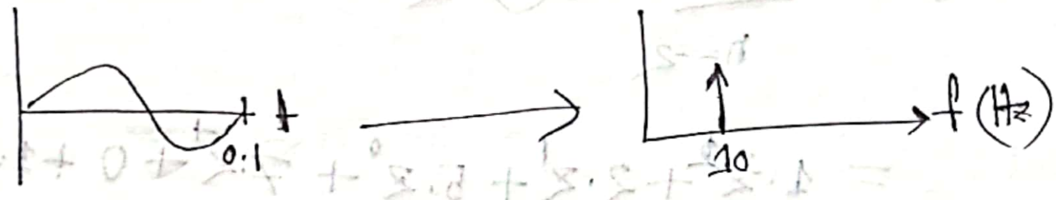
Chapter-3 (The z-transform)

যখন আমাদের যেটা দরকার তখন সেটা use করব। নিচে ~~দুটি~~ ~~signal~~ figure same frequency. তাকে নিচের ভাবে represent করা যায়।



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→ The z-transform of DT signal $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Where z is a complex variable

$$z = re^{j\omega}$$

If $r=1$, that's Fourier Transform

Ex: 3.1.1:

a) $x_1(n) = \{ \underset{\uparrow}{1}, 2, 5, 7, 0, 1 \}$ (indices 1 2 3 4 5)

$$X(z) = \sum_{n=0}^{5} x(n) z^{-n}$$

$$= 1 \cdot z^{0} + 2 \cdot z^{-1} + 5 \cdot z^{-2} + 7 \cdot z^{-3} + 0 + 1 \cdot z^{-5}$$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

b) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$, for $n = -2, -1, 0, 1, 2, 3$ (arrow marks $n = 0$)

$$X(z) = \sum_{n=-2}^{3} x(n) z^{-n}$$

$$= 1 \cdot z^{2} + 2 \cdot z^{1} + 5 \cdot z^{0} + 7 \cdot z^{-1} + 0 + 1 \cdot z^{-3}$$

$$= z^{2} + 2z + 5 + 7z^{-1} + z^{-3}$$

এমন একটা point যে point এ যাওয়ার পর আর তেমন কোনো value বাড়ে বা কমে না

ROC → Region of Convergence

→ Is the set of all values of z for which $X(z)$ attains a finite value.

→ all value of z except $z = 0, \infty$

Ex: b) $x_1(n) = 0$ বাদে বাকি সব roc → $-2, -1, 1, 2,$

$x(n) = \delta(n) = \{1, 0, 0, \ldots\}$, for $n = 0, 1, 2, \ldots$ (arrow marks $n = 0$)

$$X(z) = 1 \cdot z^{0} + 0$$

$$= 1 \cdot 1$$

$$= 1$$

$$x(n) = \delta(n-k) \quad \delta(n) \{1, 0, 0, 0\}$$

$$\{0, 0, 1, 0, 0, 0\}_{n=k}$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= z^{-k}$$

$$x(n) = \delta(n+k)$$

$$\{0, 0; 0, 1, 0,\}_{n=-k}$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= z^{k}$$

EX: Determine the z-transform of

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$= \left(\frac{1}{2}\right)^0 \cdot 1, \left(\frac{1}{2}\right)^1 \cdot 1, \left(\frac{1}{2}\right)^2 \cdot 1$$

$$= \left[1, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \ldots\right]$$

$$X(z) = 1 \cdot z^0 + \frac{1}{2} z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} \quad \text{Geometric series} \left(\frac{1}{1-r}\right)$$

$A < 1$

$\left|\frac{1}{2} z^{-1}\right| < 1$

$\Rightarrow |z| > \frac{1}{2}$ roc ~~হচ্ছে~~ $\rightarrow \frac{1}{2}$ ~~হওয়া~~ ~~মানে না~~।

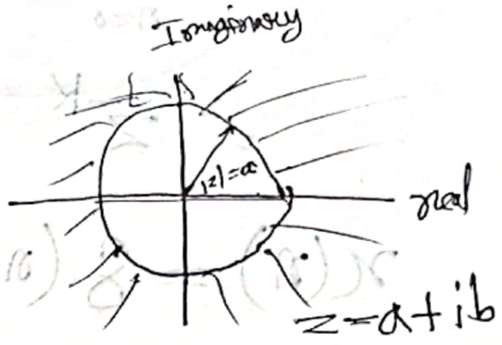
Ex: 3.1.3:

$x(n) = \alpha^{n} u(n)$

$A < 1$

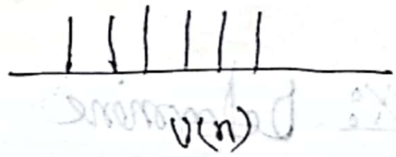
$|\alpha^{-1} z| < 1$

$X(z) = \frac{1}{1 - \alpha z^{-1}}$



Roc: $|z| > \alpha$ alpha

Ex: 3.1:



$x(n) = -\alpha^{n} u(-n-1)$

$$X(z) = \sum_{n=-\alpha}^{\alpha} x(n) z^{-n}$$

$$= \sum_{n=-\alpha}^{-1} -\alpha^{n} \cdot z^{-n}$$

$l = -n$

$$= \sum_{l=1}^{\alpha} -\alpha^{-l} \cdot z^{l} \qquad = \frac{1}{1-A}$$

$$= -\sum_{l=1}^{\alpha} \underbrace{(\alpha^{-1} z)}_{A}^{l} = -(A^1 + A^2 + A^3 + A^4 + \ldots)$$

