Chapter 0

Mathematical preliminaries

0.1 Even and odd functions

Definition 0.1 *The function* $f : \mathbb{R} \to \mathbb{R}$ *is an even function iff*

$$f(-x) = f(x), \quad \forall x \in \mathbb{R}. \tag{1}$$

The graph of any even function is symmetric with respect to the y-axis.

Lemma 0.1 *Let* $f(x)$ *be an integrable even function. Then,*

$$\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(x)dx, \quad \forall a \in \mathbb{R}, \tag{2}$$

and therefore

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx, \quad \forall a \in \mathbb{R}. \tag{3}$$

Moreover, if $\int_0^\infty f(x)dx$ *exists, then*

$$\int_{-\infty}^{0} f(x)dx = \int_{0}^{\infty} f(x)dx, \tag{4}$$

and

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx. \tag{5}$$

Definition 0.2 *The function* $f : \mathbb{R} \to \mathbb{R}$ *is an odd function iff*

$$f(-x) = -f(x), \quad \forall x \in \mathbb{R}. \tag{6}$$

If we let $x = 0$ in (6), we find that $f(0) = 0$ for any odd function $f(x)$. Also, the graph of any odd function is symmetric with respect to the point (0, 0).

Lemma 0.2 *Let* $f(x)$ *be an integrable odd function. Then,*

$$\int_{-a}^{a} f(x)dx = 0, \quad \forall a \in \mathbb{R}. \tag{7}$$

Moreover, if $\int_0^\infty f(x)dx$ *exists, then*

$$\int_{-\infty}^{\infty} f(x)dx = 0. \tag{8}$$

0.2 Useful sums with interesting proofs

The following sums occur frequently when estimating operation counts of numerical algorithms:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \tag{9}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \tag{10}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{11}$$

0.3 Sequences satisfying linear recursions

Definition 0.3 *A sequence $(x_n)_{n\geq 0}$ satisfies a linear recursion of order k iff there exist constants a_i, $i = 0 : k$ with $a_k \neq 0$, such that*

$$\sum_{i=0}^{k} a_i x_{n+i} = 0, \quad \forall n \geq 0. \tag{12}$$

The recursion (12) is called a linear recursion because of the following linearity properties:
(i) If the sequence $(x_n)_{n\geq 0}$ satisfies the linear recursion (12), then the sequence $(z_n)_{n\geq 0}$ given by

$$z_n = C x_n, \quad \forall n \geq 0, \tag{13}$$

where C is an arbitrary constant, also satisfies the linear recursion (12).
(ii) If the sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ satisfies the linear recursion (12), then the sequence $(z_n)_{n\geq 0}$ given by

$$z_n = x_n + y_n, \quad \forall n \geq 0, \tag{14}$$

also satisfies the linear recursion (12).

Definition 0.4 *The characteristic polynomial $P(z)$ corresponding to the linear recursion $\sum_{i=0}^{k} a_i x_{n+i} = 0$, for all $n \geq 0$, is defined as*

$$P(z) = \sum_{i=0}^{k} a_i z^i. \tag{15}$$

$P(z)$ is a polynomial of degree k, i.e., $\deg(P(z)) = k$. If $P(z)$ has p different roots, λ_j, $j = 1 : p$, with $p \leq k$, and if $m(\lambda_j)$ denotes the multiplicity of the root λ_j, then $\sum_{j=1}^{p} m(\lambda_j) = k$ where λ_j can be a complex number.

Theorem 0.1 *Let $(x_n)_{x\geq 0}$ be a sequence satisfying the linear recursion*

$$\sum_{i=0}^{k} a_i x_{n+i} = 0, \quad \forall n \geq 0, \tag{16}$$

with $a_k \neq 0$, and let $P(z) = \sum_{i=0}^{k-1} a_i z^i$ be the characteristic polynomial associated with recursion (16). Let λ_j, $j = 1 : p$, where $p \leq k$, be the roots of $P(z)$, and let $m(\lambda_j)$ be the multiplicity of λ_j. The general form of the sequence $(x_n)_{n\geq 0}$ satisfying the linear recursion (16) is

$$x_n = \sum_{j=1}^{p} \left(\sum_{i=0}^{m(\lambda_j)-1} C_{i,j} n^i \right) \lambda_j^n, \quad \forall n \geq 0, \tag{17}$$

where $C_{i,j}$ are constant numbers.

