1. $[t^n](\alpha f(t) + \beta g(t)) = \alpha[t^n]f(t) + \beta[t^n g(t)]$
2. $[t^n]t^S f(t) = [t^{n-s}]f(t)$
3. $[t^n]f'(t) = (n+1)f_{n+1}, \quad [t^n]f(t) = \frac{1}{n}[t^{n-1}]f(t)$
4. $[t^n]f(t)g(t) = \sum_{k=0}^{n} f_k g_{n-k}$
5. $[t^n]f(t) \circ g(t) = [t^n]\sum_{k\geq 0} f_k g(t)^k = \sum_{k\geq 0} f_k [t^n] g(t)^k$

Regola di newton

$[t^n](1+\alpha t)^r = \binom{r}{n}\alpha^n$

$f(t) = \frac{1}{1-t}$

$[t^n]\frac{1}{1-t} = [t^n](1-t)^{-1} = \binom{-1}{n}(-1)^n$

> Proprietà binomiali
> $\binom{n}{k} = \binom{n}{n-k}$

> $\binom{-n}{k} = \binom{n+k-1}{k}(-1)^k$

$\binom{-1}{n}(-1)^n = \binom{1+n-1}{n} = \binom{n}{n} = 1$

$f(t) = \frac{1}{1-ct}$

$[t^n]\frac{1}{1-ct} = [t^n](1-ct)^{-1} = \binom{-1}{n}(-c)^n = \binom{1+n-1}{n}(-1)^n(-c)^n = \binom{n}{n}c^k = c^n$

$f(t) = \frac{t}{(1-t)^2}$

$[t^n]\frac{t}{(1-t)^2} = [t^{n-1}](1-t)^2 = \binom{-2}{n-1}(-1)^{n-1} = \binom{2+n-1-1}{n-1} = \binom{n}{n-1} = \frac{n!}{(n-1)!()}$

$[t^n](\frac{t^2}{(1-t)^2} + \frac{2t}{(1-t)}) = [t^{n-2}(1-t)^{-2}] + 2[t^n - 1](1-t)^{-1} =$
$\binom{-2}{n-2}(-1)^{n-2} + 2\binom{-1}{n-1}(-1)^{n-1} =$
$\binom{2+n-2-1}{n-2} + 2\binom{1+n-1-1}{} = \binom{n-1}{n-2} + 2\binom{n-1}{n-1} =$

Fibonacci in formula chiusa

$f(t) = \frac{t}{1-t-t^2}$
Trattiamo una funzione razionale fratta. Dobbiamo trovare $[t^n] = f(t)$

- Si può trovare la ricorrenza per F_n
- Oppure la formula esplicita per F_n

$(1-t-t^2)f(t) = t$
La condizione iniziale è $f(t) = 0$ se $t = 0$

$[t^n](1-t-t^2)f(t) = [t^n]t$
$[t^n]f(t) - [t^{n-1}]f(t) - [t^{n-2}]f(t) = \delta_{n,1}$

$F_n - F_{n-1} - F_{n-2} = \delta_{n,1}$

$\forall n \geq 2 \quad F_n = F_{n-1} + F_{n-2}$ perché il delta vale zero

$F_0 = 0 \quad F_1 = 1$ per le condizioni iniziali

In questo modo abbiamo trovato la ricorrenza.

Altro esempio per trovare la ricorrenza di $f(t)$
$f(t) = \frac{t+t^2}{1-2t-3t^3}$

$(1-2t-3t^3)f(t) = t + t^2$
$[t^n](1-2t-3t^3)f(t) = [t^n](t+t^2)$
$[t^n]f(t) - 2[t^{n-1}]f(t) - 3[t^{n-3}]f(t) = \delta_{n,1} + \delta_{n,2}$
Scriviamo $[t^{n-3}]$ perchè l'esponente di t è 3
$f_n - 2f_{n-1} - 3f_{n-3} = \delta_{n,1} + \delta_{n,2}$
$\forall n \geq 3 \quad f_n = 2f_{n-1} + 3f_{n-3}$

$f_0 = 0$
$f_1 - 2f_0 - 3f_{-2} = 1 \Rightarrow \quad f_1 = 1 + 2f_0 = 1$
$f_{-2} = 0$ perchè per i numeri negativi si prende 0
$f_2 - 2f_1 - 3f_{-1} = 1 \Rightarrow \quad f_2 = 1 + 2f_1 = 1$

Troviamo ora la formula esplicita per F_n

$f(t) = \frac{t}{1-t-t^2}$

Vogliamo trasformare il denominatore in:

$1 - t - t^2 = (1 - \phi t)(1 - \hat{\phi} t)$

Con

$$\phi = \frac{1 - \sqrt{5}}{2} \qquad \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

$f(t) = \frac{t}{(1-\phi t)(1-\hat{\phi} t)} = \frac{A}{(1-\phi t)} + \frac{B}{(1-\hat{\phi} t)}$

Si trovano le radici del denominatore.

