Emmy Noether-Programm
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Max Planck Institute for Biogeochemistry

IMPRS gBGC
International Max Planck Research School for Global Biogeochemical Cycles

We find general formulas for the transit time and age distributions in well-mixed compartmental systems. As an example we consider a simple three-pool global carbon model with nonlinearities in two different scenarios:

steady state	pre-industrial system before 1765
perturbed	adding fossil fuels to the atmosphere, considering land use change, 1765-2500

transit time	time that a particle needs to travel through the system exit time − entry time
system age	for particles in the system current time − entry time
compartment age	system age of particles in a compartment

$$\Phi(t, t_0) = \exp\left[(t - t_0)\, \mathrm{A}(t)\right]$$

$$\tfrac{d}{dt}\, \Phi(t, t_0) = \mathrm{A}(t)\, \Phi(t, t_0)$$

one-dimensional	$\Phi(t, t_0) = \exp\left[(t - t_0)\, A(t)\right]$	exponential function
multi-dimensional, autonomous		matrix exponential
multi-dimensional, nonautonomous		only numerical solution

Well-mixed compartmental systems can be descibed by a system of generally **nonlinear** ordinary differential equations of the form

$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}(\mathbf{x}(t), t)\,\mathbf{x}(t) + \mathbf{u}(t), \quad t > t_0, \tag{1}$$

with a given initial value $\mathbf{x_0}$.

We assume to know (at least numerically) the unique solution of (1) and denote it by $\mathbf{x}$. We then plug it into $\mathrm{A}(\mathbf{x}(t), t)$ and obtain the **linear** system of ordinary differential equations

$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}(t)\,\mathbf{x}(t) + \mathbf{u}(t), \quad t > t_0.$$

This equation has the general solution formula

$$\mathbf{x}(t) = \underbrace{\Phi(t, t_0)\,\mathbf{x_0}}_{\mathrm{age}(t) = t - t_0 + \text{initial age}} + \int\limits_{t_0}^{t} \underbrace{\Phi(t, \tau)\,\mathbf{u}(\tau)}_{\mathrm{age}(t) = t - \tau}\,d\tau,$$

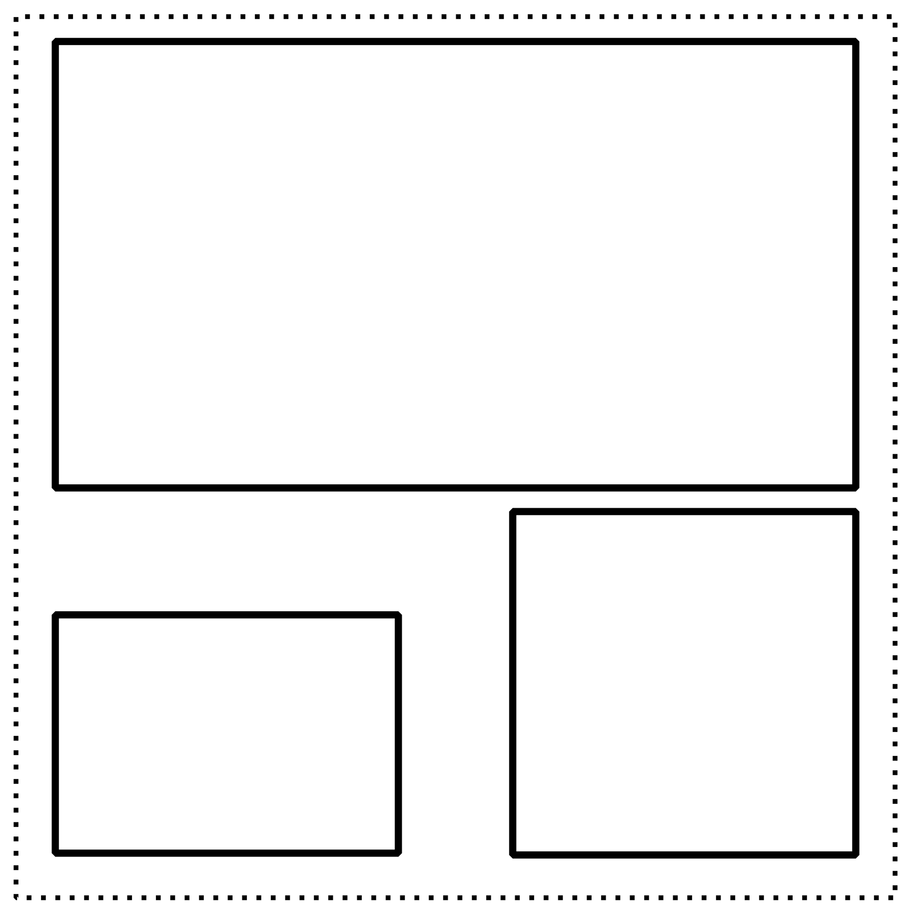
where Φ is the so-called state transition matrix. This leads immediately to the vector of age densities

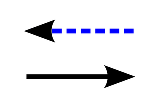
$$\mathbf{p}(a, t) = \begin{cases} \Phi(t, t_0)\,\mathbf{p_0}(a - (t - t_0)), & a \geq t - t_0, \\ \Phi(t, t - a)\,\mathbf{u}(t - a), & a < t - t_0, \end{cases}$$

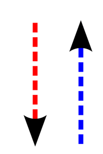
where $\mathbf{p_0}$ is the initial age distribution.

