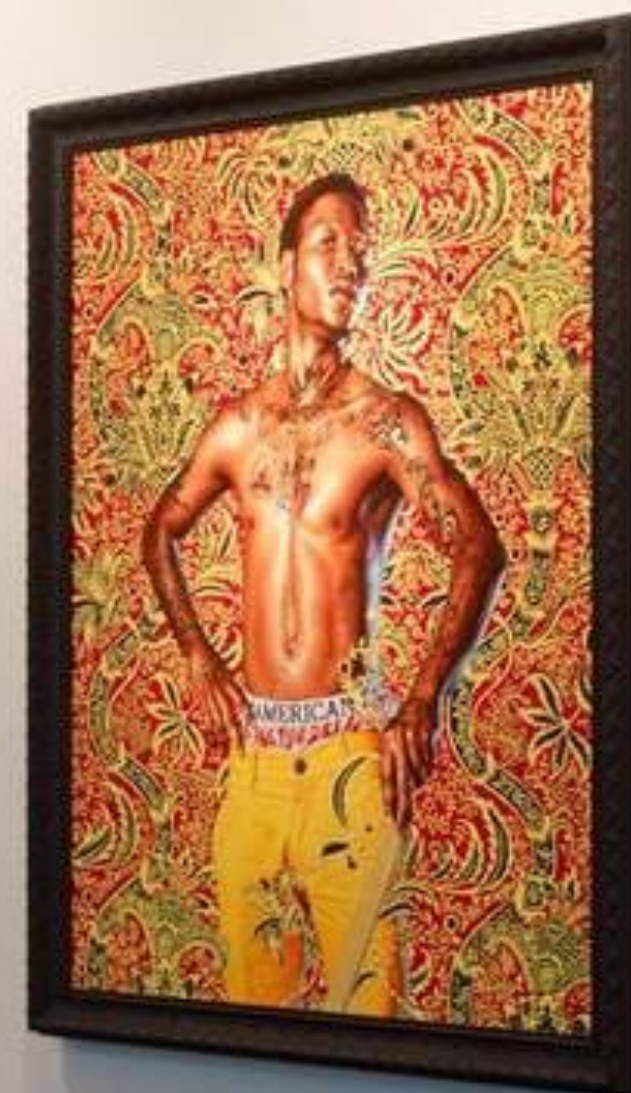Definition

Applications

Motivation

Derivation

μ, σ

Example

Poisson distribution
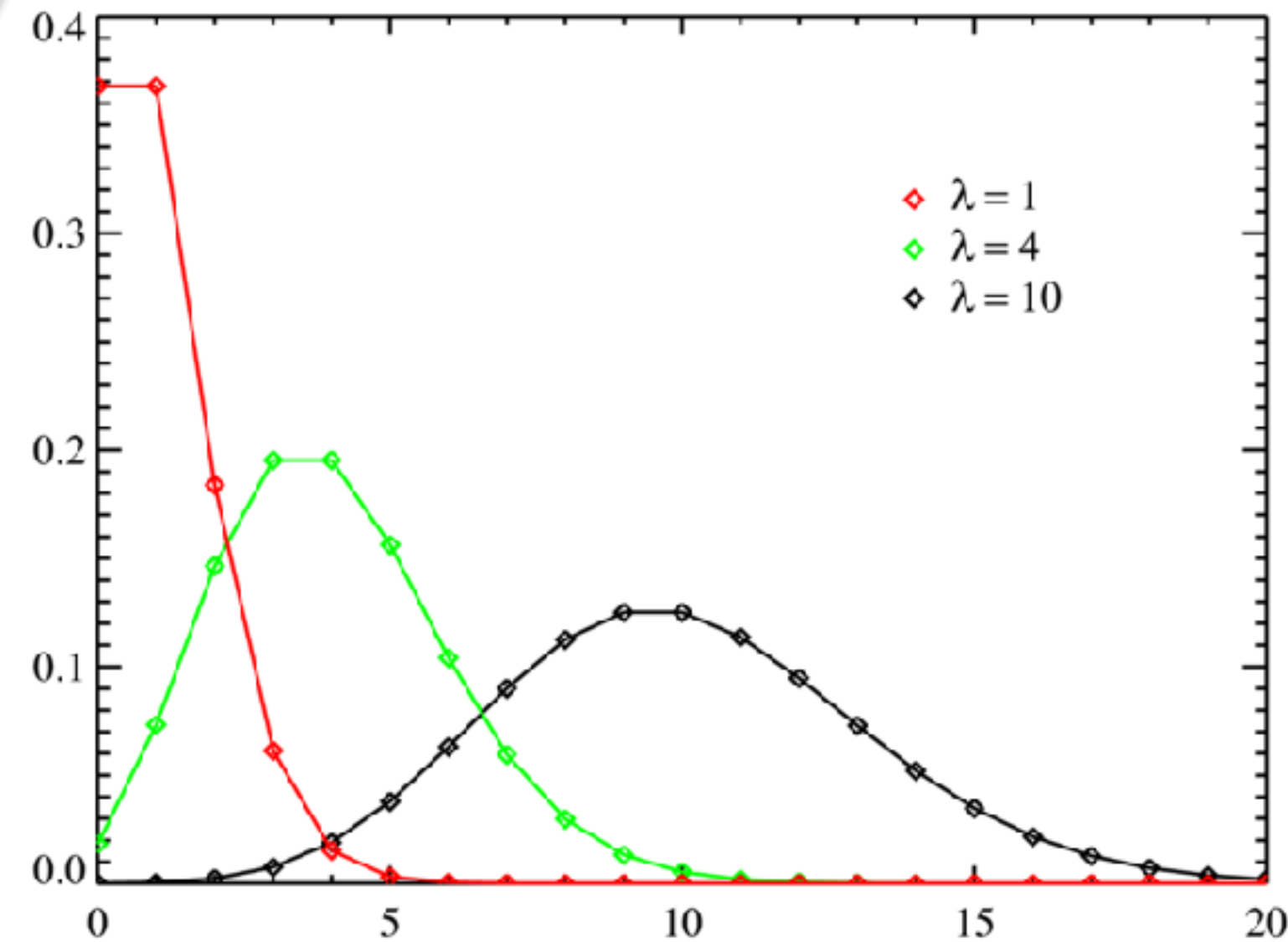
The Poisson Distribution

Parameter $\lambda \geq 0$ (some > 0)

Support $\mathbb{N}$

PMF
$$P_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

More in notebook

Significance Approximates $B_{p,n}$ for large n and small p so that np = λ is moderate

We are Poisson

P_λ approximates $B_{p,n}$ for small p, large n

Numerous applications

People clicking ad

Responses to spam

Rare-disease infections

Daily 911 calls

Daily store customers

Gallery purchasing customers

Flight no shows

Typos in a page

$450M

Small k

$$\overbrace{}^{k}$$

λ	$P_\lambda(k)$	0	1	2	3
General	$e^{-\lambda}\dfrac{\lambda^k}{k!}$	$\dfrac{1}{e^\lambda}$	$\dfrac{\lambda}{e^\lambda}$	$\dfrac{\lambda^2}{2e^\lambda}$	$\dfrac{\lambda^3}{6e^\lambda}$
1	$\dfrac{1}{ek!}$	$\dfrac{1}{e}$	$\dfrac{1}{e}$	$\dfrac{1}{2e}$	$\dfrac{1}{6e}$
2	$\dfrac{2^k}{e^2 k!}$	$\dfrac{1}{e^2}$	$\dfrac{2}{e^2}$	$\dfrac{2}{e^2}$	$\dfrac{4}{3e^2}$
0	$\dfrac{0^k}{k!}$	1	0	0	0