$$= A(A + A^2 + A^3 + \ldots)$$

$$= -\frac{\alpha^{-1} z}{1 - \alpha^{-1} z}$$

$$= -\frac{\alpha^{-1}z/\alpha^{-1}z}{(1-\alpha^{-1}z)/\alpha^{-1}z}$$

$$= -\frac{1}{\alpha z^{-1}-1} = \frac{1}{1-\alpha z^{-1}}$$

$x(n) = -\alpha^n u(-n-1)$

$$X(z) = \frac{1}{1-\alpha^{-1}z}$$

ROC: $|A| < 1$

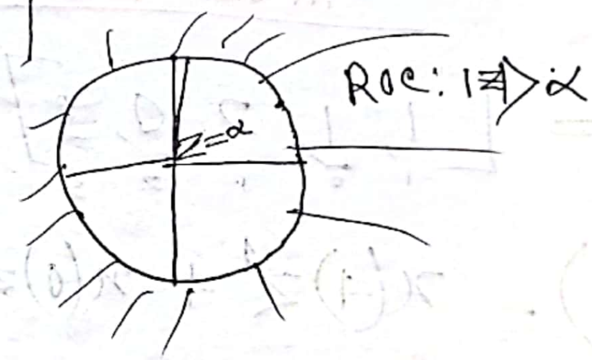
$|\alpha^{-1}z| < 1$

$|z| < \alpha$

$x(n) = \alpha^n u(n)$

$$X(z) = \frac{1}{1-\alpha z^{-1}}$$

ROC: $|z| > \alpha$



Ex: 3.1.5

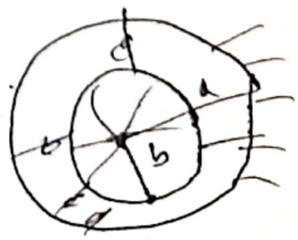
$x(n) = a^n u(n) + b^n u(-n-1)$

$$X(z) = \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}}$$

ROC: $|z| > a$

ROC: $|z| < b$

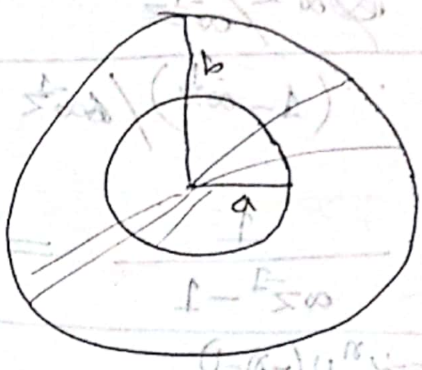
If $a > b$



roc is absent, no common part on a and b.

$a<b$

$X(z) \cdots$



$|a|<|z|<|b|$

Table - 3.1: Very Important

After Mid

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$$x(n) = [\underset{-1}{1}, \underset{0}{1}, \underset{1}{2}, \underset{2}{0}, \underset{3}{3}]$$

$$X(z) = x(-1)z^{1} + x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= 1z^{1} + 1 + 2z^{-1} + 0 + 3z^{-3}$$

$$= z^{1} + 1 + 2z^{-1} + 3z^{-3}$$

3.2 Properties of the z-transformation:

① Linearity

if $x_1(n) \xleftrightarrow{z} X_1(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$, then

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{z} X(z) = a_1 X_1(z) + a_2 X_2(z)$$

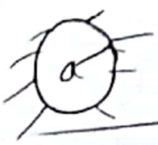
Ex: 3.2.1 Determine the z-transformation and the ROC of the signal :

$$x(n) = [3(2^n) - 4(3^n)]\, u(n)$$

$$= 3(2^n)\, u(n) - 4(3^n)\, u(n)$$

$a^n u(n) \rightarrow \dfrac{1}{1-az^{-1}}$

$|z| > a$ = ROC



$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$$

ROC = $|z|>2$ $\quad\quad$ $|z|>3$

∴ তাদের সম্মিলিত ROC হবে $|z|>3$

③ Time shifting

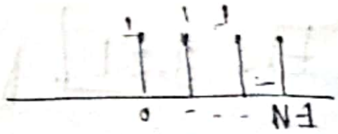
if $x(n) \overset{z}{\longleftrightarrow} X(z)$

then, $x(n-K) \overset{z}{\longleftrightarrow} z^{-K} \cdot X(z)$

Ex: 3.2.3 H.W

Ex: 3.2.4

$$x(n) = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{elsewhere} \end{cases}$$



$$x(n) = u(n) - u(n-N)$$

$$X(z) = \frac{1}{1-z^{-1}} - \frac{z^{-N}}{1-z^{-1}}$$

$$= \frac{1-z^{-N}}{1-z^{-1}}$$

③ Time reversal

if $x(n) \xleftrightarrow{z} X(z)$

then, $x(-n) \xleftrightarrow{z} X(z^{-1})$

Ex $x(n) = u(-n)$

$= \dfrac{1}{1-z}$

$z(u(n)) = \dfrac{1}{1-z^{-1}}$

Convolution of two sequences:

important

if $x_1(n) \xleftrightarrow{z} X_1(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$

convolution থেকে multiply তে রূপান্তর হয়ে যায়।

then,

$x(n) = x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z) \cdot X_2(z)$

convolution

* Convolution by means of z-transformation

① Compute the z-transforms of the signals to be convolved

$$X_1(z) = Z[x_1(n)]$$

$$X_2(z) = Z[x_2(n)]$$

② Multiply the two z-transforms

$$X(z) = X_1(z)\,X_2(z)$$

③ find the inverse z-transforms of $X(z)$

$$x(n) = Z^{-1}[X(z)]$$

Ex: 3.2.9 | $x_1(n) = \{1, -2, 1\}$ and $x_2(n) = \{1, 1, 1, 1, 1, 1\}$

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$X(z) = X_1(z)\,X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

$$\xleftarrow{\text{inverse of } X(z)} \{1, -1, 0, 0, 0, 0, -1, 1\}$$

3) Rational Z-transformation

Poles and Zeros:

$$X(z) = \frac{B(z)}{A(z)}$$

→ The zeros of a z-transform $X(z)$ are the value of z for which $X(z) = 0$.

→ The poles ..... for which $X(z) = \infty$ ∞ → infin

$$X(z) = \frac{B(z)}{A(z)} = c \cdot \frac{(z-z_1)(z-z_2)\cdots\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

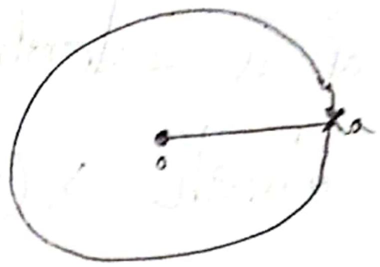
Zeros: $z_1, z_2, \ldots z_M$
Poles: p_1, p_2, p_N

Ex: 3.3.1 | Determine the pole-zero plot for the signal

$$x(n) = a^n u(n)$$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} = \frac{z-0}{z-a}$$

zero : 0
pole : a



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$A(z) =$ যে z value এর জন্য $A(z)$ এর value zero হয় তাকে pole বলে। $a-3=0$

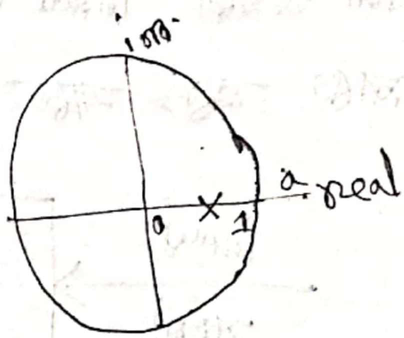
$a=3$

$B(z) =$

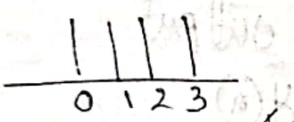
3.3.2 Pole location and Time domain ~~Baiso~~ Bahavior for ~~Casat~~ Causal system.

