DSP (cls-11) Chapter-3 14/3/23

Z-transform

→ Fourier transform

→ Z-Transform is the discrete version of laplace Transform.

Z-Transform :-

The z-transform of a DT signal $x(n)$ is defined as the power series.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Z is a complex variable. $\boxed{z = re^{j\omega}}$

$$x(n) \xleftrightarrow{Z} X(z)$$

converge diverge

Z-Transform exists only for those values of z for which the series converges.

The region of converge (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

Ex: 3.1.1 $x_1(n) = \{\underline{1}, 2, 5, 7, 0, 1\}$ $\sum_{n=0}^{5} x(n) z^{-n}$

$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5}$

$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$

ROC:- entire z-plane except $z=0$.

$x_2(n) = \{1, 2, \underline{5}, 7, 0, 1\}$

$= z^2 + 2z^1 + 5 + 7z^{-1} + z^{-3}$

ROC:- entire z plane except $z=0$ and $z=\alpha$.

$x_c(n) = \delta(n)$

$x(z) = 1$, ROC:- entire z-plane.

$x_c(n) = \delta(n-k)$

$X(z) = z^{-k}$

$x_7(n) = \delta(n+k)$

$X(z) = z^{k}$

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$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad , \quad z = re^{j\omega}$$

Ex-3.1.1 $x(n) = \{1, 2, \underset{0}{5}, 7, 0, 1\}$

$$x(z) = x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= z^{2} + 2.z^{1} + 5 + 7z^{-1} + 1 z^{-3}$$

ROC = all values of z except $z = 0, \infty$

Ex-3.1.2 Determine the z-transform of the signal

$$x(n) = \left(\frac{1}{2}\right)^{n} u(n)$$

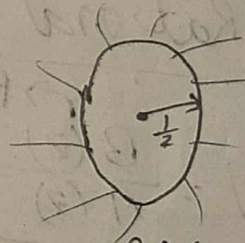
$$= \left\{ \underset{0}{1}, \frac{1}{2}, \left(\frac{1}{2}\right)^{2}, \left(\frac{1}{2}\right)^{3} - - \right\} \qquad u(n) = \{1, 1, 1 \ldots\} \;\; n = 0 \; 1 \; 2$$

powerseries

$$x(z) = 1 + \frac{1}{2} z^{-1} + \left(\frac{1}{2}\right)^{2} z^{-2} + \left(\frac{1}{2}\right)^{3} z^{-3} + - -$$

$$= 1 + A + A^{2} + A^{3} + - -$$

$$= \frac{1}{1-A} \quad , \; A < 1$$



ROC

$$x(z) = \frac{1}{1 - \left(\frac{1}{2}\right) z^{-1}} \quad , \quad ROC : |z| > \frac{1}{2}$$

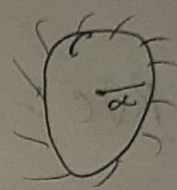
→ signal R.H.S হলে ROC = outer side.

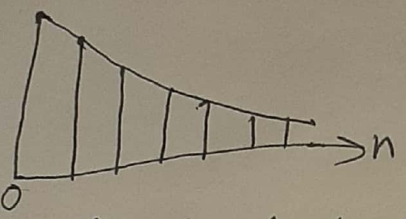
* For a RHS signal the ROC is outside of the circle.

Ex-3.1.3 Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \rightarrow \text{RHSS}$$

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$





$$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad \alpha z^{-1} < 1.$$

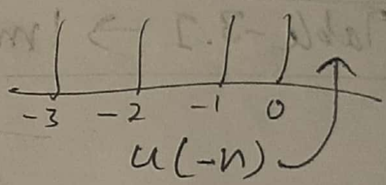
ROC $= |z| > \alpha$

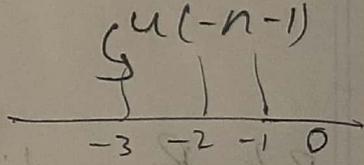
α হতে বেশি হলে convergence,

– delay = ডানে
fold = বামে

Ex-3.1.4 $x(n) = -\alpha^n u(-n-1)$

$$= \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$





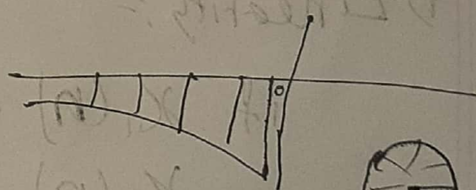
$$X(z) = \sum_{n=-\infty}^{-1} (-\alpha^n) z^{-n}$$

