

# Chapter 0

## Mathematical preliminaries

### 0.1 Even and odd functions

**Definition 0.1** *The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is an even function iff*

$$f(-x) = f(x), \quad \forall x \in \mathbb{R}. \quad (1)$$

The graph of any even function is symmetric with respect to the  $y$ -axis.

**Lemma 0.1** *Let  $f(x)$  be an integrable even function. Then,*

$$\int_{-a}^0 f(x)dx = \int_0^a f(x)dx, \quad \forall a \in \mathbb{R}, \quad (2)$$

*and therefore*

$$\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx, \quad \forall a \in \mathbb{R}. \quad (3)$$

*Moreover, if  $\int_0^\infty f(x)dx$  exists, then*

$$\int_{-\infty}^0 f(x)dx = \int_0^\infty f(x)dx, \quad (4)$$

*and*

$$\int_{-\infty}^\infty f(x)dx = 2 \int_0^\infty f(x)dx. \quad (5)$$

**Definition 0.2** *The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is an odd function iff*

$$f(-x) = -f(x), \quad \forall x \in \mathbb{R}. \quad (6)$$

If we let  $x = 0$  in (6), we find that  $f(0) = 0$  for any odd function  $f(x)$ . Also, the graph of any odd function is symmetric with respect to the point  $(0, 0)$ .

**Lemma 0.2** *Let  $f(x)$  be an integrable odd function. Then,*

$$\int_{-a}^a f(x)dx = 0, \quad \forall a \in \mathbb{R}. \quad (7)$$

*Moreover, if  $\int_0^\infty f(x)dx$  exists, then*

$$\int_{-\infty}^\infty f(x)dx = 0. \quad (8)$$

## 0.2 Useful sums with interesting proofs

The following sums occur frequently when estimating operation counts of numerical algorithms:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}; \quad (9)$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}; \quad (10)$$

$$\sum_{k=1}^n k^3 = \left( \frac{n(n+1)}{2} \right)^2. \quad (11)$$

## 0.3 Sequences satisfying linear recursions

**Definition 0.3** A sequence  $(x_n)_{n \geq 0}$  satisfies a linear recursion of order  $k$  iff there exist constants  $a_i$ ,  $i = 0 : k$  with  $a_k \neq 0$ , such that

$$\sum_{i=0}^k a_i x_{n+i} = 0, \quad \forall n \geq 0. \quad (12)$$

The recursion (12) is called a linear recursion because of the following linearity properties:

(i) If the sequence  $(x_n)_{n \geq 0}$  satisfies the linear recursion (12), then the sequence  $(z_n)_{n \geq 0}$  given by

$$z_n = Cx_n, \quad \forall n \geq 0, \quad (13)$$

where  $C$  is an arbitrary constant, also satisfies the linear recursion (12).

(ii) If the sequences  $(x_n)_{n \geq 0}$  and  $(y_n)_{n \geq 0}$  satisfies the linear recursion (12), then the sequence  $(z_n)_{n \geq 0}$  given by

$$z_n = x_n + y_n, \quad \forall n \geq 0, \quad (14)$$

also satisfies the linear recursion (12).

**Definition 0.4** The characteristic polynomial  $P(z)$  corresponding to the linear recursion  $\sum_{i=0}^k a_i x_{n+i} = 0$ , for all  $n \geq 0$ , is defined as

$$P(z) = \sum_{i=0}^k a_i z^i. \quad (15)$$

$P(z)$  is a polynomial of degree  $k$ , i.e.,  $\deg(P(z)) = k$ . If  $P(z)$  has  $p$  different roots,  $\lambda_j$ ,  $j = 1 : p$ , with  $p \leq k$ , and if  $m(\lambda_j)$  denotes the multiplicity of the root  $\lambda_j$ , then  $\sum_{j=1}^p m(\lambda_j) = k$  where  $\lambda_j$  can be a complex number.

