Covariance

Do Expectations Multiply?

$$E(XY) = \sum_{x,y} xy \cdot p(x,y)$$

$$E(XY) \stackrel{?}{=} EX \cdot EY$$

$$X = Y = \begin{cases} -1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{cases}$$

	-1	1	y
-1	½	0	½
1	0	½	½
x	½	½	

$$EX = EY = 0$$

$$EX \cdot EY = 0$$

$$E(XY) = EX^2 = E(1) = 1$$

$$E(XY) \neq EX \cdot EY$$

Expectations do not always multiply!

Satisfy any relation?

Wild World of Product Expectations

For any α, β, γ ∃ X, Y with: $EX = \alpha$ $\quad EY = \beta$ $\quad E(XY) = \gamma$

$$Y' = X' = \begin{cases} -1 & \frac{1}{2} \\ +1 & \frac{1}{2} \end{cases} \qquad EX' = EY' = 0 \qquad E(X'Y') = E[(X')^2] = 1$$

$$X = (\gamma - \alpha\beta)X' + \alpha \qquad Y = Y' + \beta$$

$$EX = \alpha \qquad EY = \beta$$

$$\begin{aligned} E(XY) &= E((\gamma - \alpha\beta)X' + \alpha)(Y' + \beta) \\ &= (\gamma - \alpha\beta)\underbrace{E(X'Y')}_{1} + \alpha\underbrace{EY'}_{0} + (\gamma - \alpha\beta)\beta\underbrace{EX'}_{0} + \alpha\beta \\ &= \gamma \end{aligned}$$

Can we still say something about E(XY)?

Covariance

Sufficient, and easier, to understand 0-mean variables

"Centralize" X, Y, consider expectation of centralized product

$$\sigma_{X,Y} \triangleq \mathrm{Cov}(X,Y) \triangleq E[(X-\mu_X)\cdot(Y-\mu_Y)]$$

$$= E(XY) - E(X\mu_Y) - E(\mu_X Y) + E(\mu_X\mu_Y)$$

$$= E(XY) - E(X)\mu_Y - \mu_X E(Y) + \mu_X\mu_Y$$

$$= E(XY) - \mu_X\mu_Y$$

If seems complex, think of E(XY) for 0-mean variables

Amount X and Y vary together

Properties

$$\mathrm{Cov}(X,X) = EX^2 - \mu_X^2 = V(X)$$

$$\mathrm{Cov}(X,Y) = E((X-\mu_X)(Y-\mu_Y)) = \mathrm{Cov}(Y,X)$$

$$\mathrm{Cov}(aX,Y) = E(aXY) - \mu_{aX}\mu_Y = aE(XY) - a\mu_X\mu_Y = a\mathrm{Cov}(X,Y)$$

$$\begin{aligned}\mathrm{Cov}(X+a,y) &= E[((X+a)-\mu_{X+a})(Y-\mu_Y)] \\ &= E(X-\mu_X)(Y-\mu_Y) = \mathrm{Cov}(X,Y)\end{aligned}$$

Intuitively if X changes by σ_X, Y grows by $\sigma_{X,Y} \cdot \sigma_X \cdot \sigma_Y$

Correlation Coefficient

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

Pearson's correlation coefficient

Properties:

$$\rho_{X,X} = 1 \qquad \rho_{X,-X} = -1$$

$$\rho_{X,Y} = \rho_{Y,X}$$

$$\rho_{aX+b,cY+d} = \text{sign}(ac) \cdot \rho_{X,Y}$$

$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & 0 \\ -1 & x < 0 \end{cases}$$

If X $\nearrow$ by σ_X, by how many σ_Y do we expect Y to $\nearrow$

Bounds on $\rho_{X,Y}$?