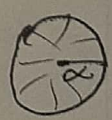
$$= \sum_{n=-1}^{\infty} (\alpha^{-1} z)^n = -\sum_{l=1}^{\infty} (\alpha^{-1} z)^l \quad [l = -n]$$

L.H.S.S.

$$= A + A^2 + A^3 + \cdots$$

$$= A(1 + A + A^2 + \cdots)$$





ROC

$$X(z) = -\frac{A}{1-A}$$

$$= -\frac{\alpha^{-1} z}{1-\alpha^{-1} z} = -\frac{1}{\frac{1}{\alpha^{-1} z} - 1} = \frac{1}{1-\alpha z^{-1}}, \quad |\alpha^{-1} z| < 1$$

ROC $= |z| < \alpha$

α ROC of a L.H.S. signal is the interior of the circle.

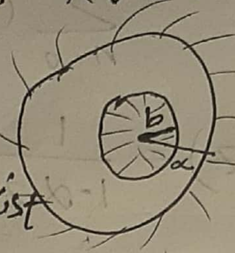
Ex-3.1.5. $x(n) = \alpha^n u(n) + b^n u(-n-1)$

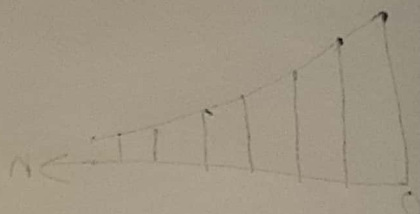
$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}}$$

ROC :- $|z| > \alpha$ ∴ ROC :- $|z| < b$

case-1 :- $|b| < |\alpha|$

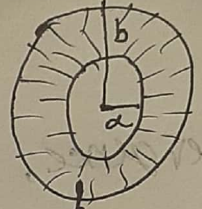
· ROC doesn't exist



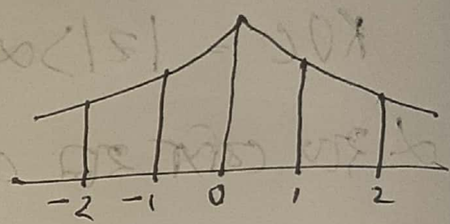


case-2 :- $|b| > |\alpha|$

ROC $= |\alpha| < |z| < |b|$



common part = ROC



$\boxed{\text{Table-3.2} \rightarrow \text{Imp}}$

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3.2 Properties of z transform

পৃথকভাবে * বা: + করার পর
+ " * " তো → Lin

1) Linearity :-

$$\text{if } x_1(n) \longleftrightarrow X_1(z)$$
$$x_2(n) \longleftrightarrow X_2(z)$$

then $x(n) = a_1 x_1(n) + a_2 x_2(n) \longleftrightarrow a_1 X_1(z) + a_2 X_2(z)$

$\underline{Ex\text{-}3.2.1}$ $x(n) = [3(2^n) - 4(3^n)]u(n)$

$= 3(2^n)u(n) - 4(3^n)u(n)$

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$$

Time shifting:-

if $x(n) \xrightarrow{z} X(z)$

$x(n-k) \xrightarrow{z} z^{-k}X(z)$

$\delta(n) \xrightarrow{z} 1$

$\delta(n-k) \longrightarrow z^{-k}$

Ex:- 3.2.3 — Hw.

