

Linear ordinary differential equations with constant coefficients. Revisiting the impulsive response method using factorization

Roberto Camporesi

Dipartimento di Matematica, Politecnico di Torino

Corso Duca degli Abruzzi 24, 10129 Torino Italy

e-mail: camporesi@polito.it *

May 19, 2011

Abstract

We present an approach to the impulsive response method for solving linear constant-coefficient ordinary differential equations based on the factorization of the differential operator. The approach is elementary, we only assume a basic knowledge of calculus and linear algebra. In particular, we avoid the use of distribution theory, as well as of the other more advanced approaches: Laplace transform, linear systems, the general theory of linear equations with variable coefficients and the variation of constants method. The approach presented here can be used in a first course on differential equations for science and engineering majors.

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doi:10.1080/0020739X.2010.543162 (<http://dx.doi.org/10.1080/0020739X.2010.543162>)

1 Introduction

Linear constant-coefficient differential equations constitute an important chapter in the theory of ordinary differential equations, also in view of their many applications in various fields of science.

In introductory courses on differential equations, the treatment of second or higher order non-homogeneous equations is usually limited to illustrating the method of undetermined coefficients. Using this, one finds a particular solution when the forcing term is a polynomial, an exponential, a sine or a cosine, or a product of terms of this kind.

It is well known that the impulsive response method gives an explicit formula for a particular solution in the more general case in which the forcing term is an arbitrary continuous function. This method is generally regarded as too difficult to implement in a first course on differential equations. Students become aware of it only later, as an application of the theory of the Laplace transform [4] or of distribution theory [7].

An alternative approach which is sometimes used consists in developing the theory of linear systems first, considering then linear equations of order n as a particular case of this theory. The problem with this approach is that one needs to “digest” the theory of linear systems, with all the issues related to the diagonalization of matrices and the Jordan form ([2], chapter 3).

Another approach is by the general theory of linear equations with variable coefficients, with the notion of Wronskian and the method of the variation of constants. This approach can be implemented also in the case of constant coefficients ([3], chapter 2). However in introductory courses, the variation of constants method is often limited to first-order equations. Moreover, this method may be very long to implement in specific calculations, even for rather simple equations. Finally, within this approach, the occurrence of the particular solution as a convolution integral is rather indirect, and appears only at the end of the theory (see, for example, [3], exercise 4 p. 89).

The purpose of these notes is to give an elementary presentation of the impulsive response method using only very basic results from linear algebra and calculus in one or many variables. We discuss in detail the case of second-order equations, but our approach can easily be generalized to linear equations of any order (see remark 5 in section 4).

Consider the second-order equation

$$y'' + ay' + by = f(x), \quad (1.1)$$

where we use the notation $y(x)$ in place of $x(t)$ or $y(t)$, $y^{(k)}$ denotes as usual the derivative of order k of y , a, b are real constants, and the *forcing term* $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function in an interval I . When $f \neq 0$ the equation is called *non-homogeneous*. When $f = 0$ we get the *associated homogeneous equation*

$$y'' + ay' + by = 0. \quad (1.2)$$

The basic tool of our investigation is the so called *impulsive response*. This is the function defined as follows. Let $p(\lambda) = \lambda^2 + a\lambda + b$ ($\lambda \in \mathbb{C}$) be the *characteristic polynomial*, and let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the roots of $p(\lambda)$ (not necessarily distinct). We define

the impulsive response $g = g_{\lambda_1 \lambda_2}$ by the following formula:

$$g_{\lambda_1 \lambda_2}(x) = e^{\lambda_2 x} \int_0^x e^{(\lambda_1 - \lambda_2)t} dt \quad (x \in \mathbb{R}).$$

It turns out that g solves the homogeneous equation (1.2) with the initial conditions

$$y(0) = 0, \quad y'(0) = 1.$$

Moreover, the impulsive response allows one to solve the non-homogeneous equation with an arbitrary continuous forcing term and with arbitrary initial conditions. Indeed, if $0 \in I$, we shall see that the general solution of (1.1) in the interval I can be written as

$$y = y_p + y_h, \tag{1.3}$$

where the function y_p is given by the *convolution integral*

$$y_p(x) = \int_0^x g(x-t)f(t) dt, \tag{1.4}$$

and solves (1.1) with trivial initial conditions at the point $x = 0$, i.e.,

$$y_p(0) = y'_p(0) = 0,$$

whereas the function

$$y_h(x) = c_0 g(x) + c_1 g'(x) \tag{1.5}$$

gives the general solution of the associated homogeneous equation (1.2) as the coefficients c_0, c_1 vary in \mathbb{R} . In other words, the two functions g, g' are linearly independent solutions of this equation and form a basis of the vector space of its solutions.

The linear relation between the coefficients c_0, c_1 in (1.5) and the initial data $b_0 = y(0) = y_h(0)$ and $b_1 = y'(0) = y'_h(0)$ is easily obtained. If we impose the initial conditions at an arbitrary point $x_0 \in I$, we can just replace \int_0^x with $\int_{x_0}^x$ in (1.4), and $g(x), g'(x)$ with $g(x - x_0), g'(x - x_0)$ in (1.5). The function y_p satisfies then $y_p(x_0) = y'_p(x_0) = 0$, and the relation between c_k and $b_k = y^{(k)}(x_0) = y_h^{(k)}(x_0)$ ($k = 0, 1$) remains the same as before.

The proof of (1.4) that we give is based on the factorization of the differential operator acting on y in (1.1) into first-order factors, along with the formula for solving first-order linear equations. It requires, moreover, the interchange of the order of integration in a double integral, that is, the Fubini theorem.

The proof is constructive in that it produces directly the particular solution y_p as a convolution integral between the impulsive response g and the forcing term f . In particular if we take $f = 0$, we get that the unique solution of the homogeneous initial value problem with all vanishing initial data is the zero function. By linearity, this implies the uniqueness of the solutions of the initial value problem (homogeneous or not) with arbitrary initial data.

In general, the factorization method provides an elementary proof of existence, uniqueness, and extendability of the solutions of a linear initial value problem with constant

coefficients (homogeneous or not). We thus obtain a foundation for a complete theory of linear constant-coefficient differential equations.

