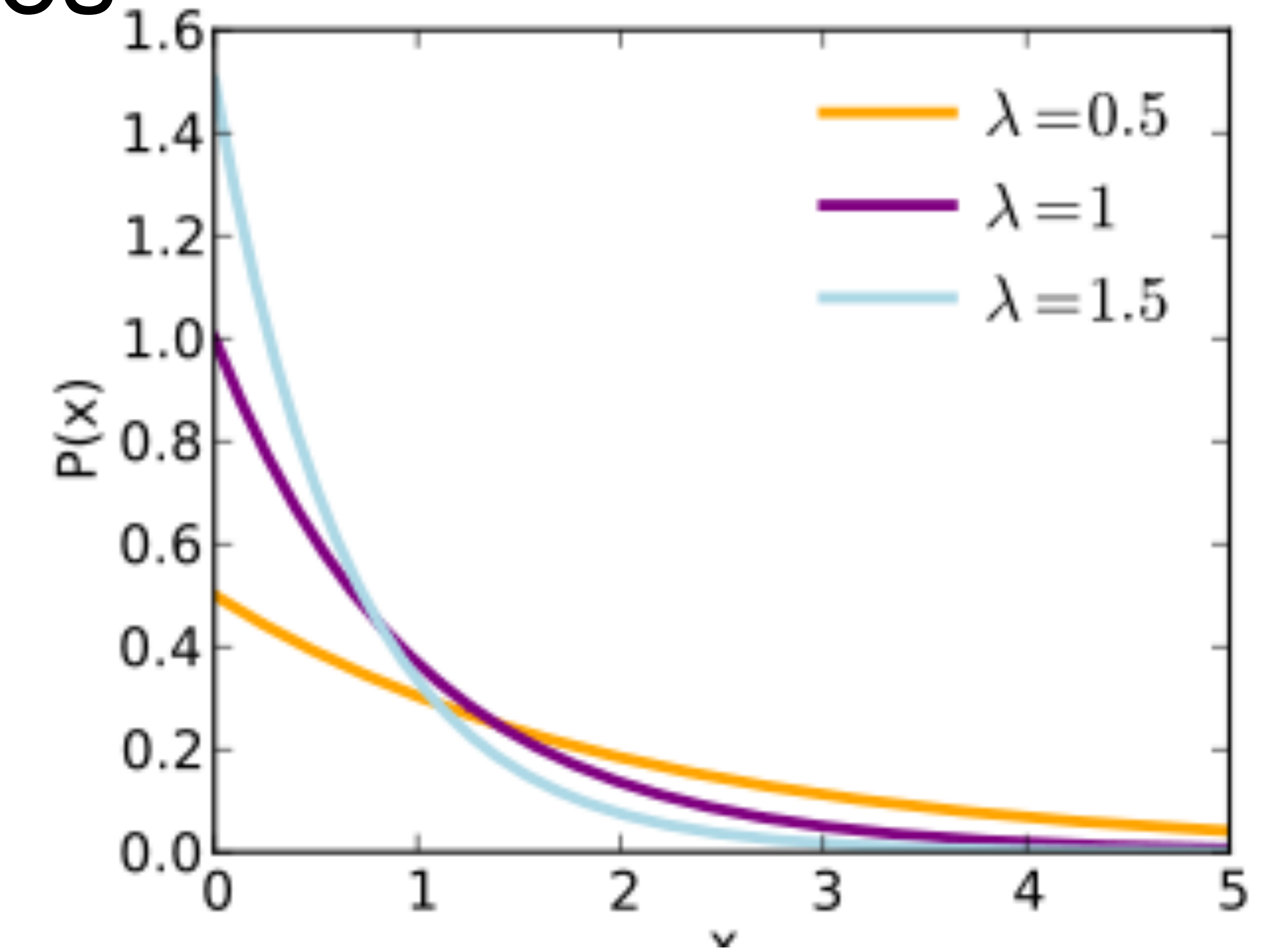
Definition

Extends geometric distribution to continuous values

$\lambda > 0$

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

≥ 0 ✅



$$\int f(x)dx = \int_0^\infty \lambda e^{-\lambda x}\, dx$$
$$= -\,e^{-\lambda x}\Big|_0^\infty$$
$$= 0 - (-1)$$
$$= 1$$