• Ex: 3.2.4 / $x(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0 & \text{elsewhere} \end{cases}$

$$\boxed{\begin{array}{l} \text{finite} = \frac{1-a^N}{1-a} \\ \text{Infinite} = \frac{1}{1-a} \end{array}}$$

$$X(z) = \sum_{n=0}^{N-1} 1.z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots z^{-(N-1)}$$

$$= \frac{1-z^{-N}}{1-z^{-1}}$$

Find the z transform:-

$x(n) = u(n) - u(n-N)$

$Z(z) = Z[u(n) - u(n-N)]$

$= Z[u(n)] - Z[u(n-N)]$

$$= \frac{1}{1-z^{-1}} - z^{-N}\frac{1}{1-z^{-1}}$$

$$= (1-z^{-N}) \times \frac{1}{1-z^{-1}}$$

Differentiation in the z-domain

if $x(n) \xleftrightarrow{z} X(z)$

then $nx(n) \xleftrightarrow{z} -z \dfrac{dX(z)}{dz}$

Proof — H.W.

Ex-3.2.7 $x(n) = na^n u(n) \longrightarrow \dfrac{1}{1-az^{-1}}$

$$= -z \frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right] \quad (1-az^{-1})^{-1}$$

$$= -z \cdot \frac{-az^{-2}}{(1-az^{-1})^2} = \frac{az^{-1}}{(1-az^{-1})^2}$$

Convolution of two sequences:-

(Z-t এ মধ্যে convolution করা যায় & faster)

if $x_1(n) \longrightarrow X_1(z)$

$x_2(n) \longrightarrow X_2(z)$

then $x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z) \cdot X_2(z)$

inverse $x_1(n) * x_2(n)$

Ex-3.2.9] $x_1(n) = \{1, -2, 1\}$

$x_2(n) = \{1, 1, 1, 1, 1, 1\}$

compute the convolution $x(n)$ of the signal.

$X_1(z) = 1 - 2z^{-1} + z^{-2}$

$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

$X(z) = X_1(z) \cdot X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$

$x(n) = \{1, -1, 0, 0, 0, 0, -1, 1\}$

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Steps for computing convolution using z-transform

1) Complete the z-transform of the signals to be convolved.

$$X_1(z) = Z\{x_1(n)\}$$
$$X_2(z) = Z\{x_2(n)\}$$

2) Multiply the two z transforms.

$$X(z) = X_1(z)\, X_2(z)$$

3) Find the inverse z-transforms of $X(z)$

$$x(n) = Z^{-1}\{X(z)\}$$

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3.3 Rational z-Transform

Polinomial

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

roots

$$= C\,\frac{(z-z_1)(z-z_2)\dots(z-z_M)}{(z-p_1)(z-p_2)\dots(z-p_N)}$$

2nd order
$ax^2 + bx + c = 0$
$(x-x_1)(x-x_2) = 0$
$x = x_1$, $x = x_2$

roots

$$X(z) = C\,\frac{(z-z_1)(z-z_2)\dots(z-z_M)}{(z-p_1)(z-p_2)\dots(z-p_N)}$$

Poles & Zeros

The zeros of z-transforms $X(z)$ are the values of z for which

$X(z)=0$. [Zeros:- $z_1, z_2, z_3 \dots z_m$]
↳ roots of the numerator (লব)

The poles of a z-transforms are the values of z for which

$X(z)=\infty$. [Poles:- $P_1, P_2, P_3 \dots P_N$]
↳ roots of the denominator (হর)

Ex.3.3.1/ Determine the pole-zero plot for the signal

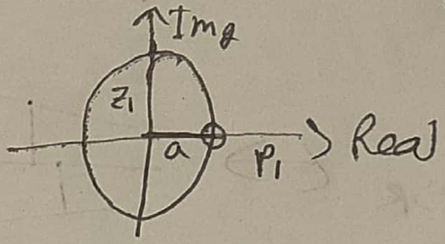
$x(n)=a^n u(n)$

Sln: $X(z)=\dfrac{1}{1-az^{-1}} = \dfrac{1}{1-\frac{a}{z}} = \dfrac{1}{\frac{z-a}{z}} = \dfrac{z}{z-a} = \dfrac{z-0}{z-a}$

