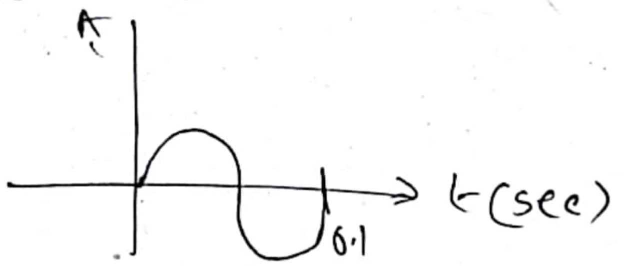
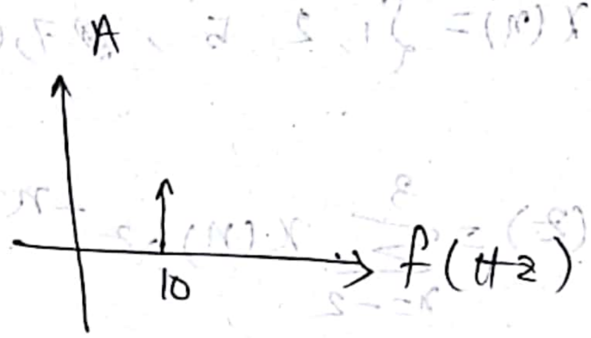
Chapter : 3

The z - transform

*** Time domain Signal:**





*** z-transform:** The z-transform of a DT signal $x(n)$ is define as :

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

Where, z is a complex variant.

$z = re^{j\omega}$

$\hookrightarrow r = 1$, fourier Transform

*** Example : 3.1.1**

(a) $x_1(n) = \{ \underset{\uparrow}{1}, 2, 5, 7, 0, 1 \}$.

origin

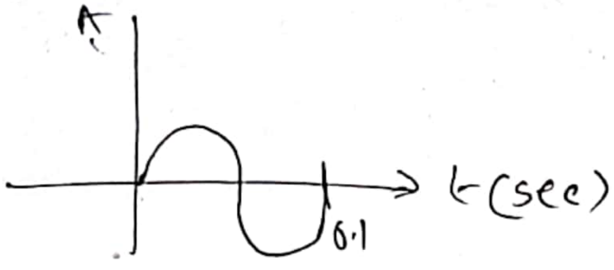
$$x(z) = \sum_{n=0}^{5} x(n) \cdot z^{-n}$$

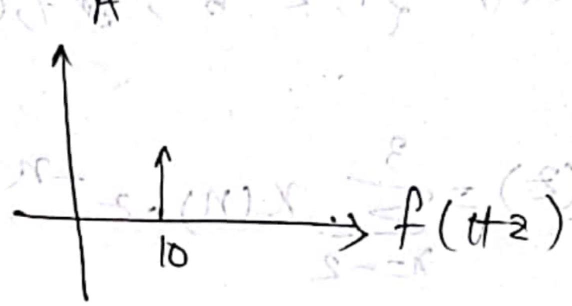
$x(z) = 1 \cdot z^{-0} + 2 \cdot z^{-1} + 5 \cdot z^{-2} + 7 \cdot z^{-3} + 0 \cdot z^{-4} + 1 \cdot z^{-5}$

Chapter: 3

The z-transform

***Time domain signal:**





***z-transform:** The z-transform of a DT signal $x(n)$ is define as:

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

Where, z is a complex variant.

$$z = re^{j\omega}$$

$\hookrightarrow r = 1$, fourier Transform

***Example: 3.1.1**

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

origin

$$x(z) = \sum_{n=0}^{5} x(n) \cdot z^{-n}$$

$x(z) = 1 \cdot z^{-0} + 2 \cdot z^{-1} + 5 \cdot z^{-2} + 7 \cdot z^{-3} + 0 \cdot z^{-4} + 1 \cdot z^{-5}$

$$x(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

$\boxed{\text{origin} = 0}$

(b) $x(n) = \{\overset{-2}{1}, \overset{-1}{2}, \overset{0}{5}, \overset{1}{7}, \overset{2}{0}, \overset{3}{1}\}$

$$x(z) = \sum_{n=-2}^{3} x(n) \cdot z^{-n}$$

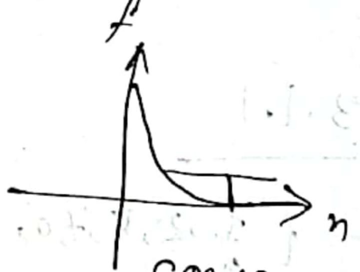
$$= 1 \cdot z^{-(-2)} + 2 \cdot z^{-(-1)} + 5z^{-0} + 7z^{-1} + 0 \cdot z^{-2} + 1 \cdot z^{-3}$$

$$= z^{2} + 2z + 5 + 7z^{-1} + z^{-3}$$

* ROC: Region of convergence:

যে Point এ যাওয়ার পর Graph এর কোন Change হবে না সেই point কে ROC বলে।

Is the set of all values of z for which $x(z)$ attaies of a finite value.



convergence point.

[Z transforms for all values of z except $z = 0, \infty$]

[z এর negative value এর জন্য 0
z " positive " " " ∞]

$\delta(n)$ - impulse.

$x(n) = \delta(n) = \{1, 0, 0, \ldots\}$

$\therefore X(z) = 1$

$\sum x(n) z^{-n}$

$= 1 \cdot z^{0} + 0 + 0 \ldots\ldots = 1$

$x(n) = \delta(n-k)$ $\quad \{0, 0, \overrightarrow{1}, 0, 0, \ldots\}$ ($\uparrow$ $n = k$)

$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$= 0 + 0 + 1 \cdot z^{-k} + 0 + 0 = z^{-k}$

ROC → except $z = 0$

$x(n) = \delta(n+k)$

$\overleftarrow{\{1\ 0\ 0\ 0\ 0\ 0\ 0 \ldots\ldots}$ $= 1 \cdot z^{k} + 0 + 0 \ldots = z^{k}$

ROC → except $z = \infty$

Example:

Determine z-transform of

$x(n) = (1/2)^n\, u(n)$

[unitstep, u → signal (0 – ∞)
∞ length signal.]

$= [1, \frac{1}{2}, (\frac{1}{2})^2 + (\frac{1}{2})^3 + (\frac{1}{3})^4 \ldots\ldots]$

↑
origin

$$x(z) = 1 \cdot z^0 + \frac{1}{2} \cdot z^{-1} + \left(\frac{1}{2}\right)^2 \cdot z^{-2} + \left(\frac{1}{2}\right)^3 \cdot z^{-3} + \ldots$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} \cdot z^{-1}\right)^n = \sum_{n=0}^{\infty} (A)^n$$

$$= \frac{1}{1-A} = \frac{1}{1-\frac{1}{2}z^{-1}} \quad [\text{যখন } A<1]$$

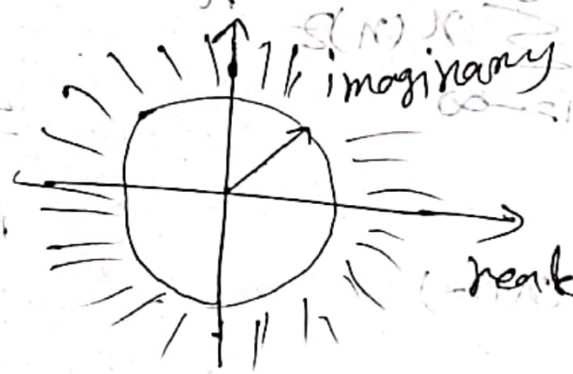
ROC: $\left|\frac{1}{2} z^{-1}\right| < 1 = |z| > \frac{1}{2}$

[½ এর চেয়ে বড় সব value এর জন্য exist করে]

Example: 3.1.3

$$x(n) = \alpha^n u(n)$$

$$x(z) = \frac{1}{1-\alpha z^{-1}}$$



ROC: $|z| > \alpha$

Example 3.1

$$x(n) = -\alpha^n u(-n-1)$$

$(-n-1)$

→ মানে বামে যাওয়া মানে শুরু -1 ঘর সরবে

$$x(z) = \sum_{n=-\infty} \alpha^n \cdot z^{-n}$$

$$l = -n \quad = \sum_{l=1}^{\infty} -\alpha^{-l} \cdot z^{-l}$$

$$= \sum_{l=1}^{\infty} \frac{(\alpha^{-1}z)^l}{A} = (A + A^2 + A^3 + A^4 + \dots)$$

$$= -A(1 + A + A^2 + A^3 + \dots)$$

$$z\text{-transform} = -\frac{A}{1-A}$$

$$= \frac{\alpha^{-1}z}{1-\alpha^{-1}z}$$

$$\frac{1}{\alpha^{-1}z} = \alpha^{-1}z$$

$$= \frac{(\alpha^{-1}z)(\alpha^{-1}z)}{\frac{1}{\alpha^{-1}z} - \frac{\alpha^{-1}z}{\alpha^{-1}z}}$$

$$= -\frac{1}{\alpha^{-1}z - 1}$$

$$= \frac{1}{1-\alpha^{y}z^{-1}}$$

Left hand signal	Right hand signal:
$x(n) = -\alpha^n U(-n-1)$	$x(n) = \alpha^n U(n)$
$X(z) = \frac{1}{1-\alpha z^{-1}}$	$X(z) = \frac{1}{1-\alpha z^{-1}}$
ROC:	ROC:
$\lvert A \rvert < 1$	$\lvert z \rvert > \alpha$
$= \lvert \alpha^{-1} z \rvert < 1$	
$= \lvert z \rvert < \alpha$	

Example :3.1.5

$x(n) = a^n U(n) + n^n U(-n-1)$

$$x(z) = \sum_{n=-\infty}^{\infty} a^n U(n) + \sum_{n=-\infty}^{\infty} b^n U(-n-1)$$

$$= \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\alpha z^{-1}}$$ hoy nai

$$= \frac{1-\alpha z^{-1} + 1-\alpha z^{-1}}{1-\alpha z^{-1}} = \frac{2-2(\alpha z^{-1})}{1-\alpha z^{-1}}$$

$$= \frac{2(1-\alpha z^{-1})}{1-\alpha z^{-1}} = 2$$

Example 3.1.5 :

$x(n) = a^n U(n) + b^n U(-n-1)$

$$x(k) = \frac{1}{1-\alpha z^{-1}} - b\frac{1}{1-bz^{-1}}$$

ROC:

$|z| < b$

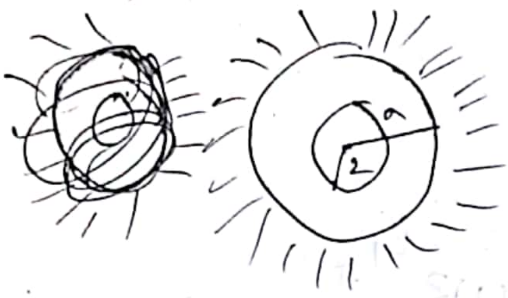
ROC:

$|z| > a$

ROC of $x(k)$:

Exact doesn't becaues common point nei.