$$x(n) = a^n u(n)$$

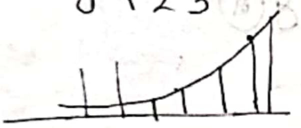
$$X(z) = \frac{1}{1 - az^{-1}}$$

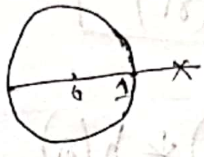


$a = 1$

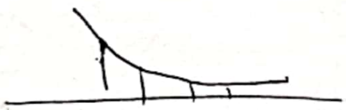


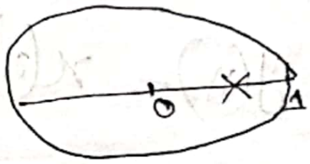
$a > 1$



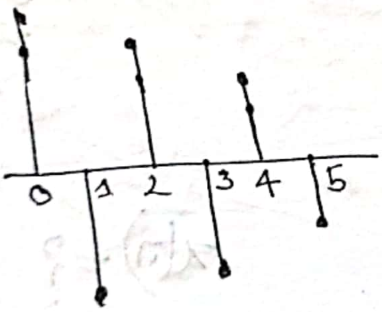


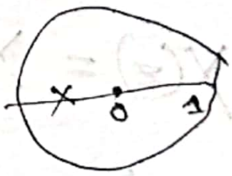
$a < 1$

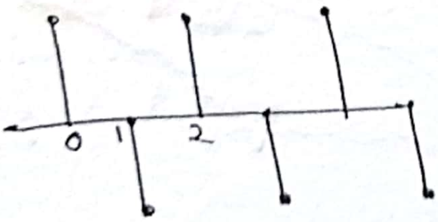


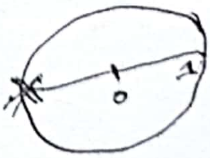


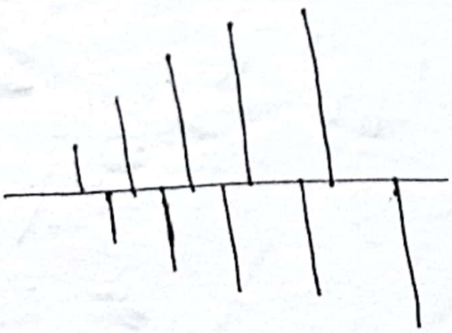
$a < 0$

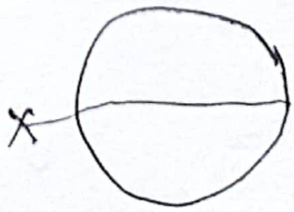






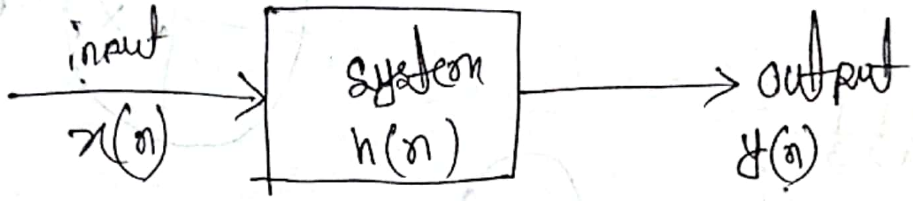






3.3.3° System function of an ~~line~~ LTI system

কখনো কখনো input এবং output জানি, system কে খুঁজে বের করতে হবে, যাকে বলে system identification.



$$y(n) = x(n) * h(n)$$

z-transform, $Y(z) = X(z) \cdot H(z)$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)}$$

আমাদের আছে z domain থেকে time domain এ যেতে হলে ~~inverse~~ z-transform inverse করতে হবে

$x(n) = ?$ দরকার, যা time domain ~~থেকে~~

3.3.4)

$$y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

$$\Rightarrow Y(z) \cdot \left(1 - \frac{1}{2} z^{-1}\right) = 2X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

$$a^n u(n) = \frac{1}{1 - a z^{-1}}$$

$$\Rightarrow H(z) = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

$$= 2 \cdot \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$$h(n) = 2 \cdot \left(\frac{1}{2}\right)^n \cdot u(n)$$

3.4 Inverse z-transform:

Inverse z-transform by Power series Expansion:

3.4.2 Determine the inverse z-transform of

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-1}}$$

a) ROC : $|z|>1$

b) ROC: $|z|<0.5$

$x(n) = [2, 3, 1]$ (indices 0, 1, 2)

$X(z) = 2+3z^{-1}+z^{-2}$

$1-1.5z^{-1}+0.5z^{-2}$) 1 ($1+1.5z^{-1}+1.65z^{-2}$

$1-1.5z^{-1}+0.5z^{-2}$

⇓

$[1, 1.5, 1.6]$

causal হলে, তার series হবে negative

$|z|>1$

* non-causal; তার series হবে positive,

$|z|<0.5$

$0.5z^{-2}-1.5z^{-1}+1$) 1 ($2z^2+6z^3+14z^4+\ldots$

$1-3z+2z^2$

$3z-2z^2$

$3z-9z^2+6z^3$

$7z^2-6z^3$

$[\ldots 14, 6, 2, 0, 0]$ (indices $\ldots, 4, 3, 2, 1, 0$)

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3.4.3 Inverse z-transform by ~~partial function~~ partial fraction Expansion ***

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$= \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}} \qquad a^n u(n) = \frac{1}{1-az^{-1}}$$

a) $|z| > 1 \rightarrow$ causal

b) $|z| < 0.5 \rightarrow$ non-causal

c) $0.5 < |z| < 1$ $\swarrow$ causal and non-causal

a) $x(n) = 2(1)^n u(n) - (0.5)^n u(n)$

b) $x(n) = -2(1)^n u(-n-1) + (0.5)^n u(-n-1)$

c) $x(n) = \underbrace{-2(1)^n u(-n-1)}_{\text{non-causal}} - \underbrace{(0.5)^n u(n)}_{\text{causal}}$

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3.5 Analysis of LTI System in z-domain

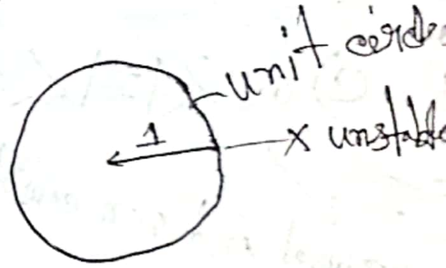
Causality and Stability

Causality

$h(n) = 0, \quad n < 0$

Stability:

A LTI system is BIBO stable if and only if the ROC of the system functions include the unit circle.



