ELEC70066

Advanced Applied Optimisation

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> **Definition 12.0.1: Limit of Sequence in $\mathbb{R}$**
>
> Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say
> $$\lim_{n \to \infty} s_n = s$$
> where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$
> $$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

> **Question 1**
>
> Is the set $x-\text{axis}\backslash\{\text{Origin}\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

> **Note:-**
>
> We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space

> **Claim 12.0.1** Topology
>
> Topology is cool

> **Example 12.0.1** (Open Set and Close Set)
>
> Open Set:
> - ϕ
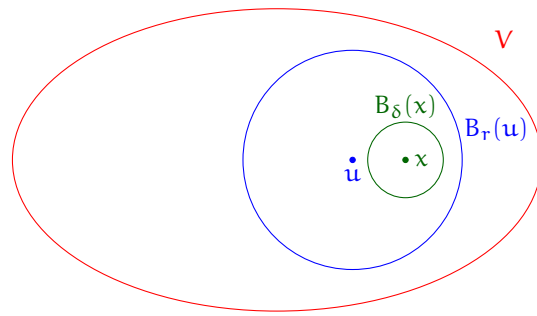> - $\bigcup_{x \in X} B_r(x)$ (Any $r > 0$ will do)
> - $B_r(x)$ is open
>
> Closed Set:
> - X, ϕ
> - $\overline{B_r(x)}$
>
> $x-\text{axis} \cup y-\text{axis}$

> **Theorem 12.0.1**
>
> If $x \in$ open set V then $\exists\, \delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V, $x \in B_r(u) \subset V$

Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leqslant d(u, x) + d(x, y) < d + \delta < r$$

☺

Corollary 12.0.1

By the result of the proof, we can then show...

Lenma 12.0.1

Suppose $v_1, \ldots, v_n \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

Proposition 12.0.1

$1 + 1 = 2$.

Random

Definition 12.0.2: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\|\cdot\| : V \to \mathbb{R}_{\geqslant 0}$ satisfying

① $\|x\| = 0 \iff x = 0 \; \forall \, x \in V$

② $\|\lambda x\| = |\lambda| \|x\| \; \forall \, \lambda \in \mathbb{R} (\text{or } \mathbb{C}), x \in V$

③ $\|x + y\| \leqslant \|x\| + \|y\| \; \forall \, x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.
• Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \to \mathbb{R}_{\geqslant 0}$) where ② becomes $\|\lambda x\| = |\lambda| \|x\|$ $\forall \, \lambda \in \mathbb{C}, x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

Example 12.0.2 (p−Norm)

$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geqslant 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_m|^p \right)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.

Special Case $p \to \infty$ ($\mathbb{R}^m$ **with** $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

> **Question 2**
>
> Prove that triangle inequality is true if $p \geqslant 1$ for $p-$norms. (What goes wrong for $p < 1$?)

Solution: **For Property ③ for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\sum_i (x_i + y_i)^2 \leqslant \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2} \right)^2$$

$$\implies \sum_i (x_i^2 + 2x_i y_i + y_i^2) \leqslant \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2$$

$$\implies \left[\sum_i x_i y_i\right]^2 \leqslant \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]$$

So in other words prove $\langle x, y \rangle^2 \leqslant \langle x, x \rangle \langle y, y \rangle$ where

$$\langle x, y \rangle = \sum_i x_i y_i$$

> **Note:-**
>
> - $\|x\|^2 = \langle x, x \rangle$
>
> - $\langle x, y \rangle = \langle y, x \rangle$
>
> - $\langle \cdot, \cdot \rangle$ is $\mathbb{R}-$linear in each slot i.e.
>
> $$\langle rx + x', y \rangle = r\langle x, y \rangle + \langle x', y \rangle \text{ and similarly for second slot}$$
>
>   Here in $\langle x, y \rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y \rangle^2 \leqslant \langle x, x \rangle \langle y, y \rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y \rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}
\langle x - \lambda y, x - \lambda y \rangle &= \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle \\
&= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle \\
&= \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle
\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y \rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y \rangle = \sum_i \overline{x_i} y_i$$

Then we still have $\langle x, x \rangle \geqslant 0$

Algorithms

Algorithm 1: what

Input: This is some input
Output: This is some output
/* This is a comment */

1 some code here;
2 $x \leftarrow 0$;
3 $y \leftarrow 0$;
4 **if** $x > 5$ **then**
5   x is greater than 5 ;   // This is also a comment
6 **else**
7   x is less than or equal to 5;
8 **end**
9 **foreach** *y in 0..5* **do**
10    $y \leftarrow y + 1$;
11 **end**
12 **for** y *in* $0..5$ **do**
13    $y \leftarrow y - 1$;
14 **end**
15 **while** $x > 5$ **do**
16    $x \leftarrow x - 1$;
17 **end**
18 **return** Return something here;