DSP (cls-11) Chapter-3 14/3/23

Z-transform

→ Forier transform

→ Z-Transform is the discrete version of laplace Transform.

Z Transform :-

The Z-transform of a DT signal $x(n)$ is defined as the power series.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Z is a complex variable. $\boxed{z = re^{j\omega}}$

$$x(n) \xleftrightarrow{Z} X(z)$$

converge diverge

Z-Transform exists only for those values of z for which the series converges.

The region of converge (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

Ex: 3.1.1 $x_1(n) = \{\underline{1}, 2, 5, 7, 0, 1\}$ $\sum_{n=0}^{5} x(n) z^{-n}$

$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5}$$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

ROC:- entire z-plane except $z=0$.

$x_2(n) = \{1, 2, \underline{5}, 7, 0, 1\}$

$$= z^2 + 2z^1 + 5 + 7z^{-1} + z^{-3}$$

ROC:- entire z plane except $z=0$ and $z=\alpha$.

$x_6(n) = \delta(n)$

$x(z) = 1$, ROC:- entire z-plane.

$x_6(n) = \delta(n-k)$

$X(z) = z^{-k}$

$x_7(n) = \delta(n+k)$

$X(z) = z^{k}$

DSP (cls-12)

Online

4/12/23 ①

Online

$$x(z) = \sum_{n=-\alpha}^{\alpha} x(n) z^{-n}, \quad z = re^{j\omega}$$

Ex-3.1.1 $x(n) = \{1, 2, \underset{0}{5}, 7, 0, 1\}$

$$x(z) = x(-2)z^2 + x(-1)z^1 + x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= z^2 + 2 \cdot z^1 + 5 + 7z^{-1} + z^{-3}$$

ROC = all values of z except $z = 0, \alpha$

Ex-3.1.2 Determine the z-transform of the signal

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n)$$

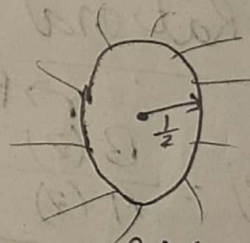
$$= \left\{\underset{0}{1}, \tfrac{1}{2}, \left(\tfrac{1}{2}\right)^2, \left(\tfrac{1}{2}\right)^3, - - \right\} \qquad u(n) = \{1, 1, 1 \dots\}, \; n = 0, 1, 2$$

power series

$$x(z) = 1 + \tfrac{1}{2} z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \left(\tfrac{1}{2}\right)^3 z^{-3} + - - .$$

$$= 1 + A + A^2 + A^3 + - -$$

$$= \frac{1}{1-A}, \; |A| < 1$$



ROC

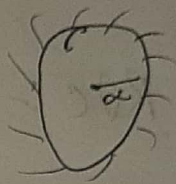
$$x(z) = \frac{1}{1 - \left(\frac{1}{2}\right) z^{-1}}, \quad \text{ROC}: |z| > \tfrac{1}{2}$$

→ signal R.H.S হলে ROC = outer side.

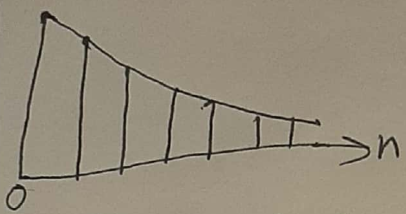
* For a RHS signal the ROC is outside of the circle.

Ex-3.1.3 Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \rightarrow \text{RHSS}$$



$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$



$$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad \alpha z^{-1} < 1.$$

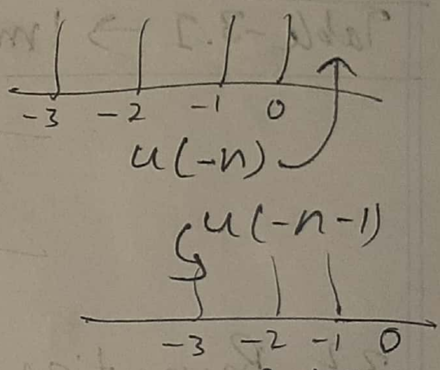
ROC = $|z| > \alpha$

α হতে বেশি হলে convergence.

– delay = ডান
fold = বাম

Ex-3.1.4 $x(n) = -\alpha^n u(-n-1)$

$$= \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$



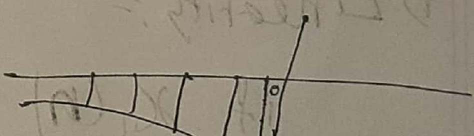
$$X(z) = \sum_{n=-\infty}^{-1} (-\alpha^n) z^{-n}$$

$$= \sum_{n=-1}^{-\infty} (\alpha^{-1} z)^n = -\sum_{l=1}^{\infty} (\alpha^{-1} z)^l \quad [l = -n]$$

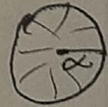
L.H.S.S.

$$= A + A^2 + A^3 + \cdots$$

$$= A(1 + A + A^2 + \cdots)$$



$$X(z) = -\frac{A}{1-A}$$



ROC

$$= -\frac{\alpha^{-1} z}{1-\alpha^{-1} z} = -\frac{1}{\frac{1}{\alpha^{-1} z} - 1} = \frac{1}{1-\alpha z^{-1}}, \quad |\alpha^{-1} z| < 1$$

ROC = $|z| < \alpha$

* ROC of a L.H.S. signal is the interior of the circle.

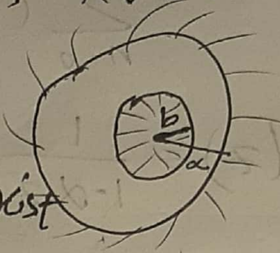
Ex-3.1.5. $x(n) = \alpha^n u(n) + b^n u(-n-1)$

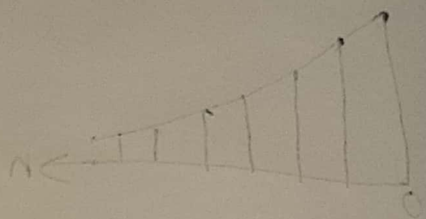
$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}}$$

ROC :- $|z| > \alpha$ ∴ ROC :- $|z| < b$

case-1 :- $|b| < |\alpha|$

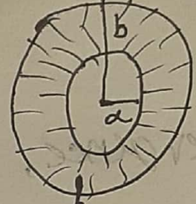
· ROC doesn't exist

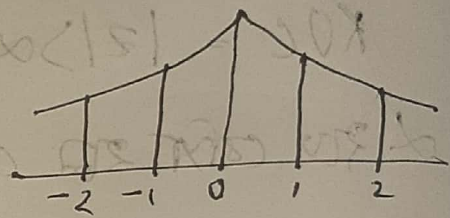




case-2 :- $|b| > |\alpha|$

ROC $= |\alpha| < |z| < |b|$





$\boxed{\text{Table-3.1} \rightarrow \text{Imp}}$

DSP (T-13)

14/1/23
online

3.2 Properties of z transform

পৃথকভাবে * করে + করলে যা
+ " * " তা → Lin

1) **Linearity :-**

if $x_1(n) \longleftrightarrow X_1(z)$

$x_2(n) \longleftrightarrow X_2(z)$

then $x(n) = a_1 x_1(n) + a_2 x_2(n) \longleftrightarrow a_1 X_1(z) + a_2 X_2(z)$

Ex-3.2.1 $x(n) = [3(2^n) - 4(3^n)]u(n)$

$= 3(2^n)u(n) - 4(3^n)u(n)$

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$$

Time shifting:-

if $x(n) \xrightarrow{z} X(z)$

$x(n-k) \xrightarrow{z} z^{-k} X(z)$

$\delta(n) \xrightarrow{z} 1$

$\delta(n-k) \longrightarrow z^{-k}$

Ex:- 3.2.3 — Hw.

- Ex: 3.2.4 / $x(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0 & \text{elsewhere} \end{cases}$

$\boxed{\text{finite} = \frac{1-a^N}{1-a} \quad \text{Infinite} = \frac{1}{1-a}}$

$$X(z) = \sum_{n=0}^{N-1} 1 \cdot z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots z^{-(N-1)}$$

$$= \frac{1-z^{-N}}{1-z^{-1}}$$

Find the z transform:-

$x(n) = u(n) - u(n-N)$

$Z(z) = Z[u(n) - u(n-N)]$

$= Z[u(n)] - Z[u(n-N)]$

$= \frac{1}{1-z^{-1}} - z^{-N} \frac{1}{1-z^{-1}}$

$= (1-z^{-N}) \times \frac{1}{1-z^{-1}}$

Differentiation in the z-domain

if $x(n) \overset{z}{\longleftrightarrow} X(z)$

then $nx(n) \overset{z}{\longleftrightarrow} -z \dfrac{dX(z)}{dz}$

Proof — H.W.

Ex-3.2.7 $x(n) = na^n u(n)$ $\longrightarrow \dfrac{1}{1-az^{-1}}$

$$= -z \frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right] \rightarrow (1-az^{-1})^{-1}$$

$$= -z \cdot \frac{-az^{-2}}{(1-az^{-1})^2} = \frac{az^{-1}}{(1-az^{-1})^2}$$

Convolution of two sequences:-

(z-t এ সহজে convolution করা যায় & faster)

if $x_1(n) \longrightarrow X_1(z)$

$x_2(n) \longrightarrow X_2(z)$

then $x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X_1(z) \cdot X_2(z)$

inverse $x_1(n) * x_2(n)$

Ex-3.2.9 $x_1(n) = \{1, -2, 1\}$

$x_2(n) = \{1, 1, 1, 1, 1, 1\}$

compute the convolution $x(n)$ of the signal.

