Combinatorics

Permutations

Permutations

abc, acb, bac, bca, cab, cba

Factorials

n!

1·2·…n

Anagrams

Constrained

Circular arrangements

Musical Chairs, Bryant Park, NYC, Summer 2013

Permutations

A permutation is an ordering of a set of objects

permutations of n objects?

Objects can be anything

For most excitement

Letters!

# letters	permu-tations	# permu-tations
1	a	1
2	a b b a	2
3	a b c a c b b a c b c a c a b c b a	6

General n?

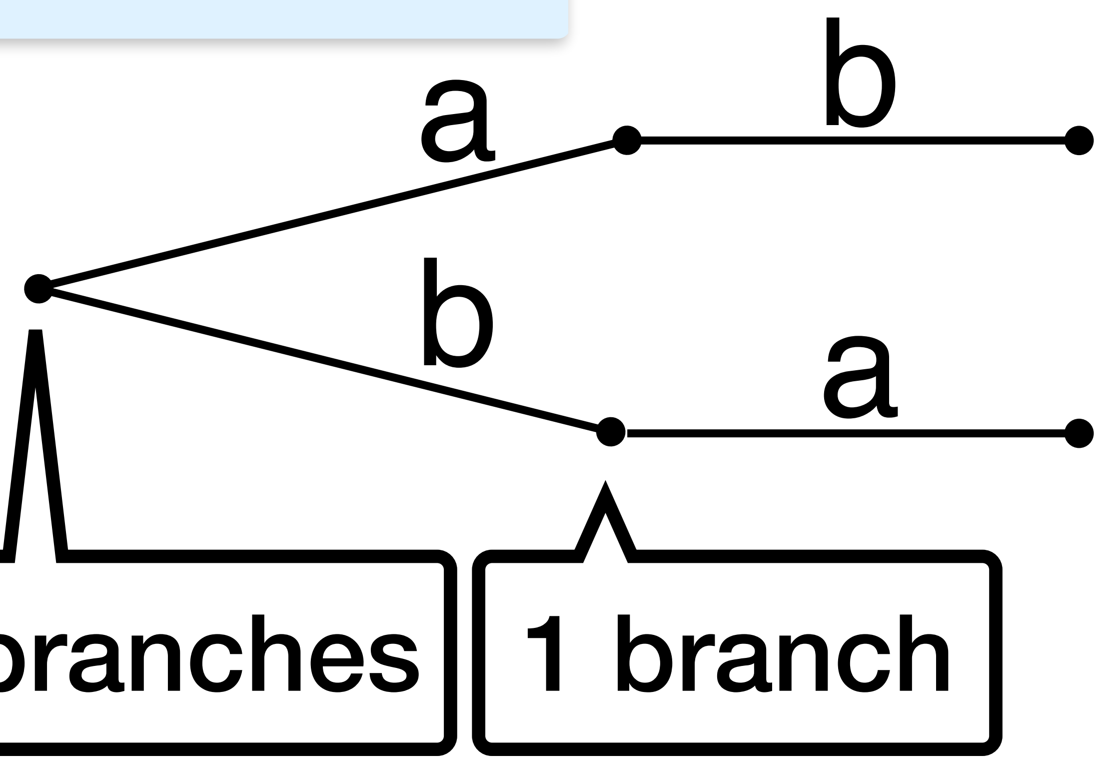
Counting Permutations

2 objects

2 choices

1 choice

a b
b a



$2 \times 1 = 2$

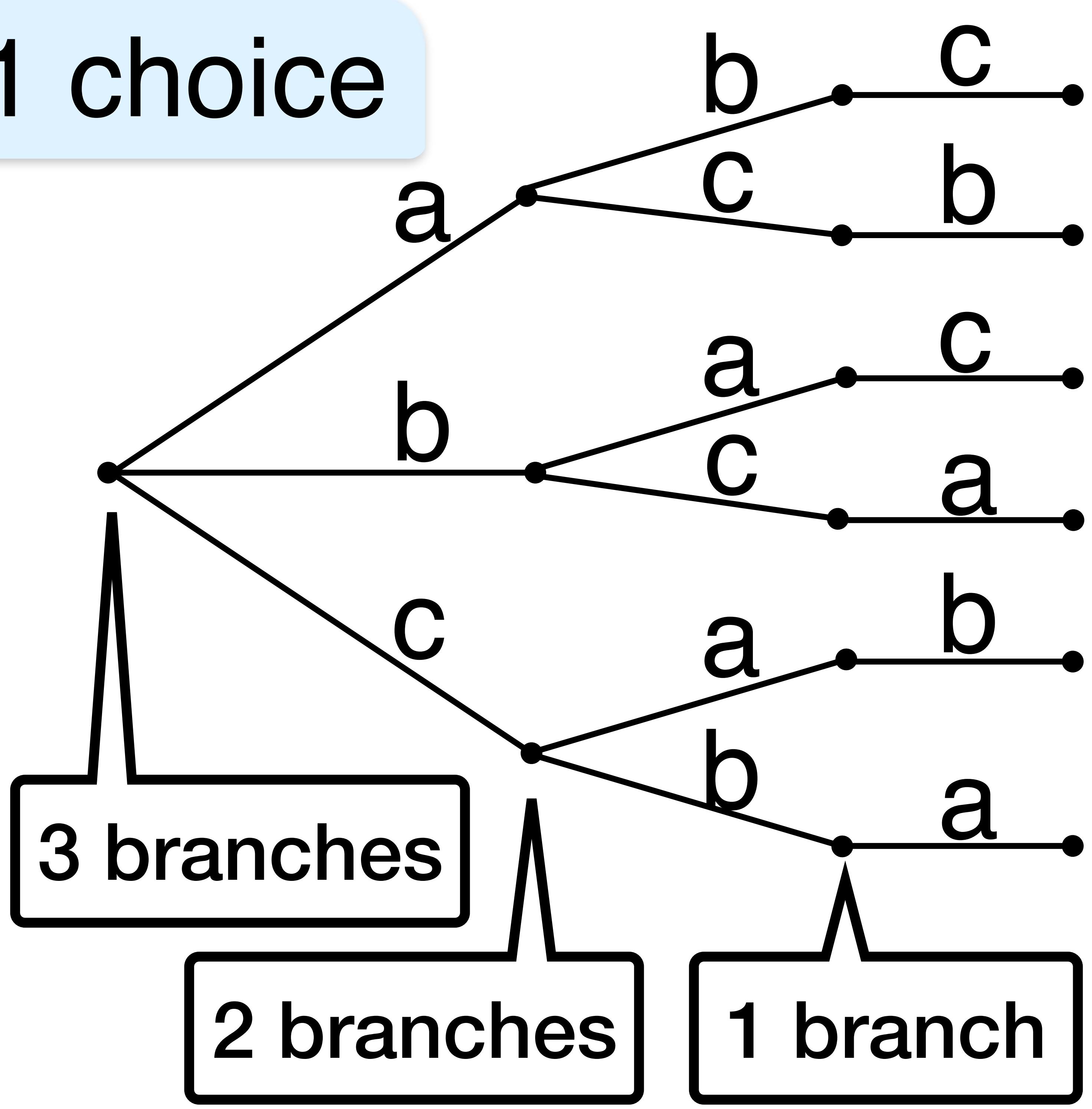
3 objects

3 choices

2 choices

1 choice

a b c
a c b
b a c
b c a
c a b
c b a



“X”

