**Neural Network Framework: Bubble Detection using Option Prices**

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**Abstract:** *The three-step approach*(Fusari et al., 2020) *calibrates a generalised stochastic volatility jump diffusion (GSVJD) model to daily option prices, for bubble detection. Unlike traditional methods, it overcomes joint-hypothesis related issues and captures the forward-looking nature of bubbles. However, reliance on Monte Carlo simulations makes the high-dimensional calibration procedure highly inefficient. Practitioners are averse to such implementations, given the recurring re-computation needs for detecting bubbles during frequently changing market scenarios. Furthermore, compromises regarding the inclusion of options are made to accommodate for computational feasibility. Hence, neural networks were introduced as numerical solvers, in place of Monte Carlo simulation, in a two-step calibration procedure, inspired by Liu et al. (2019) and Horvath et al. (2021). The neural network framework allows for daily calibrations of the GSVJD model and subsequent bubble detection to be carried out within seconds, even with the inclusion of all traded options. The tractability and performance of the bubble detection method, along with overall option pricing, is improved, enabling for superior and efficient risk management. The benchmark three-step approach, along with its enhanced neural network versions are applied to the S&P 500 index between 2019 and 2022.*

**Keywords**: *Neural Network, Asset Price Bubbles, Stochastic Volatility Model*

1. **Introduction**

Bubbles occur when the asset price deviates from its fundamental value, capturing willingness of investors to purchase at exaggerated values, for reselling at a higher price. The inevitable collapse results in mass socioeconomic and financial losses. Recently, frequency at which exuberance occur, has significantly increased. The growing integration of global financial markets, and interlinkage across different asset classes, has vastly scaled the potential of devastation. Hence, there is need for developing a robust and accurate early method, to ensure implementation of timely damage control measures. Traditional methods suffered from joint-hypothesis issues, given their focus on modelling the fundamental value (Protter, 2013; Jarrow, 2015). The past decade, witnessed popularity amongst recursive regression techniques (Phillips et al., 2011, 2015) and the LPPLS model (Johansen et al., 1999, 2000), given their improved ability to detect and date bubbles. Nevertheless, they were also plagued by joint hypothesis issues, and the latter ignoring impacts of exogenous factors (Shu & Song, 2024). Furthermore, large timeseries data are required, creating vulnerability to structural breaks (Fusari et al., 2020). On this note, preferences shifted towards the local martingale theory of bubbles.

The theory detects short-term bubbles (Jarrow et al., 2007, 2010), reducing dependency on large timeseries data. By seeking for strict local martingale tendencies, estimation of fundamental values is made redundant (Protter, 2013). Exuberance is observed by modelling the asset’s volatility (Jarrow et al., 2011). It was not till the application of a Hidden Markov Model, in Obayashi et al. (2017), that bubbles could be dated. Time-stamping can also be achieved by calibrating stochastic volatility models to daily market observations, as seen in Chaim and Laurini (2019) and Laurini and Chaim (2021). However, calibrations to spot prices are uncapable of capturing forward looking expectations. Literature exploring option prices for identifying bubbles in the underlying price has gained traction, given that the derivative excels at revealing such expectations. The SABR model is calibrated to daily volatility smiles in Piiroinen et al. (2018) and Stahl and Blauth (2024) with the latter extending the application to the entire surface. Biagini et al. (2024) train a deep neural network to learn various stochastic volatility models for identifying bubbles in the underlying, from call option prices. Whereas, Jarrow and Kwok (2021, 2024) employ a nonparametric method to infer bubbles from option data. The three-step approach (Fusari et al., 2020), provides a solid foundation for overcoming the joint-hypothesis issues, by calibrating a generalised stochastic volatility model (GSVJD) to market put options, and using a statistical test designed to observe bubbles. Hence, it is selected as the benchmark in this paper.

The GSVJD lacks a closed form solution, forcing reliance on Monte Carlo simulation during calibrations. This created a large computational burden, and to strike balance between robustness and efficiency, Fusari et al. (2020) were restricted to calibrating parameters from the daily most liquid option smile. Consideration of various option maturities improves comprehension of forward-looking expectations, especially with regards to changes in the underlying price (Ulrich & Walther, 2020; Funahashi, 2023), and, betters the quality of bubble detection. The GSVJD is a highly sophisticated model, however, industrial application of stochastic processes reveal strong preference for tractability over accuracy (Horvath et al., 2021). Nevertheless, compromising accuracy during bubble detection can be devastating. Hence, to improve efficiency, without sacrificing accuracy, this study proposes the application of neural networks.

