Chapter 0

Mathematical preliminaries

0.1 Even and odd functions

Definition 0.1 *The function* $f : \mathbb{R} \to \mathbb{R}$ *is an even function if and only if*

$$f(-x) = f(x), \quad \forall x \in \mathbb{R}. \tag{1}$$

The graph of any even function is symmetric with respect to the y-axis.

Lemma 0.1 *Let* $f(x)$ *be an integrable even function. Then,*

$$\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(x)dx, \quad \forall a \in \mathbb{R}, \tag{2}$$

and therefore

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx, \quad \forall a \in \mathbb{R}. \tag{3}$$

Moreover, if $\int_0^\infty f(x)dx$ *exists, then*

$$\int_{-\infty}^{0} f(x)dx = \int_{0}^{\infty} f(x)dx, \tag{4}$$

and

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx. \tag{5}$$

Definition 0.2 *The function* $f : \mathbb{R} \to \mathbb{R}$ *is an odd function if and only if*

$$f(-x) = -f(x), \quad \forall x \in \mathbb{R}. \tag{6}$$

If we let $x = 0$ in (6), we find that $f(0) = 0$ for any odd function $f(x)$. Also, the graph of any odd function is symmetric with respect to the point $(0, 0)$.

Lemma 0.2 *Let* $f(x)$ *be an integrable odd function. Then,*

$$\int_{-a}^{a} f(x)dx = 0, \quad \forall a \in \mathbb{R}. \tag{7}$$

Moreover, if $\int_0^\infty f(x)dx$ *exists, then*

$$\int_{-\infty}^{\infty} f(x)dx = 0. \tag{8}$$

0.2 Useful sums with interesting proofs

The following sums occur frequently when estimating operation counts of numerical algorithms:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \tag{9}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \tag{10}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{11}$$

0.3 Sequences satisfying linear recursions

Definition 0.3 *A sequence $(x_n)_{n\geq 0}$ satisfies a linear recursion of order k if and only if there exist constants a_i, $i = 0 : k$ with $a_k \neq 0$, such that*

$$\sum_{i=0}^{k} a_i x_{n+i} = 0, \quad \forall n \geq 0. \tag{12}$$

The recursion (12) is called a linear recursion because of the following linearity properties:
(i) If the sequence $(x_n)_{n\geq 0}$ satisfies the linear recursion (12), then the sequence $(z_n)_{n\geq 0}$ given by

$$z_n = Cx_n, \quad \forall n \geq 0, \tag{13}$$

where C is an arbitrary constant, also satisfies the linear recursion (12).
(ii) If the sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ satisfies the linear recursion (12), then the sequence $(z_n)_{n\geq 0}$ given by

$$z_n = x_n + y_n, \quad \forall n \geq 0, \tag{14}$$

also satisfies the linear recursion (12).

Definition 0.4 *The characteristic polynomial $P(z)$ corresponding to the linear recursion $\sum_{i=0}^{k} a_i x_{n+i} = 0$, for all $n \geq 0$, is defined as*

$$P(z) = \sum_{i=0}^{k} a_i z^i. \tag{15}$$

$P(z)$ is a polynomial of degree k, i.e., $\deg(P(z)) = k$. If $P(z)$ has p different roots, λ_j, $j = 1 : p$, with $p \leq k$, and if $m(\lambda_j)$ denotes the multiplicity of the root λ_j, then $\sum_{j=1}^{p} m(\lambda_j) = k$ where λ_j can be a complex number.

Theorem 0.1 *Let $(x_n)_{x\geq 0}$ be a sequence satisfying the linear recursion*

$$\sum_{i=0}^{k} a_i x_{n+i} = 0, \quad \forall n \geq 0, \tag{16}$$

with $a_k \neq 0$, *and let* $P(z) = \sum_{i=0}^{k-1} a_i z^i$ *be the characteristic polynomial associated with recursion* (16). *Let* λ_j, $j = 1 : p$, *where* $p \leq k$, *be the roots of* $P(z)$, *and let* $m(\lambda_j)$ *be the multiplicity of* λ_j. *The general form of the sequence* $(x_n)_{n \geq 0}$ *satisfying the linear recursion* (16) *is*

$$x_n = \sum_{j=1}^{p} \left(\sum_{i=0}^{m(\lambda_j)-1} C_{i,j} n^i \right) \lambda_j^n, \quad \forall n \geq 0, \tag{17}$$

where $C_{i,j}$ *are constant numbers.*

0.4 The "Big O" and "little o" notations

Definition 0.5 *Let* $f, g : \mathbb{R} \to \mathbb{R}$. *We write that* $f(x) = O(g(x)$, *as* $x \to \infty$, *if and only if there exist constants* $C > 0$ *and* $M > 0$ *such that* $\left| \frac{f(x)}{g(x)} \leq C \right|$, *for any* $x \geq M$. *This can be written equivalently as*

$$f(x) = O(g(x)), \quad \textit{as} \quad x \to \infty, \quad \textit{iff} \quad \limsup_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| < \infty. \tag{18}$$

The "little o" notation refers to functions whose ratios tend to 0 at certain points, and can be defined for $x \to \infty$, $x \to a$, and $x \to -\infty$ as follows:

Definition 0.6 *Let* $f, g : \mathbb{R} \to \mathbb{R}$. *Then*

$$f(x) = o(g(x)), \quad \textit{as} \quad x \to \infty, \quad \textit{iff} \quad \lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| = 0; \tag{19}$$

$$f(x) = o(g(x)), \quad \textit{as} \quad x \to -\infty, \quad \textit{iff} \quad \lim_{x \to -\infty} \left| \frac{f(x)}{g(x)} \right| = 0; \tag{20}$$

$$f(x) = o(g(x)), \quad \textit{as} \quad x \to a, \quad \textit{iff} \quad \lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = 0. \tag{21}$$

0.5 Exercises

0.5.1 Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function.

Part (i)

Show that $xf(x)$ is an even function and $x^2 f(x)$ is an odd function.

Answer Let $g(x) = xf(x)$, using definition (6), we have

$$\begin{aligned} g(-x) &= -xf(-x) \\ &= -x(-f(x)) \\ &= xf(x) \\ &= g(x). \end{aligned}$$

Therefore, $xf(x)$ is an even function.