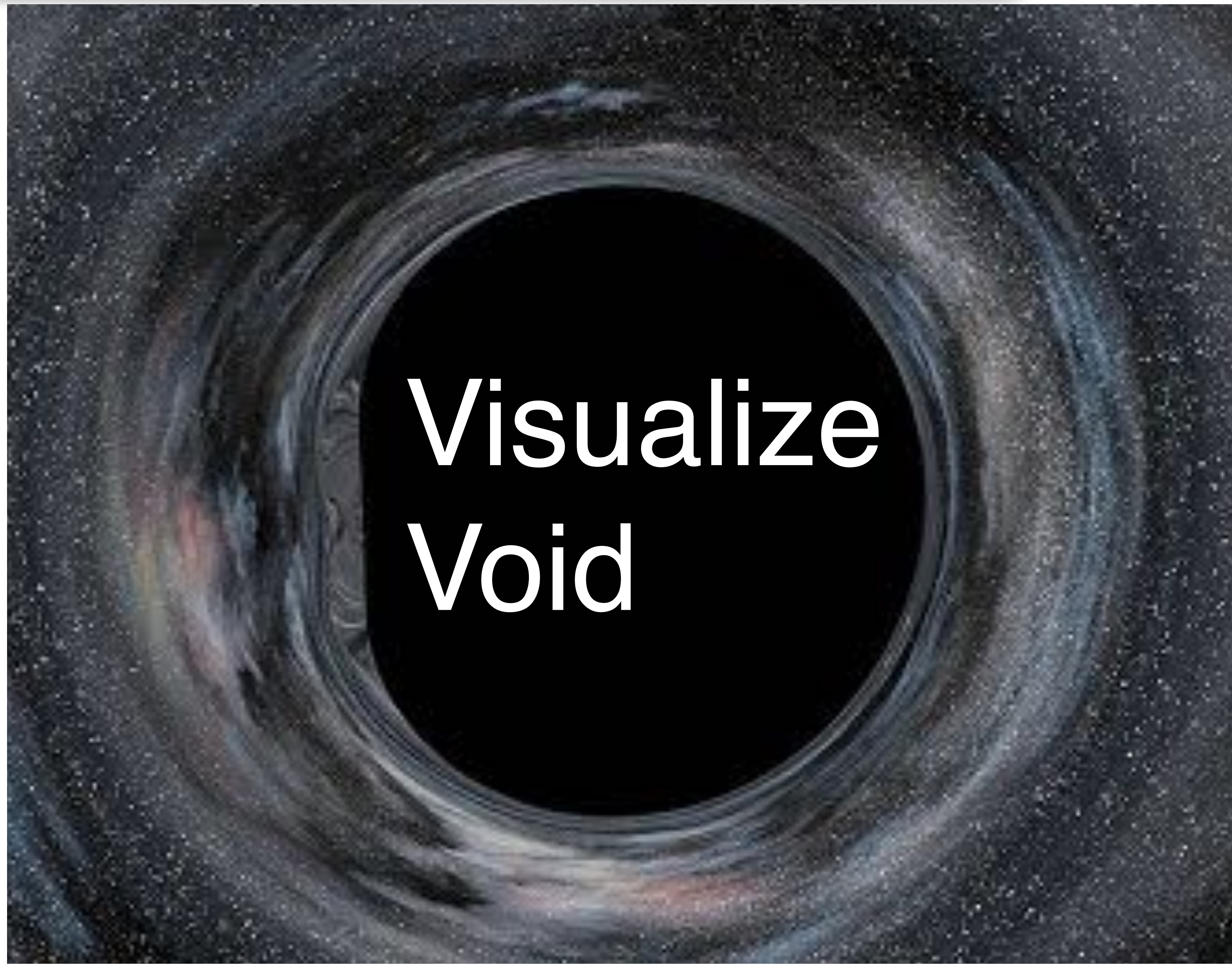
$3 \times 2 \times 1 = 6$

permutations of n objects $= n \times (n-1) \times \ldots \times 2 \times 1 \triangleq n!$

n factorial

0 Factorial

For $n \geq 1$ $n! = \#$ permutations of n objects $= n \times (n-1) \times \ldots \times 2 \times 1$



What about 0! ?

How many ways can you permute 0 objects?

a,b: ab, ba a: a : ?

a,b: (ab), (ba) a: (a) : ( )

$0! = 1$

Exact same reason as $2^0=1$

Alternative Factorial View

Write Left to right → Smallest to largest

One position for the 1st object

Two for 2nd • 1 •

Three for 3rd • 2 • 1 •

Four for 4th • 2 • 1 • 3 •

⋮

1 x 2 x 3 x… x n = n!

Recursive Definition

n! can be defined recursively

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$$

$$= n \cdot [(n-1) \cdot \ldots \cdot 2 \cdot 1]$$

$$= n \cdot (n-1)! \qquad \forall n \geq 1$$

1!=1x0!
Extends to negatives

n	Product	n!	
0	1	1	
			x 1
1	1	1	
			x 2
2	2 × 1	2	
			x 3
3	3 × 2 × 1	6	
			x 4
4	4 × 3 × 2 × 1	24	
			x 5
5	5 × 4 × 3 × 2 × 1	120	
			x 6
6	6 × 5 × 4 × 3 × 2 × 1	720	

Examples and applications

Basic Permutations

orders to visit 3 cities

LA, SD, SF

$$\left.\begin{array}{l} \text{LA SD SF} \\ \cdots\cdots \\ \text{SF SD LA} \end{array}\right\} 3! = 3 \times 2 \times 1 = 6$$

anagrams of 5 distinct letters

PEARS

$$\left.\begin{array}{l} \text{SPEAR} \\ \cdots\cdots \\ \text{EAPRS} \end{array}\right\} 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Constrained Anagrams of PEARS

A,R stay adjacent in order

PARSE
.....
SEPAR

Permutations of P E AR S

$4! = 4 \times 3 \times 2 \times 1 = 24$

A,R are adjacent in any order

SPARE
.....
RAESP

2 orders

24 anagrams each

"X"