Who’s Exponential

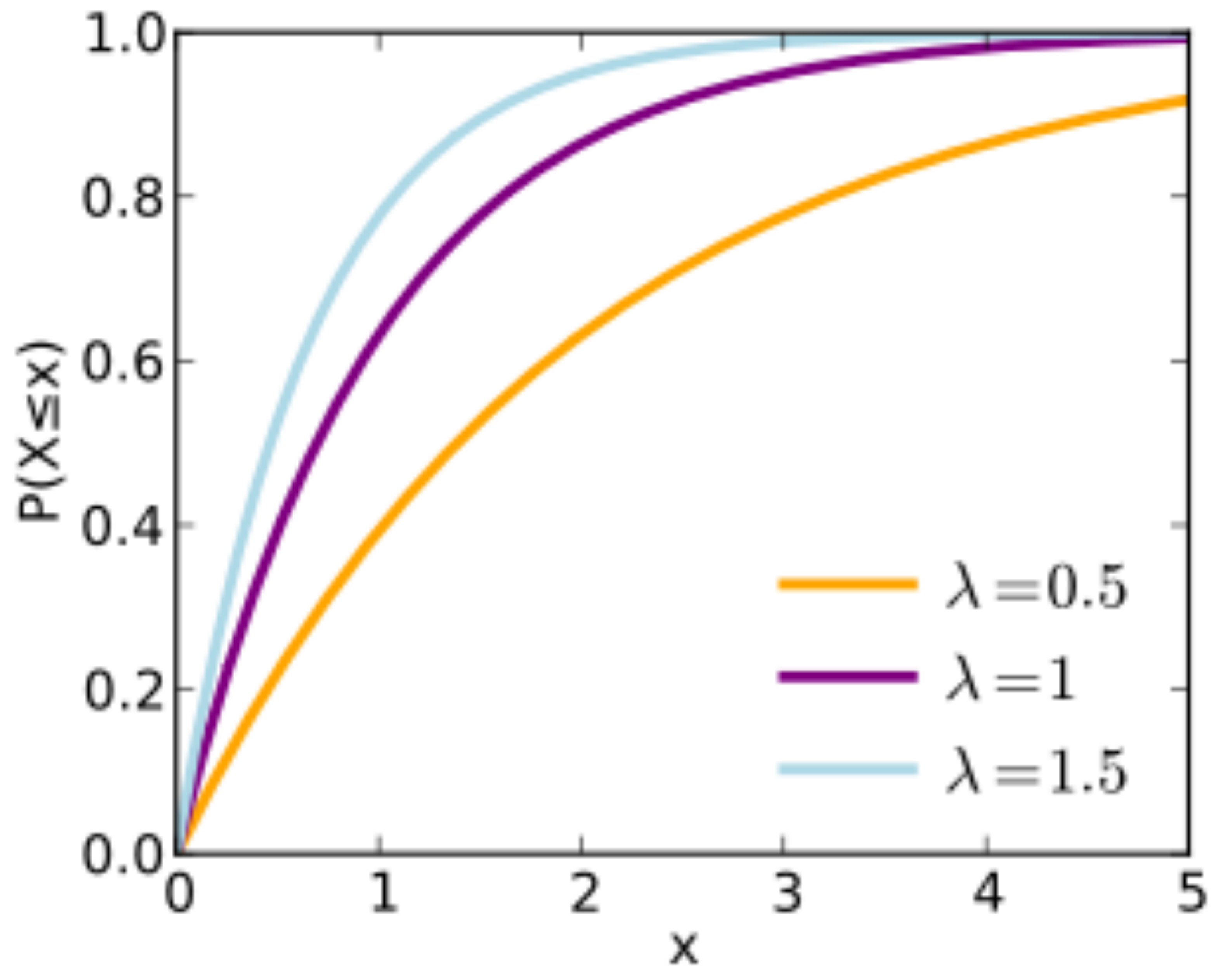
Duration of a phone call

Wait time when you call an airline

Lifetime of a car

Time between accidents

CDF



$x \geq 0$