Let $g(x) = x^2 f(x)$, using definition (6), we have

$$\begin{aligned} g(-x) &= (-x)^2 f(-x) \\ &= x^2(-f(x)) \\ &= -x^2 f(x) \\ &= -g(x) \end{aligned}$$

Therefore, $x^2 f(x)$ is an odd function.

Part (ii)

Show that the function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function and that the function $g_2 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function.

Answer

$$\begin{aligned} g_1(-x) &= f((-x)^2) \\ &= f(x^2) \\ &= g_1(x) \end{aligned}$$

Therefore g_1 is an even function.
Let $y = x^3$:

$$\begin{aligned} g_2(-x) &= f((-x)^3) \\ &= f(-x^3) \\ &= f(-y) \\ &= -f(y) \\ &= -f(x^3) \\ &= -g_2(x) \end{aligned}$$

Therefore g_2 is an odd function.

Part (iii)

Let i be even, j be odd, and $y = x^j$:

$$\begin{aligned} h(-x) &= (-x)^i f((-x)^j) \\ &= x^i f(-x^j) \\ &= x^i f(-y) \\ &= -x^i f(y) \\ &= -x^i f(x^j) \\ &= -h(x) \end{aligned}$$

Let i be odd, j be even:

$$\begin{aligned} h(-x) &= (-x)^i f((-x)^j) \\ &= -x^i f(x^j) \\ &= -h(x) \end{aligned}$$

When $i + j$ is odd, $h(x)$ is an odd function.

0.5.2 Question 2

Let $S(n,2)=\sum_{k=1}^{n}k^2$ and $S(n,3)=\sum_{k=1}^{n}k^3$.

Part (i)

Let $T(n,2,x)=\sum_{k=1}^{n}k^2x^k$. Use formulas,

$$T(n,2,x)=x\frac{d}{dx}(T(n,1,x)),$$

and

$$T(n,1,x)=\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2},$$

to show that

$$T(n,2,x)=\frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3}.$$

Answer Using quotient rule,

$$\begin{aligned}
T(n,2,x)&=x\frac{d}{dx}(T(n,1,x))\\
&=x\frac{d}{dx}\left(\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}\right)\\
&=x\left(\frac{\begin{array}{l}\frac{d}{dx}(x-(n+1)x^{n+1}+nx^{n+2})(1-x)^2\\ \qquad-(x-(n+1)x^{n+1}+nx^{n+2})\frac{d}{dx}(1-x)^2\end{array}}{(1-x)^4}\right)\\
&=x\left(\frac{\begin{array}{l}(1-(n+1)^2x^n+(n^2+2n)x^{n+1})(1-x)^2\\ \qquad-(x-(n+1)x^{n+1}+nx^{n+2})(-2(1-x))\end{array}}{(1-x)^4}\right)\\
&=x\left(\frac{\begin{array}{l}(1-(n+1)^2x^n+(n^2+2n)x^{n+1})(1-x)^2\\ \qquad\qquad+2(x-(n+1)x^{n+1}+nx^{n+2})(1-x)\end{array}}{(1-x)^4}\right)\\
&=x\left(\frac{(1-(n+1)^2x^n+(n^2+2n)x^{n+1})(1-x)+2(x-(n+1)x^{n+1}+nx^{n+2})}{(1-x)^3}\right)\\
&=x\left(\frac{\begin{array}{l}(1-(n+1)^2x^n+(n^2+2n)x^{n+1})\\ \qquad-(x-(n+1)^2x^{n+1}+(n^2+2n)x^{n+2})+\cdots\end{array}}{(1-x)^3}\right)\\
&=x\left(\frac{1-x-(n+1)^2x^n+(2n^2+4n+1)x^{n+1}-(n^2+2n)x^{n+2}+\cdots}{(1-x)^3}\right)\\
&=x\left(\frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}\right)\\
&=\frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3}
\end{aligned}$$

Part (ii)

Note that $S(n,2) = T(n,2,1)$. Use l'Hôpitals's rule to evaluate $T(n,2,1)$, and conclude that $S(n,2) = \frac{n(n+1)(2n+1)}{6}$.

Answer

$$\begin{aligned}
\lim_{x\to 1} T(n,2,x) &= \lim_{x\to 1} \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \\
&= \lim_{x\to 1} \frac{\begin{aligned}(n-1)n(n+1)^3x^{n-2} - n(n+1)(n+2)(2n^2+2n-1)x^{n-1} \\ + (n+1)(n+2)(n+3)n^2x^n\end{aligned}}{6} \\
&= \frac{\begin{aligned}(n-1)n(n+1)^3 - n(n+1)(n+2)(2n^2+2n-1) \\ + (n+1)(n+2)(n+3)n^2\end{aligned}}{6} \\
&= \frac{n(n+1)(2n+1)}{6} \\
S(n,2) &= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Part (iii)

Compute $T(n,3,x) = \sum_{k=1}^{n} k^3 x^k$ using the formula

$$T(n,3,x) = x\frac{d}{dx}(T(n,2,x)).$$

Answer

$$\begin{aligned}
T(n,3,x) &= x\frac{d}{dx}(T(n,2,x)) \\
&= x\frac{d}{dx}\frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \\
&= \frac{\begin{aligned}x((n^3(x-1)^3 - 3n^2(x-1)^2 + 3n(x^2-1) \\ - x(x+4) - 1)x^n + (x+4)x + 1)\end{aligned}}{(1-x)^4}
\end{aligned}$$

Part (iv)

Note that $S(n,3) = T(n,3,1)$. Use l'Hôpital's rule to evaluate $T(n,3,1)$, and conclude that $S(n,3) = \left(\frac{n(n+1)}{2}\right)^2$.

$$\lim_{x\to 1} T(n,3,x) = \lim_{x\to 1} \frac{\begin{aligned}x((n^3(x-1)^3 - 3n^2(x-1)^2 + 3n(x^2-1) \\ - x(x+4) - 1)x^n + (x+4)x + 1)\end{aligned}}{(1-x)^4}$$

0.5.3 Question 3

Compute $S(n,4) = \sum_{k=1}^{n} k^4$ using the recursion formula for $i = 4$, the fact that $S(n,0) = n$, and formulas for $S(n,1)$, $S(n,2)$, and $S(n,3)$.

