

Chapter 0

Mathematical preliminaries

0.1 Even and odd functions

Definition 0.1 *The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is an even function iff*

$$f(-x) = f(x), \quad \forall x \in \mathbb{R}. \quad (1)$$

The graph of any even function is symmetric with respect to the y -axis.

Lemma 0.1 *Let $f(x)$ be an integrable even function. Then,*

$$\int_{-a}^0 f(x)dx = \int_0^a f(x)dx, \quad \forall a \in \mathbb{R}, \quad (2)$$

and therefore

$$\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx, \quad \forall a \in \mathbb{R}. \quad (3)$$

Moreover, if $\int_0^\infty f(x)dx$ exists, then

$$\int_{-\infty}^0 f(x)dx = \int_0^\infty f(x)dx, \quad (4)$$

and

$$\int_{-\infty}^\infty f(x)dx = 2 \int_0^\infty f(x)dx. \quad (5)$$

Definition 0.2 *The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is an odd function iff*

$$f(-x) = -f(x), \quad \forall x \in \mathbb{R}. \quad (6)$$

If we let $x = 0$ in (6), we find that $f(0) = 0$ for any odd function $f(x)$. Also, the graph of any odd function is symmetric with respect to the point $(0, 0)$.

Lemma 0.2 *Let $f(x)$ be an integrable odd function. Then,*

$$\int_{-a}^a f(x)dx = 0, \quad \forall a \in \mathbb{R}. \quad (7)$$

Moreover, if $\int_0^\infty f(x)dx$ exists, then

$$\int_{-\infty}^\infty f(x)dx = 0. \quad (8)$$

0.2 Useful sums with interesting proofs

The following sums occur frequently when estimating operation counts of numerical algorithms:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}; \quad (9)$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}; \quad (10)$$

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2. \quad (11)$$

0.3 Sequences satisfying linear recursions

Definition 0.3 A sequence $(x_n)_{n \geq 0}$ satisfies a linear recursion of order k iff there exist constants a_i , $i = 0 : k$ with $a_k \neq 0$, such that

$$\sum_{i=0}^k a_i x_{n+i} = 0, \quad \forall n \geq 0. \quad (12)$$

The recursion (12) is called a linear recursion because of the following linearity properties:

(i) If the sequence $(x_n)_{n \geq 0}$ satisfies the linear recursion (12), then the sequence $(z_n)_{n \geq 0}$ given by

$$z_n = Cx_n, \quad \forall n \geq 0, \quad (13)$$

where C is an arbitrary constant, also satisfies the linear recursion (12).

(ii) If the sequences $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ satisfies the linear recursion (12), then the sequence $(z_n)_{n \geq 0}$ given by

$$z_n = x_n + y_n, \quad \forall n \geq 0, \quad (14)$$

also satisfies the linear recursion (12).

Definition 0.4 The characteristic polynomial $P(z)$ corresponding to the linear recursion $\sum_{i=0}^k a_i x_{n+i} = 0$, for all $n \geq 0$, is defined as

$$P(z) = \sum_{i=0}^k a_i z^i. \quad (15)$$

$P(z)$ is a polynomial of degree k , i.e., $\deg(P(z)) = k$. If $P(z)$ has p different roots, λ_j , $j = 1 : p$, with $p \leq k$, and if $m(\lambda_j)$ denotes the multiplicity of the root λ_j , then $\sum_{j=1}^p m(\lambda_j) = k$ where λ_j can be a complex number.

Theorem 0.1 Let $(x_n)_{n \geq 0}$ be a sequence satisfying the linear recursion

$$\sum_{i=0}^k a_i x_{n+i} = 0, \quad \forall n \geq 0, \quad (16)$$

with $a_k \neq 0$, and let $P(z) = \sum_{i=0}^{k-1} a_i z^i$ be the characteristic polynomial associated with recursion (16). Let λ_j , $j = 1 : p$, where $p \leq k$, be the roots of $P(z)$, and let $m(\lambda_j)$ be the multiplicity of λ_j . The general form of the sequence $(x_n)_{n \geq 0}$ satisfying the linear recursion (16) is

$$x_n = \sum_{j=1}^p \left(\sum_{i=0}^{m(\lambda_j)-1} C_{i,j} n^i \right) \lambda_j^n, \quad \forall n \geq 0, \quad (17)$$

where $C_{i,j}$ are constant numbers.