We would like to mention that the approach by factorization to linear constant-coefficient differential equations is certainly not new (see, for example, [6], and the references therein). The use of formula (1.4) for finding a particular solution of the non-homogeneous equation is also known as the Cauchy or Boole integral method, and has been discussed in general in [5]. (See also [1], for an approach using factorization.)

The plan of this paper is as follows. In section 2 we briefly review the case of first-order linear equations within the framework of the impulsive response method.

We then proceed, in section 3, with the basic case of second-order equations. Some examples are given to illustrate the method.

Finally, in section 4, we collect some general remarks, including a brief discussion of higher-order linear equations with constant coefficients.

2 First-order equations

Consider the first-order linear differential equation

$$y' + ay = f(x), \quad (2.1)$$

where $y' = \frac{dy}{dx}$, a is a real constant, and the forcing term f is a continuous function in an interval $I \subset \mathbb{R}$. It is well known that the general solution of (2.1) is given by

$$y(x) = e^{-ax} \int e^{ax} f(x) dx, \quad (2.2)$$

where $\int e^{ax} f(x) dx$ denotes the set of all primitives of the function $e^{ax} f(x)$ in the interval I (i.e., its indefinite integral).

Suppose that $0 \in I$, and consider the integral function $\int_0^x e^{at} f(t) dt$. By the Fundamental Theorem of Calculus, this is the primitive of $e^{ax} f(x)$ that vanishes at 0. The theorem of the additive constant for primitives implies that

$$\int e^{ax} f(x) dx = \int_0^x e^{at} f(t) dt + k \quad (k \in \mathbb{R}),$$

and we can rewrite (2.2) in the form

$$\begin{aligned} y(x) &= e^{-ax} \int_0^x e^{at} f(t) dt + ke^{-ax} \\ &= \int_0^x e^{-a(x-t)} f(t) dt + ke^{-ax} \\ &= \int_0^x g(x-t) f(t) dt + kg(x), \end{aligned} \quad (2.3)$$

where $g(x) = e^{-ax}$. The function g is called the *impulsive response* of the differential equation $y' + ay = 0$. It is the unique solution of the initial value problem

$$\begin{cases} y' + ay = 0 \\ y(0) = 1. \end{cases}$$

Formula (2.3) illustrates a well-known result in the theory of linear differential equations. Namely, the general solution of (2.1) is the sum of the general solution of the associated homogeneous equation $y' + ay = 0$ and of any particular solution of (2.1). In (2.3) the function

$$y_p(x) = \int_0^x g(x-t)f(t) dt \quad (2.4)$$

is the particular solution of (2.1) that vanishes at $x = 0$.

If x_0 is any point of I , it is easy to verify that

$$y(x) = \int_{x_0}^x g(x-t)f(t) dt + y_0 g(x-x_0) \quad (x \in I)$$

is the unique solution of (2.1) in the interval I that satisfies $y(x_0) = y_0$ ($y_0 \in \mathbb{R}$).

We shall now see that formula (2.4) gives a particular solution of the non-homogeneous equation also in the case of second-order linear constant-coefficient differential equations, by suitably defining the impulsive response g .

3 Second-order equations

Consider the second-order non-homogeneous linear differential equation

$$y'' + ay' + by = f(x), \quad (3.1)$$

where $y'' = \frac{d^2y}{dx^2}$, $a, b \in \mathbb{R}$, and the forcing term $f : I \rightarrow \mathbb{R}$ is a continuous function in the interval $I \subset \mathbb{R}$, i.e., $f \in C^0(I)$. For $f = 0$ we get the associated homogeneous equation

$$y'' + ay' + by = 0. \quad (3.2)$$

We will write (3.1) and (3.2) in operator form as $Ly = f(x)$ and $Ly = 0$, where L is the *linear second-order differential operator with constant coefficients* defined by

$$Ly = y'' + ay' + by,$$

for any function y at least twice differentiable.

Denoting by $\frac{d}{dx}$ the differentiation operator, we have

$$L = \left(\frac{d}{dx}\right)^2 + a \frac{d}{dx} + b. \quad (3.3)$$

L defines a map $C^2(\mathbb{R}) \rightarrow C^0(\mathbb{R})$ that to each function y at least twice differentiable over \mathbb{R} with continuous second derivative associates the continuous function Ly . The fundamental property of L is its linearity, that is,

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2,$$

$\forall c_1, c_2 \in \mathbb{R}, \forall y_1, y_2 \in C^2(\mathbb{R})$. This formula implies some important facts. First of all, if y_1 and y_2 are any two solutions of the homogeneous equation, then any linear combination $c_1y_1 + c_2y_2$ is also a solution of this equation. In other words, the set

$$V = \{y \in C^2(\mathbb{R}) : Ly = 0\}$$

is a vector space over \mathbb{R} . We shall see that this vector space has dimension two, and that the solutions of (3.2) are defined in fact on the whole of \mathbb{R} and are of class C^∞ there.

Secondly, if y_1 and y_2 are two solutions of (3.1) (in a given interval $I' \subset I$), then their difference $y_1 - y_2$ solves (3.2). It follows that if we know a particular solution y_p of the non-homogeneous equation (in an interval I'), then any other solution of (3.1) in I' is given by $y_p + y_h$, where y_h is a solution of the associated homogeneous equation. We shall see that the solutions of (3.1) are defined on the whole of the interval I in which f is continuous (and are of course of class C^2 there).

The fact that L has constant coefficients (i.e., a and b in (3.3) are constants) allows one to find explicit formulas for the solutions of (3.1) and (3.2). To this end, it is useful to consider complex-valued functions, $y : \mathbb{R} \rightarrow \mathbb{C}$. If $y = y_1 + iy_2$ (with $y_1, y_2 : \mathbb{R} \rightarrow \mathbb{R}$) is such a function, the derivative y' may be defined by linearity as $y' = y'_1 + iy'_2$. It follows that $L(y_1 + iy_2) = Ly_1 + iLy_2$. In a similar way one defines the integral of y :

$$\int y(x) dx = \int y_1(x) dx + i \int y_2(x) dx, \quad \int_c^d y(x) dx = \int_c^d y_1(x) dx + i \int_c^d y_2(x) dx.$$

The theorem of the additive constant for primitives and the Fundamental Theorem of Calculus extend to complex-valued functions.

