

Z-transform

→ Fourier transform

→ Z-Transform is the discrete version of Laplace Transform.

Z Transform :-

The Z-transform of a DT signal $x(n)$ is defined as the power series.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

z is a complex variable. $\boxed{z = re^{j\omega}}$

$$x(n)z \xrightarrow{\text{Z}} X(z)$$

converge \rightarrow diverge

Z-Transform exists only for those values of z for which the series converges.

The region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

$$\text{Ex: 3.1.1} \quad x_1(n) = \{1, 2, 5, 3, 0, 1\} \quad \sum_{n=0}^{\infty} x_1(n) z^{-n}$$

$$x_1(z) = x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \\ x(4)z^{-4} + x(5)z^{-5}$$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-4}$$

R_{OC}: entire z-plane except z=0.

$$x_2(n) = \{1, 2, 5, 7, 0, 1\} \quad \sum_{n=0}^{\infty} x_2(n) z^{-n} = (z) x_1(z)$$

$$= z^2 + 2z^1 + 5 + 7z^{-1} + z^{-3}$$

R_{OC}: entire z-plane except z=0 and z=2.

$$x_c(n) = s(n)$$

x(z) = 1, R_{OC}: entire z-plane.

$$x_c(n) = s(n-k)$$

$$x(z) = z^{-k}$$

$$x_c(n) = s(n+k)$$

$$x(z) = z^{k+1}$$

Master (s)x will not go poles or

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$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \quad z = re^{j\omega}$$

Ex-3.1.1 $x(n) = \{1, 2, \frac{5}{2}, 7, 0, 1\}$

$$\begin{aligned} x(z) &= x(-2)z^2 + x(-1)z^1 + x(0)z^0 + x(1)z^{-1} + \\ &\quad x(2)z^{-2} + x(3)z^{-3} \\ &= 1 + 2z^1 + 5 + 7z^{-1} + z^{-3} \end{aligned}$$

ROC = all values of z except $z=0, \infty$

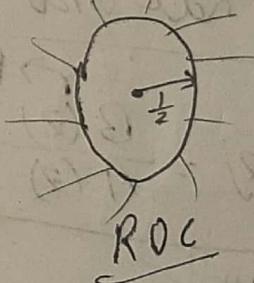
Ex-3.1.2, Determine the z -transform of the signal

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$= \left\{ \frac{1}{0}, \frac{1}{2}, \left(\frac{1}{2}\right)^1, \left(\frac{1}{2}\right)^2, \dots \right\} \quad u(n) = \left\{ 1, 1, 1, \dots \right\}$$

power series,

$$\begin{aligned} x(z) &= 1 + \frac{1}{2} z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \left(\frac{1}{2}\right)^3 z^{-3} + \dots \\ &= 1 + A + A^2 + A^3 + \dots \quad A = \frac{1}{2} \\ &= \frac{1}{1-A} \end{aligned}$$



$$x(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^1}, \quad ROC \therefore |z| > \frac{1}{2}$$

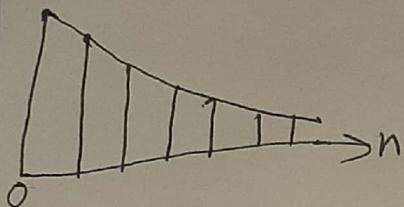
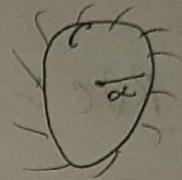
→ signal R.H.S \Rightarrow ROC = outer side.

* For a RHS signal the ROC is outside of the circle.

Ex-31.3 Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \rightarrow \text{RHSS}$$

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$



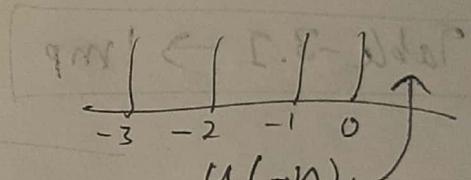
$$X(z) = \frac{1}{1 - \alpha z^{-1}}, |\alpha z^{-1}| < 1.$$

$$\text{ROC} = |z| > \alpha$$

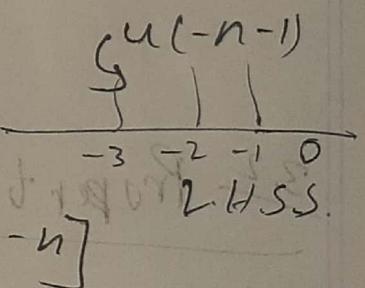
λ \Rightarrow ROC \Rightarrow convergence, delay = ∞
 $\delta/\lambda = 80^\circ$

Ex-3.1.4 $x(n) = -\alpha^n u(-n-1)$

$$= \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$



$$X(z) = \sum_{n=-\infty}^{-1} (-\alpha^n) z^{-n}$$



$$= \sum_{n=-\infty}^{-1} (\alpha^{-1} z)^n = -\sum_{n=1}^{\infty} (\alpha^{-1} z)^n [1 = -n]$$

$$= A + A^2 + A^3 + \dots$$

$$= A(1 + A + A^2 + \dots)$$

$$X(z) = -\frac{A}{1 - A}$$

$$\frac{A}{1 - \alpha^{-1} z} \leftrightarrow (N)_s(s) + (N)_s \omega_0 = (N)_s$$

$$-\frac{1}{\alpha^{-1} z - 1} = -\frac{1}{\alpha^{-1} z - 1} = \frac{1}{1 - \alpha z^{-1}}, |\alpha z^{-1}| < 1$$

* ROC of a L.H.S. signal is the interior of the circle. $\text{ROC} = |z| < \alpha$

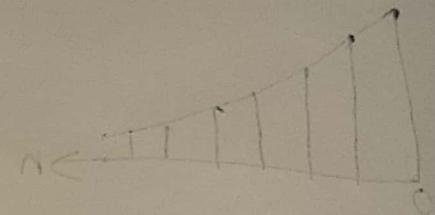
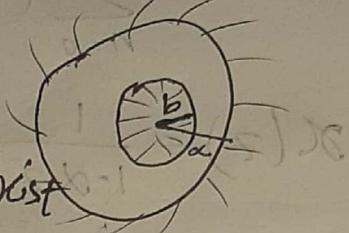
$$\text{Ex-3.1.5. } x(n) = \alpha^n u(n) + b^n u(-n-1)$$

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}}$$

$$\text{ROC: } |z| > |\alpha| \quad \text{ROC: } |z| > |b|$$

case-1: $|b| < |\alpha|$

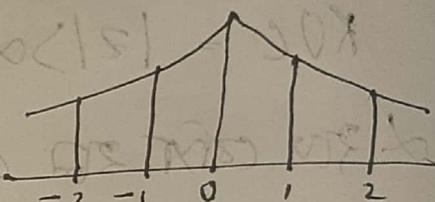
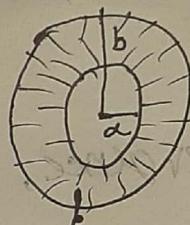
• ROC doesn't exist



case-2: $|b| > |\alpha|$

$$\text{ROC: } |\alpha| < |z| < |b|$$

$$z = 6j\theta$$



common part = ROC

Table-3.1 \rightarrow Imp

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3.2 Properties of Z transform

1) Linearity:-

$$\text{if } x_1(n) \leftrightarrow X_1(z)$$

$$x_2(n) \leftrightarrow X_2(z)$$

$$\text{then } x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) \leftrightarrow \alpha_1 X_1(z) + \alpha_2 X_2(z)$$

$$\frac{1}{(1-\alpha z^{-1})} + \frac{1}{(1-\beta z^{-1})} = \frac{1}{1-(\alpha+\beta)z^{-1}}$$

$$z = 6j\theta$$

Note: It is maintained because $\alpha + \beta < 1$ so $z > 1$

$$\text{Ex-3.2.1 } x(n) = [3(2^n) - 4(3^n)] u(n)$$

$$= 3(2^n) u(n) - 4(3^n) u(n)$$

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$$

Time shifting :-

$$\begin{array}{ccc} \text{if } x(n) & \xrightarrow{z} & X(z) \\ x(n-k) & \xrightarrow{z} & z^k * X(z) \end{array} \quad \left| \begin{array}{l} s(n) \xrightarrow{z} 1 \\ s(n-k) \xrightarrow{z} z^{-k} \end{array} \right.$$