The universal approximation theorem (Cybenko, 1989; Hornik et al., 1989; Hornik, 1991) reveals that neural networks can estimate any function, with a given level of accuracy. During model calibrations, two approaches have been employed. First, the one-step approach (Hernandez, 2016; Stone, 2020), directly estimates parameters as outputs, from inputting market observations. Second, the two-step approach (Liu et al., 2019; Horvath et al., 2021), which trains the network to estimate market observations, and employs an optimizer for calibration. The additional step provides real-time validation to the calibrated parameters, which is satisfactory for regulatory bodies, and abides by responsible application of machine learning for sustainable business practices. In context to bubble detection, it ensures real-time testing of potential joint-hypothesis issues. The two-step calibration allows training and testing on synthetic datasets, giving the practitioner control over its size, and variations of hypothetical market regimes. It improves robustness of the network and reduces/eliminates the need for retraining, making performances immune to changing market regime, and solely dependent on the selected stochastic process.

This paper proposes a two-step neural network calibration framework for improving the efficiency of the three-step approach. It allows for calibrating the GSVJD model to the entire surface, which adds to the main contribution of this paper. The contributions are relevant for ensuring timely and appropriate implementation of measures, to reduce/curb damages from of a collapsing bubble. In the rest of the paper, Section 2 provides a theoretical background, whereas Section 3 discusses the neural network framework. Data and results from the empirical analysis are respectively, presented in Section 4 and 5, with Section 6 concluding the paper.

1. **Theoretical Background**

The local martingale theory of bubbles was developed within Loewenstein and Willard (2000), Cox and Hobson (2005), Heston et al. (2007), Jarrow et al. (2007), Jarrow et al. (2010), and Protter (2013). Consider an economy being characterized by , in a continuous time model, with interval . Markets are competitive and frictionless, comprising of a risky asset, and riskless money market account (MMA). Under No Free Lunch Vanishing Risk (NFVLR), an equivalent local martingale measure (ELMM), exists for probability measure (Delbaen & Schachermayer, 1994). The fundamental value is the price a trader is willing to pay to purchase the risky asset, and hold it till liquidation. It incorporates the expected discounted liquidation and cash flow values, under .

Bubbles exist if the market price exceeds the fundamental value. The presence of bubbles violates the put-call parity, hence requiring the No Dominance (ND) condition from Merton (1973), to be satisfied. Markets are incomplete, indicating infinite possible ELMMs. The switch in ELMMs across time gives rise to the existence and birth/restarting of bubbles. Option contracts have bounded lives and reveal type III bubbles when the underlying process exhibits strict local martingale tendencies. Through backward induction, traders realise bubbles cannot exist in assets with bounded payoffs. Hence, the fundamental value of put options is equivalent to market prices. When the underlying asset displays a bubble, put options can be priced using standard option pricing methodologies. Call options have unbounded payoffs and can inherit bubbles from the underlying, when their market price exceeds the fundamental value. Therefore, standard risk-neural measures for pricing are not favorable for pricing call options. Under the ND condition, call option and underlying price bubbles share a linear relationship, with the former signaling towards the presence of the latter.

The three-step approach (Fusari et al., 2020) detects bubbles by calibrating to market put options for computing the fundamental value of call options. Accurate calibration overcomes the joint hypothesis issue and capture forward-looking nature of bubbles. First, a pricing model must be selected, such that it admits martingale and strict local martingale properties. Considering and to denote market price of the risky asset, and strike price in units of MMA, respectively. The GSVJD process is favoured and displayed in (1) and (2), comprising of 9 parameters represented by . For , the process is a strict local martingale when . In such a scenario, explosive behaviour is present in the volatility is greater, relative to the underlying asset’s price hike. The process lacks a closed form solution, and hence calibration, in the second step, relies on Monte Carlo simulations. Parameters are calibrated by minimising (3), using Differential Evolution (Storn & Price, 1997)**,** such that market, *,* and modeled, *,*  put option implied volatilities align. Implied volatilities allow for the normalization of the objective function, and interpretation of error terms in percentage.

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|  | 3 |

Next, fundamental call values are computed using . For strike prices, , the bubble in an option is denoted by , with representing the option lot size. Given determination of the *true* market price is vague, , when , and if , or , is computed according to(4). Since there are call options, it isrecommended to take the average of across the strike prices. The average magnitude of the call option bubble, , can be interpretated as a percentage of the market price, , and lower bound for the size of exuberance in the underlying price.

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|  | 6 |
|  | 7 |

The conditional test relies of negative bubbles ( occurring due to potential model misspecifications or observation errors. The null hypothesis, , implies nonexistence of bubbles. From (*6*), if , and , then , indicating that the observed bubble is an error. On these foundations, an estimator for the unbiased variance, is constructed, using a small window. Daily bubble estimates, at observation times , where , are collected and stored in , consisting of elements. The null hypothesis is rejected, if , with latter, the time-varying threshold. Observe, is constructed using all available information on a given day, and therefore allows for real-time bubble detection. For more information on the three-step approach, the reader is referred to Fusari et al. (2020).