It is then easy to verify that

$$\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x}, \quad \forall \lambda \in \mathbb{C}.$$

It follows that the complex exponential $e^{\lambda x}$ is a solution of (3.2) if and only if λ is a root of the characteristic polynomial $p(\lambda) = \lambda^2 + a\lambda + b$. Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the roots of $p(\lambda)$, so that

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2).$$

The operator L factors in a similar way as a product (composition) of first-order differential operators:

$$L = \left(\frac{d}{dx} - \lambda_1 \right) \left(\frac{d}{dx} - \lambda_2 \right). \quad (3.4)$$

Indeed we have

$$\begin{aligned} \left(\frac{d}{dx} - \lambda_1 \right) \left(\frac{d}{dx} - \lambda_2 \right) y &= \left(\frac{d}{dx} - \lambda_1 \right) (y' - \lambda_2 y) \\ &= y'' - (\lambda_1 + \lambda_2)y' + \lambda_1 \lambda_2 y, \end{aligned}$$

which coincides with Ly since $\lambda_1 + \lambda_2 = -a$, and $\lambda_1 \lambda_2 = b$. Note that in (3.4) the order with which the two factors are composed is unimportant. In other words, the two operators $\left(\frac{d}{dx} - \lambda_1 \right)$ and $\left(\frac{d}{dx} - \lambda_2 \right)$ commute:

$$\left(\frac{d}{dx} - \lambda_1 \right) \left(\frac{d}{dx} - \lambda_2 \right) = \left(\frac{d}{dx} - \lambda_2 \right) \left(\frac{d}{dx} - \lambda_1 \right).$$

The idea is now to use (3.4) to reduce the problem to first-order differential equations. It is useful to consider linear differential equations with complex coefficients, whose solutions will be, in general, complex-valued. For example the first-order homogeneous equation $y' - \lambda y = 0$ with $\lambda \in \mathbb{C}$ has the general solution

$$y(x) = k e^{\lambda x} \quad (k \in \mathbb{C}).$$

(Indeed if $y' = \lambda y$, then $\frac{d}{dx}(y(x)e^{-\lambda x}) = 0$, whence $y(x)e^{-\lambda x} = k$.) The first-order non-homogeneous equation

$$y' - \lambda y = \left(\frac{d}{dx} - \lambda\right) y = f(x) \quad (\lambda \in \mathbb{C}),$$

with complex forcing term $f : I \subset \mathbb{R} \rightarrow \mathbb{C}$ continuous in $I \ni 0$, has the general solution

$$\begin{aligned} y(x) &= e^{\lambda x} \int e^{-\lambda x} f(x) dx \\ &= e^{\lambda x} \int_0^x e^{-\lambda t} f(t) dt + k e^{\lambda x} \\ &= \int_0^x g_\lambda(x-t) f(t) dt + k g_\lambda(x) \quad (x \in I, k = y(0) \in \mathbb{C}). \end{aligned} \quad (3.5)$$

Here $g_\lambda(x) = e^{\lambda x}$ is the impulsive response of the differential operator $(\frac{d}{dx} - \lambda)$. It is the (unique) solution of $y' - \lambda y = 0$, $y(0) = 1$. Formula (3.5) can be proved as (2.3) in the real case. In particular, the solution of the first-order problem

$$\begin{cases} y' - \lambda y = f(x) \\ y(0) = 0 \end{cases}$$

($\lambda \in \mathbb{C}$) is unique and is given by

$$y(x) = \int_0^x e^{\lambda(x-t)} f(t) dt. \quad (3.6)$$

The following result gives a particular solution of (3.1) as a convolution integral.

Theorem 3.1. *Let $f \in C^0(I)$, and suppose that $0 \in I$. Then the initial value problem*

$$\begin{cases} y'' + ay' + by = f(x) \\ y(0) = 0, \quad y'(0) = 0 \end{cases} \quad (3.7)$$

has a unique solution, defined on the whole of I , and given by the formula

$$y(x) = \int_0^x g(x-t) f(t) dt \quad (x \in I), \quad (3.8)$$

where g is the function defined by

$$g(x) = \int_0^x e^{\lambda_2(x-t)} e^{\lambda_1 t} dt \quad (x \in \mathbb{R}). \quad (3.9)$$

In particular if we take $f = 0$, we get that the only solution of the homogeneous problem

$$\begin{cases} y'' + ay' + by = 0 \\ y(0) = 0, \quad y'(0) = 0 \end{cases} \quad (3.10)$$

is the zero function $y = 0$.

Proof. We rewrite the differential equation (3.1) in the form

$$\left(\frac{d}{dx} - \lambda_1\right) \left(\frac{d}{dx} - \lambda_2\right) y = f(x).$$

Letting

$$h = \left(\frac{d}{dx} - \lambda_2\right) y = y' - \lambda_2 y,$$

we see that y solves the problem (3.7) if and only if

$$h \text{ solves } \begin{cases} h' - \lambda_1 h = f(x) \\ h(0) = 0 \end{cases} \quad \text{and} \quad y \text{ solves } \begin{cases} y' - \lambda_2 y = h(x) \\ y(0) = 0. \end{cases}$$

From (3.6) we get

$$h(x) = \int_0^x e^{\lambda_1(x-t)} f(t) dt,$$

$$y(x) = \int_0^x e^{\lambda_2(x-t)} h(t) dt.$$

Substituting h from the first formula into the second, we obtain $y(x)$ as a repeated integral (for any $x \in I$):

$$y(x) = \int_0^x e^{\lambda_2(x-t)} \left(\int_0^t e^{\lambda_1(t-s)} f(s) ds \right) dt. \quad (3.11)$$