$X_1(z) = 1 - 2z^{-1} + z^{-2}$

$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

$X(z) = X_1(z) \cdot X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$

$x(n) = \{1, -1, 0, 0, 0, 0, -1, 1\}$

CamScanner

Steps for computing convolution using z-transform

1) Complete the z-transform of the signals to be convolved.

$$X_1(z) = Z\{x_1(n)\}$$
$$X_2(z) = Z\{x_2(n)\}$$

2) Multiply the two z transforms.

$$X(z) = X_1(z)\, X_2(z)$$

3) Find the inverse z-transforms of $X(z)$

$$x(n) = Z^{-1}\{X(z)\}$$

DSP (T-14)

3/4/23
Online

3.3 Rational z-Transform

Polinomial

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

roots

$$= C\,\frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

2nd order

$ax^2 + bx + c = 0$

$(x-x_1)(x-x_2) = 0$

$x = x_1$, $x = x_2$

roots

$$X(z) = C\,\frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

Poles & Zeros

The zeros of z-transforms $X(z)$ are the values of z for which

$X(z) = 0$. [Zeros :- $z_1, z_2, z_3 \dots z_n$]
↳ roots of the numerator (লব)

The poles of a z-transforms are the values of z for which

$X(z) = \alpha$. [Poles :- $P_1, P_2, P_3, \dots P_N$]
↳ roots of the denominator (হর)

Ex.3.3.1/ Determine the pole-zero plot for the signal

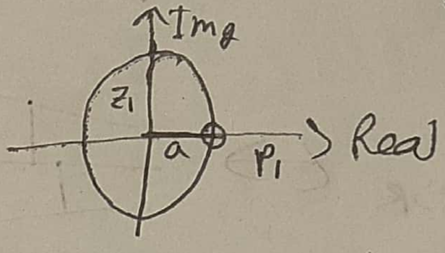
$x(n) = a^n u(n)$

Sln: $X(z) = \dfrac{1}{1-az^{-1}} = \dfrac{1}{1-\frac{a}{z}} = \dfrac{1}{\frac{z-a}{z}} = \dfrac{z}{z-a} = \dfrac{z-0}{z-a}$

↓

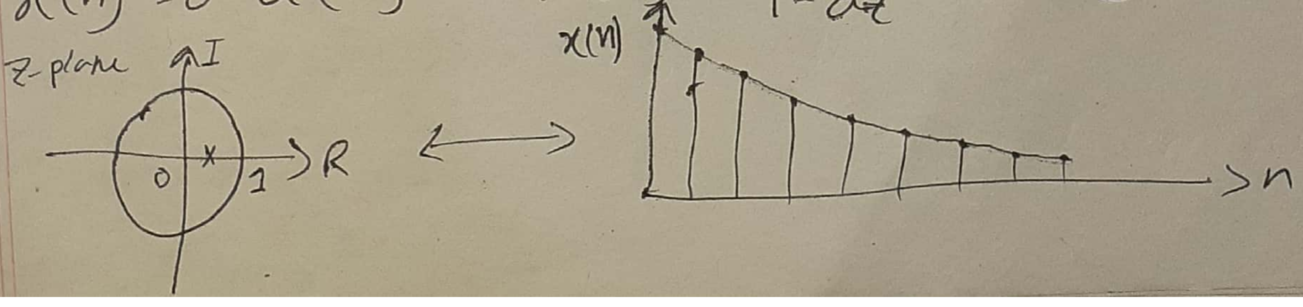
Zero :- $Z_1 = 0$

Pole :- $P_1 = a$

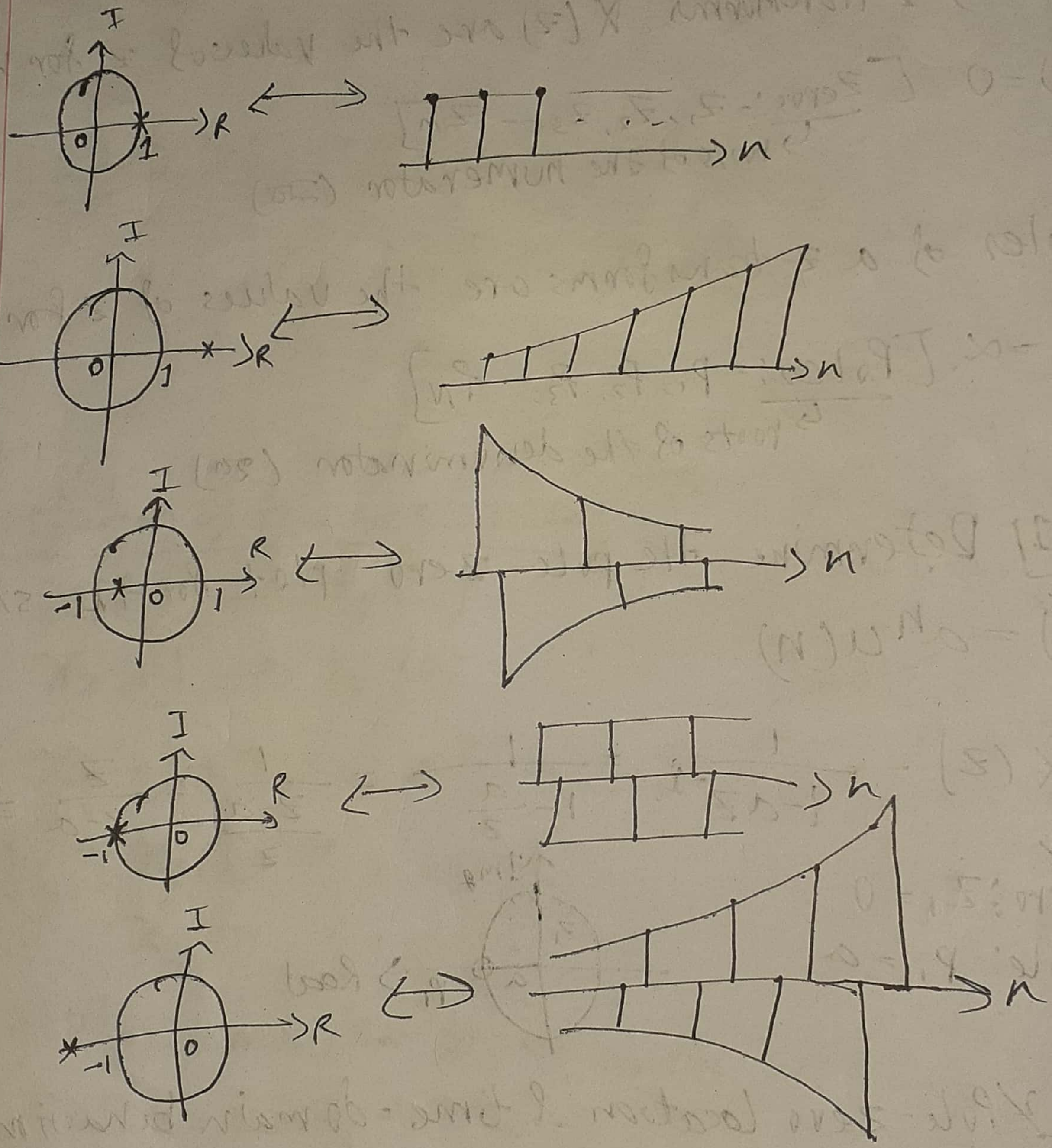


Ex-3.3.2/ Pole-zero location & time-domain behavior for causal signal

$x(n) = a^n u(n) \longleftrightarrow X(z) = \dfrac{1}{1-az^{-1}} = \dfrac{z}{z-a}$



$$x(n) = a^n u(n) \longleftrightarrow X(z) = \frac{z}{z-a}$$



3.3.3 The system function of an LTI system

$y(n) = x(n) * h(n)$ [time-domain → after convolution

Z-domain → after multiplication

$Y(z) = X(z) \cdot H(z)$

$H(z) = \dfrac{Y(z)}{X(z)}$ → output (numerator), → input (denominator)

System function :- Use when we know input & output. But D.K SF.

Ex-3.3.4 Determine the system function & the unit sample response

$y(n) = \frac{1}{2} y(n-1) + 2x(n)$

$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$

$2X(z) = Y(z) \left[1 - \frac{1}{2} z^{-1}\right]$

$\dfrac{Y(z)}{X(z)} = \dfrac{2}{1 - \frac{1}{2} z^{-1}}$

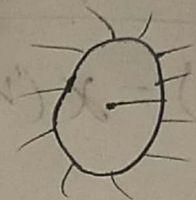
$\Rightarrow H(z) = \dfrac{2}{1 - \frac{1}{2} z^{-1}}$

$h(n) = 2\left(\frac{1}{2}\right)^n u(n)$

DSP (Cls-15) 14/23

Online

Inverse Z-Transform



Ex: $X(z) = \frac{1}{1-az^{-1}}$, $|z| > |a|$ ↳ R.H.S (causal)

↓ inverse

$x(n) = a^n u(n)$ ✓ / $a^n u(-n-1)$ ✗ anti-causal

The inverse z-Transform by power series expression

Ex-3.4.2 Determine the inverse z-transformation of

$$\boxed{X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}} \rightarrow z^{-1}+z^{-2}+z^{-3}$$

When a) ROC :- $|z| > 1 \rightarrow x(n)$ should be causal

b) ROC :- $|z| < 0.5 \rightarrow x(n)$ " " anti-causal.