Ex-3.2.3

$$\text{Ex-3.2.4 } x(n) = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} \text{finite} &= \frac{1-a^N}{1-a} \\ \text{Infinite} &= \frac{1}{1-a} \end{aligned} \quad X(z) = \sum_{n=0}^{N-1} 1 \cdot z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \dots + z^{-(N-1)}$$

Find the z transform :-

$$x(n) = u(n) - u(n-N)$$

$$\begin{aligned} Z(z) &= Z[u(n)] - Z[u(n-N)] \\ &= Z[u(n)] - z^{-N} Z[u(n)] \end{aligned}$$

$$\begin{aligned} z^{-N} &= \frac{1}{1-z^{-1}} - z^{-N} \frac{1}{1-z^{-1}} = (s) \cancel{x} \\ &= (1-z^{-N}) \cancel{x} \frac{1}{1-z^{-1}} \end{aligned}$$

Differentiation in the z-domain

$$\text{if } x(n) \xrightarrow{z} X(z)$$

$$\text{then } n x(n) \xrightarrow{z} -z \frac{dX(z)}{dz}$$

Proof - H.W.

$$\begin{aligned} \text{Ex-3.2.7 } X(z) &= \underline{n a^n u(n)} \xrightarrow{z} \frac{1}{1-az^{-1}} \\ &= -z \frac{d}{dz} \left[\frac{1}{1-az^{-1}} \right] \xrightarrow{(1-az^{-1})^{-1}} \\ &= -z \cdot \frac{-az^{-2}}{(1-az^{-1})^2} = \frac{az^{-1}}{(1-az^{-1})^2} \end{aligned}$$

~~Convolution of two sequences:~~ (Z + ?) ~~convolution~~ convolution ~~& faster~~

$$\text{if } x_1(n) \rightarrow X_1(z)$$

$$x_2(n) \rightarrow X_2(z)$$

$$\text{then } x_1(n) * x_2(n) \xrightarrow{z} X_1(z) \cdot X_2(z)$$

$$\text{Ex-3.2.9 } X_1(n) = \{1, -2, 1\}$$

$$X_2(n) = \{1, 1, 1, 1, 1\}$$

Compute the convolution $x(n)$ of the signals.

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$X(z) = X_1(z) \cdot X_2(z) = 1 - z^{-1} - z^{-5} + z^{-7}$$

$$x(n) = \{1, -1, 0, 0, 0, 0, -1, 1\}$$

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Steps for computing convolution using z-transform

1) Complete the z-transform of the signals to be convolved.

$$X_1(z) = Z\{x_1(n)\}$$

$$X_2(z) = Z\{x_2(n)\}$$

2) Multiply the two z-transforms.

$$X(z) = X_1(z) \cdot X_2(z)$$

3) Find the inverse z-transforms of $X(z)$

$$x(n) = Z^{-1}\{X(z)\}$$

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3.3 Rational z-transform

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

2nd order

$$an^2 + bn + c = 0$$

$$(n-n_1)(n-n_2) = 0$$

$$n = n_1$$

$$n = n_2$$

$$X(z) = C \frac{(z-z_1)(z-z_2) \dots (z-z_M)}{(z-p_1)(z-p_2) \dots (z-p_N)}$$

$$X(z) = C \frac{(z-z_1)(z-z_2) \dots (z-z_M)}{(z-p_1)(z-p_2) \dots (z-p_N)}$$

Poles & Zeros

The zeros of z-transforms $X(z)$ are the values of z for which

$$X(z) = 0. \quad [\text{Zeros: } -z_1, -z_2, -z_3, \dots, -z_n]$$

\hookrightarrow roots of the numerator ($\neq 0$)

The poles of a z-transforms are the values of z for which

$$X(z) = \infty. \quad [\text{Poles: } p_1, p_2, p_3, \dots, p_N]$$

\hookrightarrow roots of the denominator ($\neq 0$)

Ex-3.3.1] Determine the pole-zero plot for the signal

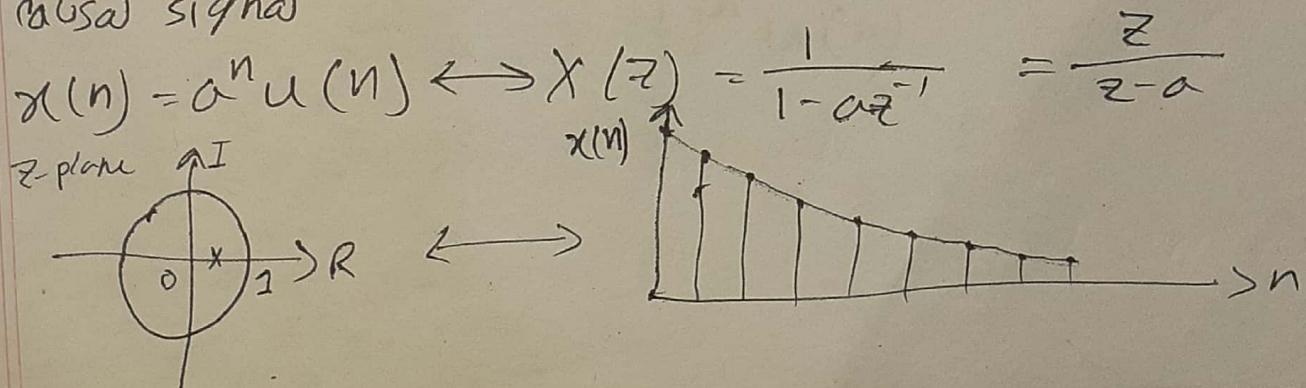
$$x(n) = a^n u(n)$$

Sln:

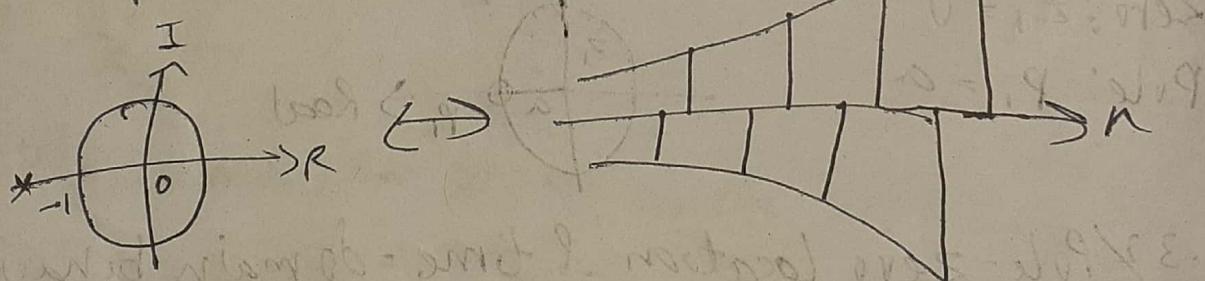
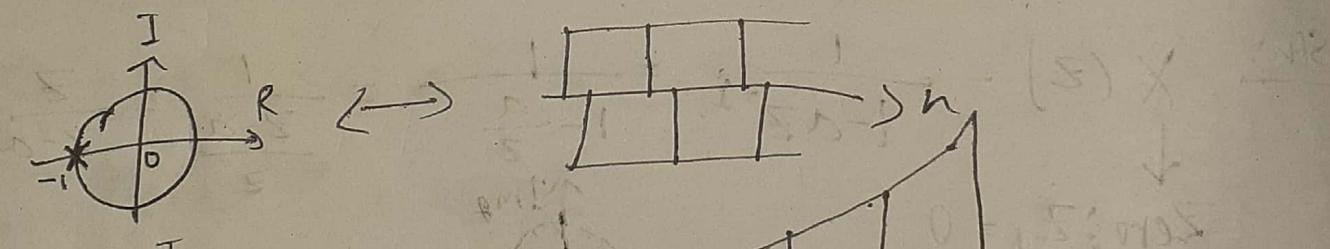
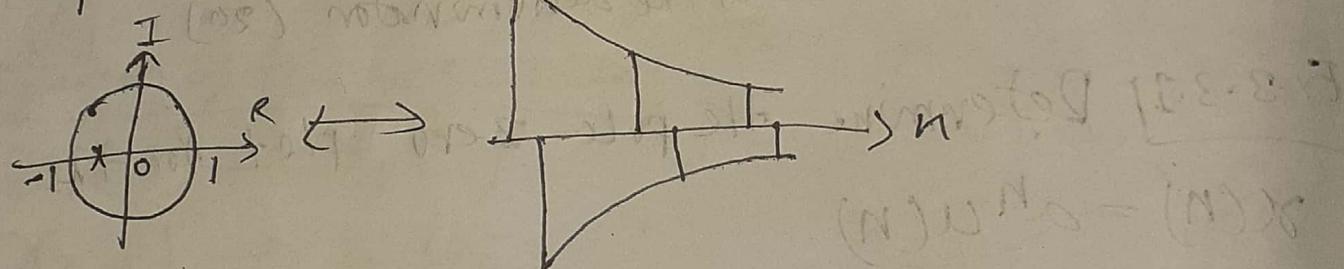
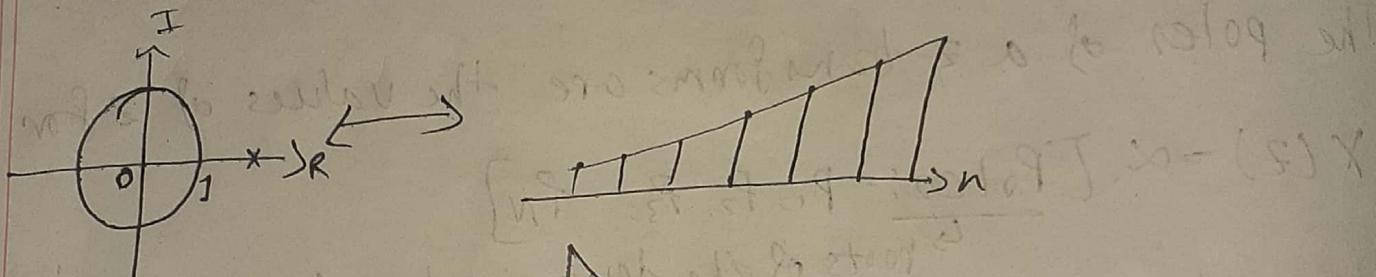
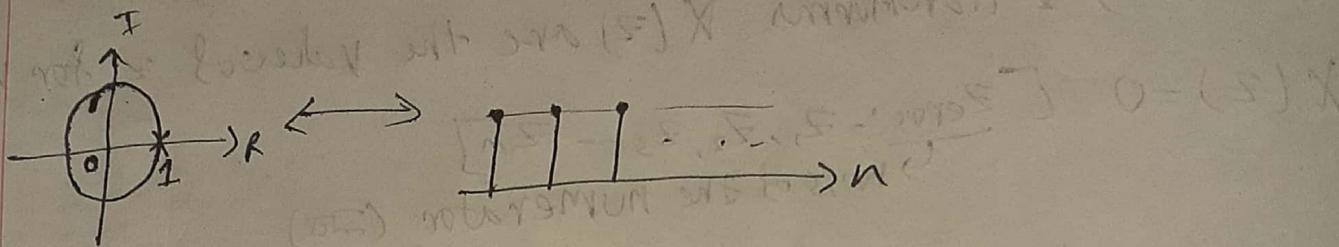
$$X(z) = \frac{1}{1 - az^{-1}} = \frac{1}{1 - \frac{a}{z}} = \frac{\frac{1}{z}}{\frac{z-a}{z}} = \frac{\frac{1}{z}}{\frac{z-a}{z}} = \frac{z}{z-a} = \frac{z-0}{z-a}$$

\downarrow
 Zero: $z_1 = 0$
 Pole: $p_1 = a$

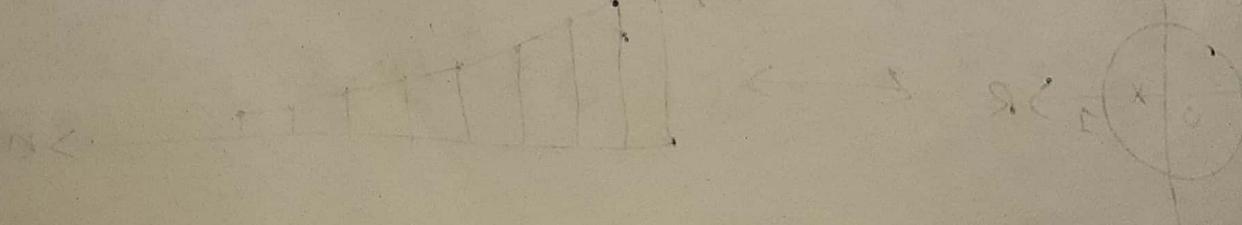
Ex-3.3.2] Pole-zero location & time-domain behavior for causal signal



$$x(n) = a^n u(n) \leftrightarrow X(z) = \frac{z}{z-a}$$



$$\frac{s}{s-5} = \frac{1}{s+1} = (s)X \Leftrightarrow (N)u^n o - (n)X$$



3.3.3 / The system function of an LTI system

$$Y(n) = x(n) * h(n)$$

[time-domain \Rightarrow convolution
 \Rightarrow \mathcal{Z} -domain \Rightarrow multiplication]

$$Y(z) = X(z) \cdot H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

↓
System function :- Use when we know input & output. But D.K SF.

Ex-3.3.4 Determine the system function & the unit sample response

$$y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

$$2X(z) = Y(z) \left[1 - \frac{1}{2} z^{-1} \right]$$

$$\frac{Y(z)}{X(z)} = \frac{(z)}{1 - \frac{1}{2} z^{-1}}$$

$$\Rightarrow H(z) = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

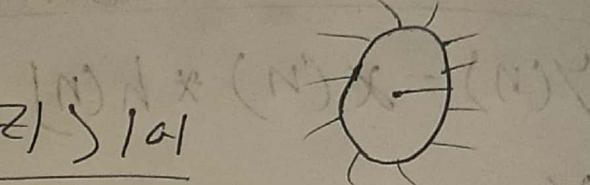
$$h(n) = 2 \left(\frac{1}{2}\right)^n u(n)$$

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Inverse Z-Transform

Ex: $X(z) = \frac{1}{1-az^{-1}}$, $|z| > |a|$

↓ inverse



↳ R.H.S (causal)

$$x(n) = a^n u(n) \quad | \quad a^n u(-n-1)$$

~~XX~~ anti-causal

The inverse z-Transform by power series expansion

Ex-3.4.2 Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} = z^{-1} + z^{-2} + z^{-3}$$

When of ROC ($-1 < z < \infty$) $\rightarrow x(n)$ should be causal

b) ROC: $|z| < 0.5 \rightarrow x(n)$ anti-causal.

