

# Paths in Graphs: Bellman-Ford Algorithm

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**Graph Algorithms**  
**Algorithms and Data Structures**

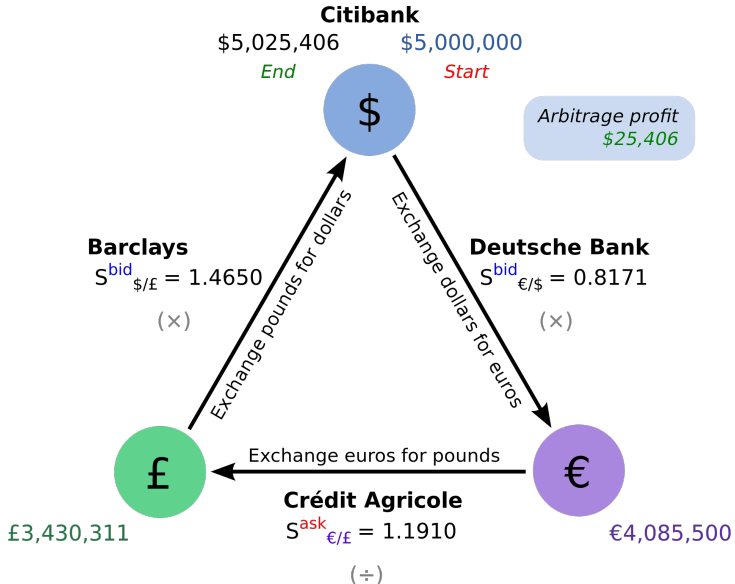
# Outline

- 1 Currency Exchange
- 2 Reduction to Shortest Paths
- 3 Bellman–Ford algorithm
- 4 Proof of Correctness
- 5 Negative Cycles
- 6 Infinite Arbitrage

## Currency Exchange

You can convert some currencies into some others with given exchange rates. What is the maximum amount in Russian rubles you can get from 1000 US dollars using unlimited number of currency conversions? Is it possible to get as many Russian rubles as you want? Is it possible to get as many US dollars as you want?

# Arbitrage





$$1 \text{ USD} \rightarrow 0.88 \cdot 0.84 \cdot \dots \cdot 8.08 \text{ RUR}$$

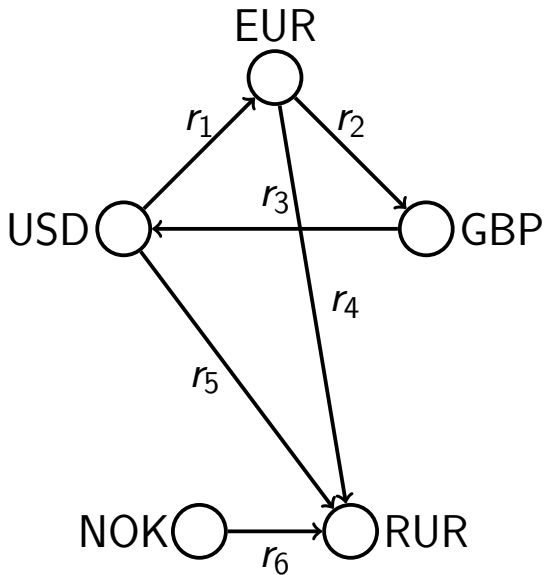
EUR  
○

USD○

○GBP

NOK○

○RUR



# Maximum product over paths

**Input:** Currency exchange graph with weighted directed edges  $e_i$  between some pairs of currencies with weights  $r_{e_i}$  corresponding to the exchange rate.

**Output:** Maximize  $\prod_{j=1}^k r_{e_j} = r_{e_1} r_{e_2} \dots r_{e_k}$  over paths  $(e_1, e_2, \dots, e_k)$  from USD to RUR in the graph.