$x(n) = \{\underline{1}, 2, 3\}$

$X(z) = 1 + 2z^{-1} + 3z^{-2}$

$x(n) = \{1, 2, \underline{3}\}$

$X(z) = z^2 + 2z^1 + 3$

a) $X(z) = \frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} \rightarrow z^{-1}+z^{-2}+z^{-3}+\_$

$$1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} \,\big)\, 1 \quad \Big(1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+$$

$$1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}$$

$$\frac{3}{2}z^{-1}-\frac{1}{2}z^{-2}$$

$$\frac{3}{2}z^{-1}-\frac{9}{4}z^{-2}+\frac{3}{4}z^{-3}$$

$$\frac{7}{4}z^{-2}+\frac{3}{4}z^{-3}$$

$$x(n)=\left\{1, \frac{3}{2}, \frac{7}{4}, \ldots\right\}$$

Ex:- 3.4.8 $\quad X(z)=\dfrac{1}{1-1.5z^{-1}+0.5z^{-2}}$, $x(n)=?$

a) ROC:- $|z|>1 \rightarrow$ causal

b) ROC:- $|z|<0.5 \rightarrow$ anti-causal

c) ROC:- $0.5<|z|<1 \rightarrow$ contain both causal & anti-causal part.

$$X(z)=\frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$=\frac{2}{1-z^{-1}}-\frac{1}{1-0.5z^{-1}}$$

$X(z)=\dfrac{1}{1-az^{-1}}$

$x(n)=a^n u(n)$

a) $x(n)=2(1)^n u(n)-(0.5)^n u(n)$

b) $x(n)=2(1)^n u(-n-1)-(0.5)^n u(-n-1)$

c) $x(n)=-2(1)^n u(-n-1)-(0.5)^n u(n)$

DSP (cls-16)

27/5/23

$$\boxed{x(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}}$$

① $x(n) = \{\underset{0}{\underline{1}}, 2, 5, 7, 0, 1\}$

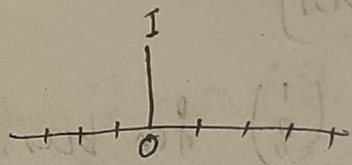
$$X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

② $x(n) = \{1, 2, \underline{5}, 7, 0, 1\}$

$$X(z) = z^{2} + 2z^{1} + 5 + 7z^{-1} + z^{-3}$$

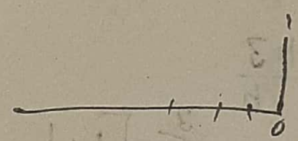
③ $x(n) = \delta(n)$

$$X(z) = 1 \cdot z^{-0} = 1$$



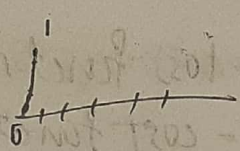
④ $x(n) = \delta(n-k)$

$$X(z) = z^{-k}$$



⑤ $x(n) = \delta(n+k)$

$$X(z) = z^{k}$$



⑥ $x(n) = a^{n} u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} a^{n} z^{-n} = \sum_{n=-\infty}^{\infty} (a z^{-1})^{n}$$

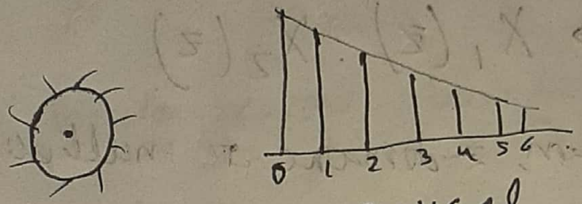
$$= \frac{1}{1 - az^{-1}}, \quad |az^{-1}| < 1$$

<u>ROC :-</u> $z > |a|$

⑦ $x(n) = a^n u(-n-1)$

$$X(z) = -\frac{1}{1-az^{-1}}, \quad |z| < a.$$

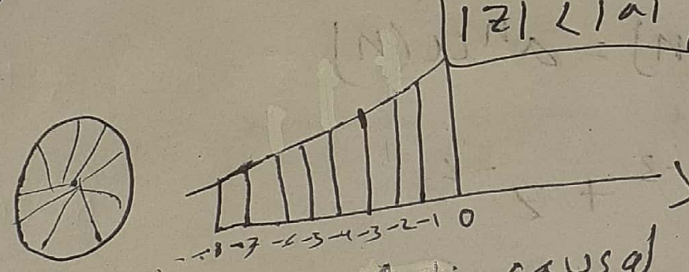
⑧ $x(n) = a^n u(n)$ $\boxed{a<1}$



ROC exist

causal

⑨ $x(n) = a^n u(-n-1)$ $\boxed{|z| < |a|}$



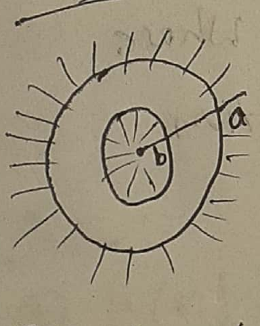
ROC:

Anti-causal

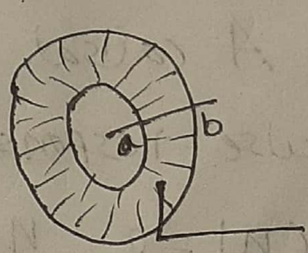
⑩ $x(n) = a^n u(n) + b^n u(-n-1)$

a<b → common part = ROC

a>b



ROC doesn't exist



ROC exist

Convolution Property:-

$x_1(n) \xrightarrow{z} X_1(z)$

$x_2(n) \xrightarrow{z} X_2(z)$

$x_1(n) * x_2(n) \xrightarrow{z} X_1(z) \cdot X_2(z)$

side note (time domaine এ convolution, z-domaine এ multiplication হয়ে যায়)

(11) $X(z) = \dfrac{1}{1-az^{-1}} \quad |z| > |a|$

$x(n) = ?$, $x(n) = a^n u(n)$

(12) $X(z) = 1 + z^{-3} + z^{-6}$

$x(n) = \{1, 0, 0, 1, 0, 0, 1\}$

3.5.3 Causality & Stability

Causal:- A causal LTI system is one whose unit impulse response satisfies the condition

$h(n) = 0, \quad n < 0$

side note (for neg. n, impulse response = 0)

Stable:- A stable LTI system is one whose unit impulse response satisfies the condition

$$\sum_{n=-\alpha}^{\alpha} |h(n)| < \alpha$$

Sidenote (impulse response bounded for stable)

Ex:- 3.5.2 An LTI system is characterized by the system function

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-3z^{-1}}$$

Specify the ROC of $H(z)$ & determine $h(n)$ for the following conditions:-

a) The system is stable

b) " " " causal

c) " " " anti-causal

Sln:-

(b) $h(n) = \left(\frac{1}{2}\right)^n u(n) + 2(3)^n u(n)$

(c) $h(n) = -\left(\frac{1}{2}\right)^n u(-n-1) - 2(3)^n u(-n-1)$

(a) $\frac{1}{2} < |z| < 3$

$$h(n) = \left(\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)$$

<u>Ex:- 3.5.4</u> Determine the response of the system $y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n)$ to the input signal $x(n) = \delta(n) - \frac{1}{3}\delta(n-1)$

<u>Sln:-</u>

$$\boxed{\begin{array}{l} Y(z) = H(z) X(z) \\ \downarrow \\ y(n) \end{array}}$$

$$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n)$$

$$\cancel{Y(z) = \frac{5}{6} z^{-1} Y(z) - \frac{1}{6} z^{-2} Y(z) + X(z)}$$

$$Y(z) = \frac{5}{6} z^{-1} Y(z) - \frac{1}{6} z^{-2} Y(z) + X(z)$$

$$Y(z) - \frac{5}{6} z^{-1} Y(z) + \frac{1}{6} z^{-2} Y(z) = X(z)$$

$$Y(z)\left(1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}\right) = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

$$H(z) = \frac{1}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

$$= \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - \frac{1}{3} z^{-1}\right)}$$

$$X(z) = 1 - \frac{1}{3} z^{-1}$$

$$\therefore Y(z) = H(z)\,X(z)$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$$y(n) = \left(\frac{1}{2}\right)^n u(n)$$

* One-sided z-transform

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

TT-3:-

9.5.23

ch-3

9/5/23

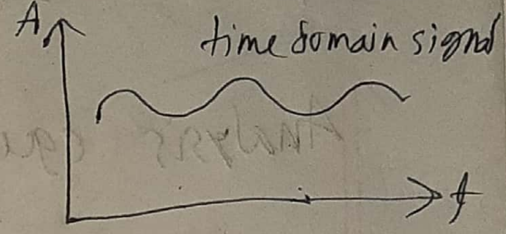
DSP (CS-17)

Ch-4 (Frequency Analysis of Signals)

→ convolution, Fourier Transform

1) Fourier Transform:- finite energy signal. (aperiodic)

2) Fourier series:- Periodic signal.

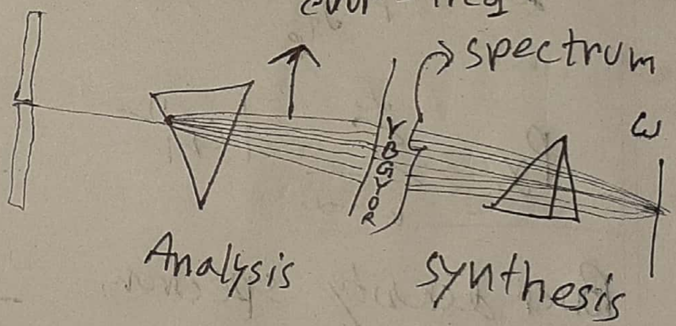


অর্থাৎ signal composite of many nb of sinusoid.