Let, $a>b$ হলে হয় তখন ROC absent।



আবার, $a<b$

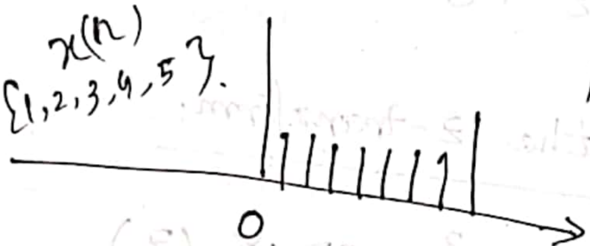
$x(z)$ exist ~~for |a| < |b|~~ for $|a| < |z| < |b|$

common point.

Book table 3.1

Summary :

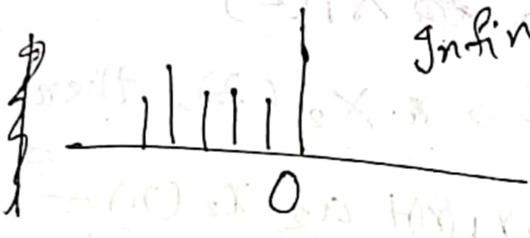
Right signal: $x(n)$ $\{1,2,3,4,5\}$.



finite $x(z)$ all value except $|z|=0$

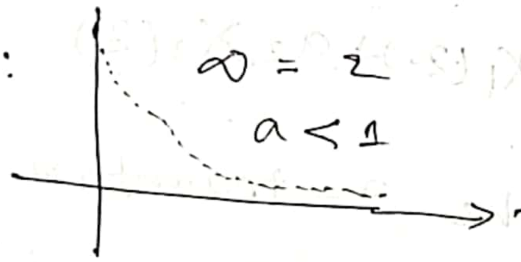
Infinite.

Left signal:



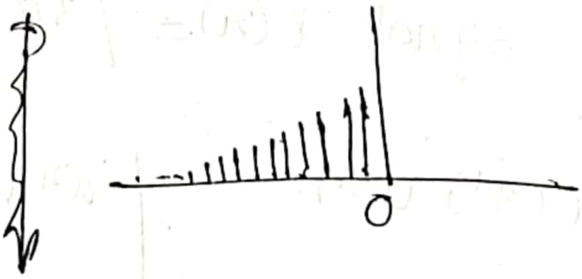
$|z| = z^{+ve}$ all value except $|z| = \infty$

Positive finite:



ROC: $|z| > a$.

Negative finite:



$|z| < \infty$

After mid

Z-transform example:

$x(n) = [1, \underline{1}, 2, 0, 3]$

$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$X(z) = x(-1)z^{1} + x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$

$= z^{1} + 1 + 2z^{-1} + 0 + 3z^{-3}$

3.2: Properties of the z-transform:

1. Linearity: If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$

$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$, then

$x(n) = a_1 x_1(n) + a_2 x_2(n) \overset{z}{\longleftrightarrow}$

$X(z) = a_1 X_1(z) + a_2 X_2(z)$

Example: 3.2.1: Determine the z-transform and the ROC of the signal $x(n) = [3(2^n) - 4(3^n)]u(n)$

$= 3(2^n)u(n) - 4(3^n)u(n)$

$a^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}$

$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$

ROC $|z| > a$

$|z| > 2$; $|z| > 3$

$\therefore$ ROC ; $|z| > 3$

2. Time-shifting:

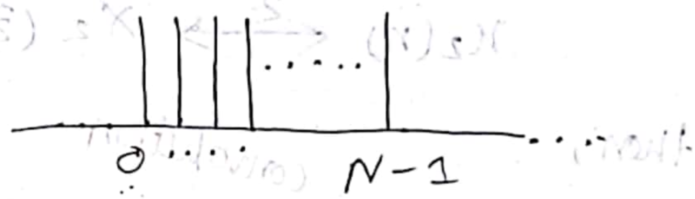
If $x(n) \overset{z}{\longleftrightarrow} X(z)$

Then, $x(n-k) \overset{z}{\longleftrightarrow} z^{-k} X(z)$

Example: 3.2.3: Hw.

Example 3.2.4:

$$x = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$



$$x(n) = u(n) - u(n-N)$$

$$X(z) = \frac{1}{1-z^{-1}} - \frac{z^{-N}}{1-z^{-1}}$$

$$= \frac{1-z^{-N}}{1-z^{-1}}$$

3. Time reversal:

If $x(n) \xleftrightarrow{z} X(z)$

then, $x(-n) \xleftrightarrow{z} X(z^{-1})$

Example: 3.2.6:

$x(n) = u(-n)$

$X(z) = \frac{1}{1-z}$

$z(u(n)) = \frac{1}{1-z^{-1}}$

* Convolution of two squares:

If, $x_1(n) \xleftrightarrow{z} X_1(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$

then,

convolution

$x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z) X_2(z)$

$y(n) = \sum x(k) h(n-k)$

* Convolution by means of z-Transform:

1. Compute the z-transforms of the signals to be convoluted.

$X_1(z) = z[x_1(n)]$

$X_2(z) = z[x_2(n)]$

2. multiply the two z-~~into~~ transforms,

~~$X(z) = X_1(z) X_2(z)$~~

$X(z) = X_1(z) X_2(z)$

3. find the impulse Z-Transform of $X(z)$

$x(n) = z^{-1}[X(z)]$

Example: 3.2.9:

$x_1(n) = \{1, -2, 1\}$

$x_2(n) = \{1, 1, 1, 1, 2\}$

$X_1(z) = 1 - 2z^{-1} + z^{-2}$

$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

$X(z) = X_1(z) X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$

$x(n) = \{1, -1, 0, 0, 0, 0, -1, 1\}$

* 3.3: Rational z-transform:

Poles and zeros:

$$X(z) = \frac{B(z)}{A(z)}$$

B(z) এর roots → ~~poles~~ zeros

A(z) " " → poles.

The zeros of a a z-transform X(z) are the values of z for which X(z) = 0.

The poles ... for which X(z) = ∞

$$X(z) = \frac{B(z)}{A(z)} = c\frac{(z-z_1)(z-z_2)(z-z_3)(z-z_m)}{(z-P_1)(z-P_2)(z-P_3)(z-P_m)}$$

Zeros : $z_1, z_2, z_3, \ldots z_m$

Poles : $P_1, P_2, P_3, \ldots P_m$.

Example; 3.3.1

Determine the pole-zero plot for the signal

$$x(n) = a^n u(n)$$

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} = \frac{(z-0)}{(z-a)}$$

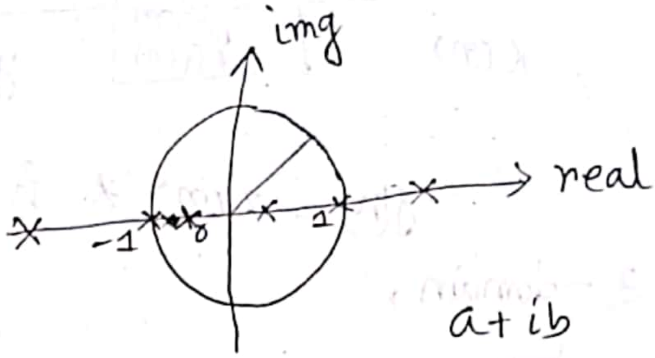
zero : 0

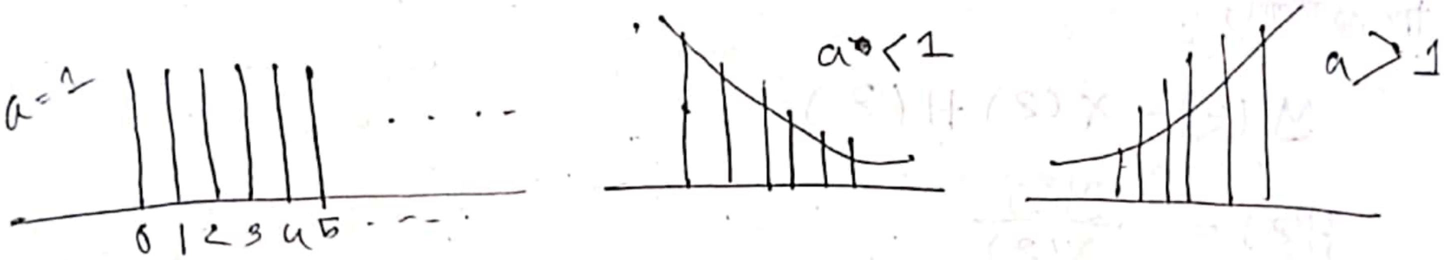
pole : a

3.3.2 :
* Pole location and Time domain behavior for causal signals.