Answer

$$\begin{aligned}
S(n,4) &= \frac{1}{5}\left((n+1)^5 - 1 - \sum_{j=0}^{3}\binom{5}{j} xS(n,j)\right) \\
&= \frac{1}{5}\left((n+1)^5 - 1 - S(n,0) - 5S(n,1) - 10S(n,2) - 10S(n,3)\right) \\
&= \frac{1}{5}\left((n+1)^5 - 1 - n - 5\frac{n(n+1)}{2} - 10\frac{n(n+1)(2n+1)}{6} - 10\left(\frac{n(n+1)}{2}\right)^2\right) \\
&= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}
\end{aligned}$$

0.5.4 Question 4

It is easy to see that the sequence $(x_n)_{n\geq 1}$ given by $x_n = \sum_{k=1}^{n} k^2$ satisfies the recursion

$$x_{n+1} = x_n + (n+1)^2, \quad \forall n \geq 1, \tag{22}$$

with $x_1 = 1$.

Part (i)

By substituting $n+1$ for n in (22), obtain that

$$x_{n+2} = x_{n+1} + (n+2)^2. \tag{23}$$

Subtract (22) from (23) to find that

$$x_{n+2} = 2x_{n+1} - x_n + 2n + 3, \quad \forall n \geq 1, \tag{24}$$

with $x_1 = 1$ and $x_2 = 5$.

Answer

$$\begin{aligned}
x_{(n+1)+1} &= x_{n+1}((n+1)+1)^2 \\
x_{n+2} &= x_{n+1}(n+2)^2
\end{aligned}$$

Subtract (22) from (23):

$$\begin{aligned}
x_{n+2} - x_{n+1} &= x_{n+1} + (n+2)^2 - x_n - (n+1)^2 \\
x_{n+2} &= 2x_{n+1} - x_n + n^2 + 4n + 4 - n^2 - 2n - 1 \\
&= 2x_{n+1} - x_n + 2n + 3
\end{aligned}$$

Part (ii)

Similarly, substitute $n+1$ for n in (24) and obtain that

$$x_{n+3} = 2x_{n+2} - x_{n+1} + 2(n+1) + 3. \tag{25}$$

Subtract (24) from (25) to find that

$$x_{n+3} = 3x_{n+2} - 3x_{n+1} + x_n + 2, \quad \forall n \geq 1, \tag{26}$$

with $x_1 = 1$, $x_2 = 5$, and $x_3 = 14$.

Answer

$$\begin{aligned} x_{(n+1)+2} &= 2x_{(n+1)+1} - x_{n+1} + 2(n+1) + 3 \\ x_{n+3} &= 2x_{n+2} - x_{n+1} + 2(n+1) + 3 \end{aligned}$$

Subtract (24) from (25)

$$\begin{aligned} x_{n+3} - x_{n+2} &= 2x_{n+2} - x_{n+1} + 2(n+1) + 3 - 2x_{n+1} + x_n - 2n - 3 \\ x_{n+3} &= 3x_{n+2} - 3x_{n+1} + x_n + 2 \end{aligned}$$

Part (iii)

Use a similar method to prove that the sequence $(x_n)_{n\geq 0}$ satisfies the linear recursion

$$x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n = 0, \quad \forall n \geq 1. \tag{27}$$

The characteristic polynomial associated to the recursion (27) is

$$P(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 = (z-1)^4.$$

Use the fact that $x_1 = 1$, $x_2 = 5$, $x_3 = 14$, and $x_4 = 30$ to show that

$$x_n = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1,$$

and conclude that

$$S(n, 2) = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1.$$

Answer Substitute $n+1$ for n in (26) to obtain

$$x_{n+4} = 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2. \tag{28}$$

Subtract (26) from (28) to obtain that

$$\begin{aligned} x_{n+4} - x_{n+3} &= 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2 \\ &\quad - (3x_{n+2} - 3x_{n+1} + x_n + 2) \\ x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} - x_n &= 0 \end{aligned}$$

The characteristic polynomial has root $\lambda = 1$ with multiplicity 4. The linear recursion can be expressed as

$$\begin{aligned} x_n &= \sum_{j=1}^{p} \left(\sum_{i=0}^{3} C_{i,j} n^i \right) \lambda_j^n \\ &= \sum_{j=1}^{p} \left(C_{0,j} + C_{1,j} n + C_{2,j} n^2 + C_{3,j} n^3 \right) \lambda_j^n \\ &= C_1 + C_2 n + C_3 n^2 + C_4 n^3 \end{aligned}$$

Since $x_1 = 1$, $x_2 = 5$, $x_3 = 14$, and $x_4 = 30$, C_1, C_2, C_3, and C_4 must solve the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \\ 30 \end{pmatrix}.$$

We obtain that $C_1 = 0$, $C_2 = \frac{1}{6}$, $C_3 = \frac{1}{2}$, and $C_4 = \frac{1}{3}$ and therefore

$$x_n = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} = \frac{n(n+1)(n+2)}{6}$$

0.5.5 Question 5

Find the general form of the sequence $(x_n)_{n \geq 0}$ satisfying the linear recursion

$$x_{n+3} = 2x_{n+1} + x_n, \quad \forall n \geq 0,$$

with $x_0 = 1$, $x_1 = -1$, and $x_2 = 1$.

Answer Rewrite the recursion in the form (12) as

$$x_{n+3} - 2x_{n+1} - x_n = 0, \quad \forall n \geq 0.$$

The characteristic polynomial associated to the linear recursion is

$$\begin{aligned} P(z) &= z^3 - 2z - 1 \\ &= (z+1)(z^2 - z - 1) \end{aligned}$$

and the roots of $P(z)$ are

$$\lambda_1 = -1, \quad \lambda_2 = \frac{1+\sqrt{5}}{2}, \quad \lambda_3 = \frac{1-\sqrt{5}}{2}.$$

From Theorem 0.1, we find that

$$x_n = C_1 \lambda_1^n + C_2 \lambda_2^n + C_3 \lambda_3^n, \quad \forall n \geq 0.$$

Given $x_0 = 1$, $x_1 = -1$, and $x_2 = 1$, we obtain the linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

By solving the linear system, we find that $C_1 = 1$, $C_2 = 0$, and $C_3 = 0$. The general formula for is

$$x_n = (-1)^n, \quad \forall n \geq 0.$$

0.5.6 Question 6

The sequence $(x_n)_{n\geq 0}$ satisfies the recursion

$$x_{n+1} = 3x_n + 2, \quad \forall n \geq 0,$$

with $x_0 = 1$.