1. **Methodology**

To boost efficiency during GSVJD calibrations in the three-step approach, inspired by Liu et al. (2019) and Horvath et al. (2021), a Multi-Layer Perceptron (MLP) neural network is employed in a two-step calibration framework. First, in the *forward pass*,the network is trained to learn the dynamics of the GSVJD process. The architecture of an MLP network, along with mechanisms of a neuron are revealed in Figure 1, and are used to explain the training stage. Inputs () enter the network, and pass through each neuron in the hidden layers, to generate output (), by assigning weights () and bias (), such that , where represents an activation function. Activation functions play a crucial role in understanding complex nonlinear input-output dependencies. The outputs, enter the next layer as inputs, until the last layer, which provides the network output, . The aim is to minimise the error between , and the true output, by readjusting weights and biases, within the hidden layers, at each epoch.

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| --- | --- |
| A diagram of a neural network  Description automatically generated | A diagram of a function  Description automatically generated |

MLP Architecture Mechanisms of a Neuron

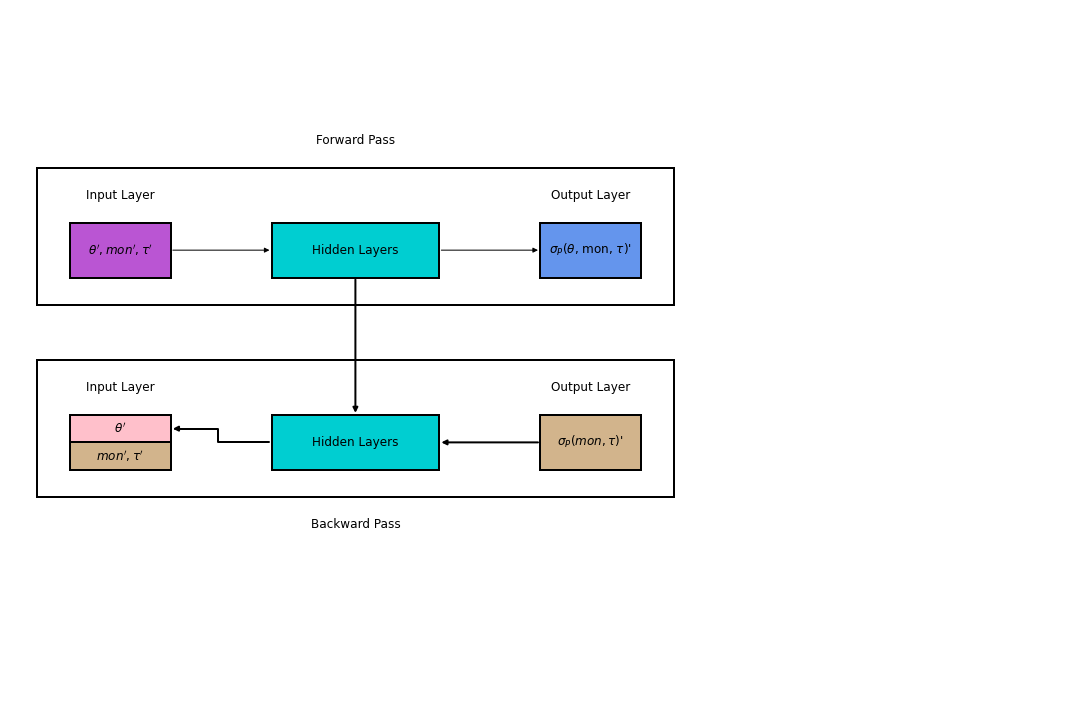
**Figure 1: Architecture of a MLP neural network, and mechanism of a neuron.**

The network is fed all components of , along with moneyness () and maturity (), as inputs, such that the dynamics of the GSVJD process are well understood to estimate put option implied volatilities, . Optimal architecture is determined by conducting a random search over 720 unique hyperparameter combinations presented in Table 2. Every combination undergoes 3-fold cross-validation training on a synthetic dataset of 1 million options, over 200 epochs. The dataset is created by using Latin Hypercube Sampling to generate input, and employing discretized versions of (1) and (2) from Fusari et al. (2020), for pricing. The mean squared error (MSE) loss function is used, but performances are also measured by analysing average RMSE and MAE metrics, across each fold. Activation function in the output layer, and optimizer are fixed to *‘linear’* and *‘Adam’*, respectively. Next, scaled inputs () and outputs () are computed, as in Horvath et al. (2021), to ensure improved convergence, and more efficient calibrations in the second step. Upon completion of the random search, the top architectures are selected for optimal training on a larger synthetic dataset, comprising of 10 million options over 200 epochs, with a scheduled learning rate, starting at and being halved after every 50 epochs. All architectures from the optimal training stage are carried to the next step, *backward pass*.

**Table 1: Hyperparameter range comprising of 720 unique architecture combinations for the random search.**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Neurons** | **Hidden Layers** | **Batch Size** | **Learning Rate** | **Activation Function** | **regularization** |
| [10,20,30,40] | [2,3, 4] | [2048,4096, 8192] | [] | ReLU, ELU | [0, ] |

The *backward pass* provides real-time validation for overcoming the joint hypothesis issue. The trained hidden layers from the *forward pass* are carried to this step, as seen in Figure 2. The weights and bias at each neuron are known, hence only matrix multiplications required to map daily to market observations. Scaled market put option implied volatilities, , are introduced as target outputs in the loss function, which is similar to (3). Parameters are obtained from estimating , such that (3) is minimized. The computations are received in the input layer nodes, corresponding to components. These nodes become learnable, whereas those assigned to relevant market information on , and, remain fixed.



