1) To prove the given problem is NP complete, we need to prove it two steps:

1. This problem belongs to NP.
2. An NP complete problem can be reduced to Subset Sum problem.

Step 1: Is the subset sum (SS) problem NP?

This problem is short and verifiable certificate. The algorithm of this problem looks for a a set B which is basically a subset of $A = \{x_1, x_2, \cdots x_n\}$, ~~the~~ sum of the elements in B can be equal to ω (the given target integer). So, its easy to understand that there exists a polynomial-time algorithm that verifies whether B is a set of numbers, whose sum is ω:

It just verifies that $\sum_{x_i \in B} x_i = \omega$.

Step 2: Reduction of a NP-C problem to SS.

In this step we are going to reduce 3-CNF SAT problem to Subset Sum.

In 3-CNF SAT problem, lets there are n variables x_i and m number of clauses c_j

For each variable x_i, the 3CNF SAT algorithm constructs numbers t_i and f_i of $n+m$ digits, by following the steps as below:

a) The ith digit of t_i and f_i is equal to 1.

b) For $n+1 \leq j \leq n+m$, the jth digit of t_i is equal to 1 if x_i is present in the clause c_{j-n}

c) For $n+1 \leq j \leq n+m$, the jth digit of f_i is equal to 1 if $\overline{x_i}$ is present in c_{j-n}

d) assign 0 for all other t_is and f_is.

For each clause c_j, the algorithm finds numbers x_j and y_j of $n+m$ digits, by following the two steps below:

a) Assign 1 to the $(n+j)$ th digit of x_j and y_j.

b) Assign 0 to others.

Finally, the algorithm constructs a sum number w of $n+m$ digits by:

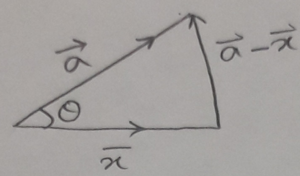
a) For $1 \leq j \leq n$, 1 is assigned to the j-th digit of ω

b) For $n+1 \leq j \leq n+m$, the j-th digit of ω is assigned to 3.

This algorithm also finds a subset to satisfy a target value.
It includes those t_i's and f_i's which can make the sum n. Thus, the 3CNF-SAT can be reduced to subset sum problem.

Thus, it is now proved that the subset sum problem is NP complete.

2.a)



Using the law of cosine, we get,

$$|\bar{a}-\bar{x}|^2 = |\bar{a}|^2 + |\bar{x}|^2 - 2|\bar{a}||\bar{x}|\cos\theta$$

$$\Rightarrow 2|\bar{a}||\bar{x}|\cos\theta = -|\bar{a}-\bar{x}|^2 + |\bar{a}|^2 + |\bar{x}|^2$$

We can write $|\bar{a}-\bar{x}|^2$ as the inner product of the vector $|\bar{a}-\bar{x}|$ with itself

So,

$$2|\bar{a}||\bar{x}|\cos\theta = -(\bar{a}-\bar{x})\cdot(\bar{a}-\bar{x}) + \bar{a}\cdot\bar{a} + \bar{x}\cdot\bar{x}$$

$$= -(\bar{a}\cdot\bar{a} - \bar{a}\,\bar{x} - \bar{x}\cdot\bar{a} + \bar{x}\cdot\bar{x}) + \bar{a}\cdot\bar{a} + \bar{x}\cdot\bar{x}$$

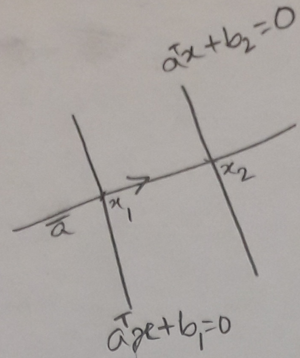
as the properties of inner product, $\bar{a}\cdot\bar{x} = \bar{x}\cdot\bar{a}$,

$$2|\bar{a}||\bar{x}|\cos\theta = -(\bar{a}\cdot\bar{a} - 2\,\bar{a}\cdot\bar{x} + \bar{x}\cdot\bar{x}) + \bar{a}\cdot\bar{a} + \bar{x}\cdot\bar{x}$$

$$= 2\,\bar{a}\cdot\bar{x}$$

$$\Rightarrow |\bar{a}||\bar{x}|\cos\theta = \bar{a}\cdot\bar{x}$$

2 b)



Suppose in the above figure, $\bar{a}$ is a normal vector, x_1 and x_2 are the points where the hyperplanes intersects the vector $\vec{a}$.