$2 \times 24 = 48$

A,R are not adjacent

AESPR
.....
SRPAE

$-$

$5! - 48 = 120 - 48 = 72$

More Constrained Permutations

ways 3 distinct boys and 2 distinct girls can stand in a row

Unconstrained

$(3+2)! = 5! = 120$

Alternating boys and girls

$3! \times 2! = 6 \times 2 = 12$

Boys together and girls together

$2 \times 3! \times 2! = 24$

Unconstrained, but orientation (left to right) doesn’t matter

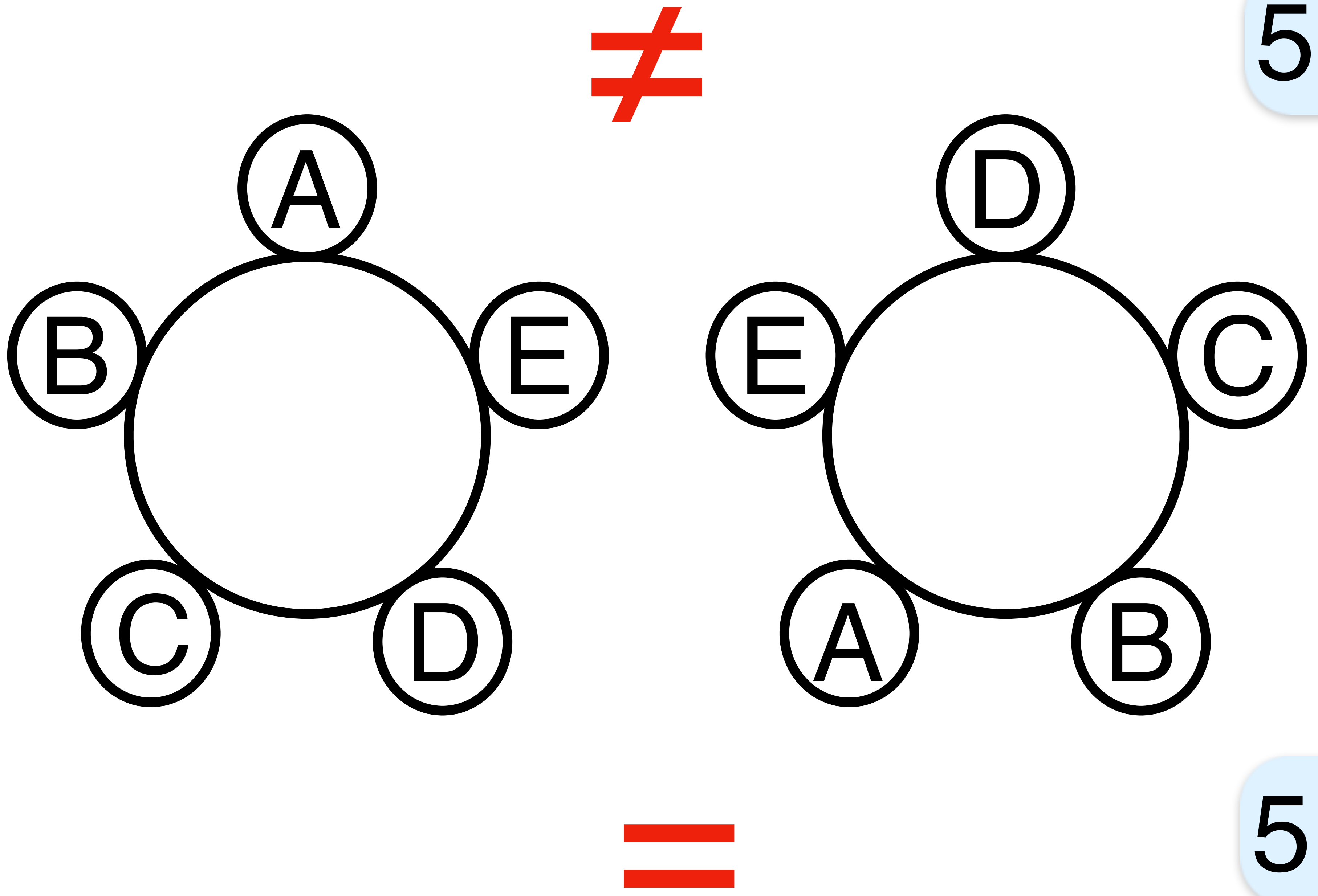
$5! / 2 = 60$

Circular Arrangements

ways 5 people can sit at a round table = ?

Rotations matter

Rotations don't matter



5! = 120

5! / 5 = 4! = 24

Combinatorics

Permutations

Permutations — abc, acb, bac, bca, cab, cba

Factorials — $n!$ — $1 \cdot 2 \cdot \ldots n$

Anagrams

Constrained

Circular arrangements

Musical Chairs, Bryant Park, NYC, Summer 2013