ELEC70066

Advanced Applied Optimisation

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Contents

Chapter 1

Convex Sets

Chapter 2

Convex Functions

Chapter 3

Convex Optimisation Problems

Chapter 4

Duality

Chapter 5

Approximation and Fitting

Chapter 6

Statistical Estimation

Chapter 7

Geometric Problems

Chapter 8

Interior Point Methods

Chapter 9

Integer Programming

Chapter 10

Multi Objective Programming

Chapter 11

Pareto Optimality

Chapter 12

Complexity Analysis

Random Examples

Definition 12.0.1: Limit of Sequence in $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say

$$\lim_{n\to\infty} s_n = s$$

where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$

$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Question 1

Is the set $x-$axis$\setminus\{$Origin$\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

Note:-

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space

Claim 12.0.1 Topology

Topology is cool

Example 12.0.1 (Open Set and Close Set)

Open Set:
- ϕ
- $\underset{x\in X}{} B_r(x)$ (Any $r > 0$ will do)
- $B_r(x)$ is open

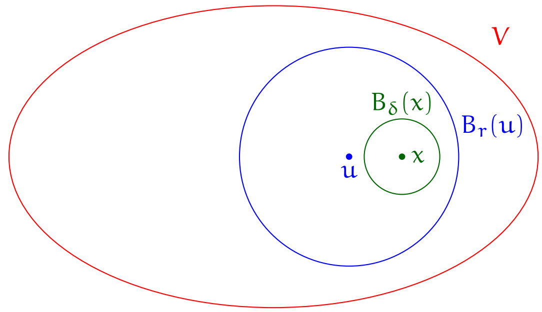
Closed Set:
- $X, \ \phi$
- $\overline{B_r(x)}$

$x-$axis $\cup$ $y-$axis

Theorem 12.0.1

If $x \in$ open set V then $\exists$ $\delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leqslant d(u, x) + d(x, y) < d + \delta < r$$

😄

Corollary 12.0.1

By the result of the proof, we can then show...

Lenma 12.0.1

Suppose $v_1, \ldots, v_n \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

Proposition 12.0.1

$1 + 1 = 2$.

Random

Definition 12.0.2: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\|\cdot\|$ $V \to \mathbb{R}_{\geqslant 0}$ satisfying

1. $\|x\| = 0 \iff x = 0 \; \forall \; x \in V$
2. $\|\lambda x\| = |\lambda| \|x\| \; \forall \; \lambda \in \mathbb{R}(\text{or } \mathbb{C}), x \in V$
3. $\|x + y\| \leqslant \|x\| + \|y\| \; \forall \; x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.
• Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \to \mathbb{R}_{\geqslant 0}$) where 2 becomes $\|\lambda x\| = |\lambda| \|x\|$ $\forall \; \lambda \in \mathbb{C}, x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

Example 12.0.2 (p—Norm)

$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geqslant 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_m|^p\right)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.

Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

Question 2

Prove that triangle inequality is true if $p \geqslant 1$ for $p-$norms. (What goes wrong for $p < 1$?)

Solution: **For Property ③ for norm-2**

When field is $\mathbb{R}$:

We have to show

$$
\begin{aligned}
&{}_i(x_i+y_i)^2 \leqslant \left(\sqrt{{}_i\, x_i^2} + \sqrt{{}_i\, y_i^2}\right)^2 \\
\implies &{}_i(x_i^2 + 2x_iy_i + y_i^2) \leqslant {}_i\, x_i^2 + 2\sqrt{\left[{}_i\, x_i^2\right]\left[{}_i\, y_i^2\right]} + {}_i\, y_i^2 \\
\implies &\left[{}_i\, x_iy_i\right]^2 \leqslant \left[{}_i\, x_i^2\right]\left[{}_i\, y_i^2\right]
\end{aligned}
$$

So in other words prove $\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$ where

$$\langle x, y\rangle = {}_i\, x_iy_i$$

Note:-

- $\|x\|^2 = \langle x, x\rangle$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle\cdot,\cdot\rangle$ is $\mathbb{R}-$linear in each slot i.e.

$$\langle rx + x', y\rangle = r\langle x, y\rangle + \langle x', y\rangle \text{ and similarly for second slot}$$

  Here in $\langle x, y\rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y\rangle$ which is going to give a quadratic equation in variable λ

$$
\begin{aligned}
\langle x - \lambda y, x - \lambda y\rangle &= \langle x, x - \lambda y\rangle - \lambda\langle y, x - \lambda y\rangle \\
&= \langle x, x\rangle - \lambda\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda^2\langle y, y\rangle \\
&= \langle x, x\rangle - 2\lambda\langle x, y\rangle + \lambda^2\langle y, y\rangle
\end{aligned}
$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y\rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y\rangle = {}_i\, \overline{x_i}y_i$$

Then we still have $\langle x, x\rangle \geqslant 0$

Algorithms

Algorithm 1: what

Input: This is some input
Output: This is some output
/* This is a comment */

some code here;
$x \leftarrow 0$;
$y \leftarrow 0$;
if $x > 5$ **then**
  x is greater than 5 ; // This is also a comment
else
  x is less than or equal to 5;
end
foreach *y in 0..5* **do**
  $y \leftarrow y + 1$;
end
for y *in* $0..5$ **do**
  $y \leftarrow y - 1$;
end
while $x > 5$ **do**
  $x \leftarrow x - 1$;
end
return Return something here;