Starting equations

$$\frac{D_{NO}}{r}\frac{d}{dr}\left(r\frac{dC_{NO}}{dr}\right)+R_{NO}=0 \tag{1}$$

Simplify

$$D_{NO}\frac{d^2C_{NO}}{dr^2}+\frac{D_{NO}}{r}\frac{dC_{NO}}{dr}=-R_{NO} \tag{2}$$

Compare to form

$$u''+P(r)u'=F(r) \tag{3}$$

$$u=C_{NO},\ P(r)=\frac{1}{r},\ F(r)=-\frac{R_{NO}}{D_{NO}} \tag{4}$$

Taylor expansion

$$u_{i+1}=u_i+u_i'\Delta r+\frac{1}{2}u_i''\Delta r^2,$$
$$u_{i-1}=u_i-u_i'\Delta r+\frac{1}{2}u_i''\Delta r^2$$

Central difference

$$u_i'=\frac{u_{i+1}-u_{i-1}}{2\Delta r},$$
$$u_i''=\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta r^2}$$

Sub back into Eq. (3)

$$\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}+P_i\frac{u_{i+1}-u_{i-1}}{2h}=F_i,\ h=\Delta r \tag{5}$$

where index i starts at 1.
Rearrange

$$\left(1-\frac{h}{2}P_i\right)u_{i-1}+(-2)u_i+\left(1+\frac{h}{2}P_i\right)u_{i+1}=h^2F_i \tag{6}$$

$$(-2)u_i=-\left(1-\frac{h}{2}P_i\right)u_{i-1}-\left(1+\frac{h}{2}P_i\right)u_{i+1}+h^2F_i \tag{7}$$

$$u_i=-\frac{1}{2}\left(-\left(1-\frac{h}{2}P_i\right)u_{i-1}-\left(1+\frac{h}{2}P_i\right)u_{i+1}+h^2F_i\right) \tag{8}$$

Use imaginary node to approximate Neumann BCs:

$$\frac{u_2-u_0}{2h}=0\rightarrow u_0=u_2 \tag{9}$$

$$\frac{u_{end+1}-u_{end-1}}{2h}=0\rightarrow u_{end+1}=u_{end-1} \tag{10}$$

Substitute into Eq. (3) when $i=1$

$$(-2)u_1=-2u_2+\frac{1}{2}h^2F_1 \tag{11}$$

Substitute into Eq. (3) when $i=end$

$$(-2)u_{end}=-\left(1-\frac{h}{2}P_{end}\right)u_{end-1}-\left(1+\frac{h}{2}P_{end}\right)u_{end+1}+h^2F_{end} \tag{12}$$

$$(-2)u_{end}=-2u_{end-1}+h^2F_{end} \tag{13}$$

Solving over iterations

$$u_i^{k+1}=-\frac{1}{2}\left(-\left(1-\frac{h}{2}P_i\right)u_{i-1}^{k+1}-\left(1+\frac{h}{2}P_i\right)u_{i+1}^{k}+h^2F_i\right) \tag{14}$$

SOR

$$-\frac{2}{\omega}u_i^{k+1}=-\left(1-\frac{h}{2}P_i\right)u_{i-1}^{k+1}-\left(1+\frac{h}{2}P_i\right)u_{i+1}^{k}+h^2F_i-\left(\frac{2}{\omega}-2\right)u_i^k \tag{15}$$

Split into 5 sections. Solutions are in the general form $A\tilde{\mathbf{u}}=B$.
Let $a_i=\frac{h}{2}P_i$.
RBC core $r_0<r<r_1$
Governing equation:

$$\frac{D_{NO}}{r}\frac{d}{dr}\left(r\frac{dC_{NO}}{dr}\right)-\lambda_{core}C_{NO}=0$$

$$F_i=\frac{\lambda_{core}u_i}{D_{NO}}$$

Boundary condition:

$$
\begin{gathered}
u'(0) = 0 \to u_1 = u_2, \\
u'_{RBC}(r_1) = \sigma,\ \sigma = u'_{CFL}(r_1)
\end{gathered}
$$

Using second-order accurate one-sided difference approximation, j represents indexes in CFL domain.

$$
\begin{aligned}
\sigma &= \frac{-3u_j + 4u_{j+1} - u_{j+2}}{2h}, \\
\frac{3u_i - 4u_{i-1} + u_{i-2}}{2h} &= \sigma, \\
3u_i - 4u_{i-1} + u_{i-2} &= 2h\sigma
\end{aligned}
$$

$$
A = \begin{bmatrix}
-2 & 1+a_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1-a_3 & -2 & 1+a_3 & \ddots & & & & \vdots \\
0 & 1-a_4 & -2 & \ddots & \ddots & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & & \ddots & \ddots & -2 & 1+a_{n-3} & 0 & 0 \\
\vdots & & & \ddots & 1-a_{n-2} & -2 & 1+a_{n-2} & 0 \\
\vdots & \cdots & \cdots & \cdots & 0 & 1-a_{n-1} & -2 & 1+a_{n-1} \\
0 & \cdots & \cdots & \cdots & \cdots & 1 & -4 & 3
\end{bmatrix},\ \tilde{\mathbf{u}} = \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix},\ B = \begin{bmatrix} h^2F_2 - (1-a_2)u_1 \\ h^2F_3 \\ \vdots \\ h^2F_i \\ \vdots \\ h^2F_{n-1} \\ 2h\sigma \end{bmatrix} \tag{16}
$$

