# Paths in Graphs: Dijkstra's Algorithm

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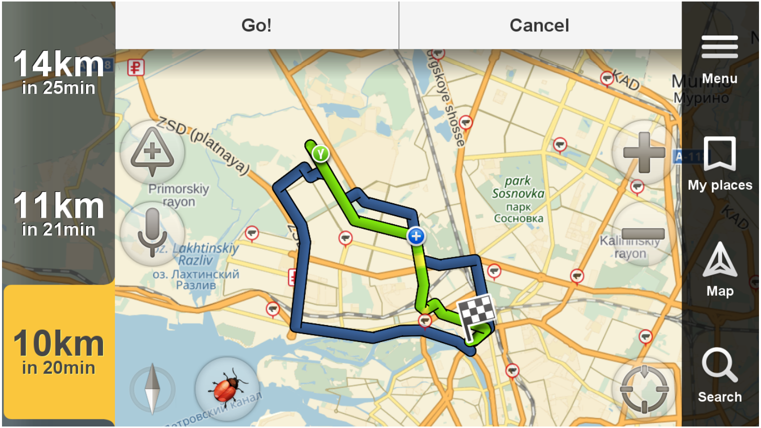
## Graph Algorithms
## Algorithms and Data Structures

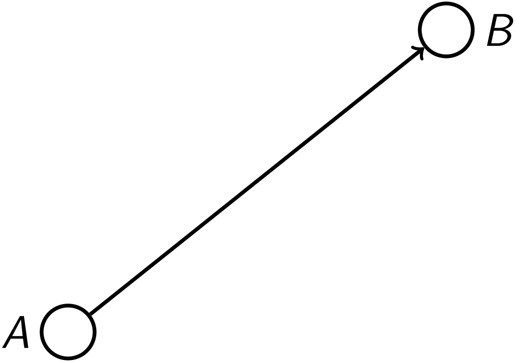
# Outline

1. Fastest Route
2. Naive Algorithm
3. Dijkstra's Algorithm
4. Dijkstra Example
5. Implementation
6. Proof of Correctness
7. Analysis

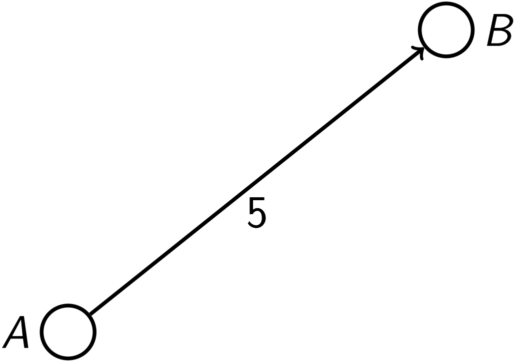
# Fastest Route

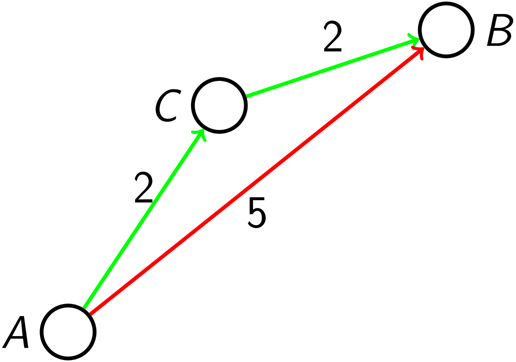
What is the fastest route to get home from work?

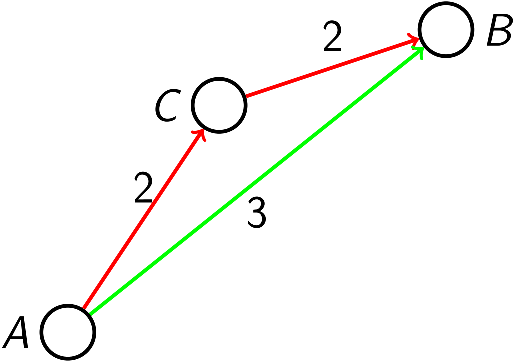




A
B

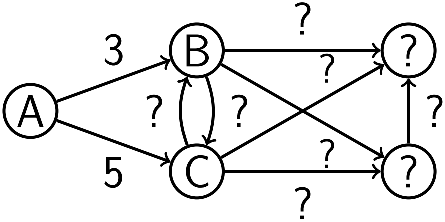

B
5
A


B
C
A
2
2
5


C
B
A
2
2
3

# Intuition

- Assume that we stay at $A$ and observe two outgoing edges:



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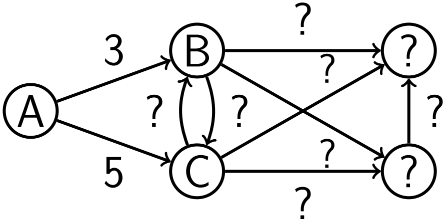


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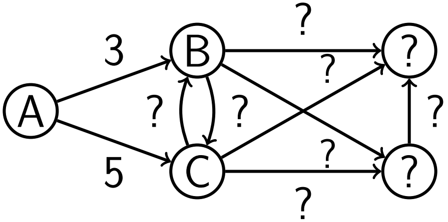

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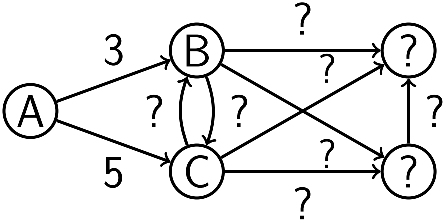


- No, because the weight of the edge $(B, C)$ might be equal to, say, 1.

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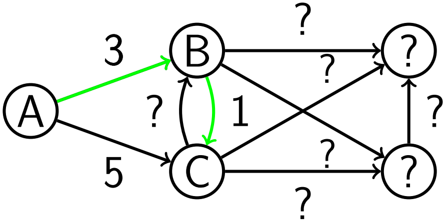


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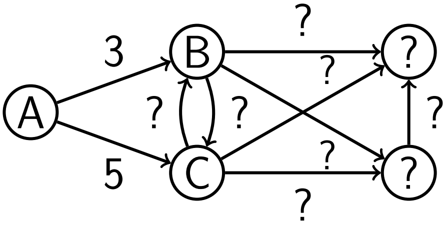
# Intuition

- Can we be sure that the distance from $A$ to $B$ is 3?



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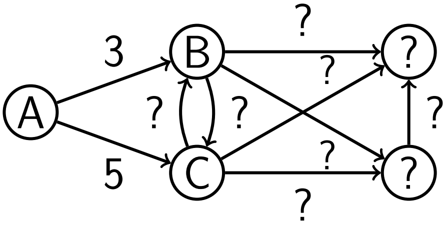



- Yes, because there are no negative weight edges.