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Use two standard approaches:

- Replace product with sum by taking logarithms of weights

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- Replace product with sum by taking logarithms of weights
- Negate weights to solve minimization problem instead of maximization problem

# Taking the Logarithm

$$xy = 2^{\log_2(x)} 2^{\log_2(y)} = 2^{\log_2(x) + \log_2(y)}$$

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$$4 \times 1 \times \frac{1}{2} = 2 = 2^1$$

$$\log_2(4) + \log_2(1) + \log_2\left(\frac{1}{2}\right) = 2 + 0 + (-1) = 1$$

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$$\prod_{j=1}^k r_{e_j} \rightarrow \max \Leftrightarrow \sum_{j=1}^k \log(r_{e_j}) \rightarrow \max$$



# Negation

$$\sum_{j=1}^k \log(r_{e_j}) \rightarrow \max \Leftrightarrow - \sum_{j=1}^k \log(r_{e_j}) \rightarrow \min$$

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$$\sum_{j=1}^k \log(r_{e_j}) \rightarrow \max \Leftrightarrow \sum_{j=1}^k (-\log(r_{e_j})) \rightarrow \min$$

# Reduction

Finally: replace edge weights  $r_{e_i}$  by  $(-\log(r_{e_i}))$  and find the shortest path between USD and RUR in the graph.

# Solved?

- Create currency exchange graph with weights  $r_{e_i}$  corresponding to exchange rates

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- Find the shortest path from USD to RUR by Dijkstra's algorithm
- Do the exchanges corresponding to the shortest path

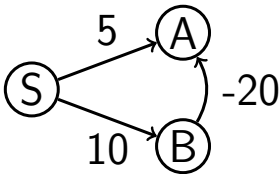
# Where Dijkstra's algorithm goes wrong?

- Dijkstra's algorithm relies on the fact that a shortest path from  $s$  to  $t$  goes only through vertices that are closer to  $s$  than  $t$ .

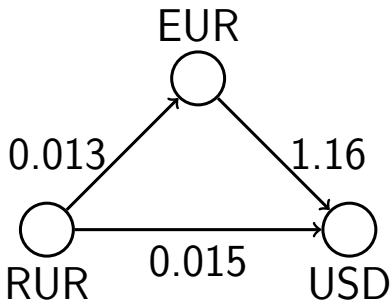


# Where Dijkstra's algorithm goes wrong?

- Dijkstra's algorithm relies on the fact that a shortest path from  $s$  to  $t$  goes only through vertices that are closer to  $s$  than  $t$ .
- This is no longer the case for graphs with negative edges:

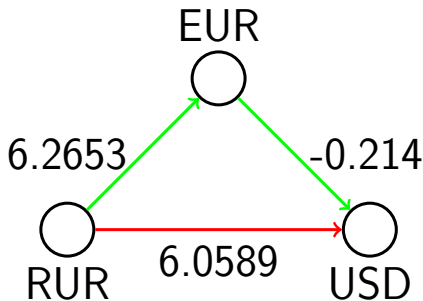


# Currency exchange example



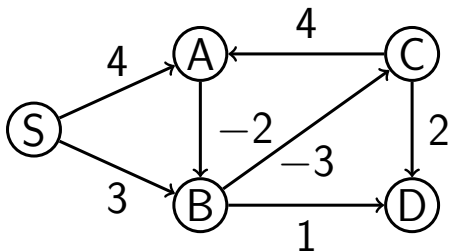
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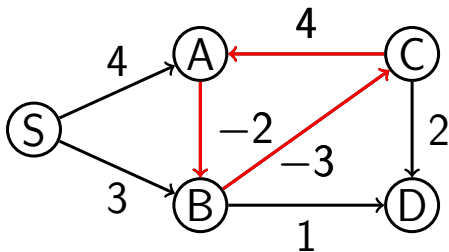


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# Negative weight cycles

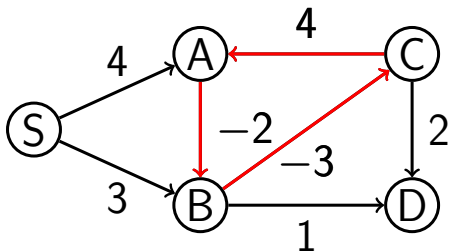


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In currency exchange, a negative cycle can make you a billionaire!

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- Remember naive algorithm from the previous lesson?



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- Remember naive algorithm from the previous lesson?
- Relax edges while dist changes
- Turns out it works even for negative edge weights!

