name,matric BN5205 Finite Element Assignment

1 Getting the weak formulation

Governing equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot (D\nabla u) + u \tag{1}$$

Boundary condition:

$$D\frac{\partial u}{\partial n}|_{\Gamma} = g \tag{2}$$

where $g = 0$.

Galerkin weighted residual method:

$$\int_{\Omega} \left(\frac{\partial u}{\partial t} - \nabla \cdot (D\nabla u) - u \right) \omega \, d\Omega = 0$$

Since $D = 0.1$ is a scalar value:

$$\int_{\Omega} \left(\frac{\partial u}{\partial t} - D\nabla^2 u - u \right) \omega \, d\Omega = 0 \tag{3}$$

Green-Gauss theorem:

$$\int_{\Omega} (f\nabla \cdot (h\nabla g) + \nabla f \cdot (h\nabla g)) \, d\Omega = \int_{\Gamma} fh\frac{\partial g}{\partial n} \, d\Gamma \tag{4}$$

Substitute $f = \omega$, $g = u$, and $h = D$ into Eq. (4)

$$\int_{\Omega} D\nabla^2 u \cdot \omega \, d\Omega = -\int_{\Omega} D\nabla u \cdot \nabla\omega \, d\Omega + \int_{\Gamma} D\frac{\partial u}{\partial n}\omega \, d\Gamma \tag{5}$$

Splitting the integral and using Eq. (5) in Eq. (3) to obtain weak formulation:

$$\int_{\Omega} \frac{\partial u}{\partial t} \, \omega \, d\Omega + \int_{\Omega} D\nabla u \cdot \nabla\omega \, d\Omega - \int_{\Omega} u \, \omega \, d\Omega = \int_{\Gamma} D\frac{\partial u}{\partial n}\omega \, d\Gamma \tag{6}$$

Let $u = \Psi_n(\xi)u_n$ and $\omega = \Psi_m(\xi)$. Represent Eq. (6) as a system of first order ODEs:

$$\mathbf{M}\frac{\partial u}{\partial t} + \mathbf{K}u = \mathbf{f} \tag{7}$$

where $\mathbf{M}$ is the global mass matrix, its element contributions are given by

$$M^e_{mn} = \int_0^1 \Psi_n(\xi)\Psi_m(\xi)J \, d\xi \tag{8}$$

$\mathbf{K}$ is the global stiffness matrix, its element contributions are given by

$$K^e_{mn} = \int_0^1 \left(D\frac{\partial \Psi_n(\xi)}{\partial \xi_i}\frac{\partial \xi_i}{\partial x_k}\frac{\partial \Psi_m(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_k} - \Psi_n(\xi)\Psi_m(\xi) \right) J \, d\boldsymbol{\xi} \tag{9}$$

and $\mathbf{f}$ is the RHS vector, its element contributions are given by

$$f_m = \int_{\Gamma} g\Psi_m(\xi) \, d\Gamma \tag{10}$$

where $g = 0$.

2 Choice of time and space step

The initial bacterial concentration creates a sharp spike in the middle of the domain. Therefore, fine mesh is needed to accurately model approximate the sharp change in density surrounding the spike. A simple mesh convergence analysis was conducted (Fig. 1). The finest mesh tested was 80×80 due to memory limitation.

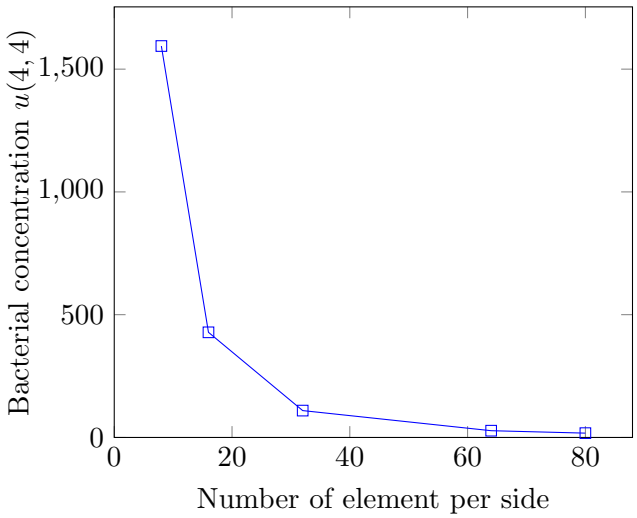

Figure 1: Mesh convergence analysis

As the difference between the results obtained 80×80 and 64×64 mesh is around 50%, a smaller space step is necessary to accurately model bacteria growth.

As the Crank-Nicolson-Galerkin scheme was used, the time step size has no effect on the stability of the simulation. However, the time step size was set to be $\Delta t = 0.05$ in a 64×64 mesh so that the Courant condition

$$D\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \tag{11}$$

could be satisfied to ensure numerical accuracy.