Part (i)

Show that the sequence $(x_n)_{n\geq 0}$ satisfies the linear recursion

$$x_{n+2} = 4x_{n+1} - 3x_n, \quad \forall n \geq 0,$$

with $x_0 = 1$ and $x_1 = 5$.

Answer Substitute $n+1$ for n to obtain

$$x_{n+2} = 3x_{n+1} + 2$$

Subtract the original recursion to get

$$\begin{aligned} x_{n+2} - x_{n+1} &= 3x_{n+1} + 2 - 3x_n - 2 \\ x_{n+2} &= 4x_{n+1} - 3x_n \end{aligned}$$

Part (ii)

Find the general formula for x_n, $n \geq 0$.

Answer The characteristic polynomial has the form

$$P(z) = z^2 - 4z + 3 = (z-1)(z-3)$$

which has roots $\lambda_1 = 1$ and $\lambda_2 = 3$. We obtain the linear system

$$\begin{cases} C_1 + C_2 = 1; \\ C_1\lambda_1 + C_2\lambda_2 = 5. \end{cases}$$

The solution to the linear system is $C_1 = -1$ and $C_2 = 2$. Therefore, the general form is

$$x_n = 2(3)^n - 1$$

0.5.7 Question 7

The sequence $(x_n)_{n\geq 0}$ satisfies the recursion

$$x_{n+1} = 3x_n + n + 2, \quad \forall n \geq 0,$$

with $x_0 = 1$.

Part (i)

Show that the sequence $(x_n)_{n\geq 0}$ satisfies the linear recursion

$$x_{n+3} = 5x_{n+2} - 7x_{n+1} + 3x_n, \quad \forall n \geq 0,$$

with $x_0 = 1$, $x_1 = 5$, and $x_2 = 18$.

Answer Substitute $n+1$ for n, we obtain

$$x_{n+2} = 3x_{n+1} + n + 3$$

Subtract

$$x_{n+2} = 4x_{n+1} - 3x_n + 1$$

Substitute $n+1$ for n, we obtain

$$x_{n+3} = 4x_{n+2} - 3x_{n+1} + 1$$

Subtract

$$x_{n+3} = 5x_{n+2} - 7x_{n+1} + 3x_n$$

Part (ii)

Find the general formula for x_n, $n \geq 0$.

Answer The characteristic polynomial is given by

$$P(z) = z^3 - 5z^2 + 7z - 3 = (z-1)^2(z-3),$$

with roots $\lambda_1 = 1$ and $\lambda_2 = 3$. The general form is

$$\begin{aligned} x_n &= \sum_{j=1}^{2} \left(\sum_{i=0}^{m(\lambda_j)-1} C_{i,j} n^i \right) \lambda_j^n \\ &= \lambda_1^n \sum_{i=0}^{1} C_{i,1} n^i + \lambda_2^n C_2 \\ &= \lambda_1^n C_{0,1} + \lambda_1^n C_{1,1} n + \lambda_2^n C_2 \\ &= C_1 + C_2 n + C_3 3^n \end{aligned}$$

Since $x_0 = 1$, $x_1 = 5$, and $x_2 = 18$, we find $C_1 = -\frac{1}{2}$, $C_2 = -\frac{5}{4}$, and $C_3 = \frac{9}{4}$. We conclude that

$$x_n = \frac{3^{n+2} - 2n - 5}{4}$$

0.5.8 Question 8

Let $P(z) = \sum_{i=0}^{k} a_i z^i$ be the characteristic polynomial corresponding to the linear recursion

$$\sum_{i=0}^{k} a_i x_{n+i} = 0, \quad \forall n \geq 0.$$

Assume that λ is a root of multiplicity 2 of $P(z)$. Show that the sequence $(y_n)_{n\geq 0}$ given by

$$y_n = Cn\lambda^n, \quad n \geq 0,$$

where C is an arbitrary constant, satisfies the recursion.

Answer

$$\begin{aligned}
\sum_{i=0}^{k} a_i y_{n+i} &= \sum_{i=0}^{k} a_i C(n+i)\lambda^{n+i} \\
&= Cn\sum_{i=0}^{k} a_i\lambda^{n+i} + C\sum_{i=0}^{k} a_i i\lambda^{n+i} \\
&= Cn\lambda^n\sum_{i=0}^{k} a_i\lambda^{i} + C\lambda^{n+1}\sum_{i=0}^{k} i a_i\lambda^{i-1} \\
&= Cn\lambda^n P(\lambda) + C\lambda^{n+1}P'(\lambda) \\
&= 0.
\end{aligned}$$

0.5.9 Question 9

Let $n > 0$. Show that

$$\begin{aligned}
O(x^n) + O(x^n) = O(x^n), \quad \text{as} \quad x \to 0; \\
o(x^n) + o(x^n) = o(x^n), \quad \text{as} \quad x \to 0.
\end{aligned}$$

Answer Let $f(x) = O(x^n)$ and $g(x) = O(x^n)$, then

$$\limsup_{x\to 0}\left|\frac{f(x)}{x^n}\right| < \infty \quad \text{and} \quad \limsup_{x\to 0}\left|\frac{g(x)}{x^n}\right| < \infty.$$

We see that

$$\limsup_{x\to 0}\left|\frac{f(x)+g(x)}{x^n}\right| \le \limsup_{x\to 0}\left|\frac{f(x)}{x^n}\right| + \limsup_{x\to 0}\left|\frac{g(x)}{x^n}\right| < \infty,$$

and therefore $O(x^n) + O(x^n) = O(x^n)$.
Let $f(x) = o(x^n)$ and $g(x) = o(x^n)$, then

$$\lim_{x\to 0}\left|\frac{f(x)}{x^n}\right| = 0 \quad \text{and} \quad \lim_{x\to 0}\left|\frac{g(x)}{x^n}\right| = 0.$$

We see that

$$\lim_{x\to 0}\left|\frac{f(x)+g(x)}{x^n}\right| \le \lim_{x\to 0}\left|\frac{f(x)}{x^n}\right| + \lim_{x\to 0}\left|\frac{g(x)}{x^n}\right| = 0,$$

and therefore $o(x^n) + o(x^n) = o(x^n)$.

