# Chapter 0

# Mathematical preliminaries

## 0.1 Even and odd functions

**Definition 0.1** *The function* $f : \mathbb{R} \to \mathbb{R}$ *is an even function iff*

$$f(-x) = f(x), \quad \forall x \in \mathbb{R}. \tag{1}$$

The graph of any even function is symmetric with respect to the $y$-axis.

**Lemma 0.1** *Let* $f(x)$ *be an integrable even function. Then,*

$$\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(x)dx, \quad \forall a \in \mathbb{R}, \tag{2}$$

*and therefore*

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx, \quad \forall a \in \mathbb{R}. \tag{3}$$

*Moreover, if* $\int_0^\infty f(x)dx$ *exists, then*

$$\int_{-\infty}^{0} f(x)dx = \int_{0}^{\infty} f(x)dx, \tag{4}$$

*and*

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx. \tag{5}$$

**Definition 0.2** *The function* $f : \mathbb{R} \to \mathbb{R}$ *is an odd function iff*

$$f(-x) = -f(x), \quad \forall x \in \mathbb{R}. \tag{6}$$

If we let $x = 0$ in (6), we find that $f(0) = 0$ for any odd function $f(x)$. Also, the graph of any odd function is symmetric with respect to the point (0, 0).

**Lemma 0.2** *Let* $f(x)$ *be an integrable odd function. Then,*

$$\int_{-a}^{a} f(x)dx = 0, \quad \forall a \in \mathbb{R}. \tag{7}$$

*Moreover, if* $\int_0^\infty f(x)dx$ *exists, then*

$$\int_{-\infty}^{\infty} f(x)dx = 0. \tag{8}$$

## 0.2 Useful sums with interesting proofs

The following sums occur frequently when estimating operation counts of numerical algorithms:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \tag{9}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \tag{10}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{11}$$

## 0.3 Sequences satisfying linear recursions

**Definition 0.3** *A sequence $(x_n)_{n\geq 0}$ satisfies a linear recursion of order $k$ iff there exist constants $a_i$, $i = 0 : k$ with $a_k \neq 0$, such that*

$$\sum_{i=0}^{k} a_i x_{n+i} = 0, \quad \forall n \geq 0. \tag{12}$$

The recursion (12) is called a linear recursion because of the following linearity properties:
(i) If the sequence $(x_n)_{n\geq 0}$ satisfies the linear recursion (12), then the sequence $(z_n)_{n\geq 0}$ given by

$$z_n = C x_n, \quad \forall n \geq 0, \tag{13}$$

where $C$ is an arbitrary constant, also satisfies the linear recursion (12).
(ii) If the sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ satisfies the linear recursion (12), then the sequence $(z_n)_{n\geq 0}$ given by

$$z_n = x_n + y_n, \quad \forall n \geq 0, \tag{14}$$

also satisfies the linear recursion (12).

**Definition 0.4** *The characteristic polynomial $P(z)$ corresponding to the linear recursion $\sum_{i=0}^{k} a_i x_{n+i} = 0$, for all $n \geq 0$, is defined as*

$$P(z) = \sum_{i=0}^{k} a_i z^i. \tag{15}$$

$P(z)$ is a polynomial of degree $k$, i.e., $\deg(P(z)) = k$. If $P(z)$ has $p$ different roots, $\lambda_j$, $j = 1 : p$, with $p \leq k$, and if $m(\lambda_j)$ denotes the multiplicity of the root $\lambda_j$, then $\sum_{j=1}^{p} m(\lambda_j) = k$ where $\lambda_j$ can be a complex number.

**Theorem 0.1** *Let $(x_n)_{x\geq 0}$ be a sequence satisfying the linear recursion*

$$\sum_{i=0}^{k} a_i x_{n+i} = 0, \quad \forall n \geq 0, \tag{16}$$

*with $a_k \neq 0$, and let $P(z) = \sum_{i=0}^{k-1} a_i z^i$ be the characteristic polynomial associated with recursion (16). Let $\lambda_j$, $j = 1 : p$, where $p \leq k$, be the roots of $P(z)$, and let $m(\lambda_j)$ be the multiplicity of $\lambda_j$. The general form of the sequence $(x_n)_{n\geq 0}$ satisfying the linear recursion (16) is*

$$x_n = \sum_{j=1}^{p} \left( \sum_{i=0}^{m(\lambda_j)-1} C_{i,j} n^i \right) \lambda_j^n, \quad \forall n \geq 0, \tag{17}$$