CFL $r_1 < r < r_2$

Governing equation

$$
\begin{gathered}
\frac{D_{NO}}{r}\frac{d}{dr}\left(r\frac{dC_{NO}}{dr}\right) = 0 \\
F_i = \frac{0}{D_{NO}}
\end{gathered}
$$

Boundary condition:

$$
\begin{gathered}
u_{CFL}(r_1) = u_{RBC}(r_1), \\
u'_{CFL}(r_2) = \phi,\ \phi = u'_{EC}(r_2),
\end{gathered}
$$

Using one-sided difference approximation again, i is CFL, j is RBC.

$$
\begin{aligned}
\phi &= \frac{-3u_k + 4u_{k+1} - u_{k+2}}{2h}, \\
\frac{3u_i - 4u_{i-1} + u_{i-2}}{2h} &= \phi, \\
3u_i - 4u_{i-1} + u_{i-2} &= 2h\phi
\end{aligned}
$$

$$A = \begin{bmatrix} -2 & 1+a_2 & 0 & & & & & 0 \\ 1-a_3 & -2 & 1+a_3 & \ddots & & & & \vdots \\ 0 & 1-a_4 & -2 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & -2 & 1+a_{n-3} & 0 & 0 \\ \vdots & & & \ddots & 1-a_{n-2} & -2 & 1+a_{n-2} & 0 \\ \vdots & \cdots & \cdots & \cdots & 0 & 1-a_{n-1} & -2 & 1+a_{n-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 & -4 & 3 \end{bmatrix}, \ \tilde{\mathbf{u}} = \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix}, \ B = \begin{bmatrix} h^2F_2 - (1-a_2)u_1 \\ h^2F_3 \\ \vdots \\ h^2F_i \\ \vdots \\ h^2F_{n-1} \\ 2h\phi \end{bmatrix} \tag{17}$$

EC $r_2 < r < r_3$ is same method as CFL, only difference is in the R_{NO}term.
Governing equation

$$\frac{D_{NO}}{r}\frac{d}{dr}\left(r\frac{dC_{NO}}{dr}\right) + R_{NO} = 0$$

$$F_i = -\frac{R_{NO_{max}}}{D_{NO}}\frac{P_{O_2}}{P_{O_2} + K_{m,eNOS}}$$

Boundary condition:

$$u_{EC}(r_2) = u_{CFL}(r_2),$$
$$u'_{EC}(r_3) = \gamma, \ \gamma = u'_{VW}(r_3)$$

VW $r_3 < r < r_4$ is same method as CFL, only difference is in the R_{O_2} term.
Governing equation

$$\frac{D_{NO}}{r}\frac{d}{dr}\left(r\frac{dC_{NO}}{dr}\right) - \lambda_{vw,t}C_{NO} = 0$$

$$F_i = \frac{\lambda_{vw,t}u_i}{D_{NO}}$$

Boundary condition:

$$u_{VW}(r_3) = u_{EC}(r_3),$$
$$u'_{VW}(r_4) = \delta, \ \delta = u'_T(r_4)$$

T $r_4 < r < r_5$. Combined since same governing equation. Governing equation

$$\frac{D_{NO}}{r}\frac{d}{dr}\left(r\frac{dC_{NO}}{dr}\right) - \lambda_{vw,t}C_{NO} = 0$$

$$F_i = \frac{\lambda_{vw,t}u_i}{D_{NO}}$$

$$u_T(r_4) = u_{VW}(r_4),$$
$$u'(r_5) = 0 \to u_n = u_{n-1}$$

$$A = \begin{bmatrix} -2 & 1+a_2 & 0 & \cdots & \cdots & \cdots & \vdots \\ 1-a_3 & -2 & 1+a_3 & \ddots & & & \vdots \\ 0 & 1-a_4 & -2 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & -2 & 1+a_{n-3} & 0 \\ \vdots & & & \ddots & 1-a_{n-2} & -2 & 1+a_{n-2} \\ 0 & \cdots & \cdots & \cdots & 0 & 1-a_{n-1} & -2 \end{bmatrix}, \ \tilde{\mathbf{u}} = \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_i \\ \vdots \\ u_{n-1} \end{bmatrix}, \ B = \begin{bmatrix} h^2F_2 - (1-a_2)u_1 \\ h^2F_3 \\ \vdots \\ h^2F_i \\ \vdots \\ h^2F_{n-1} - (1+a_{n-1})u_n \end{bmatrix} \tag{18}$$