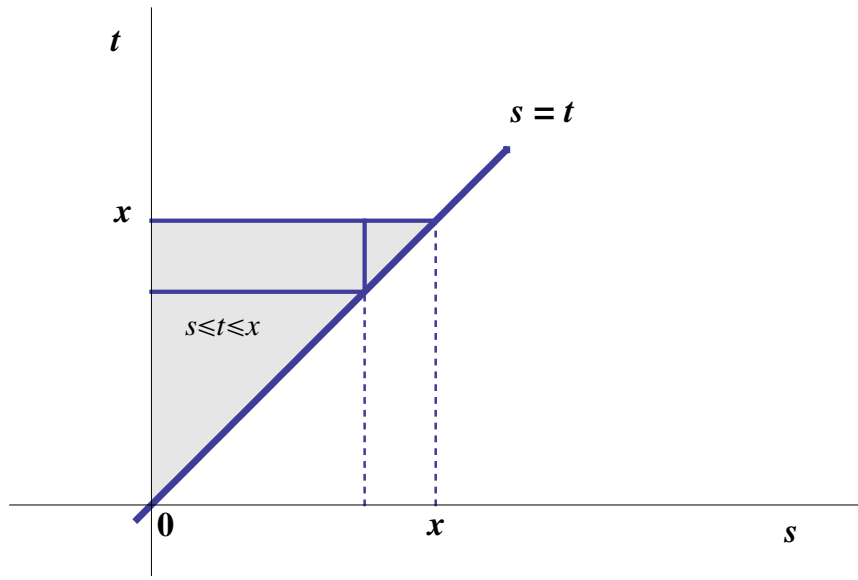
To fix ideas, let us suppose that $x > 0$. Then in the integral with respect to t we have $0 \leq t \leq x$, whereas in the integral with respect to s we have $0 \leq s \leq t$. We can then rewrite $y(x)$ as a double integral:

$$y(x) = e^{\lambda_2 x} \int_{T_x} e^{(\lambda_1 - \lambda_2)t} e^{-\lambda_1 s} f(s) ds dt,$$

where T_x is the triangle in the (s, t) plane defined by $0 \leq s \leq t \leq x$, with vertices at the points $(0, 0)$, $(0, x)$, (x, x) . In (3.11) we first integrate with respect to s and then with respect to t . Since the triangle T_x is convex both horizontally and vertically, and since the integrand function

$$F(s, t) = e^{(\lambda_1 - \lambda_2)t} e^{-\lambda_1 s} f(s)$$

is continuous in T_x , we can interchange the order of integration and integrate with respect to t first. Given s (between 0 and x) the variable t in T_x varies between s and x , see the picture below.



We thus obtain

$$y(x) = \int_0^x \left(\int_s^x e^{\lambda_2(x-t)} e^{\lambda_1(t-s)} dt \right) f(s) ds.$$

By substituting t with $t + s$ in the integral with respect to t we finally get

$$\begin{aligned} y(x) &= \int_0^x \left(\int_0^{x-s} e^{\lambda_2(x-s-t)} e^{\lambda_1 t} dt \right) f(s) ds \\ &= \int_0^x g(x-s) f(s) ds, \end{aligned} \quad (3.12)$$

which is (3.8). For $x < 0$ we can reason in a similar way and we get the same result.

The integral in formula (3.9) can be computed exactly as in the real field. We obtain the following expression of the function g :

1) if $\lambda_1 \neq \lambda_2$ ($\Leftrightarrow \Delta = a^2 - 4b \neq 0$) then

$$g(x) = \frac{1}{\lambda_1 - \lambda_2} (e^{\lambda_1 x} - e^{\lambda_2 x}); \quad (3.13)$$

2) if $\lambda_1 = \lambda_2$ ($\Leftrightarrow \Delta = 0$) then

$$g(x) = x e^{\lambda_1 x}. \quad (3.14)$$

Note that g is always a real function. Letting $\alpha = -a/2$ and

$$\beta = \begin{cases} \sqrt{-\Delta}/2 & \text{if } \Delta < 0 \\ \sqrt{\Delta}/2 & \text{if } \Delta > 0, \end{cases} \quad \text{so that } \lambda_{1,2} = \begin{cases} \alpha \pm i\beta & \text{if } \Delta < 0 \\ \alpha \pm \beta & \text{if } \Delta > 0, \end{cases}$$

we have

$$g(x) = \begin{cases} \frac{1}{2i\beta} (e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}) = \frac{1}{\beta} e^{\alpha x} \sin(\beta x) & \text{if } \Delta < 0 \\ \frac{1}{2\beta} (e^{(\alpha+\beta)x} - e^{(\alpha-\beta)x}) = \frac{1}{\beta} e^{\alpha x} \sinh(\beta x) & \text{if } \Delta > 0. \end{cases}$$

Also notice that $g \in C^\infty(\mathbb{R})$.

It is easy to check that g solves the following homogeneous initial value problem:

$$\begin{cases} y'' + ay' + by = 0 \\ y(0) = 0, \quad y'(0) = 1. \end{cases} \quad (3.15)$$

The function g is called the *impulsive response* of the differential operator L .

It is interesting to verify directly that the function y given by (3.8) solves (3.1). First let us prove the following formula for the derivative y' :

$$\begin{aligned} y'(x) &= \frac{d}{dx} \int_0^x g(x-t)f(t) dt \\ &= g(0)f(x) + \int_0^x g'(x-t)f(t) dt \quad (x \in I). \end{aligned} \quad (3.16)$$

Indeed, given h such that $x+h \in I$, we have

$$\frac{y(x+h) - y(x)}{h} = \frac{1}{h} \left(\int_0^{x+h} g(x+h-t)f(t) dt - \int_0^x g(x-t)f(t) dt \right). \quad (3.17)$$

As $g \in C^2(\mathbb{R})$, we can apply Taylor's formula with the Lagrange remainder

$$g(x_0 + h) = g(x_0) + g'(x_0)h + \frac{1}{2}g''(\xi)h^2$$

at the point $x_0 = x - t$, where ξ is some point between x_0 and $x_0 + h$. Substituting this in (3.17) and using

$$\int_0^{x+h} g(x-t)f(t) dt = \int_0^x g(x-t)f(t) dt + \int_x^{x+h} g(x-t)f(t) dt,$$

gives

$$\begin{aligned} \frac{1}{h} (y(x+h) - y(x)) &= \frac{1}{h} \int_x^{x+h} g(x-t)f(t) dt + \int_0^x g'(x-t)f(t) dt \\ &\quad + \frac{1}{2} h \int_0^{x+h} g''(\xi)f(t) dt, \end{aligned} \quad (3.18)$$

