Expectation of Modified Variables

(Law of The Unconscious Statistician)

Expectation Reminder

X

x	-2	-1	0	1	2
p(x)	⅕	⅕	⅕	⅕	⅕

$$E(X) = \sum_x p(x) \cdot x$$

$$= \underbrace{-2 \cdot \tfrac{1}{5} + \underbrace{-1 \cdot \tfrac{1}{5} + 0 \cdot \tfrac{1}{5} + 1 \cdot \tfrac{1}{5}}_{0} + 2 \cdot \tfrac{1}{5}}_{0} = 0$$

"By Symmetry"

Expectation of a Square

X

x	-2	-1	0	1	2
p(x)	⅕	⅕	⅕	⅕	⅕

$Y = X^2$

y	0	1	4
p(y)	⅕	⅖	⅖

$P(Y = 0) = P(X^2 = 0) = P(X = 0) = 1/5$

$P(Y = 1) = P(X^2 = 1) = P(X \in \{-1, 1\}) = 2/5$

$P(Y = 4) = P(X^2 = 4) = P(X \in \{-2, 2\}) = 2/5$

$E(Y) = 1/5 \cdot 0 + 2/5 \cdot 1 + 2/5 \cdot 4 = 10/5 = 2$

Alternative Formulation

$$
\begin{aligned}
E(Y) &= \sum_{y} y \cdot P(Y=y) \\
&= \sum_{y} y \cdot P(X \in g^{-1}(y)) \\
&= \sum_{y} y \sum_{x \in g^{-1}(y)} p(x) \\
&= \sum_{y} \sum_{x \in g^{-1}(y)} y \cdot p(x) \\
&= \sum_{y} \sum_{x \in g^{-1}(y)} g(x) \cdot p(x) \\
&= \sum_{x} g(x) \cdot p(x)
\end{aligned}
$$

Example

Visualize

Square Again

X

x	-2	-1	0	1	2
p(x)	⅕	⅕	⅕	⅕	⅕

$Y = X^2$

y	0	1	4
p(y)	⅕	⅖	⅖

$$E(Y) = \sum_{y=0,1,4} y \cdot p(Y=y) = \tfrac{1}{5} \cdot 0 + \tfrac{2}{5} \cdot 1 + \tfrac{2}{5} \cdot 4 = 10/5 = 2$$

