Sequential Probability

General product rule

Chain rule

$P(E \cap F) = P(E) \cdot P(F|E)$

More events

Useful in applications

Birthday paradox

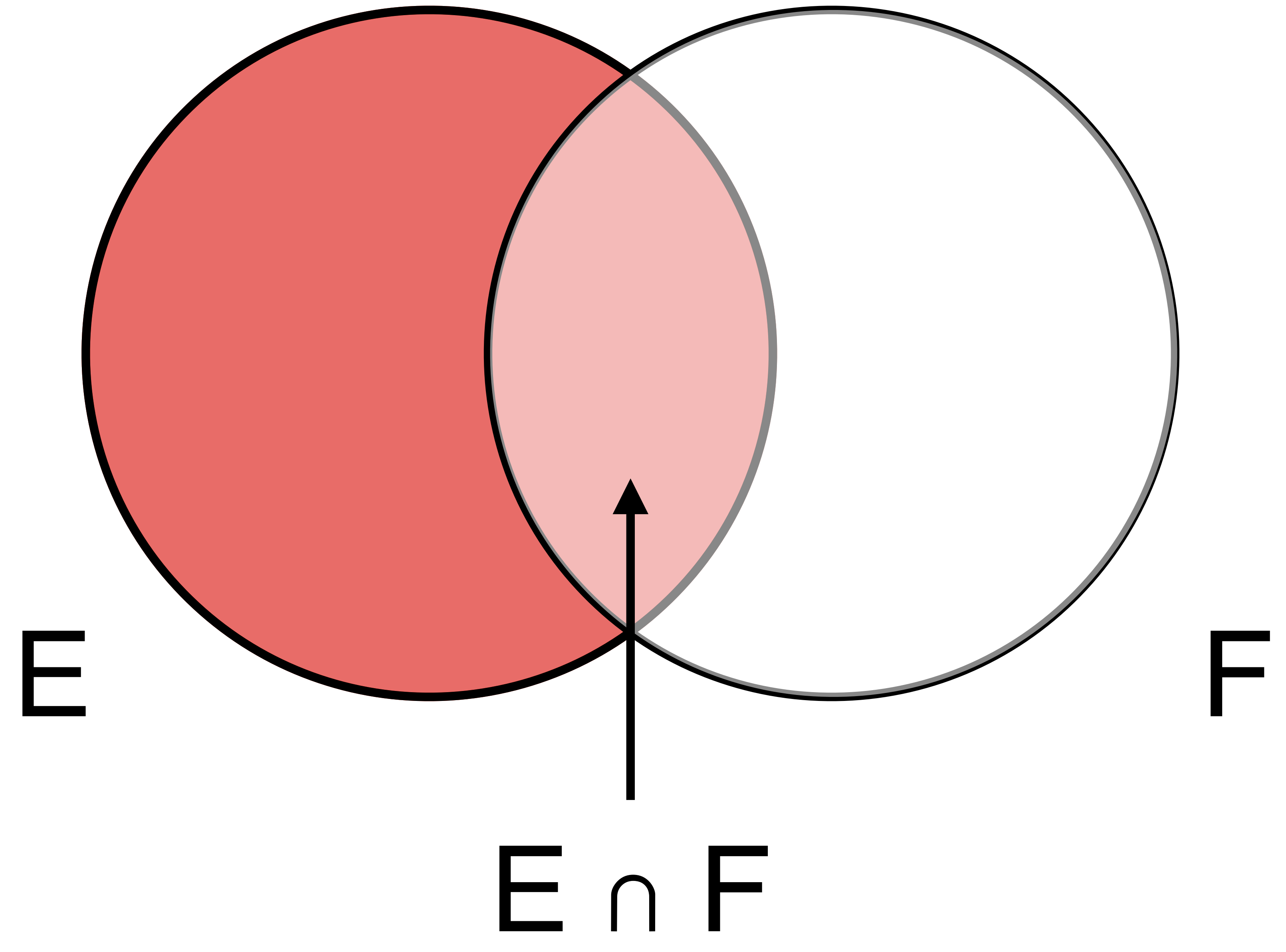
Of 23 people, $p>0.5$ that two share a birthday

Chain Rule

Conditional probability

$$P(F \mid E) = \frac{P(E \cap F)}{P(E)}$$

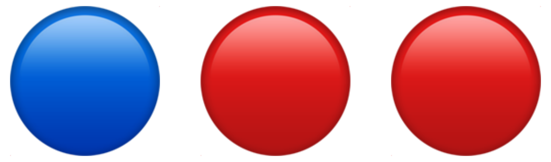
$$P(E \cap F) = P(E) \cdot P(F \mid E)$$



