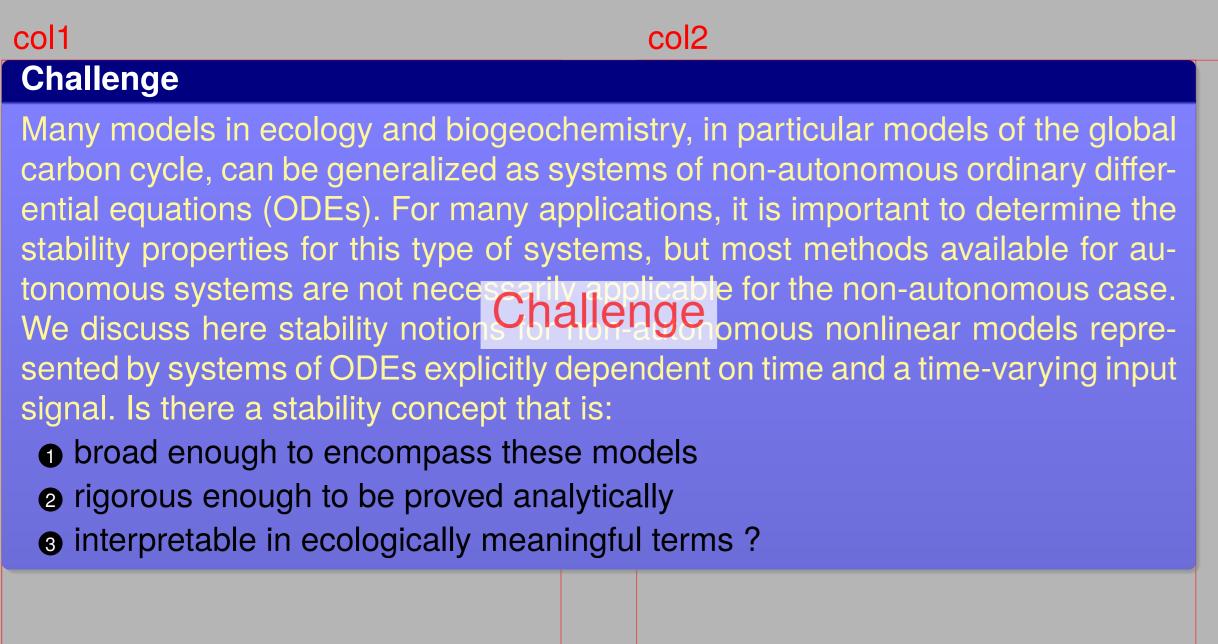
Application of Input to State Stability to ecological reservoir models

Department Biogeochemical Processes



Example: general soil model

 $C = I(t) + T(C, t) \cdot N(t, C) \cdot C(t)$ $N_{i,i}(C,t) \geq 0 \quad \forall i$ $T_{i,i}(C,t) \geq 0 \quad \forall i \neq j$ $\sum_{i} T_{i,j}(C,t) = 1 \quad \forall j$

This model structure generalizes most SOM decomposition models with any arbitrary number of pools, including those describing nonlinear interactions among state variables. It enforces mass balance and substrate dependence of decomposition, and it is flexible enough to describe:

- Heterogeneity of decomposition rates
- 2 Transformations of organic matter
- 3 Environmental variability effects
- Examples for nonlinear models are:

Organic matter interactions

• Exoenzyme models [Schimel and Weintraub, 2003, Sinsabaugh and Follstad Shah, 2012]

- 2 AWB [Allison et al., 2010]
- 3 Bacwave [Zelenev et al., 2000]
- MEND [Wang et al., 2013]
- 6 Manzoni [Manzoni and Porporato, 2007]

Also linear models fit into the general framework

- Henin's model [Henin and Dupuis, 1945, Henin et al., 1959]
- 2 ICBM [Andren and Katterer, 1997]
- RothC [Jenkinson and Rayner, general₉₇₇, Coleman and Jenkinson, 1999]
 - Century [Parton et al., 1987]
 - **5** Fontaine 1-4 [Fontaine and Barot, 2005]

One general concept to encompass especially these nonlinear models is clearly desirable.