So, the equation of the hyperplanes are $ax + b_1 = 0$ and $ax + b_2 = 0$. As two planes are parallel, $a_1 = k a_2$.

we know that in 3D-space the distance from any point $x = (x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$ is

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

So, the distance from a point x_1 to a hyperplane $ax + b_2 = 0$ is

$$\frac{|ax_1 + b_2|}{\|a\|}$$

In 3D, the distance between $ax+by+cz+d_1=0$ and $ax_1+by_1+cz_1+d_2=0$ is basically equal to the distance from a point (x_1, y_1, z_1) on the first plane to second plane:

$$\frac{|ax_1+by_1+cz_1+d_2|}{\sqrt{a^2+b^2+c^2}} = \frac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$$

Thus, the distance between two hyperplanes $ax+b_1=0$ and $ax+b_2=0$ is

$$\frac{|b_2-b_1|}{\|a\|}$$

③ Any point x can be closer to x_0 than any other x_i iff

$$\|x-x_0\| \leq \|x-x_i\|$$

here we can write $\|x-x_0\| = (x-x_0)^T \cdot (x-x_0)$

So,

$$\|x-x_0\| \leq \|x-x_i\| \Leftrightarrow (x-x_0)^T(x-x_0) \leq (x-x_i)^T(x-x_i)$$

$$\Leftrightarrow x^Tx - 2x_0^Tx + x_0^Tx_0 \leq x^Tx - 2x_i^Tx + x_i^Tx_i$$

$$\Leftrightarrow x^Tx - x^Tx - 2x_0^Tx + 2x_i^Tx \leq x_i^Tx_i - x_0^Tx_0$$

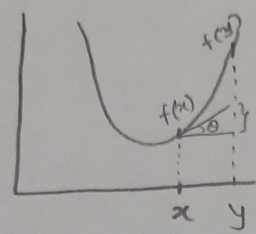
$$\Leftrightarrow 2(x_i - x_0)^Tx \leq x_i^Tx_i - x_0^Tx_0$$

This is the form of halfspace ($a^Tx \leq b$). And we can express S as $S = \{x \mid Ax \leq b\}$ with

$$A = 2\begin{bmatrix} x_1 - x_0 \\ x_2 - x_0 \\ \vdots \\ x_K - x_0 \end{bmatrix}$$

$$b = \begin{bmatrix} x_1^Tx_1 - x_0^Tx_0 \\ x_2^Tx_2 - x_0^Tx_0 \\ \vdots \\ x_n^Tx_n - x_0^Tx_0 \end{bmatrix}$$

4



From convexity, we get

$$f(x) \geq f(y) + (\nabla f(y)^T, x-y)$$
$$f(y) \geq f(x) + (\nabla f(x)^T, y-x)$$

after adding these two inequalities, we have

$$f(x) + f(y) \geq f(x) + f(y) + (\nabla f(y)^T, x-y) + (\nabla f(x)^T, y-x)$$

$$\Rightarrow -\nabla f(x)^T, y-x) - (\nabla f(y)^T, x-y) \geq 0$$

$$\Rightarrow (\nabla f(x)^T, x-y) - (\nabla f(y)^T, x-y) \geq 0$$

$$\Rightarrow (\nabla f(x) - \nabla f(y))^T (x-y) \geq 0$$

To show sufficiency, let

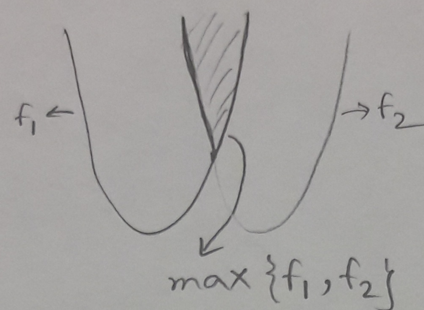
$$g(t) = f(ty + (1-t)x),$$

so, $g'(t) = \nabla f(ty + (1-t)x)^T (y-x)$

Hence,

$$f(y) = g(1) = g(0) + g'(0)$$
$$= f(x) + (\nabla f(x)^T, y-x)$$

5(a) Lets take two convex functions f_1 and f_2 from $f_i(x)$ as follows:



Lets pick any two points x and y such that both $x, y \in dom(f)$, $t \leq 0 \leq 1$

Then we can write,

$$f(tx + (1-t)y) = f_i(tx + (1-t)y) \quad \text{where } i = 1, 2, \cdots, m$$

$$\leq t f_i(x) + (1-t) f_i(y) \quad [\text{as } f_i(x) \text{ is convex}$$

$$\leq t \max\{f_1(x), f_2(x), \cdots, f_m(x)\} + (1-t) \max\{f_1(y), f_y(y), \cdots, f_m(y)\}$$

$$\leq t\, f(x) + (1-t)\, f(y)$$

$[\text{as given } f(x) = \max\{f_i x\}$

Thus, $f(x)$ is proved to be a convex function.

5(b) We can prove this by verifying the inequality for $x_1, x_2 \in \operatorname{dom} f$.

Let $\epsilon > 0$.

Then there are $y_1, y_2 \in C$ (Given C is a convex set) such that $g(x_i, y_i) \leq f(x_i) + \epsilon$ when $i \in [1,2]$.

Let $t \doteq 1, 2$.

$$f(tx_1 + (1-t)x_2) = \min_{y \in C} g(tx_1 + (1-t)x_2, y)$$

$$\Rightarrow \min_{y \in C} g(tx_1 + (1-t)x_2, y) \leq g(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2)$$
$$\leq t g(x_1, y_1) + (1-t) g(x_2, y_2)$$
$$\leq t f(x_1) + (1-t) f(x_2) + \epsilon$$

So, we can write

$$f(tx_1 + (1-t)x_2) \leq t f(x_1) + (1-t) f(x_2).$$

So, $f(x)$ is convex.

5(c) In the second derivative test for determining extrema of a function $f(x,y)$ the discrimant D is

$$H\,f(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

here $f(x,y) = \frac{x^p}{y^{p-1}}$

To show $f(x,y)$ is convex, we need to prove $Hf(x,y) \geq 0$

[note, $y>0$]

$$Hf(x,y) = \begin{bmatrix} \frac{(p^2-p)\,x^{p-2}}{y^{p-1}} & \frac{(p-p^2)\,x^{p-1}}{y^p} \\ \frac{(p-p^2)\,x^{p-1}}{y^p} & \frac{x^p\,(p^2-p)}{y^{p+1}} \end{bmatrix}$$

Here, $\frac{\partial^2 f}{\partial x^2} \geq 0$ as $y>0$, $p>1$

$\frac{\partial^2 f}{\partial y^2} \geq 0$ for same reason.

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}\cdot\frac{\partial^2 f}{\partial y \partial x} \geq 0$$

$$= \frac{p^2(p^2-2p+1)\;x^{2p-2}}{y^{2p}} - \frac{p^2(p^2-2p+1)\cdot x^{2p-2}}{y^{2p}} = 0$$

So, according to the tests of convexity, the function $Hf(x,y) \geq 0$, so, $f(x,y)$ is convex.

5(d) $f(x)$ is convex at any point x_i if $f''(x_i) > 0$

In this case we have

$$f''(x) = \frac{\partial}{\partial x} f'(x)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(x+1)\ln(x+1) - x\right)$$

$$= \frac{\partial}{\partial x}\left((x+1)\cdot\frac{1}{x+1} + 1.\ln(x+1) - 1\right)$$

$$= \frac{\partial}{\partial x}\ln(x+1)$$

$$= \frac{1}{x+1}$$

Then at $x_i = 1, 2, \cdots, m$

$$f''(x_i) = \frac{1}{x_i+1} \geq 0$$

Thus $f(x)$ is convex at x_i

(6) The functions of powers of absolute value, is convex,

For example, $|x|^p$, for $p \geq 1$, is convex.

here $|a^T x + b|$ is a function with powers (1) of absolute value. So this is a convex.

As $\log x$ is concave, we can write $\ln \frac{1}{c^T x + d}$ as $-\ln(c^T x + d)$ [according to the reciprocal rule].

So, $-\ln(c^T x + d)$ is convex as $\ln(c^T x + d)$ is concave.

and finally maximum of these two functions is definitely ~~going~~ to be either any of these two. Thus, it can be proved that $f(x) = \max\{|a^T x + b|, \ln \frac{1}{c^T x + d}\}$ is a convex function