# Outline

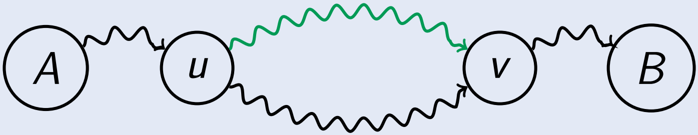
# Optimal substructure

## Observation

Any subpath of an optimal path is also optimal.

## Proof

Consider an optimal path from $A$ to $B$ and two vertices $u$ and $v$ on this path. If there were a shorter path from $u$ to $v$ we would get a shorter path from $A$ to $B$.



# Corollary

If $A \to \ldots \to u \to B$ is a shortest path from $A$ to $B$, then

$$d(A, B) = d(A, u) + w(u, B)$$

# Edge relaxation

- `dist`[$v$] will be an upper bound on the actual distance from $A$ to $v$.

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- `dist`$[v]$ will be an upper bound on the actual distance from $A$ to $v$.
- The edge relaxation procedure for an edge $(u, v)$ just checks whether going from $A$ to $v$ through $u$ improves the current value of `dist`$[v]$.

$\texttt{Relax}((u, v) \in E)$

$\texttt{if}\ \ dist[v] > dist[u] + w(u, v)\texttt{:}$

$\quad dist[v] \leftarrow dist[u] + w(u, v)$

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# Naive approach

**`Naive`$(G, A)$**

```
for all u ∈ V:
    dist[u] ← ∞
    prev[u] ← nil
dist[A] ← 0
do:
    relax all the edges
while at least one dist changes
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# Correct distances

**Lemma**

After the call to `Naive` algorithm all the distances are set correctly.