**Theorem 0.1** Let  $(x_n)_{n \geq 0}$  be a sequence satisfying the linear recursion

$$\sum_{i=0}^k a_i x_{n+i} = 0, \quad \forall n \geq 0, \quad (16)$$

with  $a_k \neq 0$ , and let  $P(z) = \sum_{i=0}^{k-1} a_i z^i$  be the characteristic polynomial associated with recursion (16). Let  $\lambda_j$ ,  $j = 1 : p$ , where  $p \leq k$ , be the roots of  $P(z)$ , and let  $m(\lambda_j)$  be the multiplicity of  $\lambda_j$ . The general form of the sequence  $(x_n)_{n \geq 0}$  satisfying the linear recursion (16) is

$$x_n = \sum_{j=1}^p \left( \sum_{i=0}^{m(\lambda_j)-1} C_{i,j} n^i \right) \lambda_j^n, \quad \forall n \geq 0, \quad (17)$$

where  $C_{i,j}$  are constant numbers.

## 0.4 The “Big O” and “little o” notations

**Definition 0.5** Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ . We write that  $f(x) = O(g(x))$ , as  $x \rightarrow \infty$ , iff there exist constants  $C > 0$  and  $M > 0$  such that  $\left| \frac{f(x)}{g(x)} \right| \leq C$ , for any  $x \geq M$ . This can be written equivalently as

$$f(x) = O(g(x)), \quad \text{as } x \rightarrow \infty, \quad \text{iff} \quad \limsup_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| < \infty. \quad (18)$$

The “little o” notation refers to functions whose ratios tend to 0 at certain fpoints, and can be defined for  $x \rightarrow \infty$ ,  $x \rightarrow a$ , and  $x \rightarrow -\infty$  as follows:

**Definition 0.6** Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ . Then

$$f(x) = o(g(x)), \quad \text{as } x \rightarrow \infty, \quad \text{iff} \quad \lim_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| = 0; \quad (19)$$

$$f(x) = o(g(x)), \quad \text{as } x \rightarrow -\infty, \quad \text{iff} \quad \lim_{x \rightarrow -\infty} \left| \frac{f(x)}{g(x)} \right| = 0; \quad (20)$$

$$f(x) = o(g(x)), \quad \text{as } x \rightarrow a, \quad \text{iff} \quad \lim_{x \rightarrow a} \left| \frac{f(x)}{g(x)} \right| = 0. \quad (21)$$

## 0.5 Exercises

### 0.5.1 Question 1

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an odd function.

**Part (i)**

Show that  $xf(x)$  is an even function and  $x^2f(x)$  is an odd function.

**Answer** Let  $g(x) = xf(x)$ , using definition (6), we have

$$\begin{aligned} g(-x) &= -xf(-x) \\ &= -x(-f(x)) \\ &= xf(x) \\ &= g(x). \end{aligned}$$

Therefore,  $xf(x)$  is an even function.

Let  $g(x) = x^2f(x)$ , using definition (6), we have

$$\begin{aligned} g(-x) &= (-x)^2f(-x) \\ &= x^2(-f(x)) \\ &= -x^2f(x) \\ &= -g(x) \end{aligned}$$

Therefore,  $x^2f(x)$  is an odd function.

**Part (ii)**

Show that the function  $g_1 : \mathbb{R} \rightarrow \mathbb{R}$  given by  $g_1(x) = f(x^2)$  is an even function and that the function  $g_2 : \mathbb{R} \rightarrow \mathbb{R}$  given by  $g_2(x) = f(x^3)$  is an odd function.

**Answer**

$$\begin{aligned}g_1(-x) &= f((-x)^2) \\&= f(x^2) \\&= g_1(x)\end{aligned}$$

Therefore  $g_1$  is an even function.

Let  $y = x^3$ :

$$\begin{aligned}g_2(-x) &= f((-x)^3) \\&= f(-x^3) \\&= f(-y) \\&= -f(y) \\&= -f(x^3) \\&= -g_2(x)\end{aligned}$$

Therefore  $g_2$  is an odd function.

**Part (iii)**

Let  $i$  be even,  $j$  be odd, and  $y = x^j$ :

$$\begin{aligned}h(-x) &= (-x)^i f((-x)^j) \\&= x^i f(-x^j) \\&= x^i f(-y) \\&= -x^i f(y) \\&= -x^i f(x^j) \\&= -h(x)\end{aligned}$$

Let  $i$  be odd,  $j$  be even:

$$\begin{aligned}h(-x) &= (-x)^i f((-x)^j) \\&= -x^i f(x^j) \\&= -h(x)\end{aligned}$$

When  $i + j$  is odd,  $h(x)$  is an odd function.