*where $C_{i,j}$ are constant numbers.*

## 0.4 The “Big O” and “little o” notations

**Definition 0.5** *Let $f, g : \mathbb{R} \to \mathbb{R}$. We write that $f(x) = O(g(x)$, as $x \to \infty$, iff there exist constants $C > 0$ and $M > 0$ such that $\left|\frac{f(x)}{g(x)} \leq C\right|$, for any $x \geq M$. This can be written equivalently as*

$$f(x) = O(g(x)), \quad \text{as} \quad x \to \infty, \quad \text{iff} \quad \limsup_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| < \infty. \tag{18}$$

The “little o” notation refers to functions whose ratios tend to 0 at certain fpoints, and can be defined for $x \to \infty$, $x \to a$, and $x \to -\infty$ as follows:

**Definition 0.6** *Let $f, g : \mathbb{R} \to \mathbb{R}$. Then*

$$f(x) = o(g(x)), \quad \text{as} \quad x \to \infty, \quad \text{iff} \quad \lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = 0; \tag{19}$$

$$f(x) = o(g(x)), \quad \text{as} \quad x \to -\infty, \quad \text{iff} \quad \lim_{x\to-\infty} \left|\frac{f(x)}{g(x)}\right| = 0; \tag{20}$$

$$f(x) = o(g(x)), \quad \text{as} \quad x \to a, \quad \text{iff} \quad \lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| = 0. \tag{21}$$

## 0.5 Exercises

### 0.5.1 Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function.

#### Part (i)

Show that $xf(x)$ is an even function and $x^2 f(x)$ is an odd function.

**Answer** Let $g(x) = xf(x)$, using definition (6), we have

$$\begin{aligned} g(-x) &= -xf(-x) \\ &= -x(-f(x)) \\ &= xf(x) \\ &= g(x). \end{aligned}$$

Therefore, $xf(x)$ is an even function.

Let $g(x) = x^2 f(x)$, using definition (6), we have

$$\begin{aligned} g(-x) &= (-x)^2 f(-x) \\ &= x^2(-f(x)) \\ &= -x^2 f(x) \\ &= -g(x) \end{aligned}$$

Therefore, $x^2 f(x)$ is an odd function.

#### Part (ii)

Show that the function $g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_1(x) = f(x^2)$ is an even function and that the function $g_2 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = f(x^3)$ is an odd function.

**Answer**

$$\begin{aligned} g_1(-x) &= f((-x)^2) \\ &= f(x^2) \\ &= g_1(x) \end{aligned}$$

Therefore $g_1$ is an even function.

Let $y = x^3$:

$$\begin{aligned} g_2(-x) &= f((-x)^3) \\ &= f(-x^3) \\ &= f(-y) \\ &= -f(y) \\ &= -f(x^3) \\ &= -g_2(x) \end{aligned}$$

Therefore $g_2$ is an odd function.

**Part (iii)**

Let $i$ be even, $j$ be odd, and $y = x^j$:

$$\begin{aligned} h(-x) &= (-x)^i f((-x)^j) \\ &= x^i f(-x^j) \\ &= x^i f(-y) \\ &= -x^i f(y) \\ &= -x^i f(x^j) \\ &= -h(x) \end{aligned}$$

Let $i$ be odd, $j$ be even:

$$\begin{aligned} h(-x) &= (-x)^i f((-x)^j) \\ &= -x^i f(x^j) \\ &= -h(x) \end{aligned}$$

When $i + j$ is odd, $h(x)$ is an odd function.

## 0.5.2 Question 2

Let $S(n, 2) = \sum_{k=1}^{n} k^2$ and $S(n, 3) = \sum_{k=1}^{n} k^3$.