0.4 The "Big O" and "little o" notations

Definition 0.5 *Let $f, g : \mathbb{R} \to \mathbb{R}$. We write that $f(x) = O(g(x)$, as $x \to \infty$, iff there exist constants $C > 0$ and $M > 0$ such that $\left|\frac{f(x)}{g(x)} \leq C\right|$, for any $x \geq M$. This can be written equivalently as*

$$f(x) = O(g(x)), \quad \text{as} \quad x \to \infty, \quad \text{iff} \quad \limsup_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| < \infty. \tag{18}$$

The "little o" notation refers to functions whose ratios tend to 0 at certain fpoints, and can be defined for $x \to \infty$, $x \to a$, and $x \to -\infty$ as follows:

Definition 0.6 *Let $f, g : \mathbb{R} \to \mathbb{R}$. Then*

$$f(x) = o(g(x)), \quad \text{as} \quad x \to \infty, \quad \text{iff} \quad \lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = 0; \tag{19}$$

$$f(x) = o(g(x)), \quad \text{as} \quad x \to -\infty, \quad \text{iff} \quad \lim_{x\to-\infty} \left|\frac{f(x)}{g(x)}\right| = 0; \tag{20}$$

$$f(x) = o(g(x)), \quad \text{as} \quad x \to a, \quad \text{iff} \quad \lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| = 0. \tag{21}$$

0.5 Exercises

0.5.1 Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function.

Part (i)

Show that $xf(x)$ is an even function and $x^2 f(x)$ is an odd function.

Answer Let $g(x) = xf(x)$, using definition (6), we have

$$\begin{aligned} g(-x) &= -xf(-x) \\ &= -x(-f(x)) \\ &= xf(x) \\ &= g(x). \end{aligned}$$

Therefore, $xf(x)$ is an even function.

Let $g(x) = x^2 f(x)$, using definition (6), we have

$$\begin{aligned} g(-x) &= (-x)^2 f(-x) \\ &= x^2(-f(x)) \\ &= -x^2 f(x) \\ &= -g(x) \end{aligned}$$

Therefore, $x^2 f(x)$ is an odd function.

Part (ii)

Show that the function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function and that the function $g_2 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function.

Answer

$$\begin{aligned} g_1(-x) &= f((-x)^2) \\ &= f(x^2) \\ &= g_1(x) \end{aligned}$$

Therefore g_1 is an even function.

Let $y = x^3$:

$$\begin{aligned} g_2(-x) &= f((-x)^3) \\ &= f(-x^3) \\ &= f(-y) \\ &= -f(y) \\ &= -f(x^3) \\ &= -g_2(x) \end{aligned}$$

Therefore g_2 is an odd function.

Part (iii)

Let i be even, j be odd, and $y = x^j$:

$$\begin{aligned} h(-x) &= (-x)^i f((-x)^j) \\ &= x^i f(-x^j) \\ &= x^i f(-y) \\ &= -x^i f(y) \\ &= -x^i f(x^j) \\ &= -h(x) \end{aligned}$$

Let i be odd, j be even:

$$\begin{aligned} h(-x) &= (-x)^i f((-x)^j) \\ &= -x^i f(x^j) \\ &= -h(x) \end{aligned}$$

When $i + j$ is odd, $h(x)$ is an odd function.

0.5.2 Question 2

Let $S(n, 2) = \sum_{k=1}^{n} k^2$ and $S(n, 3) = \sum_{k=1}^{n} k^3$.

