

1967 Mr Average

2017 Mr Average

Expectations

Collar: 14.5

Chest: 38in

Waist 34in

Weight: 11st 8lbs

Height: 5ft 7.5in

Shoe: 7

**Life expectancy
68 years**

Collar: 16

Chest: 43in

Waist 37in

Weight: 13st 3lbs

Height: 5ft 10in

Shoe: 9

**Life expectancy
81 years**

What Matters

Important random-variable properties?

Range

Min & max values of X

Lowest & highest temperature / salary

$$x_{\min} = \min \{ x \in \Omega \mid p(x) > 0 \}$$

$$x_{\max} = \max \{ x \in \Omega \mid p(x) > 0 \}$$

Average

Average temperature / salary

Range average

$$\frac{x_{\min} + x_{\max}}{2} ?$$

Element average

$$\frac{1}{|\Omega|} \sum_{x \in \Omega} x ?$$

or over x s.t. $p(x) > 0$

Sample Mean

$\Omega = \{0, \dots, 100\}$

$p(0) = .8$

$p(90) = .1$

$p(100) = .1$

all other $p(x) = 0$

Range average

$(x_{\min} + x_{\max})/2$

$(0 + 100)/2 = 50$

Element average

positive probabilities

$(0 + 90 + 100)/3 = 63.3$

Ten samples

Typical

0, 0, 0, 0, 90, 0, 0, 0, 100, 0

Sample mean

$(8 \cdot 0 + 1 \cdot 90 + 1 \cdot 100)/10 = 190/10 = 19$

More representative of what we will observe

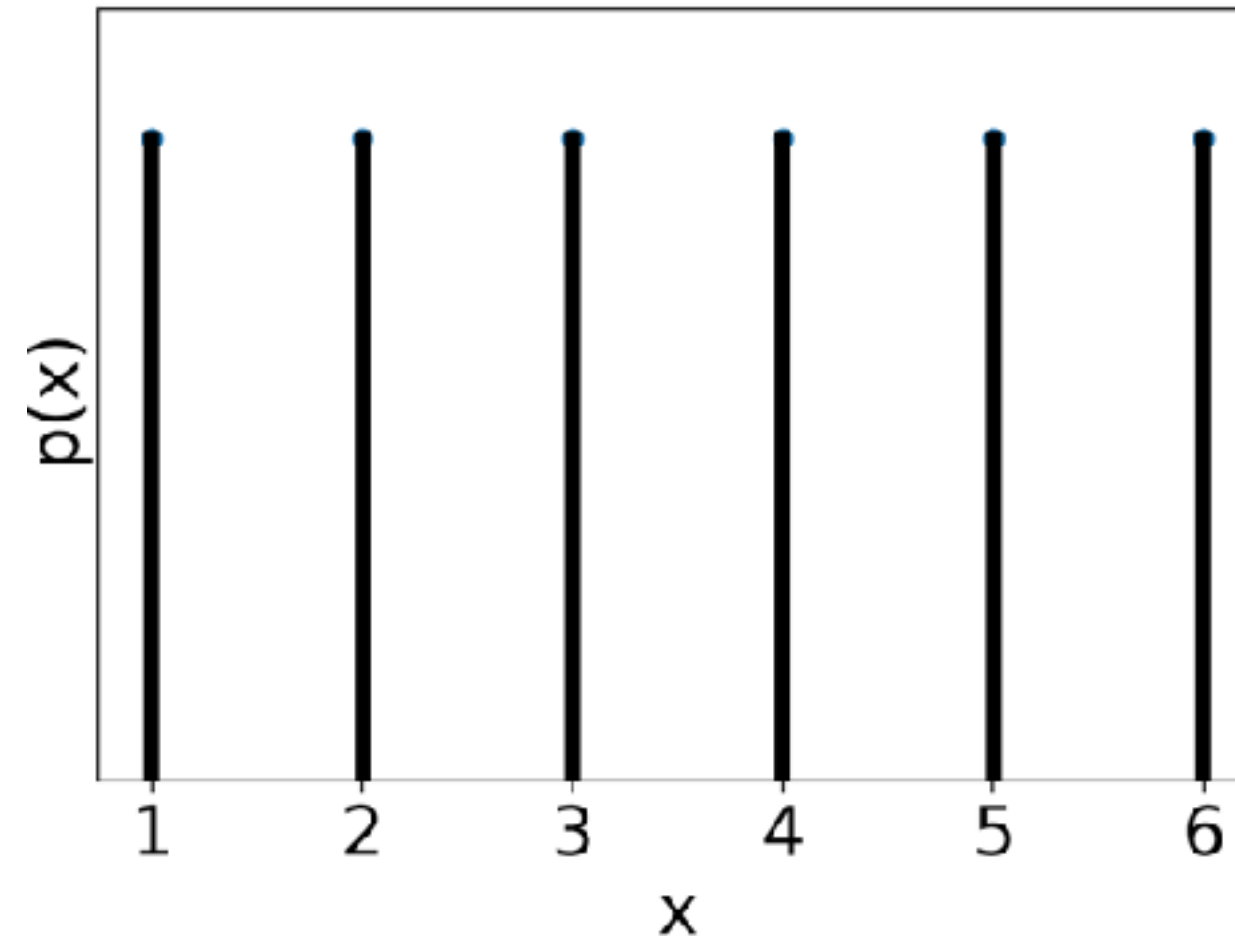
Fair Die

Roll a fair die $n \rightarrow \infty$ times

Average of the observed values = ?