Ex: 3.5.2

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

$$= \frac{1}{1 - \underbrace{\frac{1}{2}}_{\text{pole}} z^{-1}} + \frac{2}{1 - \underbrace{3}_{\text{pole}} z^{-1}}$$

a) system is stable $\qquad \frac{1}{2} < |z| < 3$

b) system is causal

c) system is anticausal

a) $h(n) = \left(\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)$

b) $h(n) = \left(\frac{1}{2}\right)^n u(n) + 2(3)^n u(n)$

c) $h(n) = -\left(\frac{1}{2}\right) u(n-1) - 2(3)^n u(-n-1)$

* one sides . z-transform

3.6 $X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

H.W EX: 3.C.1

$x(n) = [\underset{\uparrow}{1}, 2, 5, 7]$

$X^+(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3}$

Chapter-4

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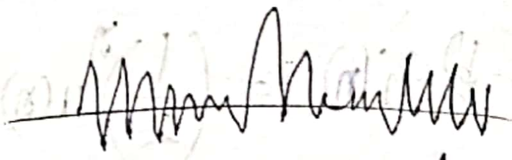
Frequency analysis of Signals

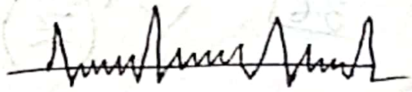
* Complex frequency বের করার জন্য freq analysis করা লাগে।



complex signal

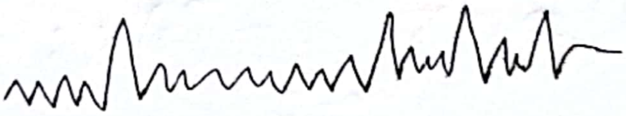
① Fourier Series → periodic signal ক্ষেত্রে

② Fourier Transform → Aperiodic " "

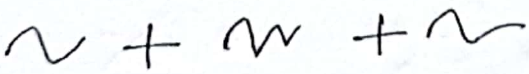


* যত complex signal হউক না কেন, that can be decomposed into a number of sinusoid.

Ex:



→ complex signal

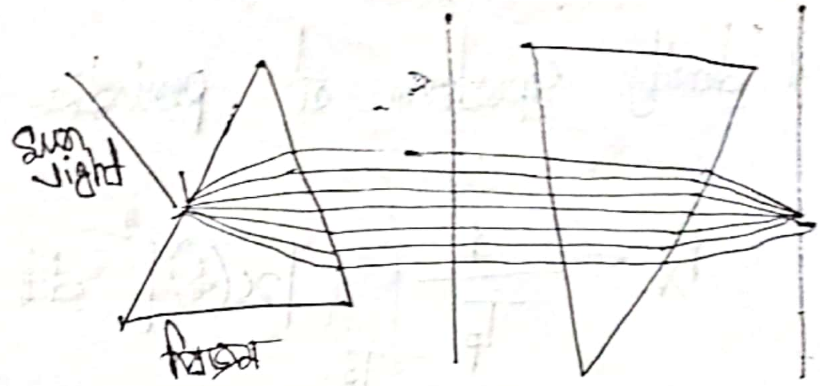


→ sinusoid

* অনেকগুলো sinusoid মিলে আবার complex signal তৈরী হয়।

Ex:

1672, Newton



4.1.1 Fourier Series for continuous-Time periodic Signals

Jean Baptiste Joseph Fourier (1768 – 1830)

complex sinusoid: $e^{i\theta} = \cos\theta + i\sin\theta$

Fourier Series Equations:

continuous Sum

Analysis Equation: $C_K = \frac{1}{T_p} \int_{T_p} x(t)\, e^{-j2\pi K F_0 t}\, dt$
(বিশ্লেষণ)

Discrete Sum

Synthesis Equations: $x(t) = \sum_{K=-\infty}^{\infty} C_K\, e^{j2\pi K F_0 t}$
(সংশ্লেষণ)

Analysis: ৪ একটা থেকে বিশ্লেষণ করবে আলাদাগুলো sinusoid বের হয়।

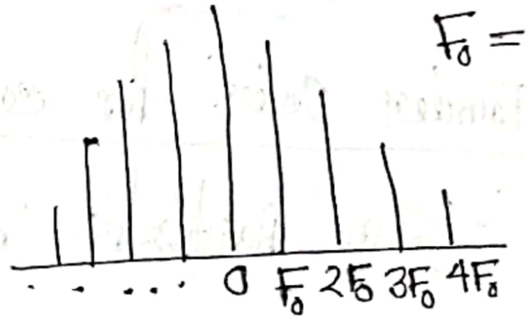
Synthesis::

4.1.2 Power density Spectrum of periodic signals

$$P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2\, dt$$

$$= \sum_{k=-\infty}^{\infty} |c_k|^2$$

$F_0 = \frac{1}{T}$



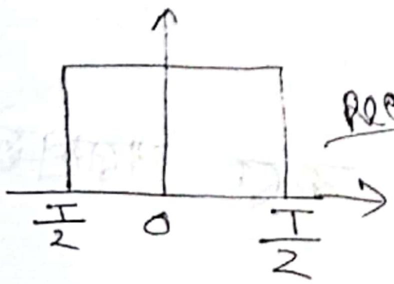
Power density spectrum

4.1.3 The Fourier Transform of Continuous-Time Aperiodic Sign[als]

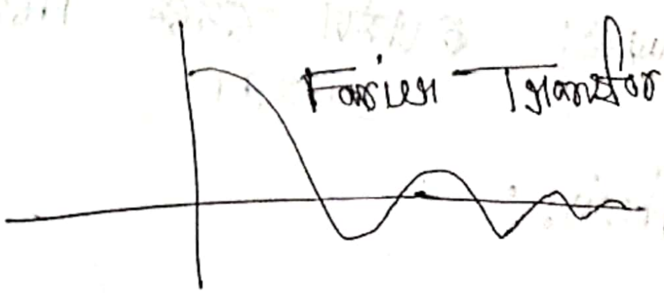
Analysis Equation: $X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi F t}\, dt$

Synthesis Equation: $x(t) = \int_{-\infty}^{\infty} X(F)\, e^{j2\pi F t}\, dF$

Ex: 4.12



Rectangular



Fourier Transform

Frequency analysis of DT signal

4.2.1] Fourier Series for DT Periodic Signal.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$ → For continuous Time signal

Synthesis equation =
$$\boxed{x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}}$$ → DT

Analysis equation =
$$\cancel{x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}}$$

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

Ex:] $x(n) = \cos \pi n/3$

$= \cos 2\pi n/6 \quad = \cos 2\pi n \cdot \frac{1}{6}$

$\frac{1}{N} = \frac{1}{6}$

4.2.2 Power density spectrum

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

$$= \sum_{k=0}^{N-1} |c_k|^2$$

DT → integration
CT → Summation

* Fourier Transform of DT signal:

$\omega = 2\pi f$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$\omega \to -\pi$ to π or
$-\frac{1}{2}$ to $\frac{1}{2}$ or
0 to 2π

* $X(\omega)$ ~~to~~ periodic? yes,

$$X(\omega + 2\pi K) = \sum x(n) e^{-j(\omega + 2\pi K)n}$$

$$= \sum x(n) e^{-j\omega n} \cdot \underbrace{e^{-j2\pi k n}}_{1}$$

$$= \sum x(n) e^{-j\omega n}$$

* Relation of Fourier Transform with Z-Transform

$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$$

$$\therefore X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$z = re^{j\omega}$, $\boxed{r=1}$