If we define the (backward) transit time of a particle as the age of the particle at the time when it exits the system, we can easily derive explicit formulas for its distribution from the age distribution.

The simple approach stock/flux for the transit time is valid only for a system in steady state. Out of steady state, stock(t)/flux(t) cannot be interpreted as a transit time.







$x_T^* = 3000$ PgC

$x_A^* = 700$ PgC

$x_S^* = 1000$ PgC

$$u_S = 45, \quad u_A(t) = \text{fossil-fuels}(t)$$

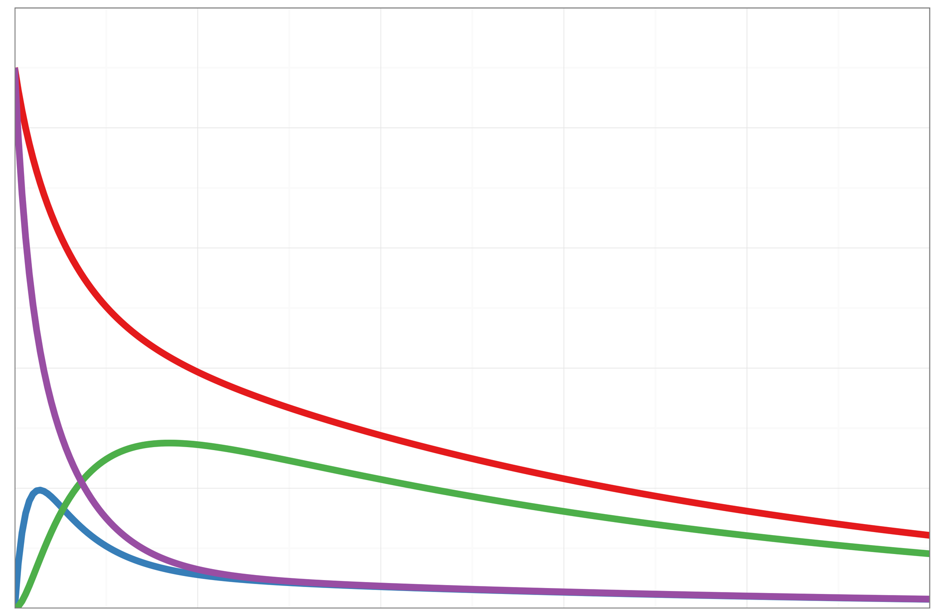
$$F_{AT} = 60\,(x_A/700)^{0.2}$$
$$F_{TA}(t) = 60\,x_T/3000$$
$$+\,\text{land-use-change}(t)$$
$$F_{AS} = 100\,x_A/700$$
$$F_{SA} = 100\,(x_S/1000)^{10}$$

