_of_dwelling	income	marital_status	age_group
1	NL	1	68000	never_married	30-34
2	NL	1	48000	never_married	25-29
3	NL	1	30000	married	35-39
4	NL	3	30000	never_married	30-34
5	NL	1	35000	married	25-29

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```
msummary(lm(clothing_expenditure~sex, data=spending_subset))
```

```
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   2353.2       42.3   55.69   <2e-16
## sexmale       -183.3       57.4   -3.19   0.0014
##
## Residual standard error: 1660 on 3345 degrees of freedom
## Multiple R-squared:  0.00304,    Adjusted R-squared:  0.00274
## F-statistic: 10.2 on 1 and 3345 DF,  p-value: 0.00142
```

```
msummary(lm(clothing_expenditure~sex.1, data=spending_subset))
```

```
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   2261.6       28.7   78.78   <2e-16
## sex.1          -91.7       28.7   -3.19   0.0014
##
## Residual standard error: 1660 on 3345 degrees of freedom
## Multiple R-squared:  0.00304,    Adjusted R-squared:  0.00274
## F-statistic: 10.2 on 1 and 3345 DF,  p-value: 0.00142
```

Recap: Sections 8.3-8.4

After Sections 8.3-8.4, you should be able to

- Implement and interpret regression using indicator (dummy) variables

Learning Objectives for Sections 8.5-8.7

After Sections 8.5-8.7, you should be able to

- Implement and interpret regression involving interactions between indicator and quantitative variables
- Implement and interpret regression involving interactions between multiple indicator variables