$\dfrac{\lambda^k}{e^\lambda}$

$\downarrow$

$\dfrac{1}{e}$

$\dfrac{2^k}{e^2}$

0^k

Binomial Approximation

$$B_{p,n}(k) = \binom{n}{k} p^k q^{n-k} \qquad q = 1 - p$$

$$p = \frac{\lambda}{n}$$

$$= \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$
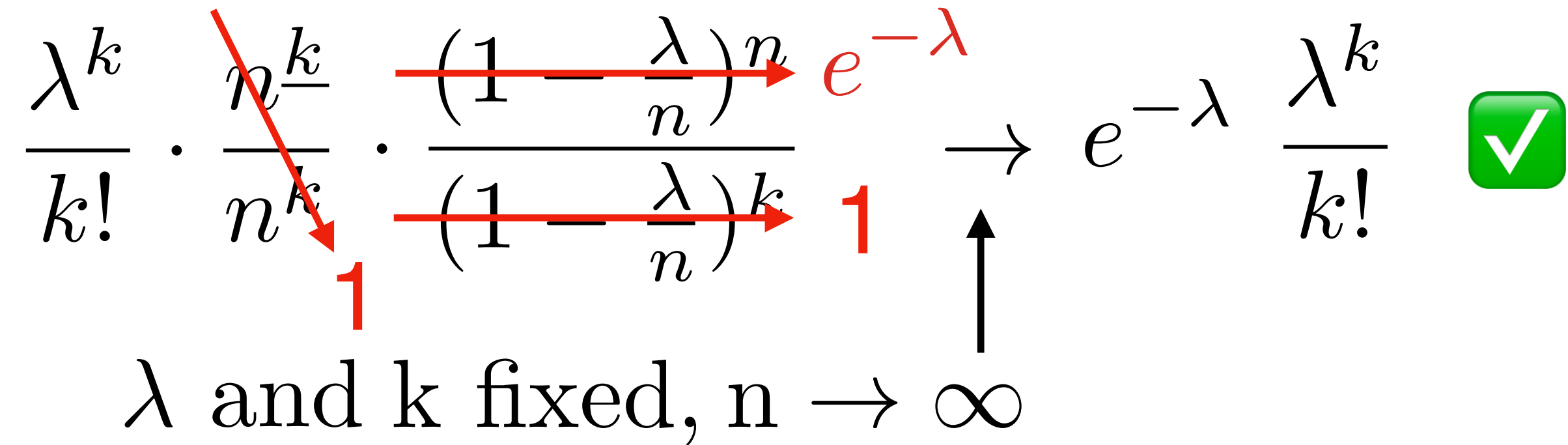
$$= \frac{n^{\underline{k}}}{k!} \cdot \frac{\lambda^k}{n^k} \cdot \frac{(1 - \frac{\lambda}{n})^n}{(1 - \frac{\lambda}{n})^k}$$

Fix k, fixe λ while n ↗ and p ↘   Derive Poisson

Limit of Binomial

$$B_{p,n}(k) = \frac{\lambda^k}{k!} \cdot \frac{n^{\underline{k}}}{n^k} \cdot \frac{(1 - \frac{\lambda}{n})^n}{(1 - \frac{\lambda}{n})^k} \to e^{-\lambda} \frac{\lambda^k}{k!} \quad ✅ \qquad \lambda = p \cdot n$$

$(1 - \frac{\lambda}{n})^n \to e^{-\lambda}$

$(1 - \frac{\lambda}{n})^k \to 1$

λ and k fixed, n $\to \infty$

① $\dfrac{n^{\underline{k}}}{n^k} = \dfrac{n}{n} \cdot \dfrac{(n-1)}{n} \cdot \ldots \cdot \dfrac{(n-k+1)}{n} \to 1$ fixed # (k) terms, each → 1

$\to 1 \quad \to 1 \quad \to 1$

② $(1 - \frac{\lambda}{n})^k \to 1$ fixed # (k) terms, each → 1

③ $(1 - \frac{\lambda}{n})^n = ((1 - \frac{\lambda}{n})^{\frac{n}{\lambda}})^\lambda \to (e^{-1})^\lambda = e^{-\lambda}$

increasing # terms, each → 1 $(1 - \frac{1}{m})^m \to e^{-1}$

Σ WILL IT ADD?

$$P_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad k\geq 0$$

≥ 0 ✅

Taylor expansion $\qquad e^\lambda = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$

$$\sum_{k=0}^{\infty}P_\lambda(k) = \sum_{k=0}^{\infty}e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^\lambda = 1$$

YES IT ADDS!

