Problem 1 (i)

Solution:

step 1: Rewriting each inequality constraint as "≤ 0"

$$\min_{x_1 \geq 0, x_2 \leq 0, x_3} \quad a_1x_1 + a_2x_2 + a_3x_3 \quad \text{------} \quad (1)$$

$$b_1x_1 + b_2x_2 + b_3x_3 - e_1 \leq 0 \quad \text{------} \quad (2)$$

$$c_1x_1 + c_2x_2 - e_2 = 0 \quad \text{------} \quad (3)$$

$$-d_3x_3 + e_3 \leq 0 \quad \text{------} \quad (4)$$

step 2: defining dual variables.

$\lambda_1 \geq 0$ for equation (2)

λ_2 for equation (3)

$\lambda_3 \geq$ for equation (4)

step 3 multiply equations 2, 3 & 4 by the dual variables & get their sum and add it to equation 1

$$\max_{\lambda_1 \geq 0, \lambda_2, \lambda_3 \geq 0} \; \min_{x_1 \geq 0, x_2 \leq 0, x_3} \; a_1x_1 + a_2x_2 + a_3x_3 + \lambda_1(b_1x_1 + b_2x_2 + b_3x_3 - e_1) + \lambda_2(c_1x_1 + c_2x_2 - e_2) + \lambda_3(-d_3x_3 + e_3)$$

step 4: Rewrite the objective so that it becomes like
(primal variable) · (expression with dual vars)

$$\max_{\lambda_1 \geq 0, \lambda_2, \lambda_3 \geq 0} \quad \min_{x_1 \geq 0, x_2 \leq 0, x_3} \quad -e_1\lambda_1 - e_2\lambda_2 + e_3\lambda_3$$
$$+ x_1(a_1 + b_1\lambda_1 + c_1\lambda_2)$$
$$+ x_2(a_2 + b_2\lambda_1 + c_2\lambda_2)$$
$$+ x_3(a_3 + b_3\lambda_1 - d_3\lambda_3)$$

step 5: Remove primal variables and adding constraints:

$$\max_{\lambda_1 \geq 0, \lambda_2, \lambda_3 \geq 0} \quad -e_1\lambda_1 - e_2\lambda_2 + e_3\lambda_3$$

such that
$$a_1 + b_1\lambda_1 + c_1\lambda_2 \geq 0$$
$$a_2 + b_2\lambda_1 + c_2\lambda_2 \leq 0$$
$$a_3 + b_3\lambda_1 - d_3\lambda_3 = 0$$

So, this is the dual linear program of the given L.P.

Problem 1 (2)

Solution: KKT conditions:

① Stationary condition:

here

$$L \triangleq f(x) + \sum_i u_i h_i(x) + \sum_j v_j l_j(x)$$

$$= a_1 x_1 + a_2 x_2 + a_3 x_3 + c_1 x_1 \lambda_2 + c_2 \lambda_2 x_2 - e_2$$

$$+ \lambda_3 e_3 - d_3 x_3 \lambda_3 + b_1 \lambda_1 x_1 + b_2 \lambda_1 x_2 + b_3 \lambda_1 x_3 - e_1 \lambda_1$$

$$\frac{\partial L}{\partial x_i} = 0$$

After getting the derivative of L respect to x_1, x_2 & x_3, we get

$$a_1 + b_1 \lambda_2 + c_1 \lambda_2 = 0$$

$$a_2 + b_2 \lambda_1 + c_2 \lambda_2 = 0$$

$$a_3 + b_3 \lambda_1 - d_3 \lambda_3 = 0.$$

② Complementary slackness:

$$\lambda_1 (b_1 x_1 + b_2 x_2 + b_3 x_3 - e_1) = 0$$

$$\lambda_2 (c_1 x_1 + c_2 x_2 - e_2) = 0$$

$$\lambda_3 (e_3 - d_3 x_3) = 0$$

③ Primal feasibility:

$$b_1 x_1 + b_2 x_2 + b_3 x_3 - e_1 \leq 0$$

$$c_1 x_1 + c_2 x_2 - e_2 = 0$$

$$e_3 - d_3 x_3 \leq 0$$

④ Dual feasibility:

$$\lambda_1 \geq 0$$

$$\lambda_3 \geq 0$$

Problem 2 (1)

To show that the given problem is a convex optimization problem, we need to prove that the objective function and all the constraints are also convex. The first part of the objective function, $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} x_{ij}$ is convex, as its linear & so affine.

To prove the second part of the objective function is convex, we need to show its second order derivative is greater or equal to zero.

$$\text{So, } f''(x) = \frac{\partial}{\partial x} f'(x)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left((b_{ij} x_{ij} + b_{ij}) \ln(x_{ij}+1) - b_{ij} x_{ij}\right)\right)$$

$$= \frac{\partial}{\partial x}\left((b_{ij} x_{ij} + b_{ij}) \frac{1}{x_{ij}+1} + b_{ij} \ln(x_{ij}+1) - b_{ij}\right)$$

$$= \frac{\partial}{\partial x}\left(b_{ij} \ln(x_{ij}+1)\right)$$

$$= \frac{b_{ij}}{x_{ij}+1}$$

as x_{ij} is at $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots n$ &

$b_{ij} > 0$, so, $f''(x) = \frac{b_{ij}}{x_{ij}+1} \geq 0$.

So the objective function is convex.

The inequality constraints given all are linear-affine functions and thus they are convex functions as well.

Thus we can prove that the problem is convex optimization problem.