Part (i)

Let $T(n, 2, x) = \sum_{k=1}^{n} k^2 x^k$. Use formulas,

$$T(n, 2, x) = x\frac{d}{dx}(T(n, 1, x)),$$

and

$$T(n, 1, x) = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2},$$

to show that

$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}.$$

Answer Using quotient rule,

$$\begin{aligned}
T(n,2,x) &= x\frac{d}{dx}(T(n,1,x)) \\
&= x\frac{d}{dx}\left(\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}\right) \\
&= x\left(\frac{\frac{d}{dx}(x-(n+1)x^{n+1}+nx^{n+2})(1-x)^2-(x-(n+1)x^{n+1}+nx^{n+2})\frac{d}{dx}(1-x)^2}{(1-x)^4}\right) \\
&= x\left(\frac{(1-(n+1)^2x^n+(n^2+2n)x^{n+1})(1-x)^2-(x-(n+1)x^{n+1}+nx^{n+2})(-2(1-x))}{(1-x)^4}\right) \\
&= x\left(\frac{(1-(n+1)^2x^n+(n^2+2n)x^{n+1})(1-x)^2+2(x-(n+1)x^{n+1}+nx^{n+2})(1-x)}{(1-x)^4}\right) \\
&= x\left(\frac{(1-(n+1)^2x^n+(n^2+2n)x^{n+1})(1-x)+2(x-(n+1)x^{n+1}+nx^{n+2})}{(1-x)^3}\right) \\
&= x\left(\frac{(1-(n+1)^2x^n+(n^2+2n)x^{n+1})-(x-(n+1)^2x^{n+1}+(n^2+2n)x^{n+2})+\cdots}{(1-x)^3}\right) \\
&= x\left(\frac{1-x-(n+1)^2x^n+(2n^2+4n+1)x^{n+1}-(n^2+2n)x^{n+2}+\cdots}{(1-x)^3}\right) \\
&= x\left(\frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}\right) \\
&= \frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3}
\end{aligned}$$

Part (ii)

Note that $S(n,2) = T(n,2,1)$. Use l'Hôpitals's rule to evaluate $T(n,2,1)$, and conclude that $S(n,2) = \frac{n(n+1)(2n+1)}{6}$.

Answer

$$\begin{aligned}
\lim_{x\to 1} T(n,2,x) &= \lim_{x\to 1}\frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3} \\
&= \lim_{x\to 1}\frac{(n-1)n(n+1)^3x^{n-2}-n(n+1)(n+2)(2n^2+2n-1)x^{n-1}+(n+1)(n+2)(n+3)n^2x^n}{6} \\
&= \frac{(n-1)n(n+1)^3-n(n+1)(n+2)(2n^2+2n-1)+(n+1)(n+2)(n+3)n^2}{6} \\
&= \frac{n(n+1)(2n+1)}{6} \\
S(n,2) &= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Part (iii)

Compute $T(n,3,x) = \sum_{k=1}^{n} k^3x^k$ using the formula

$$T(n,3,x) = x\frac{d}{dx}(T(n,2,x)).$$

Answer

$$
\begin{aligned}
T(n,3,x) &= x\frac{d}{dx}(T(n,2,x)) \\
&= x\frac{d}{dx}\frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3} \\
&= \frac{x((n^3(x-1)^3-3n^2(x-1)^2+3n(x^2-1)-x(x+4)-1)x^n+(x+4)x+1)}{(1-x)^4}
\end{aligned}
$$

Part (iv)

Note that $S(n,3) = T(n,3,1)$. Use l'Hôpital's rule to evaluate $T(n,3,1)$, and conclude that $S(n,3) = \left(\frac{n(n+1)}{2}\right)^2$.

$$
\lim_{x\to 1} T(n,3,x) = \lim_{x\to 1} \frac{x((n^3(x-1)^3-3n^2(x-1)^2+3n(x^2-1)-x(x+4)-1)x^n+(x+4)x+1)}{(1-x)^4}
$$

0.5.3 Question 3

Compute $S(n,4) = \sum_{k=1}^{n} k^4$ using the recursion formula for $i=4$, the fact that $S(n,0)=n$, and formulas for $S(n,1)$, $S(n,2)$, and $S(n,3)$.

Answer

$$
\begin{aligned}
S(n,4) &= \frac{1}{5}\left((n+1)^5 - 1 - \sum_{j=0}^{3}\binom{5}{j}S(n,j)\right) \\
&= \frac{1}{5}\left((n+1)^5 - 1 - S(n,0) - 5S(n,1) - 10S(n,2) - 10S(n,3)\right) \\
&= \frac{1}{5}\left((n+1)^5 - 1 - n - 5\frac{n(n+1)}{2} - 10\frac{n(n+1)(2n+1)}{6} - 10\left(\frac{n(n+1)}{2}\right)^2\right) \\
&= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}
\end{aligned}
$$

0.5.4 Question 4

It is easy to see that the sequence $(x_n)_{n\geq 1}$ given by $x_n = \sum_{k=1}^{n} k^2$ satisfies the recursion

$$
x_{n+1} = x_n + (n+1)^2, \quad \forall n \geq 1, \tag{22}
$$

with $x_1 = 1$.