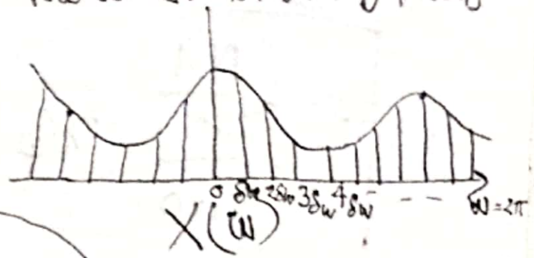
$= 1 \cdot e^{j\omega}$

$z^{-n} = (e^{j\omega})^{-n}$

$= e^{-j\omega n}$

* Discrete Fourier Transform of DT signal ***

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$ → Fourier transform DT signal



$$\delta\omega = \frac{2\pi}{N}$$

~~X(K)=~~

$$X\left(\frac{2\pi}{N}\cdot k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi kn/N}$$

~~এই দুটি~~ সমস্যা আছে

ω → continuous

x(n) → length is infinite

এখানে ω কে discrete করলে ও x(n) এখানে infinite.

$$X(K) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi kn/N}$$

$$= \cdots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} + \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \cdots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j2\pi kn/N}$$

$$x(K) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j2\pi Kn/N}$$

$$\boxed{= \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi Kn/N}}$$

Analysis equation

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

Fourier series

Synthesis

$$x_p(n) = \sum_{K=0}^{N-1} c_K\, e^{j2\pi Kn/N} \qquad n = 0 \text{ to } N-1$$

Analysis

$$c_K = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi Kn/N} \qquad K = 0 \text{ to } N-1$$

$$c_K = \frac{1}{N} X(K)$$

Synthesis Analysis

$$\therefore \boxed{x_p(n) = \frac{1}{N} \sum_{K=0}^{N-1} x(K)\, e^{j2\pi Kn/N}}$$

$x(n) = x_p(n)$, when $0 \leq n \leq N-1$

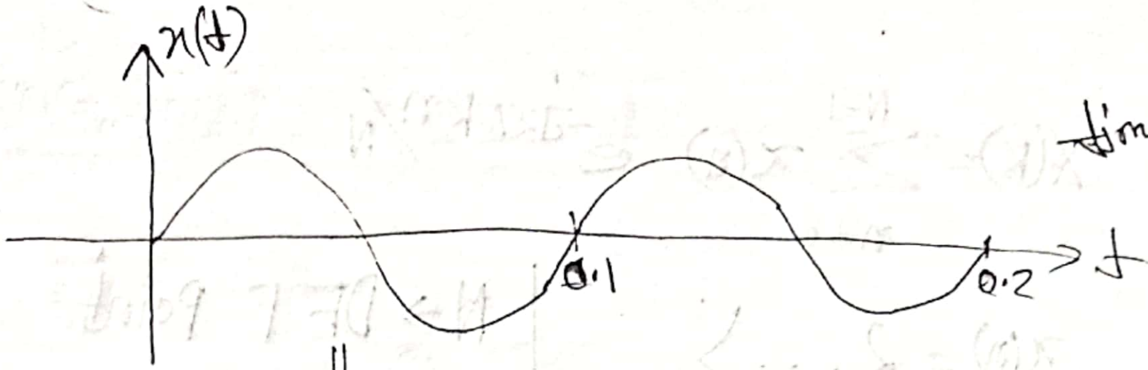
Analysis equation

$$\therefore \boxed{x(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi Kn/N}}$$

शर्त

$L \leq N$

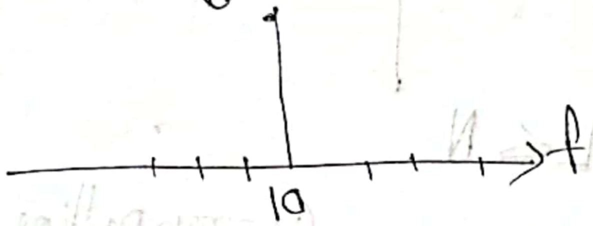
Ex



time-domain

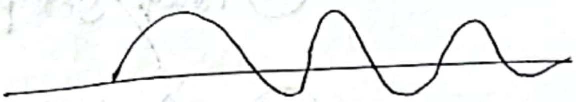
$f = 1$

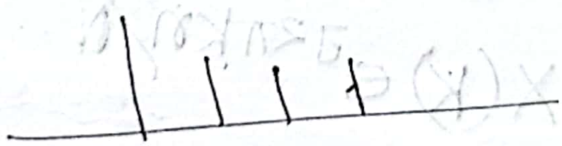
Discreate fourier Transform



frequency domain

Ex-2 $x(t) = \sin w_1 t + \sin w_2 t + \sin w_3 t$





DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

$x(n) = \{ \cdots \}$

$L = 200$

$N \to$ DFT Point

$N \to 256$

$L \leq N$

1 Sample $\to \dfrac{2\pi}{N}$ rad

$\hookrightarrow \left(\dfrac{F_s}{N}\right)$ Hz

(i) zero padding : $L \leq N$

যখন N এর মান ব

হয় তখন বড়

(ii) freq/sample : F_s/N or

(iii) if N is increased

Inverse DFT:

$$x(n) = \sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}$$

Ex: 7.1.2

$$X(K) = \sum_{n=0}^{N-1}$$

7.2 Properties of the DFT:

① Periodicity

if $x(n) \longleftrightarrow X(K)$

$x(n+N) \longrightarrow x(n)$

$X(K+N) = X(K)$

N → সংখ্যক পর পর একই signal repeat করে।

$\longleftrightarrow$ dft pair ই

② Linearity:

if $x_1(n) \longleftrightarrow X_1(K)$

$x_2(n) \longleftrightarrow X_2(K)$

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{DFT} a_1 X_1(K) + a_2 X_2(K)$$

CamScanner

3) Multiplication of Two DFTs and Circular Convolution:

$x_1(n), x_2(n)$

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi Kn/N}$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi Kn/N}$$

$$X_3(K) = X_1(K) X_2(K)$$

$$x_3(m) = \frac{1}{N} \sum_{0}^{N-1} X_3(K) e^{j2\pi Km/N}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} X_1(K) \cdot X_2(K) \cdot e^{j2\pi Km/N}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) e^{-j2\pi Kn/N} \right] \left[\sum_{l=0}^{N-1} x_2(l) e^{-j2\pi Kl/N} \right] e^{j2\pi Km/N}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{n=0}^{N-1} e^{j2\pi K(m-n-l)/N} \right]$$

$a^K = \frac{a^{\cdot}}{}$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N$$

যদি, $m-n-l = K \cdot N$

$l = ((m-n))_N$

$l = 1$, হবে

Circular Convolution

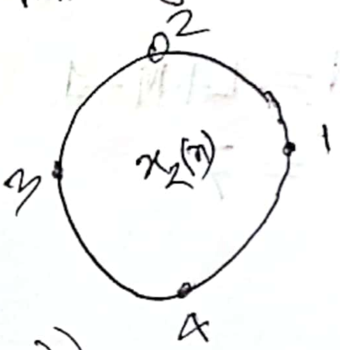
Note: যদি দুটি DFT কে multiplication করে inverse করলে circular convolution পাব