Each value $\sim n/6$ times

Average



$$\frac{\frac{n}{6} \cdot 1 + \frac{n}{6} \cdot 2 + \dots + \frac{n}{6} \cdot 6}{n} = \frac{1 + \dots + 6}{6} = \frac{1}{6} \frac{(1 + 6) \cdot 6}{2} = 3.5$$

1,...,6 \rightarrow Average = 3.5

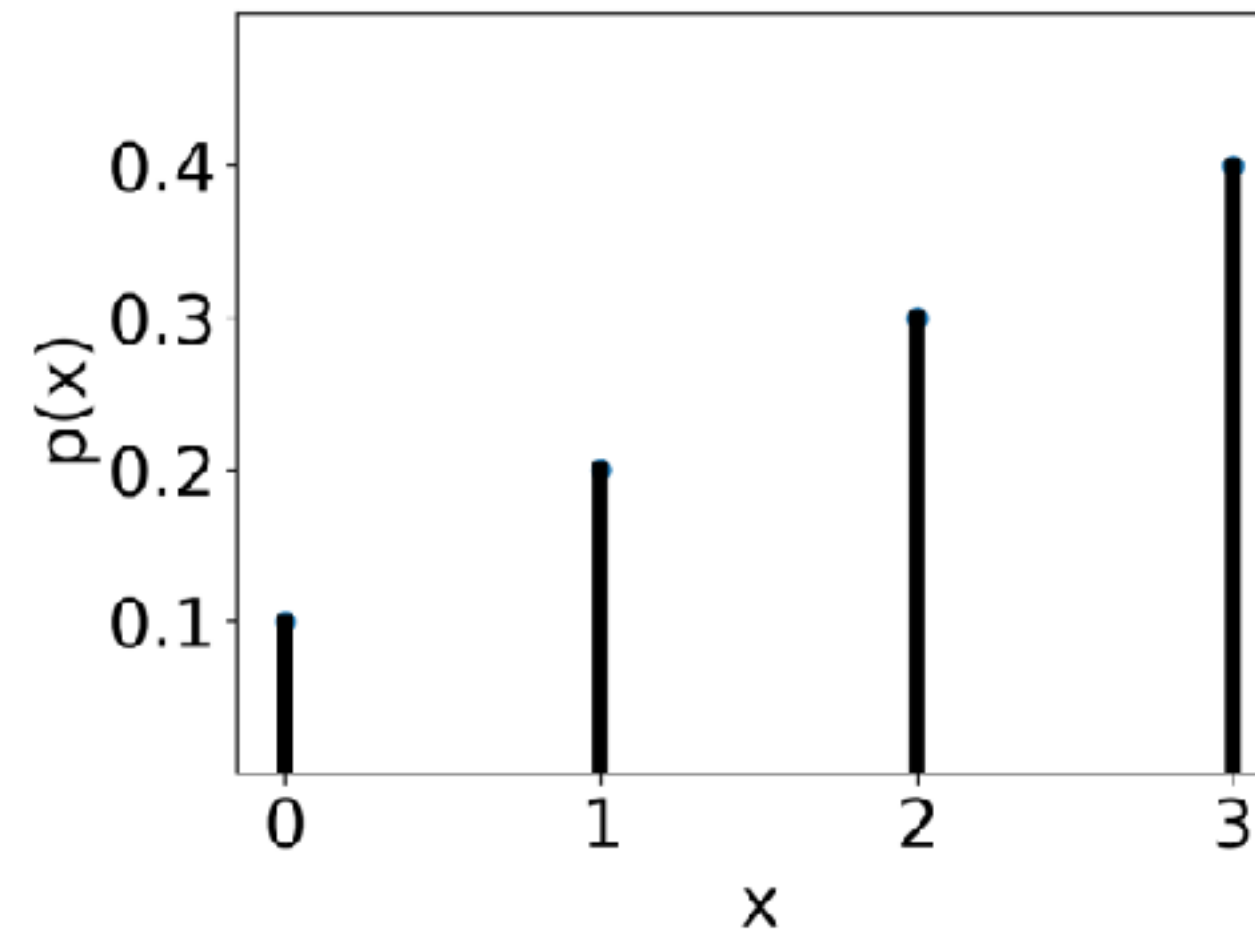


4-Sided Die

| Side | Prob | Appear |
|------|------|--------|
| 1 | .1 | .1n |
| 2 | .2 | .2n |
| 3 | .3 | .3n |
| 4 | .4 | .4n |

