DISCRETE MATHEMATICS

UNIT-1 SETS, LOGIC AND RELATION

- ## What is Discrete Mathematics?

- ✓ Discrete mathematics is mathematics that deals with discrete objects.
- ✓ Discrete objects are those which are separated from each other, which are not connected to each other.
- ✓ E.g.: - Finite mathematics, integers, people, house etc.
- ✓ The main goal of this course is to provide students with course is to provide student with an opportunity to gain an understand of theoretical foundation of computer science.
- ✓ Discrete mathematics is the study of mathematical structure that is fundamentally discrete either than continuous.
- ✓ Discrete mathematics excludes topics in continuous mathematics such as calculus of analysis.
- ✓ Set of objects studied in discrete mathematics can be finite or infinite.

- ## Define set? how it can be represented?

- ➢ **Sets: -** A set is a well defined collection of objects called elements or members of the set.
- ➢ Well defined just means that it is possible to decide if a given object belongs to the collection or not.

Representation of set

We use upper case letters such as A,B,C to denote sets and lower case letters such as a, b, c, x, y, z to denote members of a set.

We represent set in two forms: -

1. Tabular form of set: - Eg. Set v of all vowels in English

   i.e v={a, e, i ,o , u}

2. Rule method: - {x:x is an even integer, x>0}

   : denotes such

, denotes “and”

Let A= {1, 3, 5, 7} then

$1 \in A$, $3 \in A$, $2 \notin A$

• Define a universal set?

Universal set: if there are thumb sets under consideration then there happens to be a fixed set which contains each one of the given sets such a fixed set is known as a universal set. It is denoted by “U”.

If A = {1, 2, 3, 4} , B= {2, 3, 5} , C= { 3,6,7}

Then U= {1, 2, 3, 4, 5, 6, 7}

• Define Empty set?

If a set consisting No element at all is called an empty set or null set. It denotes by { }, ∅.

A = { }, ∅

• Define Subset?

Subset: - If A and B are to set such that every element in a set A is also an element of a set B then it is called a subset of B. It is Denoted by ⊆

i.e: - $A \subseteq B$

Define Powerset?

Power set: - Consider the set A= {a,b}. The Subset of A are ∅, {a}, {b} & {a,b}.

Then the family of all the subset of A is called Powerset of A, which it denoted by P(A)={ are ∅, {a}, {b} & {a,b}}.

Symbolically, P(A) = {x : x is a subset of A}

Eg, consider A=∅ then P(A) = {∅} & contain no elements.

- # Define Cardinality of a set?

  Cardinality of a set: - The number of distinct elements contained in a finite set is called the cardinality number of the set. The Cardinality of a set is denoted by various notations like, n(A) or card (A), |A|, & A.
  Eg. cardinality of empty set ∅, is 0 & is denoted by n (∅) = 0, let A = {2, 3, 4, 5, 7, 1}. Then n (A)= 5.

- # Explain Naive Set theory?

Naive Set Theory: -

- Naive set theory is one of several theories of sets used in the discussion of the foundation of mathematics.
- Unlike axiomatic set theories, which are defined using formal logic, Naive set theory is defined informally as natural language. It describes the aspects of mathematical sets familiar in discrete mathematics (for eg. venn diagram & symbolic reasoning about their Boolean algebra), I suffice for the everyday usage of set theory concept in contemporary mathematics.
- A naive Theory is considered to be a non-formalised theory, that is a theory that uses a natural language to describe sets & operations on sets.
- The words and or, if , then , not, for some for every are useful to study set naively.
- The first development of set theory was a naive set theory. It was created at the end of the 19th century by Goorg Cantor (cantorian).

- # Axiomatic theories: -

  Axiomatic set theory was developed in response to these early attempts to understand sets, with the goal of determining precisely what operations were allocated & when is allocated.

- # What are the different operations of sets?

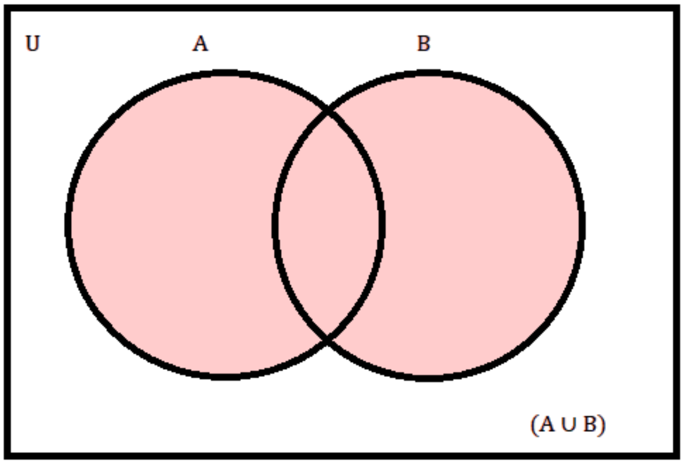
Operations on sets:-

Sets can be combined in many different ways so as to produce new sets.

1. Union of Sets: - Let A & B be two sets then the union of A & B is set of all those elemets which are neither in Set A or in Set B is set of all those element which are neither in set A or in set B or in both sets.
   Union of set A & B is denotes by AUB, which is read as ‘A union B’
   symbolically ,

$$AUB = \{x \,;\, x \in A \text{ or } X \in B\}$$

In mathematics, when we use ‘or’ i,e $x \in A$ or $e \in B$, we do not exclude the possibility that x is an element of both A & B.



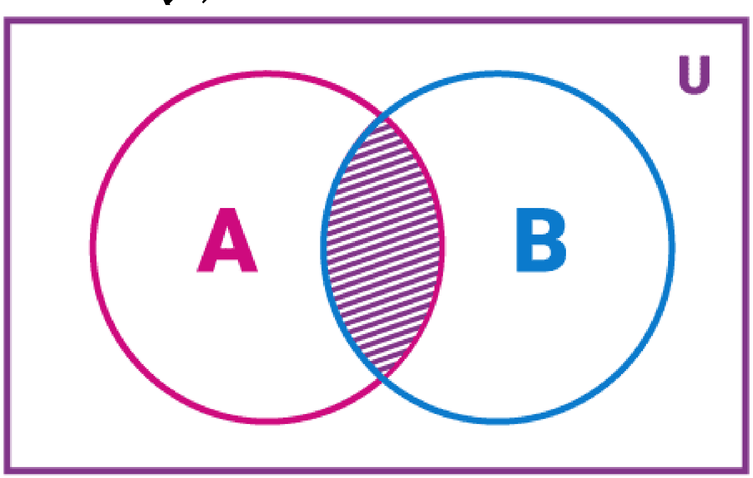
Examples: -

- A = { 1, 2, 3} , B = { x, y, z}
  AUB = {1, 2, 3, x, y, z}

- A = {1, 2, 3, 4} , B = {1, 3} , B = {1,3}, $B \leq A$
  AUB = {1, 2, 3, 4} = A

2. Intersection of sets : -
   Let A & B be two sets the intersection of A & B is the set of all elements which are in A & also B. i.e set of common elements of A & B is intersection of sets A & B. we denotes the intersection of A & B by $A \cap B$.