Helps calculate regular (not conditional) probabilities

Sequential Selection

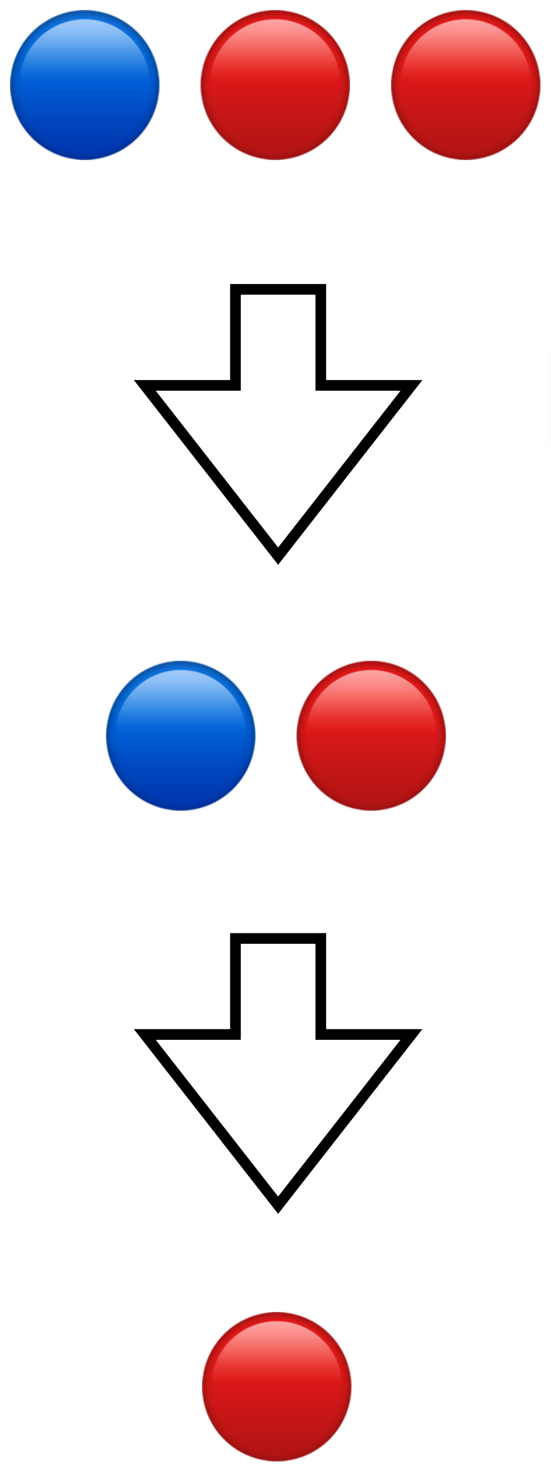
1 blue, 2 red balls



Draw 2 balls without replacement

P(both red) = ?