0.5.10 Question 10

Prove that

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= O(n^3), \quad \text{as} \quad n \to \infty; \\
\sum_{k=1}^{n} k^2 &= \frac{n^3}{3} + O(n^2), \quad \text{as} \quad n \to \infty,
\end{aligned}$$

i.e., show that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n k^2}{n^3} < \infty$$

and that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n k^2 - \frac{n^3}{3}}{n^2} < \infty.$$

Similarly, prove that

$$\sum_{k=1}^n k^3 = O(n^4), \quad \text{as} \quad n \to \infty;$$

$$\sum_{k=1}^n k^3 = \frac{n^4}{4} + O(n^3), \quad \text{as} \quad n \to \infty,$$

Answer Using (10)

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\sum_{k=1}^n k^2}{n^3} &= \limsup_{n\to\infty} \frac{\frac{n(n+1)(2n+1)}{6}}{n^3} \\
&= \limsup_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} \\
&= \frac{1}{3} < \infty \\
\limsup_{n\to\infty} \frac{\sum_{k=1}^n k^2 - \frac{n^3}{3}}{n^2} &= \limsup_{n\to\infty} \frac{\frac{2n^3+3n^2+n}{6} - \frac{n^3}{3}}{n^2} \\
&= \limsup_{n\to\infty} \frac{3n^2+n}{6n^2} \\
&= \frac{1}{2} \\
&< \infty
\end{aligned}$$

Using (11)

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\sum_{k=1}^n k^3}{n^4} &= \limsup_{n\to\infty} \frac{\left(\frac{n(n+1)}{2}\right)^2}{n^4} \\
&= \limsup_{n\to\infty} \frac{n^2(n^2+2n+1)}{4n^4} \\
&= \frac{1}{4} \\
&< \infty \\
\limsup_{n\to\infty} \frac{\sum_{k=1}^n k^3 - \frac{n^4}{4}}{n^3} &= \limsup_{n\to\infty} \frac{\frac{(n^4+2n^3+n^2)}{4} - \frac{n^4}{4}}{n^3} \\
&= \limsup_{n\to\infty} \frac{2n^3+n^2}{4n^3} \\
&= \frac{1}{2} \\
&< \infty
\end{aligned}$$

Chapter 1

Calculus review. Plain vanilla options.

1.1 Brief review of differentiation

The function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at the point $x \in \mathbb{R}$ if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, in which case the derivative $f'(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{1.1}$$

Theorem 1.1 (Product Rule.) *The product $f(x)g(x)$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable, and*

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x). \tag{1.2}$$

Theorem 1.2 (Quotient Rule.) *The quotient $\frac{f(x)}{g(x)}$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable at every point x where the function $\frac{f(x)}{g(x)}$ is well defined, and*

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}. \tag{1.3}$$

Theorem 1.3 (Chain Rule.) *The composite function $(g \circ f)(x) = g(f(x))$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable at every point x where $g(f(x))$ is well defined, and*

$$(g(f(x)))' = g'(f(x))f'(x). \tag{1.4}$$

The Chain Rule formula (1.4) can also be written as

$$\frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx},$$

where $u = f(x)$ is a function of x and $g = g(u) = g(f(x))$.

Chain Rule is often used for power functions, exponential functions, and logarithmic function:

$$\frac{d}{dx}((f(x))^n) = n(f(x))^{n-1}f'(x); \tag{1.5}$$

$$\frac{d}{dx}(e^{f(x)}) = e^{f(x)}f'(x); \tag{1.6}$$

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}. \tag{1.7}$$

Lemma 1.1 *Let $f : [a, b] \to [c, d]$ be a differentiable function, and assume that $f(x)$ has an inverse function denoted by $f^{-1}(x)$ with $f^{-1} : [c, d] \to [a, b]$. The function $f^{-1}(x)$ is differentiable at every point $x \in [c, d]$ where $f'(f^{-1}(x)) \neq 0$ and*

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}. \tag{1.8}$$

1.2 Brief review of integration

Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Recall that $F(x)$ is the antiderivative of $f(x)$ iff $F'(x) = f$, i.e.,

$$F(x) = \int f(x)dx \quad \Longleftrightarrow \quad F'(x) = f(x).$$

Theorem 1.4 (Fundamental Theorem of Calculus.) *Let $f(x)$ be a continuous function on the interval $[a, b]$, and let $F(x)$ be the antiderivative of $f(x)$. Then*

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a).$$

Integration by parts is the counterpart for integration of the product rule.

Theorem 1.5 (Integration by parts.) *Let $f(x)$ and $g(x)$ be continuous function. Then*

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx, \tag{1.9}$$

where $F(x) = \int f(x)dx$ is the antiderivative of $f(x)$. For definite integrals,

$$\int_a^b f(x)g(x)dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)dx. \tag{1.10}$$

Integration by substitution if the counterpart for integration of the chain rule.

Theorem 1.6 (Integration by substitution) *Let $f(x)$ be an integrable function. Assume that $g(u)$ is an invertible and continuously differentiable function. The substitution $x = g(u)$ changes the integration variable from x to u as follows:*

$$\int f(x)dx = \int f(g(u))g'(u)du. \tag{1.11}$$

For definite integrals,

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)du. \tag{1.12}$$

1.3 Differentiating definite integrals

If a definite integral has functions as limits of integration, e.g.,

$$\int_{a(t)}^{b(t)} f(x)dx,$$

or if the function to be integrated is a function of the integrating variable and of another variable, e.g.,

$$\int_{a}^{b} f(x,t)dx$$

then the result of the integration is a function (of the variable t in both cases above).

