Problem 1

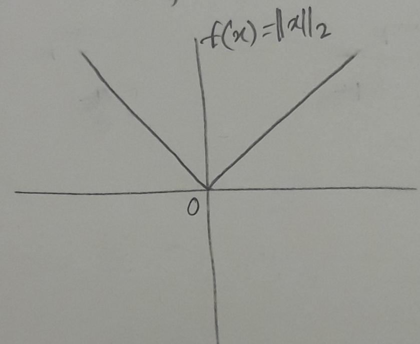
Solution:

$f(x) = \|x\|_2$ is a convex function.

For $x \neq 0$, $f(x) = \|x\|_2$ is differentiable and its gradient can be easily obtained, which is $\frac{x}{\|x\|_2}$ (derivative of $\|x\|_2$)

So, $\frac{x}{\|x\|_2}$ is the only subgradient at $x \neq 0$.

But when $x = 0$, it has many subgradients.



So the subgradient g of $f(x)$ should satisfy

$$f(y) \geq f(x) + g^T(y-x) \text{ for all } y.$$

$$\Rightarrow \|y\|_2 \geq 0 + g^T(y-0)$$

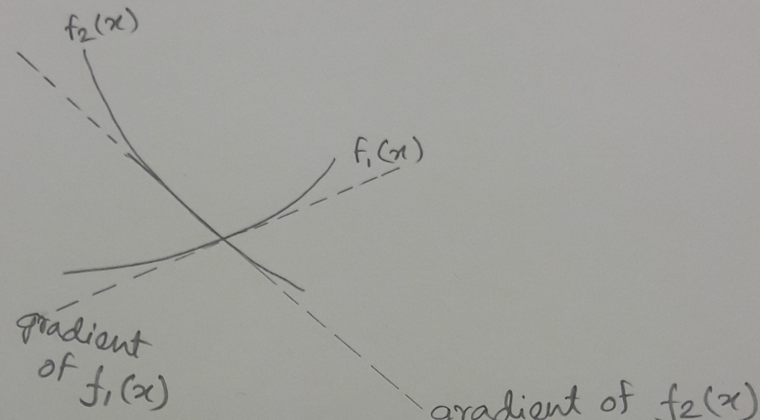
$$\Rightarrow g^T \frac{y}{\|y\|_2} \leq 1$$

To satisfy this inequality, only $\|g\|_2 \leq 1$ would make this true, since $\frac{y}{\|y\|_2}$ is an unit vector.

Problem 2

Solution: Since $f(x) = \max\{f_1(x), f_2(x)\}$ and $f_1(x)$ & $f_2(x)$ both are convex, so, $f(x)$ must be a convex function. But it may not be differentiable.

Let's draw the function $f_1(x)$ & $f_2(x)$:



In the region where $f_1(x) > f_2(x)$, the function is differentiable. Thats because we know that $f_1(x)$ is smooth (diffentiable). So, it has only one subgraddient $g = \nabla f_1(x)$.

Same logic applies for $f_2(x) > f_1(x)$. since $f_2(x)$ is differentiable, in the region where $f_2(x) > f_1(x)$, the only subgradient is $\nabla f_2(x)$.

For $f_1(x) = f_2(x)$, this the region where, f is not differentiable. So, there could be many subgradient at point x (since its not smooth anymore). And the subgradients are going to be any point on the line segment that joints the gradient of $f_1(x) = \nabla f_1(x)$ and the gradient of $f_2(x) = \nabla f_2(x)$.

And we know from the subgradient calculus, as $f(x) = \max\{f_1(x), f_2(x)\}$ & $f_1(x)$ and $f_2(x)$ both are convex, then the subdifferential of $f(x)$ will be the convex hull of the union of all the subdifferentials of $f_1(x)$ and $f_2(x)$.

$$\partial f(x) = \text{conv}\left(\cup\, \partial f_i(x)\right) \quad [\text{here } i = 1,2]$$

Problem 3 (1)

solution: $f(x) = -\sqrt{x}$

We are going to prove this by contradiction. Lets assume the $f(x)$ is subdifferentiable at $x=0$. and let g is a subgradient of f at 0.

So, The following must be hold for all $y \geq 0$.

$$f(y) \geq f(x) + g^T(y-x)$$

$$\Rightarrow -\sqrt{y} \geq 0 + g^T y$$

$$\Rightarrow g^T \leq -\frac{\sqrt{y}}{y}$$

$$\Rightarrow -g^T\sqrt{y} \geq 1$$

This inequality does not hold for $y=0$. For $y<0$, this becomes undefined, and for $y>0$, its still does not hold.