$$P(X > x) = \int_x^\infty \lambda e^{-\lambda u} du$$

$$= -e^{-\lambda u}\Big|_x^\infty$$

$$= e^{-\lambda x}$$

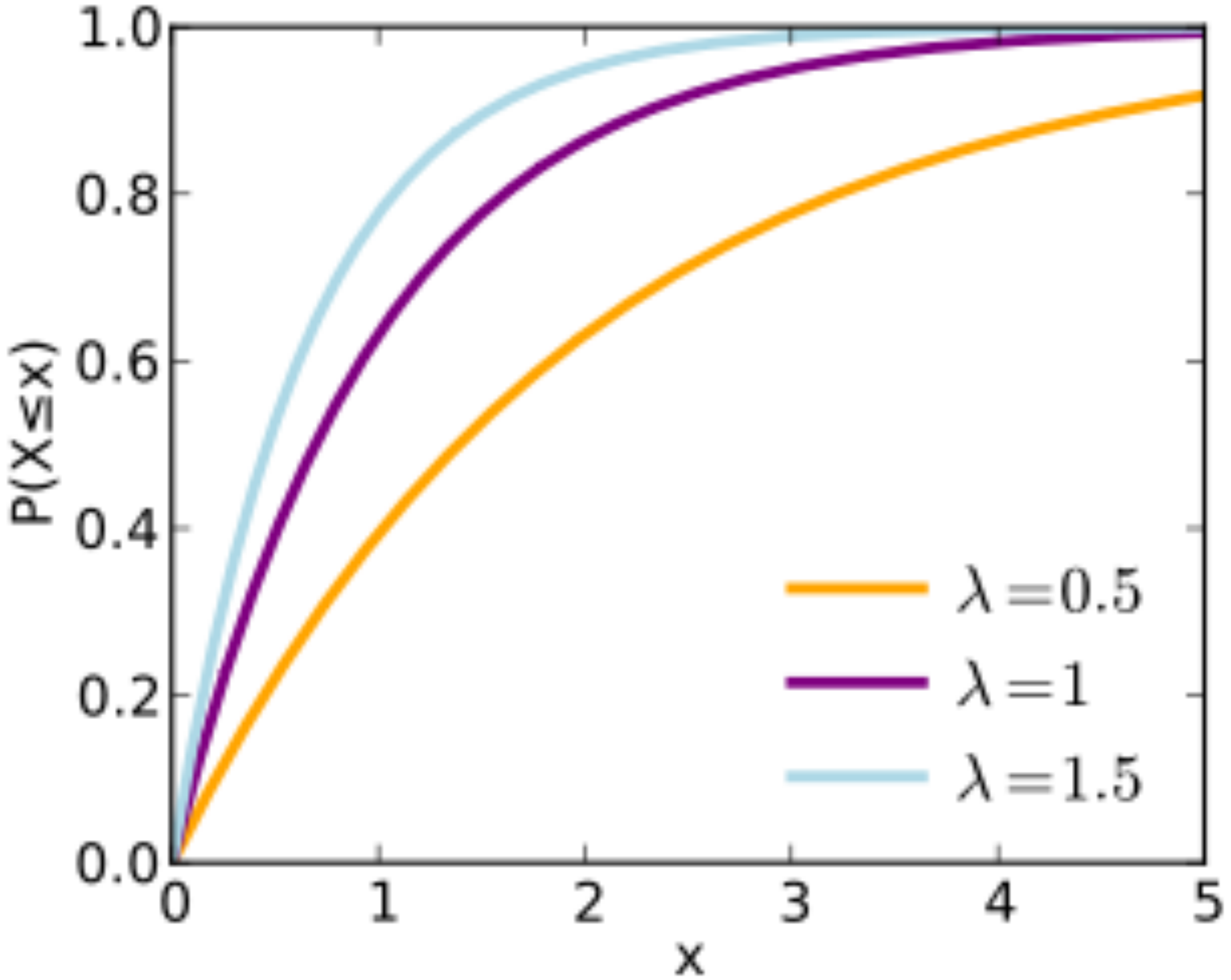
$$F(x) = P(X \leq x) \quad = 1 - P(X > x) = 1 - e^{-\lambda x} \qquad x \geq 0$$