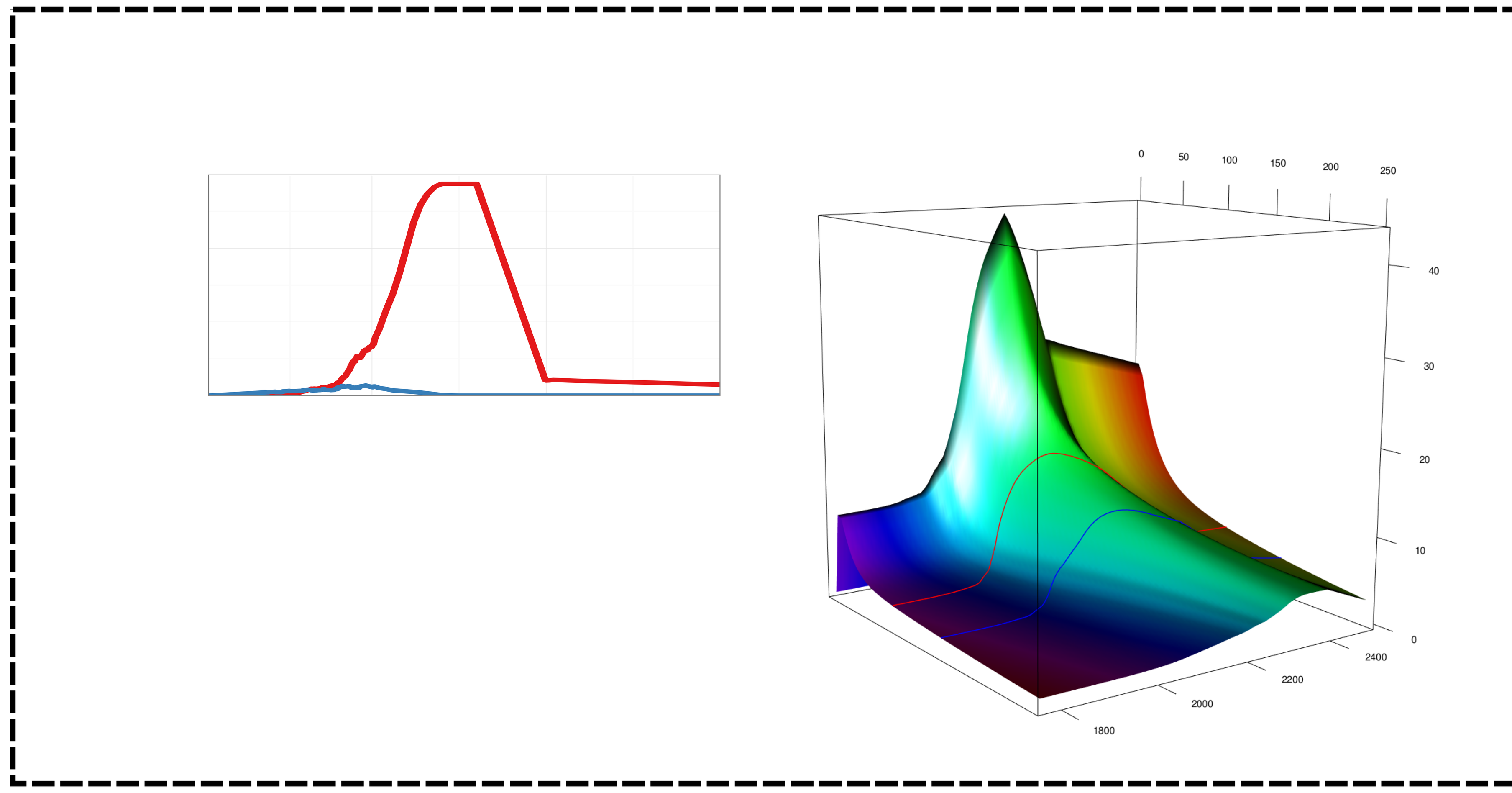
$$F_{SD} = 45\,x_S/1000$$

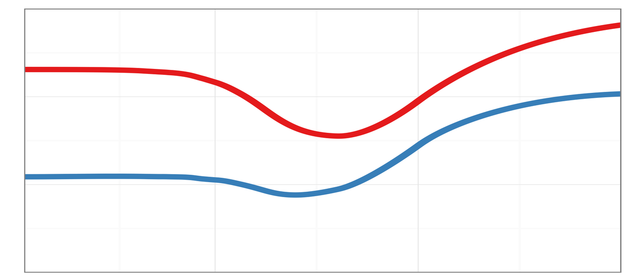
$$p_T(t) = \mathbf{z}^T \, e^{t\,\mathrm{A}} \, \mathbf{u}$$

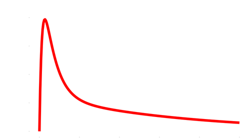
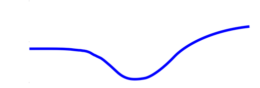
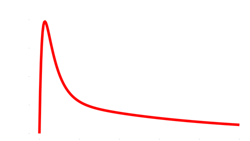
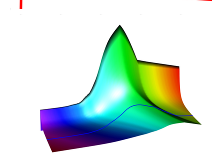
$$p_A(a) = \sum_j \left(e^{a\,\mathrm{A}}\,\mathbf{u}\right)_j$$

$$p_j(a) = \left(e^{a\,\mathrm{A}}\,\mathbf{u}\right)_j$$




0
50
100
150
200
250
40
30
20
10
0
1800
2000
2200
2400



Year	Author	System	Quantity	Geometric structure	Example
1983	Anderson	steady state	mean	point	
2009	Manzoni et al.	steady state (simple structure)	density	curve over age	
2016	Rasmussen et al.	linear time dependent	mean	curve over time	
2016	Metzler & Sierra	steady state (all structures)	density	curve over age	
2017	Metzler et al.	linear/nonlinear time dependent	density	surface over age and time	

$\mathbf{x}(t)$	vector of compartment content (e.g. C) at time t
A	compartmental matrix, describes fluxes between compartments
$\mathbf{z}(t)$	external outflux vector at time t
$\mathbf{u}(t)$	external input vector at time t

$\mathrm{A}(\mathbf{x}(t), t)$ $\mathrm{A}(\mathbf{x}(t)), \mathbf{u}(t)$	nonlinear, nonautonomous
$\mathrm{A}(\mathbf{x}(t)), \mathbf{u}$	nonlinear, autonomous
$\mathrm{A}(t), \mathbf{u}$ $\mathrm{A}, \mathbf{u}(t)$ $\mathrm{A}(t), \mathbf{u}(t)$	linear, nonautonomous
$\mathrm{A}, \mathbf{u}$	linear, autonomous