Part (i)

By substituting $n+1$ for n in (22), obtain that

$$
x_{n+2} = x_{n+1} + (n+2)^2. \tag{23}
$$

Substract (22) from (23) to find that

$$
x_{n+2} = 2x_{n+1} - x_n + 2n + 3, \quad \forall n \geq 1, \tag{24}
$$

with $x_1 = 1$ and $x_2 = 5$.

Answer

$$\begin{aligned} x_{(n+1)+1} &= x_{n+1}((n+1)+1)^2 \\ x_{n+2} &= x_{n+1}(n+2)^2 \end{aligned}$$

Substract (22) from (23):

$$\begin{aligned} x_{n+2} - x_{n+1} &= x_{n+1} + (n+2)^2 - x_n - (n+1)^2 \\ x_{n+2} &= 2x_{n+1} - x_n + n^2 + 4n + 4 - n^2 - 2n - 1 \\ &= 2x_{n+1} - x_n + 2n + 3 \end{aligned}$$

Part (ii)

Similarly, substitute $n+1$ for n in (24) and obtain that

$$x_{n+3} = 2x_{n+2} - x_{n+1} + 2(n+1) + 3. \tag{25}$$

Substract (24) from (25) to find that

$$x_{n+3} = 3x_{n+2} - 3x_{n+1} + x_n + 2, \quad \forall n \geq 1, \tag{26}$$

with $x_1 = 1$, $x_2 = 5$, and $x_3 = 14$.

Answer

$$\begin{aligned} x_{(n+1)+2} &= 2x_{(n+1)+1} - x_{n+1} + 2(n+1) + 3 \\ x_{n+3} &= 2x_{n+2} - x_{n+1} + 2(n+1) + 3 \end{aligned}$$

Substract (24) from (25)

$$\begin{aligned} x_{n+3} - x_{n+2} &= 2x_{n+2} - x_{n+1} + 2(n+1) + 3 - 2x_{n+1} + x_n - 2n - 3 \\ x_{n+3} &= 3x_{n+2} - 3x_{n+1} + x_n + 2 \end{aligned}$$

Part (iii)

Use a similar method to prove that the sequence $(x_n)_{n\geq 0}$ satisfies the linear recursion

$$x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n = 0, \quad \forall n \geq 1. \tag{27}$$

The characteristic polynomial associated to the recursion (27) is

$$P(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 = (z-1)^4.$$

Use the fact that $x_1 = 1$, $x_2 = 5$, $x_3 = 14$, and $x_4 = 30$ to show that

$$x_n = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1,$$

and conclude that

$$S(n,2) = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1.$$

Answer Substitute $n+1$ for n in (26) to obtain

$$x_{n+4} = 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2. \tag{28}$$

Substract (26) from (28) to obtain that

$$\begin{aligned} x_{n+4} - x_{n+3} &= 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2 - (3x_{n+2} - 3x_{n+1} + x_n + 2) \\ x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} - x_n &= 0 \end{aligned}$$

The characteristic polynomial has root $\lambda = 1$ with multiplicity 4. The linear recurssion can be expressed as

$$\begin{aligned} x_n &= \sum_{j=1}^{p} \left(\sum_{i=0}^{3} C_{i,j} n^i \right) \lambda_j^n \\ &= \sum_{j=1}^{p} \left(C_{0,j} + C_{1,j} n + C_{2,j} n^2 + C_{3,j} n^3 \right) \lambda_j^n \\ &= C_1 + C_2 n + C_3 n^2 + C_4 n^3 \end{aligned}$$

Since $x_1 = 1$, $x_2 = 5$, $x_3 = 14$, and $x_4 = 30$, C_1, C_2, C_3, and C_4 must solve the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \\ 30 \end{pmatrix}.$$

We obtain that $C_1 = 0$, $C_2 = \frac{1}{6}$, $C_3 = \frac{1}{2}$, and $C_4 = \frac{1}{3}$ and therefore

$$x_n = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} = \frac{n(n+1)(n+2)}{6}$$

0.5.5 Question 5

Find the general form of the sequence $(x_n)_{n \geq 0}$ satisfying the linear recursion

$$x_{n+3} = 2x_{n+1} + x_n, \quad \forall n \geq 0,$$

with $x_0 = 1$, $x_1 = -1$, and $x_2 = 1$.