## Proof

- Assume, for the sake of contradiction, that no edge can be relaxed and there is a vertex $v$ such that $\mathrm{dist}[v] > d(A, v)$.

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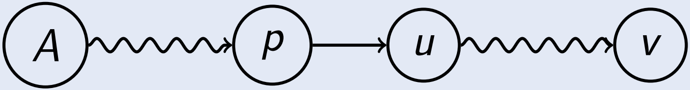
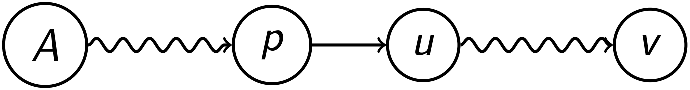
- Assume, for the sake of contradiction, that no edge can be relaxed and there is a vertex $v$ such that $\text{dist}[v] > d(A, v)$.
- Consider a shortest path from $A$ to $v$ and let $u$ be the first vertex on this path with the same property. Let $p$ be the vertex right before $u$.

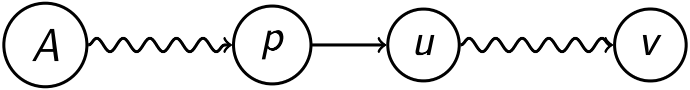


# Proof (continued)




- Then $d(A, p) = \text{dist}[p]$ and hence $d(A, u) = d(A, p) + w(p, u) = \text{dist}[p] + w(p, u)$

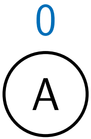
# Proof (continued)




- Then $d(A, p) = \text{dist}[p]$ and hence $d(A, u) = d(A, p) + w(p, u) = \text{dist}[p] + w(p, u)$
- $\text{dist}[u] > d(A, u) = \text{dist}[p] + w(p, u) \Rightarrow$ edge $(p, u)$ can be relaxed — a contradiction. $\square$

# Outline

# Intuition



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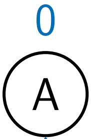


initially, we only know the distance to A

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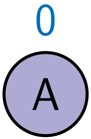