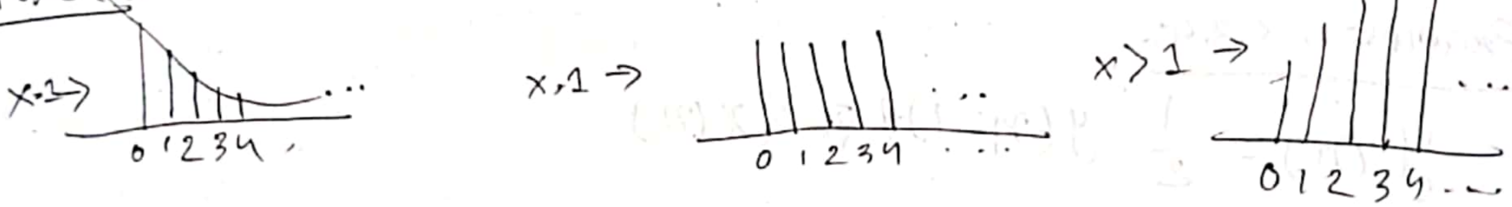
$x(n) = a^n u(n)$

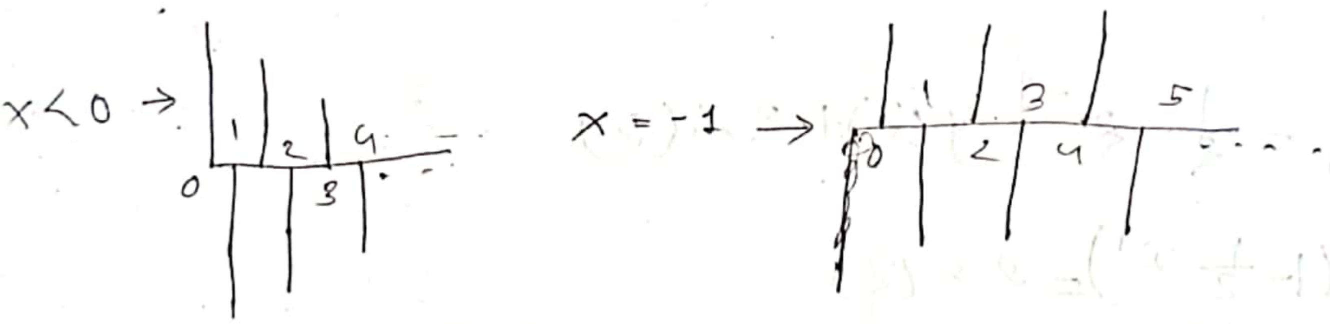
$X(z) = \frac{1}{1 - az^{-1}}$



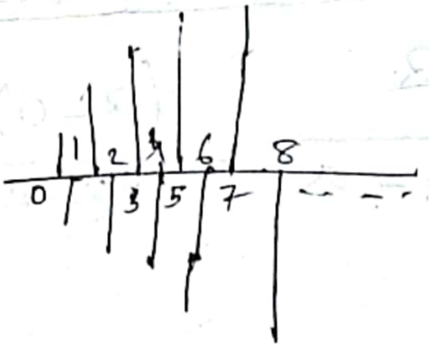


Pole (a)



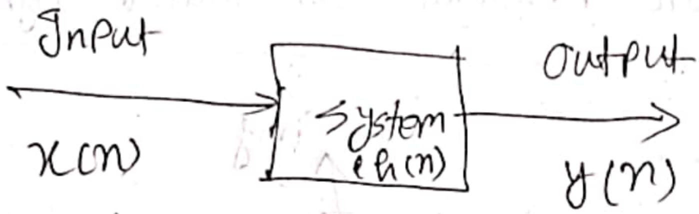


$\therefore x > -1 \Rightarrow$



3.3.3:

system function of and ~~LTI~~ LTI System:



$y(n) = x(n) * h(n) \Rightarrow h(n) =$ inverse z-transform of $H(z)$

In, z-domain,

z-transform,

$$Y(z) = X(z) H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

Example. 3.3.4:

$$y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

$$\Rightarrow Y(z)\left(1 - \frac{1}{2} z^{-1}\right) = 2X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

$$\Rightarrow \frac{y(z)}{x(z)} = \frac{4}{2 - z^{-1}}$$

$$h(n) = \left(\frac{4}{2 - z^{-1}}\right)^{-1} \Rightarrow \frac{2 - z^{-1}}{4}$$

$$\Rightarrow \frac{y(z)}{x(z)} = 2 \cdot \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$$h(n) = 2 \cdot \left(\tfrac{1}{2}\right)^{n} u(n)$$

3.4:

Inverse ~~of~~ z-Transform:

Inverse z-Transform by power series Expansion:

3.4.2: Determine the inverse z-Transform of,

$$X(z) = \frac{1}{1 - 1.5 z^{-1} + 0.5 z^{-2}}$$

(a) ROC: $|z| > 1$

(b) ROC: $|z| < 0.5$

Soln:

$[1, 1.5, 1.65 \ldots]$

$1 - 1.5 z^{-1} + 0.5 z^{-1}$

$$1 - 1.5z^{-1} + 0.5z^{-2} \overline{)\, 1 \,}\; (1 + 1.5z^{-1} + 1.65 z^{-2}$$

$1 - 1.5z^{-1} + 0.5z^{-2}$

$1.5z^{-1} + 0.5z^{-2}$

$1.5z^{-2} + 2.15z^{-1}$

$0.75z^{-2}$

~~2.15~~

$\cancel{2.15z^{-1} + 0.75z^{-2}}$

$1.65z^{-2} - 0.7z^{-3}$

(a) Causal signal,

$X(z) = 1 + 1.5z^{-1} + 1.65z^{-2} \ldots\ldots$ $[1, 1.5, 1.65\ldots]$

(b) Anti Causal signal,

$$0.5z^{-2} - 1.5z^{-1} + 1 \,\big)\, 1 \,\big(\, 2z^{2} + 6z^{3} + 14z^{4} \ldots$$

$$1 - 3z + 2z^{2}$$

$$\cancel{0.75z + 2z^{2}}$$

$$3z^{1} - 2z^{2}$$

$$3z^{1} - 9z^{2} + 6z^{3}$$

$$7z^{2} - 6z^{3}$$

$[\ldots 14, 6, 2, 0, 0]$

3.4.3:

Inverse z-transform by ~~participation~~ partial-fraction Expansion:

3.4.8: Example:

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$$\boxed{\frac{1}{1 - az^{-1}} \xrightarrow{z} a^{n}.u(n)}$$

$$= \frac{2}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}}$$

(a) $|z| > 1$

(b) $|z| < 0.5$

(c) $0.5 < |z| < 1$

a) $x(n) = 2(1)^n u(n) - (0.5)^n u(n)$

(b) $x(n) = -2(1)^n u(-n-1) + (0.5)^n u(-n-1)$

(c) $x(n) = 2(1)^n u(-n-1) - (0.5)^n u(n)$

3.5:

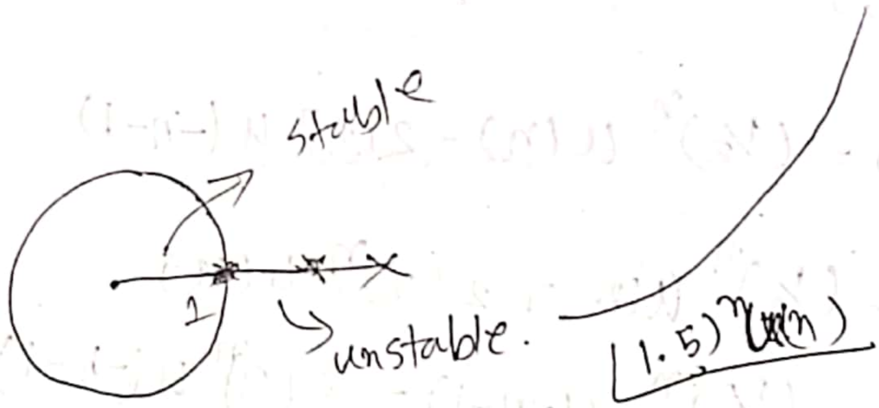
Analysis of LTI system in z-transform:

Causality and stability:

Causality:

$h(n) = 0, \quad n < 0.$

Stability:



বাইরে গেলে unstable, ভিতরে থাকলে stable.

A LTI system is BIBO stable if and only the ROC of the system functions include the unit circle.

Example : 3.5.2

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-3z^{-1}}$$

(a) System is stable.

(b) System is causal.