$x(n) = \{1, 2, 3\}$	$ x(n) = \{1, 2, 3\}$
$X(z) = 1 + 2z^{-1} + 3z^{-2}$	$ X(z) = z^2 + 2z^1 + 3$
a) $X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$	$\rightarrow z^{-1} + z^{-2} + z^{-3} + \dots$

$$= \left(1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2} \right) \frac{1}{1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}} \left(1 + \frac{3}{2} z^{-1} + \frac{7}{6} z^{-2} + \right.$$

$$\frac{\frac{3}{2} z^1 - \frac{1}{2} z^{-1}}{\frac{3}{2} z^{-1} - \frac{9}{6} z^{-2} + \frac{3}{6} z^{-3}}$$

$$\left. \frac{\frac{3}{2} z^{-2} + \frac{3}{6} z^{-3}}{} \right)$$

$$x(n) = \left\{ 1, \frac{3}{2}, \frac{7}{6}, \dots \right\}$$

Ex:- 3.4.8 $X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}, x(n) = ?$

a) ROC:- $|z| > 1 \rightarrow \text{causal}$

b) ROC:- $|z| < 0.5 \rightarrow \text{anti-causal}$

c) ROC:- $0.5 < |z| < 1 \rightarrow \text{contain both causal \& anti-causal part}$

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$$= \frac{1}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}}$$

$$\begin{cases} X(z) = \frac{1}{1 - az^{-1}} \\ x(n) = a^n u(n) \end{cases}$$

a) $x(n) = z(1)^n u(n) - (0.5)^n u(n)$

b) $x(n) = z(1)^n u(-n-1) - (0.5)^n u(-n-1)$

c) $x(n) = -z(1)^n u(-n-1) - (0.5)^n u(n)$

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

① $x(n) = \{1, 2, 5, 7, 0, 1\}$

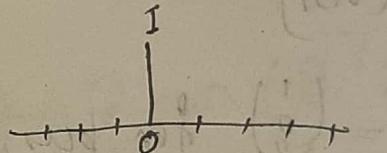
$$X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

② $x(n) = \{1, 2, 5, 7, 0, 1\}$

$$X(z) = z^2 + 2z^1 + 5 + 7z^{-1} + z^{-3}$$

③ $x(n) = S(n)$

$$X(z) = 1 \cdot z^0 (= 1)$$



④ $x(n) = S(n-k)$

$$X(z) = z^{-k}$$

⑤ $x(n) = S(n+k)$

$$X(z) = z^k$$

⑥ $x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n z^{-n} = \sum_{n=-\infty}^{\infty} (az^{-1})^n$$

$$\Rightarrow \frac{1}{1-az^{-1}} \quad |az^{-1}| < 1$$

ROC: $|z| > |a|$

$$\textcircled{7} \quad x(n) = a^n u(-n-1)$$

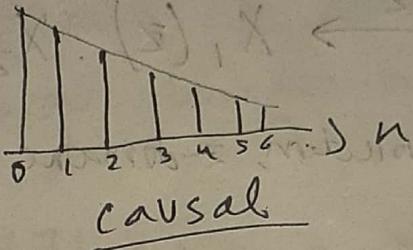
$$X(z) = -\frac{1}{1-az^{-1}}, \quad |z| < |a|$$

$$\textcircled{8} \quad x(n) = a^n u(n)$$

$$|a| < 1$$

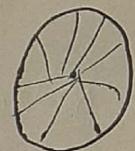


ROC exist

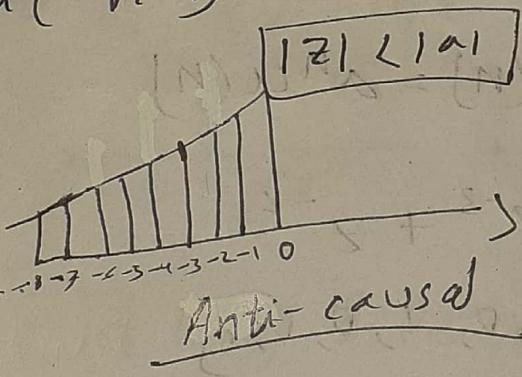


causal

$$\textcircled{9} \quad x(n) = a^n u(-n-1)$$



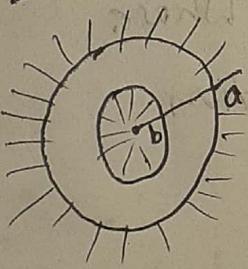
ROC:



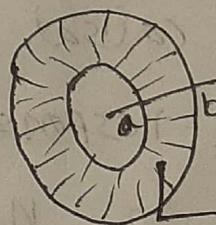
Anti-causal

$$\textcircled{10} \quad x(n) = a^n u(n) + b^n u(-n-1)$$

a > b \rightarrow common part = ROC



ROC doesn't exist



ROC exist

Convolution Property :-

$$x_1(n) \xrightarrow{z} X_1(z)$$

$$x_2(n) \xrightarrow{z} X_2(z)$$

$$x_1(n) * x_2(n) \xrightarrow{z} X_1(z) \cdot X_2(z)$$

side note (time domain \Rightarrow convolution, z-domain \Rightarrow multiplication
 \Rightarrow Z-transform)

$$\textcircled{11} \quad X(z) = \frac{1}{1 - az^{-1}} \quad |z| > |a|$$

$$x(n) = ? , \quad x(n) = a^n u(n)$$

$$\textcircled{12} \quad X(z) = 1 + z^{-3} + z^{-4}$$

$$x(n) = \{1, 0, 0, 1, 0, 0, 1\}$$

3.5.3 Causality & Stability

Causal :- A causal LTI system is one whose unit impulse response satisfies the condition

$$h(n) = 0, \quad n < 0$$

side note (for neg. n, impulse response = 0)

Stable: A stable LTI system is one whose unit impulse response satisfies the condition

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Side note (impulse response, bounded for stable)

Ex :- 3.5.2 An LTI system is characterized by the system

function $H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 15z^{-2}}$

$$= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

Specify the ROC of $H(z)$ & determine $h(n)$ for the following conditions:-

a) The system is stable

b) $h(n) \leftarrow u(n)$ causal

c) $h(n) \leftarrow u(-n)$ anti-causal

Sln :-

(b) $h(n) = \left(\frac{1}{2}\right)^n u(n) + 2(3)^n u(n)$

(c) $h(n) = -\left(\frac{1}{2}\right)^n u(-n-1) - 2(3)^n u(-n-1)$

$$\textcircled{a} \quad \frac{1}{z} < |z| < 3$$

$$h(n) = \left(\frac{1}{z}\right)^n u(n) - 2(3)^n u(-n-1)$$

Ex:- 3.5.4 Determine the response of the system

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n) \text{ to the input signal } x(n) = s(n) - \frac{1}{3}s(n-1)$$

S[n]:

$$\boxed{Y(z) = H(z)x(z)} \\ \downarrow \\ Y(n)$$

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$$

$$\boxed{Y(z) = \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) + X(z)}$$

$$Y(z) = \frac{5}{6}z^{-1}Y(z) - \frac{1}{6}z^{-2}Y(z) + X(z)$$

$$Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) = X(z)$$

$$Y(z)\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right) = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

$$H(z) = \frac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

$$= \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

$$X(z) = 1 - \frac{1}{3}z^{-1} \quad (\text{S}) \quad S = (N) \cdot \left(\frac{1}{3}\right) = (N) \cdot N \quad (1)$$

$$(1 - N) \cdot \left(\frac{1}{2}\right) \cdot S = (1 - N) \cdot N \cdot \left(\frac{1}{2}\right) = (N) \cdot N \quad (2)$$

$$\rightarrow Y(z) = H(z)X(z)$$

$$\frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$Y(n) = \left(\frac{1}{2}\right)^n u(n)$$

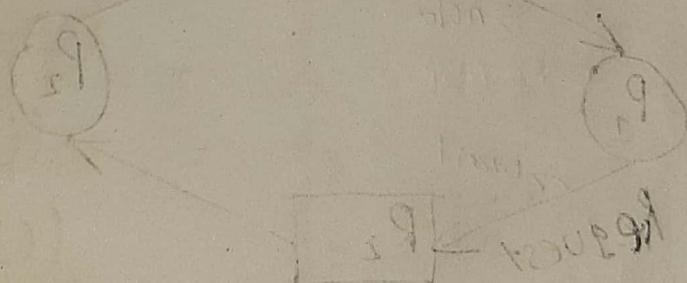
* One-sided z-Transform

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

TT-3:-

9.5.23

ch-3



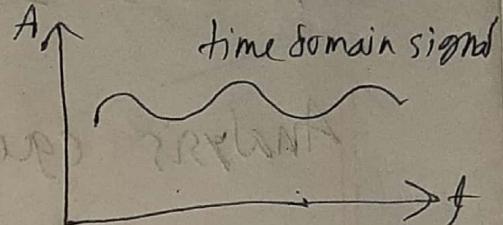
Ch-4 (Frequency Analysis of signals)

→ convolution, Fourier Transform

1) Fourier Transform:- finite energy signal. (aperiodic)

2) Fourier series:- Periodic signal.