Answer Rewrite the recursion in the form (12) as

$$x_{n+3} - 2x_{n+1} - x_n = 0, \quad \forall n \geq 0.$$

The characteristic polynomial associated to the linear recursion is

$$\begin{aligned} P(z) &= z^3 - 2z - 1 \\ &= (z+1)(z^2 - z - 1) \end{aligned}$$

and the roots of $P(z)$ are

$$\lambda_1 = -1, \quad \lambda_2 = \frac{1+\sqrt{5}}{2}, \quad \lambda_3 = \frac{1-\sqrt{5}}{2}.$$

From Theorem 0.1, we find that

$$x_n = C_1 \lambda_1^n + C_2 \lambda_2^n + C_3 \lambda_3^n, \quad \forall n \geq 0.$$

Given $x_0 = 1$, $x_1 = -1$, and $x_2 = 1$, we obtain the linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

By solving the linear system, we find that $C_1 = 1$, $C_2 = 0$, and $C_3 = 0$. The general formula for is

$$x_n = (-1)^n, \quad \forall n \geq 0.$$

0.5.6 Question 6

The sequence $(x_n)_{n \geq 0}$ satisfies the recursion

$$x_{n+1} = 3x_n + 2, \quad \forall n \geq 0,$$

with $x_0 = 1$.

Part (i)

Show that the sequence $(x_n)_{n\geq 0}$ satisfies the linear recursion

$$x_{n+2} = 4x_{n+1} - 3x_n, \quad \forall n \geq 0,$$

with $x_0 = 1$ and $x_1 = 5$.

Answer Substitute $n+1$ for n to obtain

$$x_{n+2} = 3x_{n+1} + 2$$

Subtract the original recursion to get

$$\begin{aligned} x_{n+2} - x_{n+1} &= 3x_{n+1} + 2 - 3x_n - 2 \\ x_{n+2} &= 4x_{n+1} - 3x_n \end{aligned}$$

Part (ii)

Find the general formula for x_n, $n \geq 0$.

Answer The characteristic polynomial has the form

$$P(z) = z^2 - 4z + 3 = (z-1)(z-3)$$

which has roots $\lambda_1 = 1$ and $\lambda_2 = 3$. We obtain the linear system

$$\begin{cases} C_1 + C_2 = 1; \\ C_1\lambda_1 + C_2\lambda_2 = 5. \end{cases}$$

The solution to the linear system is $C_1 = -1$ and $C_2 = 2$. Therefore, the general form is

$$x_n = 2(3)^n - 1$$

0.5.7 Question 7

The sequence $(x_n)_{n\geq 0}$ satisfies the recursion

$$x_{n+1} = 3x_n + n + 2, \quad \forall n \geq 0,$$

with $x_0 = 1$.

Part (i)

Show that the sequence $(x_n)_{n\geq 0}$ satisfies the linear recursion

$$x_{n+3} = 5x_{n+2} - 7x_{n+1} + 3x_n, \quad \forall n \geq 0,$$

with $x_0 = 1$, $x_1 = 5$, and $x_2 = 18$.

Answer Substitute $n+1$ for n, we obtain

$$x_{n+2} = 3x_{n+1} + n + 3$$

Subtract

$$x_{n+2} = 4x_{n+1} - 3x_n + 1$$

Substitute $n+1$ for n, we obtain

$$x_{n+3} = 4x_{n+2} - 3x_{n+1} + 1$$

Subtract

$$x_{n+3} = 5x_{n+2} - 7x_{n+1} + 3x_n$$

Part (ii)

Find the general formula for x_n, $n \geq 0$.