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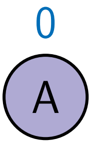


let's relax all the edges from A

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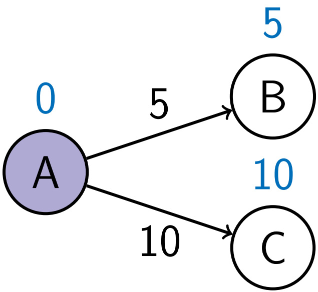


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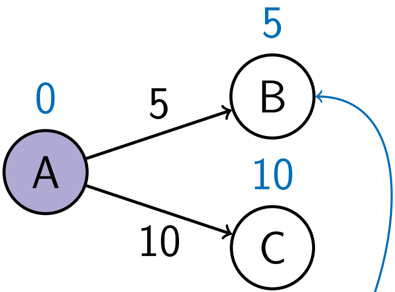
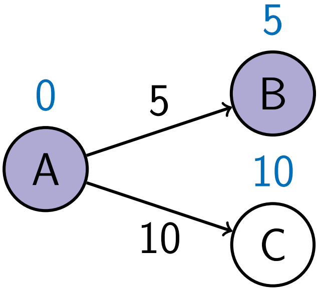



we now know the distance for B

# Intuition



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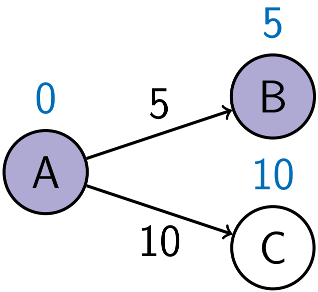


now, let's relax all the edges from B

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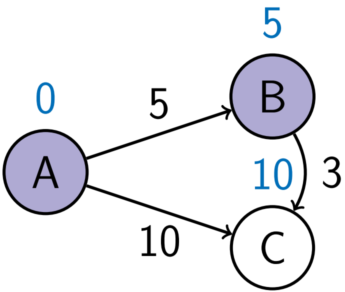


now, let's relax all the edges from B

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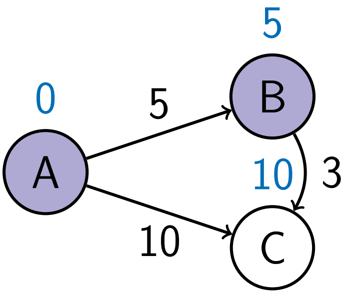


we discover an edge $(B, C)$ of weight 3 that updates dist$[C]$

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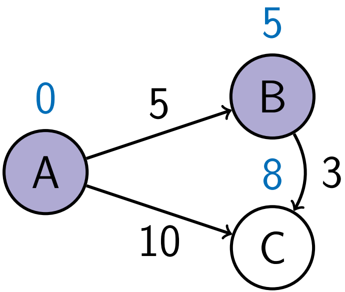

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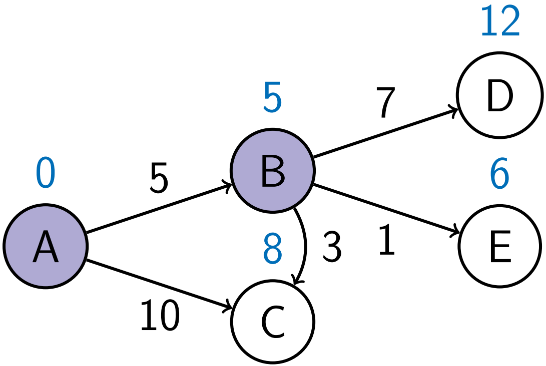


we also discover a few more outgoing edges

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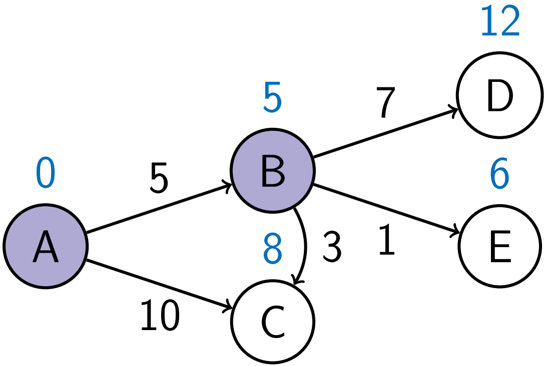


what is the next vertex for which we already know the correct distance?