0.4 The “Big O” and “little o” notations

Definition 0.5 Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$. We write that $f(x) = O(g(x))$, as $x \rightarrow \infty$, iff there exist constants $C > 0$ and $M > 0$ such that $\left| \frac{f(x)}{g(x)} \right| \leq C$, for any $x \geq M$. This can be written equivalently as

$$f(x) = O(g(x)), \quad \text{as } x \rightarrow \infty, \quad \text{iff} \quad \limsup_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| < \infty. \quad (18)$$

The “little o” notation refers to functions whose ratios tend to 0 at certain points, and can be defined for $x \rightarrow \infty$, $x \rightarrow a$, and $x \rightarrow -\infty$ as follows:

Definition 0.6 Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$. Then

$$f(x) = o(g(x)), \quad \text{as } x \rightarrow \infty, \quad \text{iff} \quad \lim_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| = 0; \quad (19)$$

$$f(x) = o(g(x)), \quad \text{as } x \rightarrow -\infty, \quad \text{iff} \quad \lim_{x \rightarrow -\infty} \left| \frac{f(x)}{g(x)} \right| = 0; \quad (20)$$

$$f(x) = o(g(x)), \quad \text{as } x \rightarrow a, \quad \text{iff} \quad \lim_{x \rightarrow a} \left| \frac{f(x)}{g(x)} \right| = 0. \quad (21)$$

0.5 Exercises

0.5.1 Question 1

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an odd function.

Part (i)

Show that $xf(x)$ is an even function and $x^2f(x)$ is an odd function.

Answer Let $g(x) = xf(x)$, using definition (6), we have

$$\begin{aligned} g(-x) &= -xf(-x) \\ &= -x(-f(x)) \\ &= xf(x) \\ &= g(x). \end{aligned}$$

Therefore, $xf(x)$ is an even function.

Let $g(x) = x^2f(x)$, using definition (6), we have

$$\begin{aligned} g(-x) &= (-x)^2f(-x) \\ &= x^2(-f(x)) \\ &= -x^2f(x) \\ &= -g(x) \end{aligned}$$

Therefore, $x^2f(x)$ is an odd function.

Part (ii)

Show that the function $g_1 : \mathbb{R} \rightarrow \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function and that the function $g_2 : \mathbb{R} \rightarrow \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function.

Answer

$$\begin{aligned}g_1(-x) &= f((-x)^2) \\&= f(x^2) \\&= g_1(x)\end{aligned}$$

Therefore g_1 is an even function.

Let $y = x^3$:

$$\begin{aligned}g_2(-x) &= f((-x)^3) \\&= f(-x^3) \\&= f(-y) \\&= -f(y) \\&= -f(x^3) \\&= -g_2(x)\end{aligned}$$

Therefore g_2 is an odd function.

Part (iii)

Let i be even, j be odd, and $y = x^j$:

$$\begin{aligned}h(-x) &= (-x)^i f((-x)^j) \\&= x^i f(-x^j) \\&= x^i f(-y) \\&= -x^i f(y) \\&= -x^i f(x^j) \\&= -h(x)\end{aligned}$$

Let i be odd, j be even:

$$\begin{aligned}h(-x) &= (-x)^i f((-x)^j) \\&= -x^i f(x^j) \\&= -h(x)\end{aligned}$$

When $i + j$ is odd, $h(x)$ is an odd function.

0.5.2 Question 2

Let $S(n, 2) = \sum_{k=1}^n k^2$ and $S(n, 3) = \sum_{k=1}^n k^3$.

Part (i)