The graphs show the reactions of a prototypical class of non- ISS proofs for a particular system. linear two pool soil models to a disturbing time varying signal. This model is a technically simple place holder for ecologically motivated nonlinear systems like the soil models mentioned above to be analyzed in the future. It is given by:

$$\dot{C}_{x} = I_{x}(t) - \left(C_{x}^{2} + C_{x}\right) k_{x}(t) \tag{1}$$

 $\dot{C}_y = I_y(t) - \left(C_y^2 + C_y\right) k_y(t)$ where C_x , C_y are the carbon contents of two unconnected

$$k_{x_{min}} \le k_x(t) \le k_{x_{max}} \tag{3}$$

$$k_{y_{min}} \le k_y(t) \le k_y \tag{4}$$

describe the seasonal changes in decomposition speed. e.g.:

$$k_{x} = \frac{k_{xmax}}{2} + \frac{k_{xmin}}{2} + \frac{1}{2}(k_{xmax} - k_{xmin})\sin(4t)$$

$$k_{y} = \frac{k_{ymax}}{2} + \frac{k_{ymin}}{2} + \frac{1}{2}(k_{ymax} - k_{ymin})\sin(4t)$$

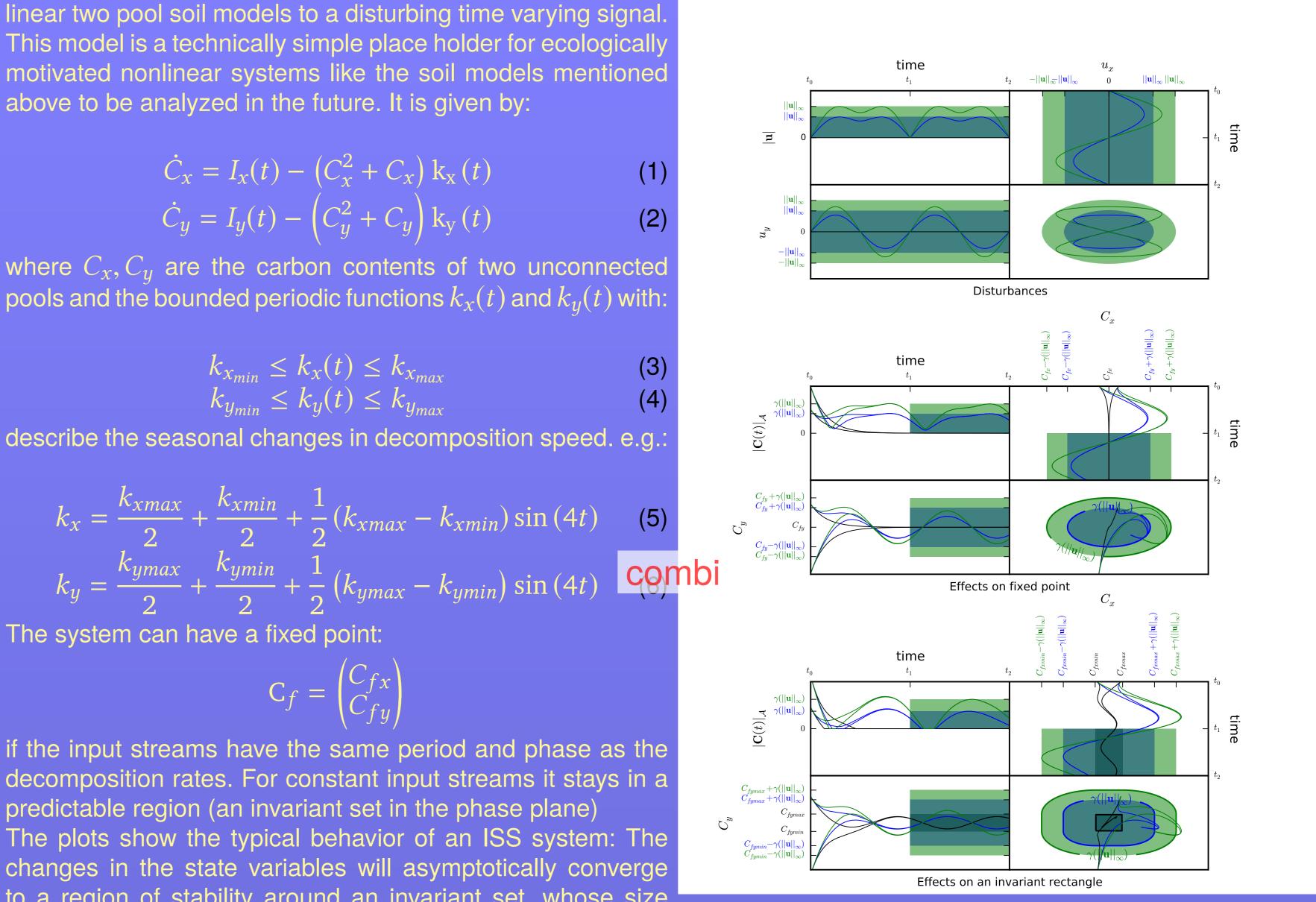
$$k_{y} = \frac{k_{ymax}}{2} + \frac{k_{ymin}}{2} + \frac{1}{2}(k_{ymax} - k_{ymin})\sin(4t)$$