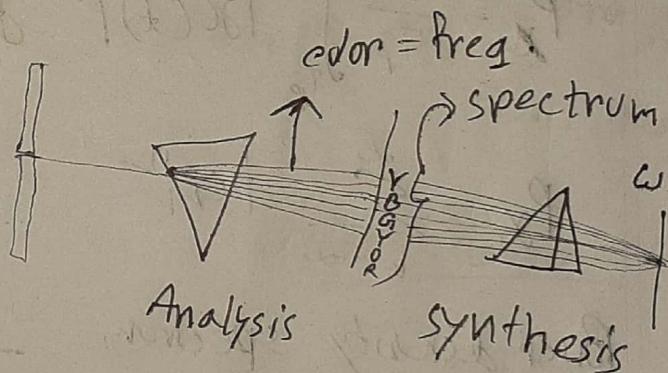
Output signal composite of many no. of sinusoids.



Frequency domain:

When a signal is decomposed in terms of sinusoidal components, a signal is said to be represented in the frequency domain.

→ Isaac Newton (1672)



U.1.1 Fourier Series of continuous time signals:-

- Jean Baptiste Joseph Fourier (1768 - 1830)
- Temperature distribution

Frequency Analysis of Continuous time Periodic signals:

Synthesis equ: $\chi(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi k f_0 t}$ [where $f_0 = \frac{1}{T_p}$]

Analysis equ: $C_k = \frac{1}{T_p} \int_{T_p} \chi(t) e^{-j2\pi k f_0 t} dt$

~~Ans~~

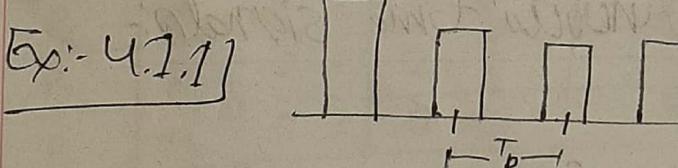
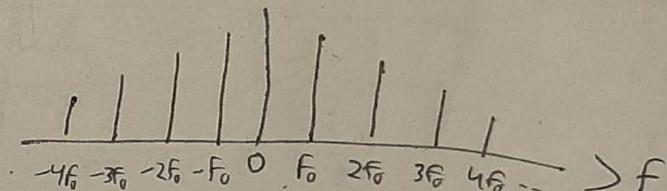
4.1.2 Power Density Spectrum of Periodic signals:

$$P_x = \frac{1}{T_p} \int_{T_p} |\chi(t)|^2 dt$$

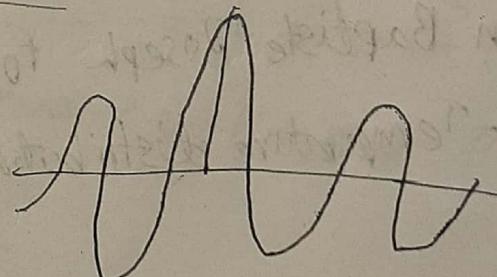
$$P_x = \sum_{k=-\infty}^{\infty} |C_k|^2$$

($k = 0, \pm 1, \pm 2, \pm 3, \dots$)

Power density spectrum
or Line spectrum



$$P.D.S = ?$$



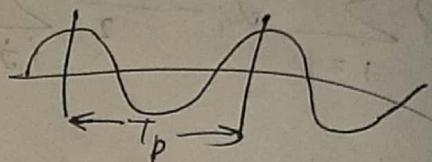
Lab A.M

23.5.23 → ch-1,2

Fourier Series of Periodic continuous time signals

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

$$c_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-j2\pi k F_0 t} dt$$



→ D.TW sample 2¹⁰

Fourier Series of Periodic Discrete-Time Signals

$$x(t) \xrightarrow{\text{freq.}} -\infty \text{ to } +\infty \rightarrow T_p$$

$$x(n) \xrightarrow{\text{freq.}} -\pi \text{ to } +\pi \rightarrow N$$

or
0 to 2π

Fourier series of Discrete Time Periodic Signals

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

$$x(n) = \sum_{k=-\infty}^{N-1} c_k e^{-j2\pi k \frac{1}{N} n}$$

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi k n / N}$$

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi k n / N}$$

Ex: 4.2.1 Determine the spectra

$$\begin{aligned} x(n) &= \cos \frac{\sqrt{2}\pi n}{\omega} \\ &= \cos 2 \frac{1}{\sqrt{2}} \pi n \end{aligned}$$

$\beta_0 = \frac{1}{\sqrt{2}}$, not a rational number.

Spectra (frequency component) : $\sqrt{2}\pi$

b) $x(n) = \cos \pi n/3$

$$= \cos 2\pi \frac{1}{6} n$$

$\beta_0 = \frac{1}{6}$ is a rational number.

Spectra (frequency component) : $\frac{1}{3}\pi$

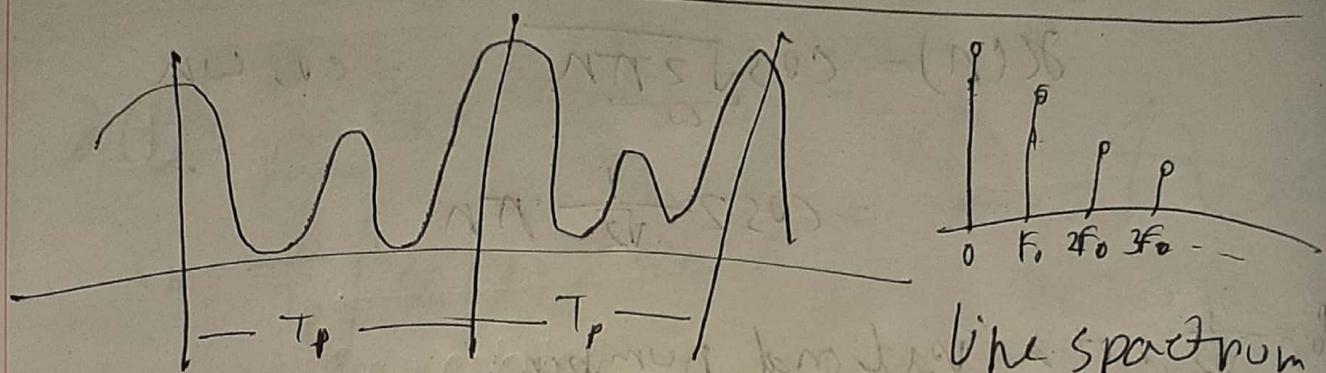
$$\begin{aligned} c_k &= \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \\ &= \frac{1}{6} \sum_{n=0}^{5} \end{aligned}$$

Power density spectrum

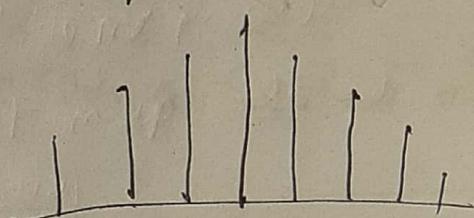
$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

$$= \sum_{n=0}^{N-1} |c_k|^2$$

Fourier Transform of Discrete Time signals



$$f_0 = \frac{1}{T_p}$$



FT of DT signals

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

\rightarrow F.T = continuous, F.S = Discrete.

4.2.6 Relation of the Fourier Transform to the z-transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z = r e^{j\omega}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

if $r=1$, z-transform becomes Fourier transform

Z -T is ~~subset~~ super set & F.T is subset.