Answer The characteristic polynomial is given by

$$P(z) = z^3 - 5z^2 + 7z - 3 = (z-1)^2(z-3),$$

with roots $\lambda_1 = 1$ and $\lambda_2 = 3$. The general form is

$$\begin{aligned} x_n &= \sum_{j=1}^{2} \left(\sum_{i=0}^{m(\lambda_j)-1} C_{i,j} n^i \right) \lambda_j^n \\ &= \lambda_1^n \sum_{i=0}^{1} C_{i,1} n^i + \lambda_2^n C_2 \\ &= \lambda_1^n C_{0,1} + \lambda_1^n C_{1,1} n + \lambda_2^n C_2 \\ &= C_1 + C_2 n + C_3 3^n \end{aligned}$$

Since $x_0 = 1$, $x_1 = 5$, and $x_2 = 18$, we find $C_1 = -\frac{1}{2}$, $C_2 = -\frac{5}{4}$, and $C_3 = \frac{9}{4}$. We conclude that

$$x_n = \frac{3^{n+2} - 2n - 5}{4}$$

0.5.8 Question 8

Let $P(z) = \sum_{i=0}^{k} a_i z^i$ be the characteristic polynomial corresponding to the linear recursion

$$\sum_{i=0}^{k} a_i x_{n+i} = 0, \quad \forall n \geq 0.$$

Assume that λ is a root of multiplicity 2 of $P(z)$. Show that the sequence $(y_n)_{n\geq 0}$ given by

$$y_n = Cn\lambda^n, \quad n \geq 0,$$

where C is an arbitrary constant, satisfies the recursion.

Answer

$$\begin{aligned} \sum_{i=0}^{k} a_i y_{n+i} &= \sum_{i=0}^{k} a_i C(n+i)\lambda^{n+i} \\ &= Cn \sum_{i=0}^{k} a_i \lambda^{n+i} + C \sum_{i=0}^{k} a_i i \lambda^{n+i} \\ &= Cn\lambda^n \sum_{i=0}^{k} a_i \lambda^i + C\lambda^{n+1} \sum_{i=0}^{k} i a_i \lambda^{i-1} \\ &= Cn\lambda^n P(\lambda) + C\lambda^{n+1} P'(\lambda) \\ &= 0. \end{aligned}$$

0.5.9 Question 9

Let $n > 0$. Show that

$$\begin{aligned} O(x^n) + O(x^n) &= O(x^n), \quad \text{as} \quad x \to 0; \\ o(x^n) + o(x^n) &= o(x^n), \quad \text{as} \quad x \to 0. \end{aligned}$$

Answer Let $f(x) = O(x^n)$ and $g(x) = O(x^n)$, then

$$\limsup_{x\to 0}\left|\frac{f(x)}{x^n}\right| < \infty \quad \text{and} \quad \limsup_{x\to 0}\left|\frac{g(x)}{x^n}\right| < \infty.$$

We see that

$$\limsup_{x\to 0}\left|\frac{f(x)+g(x)}{x^n}\right| \le \limsup_{x\to 0}\left|\frac{f(x)}{x^n}\right| + \limsup_{x\to 0}\left|\frac{g(x)}{x^n}\right| < \infty,$$

and therefore $O(x^n) + O(x^n) = O(x^n)$.

Let $f(x) = o(x^n)$ and $g(x) = o(x^n)$, then

$$\lim_{x\to 0}\left|\frac{f(x)}{x^n}\right| = 0 \quad \text{and} \quad \lim_{x\to 0}\left|\frac{g(x)}{x^n}\right| = 0.$$

We see that

$$\lim_{x\to 0}\left|\frac{f(x)+g(x)}{x^n}\right| \le \lim_{x\to 0}\left|\frac{f(x)}{x^n}\right| + \lim_{x\to 0}\left|\frac{g(x)}{x^n}\right| = 0,$$

and therefore $o(x^n) + o(x^n) = o(x^n)$.