**Figure 2: Illustration of the two-step approach, with the forward (top) and backward (bottom) pass phases.**

Differential evolution optimiser has been selected to minimise the loss function. Though faster, gradient-based approaches struggle with non-convex functions, due to the presence of multiple local-minima, and are unable to provide reliable performances without an initial guess. The preferred optimiser does not face such an issue, and when paired with neural networks, Liu et al. (2019) reveal a two-fold speed up being achieved. First, output corresponding to a set of parameters are computed at once. Next, at each iteration of the optimiser, generated parameter candidates enter the network at once, resulting in simultaneous output estimation. Performance of each architecture, prior to selecting one for bubble detection are assessed over market observations, using RMSE metrics. Parameters from the favoured network are then plugged into the discretized versions of (1) and (2), for computing call option prices, prior to applying the statistical test for bubble detection. Hence, the proposed two-step framework replaces Monte Carlo simulations, as a numerical solver for calibrating the GSVJD model, in the three-step approach.

1. **Data**

Empirical analysis is conducted over daily S&P 500 index European-styled options, between January 2, 2019, and December 30, 2022. Option prices were collected, corresponding to strikes at $5.00 increments. Given a potential issue regarding price asynchrony, following Fusari et al. (2020) and Almeida et al. (2021), implied spot prices were computed via the put-call parity. After obtaining forward prices, , the median of those belonging to the 5 most at-the-money (ATM) options is utilised in obtaining spot prices, , where and , respectively represent the risk-free rates and continuously compounded dividend yields. The risk-free rates are matched to the option maturities by linearly interpolating the Zero-Coupon Yield Curve, following Hagan and West (2006). Dividend yields are similarly matched by following the methodology from OptionMetrics[[1]](#footnote-1), under the assumption of the put-call parity. Call bubble magnitudes reduce with tenor of the contract, hence, aligning Fusari et al. (2020), options with days are discarded. Furthermore, to strike a balance between liquidity concerns and enhancing the capturing of exuberance in call prices, options with days were not considered. All market data was collected from Refinitiv Eikon.

**Table 2: Summary Statistics of daily S&P 500 options in the *HCV* and *Entire Surface* datasets. Volume (000s) is reported as accumulated values across daily contracts with similar . Mid-Prices are denoted in $, whereas , reveal the number of put /call options, across a single maturity, on a given day.**

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| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **Put (*HCV*)** | | | | **Put (*Entire Surface*)** | | | | **Call (*HCV*)** | | | |
|  | **Mean** | **P25** | **P50** | **P75** | **Mean** | **P25** | **P50** | **P75** | **Mean** | **P25** | **P50** | **P75** |
| **Mid** | 57.67 | 5.20 | 21.05 | 68.00 | 94.97 | 14.15 | 50.80 | 126.40 | 99.73 | 3.80 | 35.60 | 126.95 |
| **Volume** | 90.68 | 63.23 | 81.85 | 105.89 | 24.50 | 3.17 | 10.60 | 32.69 | 58.59 | 38.02 | 51.94 | 72.12 |
|  | -2.13 | -3.38 | -1.55 | -0.35 | -1.59 | -2.42 | -1.09 | -0.23 | 0.36 | -0.24 | 0.49 | 1.24 |
|  | 42.91 | 23.00 | 37.00 | 59.00 | 167.60 | 80.00 | 156.00 | 253.00 | 42.91 | 23.00 | 37.00 | 59.00 |
|  | 161.08 | 128.00 | 155.00 | 188.00 | 72.82 | 29.00 | 51.00 | 105.00 | 109.83 | 80.00 | 104.00 | 136.00 |

Highly illiquid contracts were filtered out on the grounds of moneyness, implied volatilities and trading volume (Fusari et al., 2020; Piiroinen et al., 2018; Stahl & Blauth, 2024). Non-traded options, and those with market implied volatilities greater than 100%, were discarded. Two criteria were established for moneyness[[2]](#footnote-2): standardised moneyness , and log-moneyness . Those options abiding with , and , were retained. Mid-prices were obtained from the average of bid-ask quotes. The trade-off between efficiency and robustness was balanced in Fusari et al., (2020) by calibrating to put option maturities with the highest cumulative volume (*HCV*). Despite their highly liquid nature, such contracts cover only a single slice of the volatility surface. The surface comprises a wide range of maturities, and therefore consists of excess information, which is not captured by *HCV* options. The neural network calibration framework overcomes the trade-off and makes it feasible to calibrate the GSVJD parameters from the daily entire surface (*Entire Surface*) of put options. The summary statistics of the *HCV* and *Entire Surface* datasets are revealed in Table 2. On average, the latter comprises of approximately 9 maturities, with each consisting of 72.82 contracting, such that nearly 677 options are considered during daily calibration. Given the dependence bubbles on , detection will be conducted on the *HCV* call options, using parameters calibrated from both datasets. The daily implied spot prices, for both datasets are estimated in abidance with the most liquid maturity. Finally, only call options abiding by the ND condition (Merton, 1973) are considered during the bubble estimation and testing phases.

