

Starting equations

$$\frac{D_{NO}}{r} \frac{d}{dr} \left(r \frac{dC_{NO}}{dr} \right) + R_{NO} = 0 \quad (1)$$

Simplify

$$D_{NO} \frac{d^2 C_{NO}}{dr^2} + \frac{D_{NO}}{r} \frac{dC_{NO}}{dr} = -R_{NO} \quad (2)$$

Compare to form

$$u'' + P(r)u' = F(r) \quad (3)$$

$$u = C_{NO}, P(r) = \frac{1}{r}, F(r) = -\frac{R_{NO}}{D_{NO}} \quad (4)$$

Taylor expansion

$$\begin{aligned} u_{i+1} &= u_i + u'_i \Delta r + \frac{1}{2} u''_i \Delta r^2, \\ u_{i-1} &= u_i - u'_i \Delta r + \frac{1}{2} u''_i \Delta r^2 \end{aligned}$$

Central difference

$$\begin{aligned} u'_i &= \frac{u_{i+1} - u_{i-1}}{2\Delta r}, \\ u''_i &= \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta r^2} \end{aligned}$$

Sub back into Eq. (3)

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + P_i \frac{u_{i+1} - u_{i-1}}{2h} = F_i, h = \Delta r \quad (5)$$

where index i starts at 1.

Rearrange

$$\left(1 - \frac{h}{2} P_i\right) u_{i-1} + (-2)u_i + \left(1 + \frac{h}{2} P_i\right) u_{i+1} = h^2 F_i \quad (6)$$

Split into 4 sections since equations for vascular wall and tissue are the same. Solutions are in the general form $A\tilde{\mathbf{u}} = B$. Let $a_i = \frac{h}{2} P_i$.

RBC core $r_0 < r < r_1$

$$\begin{aligned} u'(0) &= 0 \rightarrow u_1 = u_2, \\ u'_{RBC}(r_1) &= \sigma, \sigma = u'_{CFL}(r_1) \end{aligned}$$

Using second-order accurate one-sided difference approximation, j represents indexes in CFL domain.

$$\begin{aligned} \sigma &= \frac{-3u_j + 4u_{j+1} - u_{j+2}}{2h}, \\ \frac{3u_i - 4u_{i-1} + u_{i-2}}{2h} &= \sigma \end{aligned}$$

$$A = \begin{bmatrix} -2 - (1 - a_2) & 1 + a_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 - a_3 & -2 & 1 + a_3 & \ddots & & & & \vdots \\ 0 & 1 - a_4 & -2 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & -2 & 1 + a_{n-3} & 0 & 0 \\ \vdots & & & \ddots & 1 - a_{n-2} & -2 & 1 + a_{n-2} & 0 \\ \vdots & \cdots & \cdots & \cdots & 0 & 1 - a_{n-1} & -2 & 1 + a_{n-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & \frac{h}{2} & -2h & \frac{3h}{2} \end{bmatrix}, \tilde{\mathbf{u}} = \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix}, B = \begin{bmatrix} h^2 F_2 \\ h^2 F_3 \\ \vdots \\ h^2 F_i \\ \vdots \\ h^2 F_{n-1} \\ \sigma \end{bmatrix} \quad (7)$$

CFL $r_1 < r < r_2$

$$\begin{aligned} u'_{CFL}(r_1) &= \sigma, \sigma = u'_{RBC}(r_1), \\ u'_{CFL}(r_2) &= \phi, \phi = u'_{EC}(r_2), \end{aligned}$$

Using one-sided difference approximation again, i is CFL, j is RBC and k is EC.

$$\begin{aligned} \sigma &= \frac{3u_j - 4u_{j-1} + u_{j-2}}{2h}, \\ \frac{-3u_i + 4u_{i+1} - u_{i+2}}{2h} &= \sigma, \\ \phi &= \frac{-3u_k + 4u_{k+1} - u_{k+2}}{2h}, \\ \frac{3u_i - 4u_{i-1} + u_{i-2}}{2h} &= \phi \end{aligned}$$

$$A = \begin{bmatrix} -\frac{3h}{2} & 2h & -\frac{h}{2} & 0 & \dots & \dots & \dots & \dots & 0 \\ 1 - a_2 & -2 & 1 + a_2 & 0 & & & & & \vdots \\ \vdots & 1 - a_3 & -2 & 1 + a_3 & \ddots & & & & \vdots \\ \vdots & 0 & 1 - a_4 & -2 & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & -2 & 1 + a_{n-3} & 0 & 0 \\ \vdots & & & & \ddots & 1 - a_{n-2} & -2 & 1 + a_{n-2} & 0 \\ \vdots & \dots & \dots & \dots & \dots & 0 & 1 - a_{n-1} & -2 & 1 + a_{n-1} \\ 0 & \dots & \dots & \dots & \dots & \dots & \frac{h}{2} & -2h & \frac{3h}{2} \end{bmatrix}, \tilde{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix}, B = \begin{bmatrix} \sigma \\ h^2 F_2 \\ h^2 F_3 \\ \vdots \\ h^2 F_i \\ \vdots \\ h^2 F_{n-1} \\ \phi \end{bmatrix} \quad (8)$$

EC $r_2 < r < r_3$ is same method as CFL, only difference is in the R_{NO} term.

VW $r_3 < r < r_4$ is same method as CFL, only difference is in the R_{O_2} term.

T $r_4 < r < r_5$. Combined since same governing equation.

$$\begin{aligned} u'_T(r_4) &= \delta, \delta = u'_{VW}(r_4), \\ u'(r_5) &= 0 \rightarrow u_n = u_{n-1} \end{aligned}$$

Using one-sided difference approximation again, i is T, j is VW.

$$\begin{aligned} \delta &= \frac{3u_j - 4u_{j-1} + u_{j-2}}{2h}, \\ \frac{-3u_i + 4u_{i+1} - u_{i+2}}{2h} &= \delta \end{aligned}$$

$$A = \begin{bmatrix} -\frac{3h}{2} & 2h & -\frac{h}{2} & 0 & \cdots & \cdots & \cdots & 0 \\ 1 - a_2 & -2 & 1 + a_2 & 0 & & & & \vdots \\ \vdots & 1 - a_3 & -2 & 1 + a_3 & \ddots & & & \vdots \\ \vdots & 0 & 1 - a_4 & -2 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & -2 & 1 + a_{n-3} & 0 \\ \vdots & & & & \ddots & 1 - a_{n-2} & -2 & 1 + a_{n-2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 - a_{n-1} & -2 + (1 + a_{n-1}) \end{bmatrix}, \tilde{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_i \\ \vdots \\ u_{n-1} \end{bmatrix}, B = \begin{bmatrix} \delta \\ h^2 F_2 \\ h^2 F_3 \\ \vdots \\ h^2 F_i \\ \vdots \\ h^2 F_{n-1} \end{bmatrix} \quad (9)$$