1 Discretization

1.1 Discretizing governing equation with central difference approximation

Starting equation

$$\frac{D_{NO}}{r}\frac{d}{dr}\left(r\frac{dC_{NO}}{dr}\right) + R_{NO} = 0\tag{1}$$

Simplify

$$D_{NO}\frac{d^{2}C_{NO}}{dr^{2}} + \frac{D_{NO}}{r}\frac{dC_{NO}}{dr} = -R_{NO}$$
 (2)

Compare to form

$$u'' + P(r)u' = F(r) \tag{3}$$

$$u = C_{NO}, P(r) = \frac{1}{r}, F(r) = -\frac{R_{NO}}{D_{NO}}$$
 (4)

Taylor expansion

$$u_{i+1} = u_i + u'_i \Delta r + \frac{1}{2} u''_i \Delta r^2,$$

$$u_{i-1} = u_i - u'_i \Delta r + \frac{1}{2} u''_i \Delta r^2$$

Central difference

$$u_i' = \frac{u_{i+1} - u_{i-1}}{2\Delta r},$$

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta r^2}$$

Sub back into Eq. (3)

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + P_i \frac{u_{i+1} - u_{i-1}}{2h} = F_i, \ h = \Delta r$$
 (5)

where index i starts at 1.

Rearrange, let $a_i = \frac{h}{2}P_i$

$$(1 - a_i) u_{i-1} + (-2)u_i + (1 + a_i) u_{i+1} = h^2 F_i$$
(6)

$$(-2)u_i = -(1 - a_i)u_{i-1} - (1 + a_i)u_{i+1} + h^2 F_i$$
(7)

$$u_{i} = -\frac{1}{2} \left(-(1 - a_{i}) u_{i-1} - (1 + a_{i}) u_{i+1} + h^{2} F_{i} \right)$$
(8)

1.2 Discretization schemes for boundary condition

1.2.1 Use imaginary node and correction term

$$\frac{u_2 - u_0}{2h} = 0 \to u_0 = u_2 \tag{9}$$

$$\frac{u_{end+1} - u_{end-1}}{2h} = 0 \to u_{end+1} = u_{end-1} \tag{10}$$

Substitute into Eq. (3) when i = 1

$$(-2)u_1 = -2u_2 + \frac{1}{2}h^2F_1 \tag{11}$$

Substitute into Eq. (3) when i = end

$$(-2)u_{end} = -(1 - a_{end})u_{end-1} - (1 + a_{end})u_{end+1} + h^2 F_{end}$$
(12)

$$(-2)u_{end} = -2u_{end-1} + h^2 F_{end} (13)$$

1.2.2 Use second-order one-sided forward/backward approximation

From [1]:

$$\frac{-3u_1 + 4u_2 - u_3}{2h} = 0 \to -3u_1 + 4u_2 - u_3 = 0 \tag{14}$$

$$\frac{3u_{end} - 4u_{end-1} + u_{end-2}}{2h} = 0 \to 3u_{end} - 4u_{end-1} + u_{end-2} = 0 \tag{15}$$

2 Solution algorithm

2.1 Gauss-Seidel

From [2]:

$$u_i^{k+1} = -\frac{1}{2} \left(-(1 - a_i) u_{i-1}^{k+1} - (1 + a_i) u_{i+1}^k + h^2 F_i \right)$$
(16)

2.2 Successive overrelaxation method

$$-\frac{2}{\omega}u_i^{k+1} = -(1-a_i)u_{i-1}^{k+1} - (1+a_i)u_{i+1}^k + h^2F_i - \left(\frac{2}{\omega} - 2\right)u_i^k$$
(17)

Alternative formula [3]:

$$\mathbf{u}^{k+1} = \left(\frac{1}{\omega}D + L\right)^{-1} \left\{ \left[\left(\frac{1}{\omega} - 1\right)D - U\right] \mathbf{u}^k + \mathbf{f} \right\}$$
(18)

3 Governing equation in individual compartment

3.1 Nitric oxide

RBC core $r_0 < r < r_1$

$$\frac{D_{NO}}{r} \frac{d}{dr} \left(r \frac{dC_{NO}}{dr} \right) - \lambda_{core} C_{NO} = 0$$

$$F_i = \frac{\lambda_{core} u_i}{D_{NO}}$$

CFL $r_1 < r < r_2$

$$\frac{D_{NO}}{r} \frac{d}{dr} \left(r \frac{dC_{NO}}{dr} \right) = 0$$
$$F_i = \frac{0}{D_{NO}}$$