O_2 is solved using the same matrix as NO except in RBC where it has a constant partial pressure.
RBC core $r_0 < r < r_1$

$$P_{O_2} = 70$$

CFL $r_1 < r < r_2$
Governing equation

$$\alpha \frac{D_{O_2}}{r} \frac{d}{dr} \left(r \frac{dP_{O_2}}{dr} \right) = 0$$
$$F_i = \frac{0}{\alpha D_{O_2}}$$

Boundary condition:

$$v_{CFL}(r_1) = v_{RBC}(r_1),$$
$$v'_{CFL}(r_2) = \phi, \; \phi = v'_{EC}(r_2)$$

EC $r_2 < r < r_3$
Governing equation

$$\alpha \frac{D_{O_2}}{r} \frac{d}{dr} \left(r \frac{dP_{O_2}}{dr} \right) - R_{NO} = 0$$
$$F_i = \frac{R_{NO_{max}}}{\alpha D_{O_2}} \frac{P_{O_2}}{P_{O_2} + K_{m,eNOS}}$$

Boundary condition:

$$v_{EC}(r_2) = v_{CFL}(r_2),$$
$$v'_{EC}(r_3) = \gamma, \; \gamma = v'_{VW}(r_3)$$

VW $r_3 < r < r_4$
Governing equation

$$\alpha \frac{D_{O_2}}{r} \frac{d}{dr} \left(r \frac{dP_{O_2}}{dr} \right) - Q_{O_2\,max\,VW} \frac{P_{O_2}}{P_{O_2} + appK_m} = 0$$
$$appK_m = K_m \left(1 + \frac{C_{NO}}{C_{ref}} \right) \tag{19}$$
$$F_i = \frac{Q_{O_2\,max\,VW}}{\alpha D_{O_2}} \frac{P_{O_2}}{P_{O_2} + K_m \left(1 + \frac{u_i}{C_{ref}} \right)}$$

Boundary condition:

$$v_{VW}(r_3) = v_{EC}(r_3),$$
$$v'_{VW}(r_4) = \delta, \; \delta = v'_T(r_4)$$

T $r_4 < r < r_5$
Governing equation

$$\alpha \frac{D_{O_2}}{r} \frac{d}{dr} \left(r \frac{dP_{O_2}}{dr} \right) - Q_{O_2\,max\,T} \frac{P_{O_2}}{P_{O_2} + appK_m} = 0$$
$$appK_m = K_m \left(1 + \frac{C_{NO}}{C_{ref}} \right) \tag{20}$$
$$F_i = \frac{Q_{O_2\,max\,T}}{\alpha D_{O_2}} \frac{P_{O_2}}{P_{O_2} + K_m \left(1 + \frac{u_i}{C_{ref}} \right)}$$

Boundary condition:

$$v_T(r_4) = v_{VW}(r_4),$$
$$v'_T(r_5) = 0 \rightarrow v_n = v_{n-1}$$

$$A = \begin{bmatrix} -2 & 1+a_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1-a_3 & -2 & 1+a_3 & \ddots & & & & \vdots \\ 0 & 1-a_4 & -2 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & -2 & 1+a_{n-3} & 0 & 0 \\ \vdots & & & \ddots & 1-a_{n-2} & -2 & 1+a_{n-2} & 0 \\ \vdots & \cdots & \cdots & \cdots & 0 & 1-a_{n-1} & -2 & 1+a_{n-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 & -4 & 3 \end{bmatrix}, \ \tilde{\mathbf{v}} = \begin{bmatrix} v_2 \\ v_3 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{bmatrix}, \ B = \begin{bmatrix} h^2 G_2 - (1-a_2)v_1 \\ h^2 G_3 \\ \vdots \\ h^2 G_i \\ \vdots \\ h^2 G_{n-1} \\ 0 \end{bmatrix} \tag{21}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1-a_2 & -2 & 0 & \ddots & & & & \vdots \\ 0 & 1-a_3 & -2 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & 1-a_{n-3} & -2 & 0 & 0 & 0 \\ \vdots & & & \ddots & 1-a_{n-2} & -2 & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & 0 & 1-a_{n-1} & -2 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

$$N = \begin{bmatrix} 0 & -2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & -(1+a_2) & \ddots & & & & \vdots \\ 0 & 0 & 0 & -(1+a_3) & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & 0 & -(1+a_{n-3}) & 0 & 0 \\ \vdots & & & \ddots & 0 & 0 & -(1+a_{n-2}) & 0 \\ \vdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & -(1+a_{n-1}) \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -2 & 0 \end{bmatrix}, \ \tilde{\mathbf{u}} = \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix}, \ B = \begin{bmatrix} h^2 F_2 - (1-a_2)u_1 \\ h^2 F_3 \\ \vdots \\ h^2 F_i \\ \vdots \\ h^2 F_{n-1} \\ 2h\sigma \end{bmatrix} \tag{23}$$