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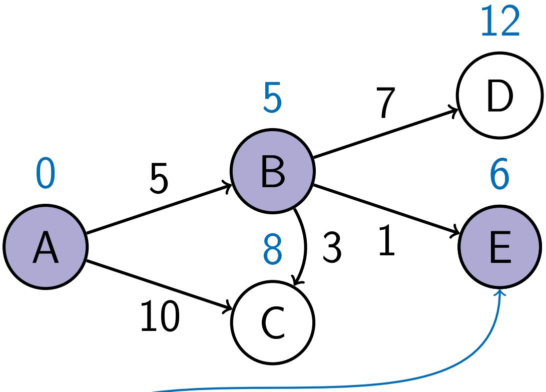


it is $E$

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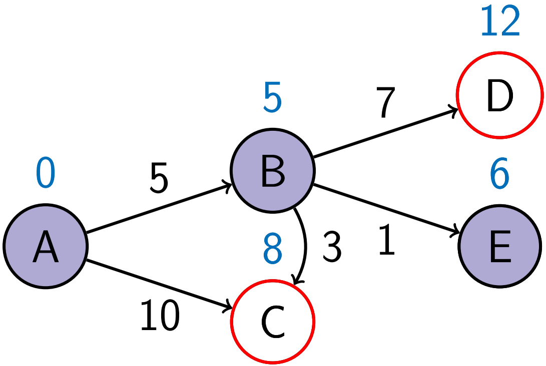


while for $C$ and $D$ it is possible that their `dist` values are larger than actual distances

# Main ideas of Dijkstra's Algorithm

- We maintain a set $R$ of vertices for which `dist` is already set correctly ("known region").

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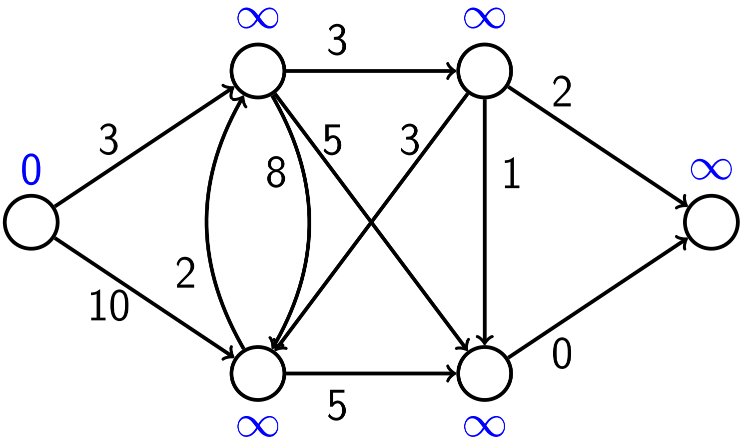
- We maintain a set $R$ of vertices for which `dist` is already set correctly ("known region").
- The first vertex added to $R$ is $A$.
- On each iteration we take a vertex outside of $R$ with the minimal `dist`-value, add it to $R$, and relax all its outgoing edges.

# Outline

# Example



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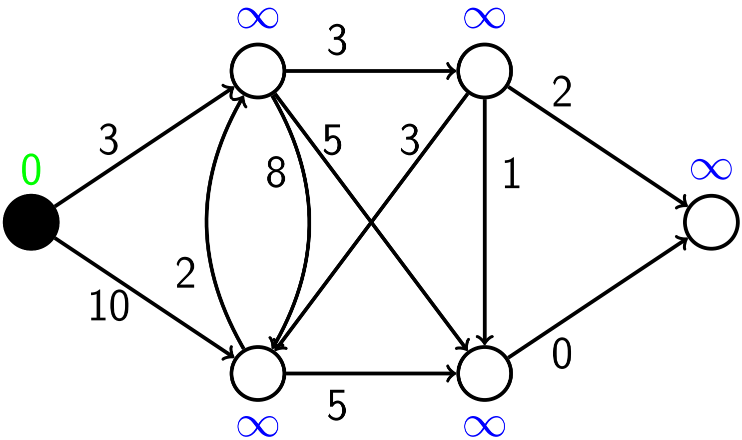

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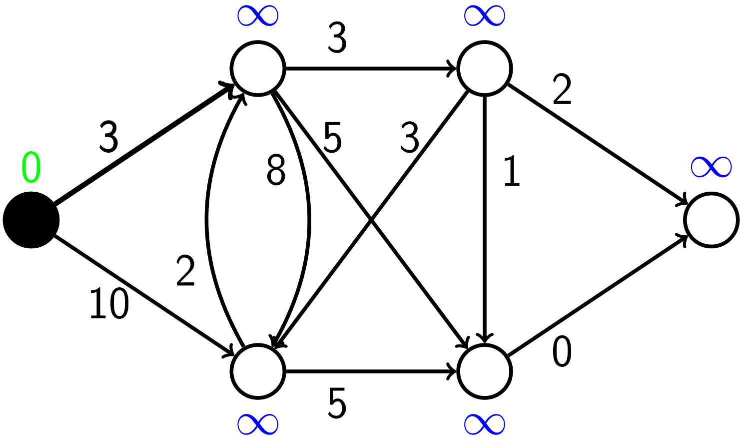

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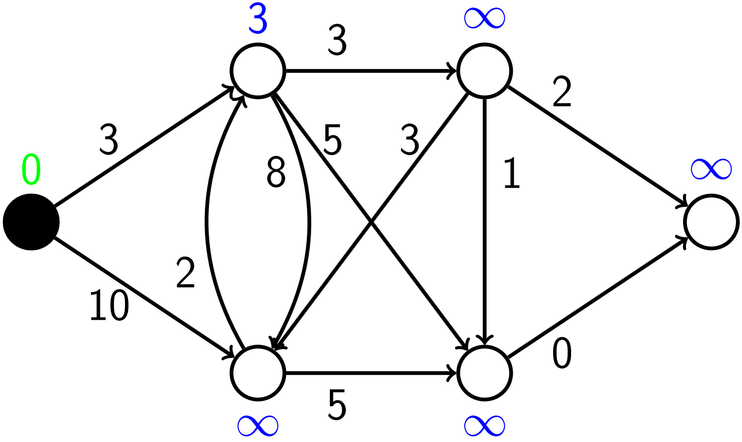

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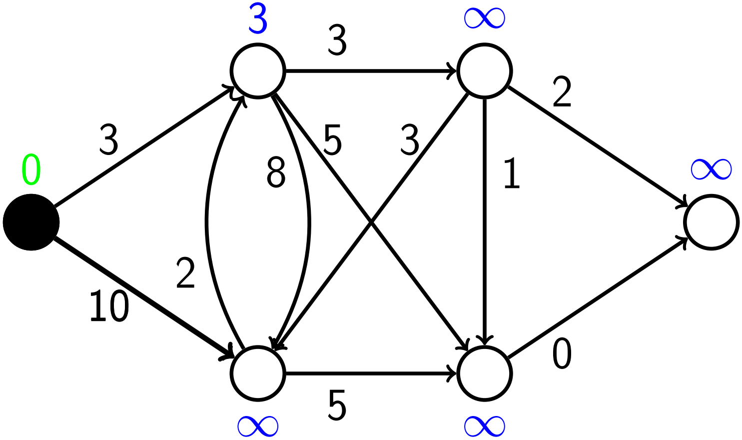

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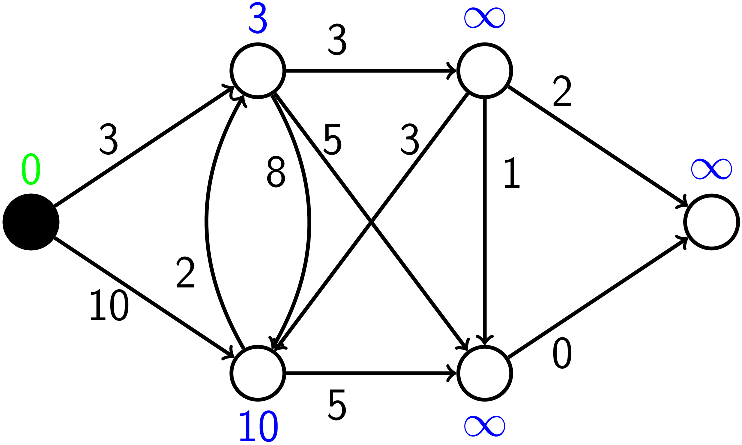

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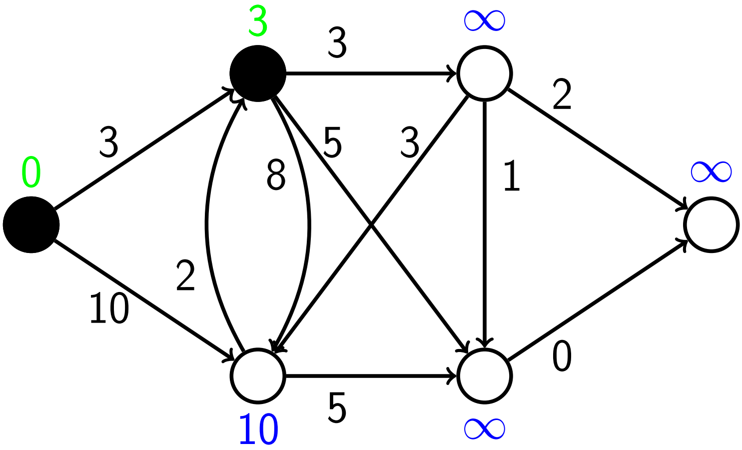