Let $T(n, 2, x) = \sum_{k=1}^n k^2 x^k$. Use formulas,

$$T(n, 2, x) = x \frac{d}{dx} (T(n, 1, x)),$$

and

$$T(n, 1, x) = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2},$$

to show that

$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}.$$

Answer Using quotient rule,

$$\begin{aligned}
T(n, 2, x) &= x \frac{d}{dx} (T(n, 1, x)) \\
&= x \frac{d}{dx} \left(\frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} \right) \\
&= x \left(\frac{\frac{d}{dx}(x - (n+1)x^{n+1} + nx^{n+2})(1-x)^2 - (x - (n+1)x^{n+1} + nx^{n+2})\frac{d}{dx}(1-x)^2}{(1-x)^4} \right) \\
&= x \left(\frac{(1 - (n+1)^2x^n + (n^2 + 2n)x^{n+1})(1-x)^2 - (x - (n+1)x^{n+1} + nx^{n+2})(-2(1-x))}{(1-x)^4} \right) \\
&= x \left(\frac{(1 - (n+1)^2x^n + (n^2 + 2n)x^{n+1})(1-x)^2 + 2(x - (n+1)x^{n+1} + nx^{n+2})(1-x)}{(1-x)^4} \right) \\
&= x \left(\frac{(1 - (n+1)^2x^n + (n^2 + 2n)x^{n+1})(1-x) + 2(x - (n+1)x^{n+1} + nx^{n+2})}{(1-x)^3} \right) \\
&= x \left(\frac{(1 - (n+1)^2x^n + (n^2 + 2n)x^{n+1}) - (x - (n+1)^2x^{n+1} + (n^2 + 2n)x^{n+2}) + \dots}{(1-x)^3} \right) \\
&= x \left(\frac{1 - x - (n+1)^2x^n + (2n^2 + 4n + 1)x^{n+1} - (n^2 + 2n)x^{n+2} + \dots}{(1-x)^3} \right) \\
&= x \left(\frac{1 + x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2}}{(1-x)^3} \right) \\
&= \frac{x + x^2 - (n+1)^2x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2x^{n+3}}{(1-x)^3}
\end{aligned}$$

Part (ii)

Note that $S(n, 2) = T(n, 2, 1)$. Use l'Hôpital's rule to evaluate $T(n, 2, 1)$, and conclude that $S(n, 2) = \frac{n(n+1)(2n+1)}{6}$.

Answer

$$\begin{aligned}
\lim_{x \rightarrow 1} T(n, 2, x) &= \lim_{x \rightarrow 1} \frac{x + x^2 - (n+1)^2x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2x^{n+3}}{(1-x)^3} \\
&= \lim_{x \rightarrow 1} \frac{(n-1)n(n+1)^3x^{n-2} - n(n+1)(n+2)(2n^2 + 2n - 1)x^{n-1} + (n+1)(n+2)(n+3)n^2x^n}{6} \\
&= \frac{(n-1)n(n+1)^3 - n(n+1)(n+2)(2n^2 + 2n - 1) + (n+1)(n+2)(n+3)n^2}{6} \\
&= \frac{n(n+1)(2n+1)}{6} \\
S(n, 2) &= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Part (iii)

Compute $T(n, 3, x) = \sum_{k=1}^n k^3 x^k$ using the formula

$$T(n, 3, x) = x \frac{d}{dx} (T(n, 2, x)).$$

Answer

$$\begin{aligned}
 T(n, 3, x) &= x \frac{d}{dx} (T(n, 2, x)) \\
 &= x \frac{d}{dx} \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \\
 &= \frac{x((n^3(x-1)^3 - 3n^2(x-1)^2 + 3n(x^2-1) - x(x+4) - 1)x^n + (x+4)x + 1)}{(1-x)^4}
 \end{aligned}$$

Part (iv)

Note that $S(n, 3) = T(n, 3, 1)$. Use l'Hôpital's rule to evaluate $T(n, 3, 1)$, and conclude that $S(n, 3) = \left(\frac{n(n+1)}{2}\right)^2$.

$$\lim_{x \rightarrow 1} T(n, 3, x) = \lim_{x \rightarrow 1} \frac{x((n^3(x-1)^3 - 3n^2(x-1)^2 + 3n(x^2-1) - x(x+4) - 1)x^n + (x+4)x + 1)}{(1-x)^4}$$

0.5.3 Question 3

Compute $S(n, 4) = \sum_{k=1}^n k^4$ using the recursion formula for $i = 4$, the fact that $S(n, 0) = n$, and formulas for $S(n, 1)$, $S(n, 2)$, and $S(n, 3)$.

Answer

$$\begin{aligned}
 S(n, 4) &= \frac{1}{5} \left((n+1)^5 - 1 - \sum_{j=0}^3 \binom{5}{j} S(n, j) \right) \\
 &= \frac{1}{5} ((n+1)^5 - 1 - S(n, 0) - 5S(n, 1) - 10S(n, 2) - 10S(n, 3)) \\
 &= \frac{1}{5} \left((n+1)^5 - 1 - n - 5 \frac{n(n+1)}{2} - 10 \frac{n(n+1)(2n+1)}{6} - 10 \left(\frac{n(n+1)}{2} \right)^2 \right) \\
 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}
 \end{aligned}$$

0.5.4 Question 4

It is easy to see that the sequence $(x_n)_{n \geq 1}$ given by $x_n = \sum_{k=1}^n k^2$ satisfies the recursion

$$x_{n+1} = x_n + (n+1)^2, \quad \forall n \geq 1, \tag{22}$$

with $x_1 = 1$.