### 0.5.2 Question 2

Let  $S(n, 2) = \sum_{k=1}^n k^2$  and  $S(n, 3) = \sum_{k=1}^n k^3$ .

**Part (i)**

Let  $T(n, 2, x) = \sum_{k=1}^n k^2 x^k$ . Use formulas,

$$T(n, 2, x) = x \frac{d}{dx} (T(n, 1, x)),$$

and

$$T(n, 1, x) = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2},$$

to show that

$$T(n, 2, x) = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}.$$

**Answer** Using quotient rule,

$$\begin{aligned}
T(n, 2, x) &= x \frac{d}{dx} (T(n, 1, x)) \\
&= x \frac{d}{dx} \left( \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} \right) \\
&= x \left( \frac{\frac{d}{dx}(x - (n+1)x^{n+1} + nx^{n+2})(1-x)^2 - (x - (n+1)x^{n+1} + nx^{n+2})\frac{d}{dx}(1-x)^2}{(1-x)^4} \right) \\
&= x \left( \frac{(1 - (n+1)^2x^n + (n^2 + 2n)x^{n+1})(1-x)^2 - (x - (n+1)x^{n+1} + nx^{n+2})(-2(1-x))}{(1-x)^4} \right) \\
&= x \left( \frac{(1 - (n+1)^2x^n + (n^2 + 2n)x^{n+1})(1-x)^2 + 2(x - (n+1)x^{n+1} + nx^{n+2})(1-x)}{(1-x)^4} \right) \\
&= x \left( \frac{(1 - (n+1)^2x^n + (n^2 + 2n)x^{n+1})(1-x) + 2(x - (n+1)x^{n+1} + nx^{n+2})}{(1-x)^3} \right) \\
&= x \left( \frac{(1 - (n+1)^2x^n + (n^2 + 2n)x^{n+1}) - (x - (n+1)^2x^{n+1} + (n^2 + 2n)x^{n+2}) + \dots}{(1-x)^3} \right) \\
&= x \left( \frac{1 - x - (n+1)^2x^n + (2n^2 + 4n + 1)x^{n+1} - (n^2 + 2n)x^{n+2} + \dots}{(1-x)^3} \right) \\
&= x \left( \frac{1 + x - (n+1)^2x^n + (2n^2 + 2n - 1)x^{n+1} - n^2x^{n+2}}{(1-x)^3} \right) \\
&= \frac{x + x^2 - (n+1)^2x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2x^{n+3}}{(1-x)^3}
\end{aligned}$$

**Part (ii)**

Note that  $S(n, 2) = T(n, 2, 1)$ . Use l'Hôpital's rule to evaluate  $T(n, 2, 1)$ , and conclude that  $S(n, 2) = \frac{n(n+1)(2n+1)}{6}$ .

**Answer**

$$\begin{aligned}
\lim_{x \rightarrow 1} T(n, 2, x) &= \lim_{x \rightarrow 1} \frac{x + x^2 - (n+1)^2x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2x^{n+3}}{(1-x)^3} \\
&= \lim_{x \rightarrow 1} \frac{(n-1)n(n+1)^3x^{n-2} - n(n+1)(n+2)(2n^2 + 2n - 1)x^{n-1} + (n+1)(n+2)(n+3)n^2x^n}{6} \\
&= \frac{(n-1)n(n+1)^3 - n(n+1)(n+2)(2n^2 + 2n - 1) + (n+1)(n+2)(n+3)n^2}{6} \\
&= \frac{n(n+1)(2n+1)}{6} \\
S(n, 2) &= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

**Part (iii)**

Compute  $T(n, 3, x) = \sum_{k=1}^n k^3 x^k$  using the formula

$$T(n, 3, x) = x \frac{d}{dx} (T(n, 2, x)).$$

**Answer**

$$\begin{aligned}
 T(n, 3, x) &= x \frac{d}{dx} (T(n, 2, x)) \\
 &= x \frac{d}{dx} \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} \\
 &= \frac{x((n^3(x-1)^3 - 3n^2(x-1)^2 + 3n(x^2-1) - x(x+4) - 1)x^n + (x+4)x + 1)}{(1-x)^4}
 \end{aligned}$$

**Part (iv)**

Note that  $S(n, 3) = T(n, 3, 1)$ . Use l'Hôpital's rule to evaluate  $T(n, 3, 1)$ , and conclude that  $S(n, 3) = \left(\frac{n(n+1)}{2}\right)^2$ .