**Relax( $(u, v) \in E$ )**

```
if  $dist[v] > dist[u] + w(u, v)$ :  
     $dist[v] \leftarrow dist[u] + w(u, v)$   
     $prev[v] \leftarrow u$ 
```

# Bellman–Ford algorithm

**BellmanFord( $G, S$ )**

{no negative weight cycles in  $G$ }

for all  $u \in V$ :

$\text{dist}[u] \leftarrow \infty$

$\text{prev}[u] \leftarrow \text{nil}$

$\text{dist}[S] \leftarrow 0$

repeat  $|V| - 1$  times:

    for all  $(u, v) \in E$ :

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# Running Time

## Lemma

The running time of Bellman–Ford algorithm is  $O(|V||E|)$ .

## Proof

- Initialize dist —  $O(|V|)$

# Running Time

## Lemma

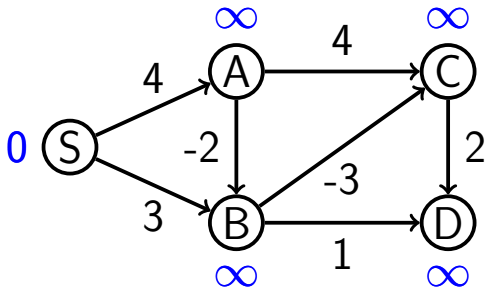
The running time of Bellman–Ford algorithm is  $O(|V||E|)$ .

## Proof

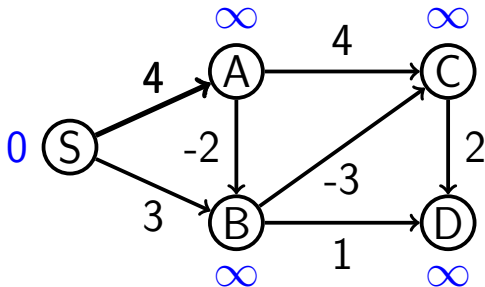
- Initialize dist —  $O(|V|)$
- $|V| - 1$  iterations, each  $O(|E|)$  —  
 $O(|V||E|)$



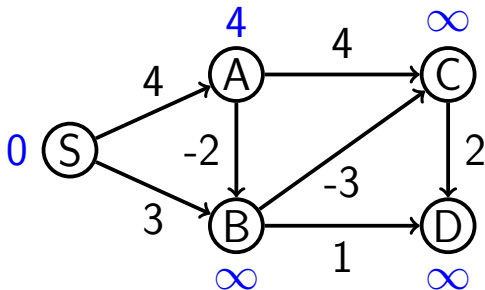
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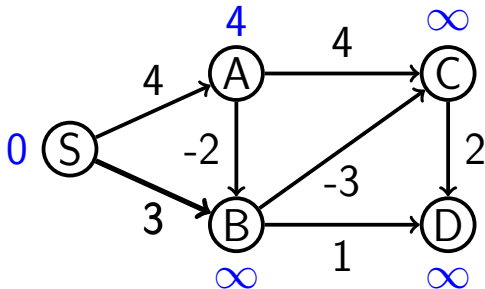


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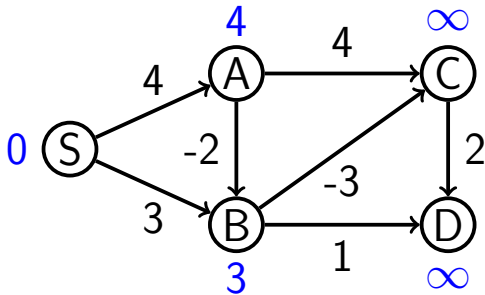