- Implement and interpret tests for differences among regression functions

8.5: Modeling Interactions between Quantitative and Qualitative Predictors

A first-order regression model with an added interaction term for the insurance innovation example is:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i1} X_{i2} + \varepsilon_i,$$

where

$$X_{i1} = \text{ size of firm } i$$

$$X_{i2} = \begin{cases} 1 & \text{if } i \text{ is a stock company} \\ 0 & \text{otherwise} \end{cases}$$

So, for stock companies, the response function would be

$$E[Y] = \beta_0 + \beta_1 X_{i1} + \beta_2(1) + \beta_3 X_{i1}(1) = (\beta_0 + \beta_2) + (\beta_1 + \beta_3) X_{i1}$$

For the other type of company (mutual companies), the response function would be

$$E[Y] = \beta_0 + \beta_1 X_{i1} + \beta_2(0) + \beta_3 X_{i1}(0) = \beta_0 + \beta_1 X_{i1}$$

FIGURE 8.14
Illustration of Meaning of Regression Coefficients for Regression Model (8.49) with Indicator Variable X_2 and Interaction Term—Insurance Innovation Example.

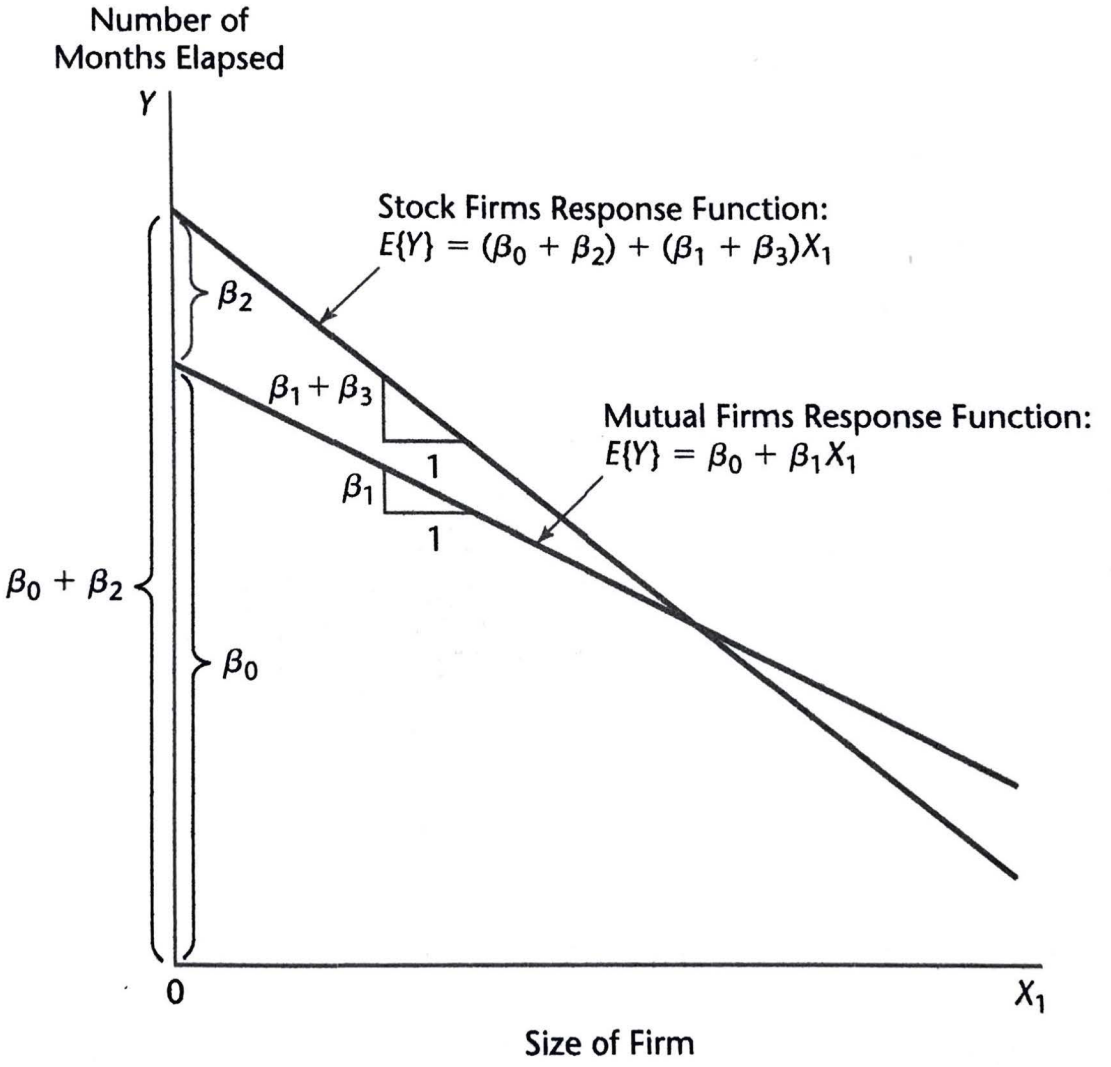
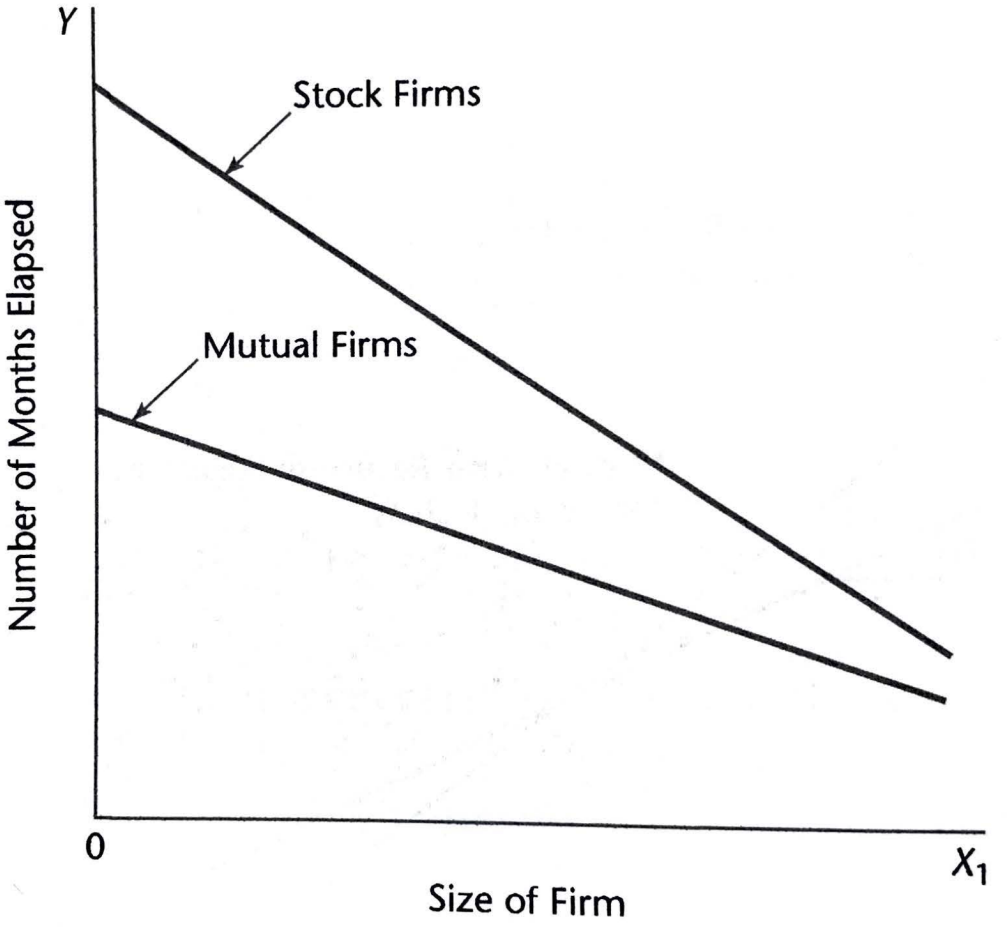


FIGURE 8.15
Another Illustration of Regression Model (8.49) with Indicator Variable X_2 and Interaction Term—Insurance Innovation Example.



Fitting regression model (8.49) yields the same response functions as would fitting separate regressions for stock firms and mutual firms.

An advantage of using model (8.49) with an indicator variable is that one regression run will yield both fitted regressions.

