

Linearity of Expectation



Expectation

$$Eg(X) = \sum_z z \cdot P(g(x) = z)$$

$$= \sum_z z \sum_{x \in g^{-1}(z)} p(x)$$

$$p(x) \rightarrow p(x, y)$$

$$= \sum_z \sum_{x \in g^{-1}(z)} z \cdot p(x)$$

$$g(x) \rightarrow g(x, y)$$

$$= \sum_z \sum_{x \in g^{-1}(z)} g(x) p(x)$$

$$\sum_x \rightarrow \sum_{x, y}$$

$$= \sum_x g(x) p(x)$$

Linearity of Expectation

$$\begin{aligned} E(X + Y) &= \sum_x \sum_y (x + y) \cdot p(x, y) \\ &= \sum_x \sum_y x \cdot p(x, y) + \sum_x \sum_y y \cdot p(x, y) \\ &= \sum_x x \sum_y p(x, y) + \sum_y y \sum_x p(x, y) \\ &= \sum_x x \cdot p(x) + \sum_y y \cdot p(y) \\ &= EX + EY \end{aligned}$$



Expectation of sum =
sum of expectations

The Hat Problem

| | | H ₁ | H ₂ | H ₃ | H |
|--|-------|----------------|----------------|----------------|---|
| 1 _i - indicator function i th student caught their own hat | 1 2 3 | 1 | 1 | 1 | 3 |
| H - # students who caught their own hat | 1 3 2 | 1 | 0 | 0 | 1 |
| | 2 1 3 | 0 | 0 | 1 | 1 |
| | 2 3 1 | 0 | 0 | 0 | 0 |
| | 3 1 2 | 0 | 0 | 0 | 0 |
| | 3 2 1 | 0 | 1 | 0 | 1 |

$$H = \sum_{i=1}^n 1_i$$

1_i - Bernoulli

$$P(1_i = 1) = \frac{\# \text{ permutations of } (\sigma_1, \dots, \sigma_n) \text{ when } \sigma_i = i}{\# \text{ permutations of } (\sigma_1, \dots, \sigma_n)} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$E(1_i) = P(1_i = 1) = \frac{1}{n}$$

$$E(H) = E\left(\sum_{i=1}^n 1_i\right) = \sum_{i=1}^n E(1_i) = \sum_{i=1}^n \frac{1}{n} = 1$$

Variance

Expectations add $E(X + Y) = EX + EY$

Do variances? $V(X + Y) \stackrel{?}{=} V(X) + V(Y)$

$$\begin{aligned} V(X + Y) &= E(X + Y)^2 - (E(X + Y))^2 \\ &= E(X^2 + 2XY + Y^2) - (EX + EY)^2 \\ &= EX^2 + 2E(XY) + EY^2 - (E^2X + 2EX \cdot EY + E^2Y) \\ &= EX^2 - E^2X + EY^2 - E^2Y + 2(E(XY) - EX \cdot EY) \\ &= V(X) + V(Y) + 2(E(XY) - EX \cdot EY) \end{aligned}$$

$$E(XY) = EX \cdot EY?$$

Do expectations multiply?

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