$1 - t - t^2 = 0$

$t_{1,2} = \frac{1 \pm \sqrt{1+4}}{-2}$

$t_1 = \frac{1+\sqrt{5}}{-2} \quad t_2 = \frac{1-\sqrt{5}}{-2}$

Sappiamo che le radici $x_1 \cdot x_2 = \frac{c}{a}$ in $ax^2 + bx + c = 0$

Possiamo inoltre scrivere $ax^2 + bx + c = 0$ in

$$a(x - x_1)(x - x_2) = 0$$

Sviluppando l'equazione

$a(-x_1(1 + \frac{x}{-x_1}))(-x_2(1 + \frac{x}{-x_2})) = 0$

$\underbrace{ax_1x_2}_{\text{C}}(1 + \frac{x}{-x_1})(1 + \frac{x}{-x_2}) = 0$

$t_1 t_2 = -1$

$\frac{1}{t_1} = -t_2 \quad \frac{1}{t_2} = -t_1$

$$\begin{aligned} 1 - t - t^2 &= -(t - t_1)(t - t_2) \\ &= (1 - \tfrac{t}{t_1})(1 - \tfrac{t}{t_2}) \\ &= (1 + t_2 t)(1 + t_1 t) \\ &= (1 - \hat{\phi} t)(1 - \phi t) \qquad \blacksquare \end{aligned}$$

Quindi abbiamo che $1 - t - t^2 = (1 + t_1 t)(1 + t_2 t)$

$$
\begin{aligned}
f(t) = \frac{t}{(1+t_1 t)(1+t_2 t)} &= \frac{A}{(1+t_1 t)} + \frac{b}{(1+t_2 t)} \\
&= \frac{A(1+t_2 t)+B((1+t_1 t))}{(1+t_1 t)(1+t_2 t)} \\
&= \frac{A+B+(At_2+Bt_1)t}{(1+t_1 t)(1+t_2 t)}
\end{aligned}
$$

$$
\begin{cases} A+B=0 \\ At_2+Bt_1=1 \end{cases}
\begin{cases} A=-B \\ -Bt_2+Bt_1=1 \end{cases}
\begin{cases} A=\frac{1}{t_2-t_1} \\ B=\frac{1}{t_1-t_2} \end{cases}
$$

$$
t_1 - t_2 = -(\tfrac{1+\sqrt{5}}{2}) + (\tfrac{1-\sqrt{5}}{2}) = -\sqrt{5}
$$

$$
B = -\frac{1}{\sqrt{5}} \qquad A = \frac{1}{\sqrt{5}}
$$

$$
\begin{aligned}
[t^n]\tfrac{1}{\sqrt{5}}(\tfrac{1}{1+t_1 t}) - \tfrac{1}{\sqrt{5}}(\tfrac{1}{1+t_2 t}) &= \tfrac{1}{\sqrt{5}}[t^n](1+t_1 t)^{-1} - \tfrac{1}{\sqrt{5}}[t^n](1+t_2 t)^{-1} \\
&= \tfrac{1}{\sqrt{5}}\tbinom{-1}{n}t_1^n - \tfrac{1}{\sqrt{5}}\tbinom{-1}{n}t_2^n \\
&= \tfrac{1}{\sqrt{5}}\tbinom{1+n-1}{n}(-1)^n t_1^n - \tfrac{1}{\sqrt{5}}\tbinom{1+n-1}{n}(-1)^n t_2^n \\
&= \tfrac{1}{\sqrt{5}}(-t_1)^n - \tfrac{1}{\sqrt{5}}(-t_2)^n \\
&= \tfrac{1}{\sqrt{5}}(\phi^n - \hat{\phi})^n \text{ forma esplicita}
\end{aligned}
$$

$$
\approx \frac{\phi^n}{\sqrt{5}}
$$

Per n grandi

$G(\binom{2n}{n}) = \frac{1}{\sqrt{1-4t}}$ $\quad$ $G(\frac{1}{n+1}\binom{2n}{n}) = \frac{1-\sqrt{1-4t}}{2t}$