Ex: 5.2.1

Perform the circular convolution of the following two sequence

$x_1(n) = \{2, 1, 2, 1\}$

$x_2(n) = \{1, 2, 3, 4\}$

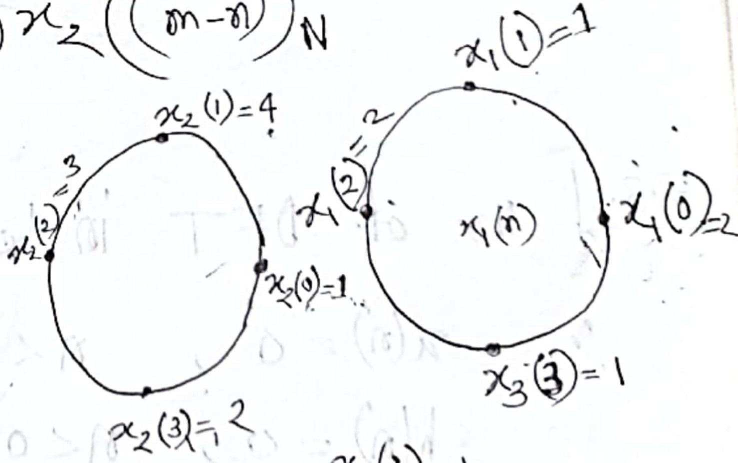


$$x_3(m) = \sum_{n=0}^{3} x_1(n)\, x_2((m-n))_N$$

$m=0$,

$$x_3(0) = \sum x_1(n)\, x_2((-n))_N$$

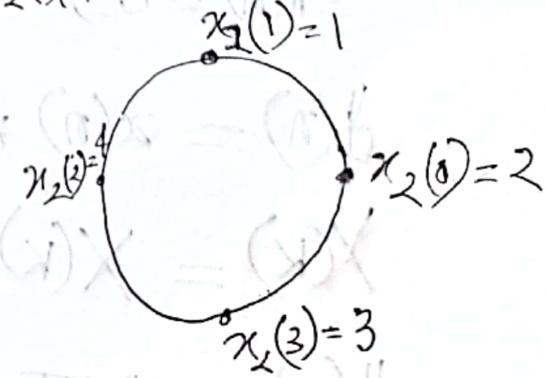
$$= 2+4+6+2 = 14$$



$m=1$,

$$x_3(1) = \sum x_1(n)\, x((1-n))_N$$

$$= 4+1+8+3 = 16$$



$m=2$,

$$x_3(2) = \sum x_1(n)\, x((2-n))_N$$

$$= 6+2+2+4 = 14$$

$x_3(3) = 8+3+4+1 = 16$

$x_3(n) = \{14, 16, 14, 16\}$

7.3 Linear filtering methods Based on the DFT:

filter করতে impulse response লাগে

$L \quad x_1(n) = \{2, 1, 2, 1, 0, 0, 0\}$

$h(n) = $

$M \quad x_2(n) = \{1, 2, 3, 4, 0, 0, 0\}$

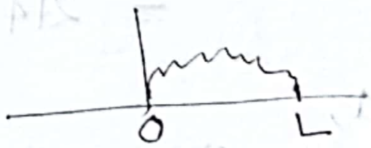
$N = L+M-1$
$= 7$

এখন Linear convolution করে shift করলে signal loss হবে না। কারণ 0 zero দিয়ে padding করার কারণে shift করলে '0' shift হবে loss হবে main signal loss হবে না।

7.3.1 Use of DFT in ~~Linear~~ Linear filtering

$x(n) = 0, \quad n < 0 \text{ and } n \geq L$

$h(n) = 0, \quad n < 0, \text{ and } n \geq M$



$y(n) = x(n) * h(n)$

$Y(K) = X(K) \cdot H(K)$

$y(n) = IDFT\left(H(K) \cdot X(K)\right)$

আর

$N \geq L+M-1$

Ex: 7.3.1 By means of DFT and IDFT, determine the response of the FIR filter with impulse response

$$h(n) = \{1, 2, 3\}$$

to the input sequence

$$x(n) = \{1, 2, 2, 1\}$$

$$y(n) = x(n) * h(n)$$

$$N = L+M-1 = 4+3-1 = 6$$

$$Y(K) = X(K) \cdot H(K)$$

$$y(n) = IDFT(Y(K))$$

বাকীগুলো H.W

7.3.2 Filtering of Long Data Sequences:

7.3.2 Filtering of Long Data Sequences

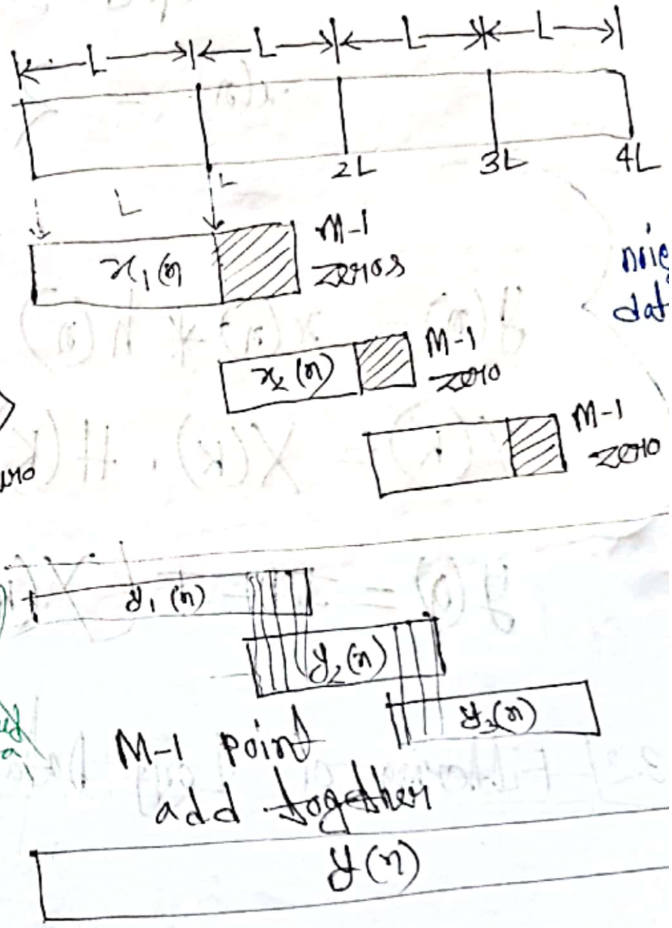
① overlap-save method

② overlap add "

Overlap add Method:

$x_1(n) = \{x(0), x(1), x(2), \ldots x(L-1), \underbrace{0,0,0,0}_{M-1 \text{ zero}}\}$