**Part (i)**

Let $T(n, 2, x) = \sum_{k=1}^{n} k^2 x^k$. Use formulas,

$$T(n, 2, x) = x\frac{d}{dx}(T(n, 1, x)),$$

and

$$T(n, 1, x) = \frac{x - (n + 1)x^{n+1} + nx^{n+2}}{(1 - x)^2},$$

to show that

$$T(n, 2, x) = \frac{x + x^2 - (n + 1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1 - x)^3}.$$

**Answer** Using quotient rule,

$$
\begin{aligned}
T(n,2,x) &= x\frac{d}{dx}(T(n,1,x)) \\
&= x\frac{d}{dx}\left(\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}\right) \\
&= x\left(\frac{\frac{d}{dx}(x-(n+1)x^{n+1}+nx^{n+2})(1-x)^2-(x-(n+1)x^{n+1}+nx^{n+2})\frac{d}{dx}(1-x)^2}{(1-x)^4}\right) \\
&= x\left(\frac{(1-(n+1)^2x^n+(n^2+2n)x^{n+1})(1-x)^2-(x-(n+1)x^{n+1}+nx^{n+2})(-2(1-x))}{(1-x)^4}\right) \\
&= x\left(\frac{(1-(n+1)^2x^n+(n^2+2n)x^{n+1})(1-x)^2+2(x-(n+1)x^{n+1}+nx^{n+2})(1-x)}{(1-x)^4}\right) \\
&= x\left(\frac{(1-(n+1)^2x^n+(n^2+2n)x^{n+1})(1-x)+2(x-(n+1)x^{n+1}+nx^{n+2})}{(1-x)^3}\right) \\
&= x\left(\frac{(1-(n+1)^2x^n+(n^2+2n)x^{n+1})-(x-(n+1)^2x^{n+1}+(n^2+2n)x^{n+2})+\cdots}{(1-x)^3}\right) \\
&= x\left(\frac{1-x-(n+1)^2x^n+(2n^2+4n+1)x^{n+1}-(n^2+2n)x^{n+2}+\cdots}{(1-x)^3}\right) \\
&= x\left(\frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}\right) \\
&= \frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3}
\end{aligned}
$$

## Part (ii)

Note that $S(n,2)=T(n,2,1)$. Use l'Hôpitals's rule to evaluate $T(n,2,1)$, and conclude that $S(n,2)=\frac{n(n+1)(2n+1)}{6}$.

**Answer**

$$
\begin{aligned}
\lim_{x\to 1} T(n,2,x) &= \lim_{x\to 1}\frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3} \\
&= \lim_{x\to 1}\frac{(n-1)n(n+1)^3x^{n-2}-n(n+1)(n+2)(2n^2+2n-1)x^{n-1}+(n+1)(n+2)(n+3)n^2x^n}{6} \\
&= \frac{(n-1)n(n+1)^3-n(n+1)(n+2)(2n^2+2n-1)+(n+1)(n+2)(n+3)n^2}{6} \\
&= \frac{n(n+1)(2n+1)}{6} \\
S(n,2) &= \frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$

## Part (iii)

Compute $T(n,3,x)=\sum_{k=1}^{n}k^3x^k$ using the formula

$$T(n,3,x)=x\frac{d}{dx}(T(n,2,x)).$$

**Answer**

$$\begin{aligned}
T(n,3,x) &= x\frac{d}{dx}(T(n,2,x)) \\
&= x\frac{d}{dx}\frac{x+x^2-(n+1)^2x^{n+1}+(2n^2+2n-1)x^{n+2}-n^2x^{n+3}}{(1-x)^3} \\
&= \frac{x((n^3(x-1)^3-3n^2(x-1)^2+3n(x^2-1)-x(x+4)-1)x^n+(x+4)x+1)}{(1-x)^4}
\end{aligned}$$

**Part (iv)**

Note that $S(n,3) = T(n,3,1)$. Use l'Hôpital's rule to evaluate $T(n,3,1)$, and conclude that $S(n,3) = \left(\frac{n(n+1)}{2}\right)^2$.

$$\lim_{x\to 1} T(n,3,x) = \lim_{x\to 1}\frac{x((n^3(x-1)^3-3n^2(x-1)^2+3n(x^2-1)-x(x+4)-1)x^n+(x+4)x+1)}{(1-x)^4}$$

### 0.5.3 Question 3

Compute $S(n,4) = \sum_{k=1}^{n} k^4$ using the recursion formula for $i=4$, the fact that $S(n,0)=n$, and formulas for $S(n,1)$, $S(n,2)$, and $S(n,3)$.