R_i - i'th ball is red



$P(R_1) = \frac{2}{3}$

$P(R_2 \mid R_1) = \frac{1}{2}$

P(both red) $= P(R_1, R_2)$

$= P(R_1) \cdot P(R_2 \mid R_1)$

$= \frac{2}{3} \cdot \frac{1}{2}$ $= \frac{1}{3}$

General Product Rule

For 3 events

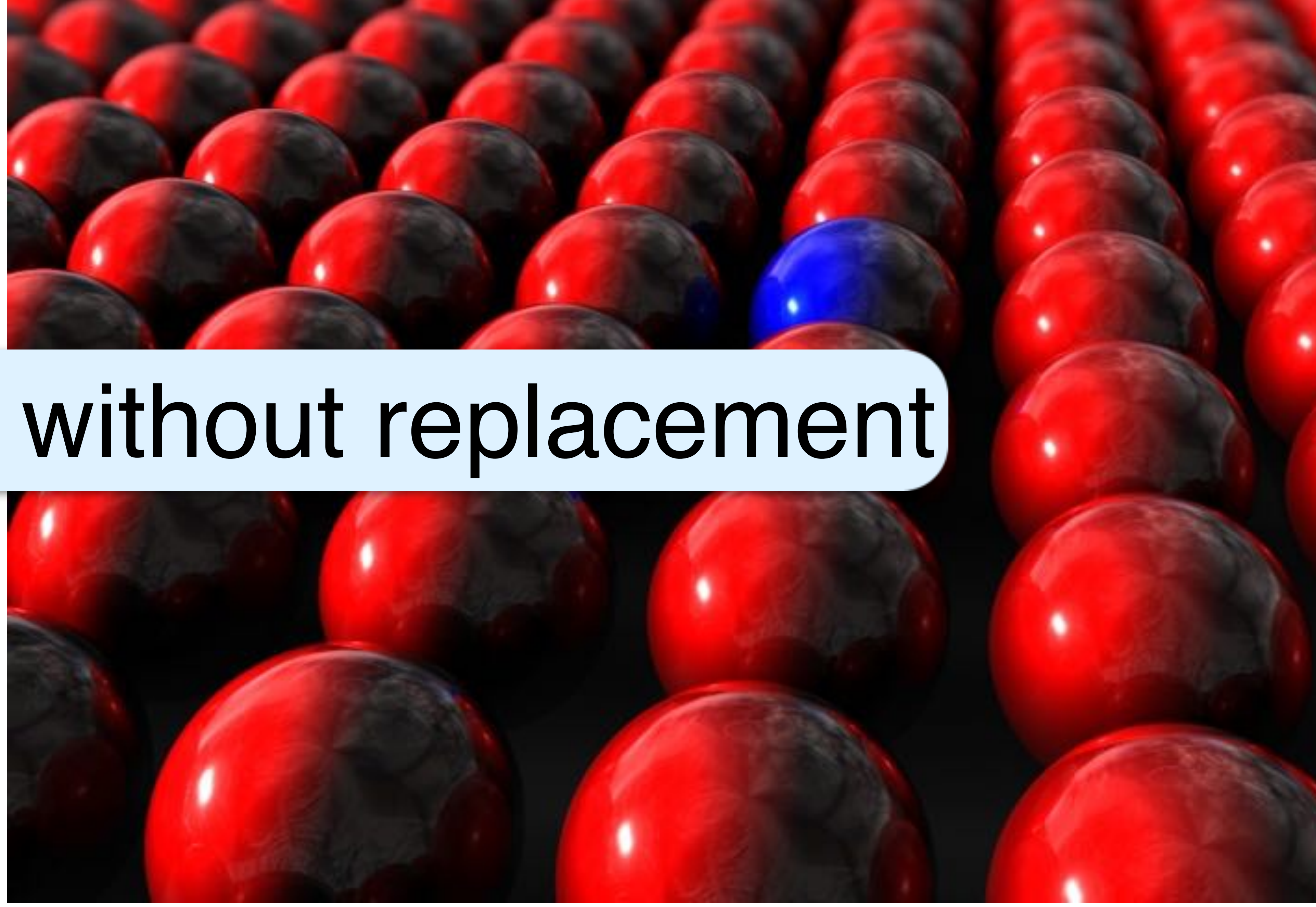
$P(E \cap F \cap G)$ $= P((E \cap F) \cap G)$

$= P(E \cap F) \cdot P(G \mid E \cap F)$

$= P(E) \cdot P(F \mid E) \cdot P(G \mid E \cap F)$

Similarly for more events

Odd Ball



n-1 red balls and one blue ball

Pick n balls without replacement

P(last ball is blue) = ?

R_i - ith ball is red

R^i - $R_1, R_2, \ldots, R_i$

$$P(\text{last ball blue}) = P(R_1)P(R_2|R_1)P(R_3|R^2)\ldots P(R_{n-1}|R^{n-2})$$

$$= \frac{n-1}{n}\,\frac{n-2}{n-1}\,\frac{n-3}{n-2}\cdots\frac{2}{3}\,\frac{1}{2} \qquad = \frac{1}{n}$$

Or..

Arrange in row, probability last ball is blue = 1/n

The Birthday Paradox

How many people does it take so that two will likely share a birthday?

- Assume that every year has 365 days
- Everyone is equally likely to be born on any given day

Probabilistically

- Choose n random integers, each $\in \{1, \dots, 365\}$, with replacement
- $B(n)$ - probability that two (or more) are the same
- For which n does $B(n)$ exceed, say, $1/2$?