(c) System is Anticausal.

pole: $\frac{1}{2}, 3$ $\qquad \frac{1}{2} < |z| < 3.$

(a) $h(n) = (\frac{1}{2})^n u(n) - 2(3)^n u(-n-1)$

(b) $h(n) = (\frac{1}{2})^n u(n) + 2(3)^n u(n)$

(c) $h(n) = -(\frac{1}{2})^n u(-n-1) - 2(3)^n u(-n-1)$

*One sided z-tran:

3.6

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

H.W

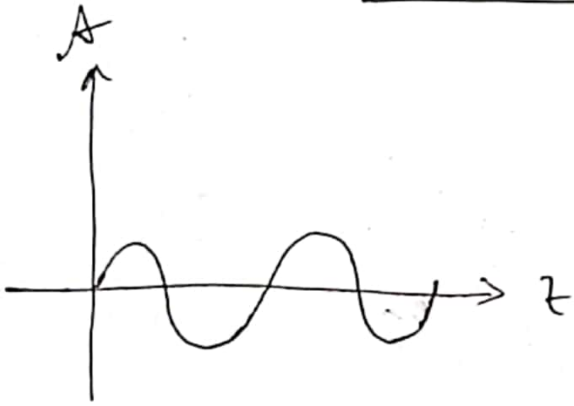
Ex: 3.6.1

$$x(n) = [1, 2, 5, 7]$$

$$X^{+}(z) = 1 + 2z^{-1} + 5z^{-1} + 7z^{-3}$$

chapter 4:

frequency Analysis of Signals



$\sin(\omega t + \theta)$

1. fourrier series → periodic signal.
2. fourrier Transform → aperiodic signal.

complex signal

Sinusoids

can be decomposed into a number of sinusoids

~~Sino~~sinusoids can be added a Complex signal.

1672, Newton

Sun light

Prisom test

4.1.1: fourier Series for continuous-time periodic signals.

Jean Baptiste joseph fourier (1768-1830)

- Heat conduction

Sinusoide $\left.\begin{array}{l}\sin \omega t \\ \cos \omega t\end{array}\right\}$ real number

Complex Sinusoid:

$$e^{i\theta} = \cos\theta + \sin\theta$$

fourier series Equations:

Analysis Equation: continuous sum

$$C_k = \frac{1}{T_p} \int x(t) e^{-j2\pi k f_0 t} dt$$

Synthesis Equation:

$$x(t) = \sum_{k=\infty}^{\infty} C_k e^{j2\pi k f_0 t}$$

discrete sum.

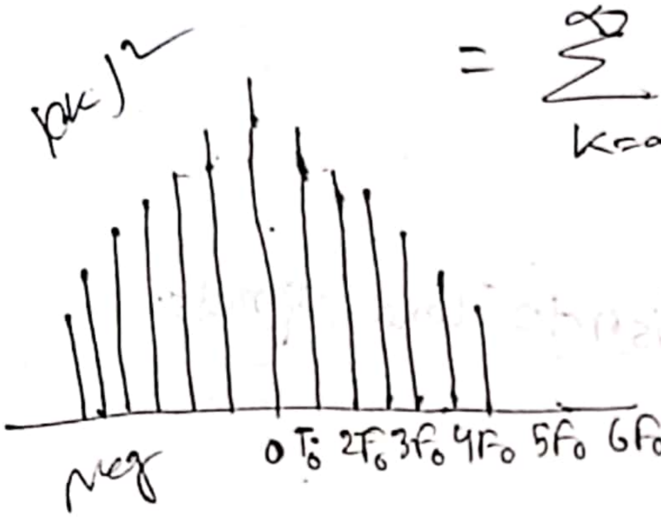
4.1.2: power density Spectrum of Periodic Signals.

$$P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2 \, dt$$

T_p ↳ Time Period.

$$\boxed{\sum |x(n)|^2}$$

$$= \sum_{k=-\infty}^{\infty} |c_k|^2$$



$$f_0 = \frac{1}{T_p}$$

↓

fundamental frequency

power density spectrum

4.1.3: The fourier transform of Continuous-Time Aperiodic Signals.

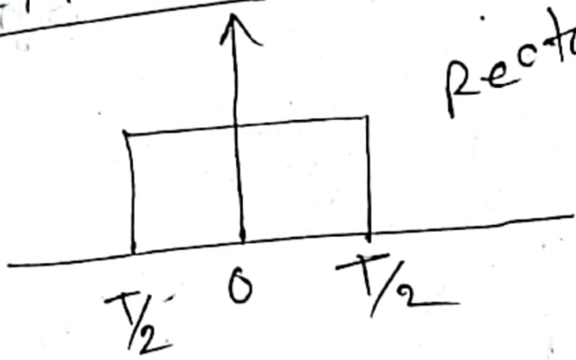
Analysis Equations:

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi F t} \, dt$$

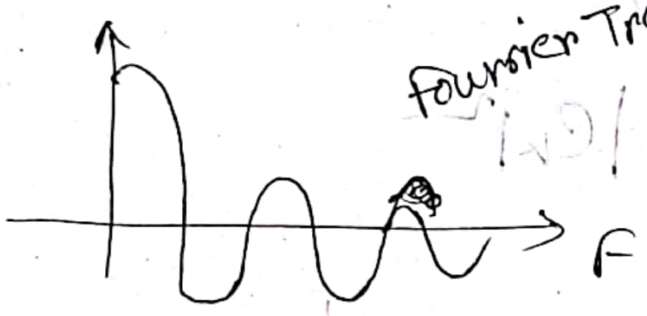
synthesis Equations:

$$x(t) = \int_{-\infty}^{\infty} X(F)\, e^{j2\pi F t} \, dF$$

Example 4.1.2 :



$|X(f)|$



4.2 freq^y analysis of Discrete-Time signals:

Ques Solution:

(1) $\delta(n) \xrightarrow{z} 1$

$\delta(n-k) \rightarrow z^{-k}$

$\delta(n+k) \rightarrow z^{k}$

(2) $x = u(n) - u(n-10)$

$\xrightarrow{z} \dfrac{1}{1-z^{-1}} - \dfrac{z^{-10}}{1-z^{-1}}$

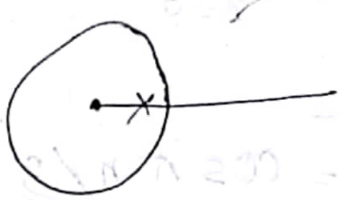
(3)

(4) } Book.

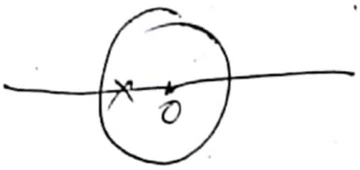
(5)

(6) (i)  $x(n) = a^n u(n)$; $a<1$

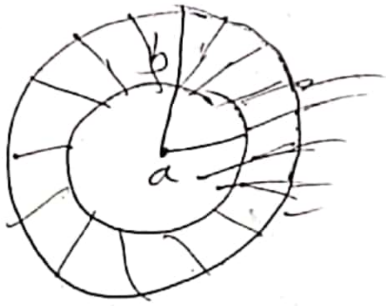
$$\rightarrow \frac{1}{1-az^{-1}} = \frac{z-0}{z-a}$$



ROC

(ii) 

ROC :

(iii) 

opposite :

— 0 —

4.2.1 fourier series for Discrete-time periodic signals :

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \rightarrow \text{continuous}$$

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N}$$ → fourrier series of DT sinusoid

Synthesis equⁿ.

Analysis equⁿ:

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

Example:

$$x(n) = \cos \pi n/3$$

$$= \cos 2\pi n/6$$

$$\frac{1}{N} = 1/6$$

$$\therefore C_k = \frac{1}{6} \sum_{n=0}^{5} x(n) e^{-j2\pi kn/6}$$

4.2.2: Power density spectrum:

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

$$= \sum_{k=0}^{N-1} |C_k|^2$$

H-W: ~~Ex 2.~~

Ex: 4.2.2

- Fourier Series $\rightarrow x(t)$
- fourier Transform $\rightarrow x(n)$

Fourier Transform of DT Signal:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$\omega = 2\pi f$
$f = -\infty$ to ∞
$\omega = -\pi$ to π
$-1/2$ to $1/2$
0 to 2π

$$X(\omega + 2\pi k) = \sum x(n) e^{-j(\omega + 2\pi k) n}.$$

$$= \sum x(n) e^{-j\omega n} e^{j 2\pi k n}.$$

Periodic with period of 2π.

Relation of fourier Transform with z-transform:

$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}; \quad z = re^{j\omega}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}.$$

$$z^{-n} = (e^{j\omega})^{-n} = e^{-j\omega n}.$$

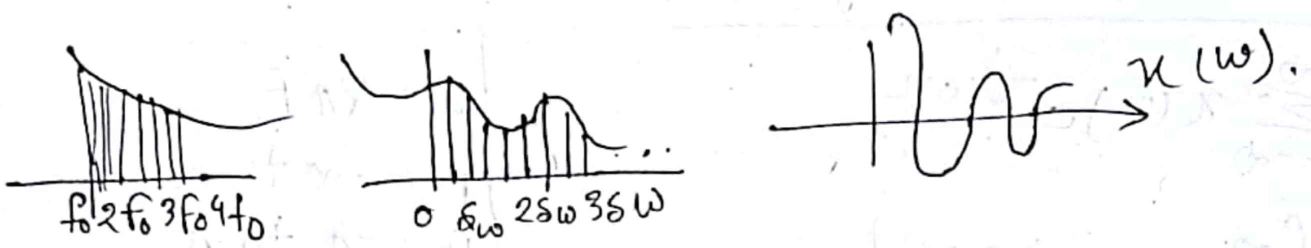
$$X(z) = \sum x(n) e^{-j\omega n}.$$

If, $r = 1$, z transform and fourier transform are same

chapter-7

* Discrete fourier transform of DT signal:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}. \qquad x_0(k) = \qquad ; \; f_0 = \frac{1}{T_p}$$



$$\delta\omega = 2\pi/N \; ; \; X(2\pi/N \cdot K) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi/N \cdot Kn}.$$

1) $\omega \Rightarrow$ continuous ; 2) $x(n)$ length is infinite.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi k n/N}$$

$$= \cdots + \sum_{n=-N}^{-1} x(n) e^{-j2\pi k n/N} + \sum_{n=0}^{N-1} x(n) e^{-j2\pi k n/N}$$

$$+ \sum_{n=N}^{2N-1} x(n) e^{-j2\pi k n/N} + \cdots$$

$$x(k) = \sum_{l=-\infty}^{\infty} \sum_{n=ln}^{lN+N-1} x(n) e^{-j2\pi kn/N}$$

$$\Rightarrow x(k) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-ln) \right] e^{-j2\pi kn/N}.$$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

$l = 0 \; x(n)$

$l = 1 \; x(n-N)$

$l = 2 \; x(n-2N)$

N sample

$$X(k) = \sum_{n=0}^{N-1} x_p(n) \, e^{-j2\pi kn/N}$$

Synthesis eqn: $x_p(n) = \sum_{k=0}^{N-1} C_k \, e^{j 2\pi kn/N}$.