$A \cap B = \{ x: x \in A \text{ and } x \in B \}$



Eg .

i. $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$

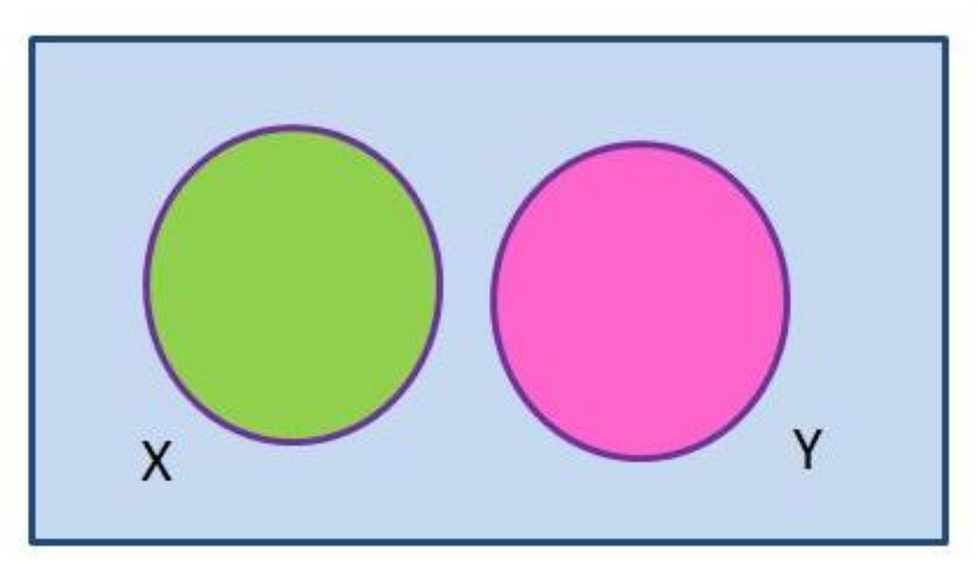
$A \cap B = \emptyset$

ii. $A = \{1, 2, 3, 4\}$, $B = \{1, 3\}$, $B \leq A$

$A \cap B = \{1, 3\} = B$

- **Define Disjoint sets?**

  Disjoint sets: - Set A & B are called disjoint set if no element is common to A and B i.e A and B are disjoint then $A \cap B = \emptyset$.



Eg .

1. $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8\}$

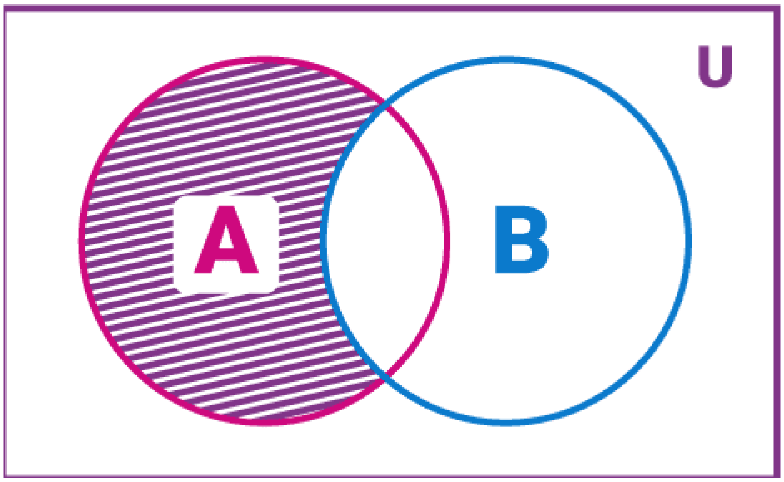
   $A \cap B = \emptyset$

- Define Difference of two sets ?

  Difference of two sets:-

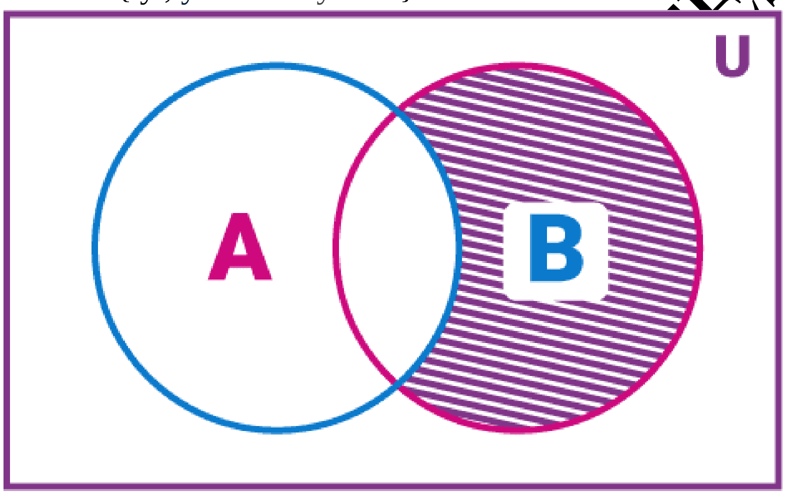
  Let A and B be two sets. The difference of A and B is denoted by 'A~B' or 'A-B' is set of all those elements of A which are not in B

$$A \sim B = \{ x ; x \in A \text{ and } x \in B \}$$



Similarly,

$B - A = \{ y ; y \in B \text{ and } y \in A \}$



Eg :- $A = \{1, 2, 3, 4\}$, $B = \{ 4, 5, 6, 7\}$

$A - B = \{ 1, 2, 3 \}$
$B - A = \{ 5, 6, 7 \}$
$A \cap B = \{ 4 \}$
$A \cup B = \{ 1, 2, 3, 4, 5, 6, 7\}$

- Explain Symmetric difference of two sets ?

  Symmetric difference of two sets:-

  Lets A & B be two sets then symmetric difference of two sets A & B is denoted by $A \triangle B$, $A \ominus B$, $A \oplus B$.

$$A \oplus B = (A - B) \cup (B - A)$$

  Hence,

  If $x \in A \oplus B$

  X is an element of exactly one of A & B or $x \in$ either A or $x \in B$ but x does not belongs to both.

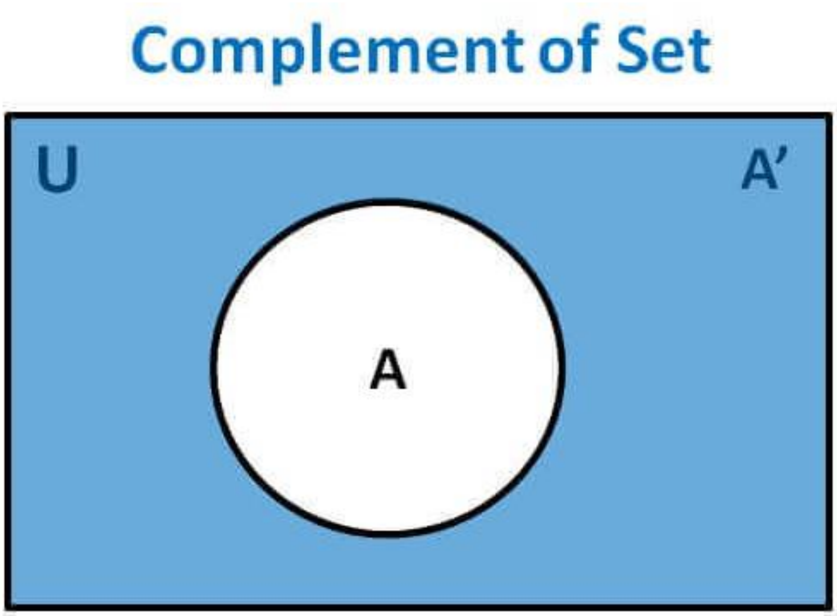
i.e $A \oplus B = (A \cup B) - (A \cap B)$

- **Define Complement of sets ?**

  Complement of sets :-

  Let A be any set, then the complement of set A is denoted by A' complement of A is set of those elements which are in universal set x but are not in set A .

$$\therefore A' = x - A$$
$$A' = \{ x : x \in X \text{ and } x \notin A \}$$



- **Explain difference types of sets?**

Types of sets: -

1. Bounded and Unbounded sets: -

We say that a subset D of the real numbers,

- Unbounded above if given any real number r we can find $d \in D$ so that $d > r$,
- Bounded above if there is a real number 0 so that $d \neq U$ for every $d \in D$, & U is said to be an upper bound for D.
- Unbounded below if given any real number r we can find $d \in D$ so that $d < r$.
- Bounded below if there is a real number L so that $d \neq L$ for every $d \in D$ and L is said to be a lower bound for D.