Part (i)

By substituting $n+1$ for n in (22), obtain that

$$x_{n+2} = x_{n+1} + (n+2)^2. \tag{23}$$

Subtract (22) from (23) to find that

$$x_{n+2} = 2x_{n+1} - x_n + 2n + 3, \quad \forall n \geq 1, \tag{24}$$

with $x_1 = 1$ and $x_2 = 5$.

Answer

$$\begin{aligned}x_{(n+1)+1} &= x_{n+1}((n+1)+1)^2 \\x_{n+2} &= x_{n+1}(n+2)^2\end{aligned}$$

Subtract (22) from (23):

$$\begin{aligned}x_{n+2} - x_{n+1} &= x_{n+1} + (n+2)^2 - x_n - (n+1)^2 \\x_{n+2} &= 2x_{n+1} - x_n + n^2 + 4n + 4 - n^2 - 2n - 1 \\&= 2x_{n+1} - x_n + 2n + 3\end{aligned}$$

Part (ii)

Similarly, substitute $n+1$ for n in (24) and obtain that

$$x_{n+3} = 2x_{n+2} - x_{n+1} + 2(n+1) + 3. \quad (25)$$

Subtract (24) from (25) to find that

$$x_{n+3} = 3x_{n+2} - 3x_{n+1} + x_n + 2, \quad \forall n \geq 1, \quad (26)$$

with $x_1 = 1$, $x_2 = 5$, and $x_3 = 14$.

Answer

$$\begin{aligned}x_{(n+1)+2} &= 2x_{(n+1)+1} - x_{n+1} + 2(n+1) + 3 \\x_{n+3} &= 2x_{n+2} - x_{n+1} + 2(n+1) + 3\end{aligned}$$

Subtract (24) from (25)

$$\begin{aligned}x_{n+3} - x_{n+2} &= 2x_{n+2} - x_{n+1} + 2(n+1) + 3 - 2x_{n+1} + x_n - 2n - 3 \\x_{n+3} &= 3x_{n+2} - 3x_{n+1} + x_n + 2\end{aligned}$$

Part (iii)

Use a similar method to prove that the sequence $(x_n)_{n \geq 0}$ satisfies the linear recursion

$$x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n = 0, \quad \forall n \geq 1. \quad (27)$$

The characteristic polynomial associated to the recursion (27) is

$$P(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 = (z-1)^4.$$

Use the fact that $x_1 = 1$, $x_2 = 5$, $x_3 = 14$, and $x_4 = 30$ to show that

$$x_n = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1,$$

and conclude that

$$S(n, 2) = \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1.$$

Answer Substitute $n+1$ for n in (26) to obtain

$$x_{n+4} = 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2. \quad (28)$$

Subtract (26) from (28) to obtain that

$$\begin{aligned}x_{n+4} - x_{n+3} &= 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2 - (3x_{n+2} - 3x_{n+1} + x_n + 2) \\x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} - x_n &= 0\end{aligned}$$

The characteristic polynomial has root $\lambda = 1$ with multiplicity 4. The linear recursion can be expressed as

$$\begin{aligned} x_n &= \sum_{j=1}^p \left(\sum_{i=0}^3 C_{i,j} n^i \right) \lambda_j^n \\ &= \sum_{j=1}^p (C_{0,j} + C_{1,j}n + C_{2,j}n^2 + C_{3,j}n^3) \lambda_j^n \\ &= C_1 + C_2n + C_3n^2 + C_4n^3 \end{aligned}$$

Since $x_1 = 1$, $x_2 = 5$, $x_3 = 14$, and $x_4 = 30$, C_1 , C_2 , C_3 , and C_4 must solve the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \\ 30 \end{pmatrix}.$$

We obtain that $C_1 = 0$, $C_2 = \frac{1}{6}$, $C_3 = \frac{1}{2}$, and $C_4 = \frac{1}{3}$ and therefore

$$x_n = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} = \frac{n(n+1)(n+2)}{6}$$