Some first think it $n \approx 365$, but in fact much smaller

First Attempt

Consider the n people in order, say alphabetically

List their birthdays

2, 10, 365, 180, 10, …

Selection with replacement

Set of all possible birthdays sequences

$\Omega = \{1,2,\ldots, 365\}^n$

$|\Omega| = 365^n$

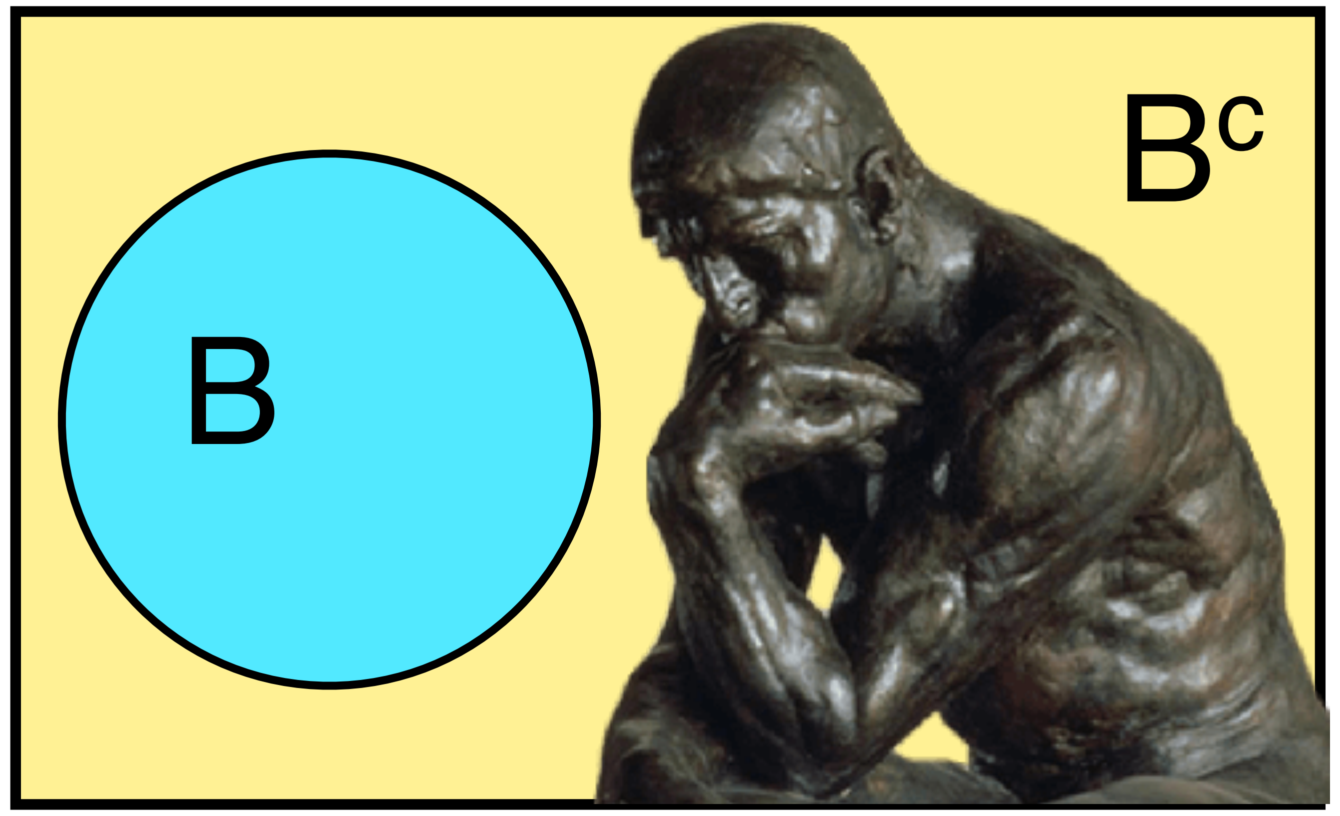
Individual birthday uniform $\rightarrow$ Ω uniform

