Problem 1.1

Lets denote z_{ij} in the linear program as $z_{ij} = x_i + x_j - 2x_ix_j$

Thus we can say $F_{ILP}(x) = F_{LP}(z)$

To prove the $F_{LP}(z)$ is the relaxation of $F_{ILP}(x)$, we need to prove that any point (x, z) is a feasible solution to the $F_{LP}(z)$.

Lets get the nontrivial two constraints from the LP program.

$$z_{ij} \le x_i + x_j$$

$$\Leftrightarrow x_i + x_j - 2x_ix_j \le x_i + x_j$$

$$\Leftrightarrow -2x_ix_j \le 0$$

$$z_{ij} \le 2 - x_i - x_j$$

$$\Leftrightarrow x_i + x_j - 2x_ix_j \le 2 - x_i - x_j$$

$$\Leftrightarrow x_i + x_j - x_ix_j - 1 \le 0$$

Both of these two inequalities hold true for all $0 \le x_i, x_j \le 1$.

And $0 \le x_i \le 1$ is the relaxation of $x_i \in \{0, 1\}, \forall i \in V$

And $0 \le z_{ij} \le 1 \Leftrightarrow 0 \le x_i + x_j - 2x_ix_j \le 1$ always holds for the same x_i, x_j range $0 \le x_i, x_j \le 1$

Problem 1.2

we need to prove

$$F_{ILP}(x) \geq \frac{1}{2} \sum_{(i,j)} w_{ij} z_{ij}$$

$$\Leftrightarrow \sum_{i,j} w_{ij} z_{ij} \leq 2 \sum_{ij} w_{ij} (x_i + x_j - 2x_i x_j)$$

To prove the above inequality, lets show that

$$z_{ij} \leq 2 (x_i + x_j - 2x_i x_j) \quad \text{holds true.}$$

Lets write,

$$(x_i + x_j) \leq 2 (x_i + x_j - 2x_i x_j) \quad \text{since } z_{ij} \leq x_i + x_j$$

$$\Leftrightarrow x_i + x_j \leq 2 (x_i + x_j) - 4 x_i x_j$$

This holds for all $0 \leq x_i + x_j \leq 1$.

And now, lets plugin the other main inequality $z_{ij} \leq 2 - x_i - x_j$

$$2 - x_i - x_j \leq 2 (x_i + x_j - 2x_i x_j)$$

$$\Leftrightarrow 2 - (x_i + x_j) \leq 2 (x_i + x_j) - 4x_i x_j$$

This also holds true for $2 \geq x_i + x_j \geq 1$.

Problem 2.1

A randomized $\frac{1}{4}$ approximation algorithm :

lets formulate the given nonlinear integer program :

$$\max \sum_{(i,j)\in E} w_{ij} (x_i - x_i x_j)$$

$$\text{s.t.} \quad x_i \in \{0,1\}, \forall i \in U$$

At first lets assign each vertex uniformly at random to either V set or to W.
For any directed edge (i,j), the probability that it appears in the cut is

$$P(i\in V \text{ and } j\in W) = \frac{1}{2}\cdot\left(1-\frac{1}{2}\right) = \frac{1}{4}$$

Let N is the total weight of the cut.
So, by the linearity of expectation,

$$E(N) = \sum_{(i,j)\in E} P(i\in V \text{ and } j\in W)\cdot w_{ij}$$

$$= \sum_{ij\in E} \frac{1}{4} w_{ij} \geq \frac{1}{4} OPT_{ILP}$$

[Let OPT_{ILP} is the maximum weight of a cut & so, its the optimal value of the given ILP.

as expected weight of the cut is at least $P(i\in V \text{ and } j\in W)$ times the optimal value and as an upper bound for the OPT_{ILP}.

$$\sum_{(i,j)\in E} w_{ij} \geq OPT_{ILP}$$

Problem 2.2

Lets denote $z_{ij} = x_i - x_i x_j$

So, we can say $F_{ILP}(x) = F_{LP}(z)$

here

$$F_{ILP}(x) = \sum_{i,j \in E} w_{ij} (x_i - x_i x_j)$$

$$F_{LP}(z) = \sum_{i,j \in E} w_{ij} z_{ij}$$

To prove the $F_{LP}(z)$ is the relaxation of the $F_{ILP}(x)$, we need to prove that (x, z) is the feasible solution to the linear program.

Lets get the main two constraints from the LP program.

$$z_{ij} \leq x_i$$

$$\Leftrightarrow x_i - x_i x_j \leq x_i$$

$$\Leftrightarrow -x_i x_j \leq 0$$

$$z_{ij} \leq 1 - x_j$$

$$\Leftrightarrow x_i - x_i x_j \leq 1 - x_j$$

$$\Leftrightarrow x_i + x_j - x_i x_j - 1 \leq 0$$

Both of these two inequalities hold for all $0 \leq x_i, x_j \leq 1$

And $0 \leq x_i \leq 1, \forall i \in V$ is the relaxation of $x_i \in \{0,1\}$

$0 \leq z_{ij} \leq 1 \Leftrightarrow 0 \leq x_i - x_i x_j \leq 1$ always holds

as $0 \leq x_i, x_j \leq 1$

Problem 2.3

Given the probability of ($i \in V$), being in the cut is $\frac{1}{2}x_i + \frac{1}{4}$.

So, the probability of being the edge (i,j) in the cut is

$$P(i \in V \text{ and } j \in W) = \left(\frac{1}{4} + \frac{x_i}{2}\right)\left(1 - \left(\frac{1}{4} + \frac{x_j}{2}\right)\right)$$

$$= \left(\frac{1}{4} + \frac{x_i}{2}\right)\left(\frac{3}{4} - \frac{x_j}{2}\right)$$

$$= \left(\frac{1}{4} + \frac{x_i}{2}\right)\left(\frac{1}{4} + \frac{1-x_j}{2}\right)$$

$$\geq \left(\frac{1}{4} + \frac{z_{ij}}{2}\right)\left(\frac{1}{4} + \frac{z_{ij}}{2}\right)$$

[as $z_{ij} \leq x_i, \forall (i,j) \in E$

$z_{ij} \leq 1 - x_j, \forall (i,j) \in E$)

$$= \frac{1}{16} + \frac{z_{ij}}{4} + \frac{z_{ij}^2}{4}$$

$$= \left(\frac{1}{16} - \frac{z_{ij}}{4} + \frac{z_{ij}^2}{4}\right) + \frac{z_{ij}}{2}$$

$$= \left(\frac{1}{4} - \frac{z_{ij}}{2}\right)^2 + \frac{z_{ij}}{2}$$

which is definitey greater than or equal to $\frac{z_{ij}}{2}$

Now, suppose, OPT_{ILP} is the optimal value of the given integer non-linear program. and N is the total weight of the cut. Then, by the linearity of expectation

$$\mathbb{E}(N) = \sum_{(ij)\in E} P(i\in V \text{ and } j\in W) \cdot w_{ij}$$

$$\geq \sum_{i,j\in E} w_{ij} \frac{z_{ij}}{2} \geq \frac{1}{2} \cdot OPT_{IZP}$$