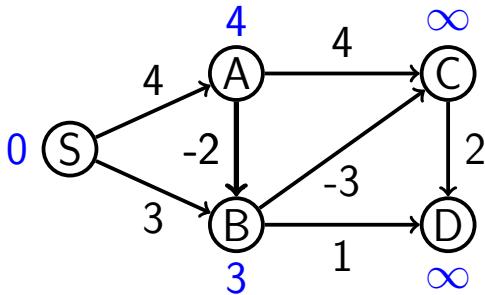
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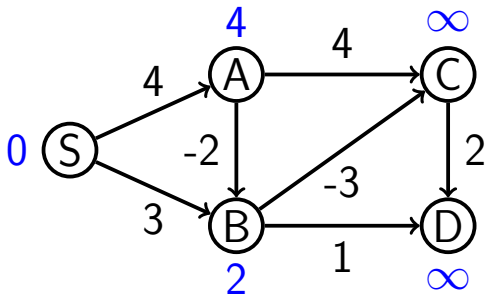
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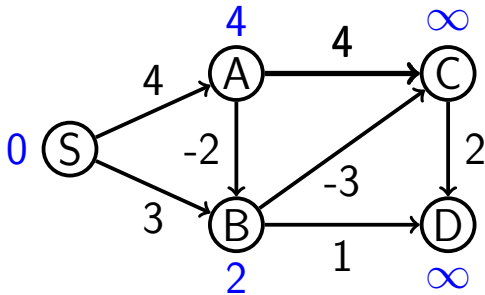
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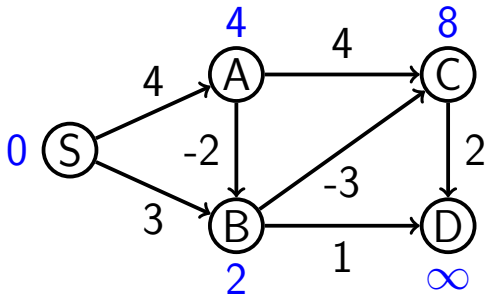
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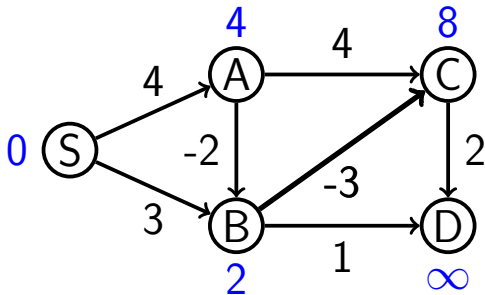
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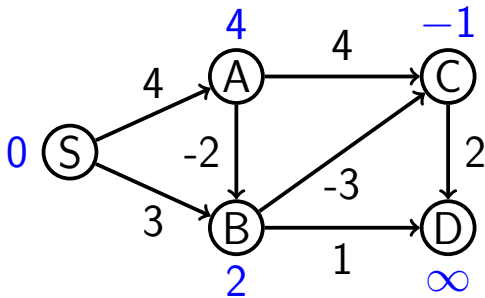
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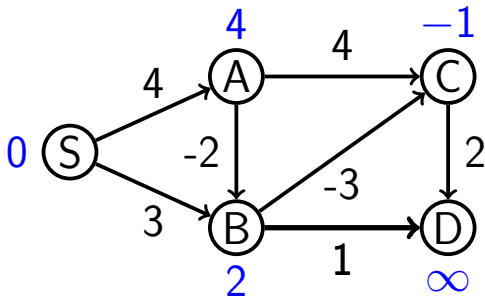


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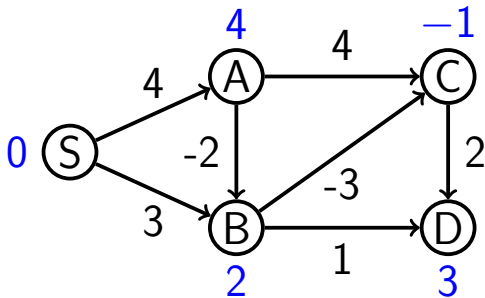