B_n {sequences with repetition}

$P(\text{ repetition }) = |B_n| / |\Omega|$

Evaluating $|B_n|$ involved

Complement



B_n n people have birthday repetition

B_n^c n people, no two share a birthday

Evaluate sequentially

Person i different b/day from all previous

Calculation

$$1 - x \le e^{-x}$$

Among n people

P(no two people share a birthday)

$$= \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{365-n+1}{365}$$

$$= \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right)$$

$$\le \prod_{i=1}^{n} e^{-\frac{i}{365}}$$

$$= \exp\left(-\frac{1}{365} \cdot \sum_{i=1}^{n-1} i\right)$$

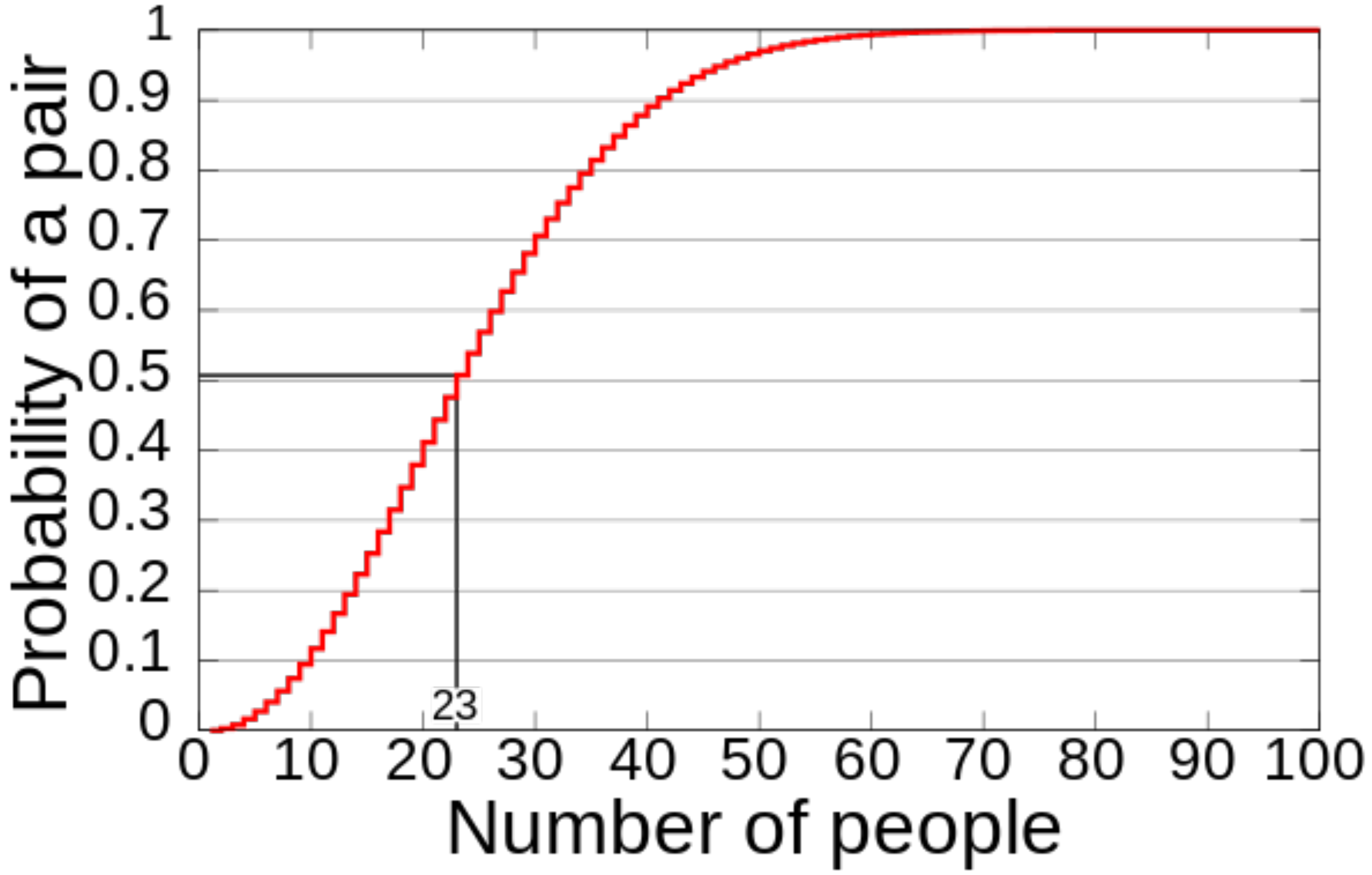
$$= \exp\left(-\frac{n(n-1)}{2 \cdot 365}\right)$$

$$\approx \exp\left(-\frac{n^2}{2 \cdot 365}\right) = 0.5$$

When the probability is 0.5

$$-\frac{n^2}{2 \cdot 365} = \ln 0.5 = -\ln 2$$

$$n \approx \sqrt{-2 \cdot 365 \cdot \ln 0.5} = 22.494$$



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This lecture: Sequential Probability

Next: Total Probability