0.5.10 Question 10

Prove that

$$\sum_{k=1}^{n} k^2 = O(n^3), \quad \text{as} \quad n \to \infty;$$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + O(n^2), \quad \text{as} \quad n \to \infty,$$

i.e., show that

$$\limsup_{n\to\infty}\frac{\sum_{k=1}^{n} k^2}{n^3} < \infty$$

and that

$$\limsup_{n\to\infty}\frac{\sum_{k=1}^{n} k^2 - \frac{n^3}{3}}{n^2} < \infty.$$

Similarly, prove that

$$\sum_{k=1}^{n} k^3 = O(n^4), \quad \text{as} \quad n \to \infty;$$

$$\sum_{k=1}^{n} k^3 = \frac{n^4}{4} + O(n^3), \quad \text{as} \quad n \to \infty,$$

Answer Using (10)

$$\begin{aligned}
\limsup_{n\to\infty}\frac{\sum_{k=1}^{n} k^2}{n^3} &= \limsup_{n\to\infty}\frac{\frac{n(n+1)(2n+1)}{6}}{n^3} \\
&= \limsup_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3} \\
&= \frac{1}{3} < \infty \\
\limsup_{n\to\infty}\frac{\sum_{k=1}^{n} k^2 - \frac{n^3}{3}}{n^2} &= \limsup_{n\to\infty}\frac{\frac{2n^3+3n^2+n}{6} - \frac{n^3}{3}}{n^2} \\
&= \limsup_{n\to\infty}\frac{3n^2+n}{6n^2} \\
&= \frac{1}{2} \\
&< \infty
\end{aligned}$$

Using (11)

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} k^3}{n^4} &= \limsup_{n\to\infty} \frac{\left(\frac{n(n+1)}{2}\right)^2}{n^4} \\
&= \limsup_{n\to\infty} \frac{n^2(n^2+2n+1)}{4n^4} \\
&= \frac{1}{4} \\
&< \infty \\
\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} k^3 - \frac{n^4}{4}}{n^3} &= \limsup_{n\to\infty} \frac{\frac{(n^4+2n^3+n^2)}{4} - \frac{n^4}{4}}{n^3} \\
&= \limsup_{n\to\infty} \frac{2n^3+n^2}{4n^3} \\
&= \frac{1}{2} \\
&< \infty
\end{aligned}
$$

Chapter 1

Calculus review. Plain vanilla options.

1.1 Brief review of differentiation

The function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at the point $x \in \mathbb{R}$ if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, in which case the derivative $f'(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{1.1}$$

Theorem 1.1 (Product Rule.) *The product $f(x)g(x)$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable, and*

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x). \tag{1.2}$$

Theorem 1.2 (Quotient Rule.) *The quotient $\frac{f(x)}{g(x)}$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable at every point x where the function $\frac{f(x)}{g(x)}$ is well defined, and*

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}. \tag{1.3}$$

Theorem 1.3 (Chain Rule.) *The composite function $(g \circ f)(x) = g(f(x))$ of two differentiable functions $f(x)$ and $g(x)$ is differentiable at every point x where $g(f(x))$ is well defined, and*

$$(g(f(x)))' = g'(f(x))f'(x). \tag{1.4}$$

The Chain Rule formula (1.4) can also be written as

$$\frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx},$$

where $u = f(x)$ is a function of x and $g = g(u) = g(f(x))$.

Chain Rule is often used for power functions, exponential functions, and logarithmic function:

$$\frac{d}{dx}((f(x))^n) = n(f(x))^{n-1}f'(x); \tag{1.5}$$

$$\frac{d}{dx}(e^{f(x)}) = e^{f(x)}f'(x); \tag{1.6}$$

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}. \tag{1.7}$$

Lemma 1.1 *Let $f : [a,b] \to [c,d]$ be a differentiable function, and assume that $f(x)$ has an inverse function denoted by $f^{-1}(x)$ with $f^{-1} : [c,d] \to [a,b]$. The function $f^{-1}(x)$ is differentiable at every point $x \in [c,d]$ where $f'(f^{-1}(x)) \neq 0$ and*

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}. \tag{1.8}$$

1.2 Brief review of integration

Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Recall that $F(x)$ is the antiderivative of $f(x)$ iff $F'(x) = f$, i.e.,

$$F(x) = \int f(x)dx \quad \Longleftrightarrow \quad F'(x) = f(x).$$

Theorem 1.4 (Fundamental Theorem of Calculus.) *Let $f(x)$ be a continuous function on the interval $[a, b]$, and let $F(x)$ be the antiderivative of $f(x)$. Then*

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a).$$

Integration by parts is the counterpart for integration of the product rule.