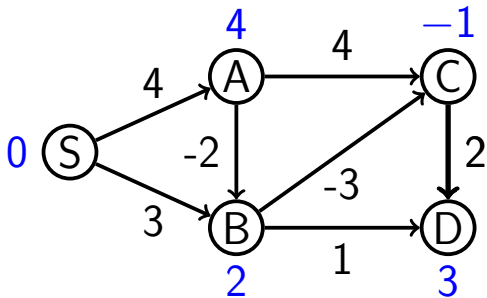
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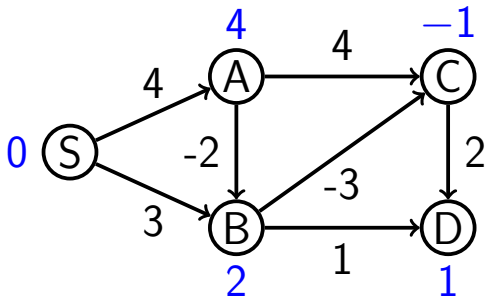
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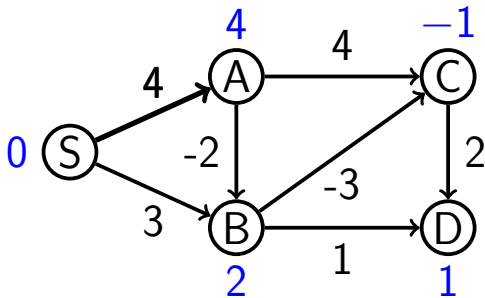
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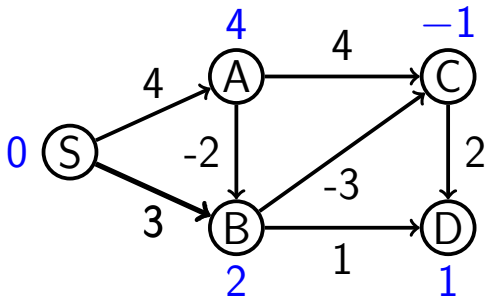
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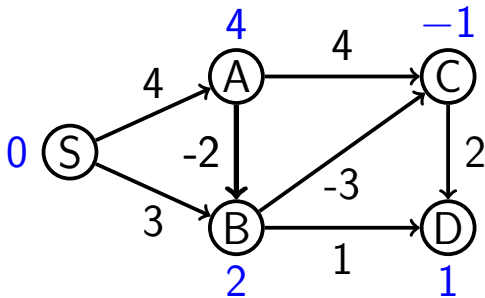
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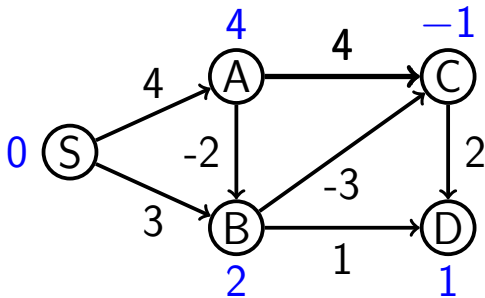
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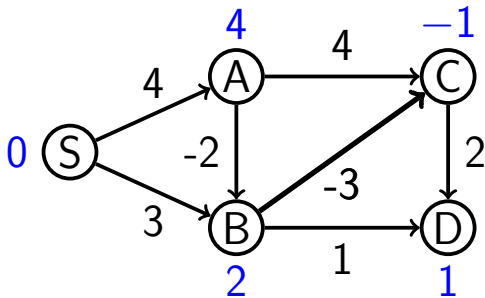


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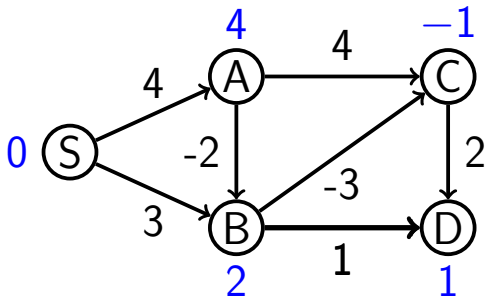