Another advantage is that tests for comparing the regression functions for the different classes of the qualitative variable can be clearly seen to involve tests of regression coefficients in a general linear model.

For instance, Figure 8.14 for the insurance innovation example shows that a test of whether the two regression functions have the same slope involves:

$$H_0 : \beta_3 = 0 \qquad vs \quad H_a : \ \beta_3 \neq 0$$

Similarly, Figure 8.14 shows that a test of whether the two regression functions are identical involves:

$$H_0 : \beta_2 = \beta_3 = 0 \qquad vs \quad H_a : \ \text{not both } \beta_3 = 0 \text{ and } \beta_3 = 0$$

8.6: More Complex Models

1. Models in which all explanatory variables are qualitative are called *analysis of variance models*.

2. Models containing some quantitative and some qualitative explanatory variables, where the chief explanatory variables of interest are qualitative and the quantitative variables are introduced primarily to reduce the variance of the error terms, are called *analysis of covariance models.*

Show 20 entries Search:

	province	type_of_dwelling	income	marital_status	age_group
1	NL	single_detached	68000	never_married	30-34
2	NL	single_detached	48000	never_married	25-29
3	NL	single_detached	30000	married	35-39
4	NL	row_house	30000	never_married	30-34
5	NL	single_detached	35000	married	25-29

Showing 1 to 20 of 3,347 entries

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```
##
## female   male
##   1534   1813

##
##       married never_married         other
##          1809           831           707
```