↓

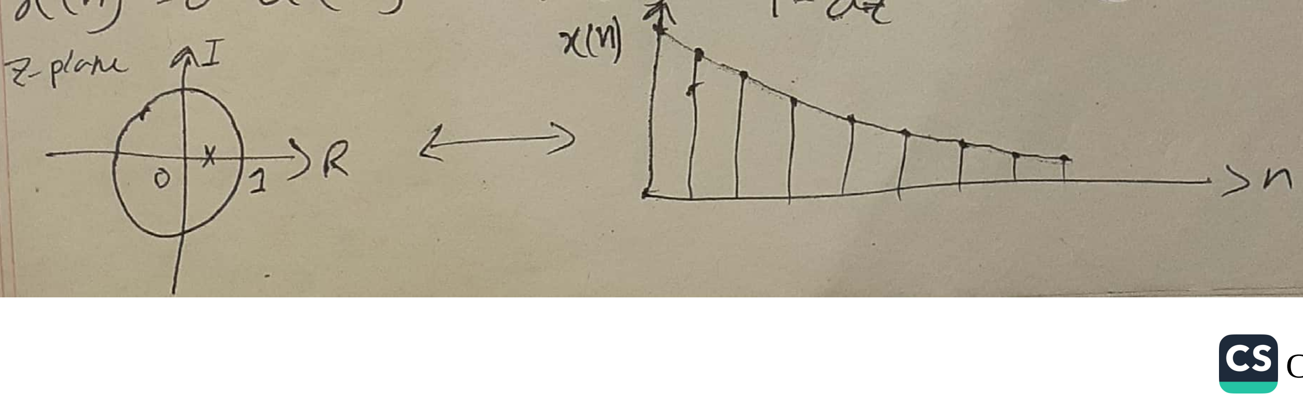
Zero:- $Z_1=0$

Pole:- $P_1=a$

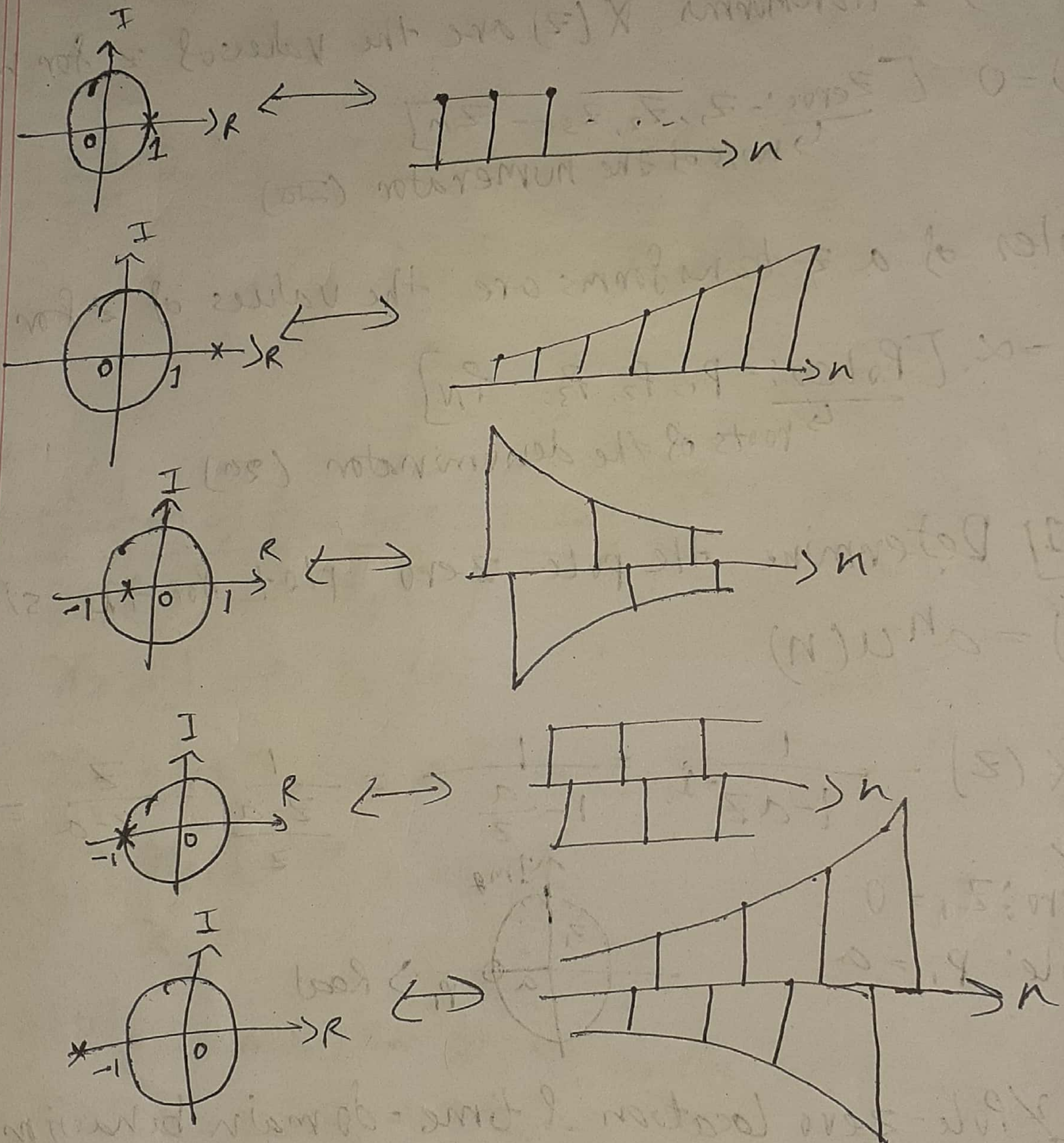


Ex-3.3.2/ Pole-zero location & time-domain behavior for causal signal

$x(n)=a^n u(n) \longleftrightarrow X(z)=\dfrac{1}{1-az^{-1}} = \dfrac{z}{z-a}$



$$x(n) = a^n u(n) \longleftrightarrow X(z) = \frac{z}{z-a}$$



3.3.3/ The system function of an LTI system

$y(n) = x(n) * h(n)$ [time-domain → after convolution

Z-domain → after multiplication

$Y(z) = X(z) \cdot H(z)$

$H(z) = \frac{Y(z)}{X(z)}$ (Y(z) → output, X(z) → input)

System function :- Use when we know input & output. But D.K SF.

Ex-3.3.4 Determine the system function & the unit sample response

$y(n) = \frac{1}{2} y(n-1) + 2x(n)$

$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$

$2X(z) = Y(z) [1 - \frac{1}{2} z^{-1}]$

$\frac{Y(z)}{X(z)} = \frac{2}{1-\frac{1}{2}z^{-1}}$

$\Rightarrow H(z) = \frac{2}{1-\frac{1}{2}z^{-1}}$

$h(n) = 2(\frac{1}{2})^n u(n)$

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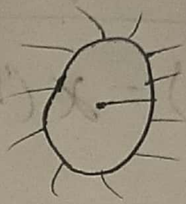
Inverse Z-Transform



Ex: $X(z) = \frac{1}{1-az^{-1}}$, $|z| > |a|$

$\downarrow$ inverse $\qquad \hookrightarrow$ R.H.S (causal)

$x(n) = a^n u(n) \;/\; a^n u(-n-1)$

anti-causal

The inverse z-Transform by power series expression

Ex-3.4.2 Determine the inverse z-transformation of

$$\boxed{X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}} \rightarrow z^{-1}+z^{-2}+z^{-3}$$

When a) ROC :- $|z| > 1 \rightarrow x(n)$ should be causal

b) ROC :- $|z| < 0.5 \rightarrow x(n)$ " " anti-causal.