Lemma 1.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then,*

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x)dx\right) = f(b(t))b'(t) - f(a(t))a'(t), \tag{1.13}$$

where $a(t)$ and $b(t)$ are differentiable functions.

Lemma 1.3 *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the partial derivative $\frac{\partial f}{\partial t}(x,t)$ exists and is continuous in both variables x and t.*

$$\frac{d}{dt}\left(\int_{a}^{b} f(x,t)dx\right) = \int_{a}^{b} \frac{\partial f}{\partial t}(x,t)dx. \tag{1.14}$$

Lemma 1.4 *Let $f(x,t)$ be a continuous function such that the partial derivative $\frac{\partial f}{\partial t}(x,t)$ exists and is continuous. Then,*

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)dx\right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t) + f(b(t),t)b'(t) - f(a(t),t)a'(t).$$

Note that Lemma 1.2 and Lemma 1.3 are special cases of Lemma 1.4.

1.4 Limits

Definition 1.1 *Let $g : \mathbb{R} \to \mathbb{R}$. The limit of $g(x)$ as $x \to x_0$ exists and is finite and equal to l iff for any $\epsilon > 0$ there exists $\delta > 0$ such that $|g(x) - l| < \epsilon$ for all $x \in (x_0 - \delta, x_0 + \delta)$, i.e.,*

$$\lim_{x \to x_0} g(x) = l, \quad \textit{iff} \quad \forall \epsilon > 0 \exists \delta > 0 \quad \textit{such that} \quad |g(x) - l| < \epsilon, \ \forall |x - x_0| < \delta.$$

Similarly,

$$\lim_{x \to x_0} g(x) = \infty, \quad \textit{iff} \quad \forall C > 0 \exists \delta > 0 \quad \textit{such that} \quad g(x) > C, \ \forall |x - x_0| < \delta.$$

$$\lim_{x \to x_0} g(x) = -\infty, \quad \textit{iff} \quad \forall C < 0 \exists \delta > 0 \quad \textit{such that} \quad g(x) < C, \ \forall |x - x_0| < \delta.$$

Theorem 1.7 *If $P(x)$ and $Q(x)$ are polynomials and $c > 1$ is a fixed constant, then*

$$\lim_{x\to\infty} \frac{P(x)}{c^x} = 0, \quad \forall c > 1; \tag{1.15}$$

$$\lim_{x\to\infty} \frac{\ln|Q(x)|}{P(x)} = 0. \tag{1.16}$$

Lemma 1.5 *Let $c > 0$ be a positive constant. Then,*

$$\lim_{x\to\infty} x^{\frac{1}{x}} = 1; \tag{1.17}$$

$$\lim_{x\to\infty} c^{\frac{1}{x}} = 1; \tag{1.18}$$

$$\lim_{x\searrow 0} x^x = 1, \tag{1.19}$$

$$\tag{1.20}$$

where the notation $x \searrow 0$ means that x goes to 0 while always being positive, i.e., $x \to 0$ with $x > 0$.

Lemma 1.6 *If k is a positive integer number, and if $c > 0$ is a positive fixed constant, then*

$$\lim_{k\to\infty} k^{\frac{1}{k}} = 1; \tag{1.21}$$

$$\lim_{k\to\infty} c^{\frac{1}{k}} = 1; \tag{1.22}$$

$$\lim_{k\to\infty} \frac{c^k}{k!} = 0, \tag{1.23}$$

$$\tag{1.24}$$

where $k! = 1 \cdot 2 \cdot ... \cdot k$.

1.5 L'Hôpital's rule and connections to Taylor expansions

Theorem 1.8 (L'Hôpital's Rule) *Let x_0 be a real number; allow $x_0 = \infty$ and $x_0 = -\infty$ as well. Let $f(x)$ and $g(x)$ be two differentiable functions.*

(i) Assume that $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = 0$. If $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists, and if there exists an interval (a, b) around x_0 such that $g'(x) \neq 0$ for all $x \in (a, b) \setminus 0$, then the limit $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ also exists and

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

(ii) Assume that $\lim_{x\to x_0} f(x)$ is either $-\infty$ or ∞, and that $\lim_{x\to x_0} g(x)$ is either $-\infty$ or ∞. If the limit $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists, and if there exists an interval (a, b) around x_0 such that $g'(x) \neq 0$ for all $x \in (a, b) \setminus 0$, then the limit $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ also exists and

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

1.6 Multivariable functions

Scalar Valued Functions Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables denoted by $x_1, x_2, ..., x_n$, and let $x = (x_1, x_2, ..., x_n)$.

Definition 1.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$. The partial derivative of the function $f(x)$ with respect to the variable x_i is denoted by $\frac{\partial f}{\partial x_i}(x)$ and is defined as*

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h\to 0} \frac{f(x_1, ..., x_{i-1}, x_i + h, x_{i+1}, ..., x_n) - f(x_1, x_2, ..., x_n)}{h}, \tag{1.25}$$

if the limit from (1.25) *exists and is finite.*

A compact formula for (1.25) can be given as follows: Let e_i be the vector with all entries equal to 0 with the exception of the i-th entry, which is equal for 1, i.e., $e_i(j) = 0$, for $j \neq i$, $1 \leq j \leq n$, and $e_i(j) = 1$. Then,

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h\to 0} \frac{f(x + he_i) - f(x)}{h}.$$

Theorem 1.9 *If all the partial derivatives of order k of the function $f(x)$ exist and are continuous, then the order in which partial derivatives of $f(x)$ of order at most k is computed does not matter.*

Definition 1.3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables and assume that $f(x)$ is differentiable with respect to all variables x_i, $i = 1 : n$. The gradient $Df(x)$ of the function $f(x)$ is the following row vector of size n:*

$$Df(x) = \left(\frac{\partial f}{\partial x_1}(x) \frac{\partial f}{\partial x_2}(x) \cdots \frac{\partial f}{\partial x_n}(x) \right). \tag{1.26}$$

Definition 1.4 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables. The Hessian of $f(x)$ is denoted by $D^2 f(x)$ and is defined as the following $n \times n$ matrix:*

$$D^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}. \tag{1.27}$$

Another commonly used notations for the gradient and Hessian of $f(x)$ are $\nabla f(x)$ and $Hf(x)$, respectively.