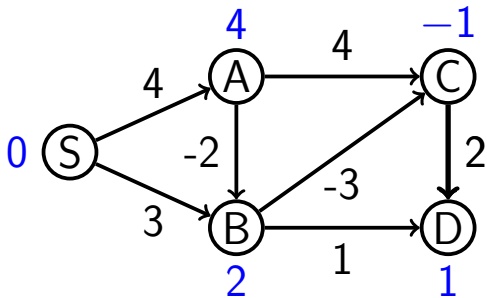
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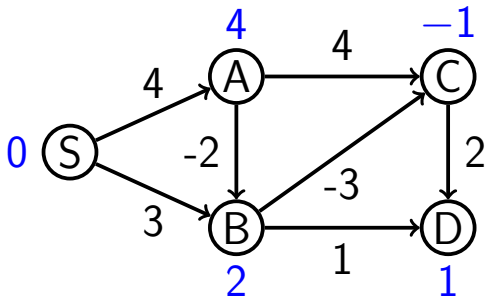
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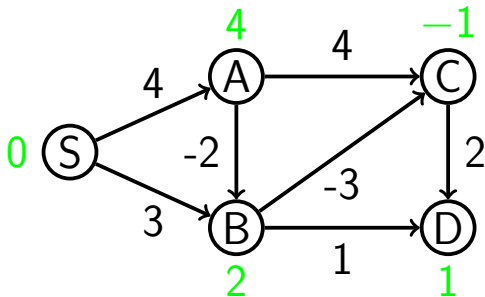
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## Lemma

After  $k$  iterations of relaxations, for any node  $u$ ,  $\text{dist}[u]$  is the smallest length of a path from  $S$  to  $u$  that contains at most  $k$  edges.

# Proof

- Use mathematical induction



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- Base: after 0 iterations, all dist-values are  $\infty$ , but for  $\text{dist}[S] = 0$ , which is correct.

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- Induction: proved for  $k \rightarrow$  prove for  $k + 1$

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- Before  $k + 1$ -th iteration,  $\text{dist}[u]$  is the smallest length of a path from  $S$  to  $u$  containing at most  $k$  edges
- Each path from  $S$  to  $u$  goes through one of the incoming edges  $(v, u)$
- Relaxing by  $(v, u)$  is comparing it with the smallest length of a path from  $S$  to  $u$  through  $v$  containing at most  $k + 1$  edge



## Corollary

*In a graph without negative weight cycles, Bellman–Ford algorithm correctly finds all distances from the starting node  $S$ .*

## Proof

- Any path with at least  $V$  edges contains a cycle

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- Any path with at least  $V$  edges contains a cycle
- This cycle can be removed without making the path longer
- Shortest path contains at most  $V - 1$  edges and will be found after  $V - 1$  iterations

## Corollary

*If there is no negative weight cycle reachable from  $S$  such that  $u$  is reachable from this negative weight cycle, Bellman–Ford algorithm correctly finds  $\text{dist}[u] = d(S, u)$ .*

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# Negative weight cycles

## Lemma

A graph  $G$  contains a negative weight cycle if and only if  $|V|$ -th (additional) iteration of  $\text{BellmanFord}(G, S)$  updates some dist-value.

## Proof

⇐ If there are no negative cycles, then all shortest paths from  $S$  contain at most  $|V| - 1$  edges, so no dist-value can be updated on  $|V|$ -th iteration.

## Proof

$\Rightarrow$  There's a negative weight cycle, say  $a \rightarrow b \rightarrow c \rightarrow a$ , but no relaxations on  $|V|$ -th iteration.

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$$\text{dist}[b] \leq \text{dist}[a] + w(a, b)$$

$$\text{dist}[c] \leq \text{dist}[b] + w(b, c)$$

$$\text{dist}[a] \leq \text{dist}[c] + w(c, a)$$

## Proof

$$\Rightarrow \text{dist}[b] + \text{dist}[c] + \text{dist}[a] \leq \\ \text{dist}[a] + w(a, b) + \text{dist}[b] + \\ w(b, c) + \text{dist}[c] + w(c, a)$$



## Proof

$$\Rightarrow \text{dist}[b] + \text{dist}[c] + \text{dist}[a] \leq \\ \text{dist}[a] + w(a, b) + \text{dist}[b] + \\ w(b, c) + \text{dist}[c] + w(c, a)$$

$w(a, b) + w(b, c) + w(c, a) \geq 0$  —  
a contradiction as  $a \rightarrow b \rightarrow c \rightarrow a$   
is a negative cycle. □

# Finding Negative Cycle

Algorithm:

- Run  $|V|$  iterations of Bellman–Ford algorithm, save node  $v$  relaxed on the last iteration