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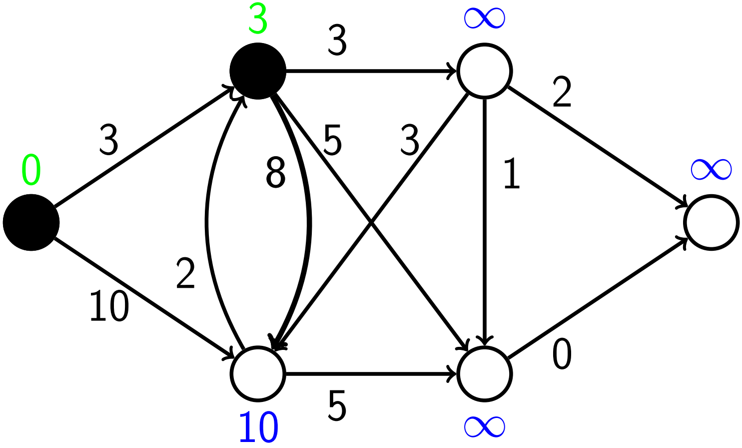

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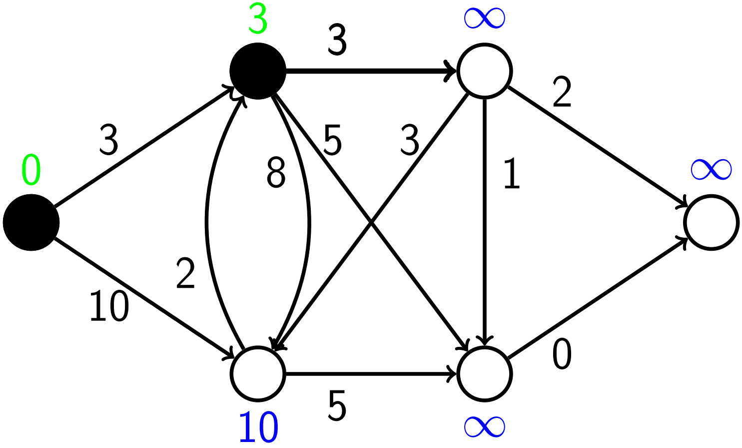

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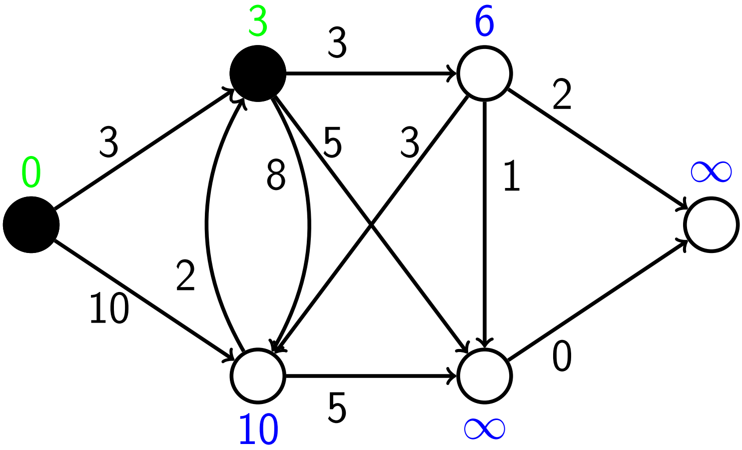

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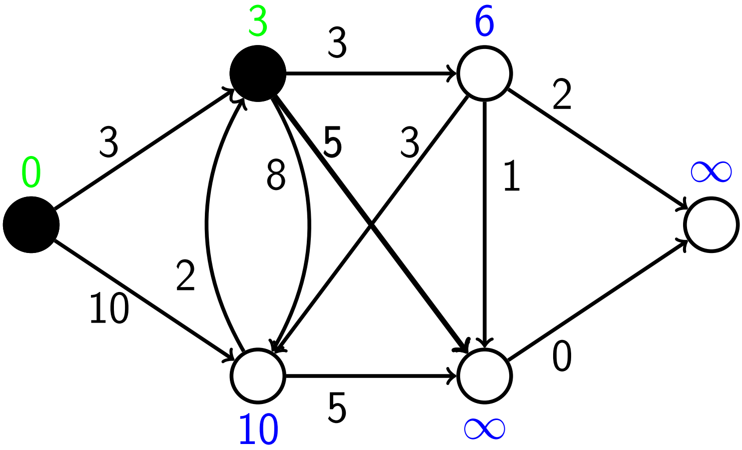

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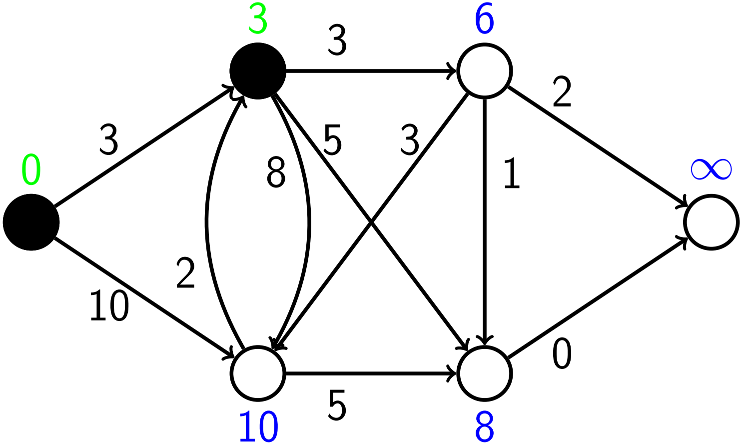

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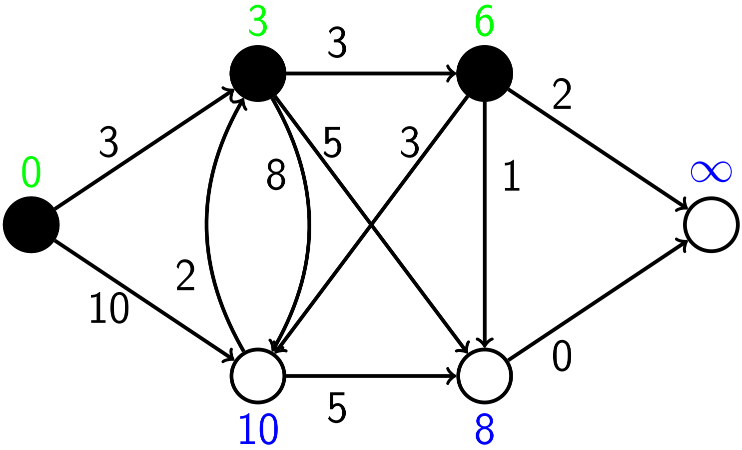

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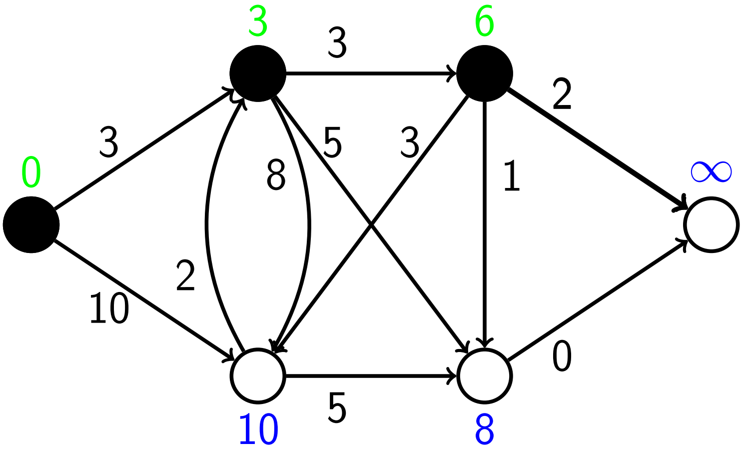

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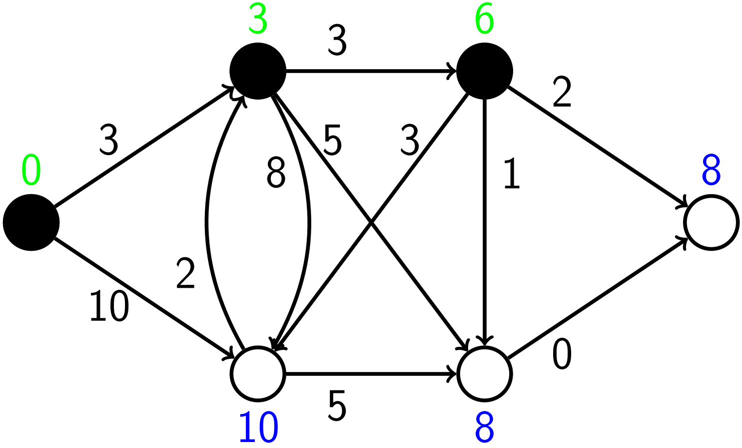

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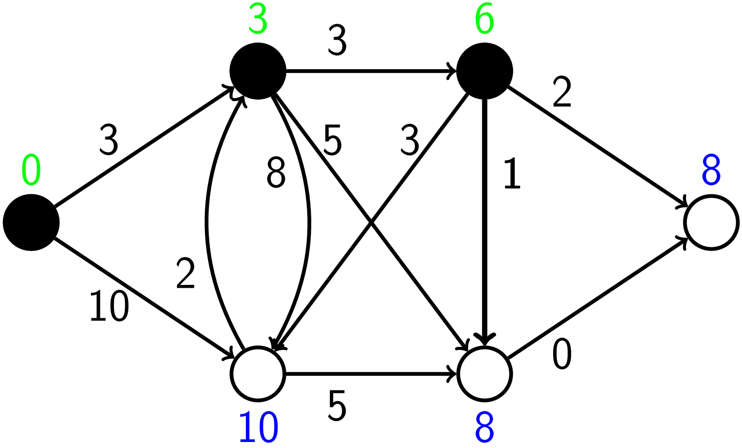

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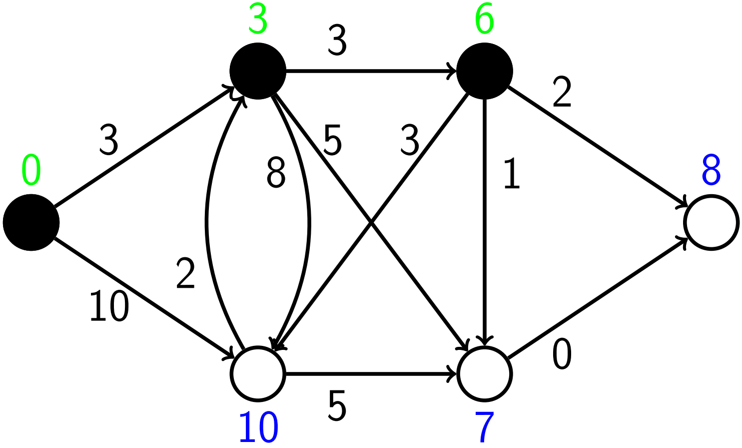

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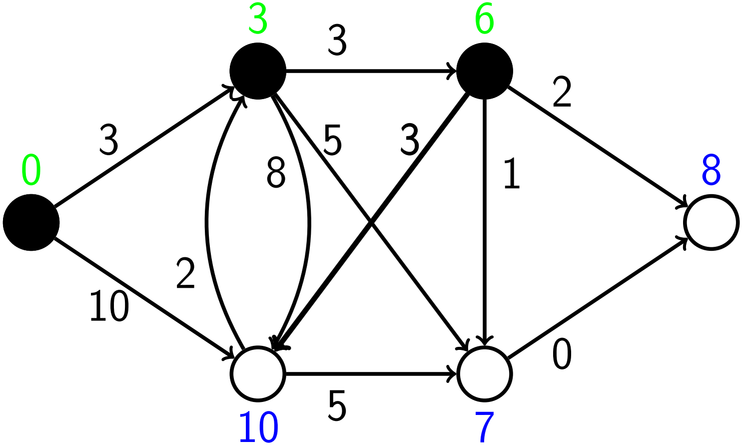

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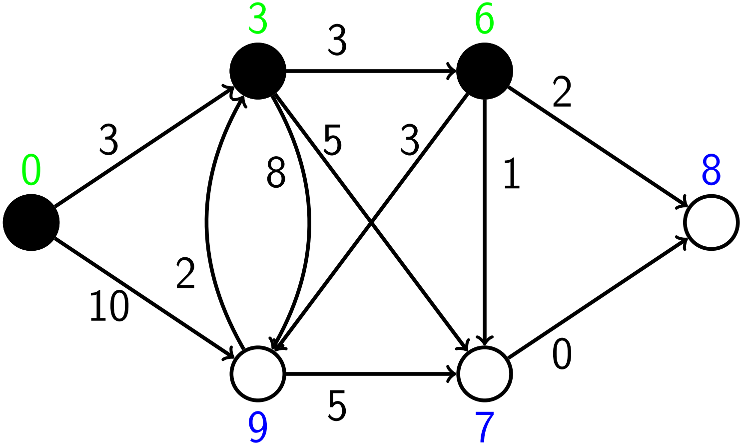

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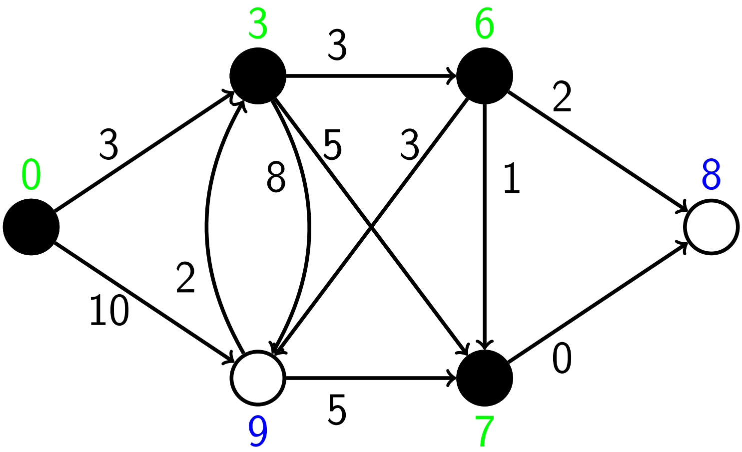