Cauchy-Schwarz Inequality

$E(X \cdot Y)$ can't take all possible values

$$|E(XY)| \leq \sqrt{EX^2} \cdot \sqrt{EY^2}$$

For any α

$$0 \leq E(\alpha X + Y)^2 = \alpha^2 EX^2 + 2\alpha E(XY) + EY^2$$

True for all α, so discriminant must be negative

$$4(EXY)^2 - 4EX^2 \cdot EY^2 \leq 0$$

$$(EXY)^2 \leq EX^2 \cdot EY^2$$

Correlation Coefficient

$$|E(X-\mu_X)(Y-\mu_Y)| \leq \sqrt{E(X-\mu_X)^2 \cdot E(Y-\mu_Y)^2}$$

Namely

$$|\sigma_{X,Y}| \leq \sigma_X \cdot \sigma_Y$$

$$\rho_{X,Y} \triangleq \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$|\rho_{X,Y}| \leq 1$$

Examples

Uncorrelated: EXY =0

$$X, Y \sim B(\tfrac{1}{2})$$

Correlation			
Positive	$X, X+Y$	$X, 2X+Y$	$\min(X,Y), \max(X,Y)$
Uncorrelated	X, Y	$3X, 4Y$	
Negative	$X, -Y$	$Y, -X$	$\lvert X-Y \rvert, \min(X,Y)$

$X = 3Y$ $\quad$ $\mathrm{Cov}(X,Y) = 3Var(X)$ $\quad$ $P = 1$

$$\perp\!\!\!\perp \rightarrow \perp$$

Independent implies uncorrelated

$$\begin{aligned} E(XY) &= \sum_x \sum_y xy \cdot p(x, y) \\ &= \sum_x \sum_y xy \cdot p(x)p(y) \\ &= \sum_x x \cdot p(x) \sum_y y \cdot p(y) \\ &= E(X) \cdot E(Y) \end{aligned}$$

$$\perp \not\leftrightarrow \perp\!\!\!\perp$$

Independent $\rightarrow$ uncorrelated

Uncorrelated $\overset{?}{\rightarrow}$ independent

$$X = \begin{cases} -1 & \frac{1}{2} \\ +1 & \frac{1}{2} \end{cases}$$

$$X = -1 \rightarrow \ Y = 0$$

$$X = +1 \rightarrow \ Y = \begin{cases} +1 \\ -1 \end{cases}$$

X \ Y	-1	0	1
-1		½	
1	¼		¼

Uncorrelated

$$EX = 0 \qquad EY = 0$$

$$E(XY) = \frac{1}{4} \cdot -1 + \frac{1}{2} \cdot 0 + \frac{1}{4} \cdot 1 = 0 = EX \cdot EY$$

Clearly dependent

Note: Uncorrelated binary random pairs are independent

Variance

$$V(X+Y) \stackrel{?}{=} V(X) + V(Y)$$

$$
\begin{aligned}
V(X+Y) &= E(X+Y)^2 - (E(X+Y))^2 \\
&= E(X^2 + 2XY + Y^2) - (EX + EY)^2 \\
&= EX^2 + 2E(XY) + EY^2 - (E^2X + 2EX \cdot EY + E^2Y) \\
&= EX^2 - \ E^2X \ + EY^2 - \ E^2Y \ + 2(E(XY) - EX \cdot EY) \\
&= V(X) + V(Y) + 2(E(XY) - EX \cdot EY) \\
&= V(X) + V(Y) + 2\mathrm{Cov}(X,Y)
\end{aligned}
$$

$$= \text{iff } \mathrm{Cov}(X,Y) = 0 \quad \text{Uncorrelated}$$

$V(X+Y)$ may be $\geq$ $=$ or $\leq$ $V(X)+V(Y)$

$$X \perp Y \underset{\longleftrightarrow}{\rightarrow} \sigma_{X,Y} = 0 \underset{\longleftrightarrow}{\rightarrow} V(X+Y) = V(X) + V(Y)$$

$$X \perp\!\!\!\perp Y \rightarrow X \perp Y \rightarrow V(X+Y) = V(X) + V(Y)$$

$$X, Y \sim B(\tfrac{1}{2})$$

$$Y = X \qquad \sigma_{X,Y} = V(X)$$

$$V(X+Y) = V(2X) = 4V(X) = V(X) + V(Y) + 2V(X)$$

$$Y = -X$$

$$V(X+Y) = V(0) - 0 = V(X) + V(X) - 2V(X)$$