Analysis eqn: $C_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j 2\pi kn/N}$; $k = 0 \cdots N$.

$\downarrow$ fourier Series.

$C_k = \frac{1}{N} X(k)$ $\therefore x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j 2\pi kn/N}$

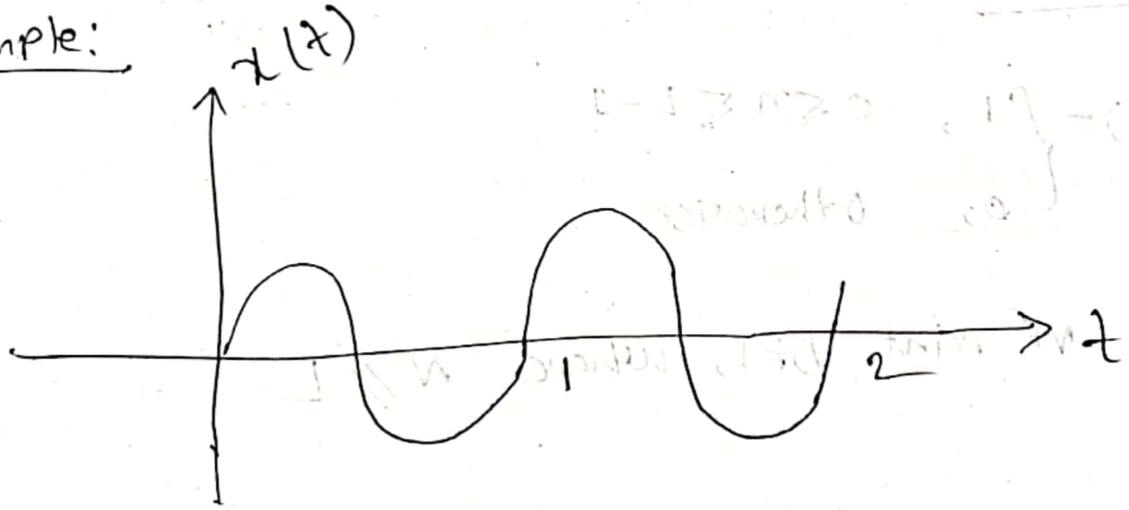
If, $x(n) = x_p(n)$, $0 \le n \le N-1$, হয় তাহলে $x_p(n)$ এর পরিবর্তে $x(n)$ নেওয়া যায়, কিন্তু অবশ্যই $x(n)$ এর length

Analysis: $X(k) = \sum_{n=0}^{N-1} x(n) \, e^{-j 2\pi kn/N}$

Synthesis: $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) e^{j 2\pi kn/N}$.

Discrete fourier Transform. (DFT pair).

Example:



DFT:

Analysis eq^n

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots\ldots N-1$$

$x(n) = \{ \ldots\ldots \}$ | N = DFT point

$L = 200$ | $N = 256$

56 - zeros ~~...~~ $L \leq N$

zero-padding

1 sample $\rightarrow \left(\frac{2\pi}{N}\right)$ rad

$L \rightarrow \left(\frac{F_s}{N}\right)$ Hz

1) zero padding $\rightarrow$ $L \leq N$

2) freq^y / per sample : F_s/N or $\left(\frac{2\pi}{N}\right)$

3) If N is increased.

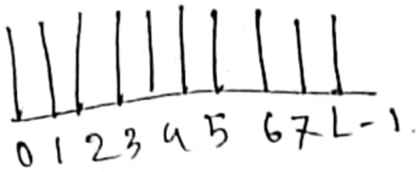
Inverse DFT:

$$x(n) = \sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}, \quad n = 0, 1, \ldots\ldots, N-1$$

Example 7.1.2:

$$x(n) = \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

Determine N point DFT, where $N \geq L$



$$X(k) = \sum_{n=0}^{L-1} x(n) e^{-j2\pi kn/N}$$

Infinite ধারা, $\frac{1}{1-a}$

finite ধারা, $\frac{1-a^N}{1-a}$

$$= \frac{1-e^{-j2\pi kL/N}}{1-e^{-j2\pi k/N}}$$

7.2 Properties of the DFT:

1. Periodicity:

If, $x(n) \longleftrightarrow X(k)$

$$x(n+N) = x(n)$$

$$X(k+N) = X(k)$$

2. Linearity:

If, $x_1(n) \longleftrightarrow X_1(k)$

$x_2(n) \longleftrightarrow X_2(k)$

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{DFT} a_1 X_1(k) + a_2 X_2(k)$$

3. Multiplication of Two DFTs and Circular Convolution:

2	1	3	5	7	8	9

$\xrightarrow{1L\ shift}$

	2	1	3	5	7	8

Circular shift $\rightarrow$

8	2	1	3	5	7

$x_1(n),\ x_2(n)$

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, e^{-j2\pi kn/N}$$

$$X_3 k = X_1(k)\, X_2(k)$$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k)\, e^{j2\pi km/N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_1(k)\, X_2(k)\, e^{j2\pi km/N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N} \right] \left[\sum_{n=0}^{N-1} x_2(l)\, e^{-j2\pi k} \right] e^{j2\pi km/N}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{n=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} \right]$$

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a=1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$

$$a = \left(e^{j2\pi(m-n-l)/N} \right)$$

$a=1$ if, $m-n-l = kN$

$l = ((m-n))_N$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N$$

↳ Circular convolution.

Ex: 7.2.1:

perform the circular convolution of the following two sequences.

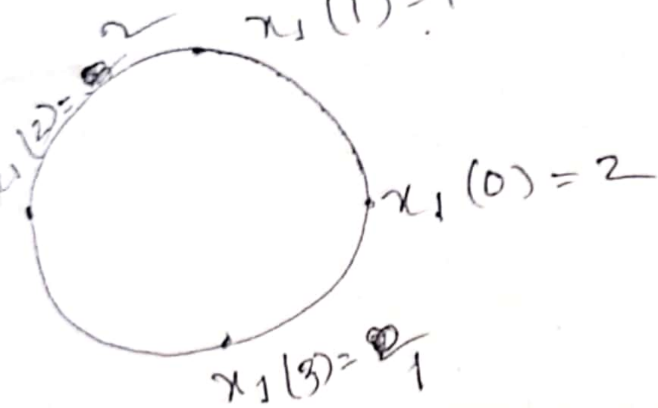
$x_1(n) = \{2, 1, 2, 1\}$

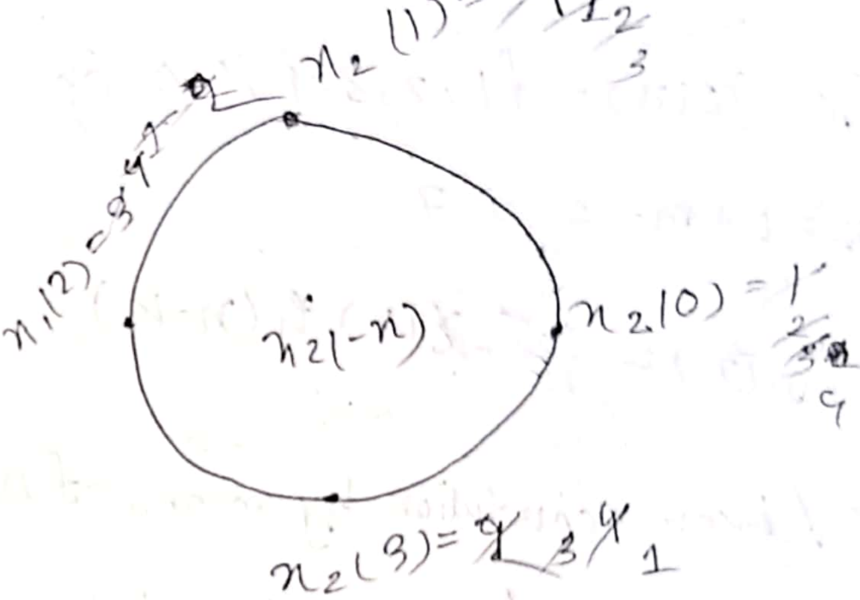
$x_2(n) = \{1, 2, 3, 4\}$

$x_3(m) = \sum_{n=0}^{3} x(n)\, x_2((m-n))_N$

~~$x_3(m) =$~~

$x(0) = \sum x_1(n)\, x_2((-n))_N$





$x_3(0) = 2 + 4 + 6 + 2 = 14$

$x_3(1) = 4 + 1 + 8 + 3 = 16$

$x_3(2) = 6 + 2 + 2 + 4 = 14$

$x_3(3) = 8 + 3 + 4 + 1 = 16$

$x_3(n) = \{1, \underset{\uparrow}{4}, 16, 14, 16\}$

1. Folding
2. Shifting
3. Multiplication
4. Summing.

7.3: ~~king~~ Linear filtering methods based on the DFT:

$L \quad x_1(n) = \{2, 1, 2, 1, 0, 0, 0\}$

$M \quad x_2(n) = \{1, 2, 3, 4, 0, 0, 0\}$

$N = L + M - 1 = 7$

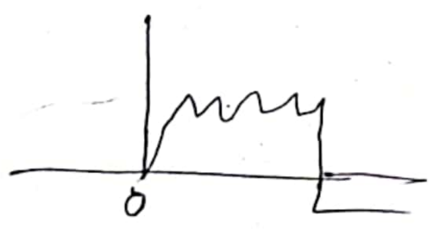
$y(n) = \sum_{k=} x(k) \cdot h(n-k)$

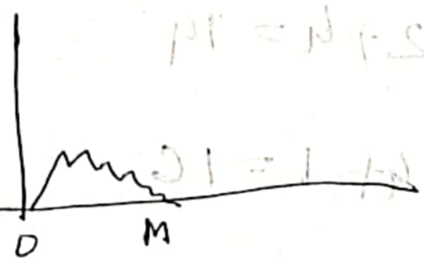
* Linear convolution by means of DFT and IDFT:

7.3.1: use of DFT in linear filtering:

$x(n) = 0, \quad n < 0$ and $n \geq L$

$h(n) = 0, \quad n < 0, \; n \geq M$





$$y(n) = \sum_{k=0}^{} h(k)\, x(n-k)$$

$$y(n) = h(n) * x(n)$$

$$Y(k) = H(k) \cdot X(k)$$

$$y(n) = \text{IDFT}\,(H(k) \cdot X(k)), \; N \geq L+m-1$$

Ex. 7.3.1: By means of DFT and IDFT determine the response of the FIR filter with impulse response

$$h(n) = \{1, 2, 3\}$$

to the input sequence,

$$x(n) = \{1, 2, 2, 1\}$$

$$y(n) = x(n) * h(n)$$

$$Y(k) = X(k) \cdot H(k)$$

$$y(n) = \text{IDFT}\,[Y(k)]$$

$$N = L+m-1 = 4+3-1 = 6$$

H.W

7.3.2: Filtering of Long Data sequences:

$$Y(k) = X(k) \cdot H(k) \xrightarrow[N \text{ Point}]{} N = L+m-1$$

1) Overlap save method.

2) Overlap-add method.

* Overlap add method:

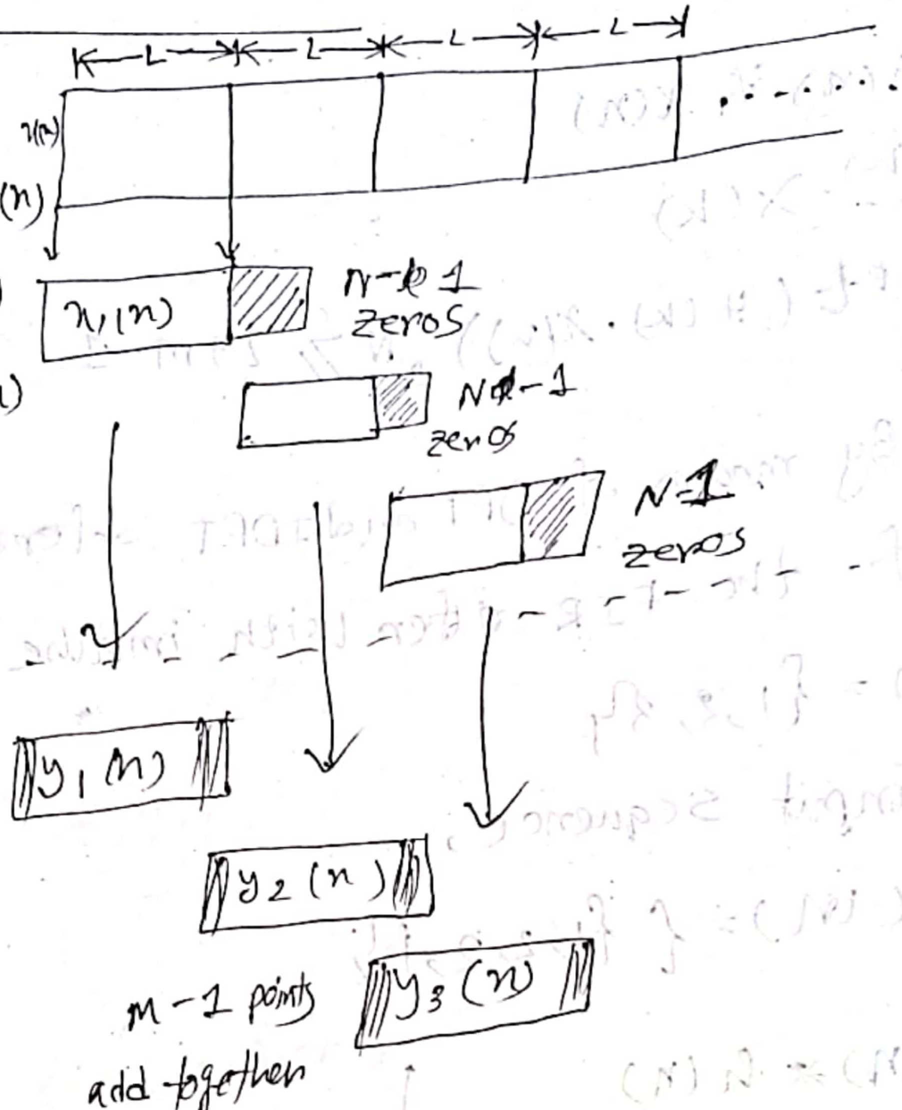
$y_1(n) = x_1(n) * h(n)$

$X_1(k) \cdot H(k) \xrightarrow{IDFT} y_1(n)$

$y_2(3) = x_2(n) * h(n)$

$X_2(k) \cdot H(k) \rightarrow y_2(n)$

⋮



$x_1(n) = \{x(0), x(1), x(2), \ldots, x(L-1), \underbrace{0, 0, 0, 0}_{m-1 \text{ zeros}}\}$

$x_2(n) = \{x(L), x(L+1), x(L+2), \ldots, x(2L-1), \underbrace{0, 0, 0 \ldots 0}_{m-1 \text{ zeros}}\}$

⋮

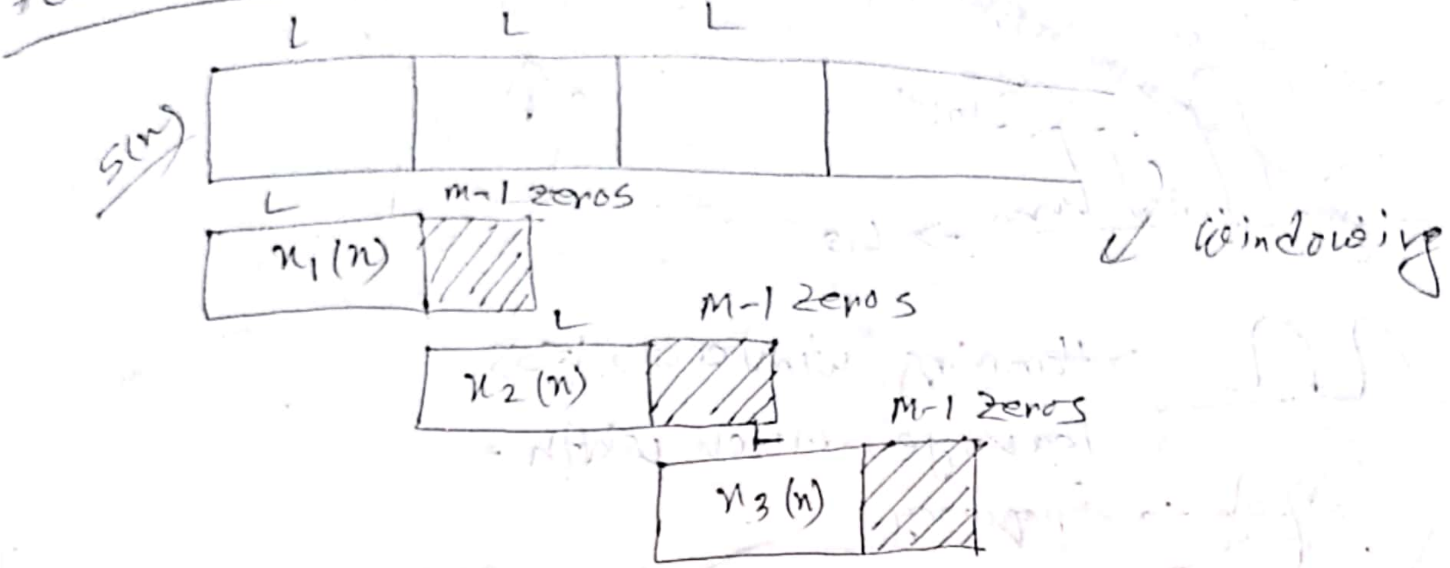
$Y_m(k) = H(k)\, X_m(k)$

$\rightarrow IDFT \rightarrow y_m(n)$

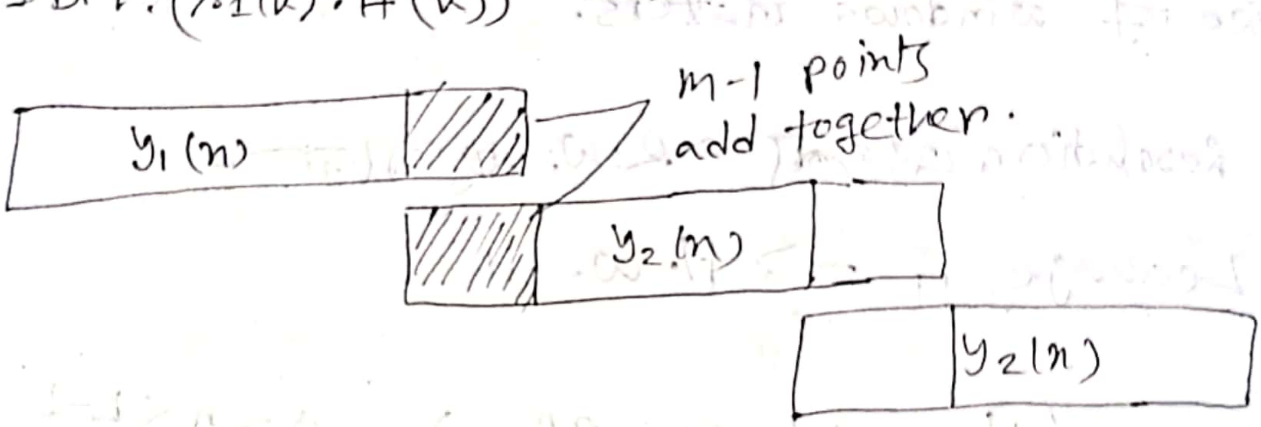
$y(n) = \{y_1(0), y_1(1), \ldots, y_1(L-1), y_1(L)+y_2(0), y_1(L+1)+y_2(1)+\ldots\}$