Frequency domain:-

When a signal is decomposed in terms of sinusoidal components, a signal is said to be represented in the frequency domain.

→ Isaac Newton (1672)



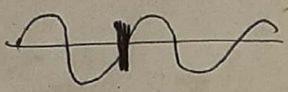
4.1.1 Fourier Series of continuous time signals:-

- Jean Baptiste Joseph Fourier (1768 — 1830)
  - Temperature distribution

Frequency Analysis of Continuous time Periodic signals:

Synthesis equ: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$ [where $F_0 = \frac{1}{T_p}$]

Analysis equ: $c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} \, dt$

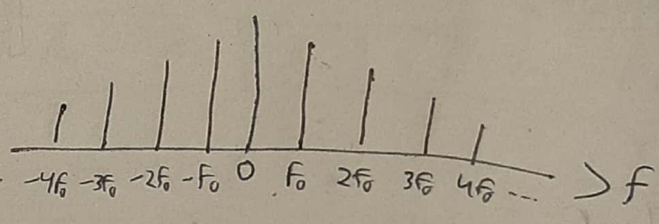


4.1.2 Power Density Spectrum of Periodic signals:

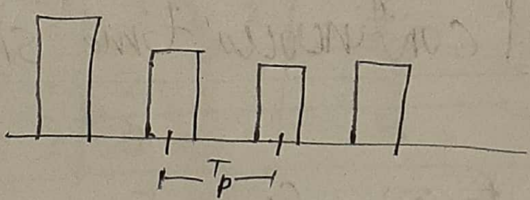
$$P_x = \frac{1}{T_p} \int_{T_p} |x(t)|^2 \, dt$$

$$P_x = \sum_{k=-\infty}^{\infty} |c_k|^2 \qquad (k = 0, \pm 1, \pm 2, \pm 3, \ldots)$$

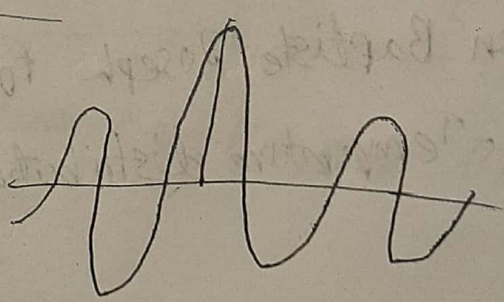
Power density spectrum or line spectrum



Ex:- 4.1.1



P.D.S = ?



Lab AM

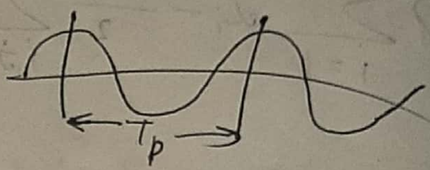
23.5.23 → ch-1,2

DSP (cls-18)

15/5/23

Fourier Series of Periodic continuous time signals

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$



$$c_k = \frac{1}{T_p}\int_{T_p} x(t) e^{-j2\pi F_0 t} \, dt$$

$0, F_0, 2F_0, 3F_0, \ldots$

→ D.T.W sample 2πs,

Fourier Series of Periodic Discrete Time signals

$x(t) \xrightarrow{\text{freq.}} -\infty$ to $+\infty \rightarrow T_p$

$x(n) \xrightarrow{\text{freq.}} -\pi$ to $+\pi \rightarrow N$

or

0 to 2π

Fourier series of Discrete Time Periodic Signals

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi k \frac{1}{N} n}$$

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi k n/N}$$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi k n/N}$$

Ex: 4.2.1 Determine the spectra

$$x(n) = \cos \underbrace{\sqrt{2}\pi}_{\omega} n \qquad = \cos \omega n$$

$$= \cos 2 \frac{1}{\sqrt{2}} \pi n$$

$f_0 = \frac{1}{\sqrt{2}}$, not a rational number.

spectra (frequency component) : $\sqrt{2}\pi$

b) $x(n) = \cos \pi n/3$

$$= \cos 2\pi \frac{1}{6} n$$

$f_0 = \frac{1}{6}$ is a rational number.

spectra (frequency component) : $\frac{1}{3}\pi$

$$c_k = \frac{1}{6} \sum_{n=0}^{5} x(n) e^{-j2\pi kn/6}$$

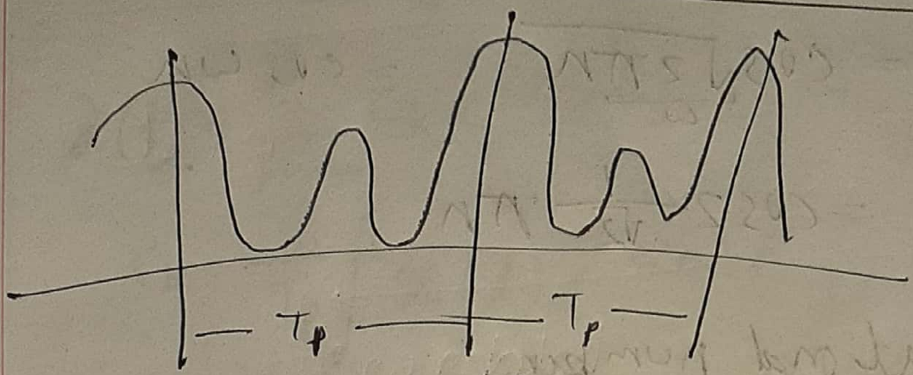
$$= \frac{1}{6} \sum_{n=0}^{5}$$

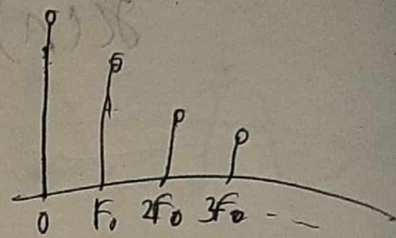
Power density spectrum

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)^2|$$

$$= \sum_{n=0}^{N-1} |c_k|^2$$

Fourier Transform of Discrete Time signals



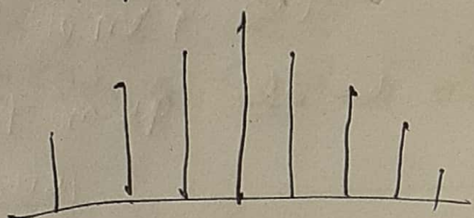


line spectrum

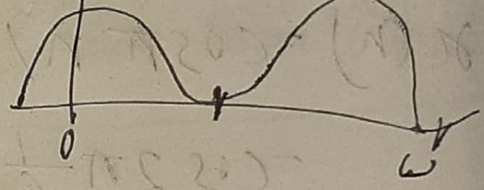
$F_0 = \frac{1}{T_p}$ $x(t)$

aperiodic

$x(\omega)$



$\xrightarrow{FT}$



FT of DT signals

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

→ F.T = continuous. F.S = Discrete

4.2.6 Relation of the Fourier Transform to the z-transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$z = re^{j\omega}$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

if $r=1$, z-transform became Fourier Transform

Z-T is ~~sub~~ super set & F.T is subset.

frequency Ranges of some natural signals.

Freq. Range (Hz)

ECG = 0 - 100 (heart)

EEG = 0 - 100 (brain)

speech = 100 - 4000 (telephone speech)

Seismic signals = ~~[illegible]~~ wind noise → 100 - 1000

seismic signal → 10 - 1000

Earthquake & nuclear explosion signal = 0.01 - 10

DSP (Cls-19) 18/5/23

Ch-7 Discrete Fourier Transform

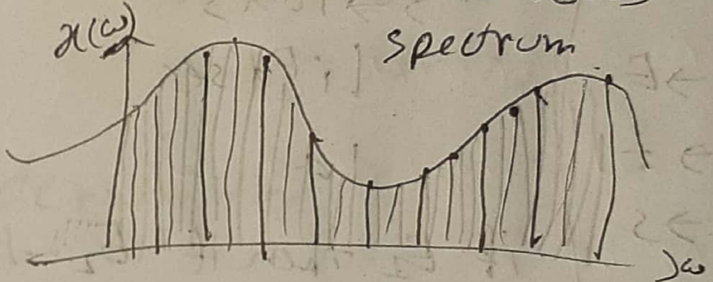
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

F.S → D.S | Periodic — $F_0 + 2F_0 + 3F_0 + \dots$

- Fourier series:- Frequency Domain - Discrete Spectrum (line)
- Fourier Transform:- Frequency Domain - continuous spectrum

$$\frac{2\pi}{N} = \delta\omega$$

$$k\frac{2\pi}{N} = \delta\omega$$



$$\omega = k\frac{2\pi}{N}$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi}{N}kn} \quad [k = 0, 1, \dots N-1]$$

$$= + \dots + \sum_{n=-N}^{-1} x(n) e^{-j2\pi kn/N} +$$

$$\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} + \sum_{n=N}^{2N-1} x(n) e^{-j2\pi kn/N} + \dots +$$

$$= \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n - lN)\right] e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}$$

Periodic x'2

synthesis $\rightarrow x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi n/N}$, $n = 0 \ldots N-1$

analysis $c_k = \frac{1}{N} \sum x_p(x) e^{-j2\pi kn/N}$ $k = 0 \ldots N-1$

for $0 \le n \le N-1$

$x(n) = x_p(n)$

$c_k = \frac{1}{N} X\left(\frac{2\pi}{N} \cdot k\right)$

analysis $\rightarrow X\left(\frac{k}{N}\right) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$, $k = 0, 1 \ldots N-1$

synthesis $\rightarrow x(\omega) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$