$$F(x) = 0 \qquad x \leq 0$$

CDF

$$P(X > x) = \begin{cases} \int_x^\infty \lambda e^{-\lambda u} du \ = -e^{-\lambda u}\Big|_x^\infty \ = e^{-\lambda x} & x \geq 0 \\ 1 & x \leq 0 \end{cases}$$

$$F(x) = P(X \leq x) = \begin{cases} 1 - P(X > x) \ = 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$



Example

$0 \le a \le b$

$$P(a \le X \le b) = P(a < X < b)$$

$$= F(b) - F(a)$$

$$= (1 - e^{-\lambda b}) - (1 - e^{-\lambda a})$$

$$= e^{-\lambda a} - e^{-\lambda b}$$

Expectation

$$EX = \int_0^\infty x\lambda e^{-\lambda x} dx$$

$$u = x \qquad dv = \lambda e^{-\lambda x} dx$$

$$\int u \, dv = uv - \int v \, du$$

$$du = 1 \qquad v = -e^{-\lambda x}$$

$$= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx$$

$$= 0 - \tfrac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty$$

$$= \frac{1}{\lambda}$$

Variance

$$EX^2 = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$u = x^2 \qquad dv = \lambda e^{-\lambda x} dx$$

$$du = 2x\ dx \qquad v = -e^{-\lambda x}$$

$$\int u\ dv = uv - \int v\ du$$

$$= -x^2 e^{-\lambda x}\Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x} dx$$

$$EX = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$= 0 + \frac{2}{\lambda} EX = \frac{2}{\lambda^2}$$

$$V(X) = EX^2 - (EX)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\sigma = \frac{1}{\lambda}$$

Memoryless

$X \sim f_\lambda$ $a, b \geq 0$

$$P(X \geq a + b | X \geq a) = \frac{P(X \geq a+b, X \geq a)}{P(X \geq a)}$$

$$= \frac{P(X \geq a+b)}{P(X \geq a)}$$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$$

$$= e^{-\lambda b}$$

$$= P(X \geq b)$$

$$P(X < a + b | x \geq a) = 1 - P(X \geq a + b | X \geq a) = 1 - P(X \geq b) = P(X < b)$$

$$f(X = a + b | X \geq a) = f(X = b)$$

While Waiting in Line

DMV has 2 clerks, each with exponential service time

When you arrive, one person is in line 🙁

While you wait, someone cuts in front of you 😡

At some point a clerk becomes available and starts serving the first person

Before first person finishes, other clerk starts serving second person

If all three of you served randomly, P(you finish last) = ⅓

P(you finish last now)?

Evaluation

A - time first person finishes

B - time second person finishes

C - time you finish

Service	P(A < B < C)
Fixed	1
Exponential	?

Orders	Probability
A < B < C	¼
A < C < B	¼
B < A < C	¼
B < C < A	¼
C < A < B	0
C < B < A	0

?

$$P(A<B<C) = \underbrace{P(A < B)}_{½} \cdot \underbrace{P(B < C \mid A < B)}_{½} = ¼$$

$$P(B<C<A) = \overbrace{P(B < A)}^{½} \cdot \overbrace{P(C < A \mid B < A)}^{½} = ¼$$

Conclusion

All three of you served randomly, P(you finish last) = ⅓

Fixed service time, P(you finish last) = 1

Exponential (memoryless) service time

- You won't finish first
- All 4 other orders equally likely
- P(you finish last) = ¼ + ¼ = ½
- Only slightly larger than ⅓

Orders	P
A < B < C	¼
A < C < B	¼
B < A < C	¼
B < C < A	¼
C < A < B	0
C < B < A	0

Summary

Exponential

PDF $f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

CDF $F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

$EX = \frac{1}{\lambda}$ $V(X) = \frac{1}{\lambda^2}$ $\sigma = \frac{1}{\lambda}$

Memoryless

Normal Distribution