* ~~lopr~~ long data-filtering:

* overlap add:



$y_1(n) = IDFT(X_1(k) \cdot H(k))$



* frequency Analysis of signals using DFT:

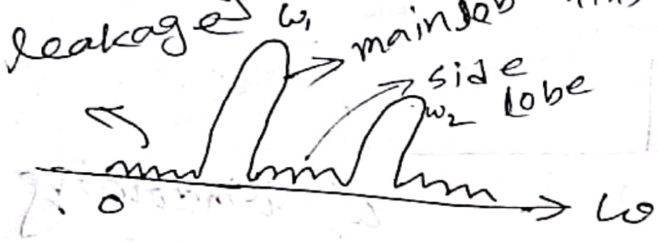
$x(n) = s(0:L) * w(n)$

$W(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} - j\omega(L-1)/2$

$$\omega(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{------} \end{cases}$$

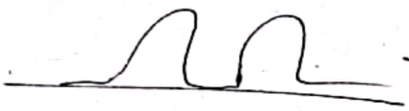
↓

Rectangular window.

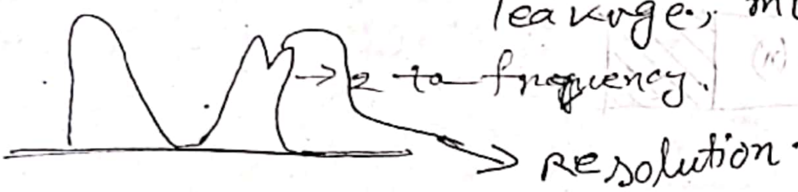


$x(n) = \cos \omega_1 n + \cos \omega_2 n$

Hamming window



→ Hamming window, less leakage, much width.



choice of windows matters.

Resolution এর জন্য → R.W. ভালো

Leakage " → H.W. "

$$w(n) = \begin{cases} \frac{1}{2}\left(1 - \cos \frac{2\pi}{L-1} n\right), & 0 \le n \le L-1 \\ 0, & \text{other} \end{cases}$$

↑

Hamming window.

Question solution:

(1) $X(k)$

(2) $X(k) = aX_1(k) + bX_2(k)$

(3) $y(n) = x(n) * h(n)$

$y(n) = IDFT(X(k) \cdot H(k))$

Chapter – 8

Efficiant computations of the DFT: fast fourier Transform Algorithms:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$k = 0 \sim N-1$

Summation: $(N-1) \cdot N$

Multiplication: $N \cdot N = N^2$

$N \approx 1024^2 \Rightarrow N =$ millions of multiplication.

$\frac{N^2}{2} + \left(\frac{N}{2}\right)^2$

$= \frac{N^2}{4} + \frac{N^2}{4} = \frac{N^2}{2}$

$$= \sum_{n=0}^{N-1} x(n) \cdot W_n^{kn}$$

$W_N = \left(e^{-j2\pi/N}\right)^{kn}$

$= e^{-j2\pi kn/N}$

* Symmetry property:

$$W_N^{k+N/2} = -W_N^{k}$$

$N = 200$ হলে, $k + \frac{N}{2} \Rightarrow 10 + 100$

$k = 10$ ", $\Rightarrow 10$

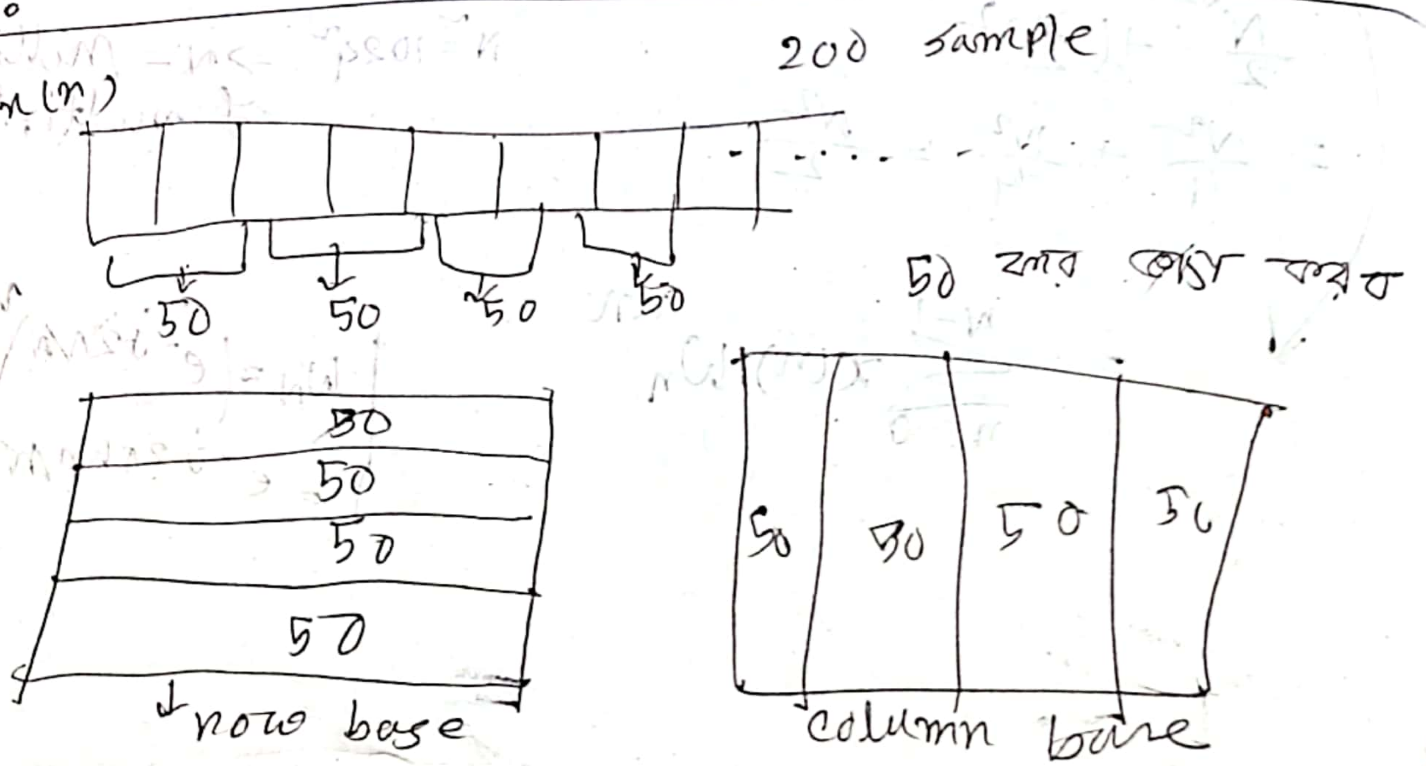
$$\left(e^{-j2\pi/N}\right)^{k+N/2}$$
$$= e^{(-j2\pi/N)(k+N/2)}$$
$$= e^{-j2\pi k/N} \cdot e^{-\frac{j2\pi N/2}{N}}$$
$$= e^{-j2\pi k/N} \cdot e^{-j\pi}$$
$$= -e^{-j2\pi k/N} = -W_N^{k}$$

* periodicity property:

$$W_N^{k+N} = W_N^{k}$$

8.1.2:

Divide and conquers Approach to computation of the DFT:



$N = LM$

$200 = 4 \cdot 50$

Row-wise:

	0	1	2	3 ------		m-1
0	$x(0)$	$x(1)$	$x(2)$	----	----	$x(m-1)$
1	$x(m)$	$x(m+1)$	--	..	.	$x(2m-1)$
2	.					$x(3m-1)$
3	:					:
:	:					:
:	:					$x(Lm-1)$
L-1	$x((L-1)m)$					

Index $n = ml + m$; $x(n) \xrightarrow{\text{এমন হয়}} x(ml+m)$

Column-wise :

$n = l + ml$

$x(n)$ হবে $x(l+ml)$.