$x_2(n) = x(L), x(L+1), \ldots x(2L-1), \underbrace{0,0,0}_{M-1 \text{ zero}}\}$

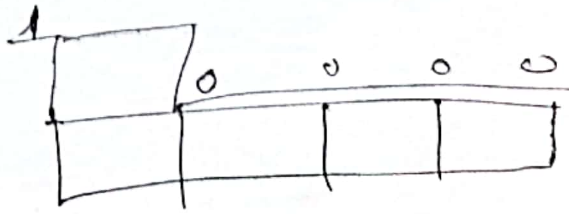


$y(n) = \{y_1(0), y_1(1), y_1(L-1), y_1(L), y_2(0) \ldots\}$

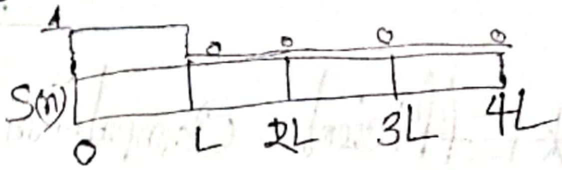
$y(n) = IDFT\left(X_1(K) \cdot H(K)\right)$

prior removed data

Windowing : এটা হচ্ছে আমরা যে Long-Data ~~তেমনি~~ নিচ্ছি মানে কী?



* Frequency Analysis of Signal using DFT:



$$w(n) = \begin{cases} 1, & 0 \leq n < L-1 \\ 0 & \text{---} \end{cases}$$

window

Resolution ভালো পাবার জন্য, Rectangular Window ব্যবহৃত হয়

Leakage কমানোর জন্য, Hamming window " " ।

Hamming window:

$$w(n) = \begin{cases} \frac{1}{2}\left(1 - \cos \frac{2\pi}{L-1} n\right), & 0 \leq n \leq L-1 \\ 0, & \text{other} \end{cases}$$

Chapter - 8

* Efficient Computation of the DFT: Fast fourier

Convolution করলে $\longrightarrow N^2$ সময় লাগে

তাহলে যদি Divide and conquer করি $\longrightarrow = \left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2$

$= \frac{N^2}{4} + \frac{N^2}{4}$

তাহলে অর্থাৎ সময় কমবে, যত ভাগ করব তত কমবে।

$= \frac{2N^2}{4} = \frac{N^2}{2}$

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi Kn/N}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

$$\boxed{W_N = e^{-j2\pi/N}}$$

- ত্রিকোণমিতির দুটি property use করে DFT কে আরো faster করা যাবে

① Symmetry property: $W_N^{K+N/2} = -W_N^K$

$$\left(e^{-j2\pi/N}\right)^{K+N/2} = e^{\left(-j2\pi/N\right)\left(K+N/2\right)}$$

(2) Periodicity property: $W_N^{K+N} = W_N^{K}$

6.1.2 Divide and Conquer Approach to Computation of the DFT:

$N = LM$ $\qquad N = 200$ ~~L = 4 M = 50~~

$= 4 \times 5 = L \times M$

Row wise:

$l \backslash m$	0	1	2	---	M-1
0	$x(0)$	$x(1)$	$x(2)$	---	$x(m-1)$
1	$x(m)$	$x(m+1)$	$x(m+2)$	---	$x(2m-1)$
2					$x(3m-1)$
⋮					$x(Lm-1)$
L-1	$x(L-1)m$				

$n = Ml + m$

Column wise:

$x(0)$	$x(L)$	$x(2L)$	--- $\cancel{x(4L)}$	$x(M-1)L$
$x(1)$	$x(L+1)$	$x(2L+1)$	$\cancel{x(4L+1)}$	$x(M-1)(L+1)$
⋮	⋮	⋮	⋮	⋮
$x(L-1)$	$x(2L-1)$	$x(3L-1)$	x --	$x(LM-1)$

$n = l + mL$

$$X(K) = \sum_{n=0}^{N} x(n)\, W_N^{kn}$$

$N = LM$

$$= \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(l, m)\, W_N^{(Mp+q)(mL+l)}$$

$$\boxed{W_N^{(Mp+q)(mL+l)} =}$$

$$\text{so } W_N^{(Mp+q)(mL+l)} = W_N^{MLmp} \cdot W_N^{mLq} \cdot W_N^{Mpl} \cdot W_N^{lq}$$

1. First, we compute the M-point DFTs

$$F(l, q) = \sum_{m=0}^{M-1} x(l, m) W_M^{mq}, \quad 0 \leq q \leq M-1$$

2. Second, we compute a new rectangular array

$$G(l, q) = W_N^{lq} \cdot F(l, q) \qquad \begin{array}{l} 0 \leq l \leq L-1 \\ 0 \leq q \leq M-1 \end{array}$$

3. Finally, we compute L-point DFTs

$$X(p, q) = \sum_{l=0}^{L-1} G(l, q) W_L^{lp}$$

<u>Ex:</u>
8.1.1

$N = LM = 5 \times 3 = 15$

<u>Column wise:</u>

	0	1	2
0	$x(0)$	$x(5)$	$x(10)$
1	$x(1)$	$x(6)$	$x(11)$
2	$x(2)$	$x(7)$	$x(12)$
3	$x(3)$	$x(8)$	$x(13)$
4	$x(4)$	$x(9)$	$x(14)$

$F(l, q)$

Algorithm-1:

1. Store the signal column wise. LM^2
2. Compute the M-point ^{M} DFT of each row. L সংখ্যক
3. Multiply the resulting array by the phase factor W_N^{lq}
4. Compute the L-point ^{L} DFT of each column. M সংখ্যক ML^2
5. Read the resulting array row wise.

DFT

$N = N^2$

$= N.N$

$= N.(LM)$

FFT

$LM^2 + LM + ML^2$

$= LM(L+M+1)$

$= N(L+M+1)$

if, $N=200$, $L=50$, $M=4$

for DFT

$N = 200(50 \times 4)$

$= 40000$

for FFT

$N = N(L+M+1)$

$= 200(50+4+1)$

$= 11000$

so, fft is faster than DFT

8.13 Radix-2 FFT algorithm:

Radix-2 = 2 box
$N = 256 = 2^8$

$\cancel{N = LM = 2^V}$

$N = r^V$, $r = 2$

$x(n)$, $f_1(n) = x(2n) \rightarrow$ even index

$f_2(n) = x(2n+1)$, $n = 0, 1, \ldots \frac{N}{2} - 1$

$\rightarrow$ odd index

$x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)$

$f_1(n) = x(0), x(2), x(4), x(6)$

$f_2(n) = x(1), x(3), x(5), x(7)$

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

$$W_N^{2m} = W_{N/2}^{m}$$

$$= \sum_{m=0}^{N/2-1} f_1(m) W_N^{K(2m)} + \sum_{m=0}^{N/2-1} f_2(m) W_N^{K(2m+1)}$$