1. **Empirical Analysis**

The random search[[3]](#footnote-3), coupled with a 3-fold cross validation, revealed preference for deeper and wider networks, with smaller batch sizes and ELU activation. Top networks were optimally trained on a larger dataset, which underwent a random 90:10 training-validation split. Performances of the top architectures from the *forward pass* are presented in Table 3. It can be witnessed that error metric from training and validation phases display convergence, reducing the likelihood of overfitting.

**Table 3: Optimal training summary of the top forward pass architectures, each uniquely identified by Arch ID.**

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Arch ID** | **Layers** | **Nodes** | **Batch Size** |  |  |  |  |  |  |  |
| 690 | 4 | 40 | 4096 | 0.007 | 0.058 | 0.007 | 0.058 | 1.22e-04 | 0.011 | 0.008 |
| 671 | 4 | 40 | 2048 | 0.007 | 0.058 | 0.007 | 0.059 | 1.25e-04 | 0.011 | 0.008 |
| 430 | 3 | 40 | 2048 | 0.007 | 0.060 | 0.007 | 0.059 | 1.29e-04 | 0.011 | 0.008 |
| 710 | 4 | 40 | 8192 | 0.007 | 0.059 | 0.007 | 0.059 | 1.29e-04 | 0.011 | 0.008 |
| 450 | 3 | 40 | 4096 | 0.007 | 0.060 | 0.007 | 0.060 | 1.33e-04 | 0.012 | 0.008 |
| 610 | 4 | 30 | 2048 | 0.007 | 0.060 | 0.007 | 0.060 | 1.33e-04 | 0.012 | 0.008 |
| 650 | 4 | 30 | 8192 | 0.007 | 0.061 | 0.007 | 0.061 | 1.38e-04 | 0.012 | 0.008 |
| 470 | 3 | 40 | 8192 | 0.008 | 0.062 | 0.008 | 0.062 | 1.40e-04 | 0.012 | 0.008 |
| 370 | 3 | 30 | 2048 | 0.008 | 0.063 | 0.008 | 0.064 | 1.46e-04 | 0.012 | 0.009 |

Each architecture was calibrated to the *HCV* dataset in the *backward pass*, and compared to the benchmark, Monte Carlo simulations[[4]](#footnote-4) in Table 4. All barring Arch ID 610 provide superior performances, with average ranging from 0.17% to 0.38%, and Arch ID 671 being superior. A computational efficiency boost, by a magnitude between and 542 is achieved[[5]](#footnote-5). A trade between accuracy and efficiency amongst the networks is observed, as faster calibrations are attributed to the optimiser being stuck at the local minima, revealing poor comprehension of the GSVJD process. Hence, justifying testing of the *backward pass* phase to market data.

**Table 4: Average performance and daily computation time (seconds) of the backward pass and benchmark.**

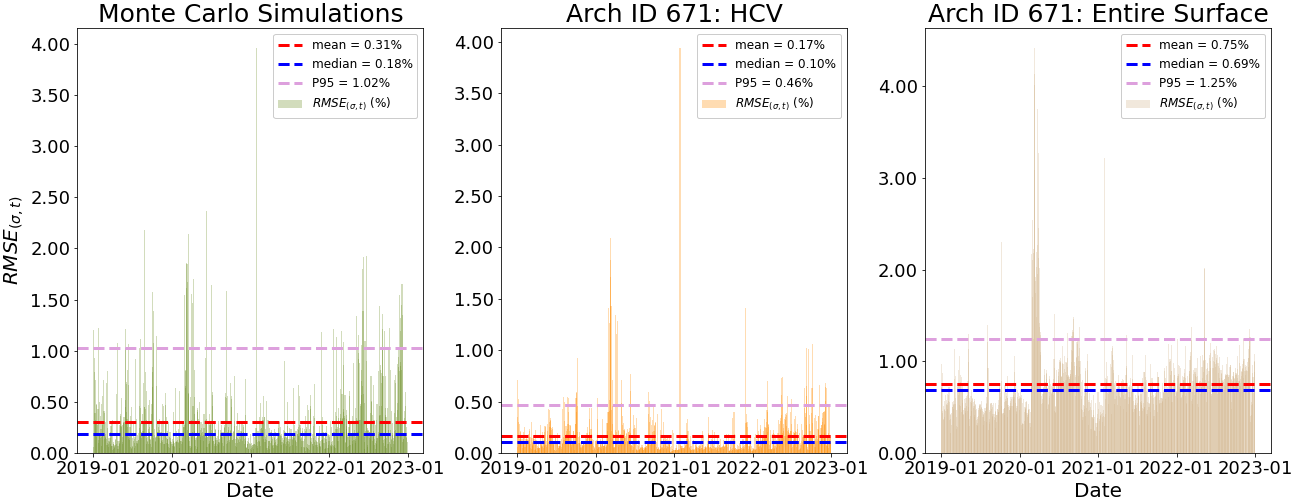
|  |  |  |
| --- | --- | --- |
| **Arch ID** |  | **Time** |
| **671** | 0.17% | 50.371 |
| **690** | 0.19% | 48.169 |
| **710** | 0.22% | 50.556 |
| **650** | 0.23% | 44.626 |
| **470** | 0.28% | 35.086 |
| **370** | 0.28% | 35.679 |
| **450** | 0.29% | 34.154 |
| **430** | 0.30% | 34.803 |
| **610** | 0.38% | 34.090 |
| **Benchmark** | 0.31% | 12.81 |

