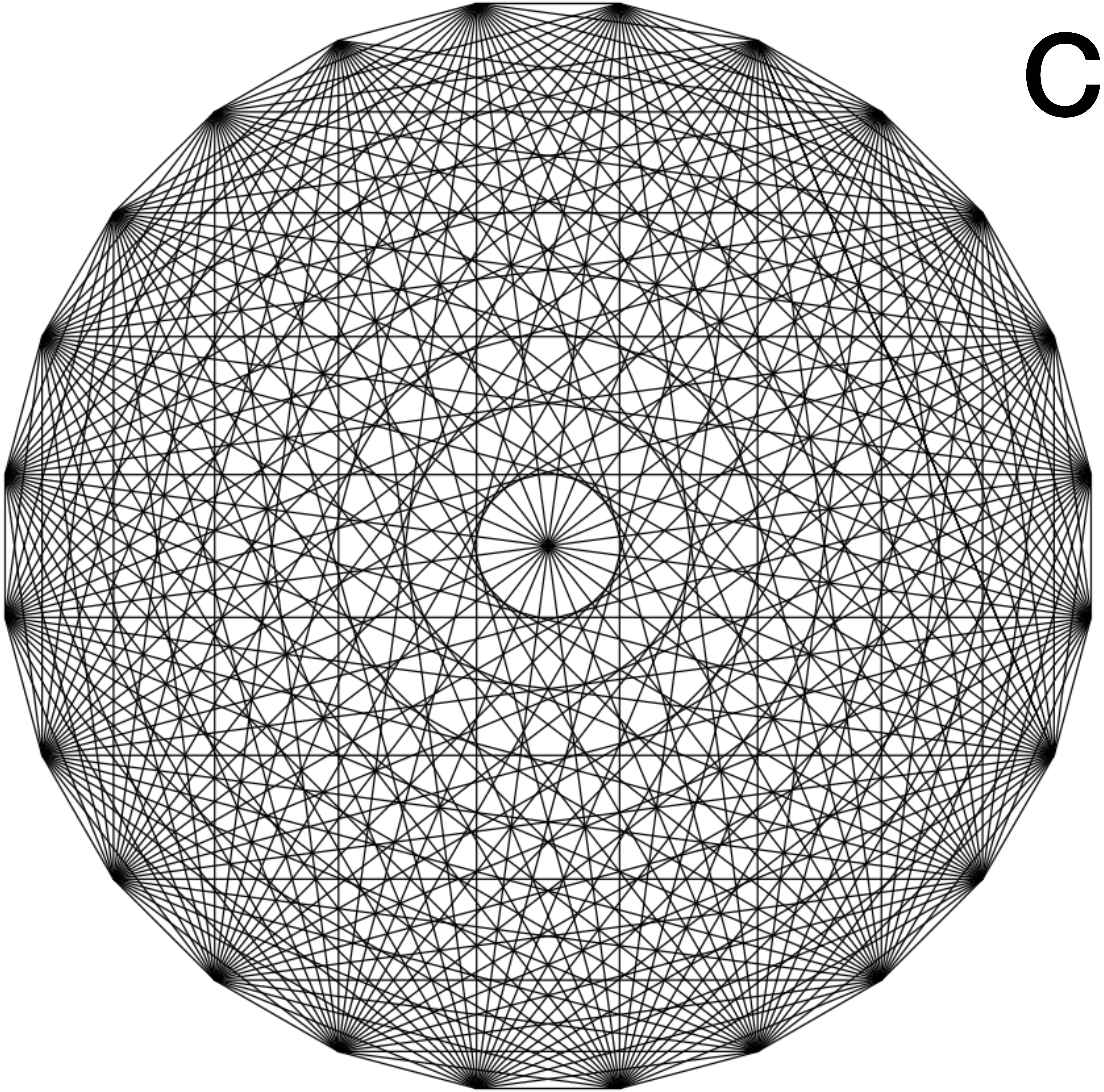


Combinations

k-subs

A k-element set is called a k-set, a k-element subset is a k-subset

$\binom{[n]}{k}$ collection of k-subsets of [n] = {1,2,…,n}

$\binom{[3]}{1}$ { {1}, {2}, {3} }

$\binom{[3]}{2}$ { {1,2}, {1,3}, {2,3} }

$\binom{[4]}{2}$ { {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4} }

Sequences with k 1's

$\binom{[n]}{k}$ collection of k-subsets of [n] = {1,2,…,n}

1-1 correspondence to n-bit sequences with k 1's

	Subsets	Binary Sequences
$\binom{[3]}{1}$	{1}, {2}, {3}	100, 010, 001
$\binom{[3]}{2}$	{1, 2}, {1, 3}, {2, 3}	110, 101, 011
$\binom{[4]}{2}$	{1, 2}, {1, 3}, …, {3, 4}	1100, 1010, …, 0011