for some ξ between $x-t$ and $x-t+h$. When h tends to zero, the first term in the right-hand side of (3.18) tends to $g(0)f(x)$, by the Fundamental Theorem of Calculus. The second term in (3.18) tends to $\int_0^x g'(x-t)f(t) dt$, by the continuity of the integral function. Finally, the third term tends to zero, since the integral that occurs in it is a bounded function of h in a neighborhood of $h = 0$. (This is easy to prove.) We thus obtain formula (3.16). Recalling that $g(0) = 0$, we finally get

$$y'(x) = \int_0^x g'(x-t)f(t) dt \quad (x \in I).$$

In the same way we compute the second derivative:

$$\begin{aligned}
y''(x) &= \left(\frac{d}{dx}\right)^2 \int_0^x g(x-t)f(t) dt \\
&= \frac{d}{dx} \int_0^x g'(x-t)f(t) dt \\
&= g'(0)f(x) + \int_0^x g''(x-t)f(t) dt \\
&= f(x) + \int_0^x g''(x-t)f(t) dt,
\end{aligned}$$

where we used $g'(0) = 1$. It follows that

$$\begin{aligned}
y''(x) + ay'(x) + by(x) &= f(x) + \int_0^x (g'' + ag' + bg)(x-t) f(t) dt \\
&= f(x), \quad \forall x \in I,
\end{aligned}$$

g being a solution of the homogeneous equation. Therefore the function y given by (3.8) solves (3.1) in the interval I . The initial conditions $y(0) = 0 = y'(0)$ are immediately verified.

We now come to the solution of the initial value problem with arbitrary initial data at the point $x = 0$.

Theorem 3.2. *Let $f \in C^0(I)$, $0 \in I$, and let y_0, y'_0 be two arbitrary real numbers. Then the initial value problem*

$$\begin{cases} y'' + ay' + by = f(x) \\ y(0) = y_0, \quad y'(0) = y'_0 \end{cases} \quad (3.19)$$

has a unique solution, defined on the whole of I , and given by

$$y(x) = \int_0^x g(x-t)f(t) dt + (y'_0 + ay_0)g(x) + y_0 g'(x) \quad (x \in I). \quad (3.20)$$

In particular (taking $f = 0$), the solution of the homogeneous problem

$$\begin{cases} y'' + ay' + by = 0 \\ y(0) = y_0, \quad y'(0) = y'_0 \end{cases}$$

is unique, of class C^∞ on the whole of \mathbb{R} , and is given by

$$y_h(x) = (y'_0 + ay_0)g(x) + y_0 g'(x) \quad (x \in \mathbb{R}). \quad (3.21)$$

Proof. The uniqueness of the solutions of the problem (3.19) follows from the fact that if y_1 and y_2 both solve (3.19), then their difference $\tilde{y} = y_1 - y_2$ solves the problem (3.10), whence $\tilde{y} = 0$ by Theorem 3.1. Now notice that the function g' satisfies the homogeneous equation (like g). Indeed, since L has constant coefficients, we have

$$Lg' = L\frac{d}{dx}g = \left[\left(\frac{d}{dx}\right)^2 + a\frac{d}{dx} + b\right]\frac{d}{dx}g = \frac{d}{dx}Lg = 0.$$

By the linearity of L and by Theorem 3.1 it follows that the function y given by (3.20) satisfies $(Ly)(x) = f(x)$, $\forall x \in I$. It is immediate that $y(0) = y_0$. Finally, since

$$y'(x) = \int_0^x g'(x-t)f(t) dt + (y'_0 + ay_0)g'(x) + y_0g''(x),$$

we have

$$\begin{aligned} y'(0) &= y'_0 + ay_0 + y_0g''(0) \\ &= y'_0 + ay_0 + y_0(-a g'(0) - b g(0)) \\ &= y'_0. \end{aligned}$$

It is also possible to give a constructive proof, analogous to that of Theorem 3.1. Indeed, by proceeding as in the proof of this theorem and using (3.5), we find that y solves the problem (3.19) if and only if y is given by

$$y(x) = \int_0^x g(x-s)f(s) ds + (y'_0 - \lambda_2 y_0)g(x) + y_0e^{\lambda_2 x}.$$

This formula agrees with (3.20) in view of the equality $e^{\lambda_2 x} = g'(x) - \lambda_1 g(x)$, which follows from (3.9) by interchanging λ_1 and λ_2 . [To see that formula (3.9) is symmetric in $\lambda_1 \leftrightarrow \lambda_2$, just make the change of variables $x-t = s$ in the integral with respect to t .]

By imposing the initial conditions at an arbitrary point of the interval I , we get the following result.

Theorem 3.3. *Let $f \in C^0(I)$, $x_0 \in I$, $y_0, y'_0 \in \mathbb{R}$. The solution of the initial value problem*

$$\begin{cases} y'' + ay' + by = f(x) \\ y(x_0) = y_0, \quad y'(x_0) = y'_0 \end{cases} \quad (3.22)$$

is unique, it is defined on the whole of I , and is given by

$$y(x) = \int_{x_0}^x g(x-t)f(t) dt + (y'_0 + ay_0)g(x-x_0) + y_0g'(x-x_0) \quad (x \in I). \quad (3.23)$$

In particular (taking $f = 0$), the solution of the homogeneous problem

$$\begin{cases} y'' + ay' + by = 0 \\ y(x_0) = y_0, \quad y'(x_0) = y'_0, \end{cases}$$

with $x_0 \in \mathbb{R}$ arbitrary, is unique, of class C^∞ on the whole of \mathbb{R} , and is given by

$$y_h(x) = (y'_0 + ay_0)g(x-x_0) + y_0g'(x-x_0) \quad (x \in \mathbb{R}). \quad (3.24)$$

