Nonlinear Dynamics and Caos
Problem 5.2.2

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consider the following matrix

$$\mathbf{M} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

We compute the eigenvalues and the eigenvalues theire multiplicity and the eigenvectors:

$$\left[\left(1 + \imath, \quad 1, \quad \left[\left[\begin{smallmatrix} \imath \\ 1 \end{smallmatrix} \right] \right] \right), \quad \left(1 - \imath, \quad 1, \quad \left[\left[\begin{smallmatrix} -\imath \\ 1 \end{smallmatrix} \right] \right] \right) \right]$$

In detail this means:

$$\lambda_1 = 1 + \imath \qquad \lambda_2 = 1 - \imath$$
$$\mathbf{e_1} = \left[\begin{smallmatrix} \imath \\ 1 \end{smallmatrix} \right] \qquad \mathbf{e_2} = \left[\begin{smallmatrix} -\imath \\ 1 \end{smallmatrix} \right]$$

So the general solution is:

$$
\begin{aligned}
\mathbf{x}(t) = \left[\begin{smallmatrix} x \\ y \end{smallmatrix} \right] &= c_1 e^{\lambda_1 t} \mathbf{e_1} + c_2 e^{\lambda_2 t} \mathbf{e_2} \\
&= \left[\begin{smallmatrix} \imath c_1 e^{t(1+\imath)} - \imath c_2 e^{t(1-\imath)} \\ c_1 e^{t(1+\imath)} + c_2 e^{t(1-\imath)} \end{smallmatrix} \right] \text{ with } c_1, c_2 \in \mathbb{C}
\end{aligned}
\tag{1}
$$

Remark:
These two complex constants actually translate into four real ones, since c_1 and c_2 both have a real and an imaginary part.

$$c_1 = \imath c_{1i} + c_{1r} \qquad c_2 = \imath c_{2i} + c_{2r}$$

We can also look at it the other way round: From a more general point of view our two real initial conditions are actually complex ones with an imaginary part of zero. Either way four constants to fulfill four conditions look promising.
Remark:
Any two dimensional vector-space can (by definition) be spanned by two linearly independent vectors. Any such set is called a base of the vector-space. To switch to another one one is free to create new base vectors as linear combinations of the old ones. For convenience we would like to have real base vectors. (We do not loose generality by this choice since we can still multiply them with complex coefficients to regain the complex solutions)
Remark:

Actually in our case of conjugate eigenvectors it is very easy to create an new (real only) base. We will proceed to do so right now. Expressing the general solution by real constants only we get:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \imath(\imath c_{1i}+c_{1r})e^{t(1+\imath)} - \imath(\imath c_{2i}+c_{2r})e^{t(1-\imath)} \\ (\imath c_{1i}+c_{1r})e^{t(1+\imath)} + (\imath c_{2i}+c_{2r})e^{t(1-\imath)} \end{bmatrix}$$

in component form:

$$
\begin{aligned}
x &= \imath\left(\imath c_{1i} + c_{1r}\right)e^{t(1+\imath)} - \imath\left(\imath c_{2i} + c_{2r}\right)e^{t(1-\imath)} \\
&= \imath\left(\imath c_{1i} + c_{1r}\right)\left(\imath\sin(t) + \cos(t)\right)e^{t} - \imath\left(\imath c_{2i} + c_{2r}\right)\left(-\imath\sin(t) + \cos(t)\right)e^{t} \\
\Re(x) &= \left((-c_{1i} + c_{2i})\cos(t) + (-c_{1r} - c_{2r})\sin(t)\right)e^{t} \\
\Im(x) &= \left((-c_{1i} - c_{2i})\sin(t) + (c_{1r} - c_{2r})\cos(t)\right)e^{t} \\
y &= \left(\imath c_{1i} + c_{1r}\right)e^{t(1+\imath)} + \left(\imath c_{2i} + c_{2r}\right)e^{t(1-\imath)} \\
&= \left(\imath c_{1i} + c_{1r}\right)\left(\imath\sin(t) + \cos(t)\right)e^{t} + \left(\imath c_{2i} + c_{2r}\right)\left(-\imath\sin(t) + \cos(t)\right)e^{t} \\
\Re(y) &= \left((-c_{1i} + c_{2i})\sin(t) + (c_{1r} + c_{2r})\cos(t)\right)e^{t} \\
\Im(y) &= \left((c_{1i} + c_{2i})\cos(t) + (c_{1r} - c_{2r})\sin(t)\right)e^{t}
\end{aligned}
$$