Two Interpretations

$\binom{[n]}{k}$

k-subsets of an n-set

n-bit sequences with k 1's

$\binom{[3]}{2}$

$\{\{1,2\}, \{1,3\}, \{2,3\}\}$

110, 101, 011

1 - 1 correspondence

Same number of elements

Mostly count sequences

Same applies to subsets

Number of n-Bit Sequences with k 1's

$$\binom{n}{k} \triangleq \left|\binom{[n]}{k}\right|$$ = # n-bit sequences with k 1's

binomial coefficient

$$\binom{3}{2} = \left|\binom{[3]}{2}\right| = |\{110, 101, 011\}| = 3$$

Locations of 1's: Ordered Pairs from {1,2,3}

$$\# = 3^{\underline{2}} = 6$$

12	110
13	101
21	110
23	011
31	101
32	011

110

101

011

$$\binom{3}{2} = \frac{3^{\underline{2}}}{2} = \frac{6}{2} = 3$$

Calculating the Binomial Coefficients

$$\binom{n}{k} \triangleq \left|\binom{[n]}{k}\right| = ?$$

number set

Each location $\in$ [n]

Specify locations of the k 1's in order

123, 531, 213, …

\# ordered locations $n^{\underline{k}}$

Every binary sequence with k 1's corresponds to k! ordered locations

10101 $\leftrightarrow$ 1,3,5 1,5,3 3,1,5 3,5,1 5,1,3 5,3,1

$$k!\binom{n}{k} = n^{\underline{k}}$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}$$

$$\left|\binom{[n]}{k}\right| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$ **for small n,k**

$$\binom{[3]}{1} = \left\{\begin{matrix} 001 \\ 010 \\ 100 \end{matrix}\right\} \qquad \binom{3}{1} = \frac{3!}{1!\,2!} = 3$$

Choose location of 1

$$\binom{[3]}{2} = \left\{\begin{matrix} 011 \\ 101 \\ 110 \end{matrix}\right\} \qquad \binom{3}{2} = \frac{3!}{2!\,1!} = 3$$

Choose location of 0

$$\binom{[4]}{2} = \left\{\begin{matrix} 0011 \\ 0101 \\ 0110 \\ 1001 \\ 1010 \\ 1100 \end{matrix}\right\} \qquad \binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

Choose location of 1's
1st location: 4 choices
2nd location: 3 choices
Each chosen twice

Simple $\binom{n}{k}$

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1$$

All-zero sequence

$$\binom{n}{n} = \frac{n!}{n!\,0!} = 1$$

All-one sequence

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = n$$

Choose location of single 1

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$$

1st location: n ways
2nd location: n-1 ways
Each sequence chosen twice

Alternative Explanation of $\binom{n}{2} = \frac{n(n-1)}{2}$

$\binom{[n]}{2} = \{\text{n-bit strings with two 1's}\}$

$A_i = \{x^n\text{: first 1 at location i}\} \quad (1 \le i \le n-1)$

$|A_i| = n - i \qquad A_i\text{'s disjoint}$

$$\underset{\substack{\uparrow \\ 1}}{0} \;\dots\; 0 \;\; \underset{\substack{\uparrow \\ i}}{1} \;\; \overbrace{x \;\; y \;\dots\; \underset{\substack{\uparrow \\ n}}{z}}^{\text{n-i locations w/ exactly one 1}}$$

$\binom{[n]}{2} = A_1 \uplus A_2 \uplus \ldots \uplus A_{n-1}$

$\binom{n}{2} = \left|\binom{[n]}{2}\right| = |A_1| + \ldots + |A_{n-1}| = (n-1) + \ldots + 1 = \frac{n(n-1)}{2}$

Binomial Coefficients by Hand

Simply calculate binomial coefficients

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

$$\binom{7}{2} = \frac{7 \cdot \cancel{6}^{\,3}}{\cancel{2} \cdot 1} = 7 \cdot 3 = 21$$

$$\binom{n}{k} = \binom{n}{n-k}, \text{ if k} > \frac{n}{2} \text{ calculate } \binom{n}{n-k}$$

$$\binom{7}{5} = \binom{7}{2} = 21$$

$$\text{Cancel terms}$$

$$\binom{12}{9} = \binom{12}{3} = \frac{\overset{2}{\cancel{12}} \cdot 11 \cdot 10}{\cancel{3} \cdot \cancel{2} \cdot 1} = 220$$