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The system can have a fixed point:

$$C_f = \begin{pmatrix} C_{fx} \\ C_{fy} \end{pmatrix}$$

if the input streams have the same period and phase as the decomposition rates. For constant input streams it stays in a predictable region (an invariant set in the phase plane) The plots show the typical behavior of an ISS system: The changes in the state variables will asymptotically converge to a region of stability around an invariant set, whose size is a monotone function of the size of the disturbance (denoted by $|u|_{\infty}$). For this particular system we proved the ISS property rigorously. The proof relies on the construction of an ISS Lyapunov function whose choice is not determined but inspired by a property of the system interpretable in ecologically terms. Expressed casually: "The system can counteract supply changes fast enough". This situation seems to be typical: The problem of establishing ISS for e.g. all the I, T, N models based on the ecologic principles they follow, is too hard. But bio-chemical of biophysical restrictions could provide clues to



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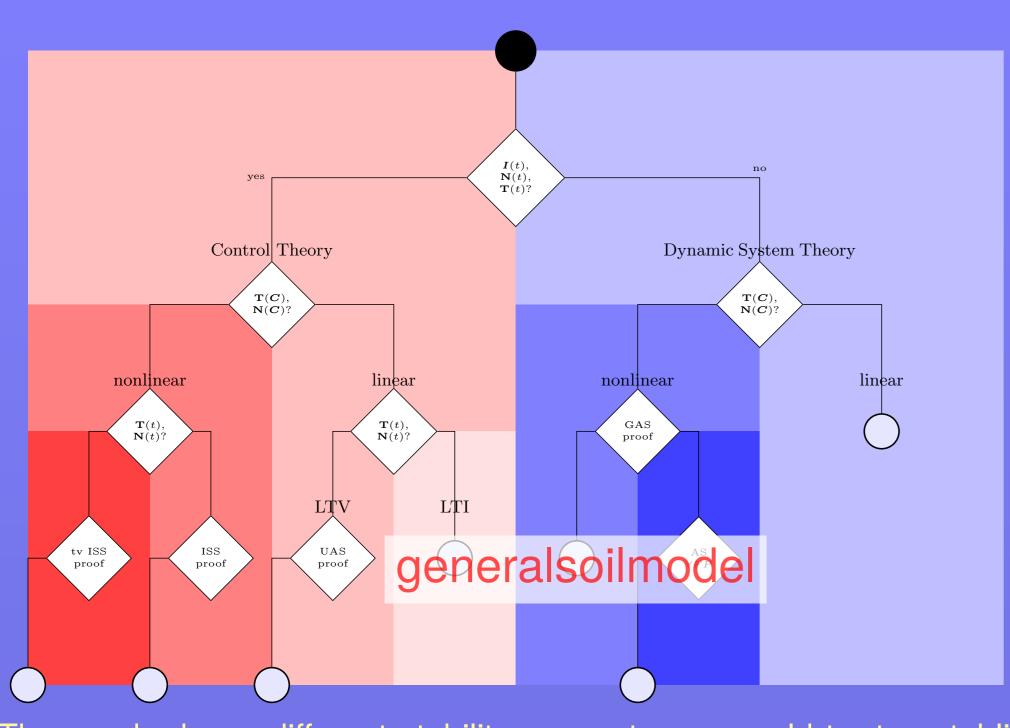
- The four plots on the top show the disturbances.
- 2 The next four plots in the middle show the effect of this disturbances on the solutions for a system with fixed point.
- 3 The next four plots in the middle show the effect of this disturbances on the solutions for the system which no longer has a fixed point, but at least an invariant set, the dark blue square in the middle.

Conclusion

We propose Input to State Stability (ISS) as candidate for the necessary generalization of the established analysis with respect to equilibria or invariant sets for autonomous systems, and showed for example systems its usefulness by applying it to reservoir models typical for element cycling in ecosystem, e.g. in soil organic matter decomposition. In a forthcoming paper we also showed how ISS generalizes existent concepts formerly only available for Linear Time Invariant (LTI) and Linear Time Variant (LTV) systems to the nonlinear case.

Intro

Results I, ISS as generalization of available stability concepts



The graph shows different stability concepts one could try to establish for the general soil model mentioned above depending on properties of its components I, T and N. The hardest to prove is Input to State Stability for time varying systems (ISStv) in the lower left corner. It turns out that ISStv also generalizes all the other concepts mentioned:

- In the case of Linear Time Invariant (LTI) systems ISS follows from the properties of the matrix.(eigenvalues)
- In the case of Linear Time Variant (LTV) systems it can be established if sufficient information about the state transition operator allows to prove uniform asymptotic stability UAS.
- For input free system (on the blue right-hand side) ISS reduces to Global Asymptotic Stability (GAS)

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