Arch ID 671 performed the best, and hence is selected for bubble detection. It improves efficiency by a magnitude of 254, and nearly doubles daily calibration accuracy, in comparison to the benchmark. As observed in Table 5, over 50% of the network values that are 3 times lower than the benchmark mean, and 75% days record metrics less than or equal to 0.19%. This documents strong capabilities of overcoming the joint-hypothesis issue. Given outperformance of the benchmark when computing daily parameters from the *HCV* dataset, , the network is employed to obtain, from *Entire Surface*. Both sets of parameters will be used to price call options for bubble detection. Average metric from *Entire Surface* is 0.75%, much greater than those observed from calibrating to the *HCV* dataset. However, the former comprises of, on average, 9 maturities each day, with several highly illiquid options being present. Overall performances are still strong, with a median of 0.69%, and metrics not exceeding 0.86% for 75% of the sample.

**Table 5: Summary Statistic for daily , and calibration time (seconds).**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | **mean** | **std** | **P25** | **P50** | **P75** |
| **Benchmark** | ***HCV*** |  | 0.31% | 0.35% | 0.12% | 0.18% | 0.33% |
| ***Time*** | 3.558 | 1.198 | 2.674 | 3.336 | 4.192 |
| **Arch ID 671** | ***HCV*** |  | 0.17% | 0.23% | 0.06% | 0.10% | 0.19% |
| ***Time*** | 50.371 | 18.154 | 36.822 | 47.972 | 60.291 |
| **Arch ID 671** | ***Entire Surface*** |  | 0.75% | 0.39% | 0.53% | 0.69% | 0.86% |
| ***Time*** | 0.013 | 0.004 | 0.010 | 0.013 | 0.016 |

Calibration performances of the network and benchmark, across the sample period are illustrated in Figure 3. Large error metrics are observed during two periods. First, aligning with the COVID-19 induced market crash (March 9 – 16, 2020), and second, January 27, 2021, corresponding to the GameStop stock short squeeze. Respectively, for , the benchmark (network) exhibits an average of 1.56% (1.29%), and 3.96% (3.94%) during these instances. Similarly, for calibrations, measures at 3.21%, on January 27, 2021, whereas between March 3, and April 4, 2020, averages at 2.19%. The metric, on March 9, 11, and 12, 2020 documented 4.01%, 4.41% and 4.11%, respectively. This is justifiable by the extremely volatile nature of markets, along with the first and last two days experiencing circuit breakers. Furthermore, the consideration of several maturities, plagued with highly illiquid options result in pricing difficulties. Nevertheless, overall performances are strong and stable, signaling overcoming of joint-hypothesis related issues during bubble detection.



**Figure 3: Daily performance of the Arch ID 671 calibration.**

Note, parametric calibrations using optimisers are vulnerable to obtaining non-unique solutions (Wshah et al., 2020). A common practice would be to use parameters from previous days as values (Büchel et al., 2022), however, this could be problematic when the market experiences extreme scenarios. Therefore, one cannot rely on the ‘’ and ‘’ conditions for detecting bubbles[[6]](#footnote-6). Instead, focus shifts to identifying call option bubbles with conditional tests, under the NFLVR and ND assumptions. Call option bubbles share a linear relationship with those in the underlying, as the magnitude of the former acts as lower bound for the latter. Hence, call options bubbles admit strict local martingale tendencies in the underlying process.

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**Figure 4: S&P 500 call option bubbles obtained from and .**

Recall, a bubble occurs when , where corresponds to the level of significance, and depends on a window of days. Bubbles detected from both and , using 180 day window, and , are displayed in Figure 4*.* The sample begins after acknowledging a burn-in period, for the initial window. Chronologically, first 3 bubbles were noticed by between February 4 and 10, 2020. Amidst concerns over an economic slowdown, due to the spread of the COVID-19 virus, following a brief dip, the S&P 500 surged with the backing of technology stocks during this period.