* Frequency ranges of some natural signals

Freq. Range (Hz)

ECG = 0 - 100 (heart)

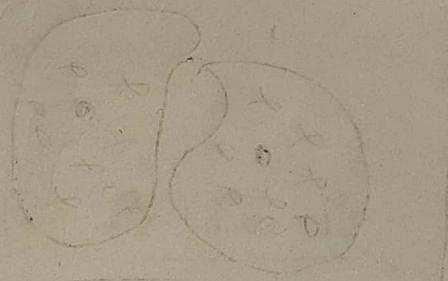
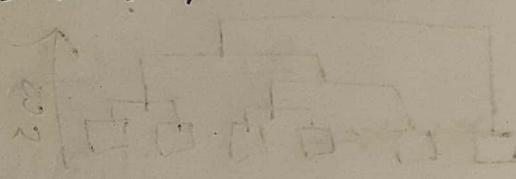
EEG = 0 - 100 (brain)

Speech = 100 - 4000 (telephone speech)

Seismic signals = ~~wind noise~~ Wind \rightarrow 100 - 1000

Seismic \rightarrow
signal

Earthquake & nuclear explosion signal = 0.01 - 10



Earthquake waves
(e.g. ground motion)

Earthquake waves

$S = S / 10$ seismic waves (1)

seismic waves + noise (2)
(distortion)

or wind noise mixed (3)

distortion

(4) - (5) \downarrow

distortion

seismic waves + noise (6) (7)

noise mixed (8)

Ch-7 Discrete Fourier Transform

$$\tilde{X}(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

F.S \rightarrow D.S | Periodic - $f_0 + 2f_0 + 3f_0 + \dots$

* Fourier series:- Frequency Domain - Discrete Spectrum (line)

* Fourier Transform:- Frequency Domain - continuous spectrum

$$\frac{2\pi}{N} = \Delta\omega$$

$$k \cdot \frac{2\pi}{N} = \Delta\omega$$

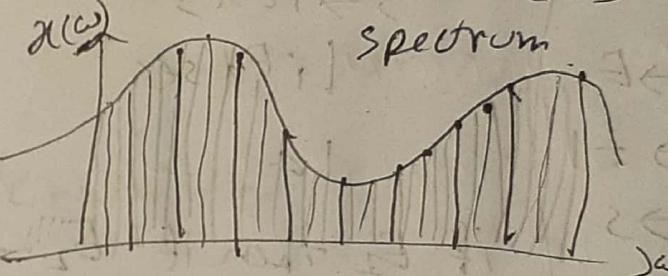
$$x(k \cdot \frac{2\pi}{N}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j \frac{2\pi}{N} kn} \quad [k=0, 1, \dots, N]$$

$$= + \dots + \sum_{n=-N}^{+1} x(n) e^{-j 2\pi k n / N} +$$

$$\sum_{n=0}^{N-1} x(n) e^{-j 2\pi k n / N} + \sum_{n=N}^{2N-1} x(n) e^{-j 2\pi k n / N} + \dots$$

$$= \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(l) \delta(n-l) \right] e^{-j 2\pi k n / N}$$

$$= \sum_{n=0}^{N-1} x_p(n) e^{-j 2\pi k n / N}$$



Periodic DT

$$\xrightarrow{\text{synthesis}} \mathcal{X}_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, n=0, \dots, N-1$$

$$\xrightarrow{\text{analysis}} c_k = \frac{1}{N} \sum_{n=0}^{N-1} \mathcal{X}_p(n) e^{-j2\pi kn/N}, k=0, \dots, N-1$$

for $0 \leq n \leq N-1$

$$x(n) = \mathcal{X}_p(n)$$

$$c_k = \frac{1}{N} \times \left(\frac{2\pi}{N} \cdot k \right)$$

analysis

$$\xrightarrow{\text{analysis}} \mathcal{X}\left(\frac{k}{N}\right) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, k=0, 1, \dots, N-1$$

synthesis

$$\xrightarrow{\text{synthesis}} \mathcal{X}(w) = \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{X}\left(\frac{k}{N}\right) e^{j2\pi kn/N}$$

$$[x(n) = \sum_{k=-\infty}^{\infty} \mathcal{X}(k) e^{-jkn}, w = \frac{2\pi}{N}]$$

$$\text{Ex: } 7.1.2 \quad x(n) = \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\mathcal{X}(k) = \sum_{n=0}^{L-1} e^{-j2\pi kn/L}$$

$$= \frac{1 - e^{-j2\pi kL/N}}{1 - e^{-j2\pi k/N}}$$

$$\frac{1 - e^{-j2\pi kL/N}}{1 - e^{-j2\pi k/N}}$$

7.2 DFT Properties

Discrete T.F $\rightarrow X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

① Periodicity: if $x(n) \xrightarrow{\text{DFT}} X(k)$,

$$\text{then } x(n+N) = x(n) \quad \{ \dots, x(n), \dots \}$$

$$X(k+N) = X(k)$$

② Linearity: if $x_1(n) \xrightarrow{\text{DFT}} X_1(k)$

$$x_2(n) \xrightarrow{\text{DFT}} X_2(k)$$

$$\text{then, } a_1 x_1(n) + a_2 x_2(n) \xrightarrow{\text{DFT}} a_1 X_1(k) + a_2 X_2(k)$$

Real-valued sequences

if $x(n)$ is real \rightarrow conjugate

$$x(N-k) = x^*(k)$$

$$= x(-k)$$

$$(N-n)x^*(k) = (N-n)x^*(N-k) = (N-n)x(k)$$

$$(N-n)x^*(k) = ((-n)N + N)x(k) = (N)x(k)$$

7.2.2 Multiplication of two DFTs & circular convol

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi kn/N}$$

$$X_3(k) = X_1(k) X_2(k)$$

$$\left(\sum_k a^k = \frac{1-a^N}{1-a} \right)$$

1D FT

$$X_3(m) = \sum_{k=0}^{N-1} X_3(k) e^{-j2\pi km/N}$$

$$= \sum_{k=0}^{N-1} X_1(k) X_2(k) e^{-j2\pi km/N}$$

$$= \sum_{k=0}^{N-1} \left[\left(\sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N} \right) \left(\sum_{l=0}^{N-1} x_2(l) e^{-j2\pi kl/N} e^{-j2\pi km/N} \right) \right]$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} \right]$$

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & \text{if } (m-n-l) \text{ is multiple of } N \\ \frac{1-a^N}{1-a}, & \text{otherwise} \end{cases}$$

$$a = e^{j2\pi(m-n-l)/N} \rightarrow a^N = 1$$

$$m-n-l=pN \Rightarrow m-n-pN = l$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \underbrace{x_2(m-n)}_N$$

∴ Multiplication of two DFTs are equivalent of circular convolution in time domain.

7.3 Linear Filtering Methods based on the DFT:

$$y(n) = x(n) * h(n)$$

↓
noise free

↓
identity of
a system

$X(k) \cdot Y(k) \rightarrow$ circular convolution

\downarrow
IDFT

$y(n)$

$x(n) \rightarrow L$

$h(n) \rightarrow M$

$FIR = ? \rightarrow$ Finite Impulse Response

Ex-7.3.1 $h(n) = \{1, 2, 3\}$

$$x(n) = \{1, 2, 2, 1\}$$

$$Y(n) = h(n) * X(n)$$

$$Y(k) = H(k) \cdot X(k)$$

$$\begin{array}{l} L=4 \\ M=3 \end{array}, N=4+3-1=6$$

linear convolution

↓ IDFT

$y(n)$

7.3 Filtering Long-Data Sequences

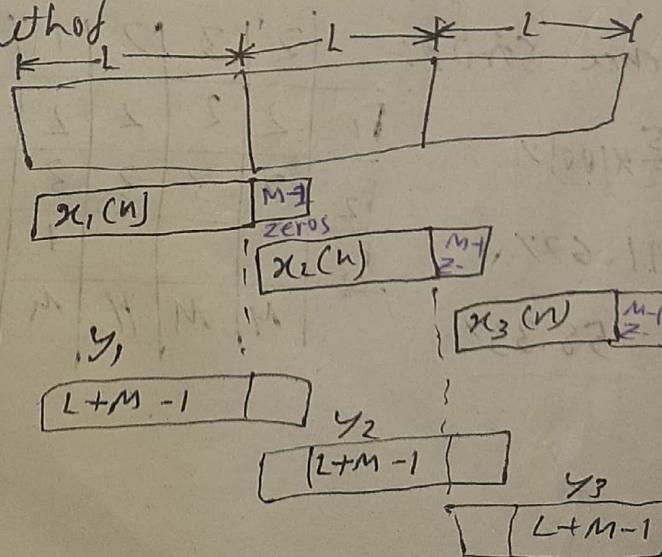
→ Overlap-add method

→ Overlap-save method

$$y(n) = x_1(n) * h(n)$$

1) segment

2) DFT apply



$$x_1(n) = \{x(0), x(1), \dots, x(L-1), \underbrace{0, 0, \dots, 0}_{M-1 \text{ zeros}}\}$$

$$x_2(n) = \{x(L-1), x(L), \dots, x(2L-1), \underbrace{0, 0, \dots, 0}_{M-1 \text{ zeros}}\}$$

$$Y_1(k) = H(k) X(k)$$

$$Y_2(k) = H(k) X_2(k)$$

$$Y(n) = \{Y_1(0), Y_1(1), \dots, Y_1(L) + Y_2(0) + Y_2(L+1) + \dots\}$$

7.4 Frequency Analysis of signals using DFT:

$$x(n) / k \cdot \frac{2\pi}{N}$$

windowing

windowing (No. of frame & the way we divide a signal).