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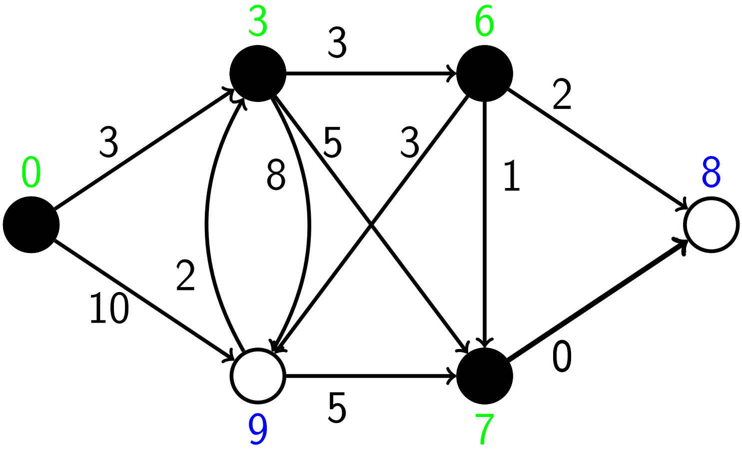

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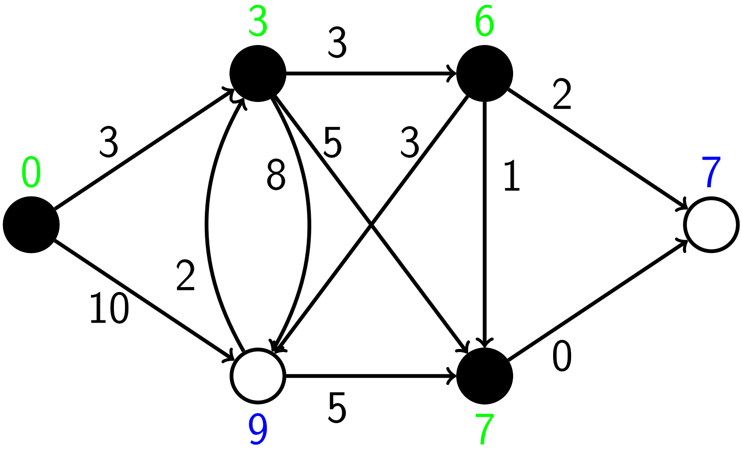

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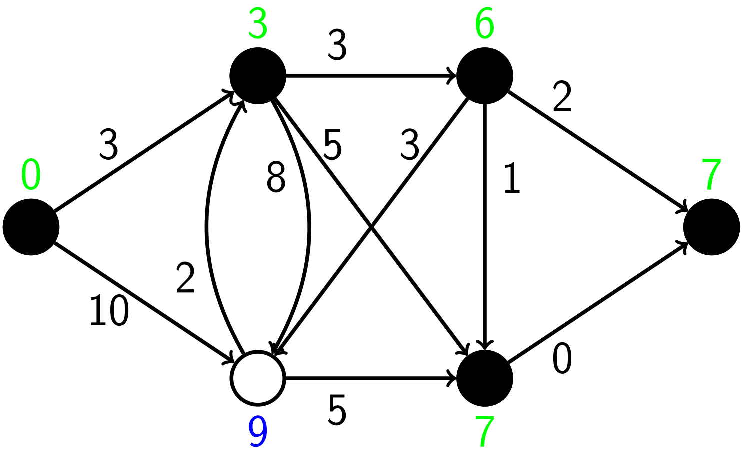
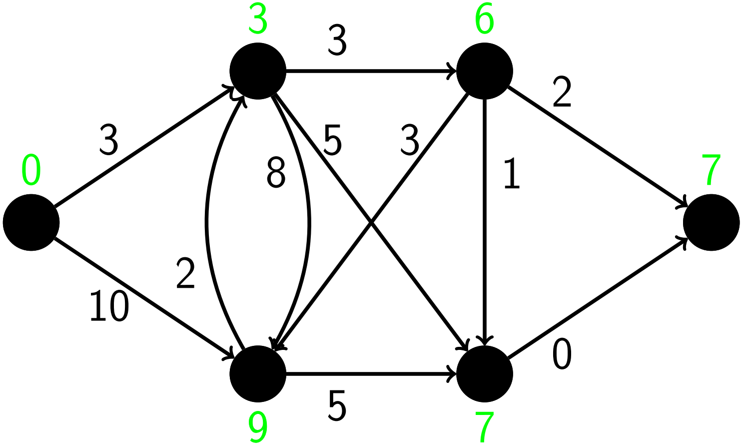


# Example



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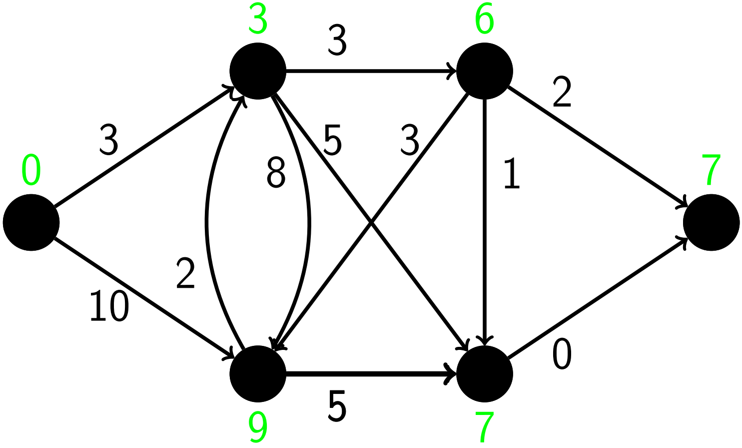

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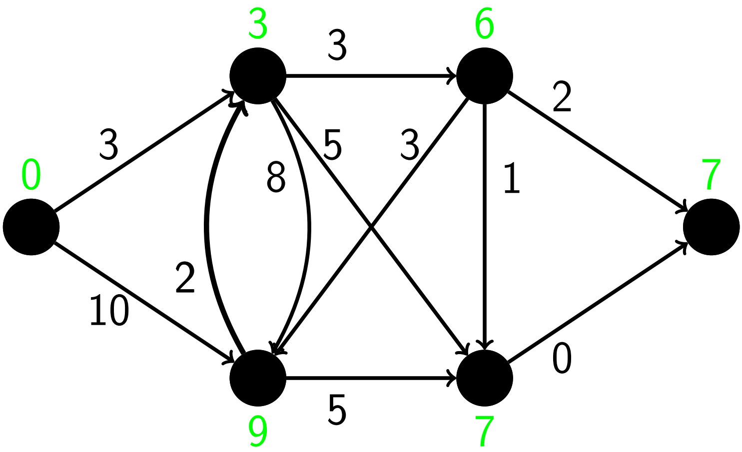

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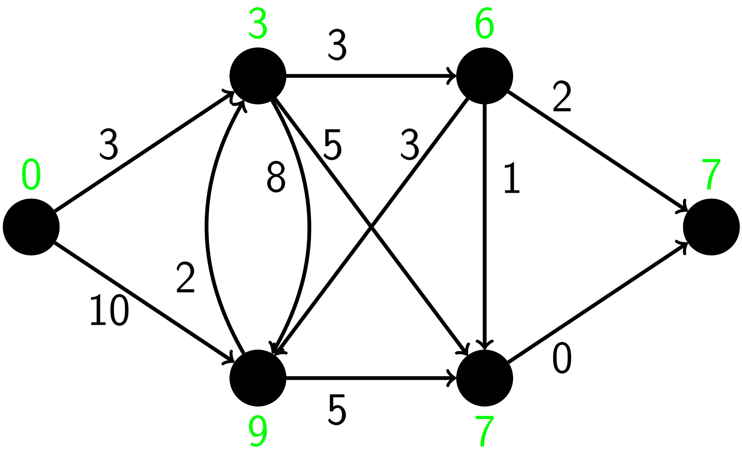

# Outline

# Pseudocode

**Dijkstra$(G, A)$**

for all $u \in V$:
  dist$[u] \leftarrow \infty$, prev$[u] \leftarrow$ nil
dist$[A] \leftarrow 0$
$H \leftarrow$ MakeQueue$(V)$ {dist-values as keys}
while $H$ is not empty:
  $u \leftarrow$ ExtractMin$(H)$
  for all $(u, v) \in E$:
    if $dist[v] > dist[u] + w(u, v)$:
      $dist[v] \leftarrow dist[u] + w(u, v)$
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      ChangePriority$(H, v, dist[v])$

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# Pseudocode

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# Pseudocode

## Dijkstra($G, A$)

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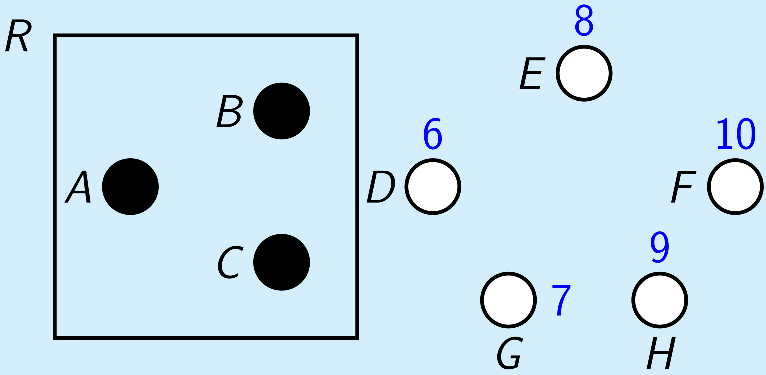
```