$\left[X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} , \quad \omega = \frac{2\pi}{N}\right]$

Ex: 7.1.2 $x(n) = \begin{cases} 1, & 0 < n \le L-1 \\ 0, & \text{otherwise} \end{cases}$

finite $\frac{1-a^L}{1-a}$

$X(k) = \sum_{n=0}^{L-1} e^{-j2\pi kn/L}$

$= \frac{1 - e^{-j2\pi kL/N}}{1 - e^{-j2\pi k/N}}$ → point

DSP (cls-20)

23/5/23

7.2 DFT Properties

Descrete T.F → $X(k) = \sum_{n=0}^{k-1} x(n) e^{-j2\pi kn/N}$

① **Periodicity**:- if $x(n) \xrightarrow{DFT} X(k)$.

then $x(n+N) = x(n)$

$X(k+N) = X(k)$

② **Linearity**:- if $x_1(n) \xrightarrow{DFT} X_1(k)$

$x_2(n) \xrightarrow{DFT} X_2(k)$

then, $a_1x_1(n) + a_2x_2(n) \xrightarrow{DFT} a_1X_1(k) + a_2X_2(k)$

Real-valued sequences

if $x(n)$ is real → conjugate

$X(N-k) = X^*(k)$

$= X(-k)$

7.2.2 Multiplication of two DFTs & circular convol

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{-2\pi kn/N}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, e^{-2\pi kn/N}$$

$$X_3(k) = X_1(k)\, X_2(k)$$

$$\left(\sum_k a^k = \frac{1-a^N}{1-a} \right)$$

IDFT

$$x_3(m) = \sum_{k=0}^{N-1} X_3(k)\, e^{-j2\pi km/N}$$

$$= \sum_{k=0}^{N-1} X_1(k)\, X_2(k)\, e^{-j2\pi km/N}$$

$$= \sum_{k=0}^{N-1} \left[\left(\sum_{k=0}^{N-1} x_1(n) e^{-j2\pi kn/N} \right) \left(\sum x_2(l) e^{-j2\pi kl/N} \right) e^{-j2\pi km/N} \right]$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} \right]$$

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & \text{if } (m-n-l) \text{ is multiply of } N \\ \frac{1-a^N}{1-a}\ 0, & \text{otherwise} \end{cases}$$

$$a = e^{j2\pi(m-n-l)/N} \rightarrow A^N = 1$$

$$m-n-l = pN \Rightarrow m-n-pN = l$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n)\, .\, x_2\overbrace{(m-n)_N}^{\text{circular}}$$

∴ Multiplication of two DFS are equivalent of circular convolution in time domain.

DSP (Cls-21) 25/5/23

7.3 Linear Filtering Methods based of the DFT:

$y(n) = x(n) * h(n)$

$y(n)$ → noise free

$h(n)$ → identaty of a system

$X_1(k) \cdot Y(k)$ → circular convolution

$\downarrow$ IDFT

$y(n)$

$\left.\begin{array}{l} x(n) \rightarrow L \\ h(n) \rightarrow M \end{array}\right\} L+M-1$

FIR = ? → Finite Impulse Response

Ex-7.3.1

$h(n) = \{1, 2, 3\}$

$x(n) = \{1, 2, 2, 1\}$

$\left.\begin{array}{l} L = 4 \\ M = 3 \end{array}\right\} N = 4+3-1 = 6$

$y(n) = h(n) * x(n)$

$Y(k) = H(k) \cdot X(k)$

$\downarrow$ IDFT

$y(n)$

Linear convolution

7.3 Filtering Long-Data Sequences

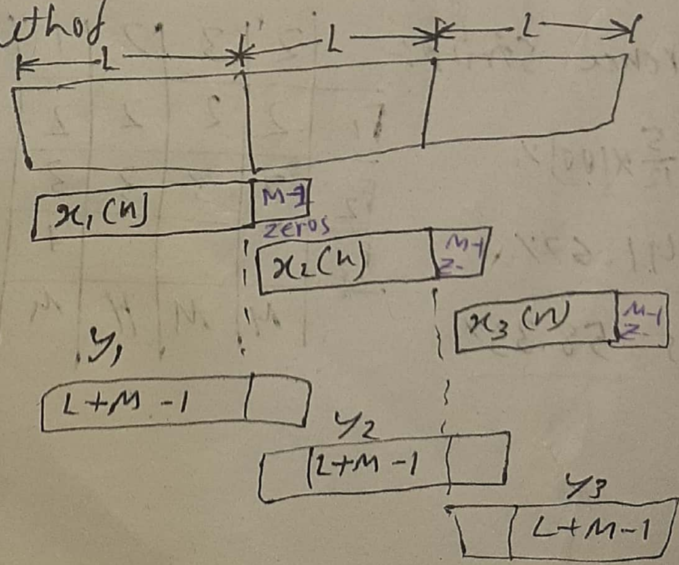
→ Overlap - add method

→ Overlap - save method

$y(n) = x_1(n) * h(n)$

1) segment

2) DFT apply



$$x_1(n) = \{x(0), x(1), \ldots\ x(L-1), \underbrace{0, 0 \ldots 0}_{M-1 \text{ zeros}}\}$$

$$x_2(n) = \{x(L-1), x(L), \ldots\ x(2L-1), \underbrace{0, 0, \ldots 0}_{M-1 \text{zeros}}\}$$

$$Y_1(k) = H(k) X(k)$$

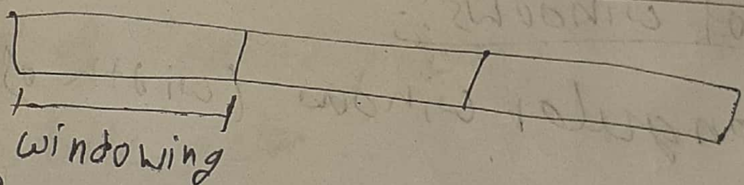
$$Y_2(k) = H(k) X_2(k)$$

$$y(n) = \{y_1(0), y_1(1), \ldots\ y_1(L) + y_2(0) + y_1(L+1) + y_2(L) \ldots\}$$

7.4 Frequency Analysis of signals using DFT:-

$x(n)$
F_s
N.

$/ k.\frac{2\pi}{N}$



windowing (No. of frame & the way we devide a signal).

$$\hat{x}(n) = x(n)\, w(n)$$

DFT

↓

frame / segment

$$\hat{x}(k) = \hat{x}(k) * H(k)$$

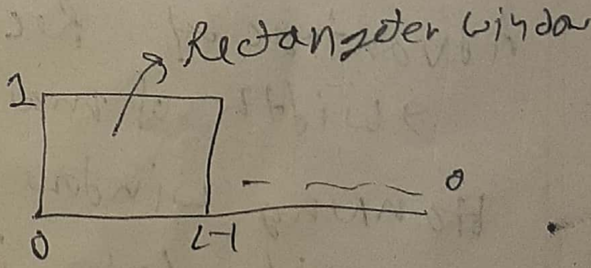
↓

$$\hat{y}(n)$$

our signal,

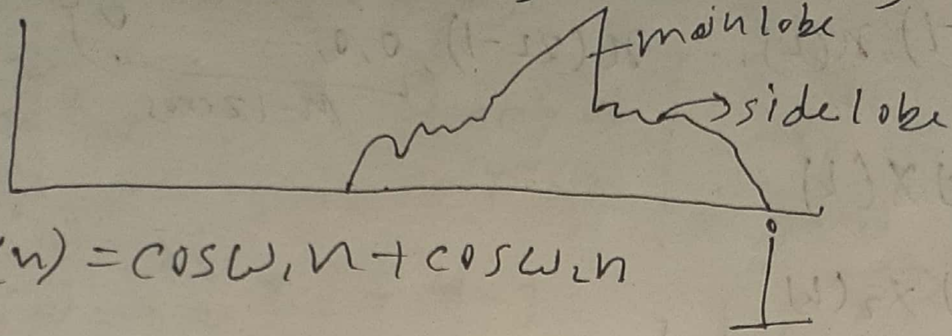
$$x(n) = \cos \omega_0 n$$

$$w(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$



$$\hat{x}(n) = x(n) * \omega(n)$$

$$\hat{X}(\omega) = \frac{1}{2}\left[W(\omega - \omega_0) + (\omega - \omega_0)\right]$$



$$x(n) = \cos\omega_1 n + \cos\omega_2 n$$

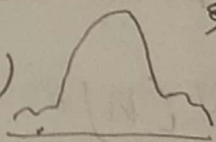
If we increase value of L than we can reduce adding windowing.

Types of windows:-

1) Rectangular window (width of main lobe is reduced)

ii) Hamming window

iii) Hanning window (H.W)



~~side~~ side lobe
= H W
main lobe width is
= R W

conclude:- (How to reduce windowing effect)

1) N should be enough larger.
2) Appropriate window should be selected.

Advantage of Rec. Window:-
→ width of main lobe is less.

Hanning window:-
→ side lobe is reduced.