$$= \underbrace{\sum_{m=0}^{N/2-1} f_1(m) W_{N/2}^{Km}} + W_N^K \underbrace{\sum_{m=0}^{N/2-1} f_2(m) W_{N/2}^{Km}}$$

$$X(K) = F_1(K) + W_N^K F_2(K)$$

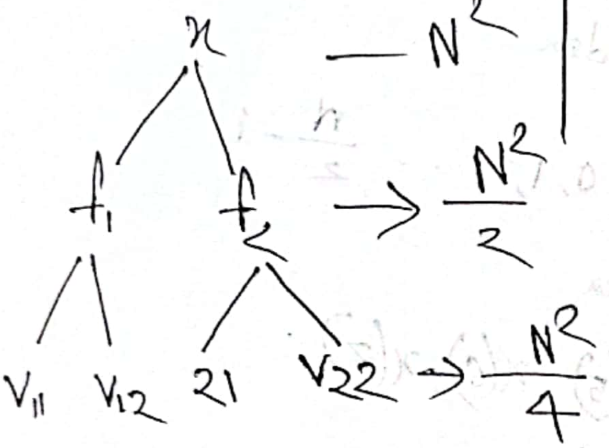
$F_1(K), F_2(K)$

Symmetry property

$$W_N^{K+N/2} = -W_N^K$$

$$X(K) = F_1(K) + W_N^K F_2(K), \quad K = 0, 1, \ldots, \frac{N}{2}$$

$$X\left(K+\frac{N}{2}\right) = F_1(K) - W_N^K F_2(K)$$



$\therefore N = \log_2 N$

$2 | 8 \rightarrow 1$
$2 | 4 \rightarrow 2$
$2 | 2 \rightarrow 3$
1

if $N = 8$, $\log_2 8 = 3$

Basic part

$$X_1(K) = F_1(K) + W_N^K F_2(K) \rightarrow A = a + W_N^r b$$

$$X_2(K) = F_1(K) - W_N^K F_2(K) \rightarrow B = a - W_N^r b$$

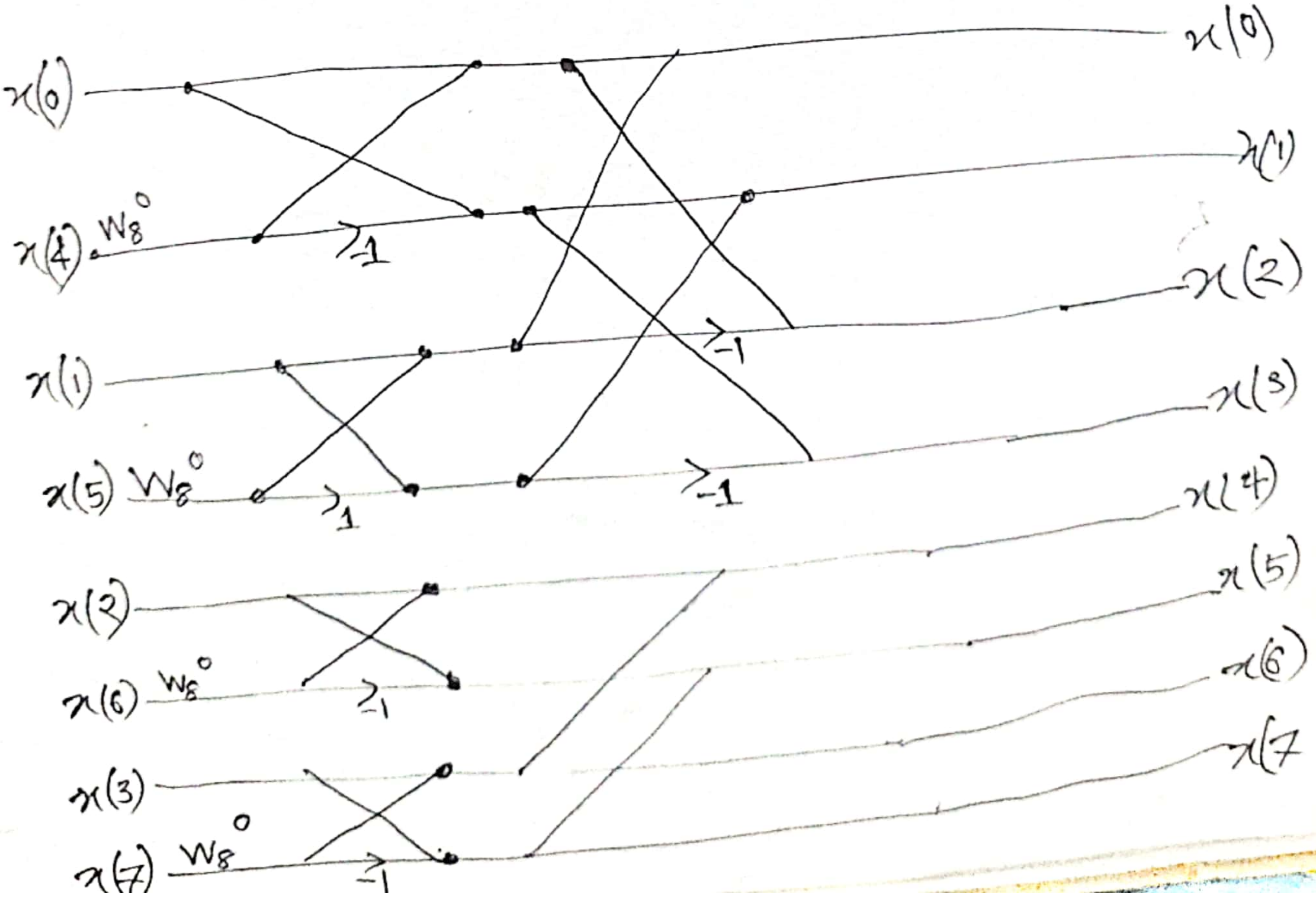
* Basic Butterfly connection

Decimation in time FFT, 8 point

* 0, 1, 2, 3, 4, 5, 6, 7 N=8

0, 2, 4, 6 1, 3, 5, 7 odd even

$\underbrace{0,4}$ $\underbrace{2,6}$ $\underbrace{1,5}$. $\underbrace{3,7}$



CSE 431 (DSP), TT # 4, Time: 30 min, Marks: 20

1. If $x(n)$ and $X(k)$ are N-point DFT pair, then $X(k+N)$ = ?
2. If $X_1(k)$ and $X_2(k)$ are the N-point DFTs of $x_1(n)$ and $x_2(n)$, respectively, then what is the N-point DFT of $x(n)=ax_1(n)+bx_2(n)$?
3. How do you compute the response of the FIR filter with impulse response $h(n)$ to the input sequence $x(n)$?
4. A finite-duration signal of length L is given as $x(n) = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...\}$. Determine the N-point DFT of this sequence.
5. Explain how the choice of window affects the spectrum estimation.
6. Describe the overlap-add method of linear filtering of long data sequence.