Vector Valued Functions Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a vector valued function given by

$$F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix},$$

where $x = (x_1, x_2, ..., x_n)$.

Definition 1.5 *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ given by $F(x) = (f_j(x))_{j=1:m)}$, and assume that the functions $f_j(x)$, $j = 1 : m$, are differentiable with respect to all variables $x_i = 1 : n$. The gradient $DF(x)$ of the function $F(x)$ is the following matrix of size $m \times n$:*

$$DF(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \cdots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}. \tag{1.28}$$

1.6.1 Functions of two variables

Scalar Valued Functions Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables denoted by x and y. The partial derivatives of the function $f(x, y)$ with respect to the variables x and y are denoted by $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$, respectively, and defined as follows:

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}; \\ \frac{\partial f}{\partial y}(x, y) &= \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}. \end{aligned}$$

The gradient of $f(x, y)$ is

$$Df(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \end{pmatrix}. \tag{1.29}$$

The Hessian of $f(x, y)$ is

$$D^2 f(x, y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x, y) & \frac{\partial^2 f}{\partial y \partial x}(x, y) \\ \frac{\partial^2 f}{\partial x \partial y}(x, y) & \frac{\partial^2 f}{\partial y^2}(x, y) \end{pmatrix}. \tag{1.30}$$

Vector Valued Functions Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}.$$

The gradient of $F(x, y)$ is

$$DF(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x, y) & \frac{\partial f_1}{\partial y}(x, y) \\ \frac{\partial f_2}{\partial x}(x, y) & \frac{\partial f_2}{\partial y}(x, y) \end{pmatrix}. \tag{1.31}$$

1.7 Plain vanilla European Call and Put options

Call Option A Call Option on an underlying asset is a contract between two parties which gives the buyer of the option the right, but not the obligation, to **buy** from the seller of the option one unit of the asset at a predetermined time T in the future, call the maturity of the option, for a predetermined price K, call the strike of the option. For this right, the buyer of the option pays $C(t)$ at time $t < T$ to the seller of the option.

Put Option A Put Option to an underlying asset is a contract between two parties which gives the buyer of the option the right, but not the obligation, to **sell** to the seller of the option one unit of the asset at a predetermined time T in the future, called the maturity of the option, for a predetermined price K, called the strike of the option. For this right, the buyer of the option pays $P(t)$ at time $t < T$ to the seller of the option.
The options described above are plain European options. An American option can be exercised at any time prior to maturity.
In an option contract, two parties exist: the buyer of the option and the seller of the option. The buyer of the option is long the option (or has a long position in the option) and the seller of the option is short the option (or has a short position in the option).
Let $S(t)$ and $S(T)$ be the price of the underlying asset at time t and at maturity T, respectively.
At time t, a call option is

$$\begin{cases} ITM & \text{if} \quad S(t) > K; \\ ATM & \text{if} \quad S(t) = K; \\ OTM & \text{if} \quad S(t) < K, \end{cases}$$

where ITM is in the money, ATM is at the money, and OTM is out of the money (OTM). A put option is

$$\begin{cases} ITM & \text{if} \quad S(t) < K; \\ ATM & \text{if} \quad S(t) = K; \\ OTM & \text{if} \quad S(t) > K. \end{cases}$$

At maturity T, a call option expires

$$\begin{cases} ITM & \text{if} \quad S(T) > K; \\ ATM & \text{if} \quad S(T) = K; \\ OTM & \text{if} \quad X(T) < K. \end{cases}$$

A put option expires

$$\begin{cases} ITM & \text{if} \quad S(T) < K; \\ ATM & \text{if} \quad S(T) = K; \\ OTM & \text{if} \quad X(T) > K. \end{cases}$$

The payoff of a call option at maturity is

$$C(T) = \max(S(T) - K, 0) = \begin{cases} S(T) - K, & \text{if} \quad S(T) > K; \\ 0, & \text{if} \quad S(T) \leq K. \end{cases}$$

The payoff of a put option at maturity is

$$P(T) = \max(K - S(T), 0) = \begin{cases} 0, & \text{if} \quad S(T) \geq K; \\ K - S(T), & \text{if} \quad S(T) < K. \end{cases}$$

1.8 Arbitrage-free pricing

An arbitrage opportunity is an investment opportunity that is guaranteed to earn money without any risk involved.
In an arbitrage-free market, we can infer relationships between prices of various securities, based on the following principle:

Theorem 1.10 (The (Generalized) Law of One Price.) *If two portfolios are guaranteed to have the same value at a future time $\tau > t$ regardless of the state of the market at time τ, then they must have the same value at time t. If one portfolio is guaranteed to be more valuable (or less valuable) than another portfolio at a future time $\tau > t$ regardless of the state of the market at time τ, then that portfolio is more valuable (or less valuable, respectively) than the other one at time $t < \tau$ as well:*
If there exists $\tau > t$ such that $V_1(\tau) = V_2(\tau)$ (or $V_1(\tau) > V_2(\tau)$, or $V_1(\tau) < V_2(\tau)$, respectively for any state of the market at time τ, then $V_1(t) = V_2(t)$ (or $V_1(t) > V_2(t)$, or $V_1(t) < V_2(t)$, respectively).

Corollary 1.1 *If the value of a portfolio of securities is guaranteed to be equal to 0 at a future time $\tau > t$ regardless of the state of the market at time τ, then the value of the portfolio at time t must have been 0 as well:*
If there exists $\tau > t$ such that $V(\tau) = 0$ for any state of the market at time τ, then $V(t) = 0$.

A consequence of Theorem 1.10 is the fact that, if the value of a portfolio at time T in the future is independent of the state of the market at that time, then the value of the portfolio in the present is the risk-neutral discounted present value of the portfolio at time T.
"Risk-neutral discounted present value" refers to the time value of money: cash can be deposited at time t_1 to be returned at time t_2 ($t_2 > t_1$), with interest. The interest rate depends on several factors, one of them being the probability of default of the party receiving the cash deposit. If this probability is zero, or close to zero, then the return is considered risk-free.
For continuously compounded interest, the value $B(t_2)$ at time $t_2 > t_1$ of $B(t_1)$ cash at time t_1 is

$$B(t_2) = e^{r(t_2-t_1)}B(t_1),$$

where r is the risk-free rate between t_1 and t_2. The value $B(t_1)$ at time $t_1 < t_2$ of $B(t_2)$ cash at time t_2 is

$$B(t_1) = e^{-r(t_2-t_1)}B(t_2).$$

Lemma 1.7 *If the value $V(T)$ of a portfolio at time T in the future is independent of the state of the market at time T, then*

$$V(t) = V(T)e^{-r(T-t)}, \tag{1.32}$$

where $t < T$ and r is the constant risk-free rate.