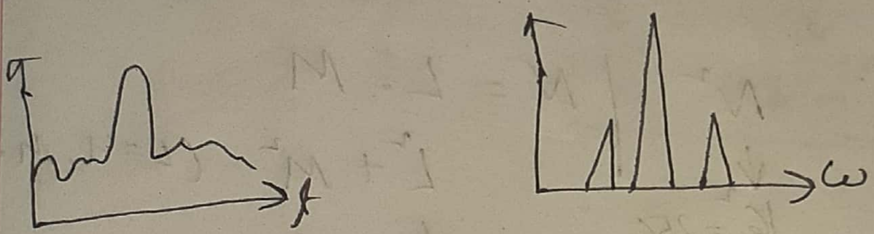
DSP(cls-22) 30/5/23

ch-8 Efficient Computation of DFT : FFT algorithm

Next Tuesday
TT-4
ch-7, 8

DFT

signal in time domain → DFT → signal in frequency domain



→ convolution এর বদলে DFT দিয়ে কাজ হয়।

FFT:- Fast Fourier Transform. এটা faster DFT কে বলে।

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad \text{(analysis)}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad (W_N = e^{-j2\pi/N})$$

$$x(n) = \frac{1}{N} \sum X(k) e^{j2\pi kn/N} \quad \text{(synthesis)}$$

$$= \frac{1}{2} \sum X(k) W_N^{-kn}$$

where, $k = 0, \ldots N-1$, & $n = 0, \ldots N-1$.

complexity:-

complex multiplication :- $N \cdot N = N^2$

complex Addition :- $(N-1) \cdot N = N^2 - N$

① **Symmetry Property :-** $W_N^{k+N/2} = -W_N^{k}$

② **Periodicity Property :-** $W_N^{k+N} = W_N^{k}$

③ **Divide & Conquer :-** $N^2 \;\Big|\; N = L \cdot M$

$N^2 \downarrow 16^2 = 256$

$L^2 + M^2 \quad (L=4, M=4)$

$\downarrow + \downarrow$

$16 + 16 = 32$

8.1.2 Divide & Conquer Approach

$N = LM$

$x(n)$: | $x0$ | $x1$ | $x2$ | —— . . . | $x(N-1)$ |

Row-Major, $n = Ml + m$

$l \backslash m$	0	1	2	3	M-1
0	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(M-1)$
1	$x(M)$	$x(M+1)$			~~x(2M)~~
2					
3					
L-1	$x(L-1)$				

Frequency :- $k = Mp + q$

$k = ql + p$

$N = L \cdot M$

$(p \Rightarrow) X(p,q) = \sum_{m=0}^{M-1}\sum_{l=0}^{L-1} x(lM)\, W_N^{(Mp+q)(mL+l)}$

$= X(p,q) = \sum_{l=0}^{L}\left\{ W_N^{lq} \underbrace{\left[\sum_{m=0}^{M-1} x(l,m)\, W_M^{mq}\right]}_{M\text{-point DFT}} \right\} W_L^{lp}$

$W^{Mp \cdot ML + Mpl + Lqm + ql}$

$= \sum_{l=0}^{L}\left\{ W_N^{lq} F(l,q)\; W_L^{lp} \right\}$

$= \underbrace{\sum_{l=0}^{L}\left\{ G(l,q)\, W_L^{lp} \right\}}_{L\text{-point DFT}}$

Column-Major, $n = l + mL$

	0	1	2	3
0	$x0$			
1	$x1$			
2	$x2$			
3	$x3$			

M point DFT = $W_N^{Mp.mL} \cdot W_N^{mpl} \cdot W_N^{lqm} \cdot W_N^{ql}$

$F(l,q) = W_N^{Mp.m} \cdot W_{N/M}^{pl} \cdot W_{N/L}^{qm} \cdot W_N^{ql}$

$\downarrow \quad 1 \qquad W_L^{pl} \qquad W_M^{qm} \cdot W_N^{ql}$

~~$M^2 MdC$~~

→ M point DFT

→ scal. multi

→ L point DFT

$LM^2 + N \cdot ML^2$

$LM(M+L+1)$

$= N(M+L)$

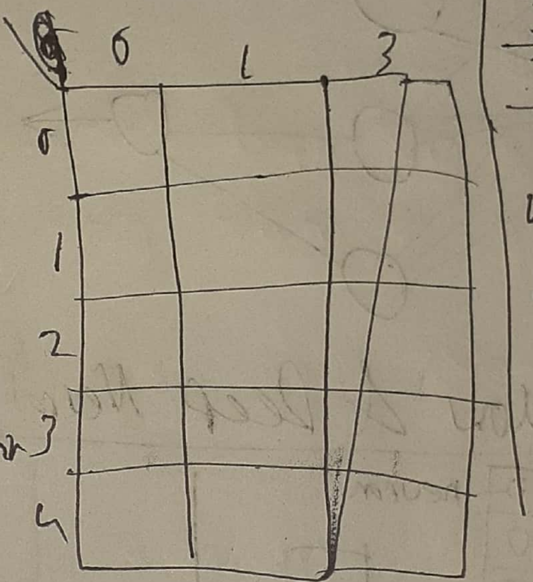
$= N \cdot N$

A. W

Ex- 8.1.1

$N=15$
$L=5$
$M=3$

1. 3 point DFT for each row
2. Scalar multiplication
3. 5 point DFT for each column



→ 11 AM online cls (Tomorrow)

Radix-2 Algorithm of FFT

$N = LM$

$= r_1 r_2 \cdots r_n = r$

$N = r.r.r \ldots r$ [ν সংখ্যক]

$N = r^{\nu}$

$N^2 = N(L.M)$

$= N.(L+M)$

$n = 15$

$x(n) = \{2, 3, \ldots, 10, 0\}$

$= 16$

zero padding

→ N কে h.c. nb দিয়ে হবে $N = r^{\nu}$, $r = 2$

$x(n) < f_1, f_2$

$f_1(n) = x(2n) \rightarrow$ জোড়

$f_2(n) = x(2n+1) \rightarrow$ বিজোড়

$(n = 0, 1, \ldots, N/2 - 1)$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \left[W_N = e^{-j2\pi/N}\right]$$

$$= \sum_{n \text{ even}} x(n) W_N^{kn} + \sum_{n \text{ odd}} x(n) W_N^{kn}$$

$$= \sum_{m=0}^{N/2-1} x(2m) W_N^{2mk} + \sum_{m=0}^{N/2-1} x(2m+1) W_N^{k(2m+1)}$$

$$= \sum_{m=0}^{N/2-1} x(2m) W_{N/2}^{mk} + W_N^{k} \sum_{m=0}^{N/2-1} x(2m+1) W_{N/2}^{km}$$

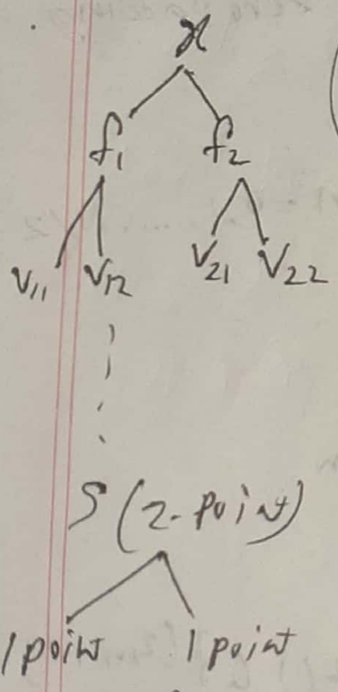
$$X(k) = \sum f_1(m) W_{N/2}^{km} + W_N^k \sum_{m=0}^{N/2-1} f_2(m) W_{N/2}^{km}$$

$$X(k) = \underbrace{F_1(k)}_{a} + W_N^k \underbrace{F_2(k)}_{b} \quad [N/2 \text{ point DFT}]$$

$$\boxed{W_N^{k+N/2} = -W_N^k} \qquad k = 0 \ldots \frac{N}{2} - 1$$

$$X\left(k + \frac{N}{2}\right) = F_1(k) - W_N^k F_2(k), \quad k = 0, \ldots \frac{N}{2} - 1$$

$$\left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 + N = \frac{N^2}{2} + N$$