$$\lim_{x \rightarrow 1} T(n, 3, x) = \lim_{x \rightarrow 1} \frac{x((n^3(x-1)^3 - 3n^2(x-1)^2 + 3n(x^2-1) - x(x+4) - 1)x^n + (x+4)x + 1)}{(1-x)^4}$$

### 0.5.3 Question 3

Compute  $S(n, 4) = \sum_{k=1}^n k^4$  using the recursion formula for  $i = 4$ , the fact that  $S(n, 0) = n$ , and formulas for  $S(n, 1)$ ,  $S(n, 2)$ , and  $S(n, 3)$ .

**Answer**

$$\begin{aligned}
 S(n, 4) &= \frac{1}{5} \left( (n+1)^5 - 1 - \sum_{j=0}^3 \binom{5}{j} S(n, j) \right) \\
 &= \frac{1}{5} ((n+1)^5 - 1 - S(n, 0) - 5S(n, 1) - 10S(n, 2) - 10S(n, 3)) \\
 &= \frac{1}{5} \left( (n+1)^5 - 1 - n - 5 \frac{n(n+1)}{2} - 10 \frac{n(n+1)(2n+1)}{6} - 10 \left( \frac{n(n+1)}{2} \right)^2 \right) \\
 &= \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{30}
 \end{aligned}$$

### 0.5.4 Question 4

It is easy to see that the sequence  $(x_n)_{n \geq 1}$  given by  $x_n = \sum_{k=1}^n k^2$  satisfies the recursion

$$x_{n+1} = x_n + (n+1)^2, \quad \forall n \geq 1, \tag{22}$$

with  $x_1 = 1$ .

**Part (i)**

By substituting  $n+1$  for  $n$  in (22), obtain that

$$x_{n+2} = x_{n+1} + (n+2)^2. \tag{23}$$

Subtract (22) from (23) to find that

$$x_{n+2} = 2x_{n+1} - x_n + 2n + 3, \quad \forall n \geq 1, \tag{24}$$

with  $x_1 = 1$  and  $x_2 = 5$ .

**Answer**

$$\begin{aligned}x_{(n+1)+1} &= x_{n+1}((n+1)+1)^2 \\x_{n+2} &= x_{n+1}(n+2)^2\end{aligned}$$

Subtract (22) from (23):

$$\begin{aligned}x_{n+2} - x_{n+1} &= x_{n+1} + (n+2)^2 - x_n - (n+1)^2 \\x_{n+2} &= 2x_{n+1} - x_n + n^2 + 4n + 4 - n^2 - 2n - 1 \\&= 2x_{n+1} - x_n + 2n + 3\end{aligned}$$

**Part (ii)**

Similarly, substitute  $n+1$  for  $n$  in (24) and obtain that

$$x_{n+3} = 2x_{n+2} - x_{n+1} + 2(n+1) + 3. \quad (25)$$

Subtract (24) from (25) to find that

$$x_{n+3} = 3x_{n+2} - 3x_{n+1} + x_n + 2, \quad \forall n \geq 1, \quad (26)$$

with  $x_1 = 1$ ,  $x_2 = 5$ , and  $x_3 = 14$ .

**Answer**

$$\begin{aligned}x_{(n+1)+2} &= 2x_{(n+1)+1} - x_{n+1} + 2(n+1) + 3 \\x_{n+3} &= 2x_{n+2} - x_{n+1} + 2(n+1) + 3\end{aligned}$$

Subtract (24) from (25)

$$\begin{aligned}x_{n+3} - x_{n+2} &= 2x_{n+2} - x_{n+1} + 2(n+1) + 3 - 2x_{n+1} + x_n - 2n - 3 \\x_{n+3} &= 3x_{n+2} - 3x_{n+1} + x_n + 2\end{aligned}$$

**Part (iii)**

Use a similar method to prove that the sequence  $(x_n)_{n \geq 0}$  satisfies the linear recursion

$$x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} + x_n = 0, \quad \forall n \geq 1. \quad (27)$$