Proof. Let y be a solution of the problem (3.22), and let τ_{x_0} denote the translation map defined by $\tau_{x_0}(x) = x + x_0$. Set

$$\tilde{y}(x) = y \circ \tau_{x_0}(x) = y(x + x_0).$$

Since the differential operator L has constant coefficients, it is *invariant under translations*, that is,

$$L(y \circ \tau_{x_0}) = (Ly) \circ \tau_{x_0}.$$

It follows that

$$(L\tilde{y})(x) = (Ly)(x + x_0) = f(x + x_0).$$

Since moreover $\tilde{y}(0) = y(x_0) = y_0$, $\tilde{y}'(0) = y'(x_0) = y'_0$, we see that y solves the problem (3.22) in the interval I if and only if \tilde{y} solves the initial value problem

$$\begin{cases} \tilde{y}'' + a\tilde{y}' + b\tilde{y} = \tilde{f}(x) \\ \tilde{y}(0) = y_0, \quad \tilde{y}'(0) = y'_0 \end{cases}$$

in the translated interval $I - x_0 = \{t - x_0 : t \in I\}$ and with the translated forcing term $\tilde{f}(x) = f(x + x_0)$. By Theorem 3.2 we have that \tilde{y} is unique and is given by

$$\tilde{y}(x) = \int_0^x g(x-s)\tilde{f}(s)ds + (y'_0 + ay_0)g(x) + y_0g'(x).$$

Therefore y is also unique and is given by

$$y(x) = \tilde{y}(x - x_0) = \int_0^{x-x_0} g(x-x_0-s)f(s+x_0)ds + (y'_0 + ay_0)g(x-x_0) + y_0g'(x-x_0).$$

Formula (3.23) follows immediately from this by making the change of variable $s + x_0 = t$ in the integral with respect to s .

We observe that the solution in (3.23) can be written as $y = y_p + y_h$, where the function $y_p(x) = \int_{x_0}^x g(x-t)f(t)dt$ solves (3.1) with the initial conditions $y_p(x_0) = y'_p(x_0) = 0$, whereas the function y_h , given by (3.24), solves the homogeneous equation with the same initial conditions as y . We can describe this fact by saying that the non-homogeneity in the differential equation (3.1) can be dealt with separately from the initial conditions.

Corollary 3.4. *The set V of real solutions of the homogeneous equation (3.2) is a vector space of dimension 2 over \mathbb{R} , and the two functions g, g' form a basis of V .*

Proof. Let y be a real solution of (3.2). Let x_0 be any point at which y is defined, and let $y_0 = y(x_0)$, $y'_0 = y'(x_0)$. Define y_h by formula (3.24). Then y and y_h both satisfy (3.2) with the same initial conditions, whence $y = y_h$. In particular, $y \in C^\infty(\mathbb{R})$. Repeating this with $x_0 = 0$, we conclude by (3.21) that *every* element of the vector space V can be written as a linear combination of g and g' in a *unique way*. (The coefficients in this combination are uniquely determined by the initial data at the point $x = 0$.) In particular g and g' are linearly independent, and form a basis of V .

Another basis of V can be obtained from the following result.

Theorem 3.5. *Every complex solution of the homogeneous equation $Ly = 0$ can be written in the form $y = c_1y_1 + c_2y_2$ ($c_1, c_2 \in \mathbb{C}$), where y_1 and y_2 are given by*

$$(y_1(x), y_2(x)) = \begin{cases} (e^{\lambda_1 x}, e^{\lambda_2 x}) & \text{if } \lambda_1 \neq \lambda_2 \\ (e^{\lambda_1 x}, xe^{\lambda_1 x}) & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Conversely, any function of this form is a solution of $Ly = 0$.

Proof. This can be shown directly using (3.4) and arguing as in the proof of Theorem 3.1. Alternatively, we observe that any complex solution of $Ly = 0$ can be written as a complex linear combination of g and g' . (Indeed if $y = \operatorname{Re} y + i \operatorname{Im} y$ solves $Ly = 0$, then $L(\operatorname{Re} y) = 0 = L(\operatorname{Im} y)$, since L has real coefficients. By Corollary 3.4, $\operatorname{Re} y$ and $\operatorname{Im} y$ are real linear combinations of g and g' , so y is a complex linear combination of g and g' .) Our claim follows then immediately from formulas (3.13) and (3.14). The last statement is easily verified.

The two functions y_1, y_2 (like g, g') form then a basis of the complex vector space $V_{\mathbb{C}}$ of complex solutions of $Ly = 0$. If λ_1 and λ_2 are real ($\Leftrightarrow \Delta \geq 0$), then y_1 and y_2 are real functions and form a basis of V as well. If instead $\lambda_{1,2} = \alpha \pm i\beta$ with $\beta \neq 0$ ($\Leftrightarrow \Delta < 0$), then

$$y_{1,2}(x) = e^{(\alpha \pm i\beta)x} = e^{\alpha x} (\cos \beta x \pm i \sin \beta x),$$

and a basis of V is given by the functions

$$\operatorname{Re} y_1(x) = e^{\alpha x} \cos \beta x, \quad \operatorname{Im} y_1(x) = e^{\alpha x} \sin \beta x. \quad (3.25)$$

Indeed if $c_1 = c + id$ and $c_2 = c' + id'$, with $c, d, c', d' \in \mathbb{R}$, then

$$\begin{aligned} c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} &= (c + id)e^{(\alpha + i\beta)x} + (c' + id')e^{(\alpha - i\beta)x} \\ &= e^{\alpha x} \left\{ (c + c') \cos \beta x + (d' - d) \sin \beta x + i \left[(d + d') \cos \beta x + (c - c') \sin \beta x \right] \right\}. \end{aligned}$$

It follows that the function $y = c_1 y_1 + c_2 y_2$ is real-valued if and only if $c = c'$ and $d = -d'$, i.e., iff $c_1 = \overline{c_2}$. In this case y is a real linear combination of the functions in (3.25).