$v_{11}(n) = f_1(2n)$

$v_{12}(n) = f_1(2n+1)$

$v_{21}(n) = f_2(2n)$

$v_{22}(n) = f_2(2n+1)$

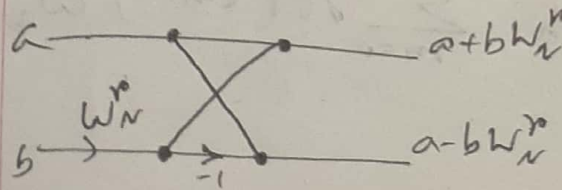
$$\boxed{V = \log_2 N}$$

8 point DFT

8 → 4 — 2 — 1, 1; 2 — 1, 1

4 — 2 — 1, 1; 2 — 1, 1

Butterfly



Basic butterfly structure

8 point DFT

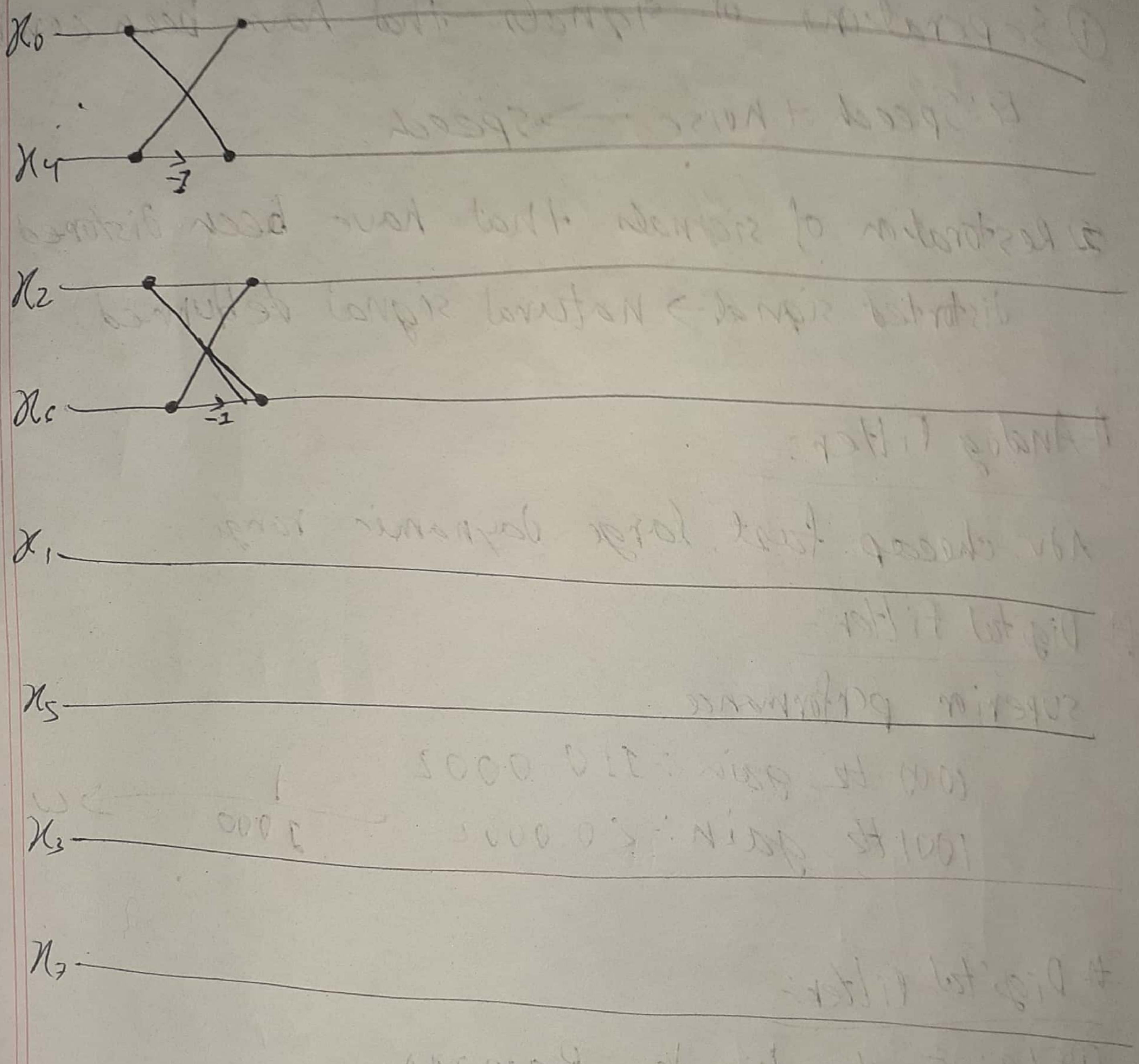
$x(0)\ x(1)\ x(2)\ x(3)\ x(4)\ x(5)\ x(6)\ x(7)$

→ $x(0)\ x(2)\ x(4)\ x(6)$ $x(1)\ x(3)\ x(5)\ x(7)$

→ $x(0)\ x(4)$ $x(2)\ x(6)$ $x(1)\ x(5)$ $x(3)\ x(7)$

p-542

$x(0)$ $x(4)$ $x(2)$ $x(6)$ $x(1)$ $x(5)$ $x(3)$ $x(7)$



H.W.

8 point FFT.

end of ch

DSP(cls-24) 2/6/23

online

Filters

Filters:-

① Seperations of signals that have been combined

Ex:-Speech + noise ⟶ speech.

② Restoration of signals that have been distored

distorted signal → natural signal deblurred.

Analog Filter:-

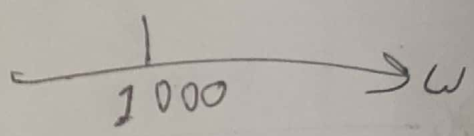
Adv:-cheeap. fast, large daynamic range.

Digital filter:-

superior performance.

1000 Hz, gain :- 1 ± 0.0002

1001 Hz, gain :- < 0.0002



Digital filter:-

① FIR:- Finite Impulse Respons.

② IIR:- Infinite Impulse Respons.

FIR filter :- → output is defined in terms of input using convolution.

$$y = x * h$$

h → impulse response of the system → filter kernal

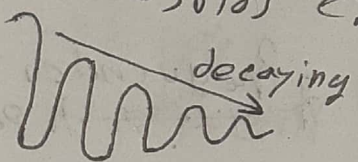
x → input signal

Ex: long data filtering in DFT.

IIR filter/Recursive filter :- → output is defined in terms of privious output & input using recursion.

$$y(n) = y(n-1) + a x(n)$$

→ Impulse response of RFs are composed of sinusoits exponentialy decaying in amplitude.



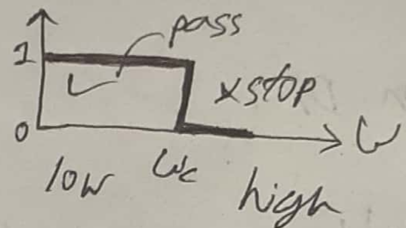
This is infinitely long

Filter classification :-

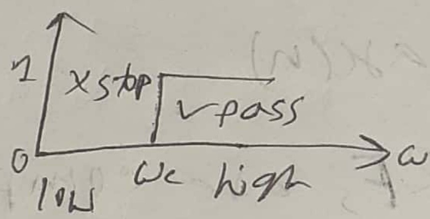
class	FIR	IIR
Timedomain filter	Moving Average	Single pole
Frequency filter	Windowed sinc	chebyshev

Frequency characteristics of filters:-

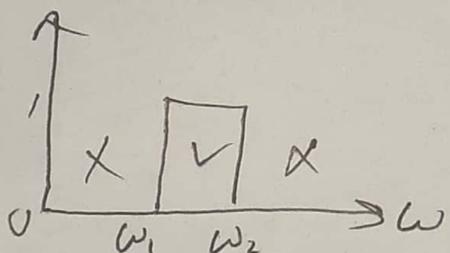
① **Low-pass filter:-** Allows the low frequencies to pass.



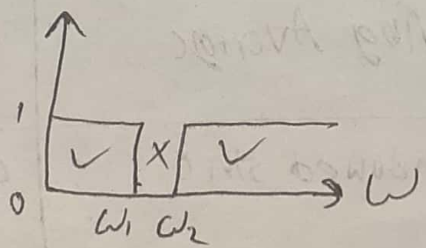
② **High-pass filter:-** Allows the high frequencies to pass.



③ **Band-pass filter:-** Allows a band of ~~frequien~~ frequencies to pass.

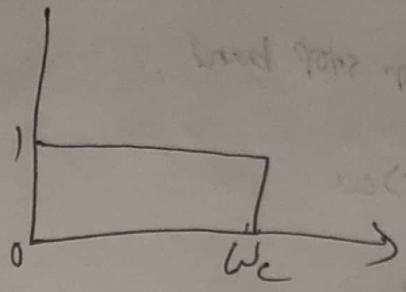


④ **Band stop/Band Reject Filter:-** stops a band of frequencies.

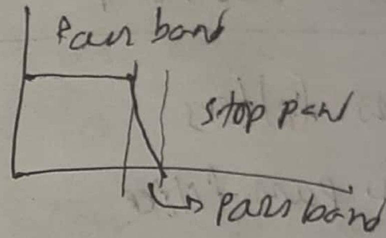


Ideal Filter Vs Practical filter

Ideal low pass frequency:



Practical low pass frequency:



Frequency domain Parameters:-

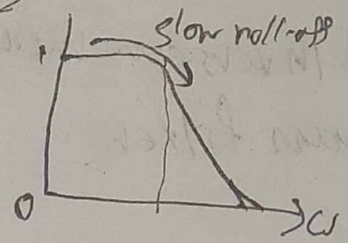
→ Roll-off

→ Ripple

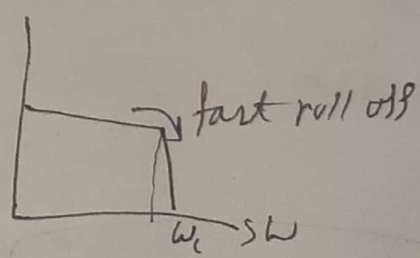
→ Stop band Attenutation.