$[t^n]\frac{1}{\sqrt{1-4t}} = [t^n](1-4t)^{-1/2} = \binom{-1/2}{n}(-4)^n$

$\Rightarrow \binom{-1/2}{n} = \frac{(-1)^n}{4^n}\binom{2n}{n}$

$$
\begin{aligned}
[t^n]\frac{1-\sqrt{1-4t}}{2t} &= \frac{1}{2}[t^n]t^{-1}(1-\sqrt{1-4t}) \\
&= \frac{1}{2}[t^{n+1}]1 - \frac{1}{2}[t^{n+1}](1-4t)^{1/2} \\
&= \frac{-1}{2}\binom{1/2}{n+1}(-4)^{n+1}
\end{aligned}
$$

> $\binom{1/2}{n} = \frac{(-1)^n}{4^n 2(n+1)}\binom{2n}{n}$

$$
\begin{aligned}
&= \frac{-1}{2}\frac{(-1)^n}{4^{n+1}(2(n+1)-1)}\binom{2(n+1)}{n+1}(-4)^{n+1} \\
&= \frac{(-1)^{n+1}}{2*4^{n+1}(2n+1)}\binom{2(n+1)}{n+1}(-4)^{n+1} \\
&= \frac{1}{2(2n+1)}\binom{2(n+1)}{n+1} \\
&= \frac{1}{2(2n+1)}\frac{(2n+1)!}{((n+1)!)^2} \\
&= \frac{1}{2(2n+1)}\frac{(2n+2)(2n+1)(2n)!}{(n+1)n!(n+1)n!} \\
&= \frac{2n!}{(n+1)n!n!} =
\end{aligned}
$$

$$\frac{1}{n+1}\binom{2n}{n}$$

Formula dei confronti del quicksort

$c(t) = \frac{2}{(1-t)^2} ln\frac{1}{1-t}$
$c_n = 2(n+1)(H_{n+1}-1)$

è un prodotto di due funzioni

$c(t) = 2\frac{1}{1-t}\frac{1}{1-t}ln(\frac{1}{1-t})$

> Chiamiamo $g(t) = \frac{1}{1-t}ln(\frac{1}{1-t})$

Si usa la convoluzione
$[t^n]\frac{1}{1-t}g(t) = \sum_{k=0}^{n} g_{n-k} = \sum_{k=0}^{n} g_k$

$c(t) = 2\frac{1}{1-t}g(t)$
$[t^n]c(t) = 2[t^n]\frac{1}{1-t}g(t) = 2\sum_{k=0}^{n} g_k$
$[t^n]g(t) = [t^n]\frac{1}{1-t}ln\frac{1}{1-t}$

> Chiamiamo $h(t) = ln\frac{1}{1-t}$

$$= [t^n]\frac{1}{1-t}h(t) = \sum_{k=0}^{n} h_k$$

$[t^n]h(t) = [t^n]ln\frac{1}{1-t}$ usiamo la derivata
$$= \frac{1}{n}[t^{n-1}](1-t)\frac{1}{(1-t)^2} =$$
$$= \frac{1}{n}[t^{n-1}]\frac{1}{1-t} = \frac{1}{n}$$

$\sum h_k = \sum_{k=0}^{n} \frac{1}{k} = H_n = g_n$
$2\sum g_k = 2\sum_{k=0}^{n} H_k$

$$c(t) = 2\sum_{k=0}^{n} H_k$$

Il risultato è un po' diverso da quello trovato ma si può manipolare.

$$c(t) = \underbrace{\frac{2}{1-t}}_{\text{f(t)}}\underbrace{\frac{1}{1-t}ln\frac{1}{1-t}}_{\text{g(t)}}$$

$$g'(t) = \frac{1}{(1-t)^2}ln\frac{1}{1-t} + \frac{1}{(1-t)}(1-t)\frac{1}{(1-t)^2}$$
$$= \frac{1}{(1-t)^2}ln\frac{1}{1-t} + \frac{1}{(1-t)^2}$$
$$= \frac{1}{2}c(t) + \frac{1}{(1-t)^2}$$

$c(t) = 2g'(t) - 2\frac{1}{(1-t)^2}$
$g(t) = G(H_n)$

$[t^n]g'(t) = (n+1)g_{n+1} = (n+1)H_{n+1}$
$[t^n]c(t) = 2[t^n]g'(t) - 2[t^n](1-t)^{-2}$

> $[t^n](1-t)^{-2} = [t^n]t^{-1}\frac{t}{(1-t)^2} = [t^{n+1}]\frac{t}{(1-t)^2} = n+1$

$$
\begin{aligned}
&= 2(n+1)H_{n+1} \\
&= 2(n+1)H_{n+1} - 2(n+1) \\
&= \boxed{2(n+1)(H_{n+1} - 1)}
\end{aligned}
$$

■