Can calculate fairly large binomial coefficients by hand

Permutations vs. Combinations

Permutations

Order matters

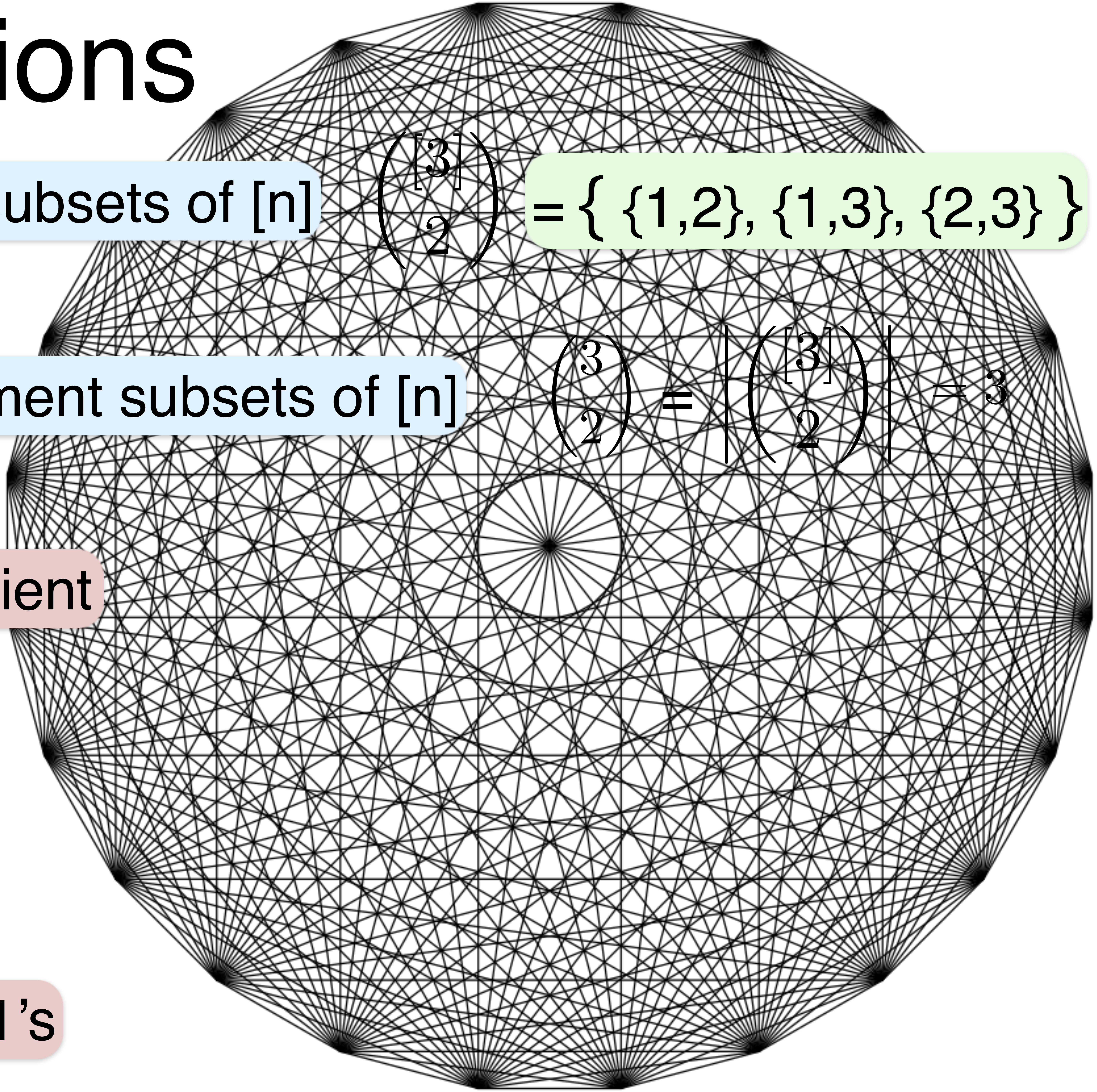
Combinations

Order doesn't matter

Type of lock you used to secure your bike?

Combinations

Combinations



$\binom{[n]}{k}$ Collection of k-element subsets of [n]

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$\binom{n}{k} \triangleq \left|\binom{[n]}{k}\right|$ Number of k-element subsets of [n]

$\binom{3}{2} = \left|\binom{[3]}{2}\right| = 3$

n choose k

Binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$\binom{n}{k}$ = # n-bit sequences with k 1’s