Next, 5 bubbles between March 13, and April 16, 2020, were spotted by , whereas revealed 8 between March 13 and April 23, 2020. A circuit breaker was triggered on March 12, 2020, as the S&P 500 witnessed its worst day since Black Monday (October 19, 1987), closing at -9.51%. The COVID-19 pandemic forced a global lockdown, and subsequent stock market crash.

Furthermore, an ongoing crude oil price war between Russia and Saudi Arabia, worsened the outlook of the global economy. Both parameter sets recorded a bubble on March 13, 2020, as the S&P 500 index closed 9.29% higher than the previous day. In the absence of the circuit breaker, the bubble may not have existed if the market price dropped to the fundamental value. This is consistent with the admissibility condition of NFLVR. Another circuit breaker was triggered on March 16, 2020, coinciding with observations from as a drop of 11.98% was experienced. Additionally, identified instances on March 17, 19, and 20, 2020 at 1%, and March 19, 2020. A highly volatile period, between March 13 to 20, 2020, witnessed daily gains of 9.29%, -11.98%, 6.00%, -5.18%, 0.47%, and -4.33%. The average over these days is 1.72%, when calibrating to the *Entire Surface* dataset. However, the detected days of exuberance are consistent with the local martingale theory of bubbles.

The S&P 500 index started recovering and eliminating its losses from the COVID-19 induced crash. Two individual bubbles were observed on June 23, and July 8, 2020, in addition to 3, between August 14 and 19, 2020, when employing . The documented occurrences on August 25, and September 30, 2020, which was followed by revealing presence on October 8, 2020. A group of 4 were discovered by between December 14, 21, 29, 2020, and January 8, 2021. During this period, spotted exuberance on December 23 and 28, 2020. The recovery was supported by major gains from technology stocks such as Meta, Apple, Amazon, Alphabet, and Microsoft, that constituted of over 20% of the S&P 500 market capitalization. Despite the economic crisis, and looming concerns over another coronavirus wave, the S&P 500 closed the year with an annual gain of 15.29%, 65.4% higher than its lowest point from March 23, 2020.

The index continued to soar through 2021, posting a record high every month since November 2020. Both parameter sets acknowledged bubbles on March 23, and April 13, 2021. Additionally, on May 14, 2021, a bubble, corresponding to a 1.5% gain in the index due to a boost from the technology sector, was observed via *HCV* calibration. September 2021 witnessed a 4.78% drop, amid growing concerns over the coronavirus delta variant, U.S. debt, and Chinese real estate market. This period also marked dips in technology, financial and energy sector stocks. Consequently, exuberance was discovered on September 7 and 10, 2021 and September 15, 2021, using and , respectively. In the same order, bubbles were spotted on October 13, and November 10, 2021. Finally, a cluster of 5 occurrences was acknowledged by the between December 14, 2021, and January 11, 2022. The bubbles on December 14, and 16, 2021 arose due to concerns over the spread of the omicron variant and weakening of tech stocks. However, the panic subdued, as research revealed lesser likelihood of hospitalization, resulting in the index bouncing back. Exuberance was noticed on December 22, whereas spotted it on the following day. The turn of the year marked the start of a downward trend in the price, with fears over an economic recession building. The observed bubbles on January 4 and 11, 2022, whereas the discovered two on January 6, and February 1, 2022. Such concerns grew throughout the year, as discovering another bubble on September 15, 2022. The final day of exuberance in the sample period was identified on December 14, 2022, according to the both parameters. It corresponded to the 0.5% rate hike introduced by the federal reserve, which resembled a slowdown of the contractionary monetary policy.

**Table 6: Statistics reported for days displaying**  **for different levels of significance (. Number of significant bubbles, and as a percentage of the sample period (excluding burn-in) are denoted by and , respectively.**

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|  | **10%** | 25.00 | 3.05% | 0.09% |
| **5%** | 17.00 | 2.07% | 0.11% |
| **1%** | 7.00 | 0.85% | 0.09% |
|  | **10%** | 24.00 | 2.93% | 1.01% |
| **5%** | 15.00 | 1.83% | 1.13% |
| **1%** | 8.00 | 0.98% | 1.32% |

Statistics for days corresponding to , are summarized in Table 6. Calibrating the neural network to the *HCV* dataset revealed 25, 17, and 7 instances of exuberance, opposed to 24, 15, and 8 from considering the *Entire Surface*, at 10%, 5%, and 1% significance, respectively. The average metrics from such days were far superior when employing the *HCV* dataset. Such performances were expected, since the *HCV* comprises of the most liquid cross section of daily options. These contracts are simpler to accurately price, given their likelihood to abide by the economic assumptions of the GSVJD. On the other hand, the *Entire Surface* considers several maturities, including illiquid ones, making daily calibration difficult.