$$x(n) = x(n) w(n)$$

frame / segment

$$\hat{x}(k) = \hat{x}(k) * H(k)$$

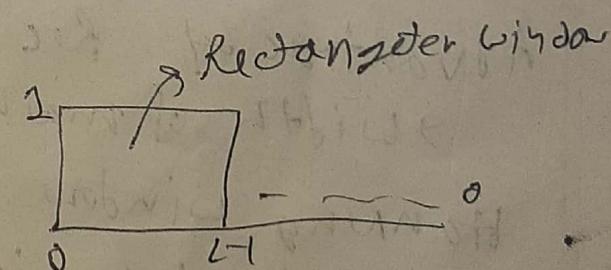
↓

$$y(n)$$

Our signal

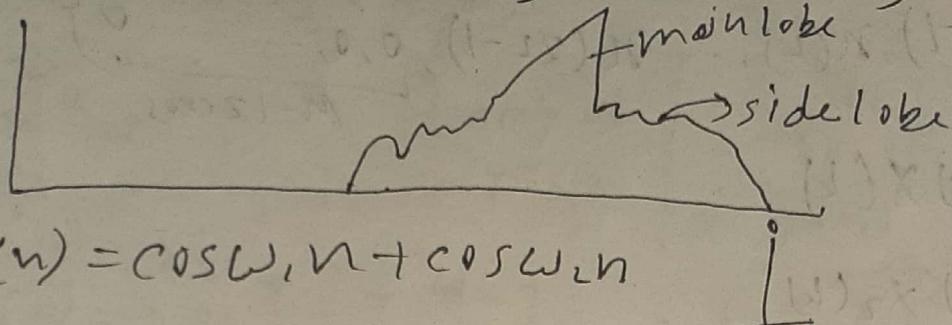
$$x(n) = \cos \omega_0 n$$

$$w(n) = \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$



$$\hat{x}(n) = x(n) * w(n)$$

$$\hat{x}(\omega) = \frac{1}{2} [\omega(\omega - \omega_0) + (\omega - \omega_0)]$$



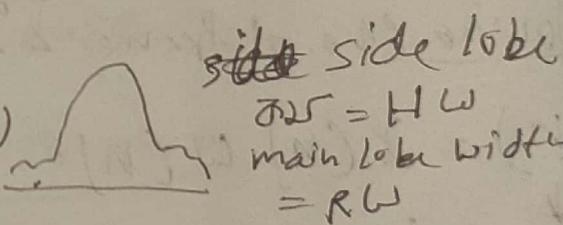
If we increase value of L than we can reduce adding windowing.

Types of windows:-

i) Rectangular window (width of main lobe is reduced)

ii) Hamming window

iii) Hanning window (H.W)



conclude:- (How to reduce windowing effect)

- 1) N should be enough larger.
- 2) Appropriate window should be selected.

Advantage of rec. window:-

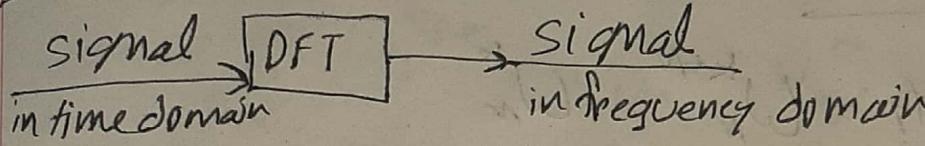
→ width of main lobe is less.

Hanning window:-

→ side lobe is reduced.

ch-8 Efficient Computation of DFT: FFT algorithm

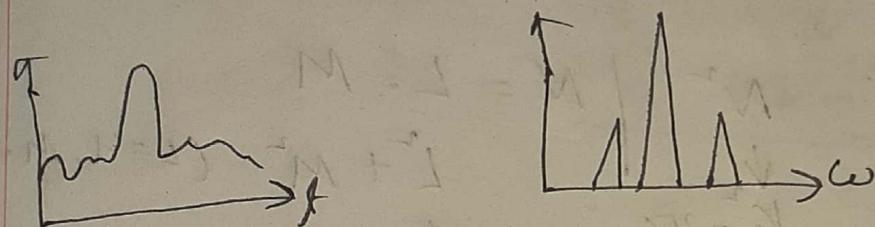
DFT



Next Tuesday

TT-4

ch-7,8



→ convolution \Rightarrow sum of DFT freq and 2 π .

FFT :- Fast Fourier Transform . for faster DFT (so fast).

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad (\text{analysis})$$

$$= \sum_{n=0}^{N-1} x(n) w_N^{-kn} \quad (w_N = e^{-j2\pi/N})$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \quad (\text{synthesis})$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) w_N^{-kn}$$

where, $k = 0, \dots, N-1$, & $n = 0, \dots, N-1$

complexity:

complex multiplication :- $N \cdot N = N^2$

complex Addition :- $(N-1) \cdot N = N^2 - N$

① Symmetry Property :- $W_N^{k+N/2} = -W_N^k$

② Periodicity Property :- $W_N^{k+N} = W_N^k$

③ Divide & Conquer :- $N^2 \mid N = L \cdot M$
 $\downarrow \quad \downarrow$
 $L^2 + M^2 \quad (L=4, M=4)$
 $\downarrow \quad \downarrow$
 $16 \cdot 16 = 256$
 $16 \cdot 16 = 32$

8.1.2 Divide & Conquer Approach

$$x(n) [x_0 | x_1 | x_2 | \dots | x(N-1)]$$

Row-Major, $n = Ml + m$

	0	1	2	3	$M-1$
0	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(M-1)$
1	$x(M)$	$x(M+1)$			
2					
3					
$M-1$					

Frequency :- $k = Mp + q$

$$k = qL + p$$

$$N = L \cdot M$$

$$(P \Rightarrow) X(Pg) = \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} X(lM) W_N^{(Mp+q)(mL+p)}$$

$$= X(Pg) = \sum_{l=0}^{L-1} \left\{ W_N^{lp} \left[\sum_{m=0}^{M-1} x(lM) W_N^{mq} \right] \right\}$$

W_L^{lp} M -point DFT

$$W^{Mp \cdot ML + MpL + Lpm + ql}$$

$$= \sum_{l=0}^{L-1} \left\{ W_N^{lp} f(l, q) W_L^{lp} \right\}$$

$$\sim \sum_{l=0}^{L-1} \left\{ f_l(l, q) W_L^{lp} \right\}$$

L -point DFT

Column-Major, $n = l + Ml$

	0	1	2	3
0	x_0			
1	x_1			
2	x_2			
3	x_3			

$$M \text{ point DFT} = W_n^{MP, ML} \cdot W_n^{MP} \cdot W_n^{LM} \cdot W_n^{11}$$

$$F(kg) = W_n^{MP, M} \cdot W_n^{P1} \cdot W_n^{LM} \cdot W_n^{11} \cdot W_n^{21}$$