- Bounded if it is bounded above and below.

  For eg the +ve integers are bounded below & unbounded above. The -ve real numbers are bounded above and bounded below. If D is bounded above, we say that the real number U is the least upper bound of D if both of the following are true;

- U is an upper bound of D;
- If U’ is any other upper bound of D then U ≤ U’.

  Similarly, if D is bounded below, So any that real number L is the greatest lower bound of D if both the following are true:
    - L is an lower bound of D;
    - If L’ is any other upper bound of D then L ≤ L’.

2. Countable and countable set
   - If a rule such that it associates with element a ∈ A, one & Only one element b ∈ B Then the rule is called one to one. Also if for each element a ∈ A, There exists exactly one element a ∈ then this rule is called one to open & Onto which is also known as one to one correspondence.
   - One that is not finite is called an infinite set and is set to be countably infinite if there exists one to one correspondence between the element in the and element in N.

- Finite & Infinite set / countably finite & Uncountable infinite set
  - Countably infinity, we can make a list of its members in such a way that a list of its members in such a way that each one corresponds uniquely to a natural number.
  - A countably infinite set we also refer to as denumerable. The sets of non negative even integers are countably infinite.
  - A set which is either finite or denumerable is not countable is called countable.
  - A set which is not countably infinite is called an uncountably infinite set.

  Consider the set given below,

  A = { polygons with less than 8 sides}
  B = { The noted is western music}
  C = { even numbers between 1 & 10 }
  D = { prime numbers }
  E = { Square numbers }

  By listing the elements of the above sets, are obtain the following.

Here, the number of numbers of elements of the sets
x is denoted by n(x)

A = { triangle, Quadrilateral, pentagon, hexagon, heptagon}
n(A)= 5
B = {P, Q, R, S, T, U, V} n(B) = 7
C = { 2, 4, 6, 8} n(C) = 4
D = { 2, 3, 5, 7, 11, 13,.....} n(D) = ?
E = { 1, 4, 9, 16, 25 ,.........} n(E) = ?
If the number of elements in a 9 set is a finite set, that set is said to be a finite set.
If the number of elements in a set is a finite set that is said to be an infinite set.

- Cantor's Diagonal Argument: -
  Cantor's diagonal method is elegant powerfully & sample. It has been the course of fundamental & fruitfas theorems as devastating & intimately, fruitfas paradoxes.
- Cantor's Proof:-
  Suppose that f : N ➞[0, 1] is any function. Make a table value of f, where the 1st row contains the decimal expansion of f(1). The 2nd row contains the decimal expansion of f(2) the nth row contains the decimal expansion of f(n).

n	f(n)
1	0.3141592683...
2	0.3737373737...
3	0.1428571428...
4	0.7071067811...
5	0.3750000000...
⋮	...

The highlighted digits are 0.37210….. Suppose that we add 1 to each of these digits, to get the number 0.48321….

PROPOSITIONAL LOGIC:-

It is a language that is used for reasoning, we start with logic of sentences called propositional logic & elements of logic, relationships between propositions & reasoning.

Logic is a set of rules (axioms) which we can use to draw valid conclusions.

PROPOSITIONS: -
A propositions or statements is a declarative sentence which is either true or false but not both
Example

1. Pune is capital of india — False
2. Mars is a planet — True
3. 9>13. — False
4. 4+8=12 — Not false not true not a statement
5. Bring that book — Not false not true not a statement
6. x ∈ A — Not false not true not a statement
7. There are 12 months in a year — True

- Open Statement:-
  Sentence that contains one or more variable such that when certain values are substituted for the variables we get statements
  Example :-
  1. X + 7 = 9
  2. X + 3 = 5
  3. 3x + 2 > 8
- What types of sentence is
  1. 4 + 3 = 15
  2. 3 - 4 = 20
- What value of x following sentence will become true statement
  1. $3x + 9 = 15$ — 2
  2. $x + 6 = 8$ — 2
  3. $x + 1 > 5$ — $x \geq 5$
  4. $x + 2 < 8$ — $x < 6$ or $x \leq 5$
  5. $5x \geq 25$ — $x \geq 5$
  6. $5x \leq 25$ — $x \leq 5$
- Which of the following statements are true:-

1. $x + 4 = 6$ when $x = 2$ — True
2. $x + 4 \neq 6$ — False
3. $X + 5 \neq 8$ when $x = 3$ — False
4. $2x + 5y = 14$ when $x = 1$, $y = 3$ — True
5. $3x + 5y = 11$ when $x = 0$, $y = 2$ — False
6. $5 \in \{ 4, 2, x\}$ when $x = 5$ — True

What are different Logical Connectives?

Logical Connectives: - Every statement must be either true or false but not both.

- Compound Statement:-
  Two or more Statements can be combined to produce a new statement. These statements are called compound statements.

❖ Negation: -
“It is not the case that”
If ‘P’ denotes a statement then negation of P is denoted by “(¬P) or (~P)”

Truth Table:-

Negation:	
p	$\sim p$
T	F
F	T

P → Gopal is intelligent
Statement: - Gopal is not Intelligent
Propositional Form = ~P

Examples:-

1. If P is statement
   “I am going for a walk” then ~P is the statement
   “I am not going for a walk”
   Or
   “It is not the case that I am going for a walk”

2. If P is statement

“I like to read”
~P “I don't like to read”

- Conjunction (And or '∧')

If p and q are the statement the compound statement “p and q” is called as the IP conjunction a”

Example

Let us consider the statements

p: The sun is shining

q: The birds the shining

then p ∧ q is the statement

The sun is shining and the birds are singing.

Truth Table : -

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

- Disjunction (‘∨’ or ‘or’)

If p & q are statements then the compound statement “p or q” is called the disjunction of p and q.

Example : -

1. Consider if

   p : I will purchase a dress

   q : I will purchase a book

   then p ∨ q

   I will purchase a dress or I will purchase a book

Consider if ,

p : a is equal to 5

q : b is equal to 7

then $p \vee q$

a is equal to 5 or b is equal to 7

$p \vee q$

Truth Table: -

Disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- Conditional (“If…….then…”)

If p and q are statements then the compound statement “if p then q” denoted by $p \rightarrow q$ is called a conditional statement.

then,

Let p : Hari works hard.

q : Hari will pass the exam.

then

$p \rightarrow q$: If Hari works hard then he will pass the exam.

