name,matric BN5205 Finite Element Assignment

Governing equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u) + u \tag{1}$$

Boundary condition:

$$D \frac{\partial u}{\partial n}|_{\Gamma} = g \tag{2}$$

where $g = 0$.

Galerkin weighted residual method:

$$\int_{\Omega} \left(\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) - u \right) \omega \, d\Omega = 0$$

Since $D = 0.1$ is a scalar value:

$$\int_{\Omega} \left(\frac{\partial u}{\partial t} - D \nabla^2 u - u \right) \omega \, d\Omega = 0 \tag{3}$$

Green-Gauss theorem:

$$\int_{\Omega} (f \nabla \cdot (h \nabla g) + \nabla f \cdot (h \nabla g)) \, d\Omega = \int_{\Gamma} f h \frac{\partial g}{\partial n} \, d\Gamma \tag{4}$$

Substitute $f = \omega$, $g = u$, and $h = D$ into Eq. (4)

$$\int_{\Omega} D \nabla^2 u \cdot \omega \, d\Omega = - \int_{\Omega} D \nabla u \cdot \nabla \omega \, d\Omega + \int_{\Gamma} D \frac{\partial u}{\partial n} \omega \, d\Gamma \tag{5}$$

Splitting the integral and using Eq. (5) in Eq. (3) to obtain weak formulation:

$$\int_{\Omega} \frac{\partial u}{\partial t} \omega \, d\Omega + \int_{\Omega} D \nabla u \cdot \nabla \omega \, d\Omega - \int_{\Omega} u \, \omega \, d\Omega = \int_{\Gamma} D \frac{\partial u}{\partial n} \omega \, d\Gamma \tag{6}$$

Let $u = \Psi_n(\xi) u_n$ and $\omega = \Psi_m(\xi)$. Represent Eq. (6) as a system of first order ODEs:

$$\mathbf{M} \frac{\partial u}{\partial t} + \mathbf{K} u = \mathbf{f} \tag{7}$$

where $\mathbf{M}$ is the global mass matrix, its element contributions are given by

$$M^e_{mn} = \int_0^1 \Psi_n(\xi) \Psi_m(\xi) J \, d\xi \tag{8}$$

$\mathbf{K}$ is the global stiffness matrix, its element contributions are given by

$$K^e_{mn} = \int_0^1 \left(D \frac{\partial \Psi_n(\xi)}{\partial \xi_i} \frac{\partial \xi_i}{\partial x_k} \frac{\partial \Psi_m(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_k} - \Psi_n(\xi) \Psi_m(\xi) \right) \, J \, d\boldsymbol{\xi} \tag{9}$$

and $\mathbf{f}$ is the RHS vector, its element contributions are given by

$$f_m = \int_{\Gamma} g \Psi_m(\xi) \, d\Gamma \tag{10}$$