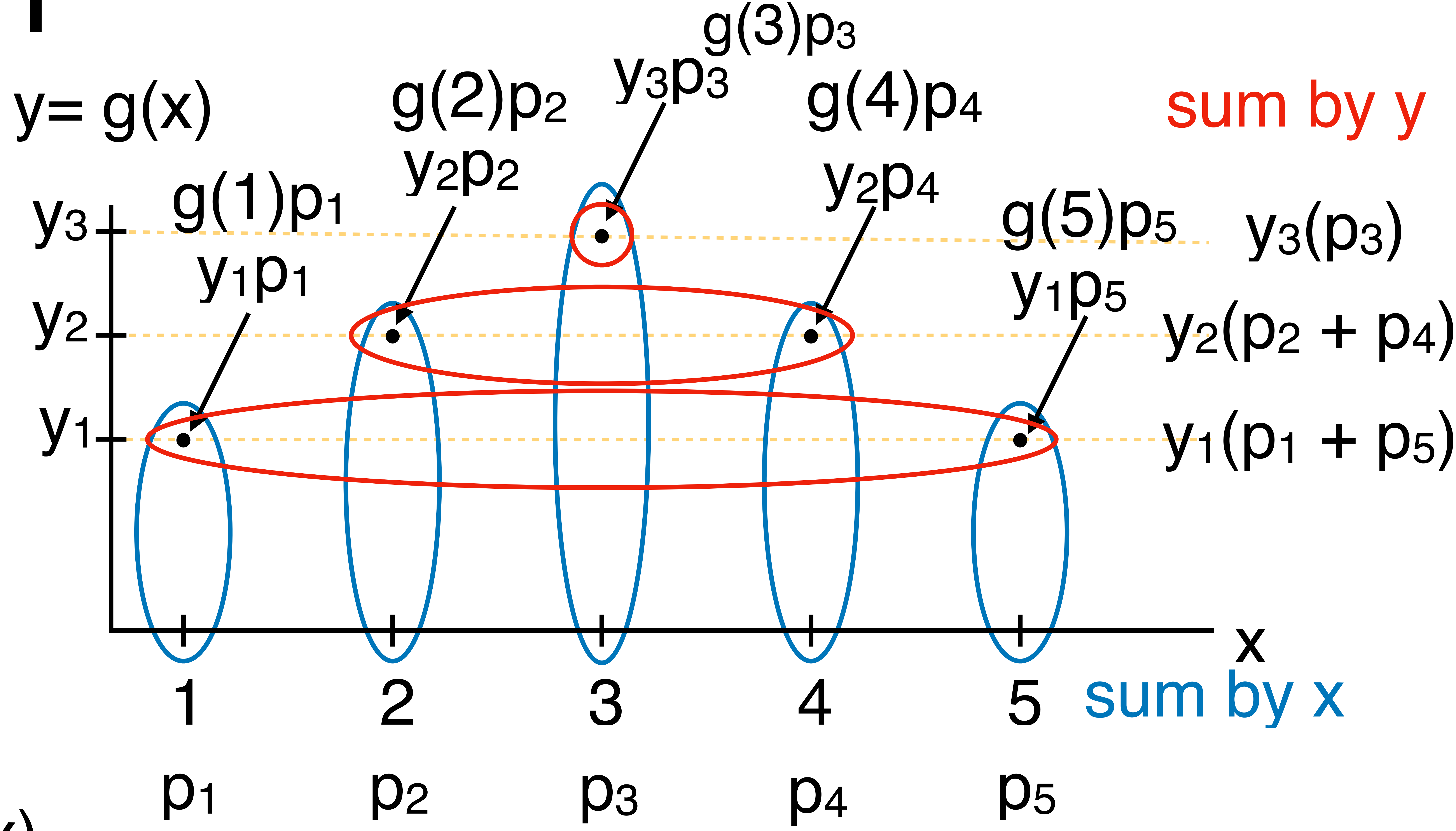
$$E(Y) = \sum_{x} x^2 \cdot p(x)$$

$$= (-2)^2 \cdot \tfrac{1}{5} + (-1)^2 \cdot \tfrac{1}{5} + 0^2 \cdot \tfrac{1}{5} + 1^2 \cdot \tfrac{1}{5} + 2^2 \cdot \tfrac{1}{5}$$

$$= \tfrac{4}{5} + \tfrac{1}{5} + \tfrac{1}{5} + \tfrac{4}{5} = 2$$

Visualization

$$
\begin{aligned}
E(Y) &= \sum_{y} y \cdot P(Y=y) \\
&= \sum_{y} y \cdot P(X \in g^{-1}(y)) \\
&= \sum_{y} y \sum_{x \in g^{-1}(y)} p(x) \\
&= \sum_{y} \sum_{x \in g^{-1}(y)} y \cdot p(x) \\
&= \sum_{y} \sum_{x \in g^{-1}(y)} g(x) \cdot p(x) \\
&= \sum_{x} g(x) \cdot p(x)
\end{aligned}
$$