$$(2)$$

Sorting for real and imaginary parts leads to the following expression for the general solution:

$$
\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{pmatrix} \Re(x) \\ \Re(y) \end{pmatrix} + i \begin{pmatrix} \Im(x) \\ \Im(y) \end{pmatrix} \\
&= \begin{pmatrix} \left((-c_{1i} + c_{2i})\cos(t) + (-c_{1r} - c_{2r})\sin(t)\right)e^{t} \\ \left((-c_{1i} + c_{2i})\sin(t) + (c_{1r} + c_{2r})\cos(t)\right)e^{t} \end{pmatrix} \\
&\quad + i \begin{pmatrix} \left((-c_{1i} - c_{2i})\sin(t) + (c_{1r} - c_{2r})\cos(t)\right)e^{t} \\ \left((c_{1i} + c_{2i})\cos(t) + (c_{1r} - c_{2r})\sin(t)\right)e^{t} \end{pmatrix} (3)
\end{aligned}
$$

It is now obvious that we can produce the new real base vectors by tuning the four real parameters. We can force the imaginary part of the sum away by the choice

$$\left\{ c_{1i} : -c_{2i}, \quad c_{1r} : c_{2r} \right\}$$

which leads to the simplified first part:

$$\begin{pmatrix} (2c_{2i}\cos(t) - 2c_{2r}\sin(t))e^{t} \\ (2c_{2i}\sin(t) + 2c_{2r}\cos(t))e^{t} \end{pmatrix}$$

We can further simplify this solution by the following choice:

$$\left\{ c_{2i} : \tfrac{1}{2}, \quad c_{2r} : 0 \right\}$$

which leads to a very simple solution indeed:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} e^{t}\cos(t) \\ e^{t}\sin(t) \end{pmatrix}$$

Lets test if it is indeed a solution of the ode:

$$\dot{\mathbf{X}} = \begin{bmatrix} -e^{t}\sin(t) + e^{t}\cos(t) \\ e^{t}\sin(t) + e^{t}\cos(t) \end{bmatrix}$$

$$\mathbf{MX} = \begin{bmatrix} -e^t \sin(t) + e^t \cos(t) \\ e^t \sin(t) + e^t \cos(t) \end{bmatrix}$$

Now we create a second real only solution by first creating a imaginary only solution and then multiplying it with i. To this end we force the real part in (3) to zero. We make the following choice:

$$\left\{ c_{1i} : c_{2i}, \quad c_{1r} : -c_{2r} \right\}$$

which leads to the simplified second part:

$$\begin{pmatrix} (-2c_{2i} \sin(t) - 2c_{2r} \cos(t)) e^t \\ (2c_{2i} \cos(t) - 2c_{2r} \sin(t)) e^t \end{pmatrix}$$

We can further simplify this solution by the following choice, that also insures that our second solution is no multiple of the first:

$$\left\{ c_{2i} : \tfrac{1}{2}, \quad c_{2r} : 0 \right\}$$

which leads to:

$$[\begin{smallmatrix} x \\ y \end{smallmatrix}] = \begin{pmatrix} -e^t \sin(t) \\ e^t \cos(t) \end{pmatrix}$$

Lets test if it is indeed a solution of the ode:

$$\dot{\mathbf{X}} = \begin{bmatrix} -e^t \sin(t) - e^t \cos(t) \\ -e^t \sin(t) + e^t \cos(t) \end{bmatrix}$$

$$\mathbf{MX} = \begin{bmatrix} -e^t \sin(t) - e^t \cos(t) \\ -e^t \sin(t) + e^t \cos(t) \end{bmatrix}$$

To test that our solutions are not lineary dependent we joint them to a matrix and compute the determinant, which must not be zero.

$$\det \begin{bmatrix} e^t \cos(t) & -e^t \sin(t) \\ e^t \sin(t) & e^t \cos(t) \end{bmatrix} = e^{2t} \sin^2(t) + e^{2t} \cos^2(t) = e^{2t} \neq 0$$

Since the new two solutions are linear independent they span the whole vector space of solutions, even the complex part that we are not interested in, since we can multiply them again with complex numbers $\tilde{c}_1$ and $\tilde{c}_2$. In fact we did not loose any generality. We just chose a differen base.