Example 1. Solve the initial value problem

$$\begin{cases} y'' - 2y' + y = \frac{e^x}{x+2} \\ y(0) = 0, \quad y'(0) = 0. \end{cases}$$

Solution. The characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, so that $\lambda_1 = \lambda_2 = 1$, and the impulsive response is

$$g(x) = x e^x.$$

The forcing term $f(x) = \frac{e^x}{x+2}$ is continuous for $x > -2$ and for $x < -2$. Since the initial conditions are posed at $x = 0$, we can work in the interval $I = (-2, +\infty)$. By Theorem 3.1 we get the following solution of the proposed problem in I :

$$\begin{aligned} y(x) &= \int_0^x (x-t) e^{x-t} \frac{e^t}{t+2} dt = e^x \int_0^x \frac{x-t}{t+2} dt \\ &= e^x \int_0^x \left(\frac{x+2}{t+2} - 1 \right) dt = e^x \left[(x+2) \log(t+2) - t \right]_0^x \\ &= e^x \left[(x+2) \log(x+2) - x - (x+2) \log 2 \right] \\ &= e^x (x+2) \log \left(\frac{x+2}{2} \right) - x e^x. \end{aligned}$$

Example 2. Solve the initial value problem

$$\begin{cases} y'' + y = \frac{1}{\cos x} \\ y(0) = 0, \quad y'(0) = 0. \end{cases}$$

Solution. We have $p(\lambda) = \lambda^2 + 1$, whence $\lambda_1 = \overline{\lambda_2} = i$, and the impulsive response is

$$g(x) = \sin x.$$

The initial data are given at $x = 0$ and we can work in the interval $I = (-\pi/2, \pi/2)$, where the forcing term $f(x) = \frac{1}{\cos x}$ is continuous. By formula (3.8) we get

$$\begin{aligned} y(x) &= \int_0^x \sin(x-t) \frac{1}{\cos t} dt \\ &= \int_0^x (\sin x \cos t - \cos x \sin t) \frac{1}{\cos t} dt \\ &= \sin x \int_0^x dt - \cos x \int_0^x \frac{\sin t}{\cos t} dt \\ &= x \sin x + \cos x \log(\cos x). \end{aligned}$$

Example 3. Solve the initial value problem

$$\begin{cases} y'' - y' = \frac{1}{\operatorname{ch} x} \\ y(0) = 0, \quad y'(0) = 0. \end{cases}$$

Solution. We have $p(\lambda) = \lambda^2 - \lambda = \lambda(\lambda - 1)$, thus $\lambda_1 = 1$, $\lambda_2 = 0$, and the impulsive response is

$$g(x) = e^x - 1.$$

The forcing term $f(x) = \frac{1}{\operatorname{ch} x}$ is continuous on the whole of \mathbb{R} . Formula (3.8) gives

$$y(x) = \int_0^x (e^{x-t} - 1) \frac{1}{\operatorname{ch} t} dt = e^x \int_0^x \frac{e^{-t}}{\operatorname{ch} t} dt - \int_0^x \frac{1}{\operatorname{ch} t} dt.$$

The two integrals are easily computed:

$$\begin{aligned} \int \frac{1}{\operatorname{ch} t} dt &= 2 \int \frac{1}{e^t + e^{-t}} dt = 2 \int \frac{e^t}{e^{2t} + 1} dt \\ &= 2 \arctan(e^t) + C, \\ \int \frac{e^{-t}}{\operatorname{ch} t} dt &= 2 \int \frac{1}{e^{2t} + 1} dt = 2 \int \frac{1 + e^{2t} - e^{2t}}{1 + e^{2t}} dt \\ &= 2t - \log(1 + e^{2t}) + C'. \end{aligned}$$

We finally get

$$\begin{aligned} y(x) &= e^x \left[2t - \log(1 + e^{2t}) \right]_0^x - 2 \left[\arctan(e^t) \right]_0^x \\ &= e^x \left[2x - \log(1 + e^{2x}) \right] + e^x \log 2 - 2 \arctan(e^x) + \frac{\pi}{2}. \end{aligned}$$

4 Discussion

Remark 1: the convolution. To better understand the structure of the particular solution (2.4)-(3.8), let us recall that if h_1 and h_2 are suitable functions (for example if h_1 and h_2 are piecewise continuous on \mathbb{R} with h_1 bounded and h_2 absolutely integrable over \mathbb{R} , or if h_1 and h_2 are two *signals*, that is, they are piecewise continuous on \mathbb{R} and vanish for $x < 0$), then their *convolution* $h_1 * h_2$ is defined as

$$h_1 * h_2(x) = \int_{-\infty}^{+\infty} h_1(x-t)h_2(t) dt.$$

The change of variables $x-t=s$ shows that convolution is commutative:

$$h_1 * h_2(x) = h_2 * h_1(x) = \int_{-\infty}^{+\infty} h_1(s)h_2(x-s) ds.$$

Moreover, one can show that convolution is associative: $(h_1 * h_2) * h_3 = h_1 * (h_2 * h_3)$.

If h_1 and h_2 are two signals, it is easy to verify that $h_1 * h_2$ is also a signal and that for any $x \in \mathbb{R}$ one has

$$h_1 * h_2(x) = \int_0^x h_1(x-t)h_2(t) dt.$$

In a similar way, if h_1 and h_2 vanish for $x > 0$, the same holds for $h_1 * h_2$ and we have

$$h_1 * h_2(x) = \int_x^0 h_1(x-t)h_2(t) dt.$$

If θ denotes the Heaviside step function, given by

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

and if we let

$$\tilde{\theta}(x) = -\theta(-x) = \begin{cases} 0 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0, \end{cases}$$

then for any two functions g and f we have

$$\int_0^x g(x-t)f(t) dt = \begin{cases} \theta g * \theta f(x) & \text{if } x \geq 0 \\ -\tilde{\theta}g * \tilde{\theta}f(x) & \text{if } x \leq 0. \end{cases} \quad (4.1)$$

In words, the integral $\int_0^x g(x-t)f(t) dt$ is the convolution of θg and θf for $x \geq 0$, it is the opposite of the convolution of $\tilde{\theta}g$ and $\tilde{\theta}f$ for $x \leq 0$.

Furthermore, if we denote g in (3.9) by $g_{\lambda_1 \lambda_2}$, we can rewrite (3.9) in the suggestive form

$$g_{\lambda_1 \lambda_2}(x) = \int_0^x g_{\lambda_2}(x-t)g_{\lambda_1}(t) dt = \begin{cases} \theta g_{\lambda_2} * \theta g_{\lambda_1}(x) & \text{if } x \geq 0 \\ -\tilde{\theta}g_{\lambda_2} * \tilde{\theta}g_{\lambda_1}(x) & \text{if } x \leq 0. \end{cases} \quad (4.2)$$

This formula relates the (complex) first-order impulsive responses to the impulsive response of order 2, and admits a generalization to linear equations of order n .