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- Start from  $x \leftarrow v$ , follow the link  $x \leftarrow \text{prev}[x]$  for  $|V|$  times — will be definitely on the cycle

# Finding Negative Cycle

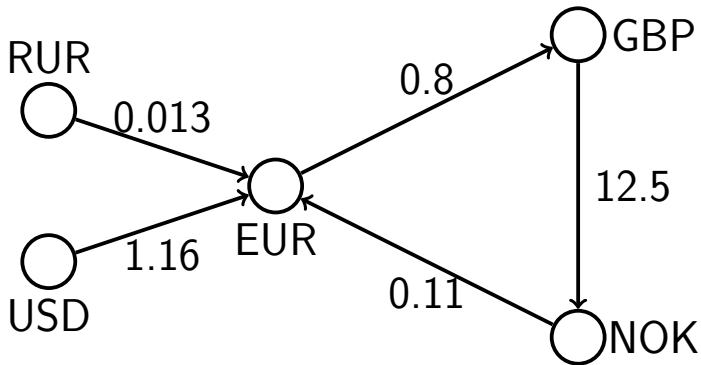
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- Save  $y \leftarrow x$  and go  $x \leftarrow \text{prev}[x]$  until  $x = y$  again

Is it possible to get as many rubles as you want from 1000 USD?

Is it possible to get as many rubles as you want from 1000 USD?

Not always, even if there is a negative cycle



Cannot exchange USD into rubles via  
(negative) cycle EUR  $\rightarrow$  GBP  $\rightarrow$  NOK.

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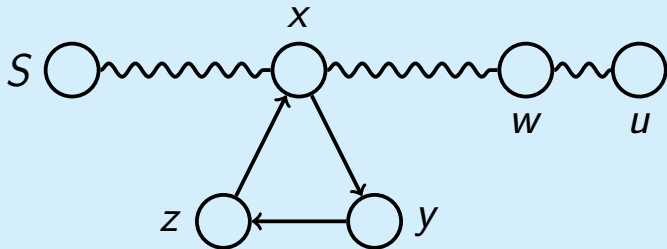
# Detect Infinite Arbitrage

## Lemma

It is possible to get any amount of currency  $u$  from currency  $S$  if and only if  $u$  is reachable from some node  $w$  for which  $\text{dist}[w]$  decreased on iteration  $V$  of Bellman-Ford.

# Proof

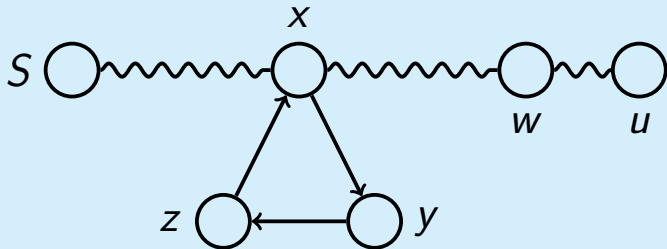
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- $\text{dist}[w]$  decreased on iteration  $V \Rightarrow w$  is reachable from a negative weight cycle
- $w$  is reachable  $\Rightarrow u$  is also reachable  $\Rightarrow$  infinite arbitrage



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- Thus  $\text{dist}[u]$  will be decreased on some iteration  $k \geq V$

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( $\Rightarrow$  continued)

- If edge  $(x, y)$  was not relaxed and  $\text{dist}[x]$  did not decrease on  $i$ -th iteration, then edge  $(x, y)$  will not be relaxed on  $i + 1$ -st iteration



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- Only nodes reachable from those relaxed on previous iterations can be relaxed
- $\text{dist}[u]$  is decreased on iteration  $k \geq V \Rightarrow u$  is reachable from some node relaxed on  $V$ -th iteration



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- All those nodes and only those can have infinite arbitrage

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- Use this negative cycle to achieve infinite arbitrage from  $S$  to  $u$

# Conclusion

- Can implement best possible exchange rate
- Can determine whether infinite arbitrage is possible
- Can implement infinite arbitrage
- Can find shortest paths in graphs with negative edge weights
- Can detect and find negative cycles in graphs