Let

$$X_1 = income$$

$$X_2 = \begin{cases} 1 & \text{if male} \\ 0 & \text{otherwise} \end{cases}$$

$$X_3 = \begin{cases} 1 & \text{if married} \\ 0 & \text{otherwise} \end{cases}$$

Consider the response function for the first-order regression model:

$$E[Y] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$$

So for a married male

$$E[Y] = (\beta_0 + \beta_2 + \beta_3) + \beta_1 X_1$$

For an unmarried male

$$E[Y] = (\beta_0 + \beta_2) + \beta_1 X_1$$

For a married female

$$E[Y] = (\beta_0 + \beta_3) + \beta_1 X_1$$

For an unmarried female

$$E[Y] = \beta_0 + \beta_1 X_1$$

Again, let

$$X_1 = income$$

$$X_2 = \begin{cases} 1 & \text{if male} \\ 0 & \text{otherwise} \end{cases}$$

$$X_3 = \begin{cases} 1 & \text{if married} \\ 0 & \text{otherwise} \end{cases}$$

But now consider the response function for the regression model including an X_2X_3 interaction:

$$E[Y] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_2 X_3$$

So for a married male

$$E[Y] = (\beta_0 + \beta_2 + \beta_3 + \beta_4) + \beta_1 X_1$$

For an unmarried male

$$E[Y] = (\beta_0 + \beta_2) + \beta_1 X_1$$

For a married female

$$E[Y] = (\beta_0 + \beta_3) + \beta_1 X_1$$

For an unmarried female

$$E[Y] = \beta_0 + \beta_1 X_1$$

Notice that the effect of being married is allowed to be different among males $(\beta_3 + \beta_4)$ and females (β_3).

```
msummary(lm(clothing_expenditure~income+sex*I(marital_status=="married")
```

```
##                                                Estimate Std. Error t value
## (Intercept)                                    1.18e+03   2.30e+02    5.11
## income                                         2.53e-02   4.11e-03    6.16
## sexmale                                       -6.97e+02   2.60e+02   -2.68
## I(marital_status == "married")TRUE             7.87e+02   2.10e+02    3.74
## sexmale:I(marital_status == "married")TRUE     1.85e+02   3.19e+02    0.58
##                                               Pr(>|t|)
## (Intercept)                                    4.5e-07
## income                                         1.5e-09
## sexmale                                         0.0076
## I(marital_status == "married")TRUE              0.0002
## sexmale:I(marital_status == "married")TRUE      0.5632
##
## Residual standard error: 1590 on 495 degrees of freedom
## Multiple R-squared:  0.113,    Adjusted R-squared:  0.106
## F-statistic: 15.8 on 4 and 495 DF,  p-value: 3.4e-12
```

```
clothing_model = lm(clothing_expenditure~(I(income-mean(income)) + I((in
msummary(clothing_model)
```

```
##                                              Estimate Std. Error t value
## (Intercept)                                  2.40e+03   1.63e+02   14.69
## I(income - mean(income))                     3.07e-02   4.64e-03    6.62
## I((income - mean(income))^2)                -4.54e-07   1.85e-07   -2.45
## sexmale                                     -7.47e+02   2.60e+02   -2.88
## I(marital_status == "married")TRUE           7.71e+02   2.09e+02    3.68
## sexmale:I(marital_status == "married")TRUE   2.15e+02   3.18e+02    0.68
##                                             Pr(>|t|)
## (Intercept)                                  < 2e-16
## I(income - mean(income))                     9.6e-11
## I((income - mean(income))^2)                 0.01447
## sexmale                                      0.00415
## I(marital_status == "married")TRUE           0.00026
## sexmale:I(marital_status == "married")TRUE   0.49897
##
## Residual standard error: 1590 on 494 degrees of freedom
## Multiple R-squared:  0.124,    Adjusted R-squared:  0.115
## F-statistic:   14 on 5 and 494 DF,  p-value: 8.35e-13
```

8.7: Comparison of Two or More Regression Functions

If we want to test whether there is a difference between the regression function between males and females, we can test whether our full model

$$\begin{aligned} E[Y] =& \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_{1,2} X_1 X_2 \\ &+ \beta_{0,3} X_3 + \beta_{1,3} X_1 X_3 + \beta_{2,3} X_2 X_3 + \beta_{1,2,3} X_1 X_2 X_3 \end{aligned}$$

is significantly better than a reduced model that assumes the effects related to sex (X_3) are all 0:

$$E[Y] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_{1,2} X_1 X_2$$

```
anova(lm(clothing_expenditure~income*I(marital_status=="married"), data=
```

```
## Analysis of Variance Table
##
## Model 1: clothing_expenditure ~ income * I(marital_status == "married")
## Model 2: clothing_expenditure ~ income * sex * I(marital_status == "married
##   Res.Df      RSS Df Sum of Sq    F Pr(>F)
## 1    496 1.29e+09
## 2    492 1.25e+09  4  42319785 4.16 0.0025
```

Recap: Sections 8.5-8.7

After Sections 8.5-8.7, you should be able to

- Implement and interpret regression involving interactions between indicator and quantitative variables
- Implement and interpret regression involving interactions between multiple indicator variables
- Implement and interpret tests for differences among regression functions

```
msummary(lm(clothing_expenditure~income*sex*I(marital_status=="married")
```

```
##                                                    Estimate Std. Error t va
## (Intercept)                                        1.18e+03   1.34e+02    8
## income                                             2.25e-02   2.73e-03    8
## sexmale                                           -6.29e+01   2.01e+02   -0
## I(marital_status == "married")TRUE                 7.13e+02   1.96e+02    3
## income:sexmale                                    -1.28e-02   4.03e-03   -3
## income:I(marital_status == "married")TRUE         -4.28e-03   4.47e-03   -0
## sexmale:I(marital_status == "married")TRUE        -1.37e+02   2.76e+02   -0
## income:sexmale:I(marital_status == "married")TRUE  1.29e-02   5.91e-03    2
##                                                   Pr(>|t|)
## (Intercept)                                        < 2e-16
## income                                             2.6e-16
## sexmale                                            0.75408
## I(marital_status == "married")TRUE                 0.00028
## income:sexmale                                     0.00150
## income:I(marital_status == "married")TRUE          0.33846
## sexmale:I(marital_status == "married")TRUE         0.61910
## income:sexmale:I(marital_status == "married")TRUE  0.02956
##
## Residual standard error: 1580 on 3339 degrees of freedom
## Multiple R-squared:  0.0957,    Adjusted R-squared:  0.0938
## F-statistic: 50.5 on 7 and 3339 DF,  p-value: <2e-16
```

```
xyplot(clothing_expenditure~income, groups=paste(ifelse(marital_status==
```