**Table *7*: metrics corresponding to the entire sample *and* days with  *and*  .**

|  |  |  |  |
| --- | --- | --- | --- |
|  | **Entire** |  |  |
|  | 0.170% | 0.192% | 0.147% |
|  | 0.799% | 0.778% | 0.830% |

The robustness of GSVJD model calibrations to both datasets can be assessed by statistics provided in Table 7. According to the local martingale theorem of bubbles, since the fundamental value acts as the lower bound for the market price, negative bubbles do not exist. However, they can arise due to model misspecification introducing a measurement error (Fusari et al., 2020). A key component to assess the validity of the bubbles would be examine metrics on days with The instance of negative magnitudes would occur due to noise in the model, which is preferred to be smaller on days with . When calibrating the *HCV* dataset, this criteria is fulfilled. The average value reads 0.170% across the entire sample, with 0.192%, and 0.147% measured, on days corresponding to and , respectively. On the other hand, calibrations from the *Entire Surface* dataset reveal to be greater for days with . At first glance, in comparison to the *HCV*, the error metrics are noticeably worse. For the entire sample, and , the respective values are provided, 0.799%, 0.778%, and 0.830%. Notice the differences between noise occurring on days with negative and positive magnitudes are very small, and all values are less than 1%, which denote reliable calibration. Recall that on average the *Entire Surface* calibrates to daily options over 9 more maturities, comprising of several illiquid contracts, in contrast to its counterpart. Moreover, between March 13 to 20, 2020, average is 1.72%, with observing bubbles in consistency with the local martingale theory. Hence, bubbles detected from *Entire Surface* calibrations can be validated.

1. **Conclusion and Discussion**

Traditional approaches suffer from joint-hypothesis related issues, along with an inability to capture the forward-looking nature of bubbles. The three-step approach (Fusari et al., 2020) relies on option pricing to eliminate such concerns. It requires accurate calibration of the GSVJD model to market put options, prior to detecting bubbles in call options prices. Put options have bounded payoffs, and cannot display bubbles, indicating their fundamental and market values must align. Hence, accurate calibration of the GSVJD to market put observations overcomes joint-hypotheses concerns. Furthermore, options reveal market expectations regarding the underlying price on a futures date. Though delivering strong performances, calibration via Monte Carlo simulations, a major computational bottleneck was faced. A two-step neural network framework for replacing Monte Carlo simulations as a numerical solver during calibrations was proposed. Post an extensive search for the optimal architecture, a preference for deeper and wider networks, with smaller batch sizes was revealed. In comparison with the benchmark, on average the speed and accuracy of daily calibrations, approximately improved by a magnitude of 254, and 2, respectively. The neural network framework was subsequently utilised identifying exuberance in the *HCV* and *Entire Surface* datasets, corresponding to the S&P 500 index.

Both datasets revealed bubbles to occur individually, or within minor clusters, with the *HCV* being preferred for capturing the forward-looking expectations of market participants. This is supported by the detection of 3 bubbles between February 4 and 10, 2020, that expressed expectations of an anticipated, which eventually started on February 20, 2020. Contrastingly, the *Entire Surface* is favoured for capturing the immediate impact of economic phenomena, based on comprehending the prevailing market regime. As seen, it revealed bubbles on March 13 and 16, 2020, coinciding with the implementation of circuit breakers. It acknowledged 5 bubbles between March 13, and 20, 2020, corresponding to a highly volatile period. It is more consistent with NFLVR assumption, highlighting how admissible strategies can force market prices to exceed fundamental values, as reflected by exuberance on days following the circuit breakers. The lower bound for admissible trade can be interpreted as constraints enforced by regulators/traders, given rational price expectations**.**

The computational boost, during much superior calibrations from neural networks, allowed for the experimentation with the two datasets to reveal the following. It is recommended to utilize the *HCV* dataset to apprehend the forward-looking view of the market regarding a potential crash, via detected bubbles. This is a direct consequence of calibrating the most liquid cross section of daily options. However, for estimating fundamental value of a given day, and subsequently assessing for bubbles based on a better comprehension of the prevailing market regime, the *Entire Surface* is preferred. It allows for daily calibrated parameters to capture information from several volatility smiles, as opposed to one within the *HCV*. Such findings are highly relevant for risk-managers, regulators, and even investors, who seek to enhance their interpretation of the prevailing market regime.

1. **References**

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1. View IvyDB\_US\_v5.4\_Reference\_Manual. [↑](#footnote-ref-1)
2. Standardised moneyness , where , represents the implied volatility of the most at-the-money (ATM) option. Log-moneyness , with , being the forward price corresponding to the option’s maturity, . [↑](#footnote-ref-2)
3. Conducted using High Performance Computer (HPC) Clusters at University of Glasgow. [↑](#footnote-ref-3)
4. Calibrations conducted with the assistance of HPC Clusters. [↑](#footnote-ref-4)
5. All network calibrations were conducted on a personal device with an Intel-i7 processor. [↑](#footnote-ref-5)
6. These observations are consistent with Fusari et al. (2020), who reveal these parameters to have a positive effect on the magnitude and occurrence of bubbles. [↑](#footnote-ref-6)