Roll off

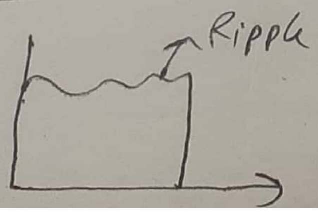
Poor

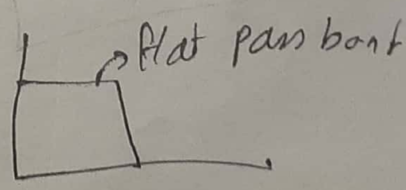


Good



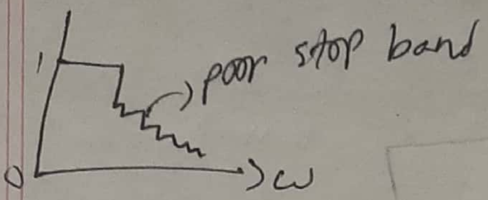
Ripple in pass band

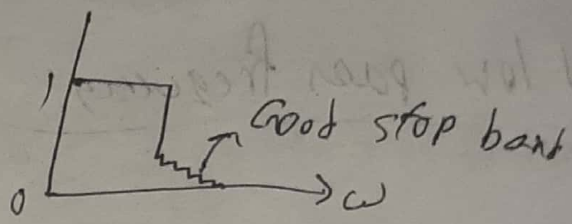




37

Stop band Attenuation :-

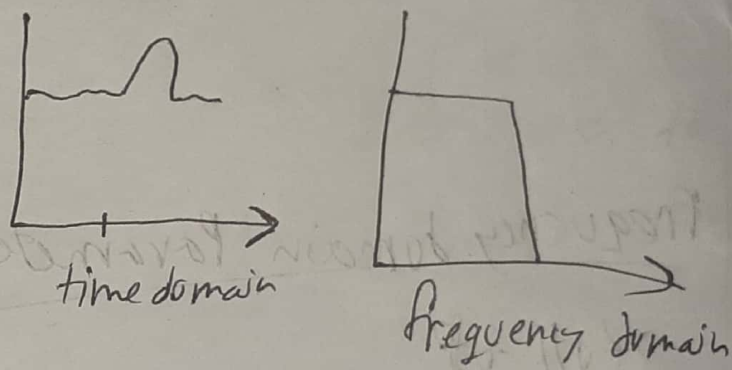




High-Pass, Band-pass :

High pass filters using low pass filter

$h(n) \rightarrow$ low-pass filter impulse response



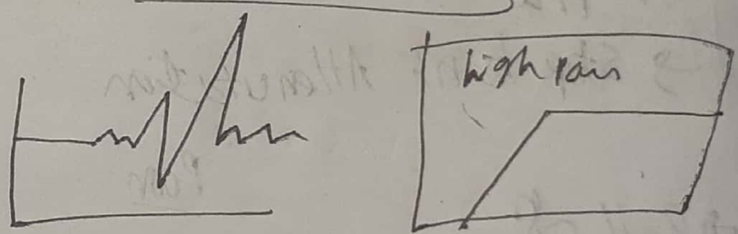
low-pass filter

$\boxed{S(n) - h(n)}$

→ high-pass filter

unit impulse

(h(n) के उपर 1 (zero))



Spectral inverseon method for high pass filter.

→ tomorrow – 11:30

DSP(CIS-25) 3/6/23 online

Digital Filters

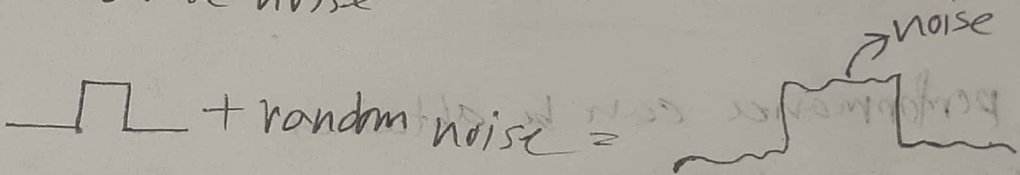
FIR :- → Moving Average Filter
→ time domain

① Moving Average Filter :-

→ Optimal for reducing random noise.
→ A premier time domain filter
→ Worst frequency domain filter with little ability to seperate one band of frequencies from another.

Random noise :-

→ white noise



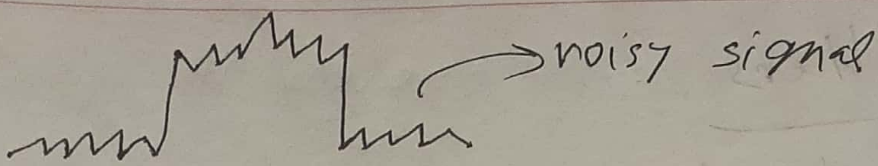
→ Random noise দূর করতে Moving average filter use হয়,

$$y[i] = \frac{1}{M}\sum_{j=0}^{M-1} x[i+j]$$

$M = 5$

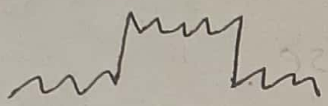
$$y[80] = \frac{x[80] + x[81] + x[82] + x[83] + x[84]}{5}$$

Ex:

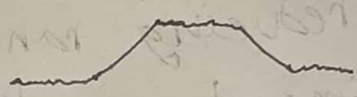


11- Point moving average filter :- (M=11)

11-points



51-points



Windowed-sinc filter (FD)

→ Used to separate one band of ~~frequency~~ frequencies from another band.

→ stable

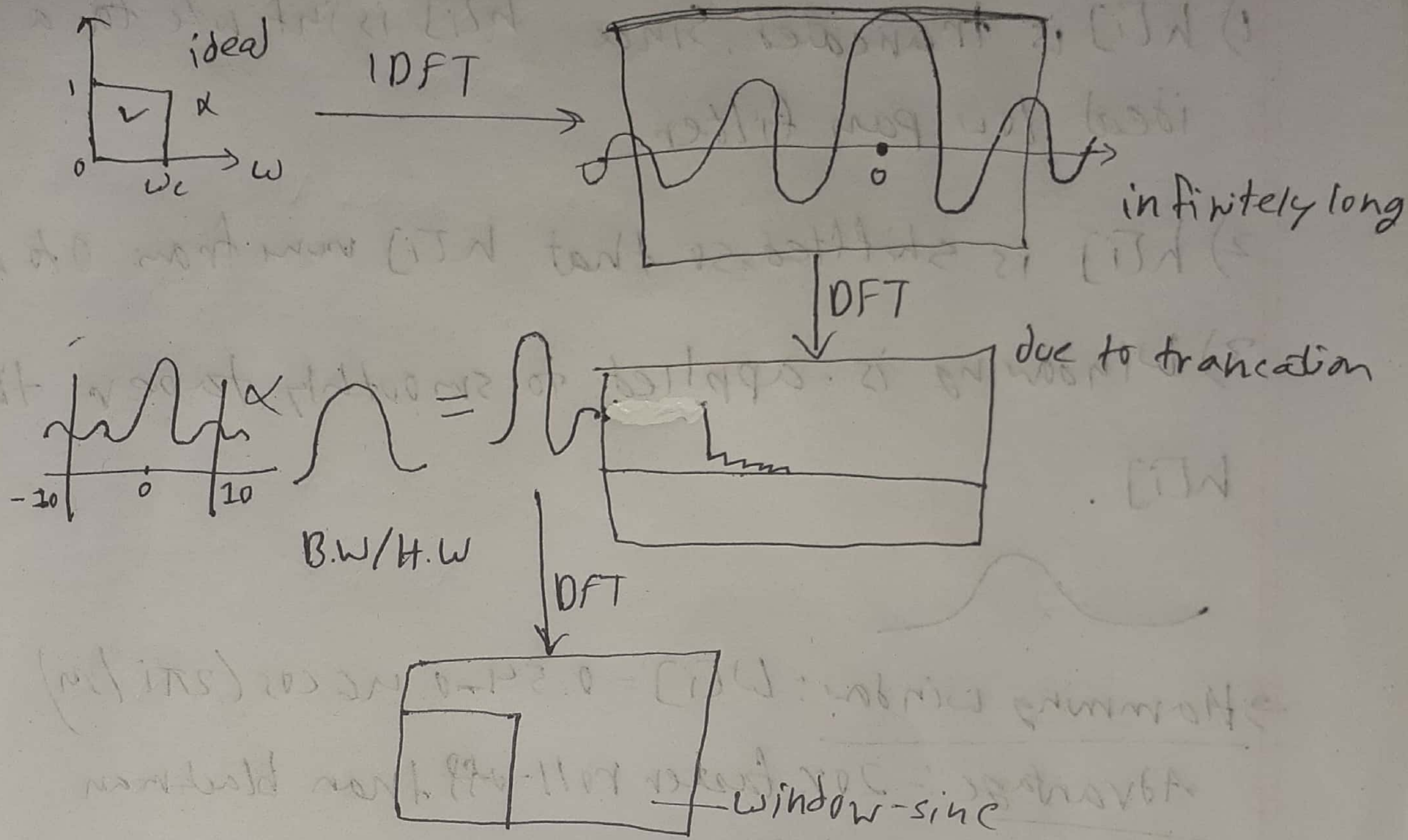
→ Incredible performance can be obtained.

→ Slow, but FFT convolution can dramatically improve the speed.

$$y(n) = x(n) * h(n)$$

slow

Frequency response of a low pass filter:-



sinc:-

impulse Response

$$G[i] = \frac{\sin(2\pi f_c i)}{i\pi} \quad \left[\frac{\sin x}{x} = \text{sinc}\right]$$

Window

→ Black man Window,

→ Hamming Window

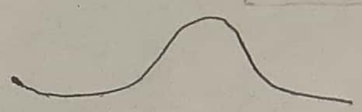
both used to taper the impulse response smoothly.

Steps for building a windowed-sinc low-pass filter:

1) h[i] is truncated. Since h[i] is infinite for a ideal low pass filter.

2) h[i] is shifted, so that h[i] runs from 0 to M.

3) Windowing is applied to smoothly taper the h[i].



→ Hamming window: $w[i] = 0.54 - 0.46\cos(2\pi i / M)$

Advantage :- 20% faster roll-off than blackman window.

→ Blackman window: $w[i] = 0.42 - 0.5\cos(2\pi i / M) + 0.08\cos(4\pi i / M)$

Advantage :- Better stopband than Hamming window

$y[i] = x[i] * h[i]$

output, input, impulse respons.

Book :- The Scientist & Engineer's Guide to DSP.

End of DSP ☺