Truth Table

$p \rightarrow q$

Biconditional:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- Biconditional (Double Implication)

(if and only if)

If p and q are two propositions then p → q and q → p is called biconditional or double implication.

Truth Table: -

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

❖ Contrapositive of Implication

The implication ~q → ~p is called the contrapositive of the implication p→q.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

p	q	p → q	~p	~q	~p → ~q
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

p	q	p → q	~q	~p	~q → ~p
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Q. using the following statements

p: mohan is rich

q: Mohan is happy

Write the following statement in symbolic form.

1. Mohan is rich but happy = p ∧ ~q
2. Mohan is poor but happy = ~p ∧ q
3. Mohan is neither rich not happy = ~(p ∧ q) ~p ∧ ~q
4. Mohan is poor or he is both rich and unhappy = ~p ∨ (p ∧ ~q)

Q. Using the following statements

P : Rajani is tall

Q : Rajani is beautiful

Write the following statement in symbolic form

1. Rajani is tall and Beautiful. P ^ Q
2. Rajani is tall but not beautiful. P ^ ~Q
3. It is false that Rajani is short or beautiful. ~(~P ∨ Q)
4. Rajani is tall or Rajani is short and beautiful. P ∨ (~P ^ Q)

Q. Express following statements in propositional form: -

1. There were many clouds in the sky but it did not rain.

   i. There are many clouds in the sky but it did not rain

   P : There are many clouds in the sky.

   Q : It rain

   Therefore, P ^ ~ Q
2. I will get first class if and only if I study well and score above 80 in mathematics.

   P : I will get first class.

   Q : I study well.

R : score above 80 in mathematics.

Therefore, $P \leftrightarrow (Q \wedge R)$

3. Computer are cheap but softwares are costly

   P : Computers are cheap.

   Q : Software is costly.

   Therefore, $P \wedge Q$

4. It is very hot and humid or Ramesh is having heart problem.

   P : It is very hot

   Q : It is very humid

   R : Ramesh is having heart problem

   Therefore, $(P \wedge Q) \vee R$

5. In small restaurants the food is good and service is poor.

   P : In small restaurant food is good

   Q : Service is poor

   Therefore, $P \wedge Q$

Q. Using the following propositions: -

P : I am bored

Q : I am waiting for one hour

R : There is no bus.

Translate the following in english

a. $(Q \vee R) \rightarrow P$

   If I am waiting for one hour or there is no bus, then I get bored.

b. $\sim Q \rightarrow \sim P$

   If I am not waiting for one hour then I am not bored.

c. $(Q \rightarrow P) \vee (R \rightarrow P)$

   If I am waiting for one hour then I am bored, or if there is no bus then I am bored.

- Propositional Calculus

  Consider the following statement

  P : He is intelligent

  Q : He is lazy

  R : he is rich

  Write in symbolic form.

  a. He is neither intelligent nor lazy. $\sim(P \wedge \sim Q)$
  b. It is false that he is intelligent but not lazy. $\sim(P \wedge \sim P)$
  c. He is intelligent or lazy but not rich. $(P \vee Q) \wedge \sim R$

d. It is false that he is intelligent or lazy but not rich. ~((P V Q) ^~ R

e. It is not true that he is not rich. ~(~R)

f. He is rich or else he is both intelligent & lazy. R V (P ^ Q)

- Propositional Calculus: -

  Construct truth table: -

1. (~P v Q) ➞Q

P	Q	~P	~P v Q	(~P v Q)➞Q
T	T	F	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

2. (P ➞ R) ^ (P ^ Q)

P	Q	R	~P	(~P➞R)	(P v R)	(P ➞ R) ^ (P ^ Q)
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	T	F
F	F	T	T	T	F	F
F	F	F	T	F	F	F

3. If P ➞ Q is false, determine the truth table of (~P (P ^ Q) ➞ Q

P	Q	P➞Q		
T	T	T		
T	F	F		
F	T	T		
F	F	T		
P	Q	P ^ Q	~P(P ^ Q)	~P(P ^ Q)➞Q
T	F	F	T	F

4. If P and Q are false proposition find the truth value of (P v Q) ^ (~P v ~Q)

P	Q	(P v Q)	~P	~Q	~P v ~Q	(P v Q)^ (~P v ~Q)
F	F	F	T	F	T	F

- Tautology: - The tautology can be described as a compound statement, which always generates the truth value. The individual part of the statement does not affect the truth value of the tautology. The tautologies can be easily translated into mathematical expressions from the ordinary language by using logical symbols.

Q. Prove that the statement (p ➞ q) ↔ (~q ➞ ~p) is a tautology: -

P	Q	P➞ Q	~P	~Q	~Q ➞ ~P	(P ➞ Q) ↔ (~Q ➞ ~P)
T	T	T	F	F	T	T
T	F	F	F	T	G	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

As the final column contains all T's. So it is tautology.

- Contradiction: - A statement that is always false is known as a contradiction. It means it contains only F in the final column of its truth table.

Q. Show that the statement p ∧~p is a contradiction.

Solution:

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Since, the last column contains all F's, so it's a contradiction.

Contingency:

A statement that can be either true or false depending on the truth values of its variables is called a contingency.

p	q	$p \to q$	$p \wedge q$	$(p \to q) \longrightarrow (p \wedge q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

Value of the final column depends on the truth value of variables. So it is a contingency..

- Logical Equivalence:-

The proposition A and B are logically equivalent if and only if they have the same truth value for every choice of the truth values of the simple proposition involved in them.

- We denoted this fact by $A \equiv B$.
- Eg. Prove that $(P \vee Q) \wedge \sim P \equiv \sim P \wedge Q$.

P	Q	~P	P v Q	(P V Q) ^ ~P	~P ^ Q
T	T	T	F	F	F
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	F	F

From the table truth values of (P V Q) ^ ~P ≡ ~P ^ Q are the same for each choice of P and Q. Hence (P V Q) ^ ~P is equivalent to ~P ^ Q.

Logical Identities: -

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

- Applications of propositional Logic: -
  - Translation of Sentence

General Rule for Translation: -

- Look for the patterns corresponding to logical connectives in the sentence & use them to define elementary propositions.
- Eg. You can have free coffee if you are a senior citizen and it is tuesday.

A : You can have free coffee.

B : You are a senior citizen.

C : It is tuesday.

- (B ^ C) ➡ A.

- if you can study well. You will get good marks and be eligible for campus placement.

A : You can study well.

B : You will get good marks.

C : Eligible for Campus Placement.

A ➡ (B ^ C)

Mathematical Induction

Mathematical induction is a concept in mathematics that is used to prove various mathematical statements and theorems. The principle of mathematical induction is sometimes referred to as PMI. It is a technique that is used to prove the basic theorems in mathematics which involve the solution up to n finite natural terms.

Working Rule

Let n_0 be a fixed integer. Suppose P (n) is a statement involving the natural number n and we wish to prove that P (n) is true for all $n \geq n_0$.

1. Basic of Induction: P (n_0) is true i.e. P (n) is true for $n = n_0$.

2. Induction Step: Assume that the P (k) is true for n = k.
Then P (K+1) must also be true.
Then P (n) is true for all $n \geq n_0$.

Example 1:

Prove the following by Mathematical Induction:

$1 + 3 + 5 +.... + 2n - 1 = n^2$.

Solution: let us assume that.