y: Fewer multiplication

x: Properties (next)
General Formulas

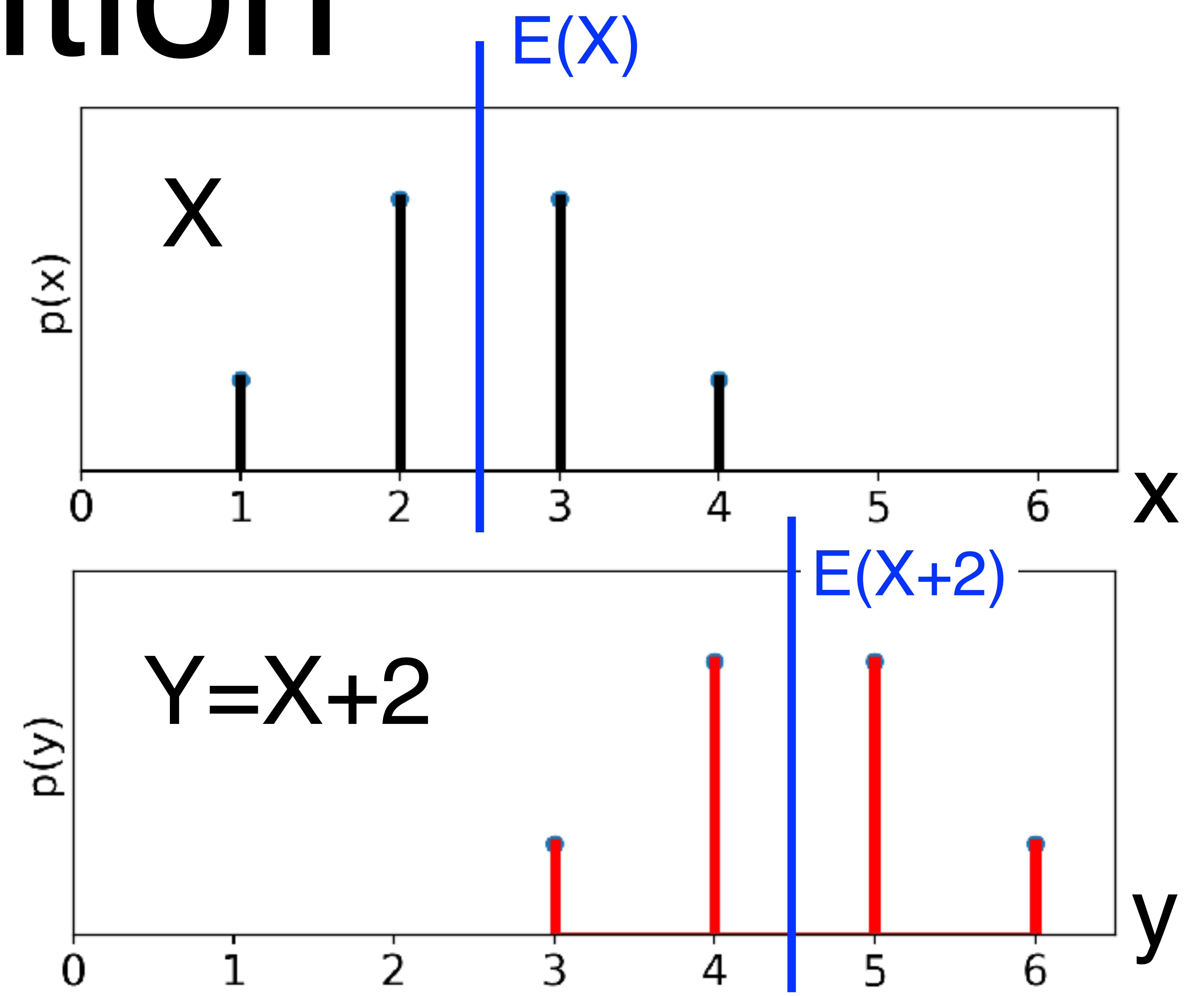
Constant Addition

$$E(X + b) = \sum p(x) \cdot (x + b)$$

$$= \sum p(x) \cdot x + \sum p(x) \cdot b$$

$$= E(X) + b \cdot \sum p(x)$$

$$= E(X) + b$$



x	0	1
p(x)	1 - p	p

Bernoulli p

$$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$$

$$E(X + 2) = (1 - p) \cdot (0 + 2) + p \cdot (1 + 2)$$

$$= 2 - 2p + 3p$$

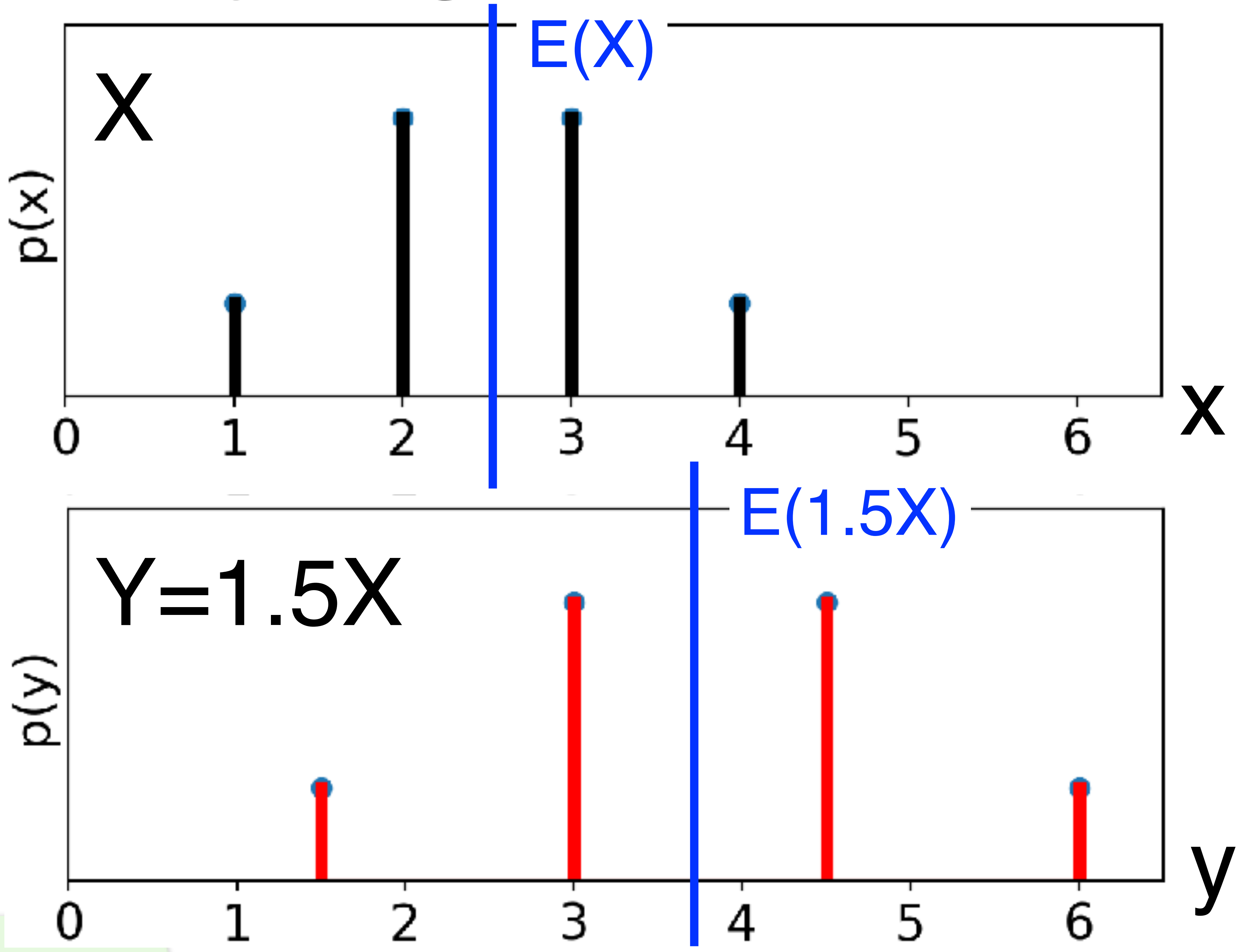
$$= p + 2 = E(X) + 2$$

Constant Multiplication

$$E(aX) = \sum p(x) \cdot (ax)$$

$$= a \sum p(x) \cdot x$$

$$= aE(X)$$



Bernoulli p

x	0	1
p(x)	1 - p	p

$$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$$

$$E(3X) = (1 - p) \cdot (3 \cdot 0) + p \cdot (3 \cdot 1)$$

$$= 3p = 3E(X)$$

Linearity of Expectation

$E(aX + b) = E(aX) + b$

$= a\,E(X) + b$

Bernoulli p

x	0	1
p(x)	1 - p	p

$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$

$E(2X + 3) = (1 - p)(2 \cdot 0 + 3) + p(2 \cdot 1 + 3)$

$= 3 - 3p + 5p$

$= 2p + 3 = 2E(X) + 3$

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