The characteristic polynomial associated to the recursion (27) is

$$P(z) = z^4 - 4z^3 + 6z^2 - 4z + 1 = (z-1)^4.$$

Use the fact that  $x_1 = 1$ ,  $x_2 = 5$ ,  $x_3 = 14$ , and  $x_4 = 30$  to show that

$$x_n = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1,$$

and conclude that

$$S(n, 2) = \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \geq 1.$$

**Answer** Substitute  $n+1$  for  $n$  in (26) to obtain

$$x_{n+4} = 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2. \quad (28)$$

Subtract (26) from (28) to obtain that

$$\begin{aligned}x_{n+4} - x_{n+3} &= 3x_{n+3} - 3x_{n+2} + x_{n+1} + 2 - (3x_{n+2} - 3x_{n+1} + x_n + 2) \\x_{n+4} - 4x_{n+3} + 6x_{n+2} - 4x_{n+1} - x_n &= 0\end{aligned}$$

The characteristic polynomial has root  $\lambda = 1$  with multiplicity 4. The linear recursion can be expressed as

$$\begin{aligned} x_n &= \sum_{j=1}^p \left( \sum_{i=0}^3 C_{i,j} n^i \right) \lambda_j^n \\ &= \sum_{j=1}^p (C_{0,j} + C_{1,j}n + C_{2,j}n^2 + C_{3,j}n^3) \lambda_j^n \\ &= C_1 + C_2n + C_3n^2 + C_4n^3 \end{aligned}$$

Since  $x_1 = 1$ ,  $x_2 = 5$ ,  $x_3 = 14$ , and  $x_4 = 30$ ,  $C_1$ ,  $C_2$ ,  $C_3$ , and  $C_4$  must solve the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \\ 30 \end{pmatrix}.$$

We obtain that  $C_1 = 0$ ,  $C_2 = \frac{1}{6}$ ,  $C_3 = \frac{1}{2}$ , and  $C_4 = \frac{1}{3}$  and therefore

$$x_n = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} = \frac{n(n+1)(n+2)}{6}$$

### 0.5.5 Question 5

Find the general form of the sequence  $(x_n)_{n \geq 0}$  satisfying the linear recursion

$$x_{n+3} = 2x_{n+1} + x_n, \quad \forall n \geq 0,$$

with  $x_0 = 1$ ,  $x_1 = -1$ , and  $x_2 = 1$ .

**Answer** Rewrite the recursion in the form (12) as

$$x_{n+3} - 2x_{n+1} - x_n = 0, \quad \forall n \geq 0.$$

The characteristic polynomial associated to the linear recursion is

$$\begin{aligned} P(z) &= z^3 - 2z - 1 \\ &= (z+1)(z^2 - z - 1) \end{aligned}$$

and the roots of  $P(z)$  are

$$\lambda_1 = -1, \quad \lambda_2 = \frac{1 + \sqrt{5}}{2}, \quad \lambda_3 = \frac{1 - \sqrt{5}}{2}.$$

From Theorem 0.1, we find that

$$x_n = C_1 \lambda_1^n + C_2 \lambda_2^n + C_3 \lambda_3^n, \quad \forall n \geq 0.$$

Given  $x_0 = 1$ ,  $x_1 = -1$ , and  $x_2 = 1$ , we obtain the linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

By solving the linear system, we find that  $C_1 = 1$ ,  $C_2 = 0$ , and  $C_3 = 0$ . The general formula for is

$$x_n = (-1)^n, \quad \forall n \geq 0.$$

### 0.5.6 Question 6

The sequence  $(x_n)_{n \geq 0}$  satisfies the recursion

$$x_{n+1} = 3x_n + 2, \quad \forall n \geq 0,$$

with  $x_0 = 1$ .



**Part (i)**

Show that the sequence  $(x_n)_{n \geq 0}$  satisfies the linear recursion

$$x_{n+2} = 4x_{n+1} - 3x_n, \quad \forall n \geq 0,$$

with  $x_0 = 1$  and  $x_1 = 5$ .

**Answer** Substitute  $n + 1$  for  $n$  to obtain

$$x_{n+2} = 3x_{n+1} + 2$$

Subtract the original recursion to get

$$\begin{aligned} x_{n+2} - x_{n+1} &= 3x_{n+1} + 2 - 3x_n - 2 \\ x_{n+2} &= 4x_{n+1} - 3x_n \end{aligned}$$

**Part (ii)**

Find the general formula for  $x_n$ ,  $n \geq 0$ .