↓ ↓ ↓ ↓ ↓
 1 W_n^{P1} W_n^{LM} W_n^{11}

A. L

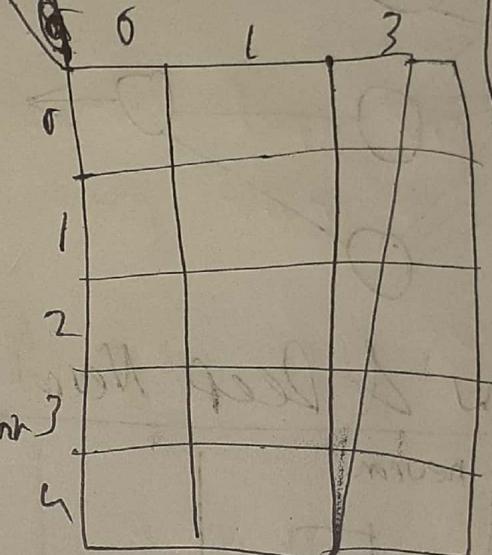
Ex - 8.1.1

1. 3 point DFT for each row

$$\begin{cases} N=15 \\ L=5 \\ M=3 \end{cases}$$

2. Scalar multiplication

3. 5 point DFT for each column



~~ML ML~~

→ M point DFT
→ scalar × m over
→ L point DFT

$$\begin{aligned} & LM^2 + N \cdot MN^2 \\ & LM(M+L+N) \\ & = N(M+L) \\ & = N \cdot N \end{aligned}$$

→ 11 AM online class (Tomorrow)

#-radix-2 Algorithm of FFT

$$N = L \cdot M$$

$$= r_1, r_2, \dots, r_n = r$$

$$N = r \cdot r \cdot r \cdots r [r \text{ is prime}]$$

$$N = r^v$$

$$N^2 = N(L \cdot M)$$

$$= N \cdot (L + M)$$

$$n = 15$$

$$\mathcal{X}(n) = \{2, 3, \dots, 10, 0\}$$

$$= 16$$

↓ zero padding

$$\Rightarrow N \text{ is h.c. nb for } 2 \text{ into } N = r^v$$

$$r = 2$$

$$\mathcal{X}(n) \begin{cases} f_1 & \mathcal{X}_1(n) = \mathcal{X}(2n) \rightarrow \text{factors} \\ f_2 & \mathcal{X}_2(n) = \mathcal{X}(2n+1) \rightarrow \text{factors} \end{cases}$$

$$X(k) = \sum_{n=0}^{N-1} \mathcal{X}(n) w_N^{kn} \quad [w_N = e^{-j2\pi/N}]$$

$$= \sum_{n \text{ even}} \mathcal{X}(n) w_N^{kn} + \sum_{n \text{ odd}} \mathcal{X}(n) w_N^{kn}$$

$$= \sum_{m=0}^{N/2-1} \mathcal{X}(2m) w_N^{2mk} + \sum_{m=0}^{N/2-1} \mathcal{X}(2m+1) w_N^{k(2m+1)}$$

$$= \sum_{m=0}^{N/2-1} \mathcal{X}(2m) w_{N/2}^{mk} + w_N^k \sum_{m=0}^{N/2-1} \mathcal{X}(2m+1) w_{N/2}^{km}$$

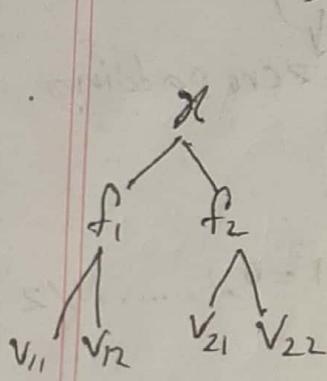
$$X(k) = \sum f_1(m) \omega_{N/2}^{km} + \omega_N^k \sum_{m=0}^{N/2-1} f_2(m) \omega_{N/2}^{km}$$

$$X(k) = \underbrace{f_1(k)}_a + \omega_N^k \underbrace{f_2(k)}_b [N/2 \text{ point DFT}]$$

$$\boxed{\omega_N^{k+N/2} = -\omega_N^k}$$

$$k = 0, \dots, \frac{N}{2} - 1$$

$$X\left(k + \frac{N}{2}\right) = f_1(k) - \omega_N^k f_2(k), \quad k = 0, \dots, \frac{N}{2} - 1$$



$$\begin{aligned} & \left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 + N \\ &= \frac{N^2}{2} + N \end{aligned}$$

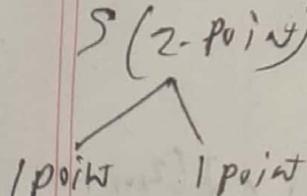
$$v_{11}(n) = f_1(2n)$$

$$v_{12}(n) = f_1(2n+1)$$

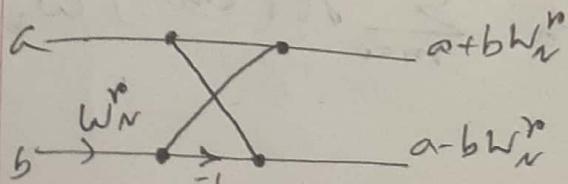
$$\boxed{V = \log_2 N}$$

$$v_{21}(n) = f_2(2n)$$

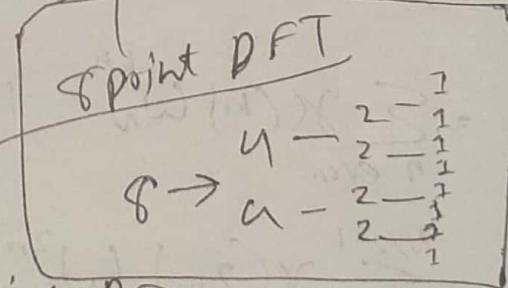
$$v_{22}(n) = f_2(2n+1)$$



Butterfly



Basic butterfly Structure



8 point DFT

$$x(0) \ x(1) \ x(2) \ x(3) \ x(4) \ x(5) \ x(6) \ x(7)$$

$$\rightarrow \underline{x(0) \ x(2) \ x(4) \ x(6)}$$

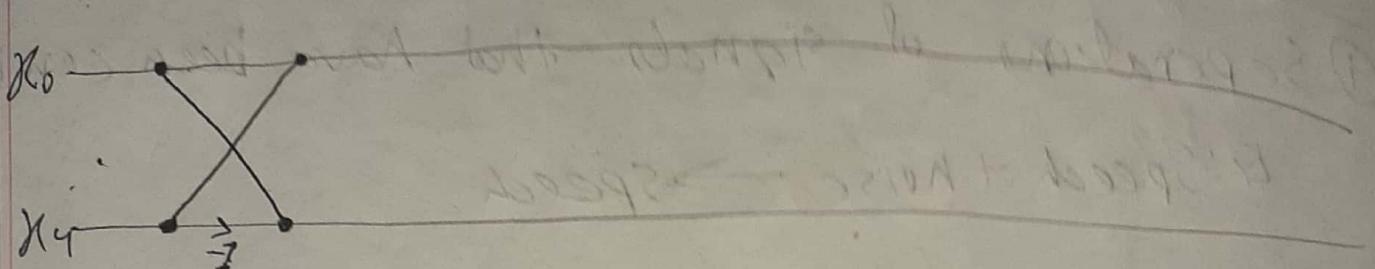
$$\underline{x(1) \ x(3) \ x(5) \ x(7)}$$

$$\rightarrow \underline{x(0) \ x(4) \ x(2) \ x(6) \ x(1) \ x(5)}$$

$$\underline{\underline{x(3) \ x(7)}}$$

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$x(0)$ $x(4)$ $x(2)$ $x(6)$ $x(1)$ $x(5)$ $x(3)$ $x(7)$



x_2 ~~compute longer duration < longer butterfly~~

x_1 ~~first stages need fewer bits than~~

x_5 ~~computing with 1000~~

x_3 ~~1000 bits~~ ~~1000 bits~~ ~~1000 bits~~

x_7 ~~1000 bits~~

H.W. ~~1000 bits~~

8 point FFT. ~~1000 bits~~

→ ∞ ~~1000 bits~~ end of ch