**Answer**

$$\begin{aligned}
S(n,4) &= \frac{1}{5}\left((n+1)^5-1-\sum_{j=0}^{3}\binom{5}{j}S(n,j)\right) \\
&= \frac{1}{5}\left((n+1)^5-1-S(n,0)-5S(n,1)-10S(n,2)-10S(n,3)\right) \\
&= \frac{1}{5}\left((n+1)^5-1-n-5\frac{n(n+1)}{2}-10\frac{n(n+1)(2n+1)}{6}-10\left(\frac{n(n+1)}{2}\right)^2\right) \\
&= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}
\end{aligned}$$

### 0.5.4 Question 4

It is easy to see that the sequence $(x_n)_{n\geq 1}$ given by $x_n = \sum_{k=1}^{n} k^2$ satisfies the recursion

$$x_{n+1} = x_n + (n+1)^2, \quad \forall n \geq 1, \tag{22}$$

with $x_1 = 1$.

**Part (i)**

By substituting $n+1$ for $n$ in (22), obtain that

$$x_{n+2} = x_{n+1} + (n+2)^2. \tag{23}$$

Substract (22) from (23) to find that

$$x_{n+2} = 2x_{n+1} - x_n + 2n + 3, \quad \forall n \geq 1, \tag{24}$$

with $x_1 = 1$ and $x_2 = 5$.

**Answer**

$$\begin{aligned} x_{(n+1)+1} &= x_{n+1}((n+1)+1)^2 \\ x_{n+2} &= x_{n+1}(n+2)^2 \end{aligned}$$

Substract (22) from (23):

$$\begin{aligned} x_{n+2} - x_{n+1} &= x_{n+1} + (n+2)^2 - x_n - (n+1)^2 \\ x_{n+2} &= 2x_{n+1} - x_n + n^2 + 4n + 4 - n^2 - 2n - 1 \\ &= 2x_{n+1} - x_n + 2n + 3 \end{aligned}$$

**Part (ii)**

Similarly, substitute $n+1$ for $n$ in (24) and obtain that

$$x_{n+3} = 2x_{n+2} - x_{n+1} + 2(n+1) + 3. \tag{25}$$

Substract (24) from (25) to find that

$$x_{n+3} = 3x_{n+2} - 3x_{n+1} + x_n + 2, \quad \forall n \geq 1, \tag{26}$$

with $x_1 = 1$, $x_2 = 5$, and $x_3 = 14$.

**Answer**

$$\begin{aligned} x_{(n+1)+2} &= 2x_{(n+1)+1} - x_{n+1} + 2(n+1) + 3 \\ x_{n+3} &= 2x_{n+2} - x_{n+1} + 2(n+1) + 3 \end{aligned}$$

Substract (24) from (25)

$$\begin{aligned} x_{n+3} - x_{n+2} &= 2x_{n+2} - x_{n+1} + 2(n+1) + 3 - 2x_{n+1} + x_n - 2n - 3 \\ x_{n+3} &= 3x_{n+2} - 3x_{n+1} + x_n + 2 \end{aligned}$$

**Part (iii)**

Use a similar method to prove that the sequence $(x_n)_{n\geq 0}$ satisfies the linear recursion

$$x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n = 0, \quad \forall n \geq 1. \tag{27}$$

The characteristic polynomial associated to the recursion (27) is

$$P(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 = (z-1)^4.$$

Use the fact that $x_1 = 1$, $x_2 = 5$, $x_3 = 14$, and $x_4 = 30$ to show that

$$x_n = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1,$$

and conclude that

$$S(n,2) = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1.$$

**Answer** Substitute $n+1$ for $n$ in (26) to obtain

$$x_{n+4} = 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2. \tag{28}$$

Substract (26) from (28) to obtain that

$$\begin{aligned} x_{n+4} - x_{n+3} &= 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2 - (3x_{n+2} - 3x_{n+1} + x_n + 2) \\ x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} - x_n = 0 \end{aligned}$$

The characteristic polynomial has root $\lambda = 1$ with multiplicity 4. There exist constants $C_i$, $i = 1:4$, such that

$$x_n = C_1 + C_2 n + C_3 n^2 + C_4 n^3.$$

Since $x_1 = 1$, $x_2 = 5$, $x_3 = 14$, and $x_4 = 30$, $C_1$, $C_2$, $C_3$, and $C_4$ must solve the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \\ 30 \end{pmatrix}.$$

We obtain that $C_1 = 0$, $C_2 = \frac{1}{6}$, $C_3 = \frac{1}{2}$, and $C_4 = \frac{1}{3}$ and therefore

$$x_n = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} = \frac{n(n+1)(n+2)}{6}$$