Finally, we observe that the proof of Theorem 3.1 can be simplified, at least at a formal level, by rewriting (3.11) in convolution form (for example for $x > 0$) and then using the associativity of this:

$$y(x) = \theta g_{\lambda_2} * (\theta g_{\lambda_1} * \theta f)(x) = (\theta g_{\lambda_2} * \theta g_{\lambda_1}) * \theta f(x). \quad (4.3)$$

This is just the equality (3.11)=(3.12) for $x > 0$. The interchange of the order of integration in the double integral considered above is then equivalent to the associative property of convolution for signals. Now (4.3) and (4.2) imply (3.8) for $x > 0$.

Remark 2: the approach by distributions. The origin of formulas (2.4)-(4.1) (or (3.8)-(4.1)) is best appreciated in the context of distribution theory. Given a linear constant-coefficient differential equation, written in the symbolic form $Ly = f(x)$, we can study the equation $LT = S$ in a suitable *convolution algebra*, for example the algebra \mathcal{D}'_+ of distributions with support in $[0, +\infty)$, which generalizes the algebra of signals. One can show that the *elementary solution* in \mathcal{D}'_+ , that is, the solution of $LT = \delta$, or equivalently, the inverse of $L\delta$ (δ the Dirac distribution), is given precisely by θg , where g is the impulsive response of the differential operator L . If y_p solves $Ly = f(x)$ with trivial initial conditions at $x = 0$, then $L(\theta y_p) = \theta f$, whence $\theta y_p = \theta g * \theta f$. This is just (2.4) for $x > 0$ and $L = \frac{d}{dx} + a$. This approach requires, however, a basic knowledge of distribution theory. (See [7], chapters II and III, for a nice presentation.)

Remark 3: existence, uniqueness and extendability of the solutions. It is well known that a linear initial value problem (with constant or variable coefficients) has a unique solution defined on the whole of the interval I in which the coefficients and the forcing term are continuous (on the whole of \mathbb{R} in the homogeneous constant-coefficient case). (See, for example, [3], Theorems 1 and 3 p.104-105.) Theorems 3.1, 3.2 and 3.3 give an elementary proof of this fact (together with an explicit formula for the solutions) for a linear second-order problem with constant coefficients.

Remark 4: complex coefficients. Theorems 3.1, 3.2, 3.3 and 3.5 remain valid if L is a linear second-order differential operator with constant *complex* coefficients, i.e., if $a, b \in \mathbb{C}$ in (3.3). In this case λ_1 and λ_2 in (3.4) are arbitrary complex numbers. The impulsive response g of L is defined again by (3.9)-(4.2) (or explicitly by formulas (3.13)-(3.14)), and solves (3.15). The solution of the initial value problem (3.22), with $a, b, y_0, y'_0 \in \mathbb{C}$ and $f : I \subset \mathbb{R} \rightarrow \mathbb{C}$ continuous in $I \ni x_0$, is given by (3.23). Finally, the set $V_{\mathbb{C}}$ of complex solutions of (3.2) is a complex vector space of dimension 2, with a basis given by $\{g, g'\}$.

Remark 5: linear equations of order n . The results obtained in section 3 for second-order equations can be generalized to linear equations of order n , by working directly with complex coefficients and using induction on n . We only give a brief outline here.

Consider the linear constant-coefficient non-homogeneous differential equation of order n

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = f(x), \quad (4.4)$$

where $y^{(k)} = \frac{d^k y}{dx^k} = \left(\frac{d}{dx}\right)^k y$, $a_1, a_2, \dots, a_n \in \mathbb{C}$, and the forcing term $f : I \subset \mathbb{R} \rightarrow \mathbb{C}$ is a continuous complex-valued function in the interval I . When $f = 0$ we obtain the associated homogeneous equation

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_{n-1} y' + a_n y = 0. \quad (4.5)$$

We can write (4.4) and (4.5) in the form $Ly = f(x)$ and $Ly = 0$, where L is the *linear differential operator of order n with constant coefficients* given by

$$L = \left(\frac{d}{dx}\right)^n + a_1 \left(\frac{d}{dx}\right)^{n-1} + \dots + a_{n-1} \frac{d}{dx} + a_n.$$

As in the case of $n = 2$, one easily proves that the complex exponential $e^{\lambda x}$, $\lambda \in \mathbb{C}$, is a solution of (4.5) if and only if λ is a root of the *characteristic polynomial*

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ be the roots of $p(\lambda)$, not necessarily all distinct, each counted with its multiplicity. The polynomial $p(\lambda)$ factors as

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

The differential operator L can be factored in a similar way as

$$L = \left(\frac{d}{dx} - \lambda_1\right) \left(\frac{d}{dx} - \lambda_2\right) \cdots \left(\frac{d}{dx} - \lambda_n\right),$$

where the order of composition of the factors is unimportant as they all commute.

We define the *impulsive response* of L , $g = g_{\lambda_1 \dots \lambda_n}$, recursively by the following formulas: for $n = 1$ we set $g_{\lambda_1}(x) = e^{\lambda_1 x}$, for $n \geq 2$ we set

$$g_{\lambda_1 \dots \lambda_n}(x) = \int_0^x g_{\lambda_n}(x-t) g_{\lambda_1 \dots \lambda_{n-1}}(t) dt \quad (x \in \mathbb{R}). \quad (4.6)$$

The impulsive response g solves the homogeneous equation (4.5) with the initial conditions

$$y(0) = y'(0) = \dots = y^{(n-2)}(0) = 0, \quad y^{(n-1)}(0) = 1.$$

It can be explicitly computed by iterating the recursive formula (4.6).

It is then easy to prove by induction on n that if $0 \in I$ and if g is the impulsive response of L , then the general solution of (4.4) in the interval I can be written in the form (1.3), where y_p is given by (1.4) and solves (4.4) with trivial initial conditions at the point $x = 0$ (i.e., $y_p^{(k)}(0) = 0$ for $k = 0, 1, \dots, n-1$), whereas the function

$$y_h(x) = \sum_{k=0}^{n-1} c_k g^{(k)}(x) \quad (4.7)$$

gives the general solution of (4.5) as the coefficients c_k vary in \mathbb{C} . Thus the n functions $g, g', g'', \dots, g^{(n-1)}$ are linearly independent solutions of this equation and form a basis of the vector space of its solutions. If L has real coefficients then g is real, and the general real solution of $Ly = 0$ is given by (4.7) with $c_k \in \mathbb{R}$.

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