**Answer** The characteristic polynomial has the form

$$P(z) = z^2 - 4z + 3 = (z - 1)(z - 3)$$

which has roots  $\lambda_1 = 1$  and  $\lambda_2 = 3$ . We obtain the linear system

$$\begin{cases} C_1 + C_2 = 1; \\ C_1\lambda_1 + C_2\lambda_2 = 5. \end{cases}$$

The solution to the linear system is  $C_1 = -1$  and  $C_2 = 2$ . Therefore, the general form is

$$x_n = 2(3)^n - 1$$

**0.5.7 Question 7**

The sequence  $(x_n)_{n \geq 0}$  satisfies the recursion

$$x_{n+1} = 3x_n + n + 2, \quad \forall n \geq 0,$$

with  $x_0 = 1$ .

**Part (i)**

Show that the sequence  $(x_n)_{n \geq 0}$  satisfies the linear recursion

$$x_{n+3} = 5x_{n+2} - 7x_{n+1} + 3x_n, \quad \forall n \geq 0,$$

with  $x_0 = 1$ ,  $x_1 = 5$ , and  $x_2 = 18$ .

**Answer** Substitute  $n + 1$  for  $n$ , we obtain

$$x_{n+2} = 3x_{n+1} + n + 3$$

Subtract

$$x_{n+2} = 4x_{n+1} - 3x_n + 1$$

Substitute  $n + 1$  for  $n$ , we obtain

$$x_{n+3} = 4x_{n+2} - 3x_{n+1} + 1$$

Subtract

$$x_{n+3} = 5x_{n+2} - 7x_{n+1} + 3x_n$$

**Part (ii)**

Find the general formula for  $x_n$ ,  $n \geq 0$ .

**Answer** The characteristic polynomial is given by

$$P(z) = z^3 - 5z^2 + 7z - 3 = (z - 1)^2(z - 3),$$

with roots  $\lambda_1 = 1$  and  $\lambda_2 = 3$ . The general form is

$$\begin{aligned} x_n &= \sum_{j=1}^2 \left( \sum_{i=0}^{m(\lambda_j)-1} C_{i,j} n^i \right) \lambda_j^n \\ &= \lambda_1^n \sum_{i=0}^1 C_{i,1} n^i + \lambda_2^n C_2 \\ &= \lambda_1^n C_{0,1} + \lambda_1^n C_{1,1} n + \lambda_2^n C_2 \\ &= C_1 + C_2 n + C_3 3^n \end{aligned}$$

Since  $x_0 = 1$ ,  $x_1 = 5$ , and  $x_2 = 18$ , we find  $C_1 = -\frac{1}{2}$ ,  $C_2 = -\frac{5}{4}$ , and  $C_3 = \frac{9}{4}$ . We conclude that

$$x_n = \frac{3^{n+2} - 2n - 5}{4}$$

**0.5.8 Question 8**

Let  $P(z) = \sum_{i=0}^k a_i z^i$  be the characteristic polynomial corresponding to the linear recursion

$$\sum_{i=0}^k a_i x_{n+i} = 0, \quad \forall n \geq 0.$$

Assume that  $\lambda$  is a root of multiplicity 2 of  $P(z)$ . Show that the sequence  $(y_n)_{n \geq 0}$  given by

$$y_n = Cn\lambda^n, \quad n \geq 0,$$

where  $C$  is an arbitrary constant, satisfies the recursion.

**Answer**

$$\begin{aligned} \sum_{i=0}^k a_i y_{n+i} &= \sum_{i=0}^k a_i C(n+i) \lambda^{n+i} \\ &= Cn \sum_{i=0}^k a_i \lambda^{n+i} + C \sum_{i=0}^k a_i i \lambda^{n+i} \\ &= Cn \lambda^n \sum_{i=0}^k a_i \lambda^i + C \lambda^{n+1} \sum_{i=0}^k i a_i \lambda^{i-1} \\ &= Cn \lambda^n P(\lambda) + C \lambda^{n+1} P'(\lambda) \\ &= 0. \end{aligned}$$

**0.5.9 Question 9**

Let  $n > 0$ . Show that

$$\begin{aligned} O(x^n) + O(x^n) &= O(x^n), \quad \text{as } x \rightarrow 0; \\ o(x^n) + o(x^n) &= o(x^n), \quad \text{as } x \rightarrow 0. \end{aligned}$$