Index

	0	1	2	3	...	M-1
0	$x(0)$	$x(L)$	$x(2L)$	-	-	$x((M-1)L)$
1	$x(1)$	$x(L+1)$				
2						
3						
⋮						
L-1	$x(L-1)$	$x(2L-1)$				$x(LM-1)$

$x(n) \rightarrow$

$X(k) \rightarrow k = Mp + q \Rightarrow$ row-wise

$k = qL + p \rightarrow$ column-wise

29.10.22

$$X(k) = \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(l,m)\, W_N^{(Mp+q)(mL+l)}$$

$$W_N^{(Mp+q)(mL+l)} = W_N^{mLMp} \cdot W_N^{Mpl} \cdot W_N^{ql} \cdot W_N^{MLq}$$

$W_N^{NmP} = 1$ $[e^{-j2\pi . mP} = 1]$

$W_N^{mqL} = W_{N/L}^{mq}$ $\left[W_N = e^{-\frac{j2\pi}{N/L} . mqL}; W_{N/L}^{mq}\right]$

$= W_M^{mq}$

$W_N^{mPL} = W_{N/N}^{Pl} = W_L^{pl}$

$$X(P,q) = \sum_{l=0}^{L-1} \left\{ W_N^{lq} \left[\sum_{m=0}^{M-1} x(l,m) W_M^{mq} \right] \right\} W_L^{lq}$$

↳ m-point DFT of a row.

summary:

1. first, we compute the M-point DFTs,

$$F(l,q) = \sum_{m=0}^{M-1} x(l,m) W_M^{mq}, \quad 0 \leq q \leq M-1$$

2. Second, we compute a new rectangular array $G(l,q)$, defined as,

$$G(l,q) = W_N^{lq} . F(l,q)$$

$$X(P,q) = \sum_{l=0}^{L-1} G(l,q) W_L^{lp}$$

3. finally we compute the L-point DFTs:

$$X(P,q) = \sum_{l=0}^{L-1} G(l,q) W_L^{lp}$$

* Example: 8.1.1 : [Page-533]

$N = 15 \Rightarrow N = 5 \times 3$

	0	1	2
0	$x(0)$	$x(5)$	$x(10)$
1	$x(1)$	$x(6)$	$x(11)$
2	$x(2)$	$x(7)$	$x(12)$
3	$x(3)$	$x(8)$	$x(13)$
4	$x(4)$	$x(9)$	$x(14)$

$F(l,q) \cdot W_N^{lq} = G_L^{lq}$

* Algorithm 1: (FFT বলে)

1. Store the signal column-wise.
2. Compute the M-point DFT of each row.
3. Multiply the resulting array by the phase factors, W_N^{lq}.
4. Compute the L-point DFT of each column.
5. Read the resulting array row-wise

* Algorithm 2: opposite:

* 8.1.3: Radix-2 FFT Algorithm:

$256 = 2$ $\quad N = LM$

$8^2 = 2^3 \quad = r_1 r_2 r_3 \cdots r_V$

$x(n) \to 41 \Rightarrow$ zero pad $\Rightarrow 64$. $\quad N = r^V$; $r = 2$

$\underline{x(n)}$ $f_1(n) = x(2n) \to$ even indexed

$f_2(n) = x(2n+1)$, $n = 0, 1, \ldots N/2 - 1$

$\to$ odd indexed

$x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7) \ldots$

$f_1 \Rightarrow x(0), x(2), x(4), x(6)$.

$f_2 \to x(1), x(3), x(5), x(7)$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} = \sum_{n \text{ even}} x(n) W_N^{kn} + \sum_{n \text{ odd}} x(n) W_N^{kn}$$

$$\cancel{= \sum_{m=0}^{N/2-1} x(2m) W_N^{kn} + \sum_{m=0}^{N/2-1} x(2m+1) W_N^{kn}}$$

$W_N^{2m} = W_{N/2}^{m}$

$$= \sum_{m=0}^{N/2-1} f_1(2m) W_N^{km} + \sum_{m=0}^{N/2-1} x(2m+1) W_N^{km}$$

$$= \sum_{m=0}^{N/2-1} f_1(m) W_{N/2}^{km} + W_N^k \sum_{m=0}^{N/2-1} f_2(m) W_{N/2}^{km}$$

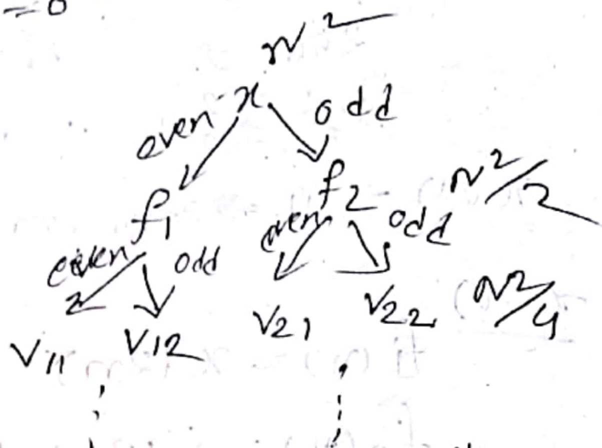
$$X(k) = F_1(k) + W_N^k \cdot F_2(k).$$

$F_1(k), F_2(k)$

$W_N^{k+N/2} = -W_N^k$

* $X(k) = F_1(k) + W_N^k F_2(k)$, $k = 0 \dots N/2$

$X(k+N/2) = F_1(k) - W_N^k F_2(k)$



until the value is 2

* complexity,

$N \cdot \log_2 N$

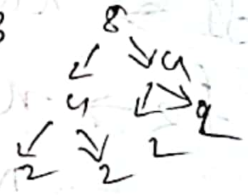
$$\boxed{\begin{aligned} X_1(k) &= F_1(k) + W_N^k F_2(k) \\ X_2(k) &= F_1(k) - W_N^k F_2(k) \end{aligned}}$$

Basic operation.

$N \Rightarrow \log_2 N$. ১ বার ভাঙ্গিতে সময়।

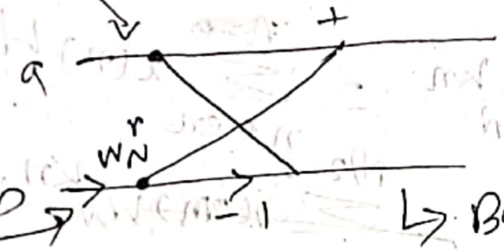
$N = 8$

$\log_2 8 = 3$



$A = a + W_N^r b$

$B = a - W_N^r b$



↳ Basic Butterfly connection of DIT of FFT algo.

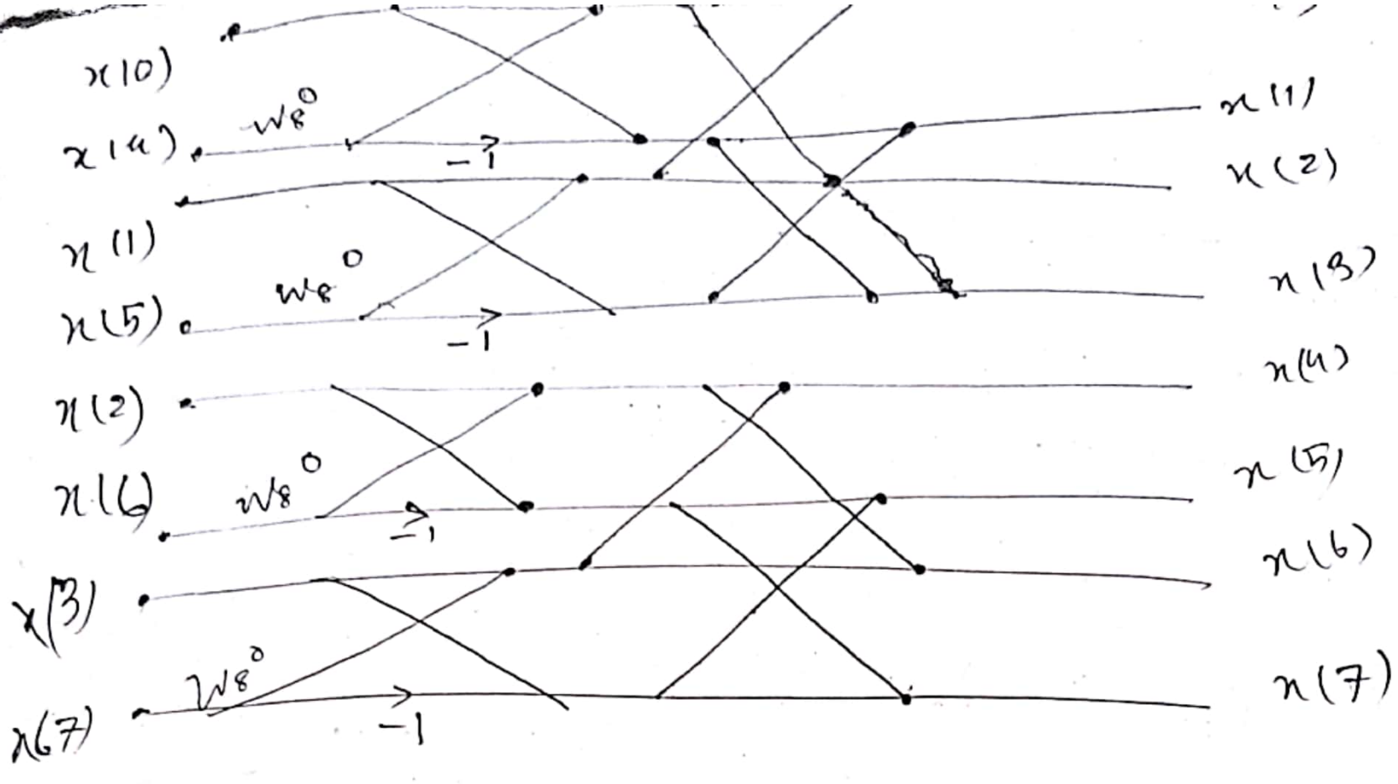
8

5

D *0, 1, 2, 3, 4, 5, 6, 7.

0, 2, 4, 6 - 1, 3, 5, 7

0, 4, 2, 6, 1, 5, 3, 7



x(0)
x(4)
W_8^0
-1
x(1)
x(5)
W_8^0
-1
x(2)
x(6)
W_8^0
-1
x(3)
x(7)
W_8^0
-1
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)