$P(n) = 1 + 3 + 5 +..... + 2n - 1 = n^2$.

For n = 1, P (1) = 1 = 1^2 = 1
It is true for n = 1................ (i)

Induction Step: For n = r,

P (r) = 1 + 3 + 5 +..... +2r-1 = r^2 is true......................... (ii)
Adding 2r + 1 in both sides
P (r + 1) = 1 + 3 + 5 +..... +2r-1 + 2r +1
= r^2 + (2r + 1) = r^2 + 2r +1 = $(r+1)^2$..................... (iii)
As P(r) is true. Hence P (r+1) is also true.
From (i), (ii) and (iii) we conclude that.
1 + 3 + 5 +..... + 2n - 1 =n^2 is true for n = 1, 2, 3, 4, 5 ....Hence Proved.

Example 2:

$1^2 + 2^2 + 3^2 + \ldots\ldots. + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution: For n = 1,

$P(1) = 1^2 = \frac{1\,(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1$

It is true for n = 1.

Induction Step: For n = r,................... (i)

$P(r) = 1^2 + 2^2 + 3^2 + \ldots\ldots.. + r^2 = \frac{r\,(r+1)(2r+1)}{6}$ is true........... (ii)

Adding $(r+1)^2$ on both sides, we get

$P(r+1) = 1^2 + 2^2 + 3^2 + \ldots\ldots. + r^2 + (r+1)^2 = \frac{r(r+1)(2r+1)}{6} + (r+1)^2$

$= \frac{r(r+1)(2r+1)+6(r+1)^2}{6} = (r+1)\left[\frac{r(2r+1)+6(r+1)}{6}\right]$

$= \frac{(r+1)}{6}\,[r(2r+1)+6(r+1)] = \frac{(r+1)}{6}\,[2r^2+7r+6]$

$= \frac{(r+1)(r+2)(2r+3)}{6}$ .... (iii)

As P (r) is true, hence P (r+1) is true.
From (i), (ii) and (iii) we conclude that

$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ is true for n = 1, 2, 3, 4, 5 ..... Hence Proved.

Example3: Show that for any integer n
$11^{n+2} + 12^{2n+1}$ is divisible by 133.

Solution:

Let P (n) = $11^{n+2}+12^{2n+1}$
For n = 1,
P (1) = $11^3+12^3=3059=133 \times 23$
So, 133 divide P (1).................. (i)

Induction Step: For n = r,

P (r) = $11^{r+2}+12^{2r+1}=133 \times s$*............ (ii)*
Now, for n = r + 1,
P (r+1) = $11^{r+2+1}+12^{2(r)+3}=11[133s-12^{2r+1}] + 144.\ 12^{2r+1}$
$= 11 \times 133s + 12^{2r+1}.133=133[11s+12^{2r+1}]=133 \times t$*........... (iii)*
As (i), (ii), and (iii) all are true, hence P (n) is divisible by 133.

Inclusion-Exclusion Principle

The Principle of Inclusion-Exclusion (PIE) is a fundamental concept in combinatorics, often used to count the elements of the union of overlapping sets. It accounts for overcounting by first summing the sizes of individual sets, then subtracting the sizes of their pairwise intersections, adding the sizes of three-way intersections, and so on. Here's the general formula:

INCLUSION-EXCLUSION PRINCIPLE

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j|$$
$$+ \sum_{1 \leqslant i < j < k \leqslant n} |A_i \cap A_j \cap A_k| - \cdots$$
$$+ (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

Example 1:

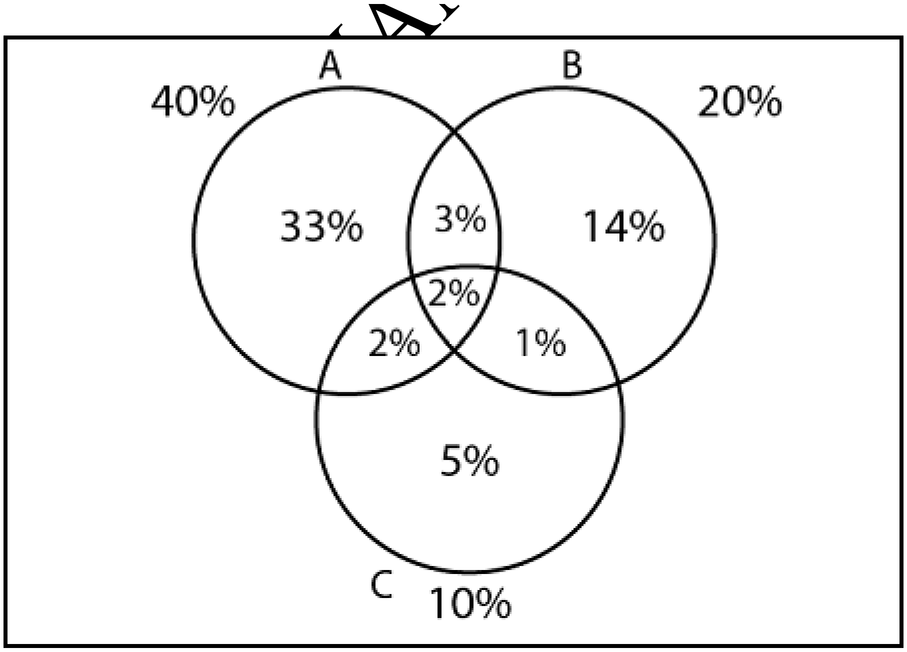
Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

Example 2:

In a town of 10000 families it was found that 40% of families buy newspaper A, 20% family buy newspaper B, 10% family buy newspaper C, 5% family buy newspaper A and B, 3% family buy newspaper B and C and 4% family buy newspaper A and C. If 2% family buy all the newspaper. Find the number of families which buy

1. *Number of families which buy all three newspapers.*
2. *Number of families which buy newspaper A only*
3. *Number of families which buy newspaper B only*
4. *Number of families which buy newspaper C only*
5. *Number of families which buy None of A, B, C*
6. *Number of families which buy exactly only one newspaper*
7. *Number of families which buy newspaper A and B only*
8. *Number of families which buy newspaper B and C only*
9. *Number of families which buy newspaper C and A only*
10. *Number of families which buy at least two newspapers*
11. *Number of families which buy at most two newspapers*
12. *Number of families which buy exactly two newspapers*

Solution:



1. *Number of families which buy all three newspapers:*

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$
$$n(A \cup B \cup C) = 40 + 20 + 10 - 5 - 3 - 4 + 2 = 60\%$$

2. *Number of families which buy newspaper A only* = 40 - 7 = 33%

3. *Number of families which buy newspaper B only* = 20 - 6 = 14%

4. *Number of families which buy newspaper C only* = 10 - 5 = 5%

5. *Number of families which buy None of A, B, and C*

$n(A \cup B \cup C)^c = 100 - n(A \cup B \cup C)$

$n(A \cup B \cup C)^c = 100 - [40 + 20 + 10 - 5 - 3 - 4 + 2]$

$n(A \cup B \cup C)^c = 100 - 60 = 40\%$

6. *Number of families which buy exactly only one newspaper* = 33 + 14 + 5 = 52%

7. *Number of families which buy newspaper A and B only* = 3%

8. *Number of families which buy newspaper B and C only* = 1%

9. *Number of families which buy newspaper C and A only* = 2%

10. *Number of families which buy at least two newspapers* = 8%

11. *Number of families which buy at most two newspapers* = 98%

12. *Number of families which buy exactly two newspapers* = 6%

The Pigeonhole Principle

If n pigeon holes are occupied by n+1 or more pigeons, then at least one pigeonhole is occupied by greater than one pigeon. Generalized pigeonhole principle is: - If n pigeon holes are occupied by kn+1 or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by k+1 or more pigeons.