**Answer** Let  $f(x) = O(x^n)$  and  $g(x) = O(x^n)$ , then

$$\limsup_{x \rightarrow 0} \left| \frac{f(x)}{x^n} \right| < \infty \quad \text{and} \quad \limsup_{x \rightarrow 0} \left| \frac{g(x)}{x^n} \right| < \infty.$$

We see that

$$\limsup_{x \rightarrow 0} \left| \frac{f(x) + g(x)}{x^n} \right| \leq \limsup_{x \rightarrow 0} \left| \frac{f(x)}{x^n} \right| + \limsup_{x \rightarrow 0} \left| \frac{g(x)}{x^n} \right| < \infty,$$

and therefore  $O(x^n) + O(x^n) = O(x^n)$ .

Let  $f(x) = o(x^n)$  and  $g(x) = o(x^n)$ , then

$$\lim_{x \rightarrow 0} \left| \frac{f(x)}{x^n} \right| = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \left| \frac{g(x)}{x^n} \right| = 0.$$

We see that

$$\lim_{x \rightarrow 0} \left| \frac{f(x) + g(x)}{x^n} \right| \leq \lim_{x \rightarrow 0} \left| \frac{f(x)}{x^n} \right| + \lim_{x \rightarrow 0} \left| \frac{g(x)}{x^n} \right| = 0,$$

and therefore  $o(x^n) + o(x^n) = o(x^n)$ .

## 0.5.10 Question 10

Prove that

$$\begin{aligned} \sum_{k=1}^n k^2 &= O(n^3), \quad \text{as } n \rightarrow \infty; \\ \sum_{k=1}^n k^2 &= \frac{n^3}{3} + O(n^2), \quad \text{as } n \rightarrow \infty, \end{aligned}$$

i.e., show that

$$\limsup_{n \rightarrow \infty} \frac{\sum_{k=1}^n k^2}{n^3} < \infty$$

and that

$$\limsup_{n \rightarrow \infty} \frac{\sum_{k=1}^n k^2 - \frac{n^3}{3}}{n^2} < \infty.$$

Similarly, prove that

$$\begin{aligned} \sum_{k=1}^n k^3 &= O(n^4), \quad \text{as } n \rightarrow \infty; \\ \sum_{k=1}^n k^3 &= \frac{n^4}{4} + O(n^3), \quad \text{as } n \rightarrow \infty, \end{aligned}$$

**Answer** Using (10)

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\sum_{k=1}^n k^2}{n^3} &= \limsup_{n \rightarrow \infty} \frac{\frac{n(n+1)(2n+1)}{6}}{n^3} \\ &= \limsup_{n \rightarrow \infty} \frac{2n^3 + 3n^2 + n}{6n^3} \\ &= \frac{1}{3} < \infty \\ \limsup_{n \rightarrow \infty} \frac{\sum_{k=1}^n k^2 - \frac{n^3}{3}}{n^2} &= \limsup_{n \rightarrow \infty} \frac{\frac{2n^3 + 3n^2 + n}{6} - \frac{n^3}{3}}{n^2} \\ &= \limsup_{n \rightarrow \infty} \frac{3n^2 + n}{6n^2} \\ &= \frac{1}{2} \\ &< \infty \end{aligned}$$

Using (11)

$$\begin{aligned}\limsup_{n \rightarrow \infty} \frac{\sum_{k=1}^n k^3}{n^4} &= \limsup_{n \rightarrow \infty} \frac{\left(\frac{n(n+1)}{2}\right)^2}{n^4} \\ &= \limsup_{n \rightarrow \infty} \frac{n^2(n^2 + 2n + 1)}{4n^4} \\ &= \frac{1}{4} \\ &< \infty\end{aligned}$$

$$\begin{aligned}\limsup_{n \rightarrow \infty} \frac{\sum_{k=1}^n k^3 - \frac{n^4}{4}}{n^3} &= \limsup_{n \rightarrow \infty} \frac{\frac{(n^4 + 2n^3 + n^2)}{4} - \frac{n^4}{4}}{n^3} \\ &= \limsup_{n \rightarrow \infty} \frac{2n^3 + n^2}{4n^3} \\ &= \frac{1}{2} \\ &< \infty\end{aligned}$$