EC $r_2 < r < r_3$

$$\frac{D_{NO}}{r}\frac{d}{dr}\left(r\frac{dC_{NO}}{dr}\right) + R_{NO} = 0$$

$$F_i = -\frac{R_{NO_{max}}}{D_{NO}} \frac{P_{O_2}}{P_{O_2} + K_{m,eNOS}}$$

 $VW r_3 < r < r_4$

$$\frac{D_{NO}}{r}\frac{d}{dr}\left(r\frac{dC_{NO}}{dr}\right) - \lambda_{vw}C_{NO} = 0$$

$$F_i = \frac{\lambda_{vw}u_i}{D_{NO}}$$

 $T r_4 < r < r_5$

$$\frac{D_{NO}}{r} \frac{d}{dr} \left(r \frac{dC_{NO}}{dr} \right) - \lambda_t C_{NO} = 0$$
$$F_i = \frac{\lambda_t u_i}{D_{NO}}$$

3.1.1 NO matrix

$$A = \begin{bmatrix} -3 & 4 & -1 & \cdots & \cdots & 0 \\ 1 - a_2 & -2 & 1 + a_2 & 0 & \cdots & \cdots & \vdots \\ 0 & 1 - a_3 & -2 & 1 + a_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & 1 - a_{n-2} & -2 & 1 + a_{n-1} \\ 0 & \cdots & \cdots & 1 & -4 & 3 \end{bmatrix}, \quad \tilde{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_i \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}, \quad \tilde{\mathbf{p}} = \begin{bmatrix} 0 \\ h^2 F_2 \\ h^2 F_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}$$

3.2Oxygen

RBC core $r_0 < r < r_1$

CFL
$$r_1 < r < r_2$$

$$P_{O_2} = 70$$

$$CFL r_1 < r < r_2$$

$$\alpha \frac{D_{O_2}}{r} \frac{d}{dr} \left(r \frac{dP_{O_2}}{dr} \right) = 0$$
$$G_i = \frac{0}{\alpha D_{O_2}}$$

EC
$$r_2 < r < r_3$$

$$\begin{split} &\alpha \frac{D_{O_2}}{r} \frac{d}{dr} \left(r \frac{dP_{O_2}}{dr} \right) - R_{NO} = 0 \\ &G_i = \frac{R_{NO_{max}}}{\alpha D_{O_2}} \frac{P_{O_2}}{P_{O_2} + K_{m,eNOS}} \end{split}$$

$$VW r_3 < r < r_4$$

$$\alpha \frac{D_{O_2}}{r} \frac{d}{dr} \left(r \frac{dP_{O_2}}{dr} \right) - Q_{O_2 \, max \, VW} \frac{P_{O_2}}{P_{O_2} + appK_m} = 0$$

$$appK_m = K_m \left(1 + \frac{C_{NO}}{C_{ref}} \right)$$

$$G_i = \frac{Q_{O_2 \, max \, VW}}{\alpha D_{O_2}} \frac{P_{O_2}}{P_{O_2} + K_m \left(1 + \frac{u_i}{C_{ref}} \right)}$$
(20)

$$T r_4 < r < r_5$$

$$\alpha \frac{D_{O_2}}{r} \frac{d}{dr} \left(r \frac{dP_{O_2}}{dr} \right) - Q_{O_2 \max T} \frac{P_{O_2}}{P_{O_2} + appK_m} = 0$$

$$appK_m = K_m \left(1 + \frac{C_{NO}}{C_{ref}} \right)$$

$$G_i = \frac{Q_{O_2 \max T}}{\alpha D_{O_2}} \frac{P_{O_2}}{P_{O_2} + K_m \left(1 + \frac{u_{i-}}{C_{C_2}} \right)}$$
(21)

3.2.1 O_2 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 - a_{r_1+1} & -2 & 1 + a_{r_1+1} & 0 & \cdots & \cdots & 0 \\ 0 & 1 - a_{r_1+2} & -2 & 1 + a_{r_1+2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \cdots & 0 & 1 - a_{n-2} & -2 & 1 + a_{n-1} \\ 0 & \cdots & \cdots & 1 & -4 & 3 \end{bmatrix}, \tilde{\mathbf{v}} = \begin{bmatrix} v_{r_1} \\ v_{r_1+1} \\ \vdots \\ v_{i} \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix}, B = \begin{bmatrix} 70 \\ h^2 G_{r_1+1} \\ \vdots \\ h^2 G_{i} \\ \vdots \\ h^2 G_{n-1} \\ 0 \end{bmatrix}$$