Example1: Find the minimum number of students in a class to be sure that three of them are born in the same month.

Solution: Here n = 12 months are the Pigeonholes

And k + 1 = 3

K = 2

Example2: Show that at least two people must have their birthday in the same month if 13 people are assembled in a room.

Solution: We assigned each person the month of the year on which he was born. Since there are 12 months in a year.

So, according to the pigeonhole principle, there must be at least two people assigned to the same month.

Inclusion-Exclusion Principle:

Let $A_1, A_2 ...... A_r$ be the subset of Universal set U. Then the number m of the element which does not appear in any subset $A_1, A_2 ...... A_r$ of U.

$$m = n\,(A_1^c \cap A_2^c \cap \ldots \ldots \cap A_r^c) = |U| - S_1 + S_2 - S_3 + \ldots\ldots\ldots (-1)^r S_r.$$

Example: Let U be the set of positive integer not exceeding 1000. Then |U|= 1000 Find |S| where S is the set of such integer which is not divisible by 3, 5 or 7?

Solution: Let A be the subset of integer which is divisible by 3

Let B be the subset of integer which is divisible by 5

Let C be the subset of integer which is divisible by 7

Then $S = A_c \cap B_c \cap C_c$ since each element of S is not divisible by 3, 5, or 7.

By Integer division,

|A|= 1000/3 = 333
|B|= 1000/5 = 200
|C| = 1000/7 = 142
|A∩B|=1000/15=66
|B∩C|=1000/21=47
|C∩A|=1000/35=28
|A∩B∩C|=1000/105=9

Thus by Inclusion-Exclusion Principle

|S|=1000-(333+200+142)+(66+47+28)-9
|S|=1000-675+141-9=457

Permutation and Combinations:

Permutation:

Any arrangement of a set of n objects in a given order is called Permutation of Object. Any arrangement of any $r \le n$ of these objects in a given order is called an r-permutation or a permutation of n object taken r at a time.

It is denoted by P (n, r)

$$P(n, r) = \frac{n!}{(n-r)!}$$

Theorem: Prove that the number of permutations of n things taken all at a time is n!.

Proof: We know that

$$n_{P_n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

Example: 4 x n_{P3}=$n+1_{P3}$

Solution: $4 \text{ x } \frac{n!}{(n-3)!} = \frac{(n+1)!}{(n+1-3)!}$

$$\frac{4 \text{ x } n!}{(n-3)!} = \frac{(n+1) \text{x } n!}{(n-2)(n-3)!}$$

$$4(n-2) = (n+1)$$
$$4n - 8 = n+1$$
$$3n = 9$$
$$n = 3.$$

Permutation with Restrictions:

The number of permutations of n different objects taken r at a time in which p particular objects do not occur is

$$\mathbf{n - p\ _{P_r}}$$

The number of permutations of n different objects taken r at a time in which p particular objects are present is

$$\mathbf{n - p\ _{P_{r-p}} \text{x } r_{P_p}}$$

Example: How many 6-digit numbers can be formed by using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 if every number is to start with '30' with no digit repeated?

Solution: All the numbers begin with '30.'So, we have to choose 4-digits from the remaining 7-digits.

∴ Total number of numbers that begins with '30' is

$7P_4 = \frac{7!}{(7-4)!} = \frac{7 X 6 X 5 X 4 X 3!}{3!}$ =840.

Permutations with Repeated Objects:

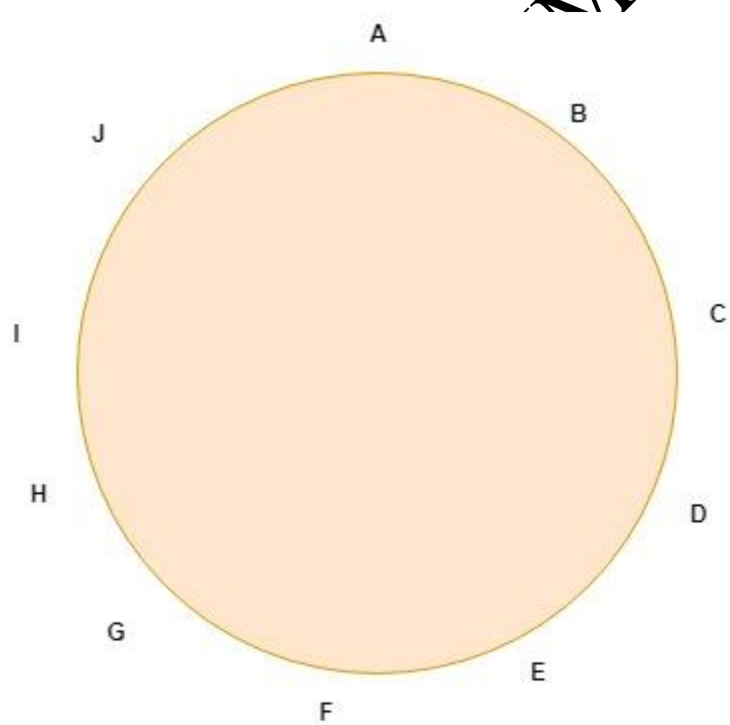
Theorem: Prove that the number of different permutations of n distinct objects taken at a time when every object is allowed to repeat any number of times is given by n^r.

Proof: Assume that with n objects we have to fill r place when repetition of the object is allowed.

Therefore, the number of ways of filling the first place is = n
The number of ways of filling the second place = n
....
....
The number of ways of filling the rth place = n
Thus, the total number of ways of filling r places with n elements is
= n. n. n.............r times =n^r.

Circular Permutations:

A permutation which is done around a circle is called Circular Permutation.



Example: In how many ways can get these letters a, b, c, d, e, f, g, h, i, j arranged in a circle?

Solution: (10 - 1) = 9! = 362880

Theorem: Prove that the number of circular permutations of n different objects is (n-1)!

Proof: Let us consider that K be the number of permutations required.

For each such circular permutations of K, there are n corresponding linear permutations. As shown earlier, we start from every object of n object in the circular permutations. Thus, for K circular permutations, we have K...n linear permutations.

Therefore, $K.\ n = n! \quad \text{or } K = \frac{n!}{n}$

$$K = \frac{n \times (n-1)!}{n}$$

$$K = (n-1)!$$

Hence Proved.

Combination:

A Combination is a selection of some or all, objects from a set of given objects, where the order of the objects does not matter. The number of combinations of n objects, taken r at a time represented by n_{C_r} or C (n, r).

$$n_{C_r} = \frac{n!}{r!\,(n-r)!}$$

Proof: The number of permutations of n different things, taken r at a time is given by

$$n_{P_r} = \frac{n!}{(n-r)!}$$

As there is no matter about the order of arrangement of the objects, therefore, to every combination of r things, there are r! arrangements i.e.,

$$n_{P_r} = r!\ n_{C_r} \quad \text{or} \quad n_{C_r} = \frac{n_{P_r}}{r!} = \frac{n!}{(n-r)!\,r!},\ n \geq r$$

Thus,

$$n_{C_r} = \frac{n!}{r!(n-r)!}$$

Example: A farmer purchased 3 cows, 2 pigs, and 4 hens from a man who has 6 cows, 5 pigs, and 8 hens. Find the number m of choices that the farmer has.