Theorem 1.5 (Integration by parts.) *Let $f(x)$ and $g(x)$ be continuous function. Then*

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx, \tag{1.9}$$

where $F(x) = \int f(x)dx$ is the antiderivative of $f(x)$. For definite integrals,

$$\int_a^b f(x)g(x)dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)dx. \tag{1.10}$$

Integration by substitution if the counterpart for integration of the chain rule.

Theorem 1.6 (Integration by substitution) *Let $f(x)$ be an integrable function. Assume that $g(u)$ is an invertible and continuously differentiable function. The substitution $x = g(u)$ changes the integration variable from x to u as follows:*

$$\int f(x)dx = \int f(g(u))g'(u)du. \tag{1.11}$$

For definite integrals,

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)du. \tag{1.12}$$

1.3 Differentiating definite integrals

If a definite integral has functions as limits of integration, e.g.,

$$\int_{a(t)}^{b(t)} f(x)dx,$$

or if the function to be integrated is a function of the integrating variable and of another variable, e.g.,

$$\int_a^b f(x, t)dx$$

then the result of the integration is a function (of the variable t in both cases above).

Lemma 1.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then,*

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x)dx\right) = f(b(t))b'(t) - f(a(t))a'(t), \tag{1.13}$$

where $a(t)$ and $b(t)$ are differentiable functions.

Lemma 1.3 *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the partial derivative $\frac{\partial f}{\partial t}(x, t)$ exists and is continuous in both variables x and t.*

$$\frac{d}{dt}\left(\int_a^b f(x, t)dx\right) = \int_a^b \frac{\partial f}{\partial t}(x, t)dx. \tag{1.14}$$

Lemma 1.4 *Let $f(x, t)$ be a continuous function such that the partial derivative $\frac{\partial f}{\partial t}(x, t)$ exists and is continuous. Then,*

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x, t)dx\right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x, t) + f(b(t), t)b'(t) - f(a(t), t)a'(t).$$

Note that Lemma 1.2 and Lemma 1.3 are special cases of Lemma 1.4.

1.4 Limits

Definition 1.1 *Let $g : \mathbb{R} \to \mathbb{R}$. The limit of $g(x)$ as $x \to x_0$ exists and is finite and equal to l iff for any $\epsilon > 0$ there exists $\delta > 0$ such that $|g(x) - l| < \epsilon$ for all $x \in (x_0 - \delta, x_0 + \delta)$, i.e.,*

$$\lim_{x \to x_0} g(x) = l, \quad \textit{iff} \quad \forall \epsilon > 0 \exists \delta > 0 \quad \textit{such that} \quad |g(x) - l| < \epsilon, \ \forall |x - x_0| < \delta.$$

Similarly,

$$\lim_{x \to x_0} g(x) = \infty, \quad \textit{iff} \quad \forall C > 0 \exists \delta > 0 \quad \textit{such that} \quad g(x) > C, \ \forall |x - x_0| < \delta.$$

$$\lim_{x \to x_0} g(x) = -\infty, \quad \textit{iff} \quad \forall C < 0 \exists \delta > 0 \quad \textit{such that} \quad g(x) < C, \ \forall |x - x_0| < \delta.$$

Theorem 1.7 *If $P(x)$ and $Q(x)$ are polynomials and $c > 1$ is a fixed constant, then*

$$\lim_{x \to \infty} \frac{P(x)}{c^x} = 0, \quad \forall c > 1; \tag{1.15}$$

$$\lim_{x \to \infty} \frac{\ln |Q(x)|}{P(x)} = 0. \tag{1.16}$$

Lemma 1.5 *Let $c > 0$ be a positive constant. Then,*

$$\lim_{x \to \infty} x^{\frac{1}{x}} = 1; \tag{1.17}$$

$$\lim_{x \to \infty} c^{\frac{1}{x}} = 1; \tag{1.18}$$

$$\lim_{x \searrow 0} x^x = 1, \tag{1.19}$$

$$\tag{1.20}$$

where the notation $x \searrow 0$ means that x goes to 0 while always being positive, i.e., $x \to 0$ with $x > 0$.