$x(n) = \{\underline{1}, 2, 3\}$

$X(z) = 1 + 2z^{-1} + 3z^{-2}$

$x(n) = \{1, 2, \underline{3}\}$

$X(z) = z^2 + 2z^1 + 3$

a) $X(z) = \frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} \rightarrow z^{-1}+z^{-2}+z^{-3}+\ldots$

$$1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\ \overline{)\ 1\ }\quad \left(1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\right.$$

$$\underline{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

$$\frac{3}{2}z^{-1}-\frac{1}{2}z^{-2}$$

$$\underline{\frac{3}{2}z^{-1}-\frac{9}{4}z^{-2}+\frac{3}{4}z^{-3}}$$

$$\frac{7}{4}z^{-2}+\frac{3}{4}z^{-3}$$

$$x(n)=\left\{1,\frac{3}{2},\frac{7}{4},\dots\right\}$$

<u>Ex:- 3.4.8</u> $X(z)=\dfrac{1}{1-1.5z^{-1}+0.5z^{-2}}$, $x(n)=?$

a) ROC:- $|z|>1$ → causal

b) ROC:- $|z|<0.5$ → anti-causal

c) ROC:- $0.5<|z|<1$ → contain both causal & anti-causal part.

$$X(z)=\frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$=\frac{2}{1-z^{-1}}-\frac{1}{1-0.5z^{-1}}$$

$X(z)=\dfrac{1}{1-az^{-1}}$

$x(n)=a^n u(n)$

a) $x(n)=2(1)^n u(n)-(0.5)^n u(n)$

b) $x(n)=2(1)^n u(-n-1)-(0.5)^n u(-n-1)$

c) $x(n)=-2(1)^n u(-n-1)-(0.5)^n u(n)$

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$$\boxed{x(z) = \sum_{n=-\alpha}^{\alpha} x(n)\, z^{-n}}$$

① $x(n) = \{\underset{0}{\underline{1}}, 2, 5, 7, 0, 1\}$

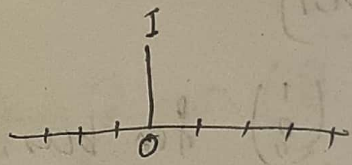
$X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$

② $x(n) = \{1, 2, \underline{5}, 7, 0, 1\}$

$X(z) = z^{2} + 2z^{1} + 5 + 7z^{-1} + z^{-3}$

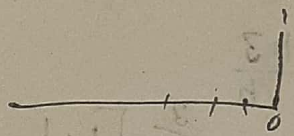
③ $x(n) = \delta(n)$

$X(z) = 1 \cdot z^{-0} = 1$



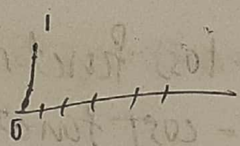
④ $x(n) = \delta(n-k)$

$X(z) = z^{-k}$



⑤ $x(n) = \delta(n+k)$

$X(z) = z^{k}$



⑥ $x(n) = a^{n} u(n)$

$X(z) = \sum_{n=-\alpha}^{\alpha} a^{n} z^{-n} = \sum_{n=-\alpha}^{\alpha} (a z^{-1})^{n}$

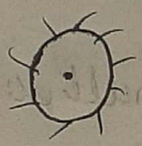
$= \frac{1}{1 - az^{-1}}, \quad |az^{-1}| < 1$

