$G(\binom{2n}{n}) = \frac{1}{\sqrt{1-4t}}$ $\qquad$ $G(\frac{1}{n+1}\binom{2n}{n}) = \frac{1-\sqrt{1-4t}}{2t}$

$[t^n]\frac{1}{\sqrt{1-4t}} = [t^n](1-4t)^{-1/2} = \binom{-1/2}{n}(-4)^n$

$\Rightarrow \binom{-1/2}{n} = \frac{(-1)^n}{4^n}\binom{2n}{n}$

$$\begin{aligned}
[t^n]\frac{1-\sqrt{1-4t}}{2t} &= \tfrac{1}{2}[t^n]t^{-1}(1-\sqrt{1-4t}) \\
&= \tfrac{1}{2}[t^{n+1}]1 - \tfrac{1}{2}[t^{n+1}](1-4t)^{1/2} \\
&= \tfrac{-1}{2}\binom{1/2}{n+1}(-4)^{n+1}
\end{aligned}$$

> $\binom{1/2}{n} = \frac{(-1)^n}{4^n 2(n+1)}\binom{2n}{n}$

$$\begin{aligned}
&= \frac{-1}{2}\frac{(-1)^n}{4^{n+1}(2(n+1)-1)}\binom{2(n+1)}{n+1}(-4)^{n+1} \\
&= \frac{(-1)^{n+1}}{2*4^{n+1}(2n+1)}\binom{2(n+1)}{n+1}(-4)^{n+1} \\
&= \frac{1}{2(2n+1)}\binom{2(n+1)}{n+1} \\
&= \frac{1}{2(2n+1)}\frac{(2n+1)!}{((n+1)!)^2} \\
&= \frac{1}{2(2n+1)}\frac{(2n+2)(2n+1)(2n)!}{(n+1)n!(n+1)n!} \\
&= \frac{2n!}{(n+1)n!n!} =
\end{aligned}$$

$$\frac{1}{n+1}\binom{2n}{n}$$

Formula dei confronti del quicksort

$c(t) = \frac{2}{(1-t)^2} ln \frac{1}{1-t}$
$c_n = 2(n+1)(H_{n+1}-1)$

è un prodotto di due funzioni

$c(t) = 2\frac{1}{1-t}\frac{1}{1-t}ln(\frac{1}{1-t})$

> Chiamiamo $g(t) = \frac{1}{1-t}ln(\frac{1}{1-t})$

Si usa la convoluzione
$[t^n]\frac{1}{1-t}g(t) = \sum_{k=0}^{n} g_{n-k} = \sum_{k=0}^{n} g_k$

$c(t) = 2\frac{1}{1-t}g(t)$
$[t^n]c(t) = 2[t^n]\frac{1}{1-t}g(t) = 2\sum_{k=0}^{n} g_k$
$[t^n]g(t) = [t^n]\frac{1}{1-t}ln\frac{1}{1-t}$

> Chiamiamo $h(t) = ln\frac{1}{1-t}$

$= [t^n]\frac{1}{1-t}h(t) = \sum_{k=0}^{n} h_k$

$[t^n]h(t) = [t^n]ln\frac{1}{1-t}$ usiamo la derivata
$= \frac{1}{n}[t^{n-1}](1-t)\frac{1}{(1-t)^2} =$
$= \frac{1}{n}[t^{n-1}]\frac{1}{1-t} = \frac{1}{n}$

$\sum h_k = \sum_{k=0}^{n} \frac{1}{k} = H_n = g_n$
$2\sum g_k = 2\sum_{k=0}^{n} H_k$

$$c(t) = 2\sum_{k=0}^{n} H_k$$

Il risultato è un po' diverso da quello trovato ma si può manipolare.

$c(t) = \underbrace{\frac{2}{1-t}}_{\text{f(t)}}\underbrace{\frac{1}{1-t}ln\frac{1}{1-t}}_{\text{g(t)}}$

$g'(t) = \frac{1}{(1-t)^2}ln\frac{1}{1-t} + \frac{1}{(1-t)}(1-t)\frac{1}{(1-t)^2}$
$= \frac{1}{(1-t)^2}ln\frac{1}{1-t} + \frac{1}{(1-t)^2}$
$= \frac{1}{2}c(t) + \frac{1}{(1-t)^2}$

$c(t) = 2g'(t) - 2\frac{1}{(1-t)^2}$
$g(t) = G(H_n)$

$[t^n]g'(t) = (n+1)g_{n+1} = (n+1)H_{n+1}$
$[t^n]c(t) = 2[t^n]g'(t) - 2[t^n](1-t)^{-2}$

> $[t^n](1-t)^{-2} = [t^n]t^{-1}\frac{t}{(1-t)^2} = [t^{n+1}]\frac{t}{(1-t)^2} = n+1$

$$
\begin{aligned}
&= 2(n+1)H_{n+1} \\
&= 2(n+1)H_{n+1} - 2(n+1) \\
&= \boxed{2(n+1)(H_{n+1}-1)} \qquad \blacksquare
\end{aligned}
$$