$$\vdots$$

4 Velocity profile

4.1 Axisymmetric

$$\frac{\dot{\gamma}}{\frac{P_g R_1}{4\mu_{\infty m}}} = \frac{\mu_{\infty m}}{\mu_{\infty}} \left[r - \alpha(1+q)\sqrt{r} + \alpha^2 + (\sqrt{r} - \alpha)C_r\right]$$
(23)

$$-\frac{dV}{dR} = \frac{P_g R_1 \mu_{\infty m}}{4\mu_{\infty m} \mu_{\infty}} \left[r - \alpha (1+q)\sqrt{r} + \alpha^2 + (\sqrt{r} - \alpha)C_r \right]$$
(24)

where

$$\alpha = \frac{\sqrt{\tau_0} + \sqrt{\mu_\infty \Lambda}}{\sqrt{\lambda \tau_w}}$$

$$q = \frac{\sqrt{\tau_0} - \sqrt{\mu_\infty \Lambda}}{\sqrt{\tau_0} + \sqrt{\mu_\infty \Lambda}}$$

$$C_r = \sqrt{r - 2\alpha q \sqrt{r} + \alpha^2}$$
(25)

where

$$\mu_{\infty} = \frac{\mu_p}{(1 - \frac{1}{2}k_{\infty}H)^2}$$

$$\Lambda = \gamma_c \left(\frac{1 - \frac{1}{2}k_0H}{1 - \frac{1}{2}k_{\infty}H}\right)^2$$

$$\tau_0 = \mu_p \gamma_c \frac{\left[\frac{1}{2}H(k_0 - k_{\infty})\right]^2}{\left(1 - \frac{1}{2}k_{\infty}H\right)^4}$$
(26)

where

$$k_0 = \exp\left\{3.874 + H[-10.41 + H(13.8 - 6.738H)]\right\}$$

$$k_0 = 0.275363 + \frac{2.0}{0.100158 + H}$$

$$k_{\infty} = \exp\left\{1.3435 + H[-2.803 + H(2.711 - 0.6479H)]\right\}$$

$$\gamma_c = \exp\left\{-6.1508 + H[27.923 + H(-25.6 + 3.697H)]\right\}$$
(27)

5 Physiological parameters

Table 1: Systemic parameters and NCFL widths [4]

			NCFL (%)	
Aggregating conditions	Hct (%)	Diameter (μm)	Outer	Inner
Non	44.0 ± 1.6	52.5 ± 4.7	14.4 ± 2.1	11.1 ± 1.1
Normal	42.8 ± 1.7	50.0 ± 4.7	21.3 ± 3.4	13.5 ± 1.2
Hyper	42.2 ± 1.6	51.8 ± 4.4	23.6 ± 2.7	15.3 ± 1.7

5.1 CFL widths from mass conservation

Hematocrit measures proportion of volume of red blood cells (RBC) to total blood volume (RBC and plasma) [5]. Cell volume fraction or tube hematocrit H_t is calculated as:

$$H_t = \frac{N_c V_{eff}}{V_t} \tag{28}$$

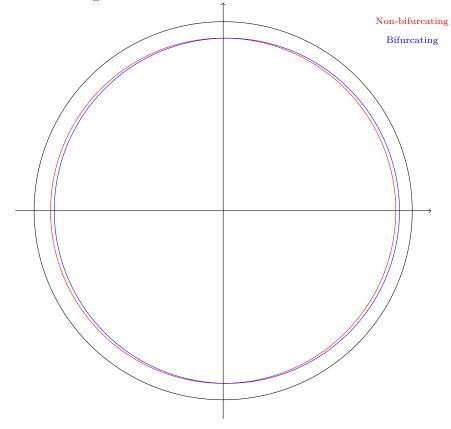
where N_c is the number of RBCs in the tube volume $V_t = \pi R^2 L$, R is the tube radius, and L is the tube length [6]. An empirical relation between H_t and H_d is given by:

$$\frac{H_t}{H_d} = H_d + (1 - H_d) \left(1 + 1.7e^{-0.35D} - 0.6e^{-0.01D} \right)$$
(29)

where D is the tube diameter in micrometers, and discharge hematocrit H_d is equal to systemic hematocrit [7].

Since systemic hematocrit in both cases (non-bifurcating and bifurcating flow) is the same, H_t is the same according to Eq. 29. Since V_t is the same due to constant vessel diameter, N_c assumed (?) to be the same, V_{eff} must be the same (Eq. 28).

Ignoring gravity effect and assuming symmetry, we need to find a egg / ellipse shaped graph with the same area as the circle in non-bifurcating flow.



Assumptions: Define normal CFL width as mean of inner and outer CFL width.

$$NCFL_{normal} = \frac{NCFL_{inner} + NCFL_{outer}}{2}$$
(30)

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