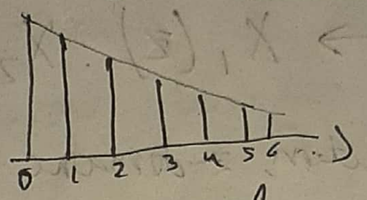
<u>ROC :-</u> $z > |a|$

⑦ $x(n) = a^n u(-n-1)$

$$X(z) = -\frac{1}{1-az^{-1}}, \quad |z| < a.$$

⑧ $x(n) = a^n u(n)$ $\quad \boxed{a<1}$

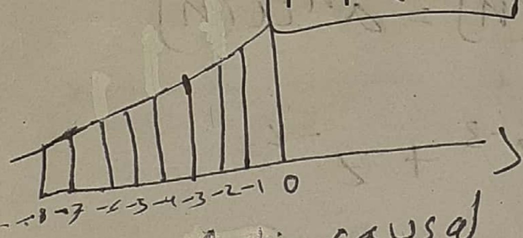




ROC exist

causal

⑨ $x(n) = a^n u(-n-1)$ $\quad \boxed{|z| < |a|}$

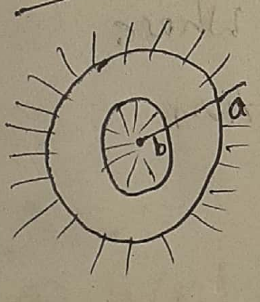


Roc :

Anti-causal

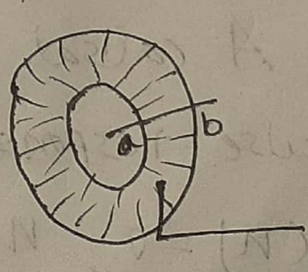
⑩ $x(n) = a^n u(n) + b^n u(-n-1)$ → common part = ROC

$a>b$



ROC doesn't exist

$a<b$



ROC exist

Convolution Property :-

$x_1(n) \xrightarrow{z} X_1(z)$

$x_2(n) \xrightarrow{z} X_2(z)$

$x_1(n) * x_2(n) \xrightarrow{z} X_1(z) \cdot X_2(z)$

side note (time domaine এ convolution, z-domaine এ multiplication এর মান)

(21) $X(z) = \frac{1}{1-az^{-1}}$ $|z| > |a|$

$x(n) = ?$, $x(n) = a^n u(n)$

(12) $X(z) = 1 + z^{-3} + z^{-6}$

$x(n) = \{1, 0, 0, 1, 0, 0, 1\}$

3.5.3 Causality & Stability

Causal :- A causal LTI system is one whose unit impulse response satisfies the condition

$h(n) = 0, \quad n < 0$

side note (for neg. n, impulse response = 0)

Stable: A stable LTI system is one whose unit impulse response satisfies the condition

$$\sum_{n=-\alpha}^{\alpha} |h(n)| < \alpha$$

Sidenote (impulse response bounded for stable)

Ex:- 3.5.2 An LTI system is characterized by the system function

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-3z^{-1}}$$

Specify the ROC of $H(z)$ & determine $h(n)$ for the following conditions:-

a) The system is stable

b) " " " causal

c) " " " anti-causal

Sln:-

(b) $h(n) = \left(\frac{1}{2}\right)^n u(n) + 2(3)^n u(n)$

(c) $h(n) = -\left(\frac{1}{2}\right)^n u(-n-1) - 2(3)^n u(-n-1)$

CamScanner

(a) $\frac{1}{2} < |z| < 3$

$h(n) = \left(\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)$

Ex:- 3.5.4 Determine the response of the system $y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n)$ to the input signal $x(n) = \delta(n) - \frac{1}{3}\delta(n-1)$

Sln:

$Y(z) = H(z) X(z)$
$\downarrow$
$y(n)$

$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n)$

$Y(z) = \frac{5}{6} z^{-1} Y(z) - \frac{1}{6} z^{-2} Y(z) + X(z)$

$Y(z) - \frac{5}{6} z^{-1} Y(z) + \frac{1}{6} z^{-2} Y(z) = X(z)$

$Y(z)\left(1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}\right) = X(z)$

$\frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$

$H(z) = \frac{1}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$

$= \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - \frac{1}{3} z^{-1}\right)}$

$X(z) = 1 - \frac{1}{3} z^{-1}$

$\therefore Y(z) = H(z)\,X(z)$

$$= \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$y(n) = \left(\frac{1}{2}\right)^n u(n)$$

* One-sided z-transform

$$X^+(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$\boxed{\begin{array}{l}\#\ \text{TT-3:-} \\ 9.5.23 \\ \text{ch-3}\end{array}}$