Mean and Variance

P_λ approximates $B_{p,n}$ for $\lambda = np$ when $n \gg 1 \gg p$

	μ	V
$B_{p,n}$	np	npq
P_λ	λ	λ

$\longleftarrow$ Expect

Calculate next

Observation

$$\frac{d}{d\lambda} \lambda^k = k\lambda^{k-1} = \frac{k}{\lambda} \lambda^k$$

$$\frac{d^2}{d\lambda^2} \lambda^k = k^{\underline{2}} \lambda^{k-2} = \frac{k^{\underline{2}}}{\lambda^2} \lambda^k$$

$$\frac{d^r}{d\lambda^r} \lambda^k = k^{\underline{r}} \lambda^{k-r} = \frac{k^{\underline{r}}}{\lambda^r} \lambda^k$$

$$k^{\underline{r}} \lambda^k = \lambda^r \frac{d^r}{d\lambda^r} \lambda^k$$

Falling Moments

$$X \sim P_\lambda$$

$$k^{\underline{r}}\lambda^k = \lambda^r \frac{d^r}{d\lambda^r}\lambda^k$$

$$E(X^{\underline{r}}) = \sum_{k=0}^{\infty} k^{\underline{r}} P_\lambda(k) = \sum_k k^{\underline{r}} e^{-\lambda}\frac{\lambda^k}{k!}$$

$$= e^{-\lambda}\sum_k k^{\underline{r}}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_k \frac{\lambda^r}{k!}\frac{d^r}{d\lambda^r}\lambda^k$$

$$= e^{-\lambda}\lambda^r\frac{d^r}{d\lambda^r}\sum_k \frac{\lambda^k}{k!} = e^{-\lambda}\lambda^r\frac{d^r}{d\lambda^r}e^\lambda$$

$$= e^{-\lambda}\lambda^r e^\lambda = \lambda^r$$

$$EX = EX^{\underline{1}} = \lambda \qquad\qquad EX(X-1) = EX^{\underline{2}} = \lambda^2$$

Mean and Variance

$$EX = EX^{\underline{1}} = \lambda \quad ✅$$

$$EX(X-1) = EX^{\underline{2}} = \lambda^2$$

$$E(X^2) = E(X(X-1)+X) = E(X(X-1)) + E(X) = \lambda^2 + \lambda$$

$$V(X) = E(X^2) - (EX)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda \quad ✅$$

$$\sigma = \sqrt{\lambda} \qquad \text{Small relative to the mean}$$

Approximation Example

P(3 defective)?

Binomial (precise) $\qquad B_{0.01,200}(3) = \binom{200}{3}(0.01)^3(0.99)^{197} \approx 0.181$

Poisson (approximation) $\quad \lambda = 200 \cdot 0.01 = 2 \qquad P_2(3) = e^{-2}\dfrac{2^3}{3!} \approx 0.18$

P(some defective)?

$B_{0.01,200}(0) = \binom{200}{0}(0.99)^{200} \approx 0.134 \qquad\qquad P_2(0) = e^{-2}\dfrac{2^0}{0!} = e^{-2} \approx 0.135$

$B_{0.01,200}(\geq 1) = 1 - 0.134 \approx 0.866 \qquad\qquad P_2(\geq 1) = 1 - 0.135 \approx 0.865$

$$P_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$\lambda \geq 0$

$k \geq 0$

Approximates $B_{p,n}$ for $\lambda = np$, when $n \gg 1 \gg p$

# of ad clicks, rare diseases, production defects

$\mu = \lambda$

$V = \lambda$

$\sigma = \sqrt{\lambda}$

Geometric Distribution

Poisson

distribution