The farmer can choose the cows in C (6, 3) ways, the pigs in C (5, 2) ways, and the hens in C (8, 4) ways. Thus the number m of choices follows:

$$m = \binom{6}{3}\binom{5}{4}\binom{8}{4} = \frac{6.5.4}{3.2.1} \times \frac{5.4}{4} \times \frac{8.7.6.5}{4.3.2.1} = 20 \times 10 \times 70 = 14000$$

Relation

n Maths, the relation is the relationship between two or more set of values.

Suppose, x and y are two sets of ordered pairs. And set x has relation with set y, then the values of set x are called domain whereas the values of set y are called range.

Example: For ordered pairs={(1,2),(-3,4),(5,6),(-7,8),(9,2)}

The domain is = {-7,-3,1,5,9}

And range is = {2,4,6,8}

Types of Relations

There are 8 main types of relations which include:

- Empty Relation
- Universal Relation
- Identity Relation
- Inverse Relation
- Reflexive Relation
- Symmetric Relation
- Transitive Relation
- Equivalence Relation

Empty Relation

An empty relation (or void relation) is one in which there is no relation between any elements of a set. For example, if set A = {1, 2, 3} then, one of the void relations can be R = {x, y} where, |x – y| = 8. For empty relation,

$R = \varphi \subset A \times A$

Universal Relation

A universal (or full relation) is a type of relation in which every element of a set is related to each other. Consider set A = {a, b, c}. Now one of the universal relations will be R = {x, y} where, $|x - y| \geq 0$. For universal relation,

$R = A \times A$

Identity Relation

In an identity relation, every element of a set is related to itself only. For example, in a set A = {a, b, c}, the identity relation will be I = {a, a}, {b, b}, {c, c}. For identity relation,

$I = \{(a, a), a \in A\}$

Inverse Relation

Inverse relation is seen when a set has elements which are inverse pairs of another set. For example if set A = {(a, b), (c, d)}, then inverse relation will be R^{-1} = {(b, a), (d, c)}. So, for an inverse relation,

$R^{-1} = \{(b, a): (a, b) \in R\}$

Reflexive Relation

In a reflexive relation, every element maps to itself. For example, consider a set A = {1, 2,}. Now an example of reflexive relation will be R = {(1, 1), (2, 2), (1, 2), (2, 1)}. The reflexive relation is given by-

(a, a) ∈ R

Symmetric Relation

In a symmetric relation, if a=b is true then b=a is also true. In other words, a relation R is symmetric only if (b, a) ∈ R is true when (a,b) ∈ R. An example of symmetric relation will be R = {(1, 2), (2, 1)} for a set A = {1, 2}. So, for a symmetric relation,

aRb ⇒ bRa, ∀ a, b ∈ A

Transitive Relation

For transitive relation, if (x, y) ∈ R, (y, z) ∈ R, then (x, z) ∈ R. For a transitive relation,

aRb and bRc ⇒ aRc ∀ a, b, c ∈ A

Equivalence Relation

If a relation is reflexive, symmetric and transitive at the same time, it is known as an equivalence relation

Representation of Types of Relations

Relation Type	Condition
Empty Relation	$R = \varphi \subset A \times A$
Universal Relation	$R = A \times A$
Identity Relation	$I = \{(a, a), a \in A\}$
Inverse Relation	$R^{-1} = \{(b, a): (a, b) \in R\}$
Reflexive Relation	$(a, a) \in R$
Symmetric Relation	$aRb \Rightarrow bRa, \forall\, a, b \in A$
Transitive Relation	aRb and $bRc \Rightarrow aRc\ \forall\, a, b, c \in A$

Binary Relation

Let P and Q be two non- empty sets. A binary relation R is defined to be a subset of P x Q from a set P to Q. If (a, b) ∈ R and R ⊆ P x Q then a is related to b by R i.e., aRb. If sets P and Q are equal, then we say R ⊆ P x P is a relation on P e.g.

(i) Let A = {a, b, c}
B = {r, s, t}
Then R = {(a, r), (b, r), (b, t), (c, s)}
is a relation from A to B.

(ii) Let A = {1, 2, 3} and B = A
R = {(1, 1), (2, 2), (3, 3)}
is a relation (equal) on A.

Example1: If a set has n elements, how many relations are there from A to A.

Solution: If a set A has n elements, A x A has n^2 elements. So, there are 2^{n^2} relations from A to A.

Example2: If A has m elements and B has n elements. How many relations are there from A to B and vice versa?

Solution: There are m x n elements; hence there are $2^{m \times n}$ relations from A to A.

Example3: If a set A = {1, 2}. Determine all relations from A to A.

Solution: There are 2^2= 4 elements i.e., {(1, 2), (2, 1), (1, 1), (2, 2)} in A x A. So, there are 2^4= 16 relations from A to A. i.e.

{(1, 2), (2, 1), (1, 1), (2, 2)}, {(1, 2), (2, 1)}, {(1, 2), (1, 1)}, {(1, 2), (2, 2)}, {(2, 1), (1, 1)},{(2,1), (2, 2)}, {(1, 1),(2, 2)},{(1, 2), (2, 1), (1, 1)}, {(1, 2), (1, 1), (2, 2)}, {(2,1), (1, 1), (2, 2)}, {(1, 2), (2, 1), (2, 2)}, {(1, 2), (2, 1), (1, 1), (2, 2)} and ∅.

Domain and Range of Relation

Domain of Relation: The Domain of relation R is the set of elements in P which are related to some elements in Q, or it is the set of all first entries of the ordered pairs in R. It is denoted by DOM (R).

Range of Relation: The range of relation R is the set of elements in Q which are related to some element in P, or it is the set of all second entries of the ordered pairs in R. It is denoted by RAN (R).

Example:

Let A = {1, 2, 3, 4}
B = {a, b, c, d}
R = {(1, a), (1, b), (1, c), (2, b), (2, c), (2, d)}.

Solution:

DOM (R) = {1, 2}
RAN (R) = {a, b, c, d}

Complement of a Relation

Consider a relation R from a set A to set B. The complement of relation R denoted by R is a relation from A to B such that

R

= {(a, b): {a, b) ∉ R}.

Example:

Consider the relation R from X to Y

X = {1, 2, 3}

Y = {8, 9}

R = {(1, 8) (2, 8) (1, 9) (3, 9)}

Find the complement relation of R.

Solution:

X x Y = {(1, 8), (2, 8), (3, 8), (1, 9), (2, 9), (3, 9)}
Now we find the complement relation R from X x Y
R = {(3, 8), (2, 9)}

Equivalence Relations

A relation R on a set A is called an equivalence relation if it satisfies following three properties:

1. Relation R is Reflexive, i.e. aRa ∀ a∈A.
2. Relation R is Symmetric, i.e., aRb ⟹ bRa
3. Relation R is transitive, i.e., aRb and bRc ⟹ aRc.

Example: Let A = {1, 2, 3, 4} and R = {(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)}.

Show that R is an Equivalence Relation.

Solution:

Reflexive: Relation R is reflexive as (1, 1), (2, 2), (3, 3) and (4, 4) ∈ R.