Problem 2 (2)

Lets take three dual variables α_i, β, γ for three constraints.

$$L \triangleq a_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}(x_{ij}+1)\ln(x_{ij}+1) - \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij} x_{ij}$$
$$+ \alpha_j c_j - \alpha_j \sum_{i=1}^{m} x_{ij} + \beta_i \sum_{j}^{n} x_{ij} + \beta_i \sum_{j}^{n} c_j - \beta_i d_i$$
$$- \beta_i \sum_{i=1}^{m} \sum_{j} x_{ij} - \gamma_{ij} x_{ij}$$

1. stationary condition:

$$\frac{\partial L}{\partial x_{ij}} = 0$$

$$\Rightarrow a_{ij} + b_{ij}\left(\frac{x_{ij}+1}{x_{ij}+1} + \ln(x_{ij}+1) - 1\right) - \alpha_j + \sum_i \beta_i - \beta_i - \gamma_{ij} = 0$$

$$\Rightarrow a_{ij} + b_{ij}\ln(x_{ij}+1) - \alpha_j + \sum_i \beta_i - \beta_i - \gamma_{ij} = 0$$

2. Complementary Slackness:

$$-\alpha_i (x_{ij} - c_j) = 0$$
$$\sum \beta_i \left(\sum_j x_{ij} + \sum_j c_j - \sum_i \sum_j x_{ij} - d_i\right) = 0$$
$$-\gamma_{ij} x_{ij} = 0$$

3. Primal feasibility:

$$x_{ij} - c_j \geq 0$$

$$\sum_i \sum_j x_{ij} + d_i \geq \sum_j x_{ij} + c_j$$

$$x_{ij} \geq 0$$

4. Dual feasibility:

$$\alpha_j \geq 0$$

$$\beta_i \geq 0$$

$$\gamma_{ij} \geq 0$$

Problem 3

Solution: Lets suppose the dual vars are u and v

$$\mathcal{L} = a^T x + u^T(Bx - c) - v^T x + x^T \mathrm{diag}(v) x$$
$$= x^T \mathrm{diag}(v) x + (a + B^T u - v)^T x - c^T u$$
$$= \sum_i v_i x_i^2 + \sum_i (a_i + B_i^T u_i - v_i) x_i - \sum_i c_i u_i$$

Taking the derivative of L over x,

$$\frac{\partial L}{\partial x} = 2 \sum_i v_i x_i + \sum_i (a_i + B_i^T u_i - v_i) = 0$$
$$\Rightarrow 2 \sum_i v_i x_i = \sum (v_i - a_i - B_i^T u_i)$$
$$\Rightarrow 2 v_i x_i = v_i - a_i - B_i^T u_i$$
$$\Rightarrow x_i = \frac{v_i - a_i - B_i^T u_i}{2 v_i}$$

Replacing x_i in $\frac{\partial L}{\partial x}$, we get,

$$\sum v_i \left(\frac{1}{4 v_i^2} (v_i - a_i - B_i^T u_i)^2 \right) - \sum (a_i + B_i^T u_i - v_i) \cdot \frac{1}{2 v_i} (v_i - a_i - B_i^T u_i) - \sum_i c_i u = 0$$

$$= -c^T u - \frac{1}{4} \sum \frac{(a_i + B_i^T u_i - v_i)^2}{v_i}$$

So, the dual function is

$$g(u,v) = -c^T u - \frac{1}{4} \sum_i^n (a_i + B_i^T u_i - v_i)^2 / v_i \; .$$

So, The dual problem is

$$\max \quad -c^T u - \frac{1}{4} \sum (a_i + B_i^T u_i - v_i)^2 / v_i$$
such that $v \geq 0$

Now lets get the derivative of the second part over v_i

$$\frac{\partial}{\partial v} \left(- \sum (a_i + B_i^T u_i - v_i)^2 / v_i \right) = 0$$

$$\Rightarrow v_i = \pm (a_i + B_i^T u_i)$$

replacing v_i by this value in the dual problem,

we get $\quad 0 \quad$ when $v_i = a_i + B_i^T u_i$

$$4 (a_i + B_i^T u_i) \text{ when } v_i = -(a_i + B_i^T u_i)$$

Thus, we are removing the v_i from the dual problem,

we get

$$\max \quad -c^T u + \sum_i \min (0, a_i + B_i^T u_i),$$
such that $u \geq 0$

and $\quad \max \sum \min (0, a_i + B_i^T u_i) = 0$

Now, lets take the $L(x,u,v)$ of the problem P2

$$L = a^T x + u^T(Bx - c) - v^T x + w^T(x-1)$$
$$= (a + B^T u - v + w)^T x - 1^T w - c^T u$$

and the dual function of P2 is

$$g(u,v,w) = -c^T u - 1^T w$$

The dual problem is

$$\max \quad -c^T u - 1^T w$$
$$\text{such that} \quad a + B^T u - v + w = 0$$
$$u \geq 0,\ v \geq 0,\ w \geq 0$$

From $a + B^T u - v + w = 0$, we get,

$$-w = a + B^T u - v$$

Lets replace w in $g(u,v,w)$, & we get

$$g(u,v,w) = -c^T u + 1^T(a + B^T u - v)$$
$$= -c^T u + \sum_i^n (B_i^T u_i - v_i + a_i)$$

since $v \geq 0$, & also $w \geq 0 \Rightarrow -w \leq 0$ and so, $B_i^T u_i - v_i + a_i \leq 0$

$\max \sum_i B_i^T u_i - v_i + a_i$ becomes zero.

as $\max \min\{0, B_i^T u_i + a_i\} = 0$

And thus, we can prove that the optimal solutions of the dual problems of P2 and P_3 are same and so, the lower bounds are also same.