Problem 3 ②

Solution:

Lets suppose $f(x) = \begin{cases} 1, & x=0 \\ 0, & x>0 \end{cases}$ is subdifferentiable at $x=0$. and g is the subgradient of f at 0. Then the following inequality for all $y \geq 0$ should hold.

$$f(y) \geq f(x) + g^T(y-x) \quad \text{for all } y \geq 0$$

If this inequality does not hold for either of $x=0$ or $x>0$, we can prove that this function is not subdifferentiable at 0.

So, for, $y=0$,

$$f(y) \geq f(0) + g^T(y-0)$$

$$\Rightarrow f(y) \geq 1 + g^T y \Rightarrow 1 \geq 1 + g^T y$$

So, this holds when $y=0$.

for $y>0$,

$$f(y) \geq f(0) + g^T(y-0)$$

$$\Rightarrow 0 \geq 1 + g^T y$$

$$\Rightarrow -g^T y \geq 1.$$

This equality does not hold for $y=0$, & $y>0$.

Thus its proved that $f(x)$ is not subdifferentiable at 0.

Problem 4:

Solution: We need to prove that

$$\text{dist}(x^+, x^*) < \text{dist}(x, x^*)$$

We know that for any subgradient g,

$$f(x) \geq f(x^*) + g^T(x - x^*)$$

$$\Rightarrow g^T(x - x^*) \leq f(x) - f(x^*) \quad \cdots\cdots \quad (1)$$

Given that,

$$\alpha < \frac{2(f(x) - f(x^*))}{\|g\|_2^2}$$

$$\Rightarrow \alpha g^T g - 2 g^T(x - x^*) < 0 \quad \left[\text{since, } \|g\| = \sqrt{g^T g} \ \& \ (1)\ g^T(x - x^*) \leq f(x) - f(x^*)\right]$$

$$\Rightarrow \alpha^2 g^T g - 2\alpha g^T(x - x^*) < 0 \quad [\text{multiplying by } \alpha \text{ and as } \alpha > 0]$$

$$\Rightarrow \|x - x^*\|_2^2 + \alpha^2 g^T g - 2\alpha g^T(x - x^*) < \|x - x^*\|_2^2$$

$$\Rightarrow x^T x - 2x^T x^* + x^{*T} x^* + \alpha^2 g^T g - 2\alpha g^T(x - x^*) < \|x - x^*\|_2^2$$

$$\Rightarrow x^T x - 2x^T x^* + x^{*T} x^* + \alpha^2 g^T g - 2\alpha g^T x + 2\alpha g^T x^* < \|x - x^*\|_2^2$$

$$\Rightarrow x^{*T} x^* - 2x^T x^* + 2\alpha g^T x^* + (x - \alpha g)^T (x - \alpha g) < \|x - x^*\|_2^2$$

$$\Rightarrow (x - \alpha g)^T (x - \alpha g) - 2(x - \alpha g) x^* + x^{*T} x^* < \|x - x^*\|_2^2$$

$$\Rightarrow \|x^+ - x^*\|_2^2 < \|x - x^*\|_2^2 \quad [\text{as } x^+ = x - \alpha g]$$

$$\Rightarrow \|x^+ - x^*\| < \|x - x^*\|$$

Problem 5

solution:

step 0: introduce dual variables for each of the constraints, α_u, β_s, $\gamma_{s,u}$

step 1: Assign a nonnegative initial value to each dual variables:

$$\alpha_u^{k-1} \geq 0, \ \beta_s^{k-1} \geq 0, \ \gamma_{s,u}^{k-1} \geq 0 .$$

step 2: Find the primal optimal variables for all $s \in S$, $u \in U$

while $s \in S$, do

while $u \in U$, do

$$x_{s,u}^{k-1} = \underset{x_{s,u}}{\operatorname{argmin}} \ L(x_{s,u}, \alpha_u^{k-1}, \beta_s^{k-1}, \gamma_{s,u}^{k-1})$$

$$= \underset{x_{s,u}}{\operatorname{argmin}} \ (a_{s,u} x_{s,u} + b_{s,u}((x_{s,u}+1)\ln(x_{s,u}+1) - x_{s,u})$$

$$+ \alpha_u c_u - \alpha_u x_{s,u} + \beta_s d_s - \beta_s x_{s,u} - \gamma_{s,u} x_{s,u}$$

$$= a_{s,u} + b_{s,u} \ln(x_{s,u}+1) - \alpha_u - \beta_s - \gamma_{s,u}$$

step 3:

while $s \in S$ do:

while $u \in U$ do:

repeat until it converges:

$$\alpha_u^k \triangleq \alpha_u^{k-1} + t\left(c_u - \sum_s x_{s,u}^{k-1}\right)^+$$

$$\beta_s^k \triangleq \beta_s^{k-1} + t\left(d_s - \sum_u x_{s,u}^{k-1}\right)^+$$

$$\gamma_{s,u}^k \triangleq \gamma_{s,u}^{k-1} + t\left(- x_{s,u}^{k-1}\right)$$