Symmetric: Relation R is symmetric because whenever (a, b) ∈ R, (b, a) also belongs to R.

Example: (2, 4) ∈ R ⟹ (4, 2) ∈ R.

Transitive: Relation R is transitive because whenever (a, b) and (b, c) belongs to R, (a, c) also belongs to R.

Example: (3, 1) ∈ R and (1, 3) ∈ R ⟹ (3, 3) ∈ R.

So, as R is reflexive, symmetric and transitive, hence, R is an Equivalence Relation.

> Note1: If $R_1$and R_2 are equivalence relation then $R_1 \cap R_2$ is also an equivalence relation.

Example: A = {1, 2, 3}

R_1 = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}

R_2 = {(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)}

$R_1 \cap R_2$ = {(1, 1), (2, 2), (3, 3)}

Note2: If R_1 and R_2 are equivalence relation then $R_1 \cup R_2$ may or may not be an equivalence relation.

Example: A = {1, 2, 3}

R_1 = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}

R_2 = {(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)}

$R_1 \cup R_2$= {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)}

Hence, Reflexive or Symmetric are Equivalence Relation but transitive may or may not be an equivalence relation.

Inverse Relation

Let R be any relation from set A to set B. The inverse of R denoted by R^{-1} is the relations from B to A which consist of those ordered pairs which when reversed belong to R that is:

$R^{-1} = \{(b, a): (a, b) \in R\}$

Example1: A = {1, 2, 3}

B = {x, y, z}

Solution: R = {(1, y), (1, z), (3, y)

R^{-1} = {(y, 1), (z, 1), (y, 3)}

Clearly $(R^{-1})^{-1}$ = R

Note1: Domain and Range of R^{-1} is equal to range and domain of R.

Example2: R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (3, 2)}

R^{-1} = {(1, 1), (2, 2), (3, 3), (2, 1), (3, 2), (2, 3)}

Note2: If R is an Equivalence Relation then R^{-1} is always an Equivalence Relation.

Example: Let A = {1, 2, 3}

R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}

R^{-1} = {(1, 1), (2, 2), (3, 3), (2, 1), (1, 2)}
R^{-1} is a Equivalence Relation.

> Note3: If R is a Symmetric Relation then R^{-1}=R and vice-versa.

Example: Let A = {1, 2, 3}
R = {(1, 1), (2, 2), (1, 2), (2, 1), (2, 3), (3, 2)}
R^{-1} = {(1, 1), (2, 2), (2, 1), (1, 2), (3, 2), (2, 3)}

> Note4: Reverse Order of Law
>
> $(SOT)^{-1} = T^{-1}$ or S^{-1}
>
> $(ROSOT)^{-1} = T^{-1}$ or S^{-1} or R^{-1}.

Closure Properties of Relations

Consider a given set A, and the collection of all relations on A. Let P be a property of such relations, such as being symmetric or being transitive. A relation with property P will be called a P-relation. The P-closure of an arbitrary relation R on A, indicated P (R), is a P-relation such that

$R \subseteq P(R) \subseteq S$

(1) Reflexive and Symmetric Closures: The next theorem tells us how to obtain the reflexive and symmetric closures of a relation easily.

Theorem: Let R be a relation on a set A. Then:

- $R \cup \Delta_A$ is the reflexive closure of R
- $R \cup R^{-1}$ is the symmetric closure of R.

Example1:

Let A = {k, l, m}. Let R is a relation on A defined by
R = {(k, k), (k, l), (l, m), (m, k)}.

Find the reflexive closure of R.

Solution: $R \cup \Delta$ is the smallest relation having reflexive property, Hence,

$R_F = R \cup \Delta = \{(k, k), (k, l), (l, l), (l, m), (m, m), (m, k)\}$.

Example2: Consider the relation R on A = {4, 5, 6, 7} defined by

R = {(4, 5), (5, 5), (5, 6), (6, 7), (7, 4), (7, 7)}

Find the symmetric closure of R.

Solution: The smallest relation containing R having the symmetric property is $R \cup R^{-1}$,i.e.

$R_S = R \cup R^{-1} = \{(4, 5), (5, 4), (5, 5), (5, 6), (6, 5), (6, 7), (7, 6), (7, 4), (4, 7), (7, 7)\}$.

(2)Transitive Closures: Consider a relation R on a set A. The transitive closure R of a relation R of a relation R is the smallest transitive relation containing R.

Recall that $R^2 = R \circ R$ and $R^n = R^{n-1} \circ R$. We define

$$\boxed{R^* = \bigcup_{i=1}^{\infty} R^i}$$

The following Theorem applies:

Theorem1: R^* is the transitive closure of R

Suppose A is a finite set with n elements.

$R^* = R \cup R^2 \cup \ldots \cup R^n$

Theorem 2: Let R be a relation on a set A with n elements. Then

$Transitive\ (R) = R \cup R^2 \cup \ldots \cup R^n$

Example1: Consider the relation R = {(1, 2), (2, 3), (3, 3)} on A = {1, 2, 3}. Then
$R^2 = R \circ R$ = {(1, 3), (2, 3), (3, 3)} and $R^3 = R^2 \circ R$ = {(1, 3), (2, 3), (3, 3)} Accordingly,
Transitive (R) = {(1, 2), (2, 3), (3, 3), (1, 3)}

Example2: Let A = {4, 6, 8, 10} and R = {(4, 4), (4, 10), (6, 6), (6, 8), (8, 10)} is a relation on set A. Determine transitive closure of R.

Solution: The matrix of relation R is shown in fig:

$$M_R = \begin{array}{cc} & \begin{array}{cccc} 4 & 6 & 8 & 10 \end{array} \\ \begin{array}{c} 4 \\ 6 \\ 8 \\ 10 \end{array} & \left\{ \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right\} \end{array}$$

Now, find the powers of M_R as in fig:

$$M_{R^2} = \begin{array}{cc} & \begin{array}{cccc} 4 & 6 & 8 & 10 \end{array} \\ \begin{array}{c} 4 \\ 6 \\ 8 \\ 10 \end{array} & \left\{ \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right\} \end{array} \quad M_{R^3} = \begin{array}{cc} & \begin{array}{cccc} 4 & 6 & 8 & 10 \end{array} \\ \begin{array}{c} 4 \\ 6 \\ 8 \\ 10 \end{array} & \left\{ \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right\} \end{array}$$

$$M_{R^4} = \begin{array}{cc} & \begin{array}{cccc} 4 & 6 & 8 & 10 \end{array} \\ \begin{array}{c} 4 \\ 6 \\ 8 \\ 10 \end{array} & \left\{ \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right\} \end{array}$$

Hence, the transitive closure of M_R is M_{R^*} as shown in Fig (where M_{R^*} is the ORing of a power of M_R).

$$M_{R^*} = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}; \quad M_{R^*} = \begin{array}{cc} & \begin{array}{cccc} 4 & 6 & 8 & 10 \end{array} \\ \begin{array}{c} 4 \\ 6 \\ 8 \\ 10 \end{array} & \left\{ \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right\} \end{array}$$

Thus, $R^* = \{(4, 4), (4, 10), (6, 8), (6, 6), (6, 10), (8, 10)\}$.

> Note: While ORing the power of the matrix R, we can eliminate M_{R^n} because it is equal to M_{R^*} if n is even and is equal to M_{R^3} if n is odd.