# Correct distances

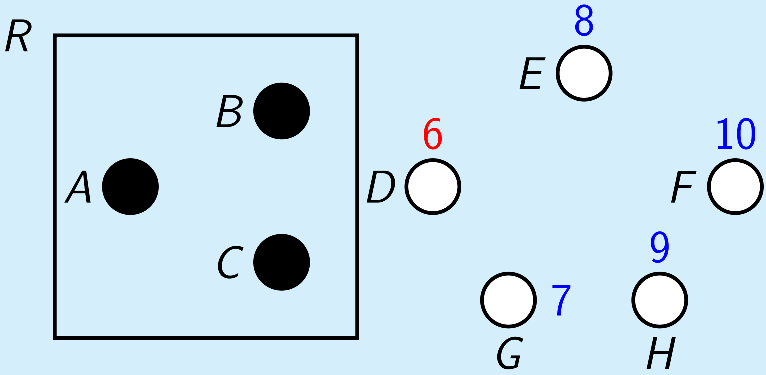
**Lemma**

When a node $u$ is selected via `ExtractMin`, $\text{dist}[u] = d(A, u)$.

**Proof**



## Proof



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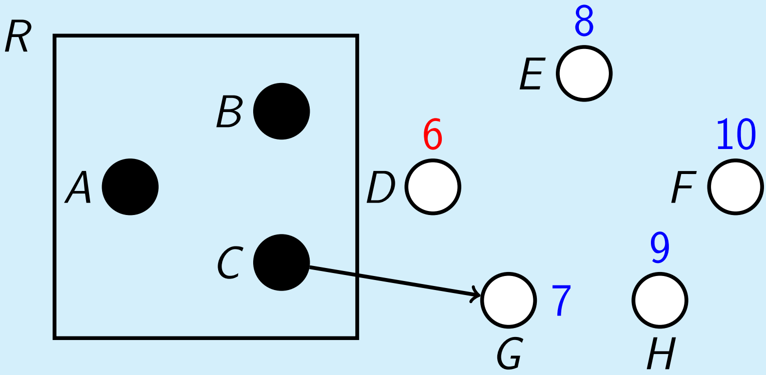

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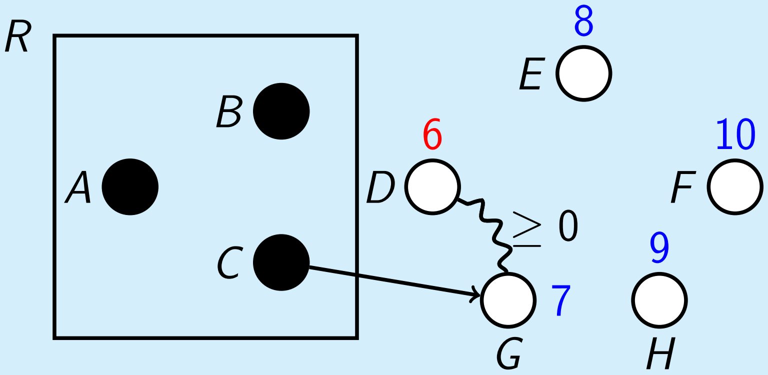
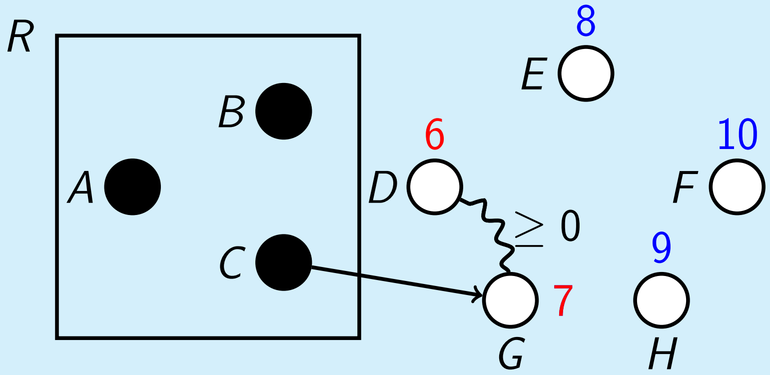


## Proof



# Outline

# Pseudocode

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```

# Running time

Total running time:

$$\begin{aligned} O(V) &+ T(\texttt{MakeQueue}) \\ &+ |V| \cdot T(\texttt{ExtractMin}) \\ &+ |E| \cdot T(\texttt{ChangePriority}) \end{aligned}$$

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Total running time:

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\begin{aligned}
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&+ |V| \cdot T(\texttt{ExtractMin}) \\
&+ |E| \cdot T(\texttt{ChangePriority})
\end{aligned}
$$

Priority queue implemented as array:
$O(|V| + |V| + |V|^2 + |E|) = O(|V|^2)$

# Running time

Total running time:

$$\begin{aligned} O(V) &+ T(\texttt{MakeQueue}) \\ &+ |V| \cdot T(\texttt{ExtractMin}) \\ &+ |E| \cdot T(\texttt{ChangePriority}) \end{aligned}$$

Priority queue implemented as binary heap:
$O(|V| + |V| + |V| \log |V| + |E| \log |V|) =$
$O((|V| + |E|) \log |V|)$

# Conclusion

- Can find the minimum time to get from work to home
- Can find the fastest route from work to home
- Works for any graph with non-negative edge weights
- Works in $O(|V|^2)$ or $O((|V| + |E|)\log(|V|))$ depending on the implementation