1.9 The Put-Call parity for European options

The Put-Call parity states that

$$P(t) + S(t) - C(t) = Ke^{-r(T-t)}, \tag{1.33}$$

where $C(t)$ and $P(t)$ are the values at time t of a European call and put option, respectively, with maturity T and strike K, on the same non-dividend paying asset with spot price $S(t)$. If the underlying asset pays dividends continuously at the rate q, the Put-Call parity has the form

$$P(t) + S(t)e^{-q(T-t)} - C(t) = Ke^{-r(T-t)}. \tag{1.34}$$

1.10 Forward and Futures contracts

Forward contract A forward contract is an agreement between two parties: one party (the long position) agrees to buy the underlying asset from the other party (the short position) at a specified time in the future and for a specified price, called the forward price. The forward price is chosen such that the forward contract has value zero at the time when the forward contract is entered into.
The contractual forward price F of a forward contract with maturity T and struck at time 0 on a non-dividend paying underlying asset with spot price $S(0)$ is

$$F = S(0)e^{rT},$$

where the interest rate r is assumed to be constant over the life of the forward contract, i.e., between times 0 and T.
If the underlying asset pays dividends continuously at the rate q, the forward price is

$$F = S(0)e^{(r-q)T}.$$

Futures contract A futures contract has a similar structure as a forward contract, but it requires the delivery of the underlying asset for the futures price. The forward and futures prices are, in theory, the same, if the risk-free interest rates are constant or deterministic, i.e., just functions of time. Major differences exist between the ways forward and futures contracts are structures, settles, and traded:

- Futures contracts trade on an exchange and have standard features, while forward contracts are over-the-counter instruments;
- Futures are marked to market and settled in a margin account on a daily basis, while forward contracts are settled in cash at maturity;
- Futures have a range of delivery dates, while forward contracts have a specified delivery date;
- Futures carry almost no credit risk, since they are settled daily, while entering into a forward contract carries some credit risk.

1.11 Exercises

1.11.1 Question 1

Use the integration by parts to compute $\int \ln(x)dx$.

Answer Let $f(x) = 1$ and $g(x) = \ln(x)$,

$$\begin{aligned}
\int \ln(x)dx &= \int 1 \cdot \ln(x)dx \\
&= x\ln(x) - \int x\frac{1}{x}dx \\
&= x\ln(x) - x + C
\end{aligned}$$

1.11.2 Question 2

Compute $\int \frac{1}{x\ln(x)}dx$ by using the substitution $u = \ln(x)$.

Answer Let $u = \ln(x)$, then $du = \frac{dx}{x}$

$$\begin{aligned}\int \frac{1}{x\ln(x)}dx &= \int \frac{1}{u}du \\ &= \ln(|u|) + C \\ &= \ln(|\ln(x)|) + C\end{aligned}$$

1.11.3 Question 3

Show that $(\tan x)' = 1/(\cos x)^2$ and

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C.$$

Note: The antiderivative of a rational function is often computed using the substitution $x = \tan(\frac{x}{2})$.

Answer Using Quotient Rule,

$$\begin{aligned}(\tan(x))' &= \left(\frac{\sin(x)}{\cos(x)}\right)' \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)}\end{aligned}$$

Let $x = \tan(u)$, then $dx = \frac{1}{\cos^2(u)}du$.

$$\begin{aligned}\int \frac{1}{1+x^2}dx &= \int \frac{1}{1+\tan^2(u)}\frac{1}{\cos^2(u)}du \\ &= \int \frac{1}{\frac{\cos^2(u)+\sin^2(u)}{\cos^2(u)}}\frac{1}{\cos^2(u)}du \\ &= \int du \\ &= u + C \\ &= \arctan(x) + C\end{aligned}$$

1.11.4 Question 4

Use l'Hôpital's rule to show that the following two Taylor approximations hold when x is close to 0:

$$\sqrt{1+x} \approx 1 + \frac{x}{2};$$
$$e^x \approx 1 + x + \frac{x^2}{2}.$$

In other words, show that the following limits exist and are constant:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-(1+\frac{x}{2})}{x^2} \quad \text{and} \quad \lim_{x\to 0}\frac{e^x-(1+x+\frac{x^2}{2})}{x^3}.$$

Answer

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{1+x}-(1+\frac{x}{2})}{x^2} &= \lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \\
&= \lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} \\
&= -\frac{1}{8} \\
\lim_{x\to 0}\frac{e^x-(1+x+\frac{x^2}{2})}{x^3} &= \lim_{x\to 0}\frac{e^x-1-x}{3x^2} \\
&= \lim_{x\to 0}\frac{e^x-1}{6x} \\
&= \lim_{x\to 0}\frac{e^x}{6} \\
&= \frac{1}{6}
\end{aligned}$$

1.11.5 Question 5

Use the definition of e, i.e., $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$, to show that

$$\frac{1}{e} = \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x.$$

Hint: Use the fact that

$$\frac{1}{1+\frac{1}{x}} = \frac{x}{x+1} = 1-\frac{1}{x+1}.$$

Answer

$$\begin{aligned}
e &= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \\
\frac{1}{e} &= \lim_{x\to\infty}\left(\frac{1}{1+\frac{1}{x}}\right)^x \\
&= \lim_{x\to\infty}\left(1-\frac{1}{x+1}\right)^x \\
&= \lim_{x\to\infty}\left(1-\frac{1}{x+1}\right)^{x+1} \\
&= \lim_{y\to\infty}\left(1-\frac{1}{y}\right)^y
\end{aligned}$$

where $y = x+1$, since

$$\lim_{x\to\infty}\left(1-\frac{1}{x+1}\right) = 1$$