1

n



$$\text{Average} = \frac{.1n \cdot 1 + .2n \cdot 2 + .3n \cdot 3 + .4n \cdot 4}{n}$$

$$= 0.1 \cdot 1 + 0.2 \cdot 2 + 0.3 \cdot 3 + 0.4 \cdot 4 = 3$$

Arithmetic average $(1+2+3+4)/4 = 2.5$

Probabilities skew to the right

Expectation

In $n \rightarrow \infty$ samples

x will appear

$p(x) \cdot n$ times

$$\text{Average} = \frac{\sum_x [P(x) \cdot n] \cdot x}{n} = \sum_x P(x) \cdot x \stackrel{\text{def}}{=} E(X)$$

Expectation

Mean

$E(X)$ also denoted

EX

μ_x

μ

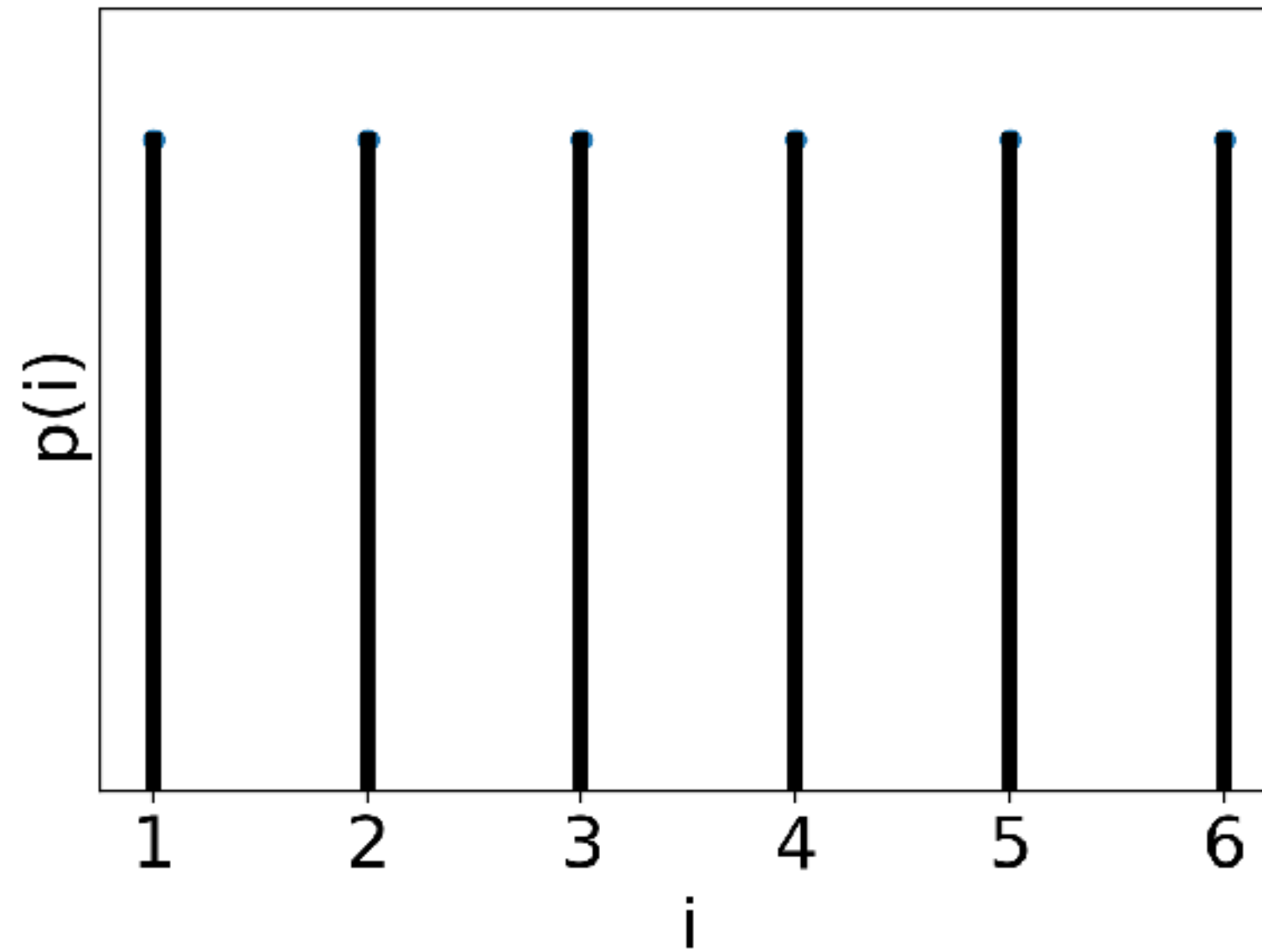
Not random

constant

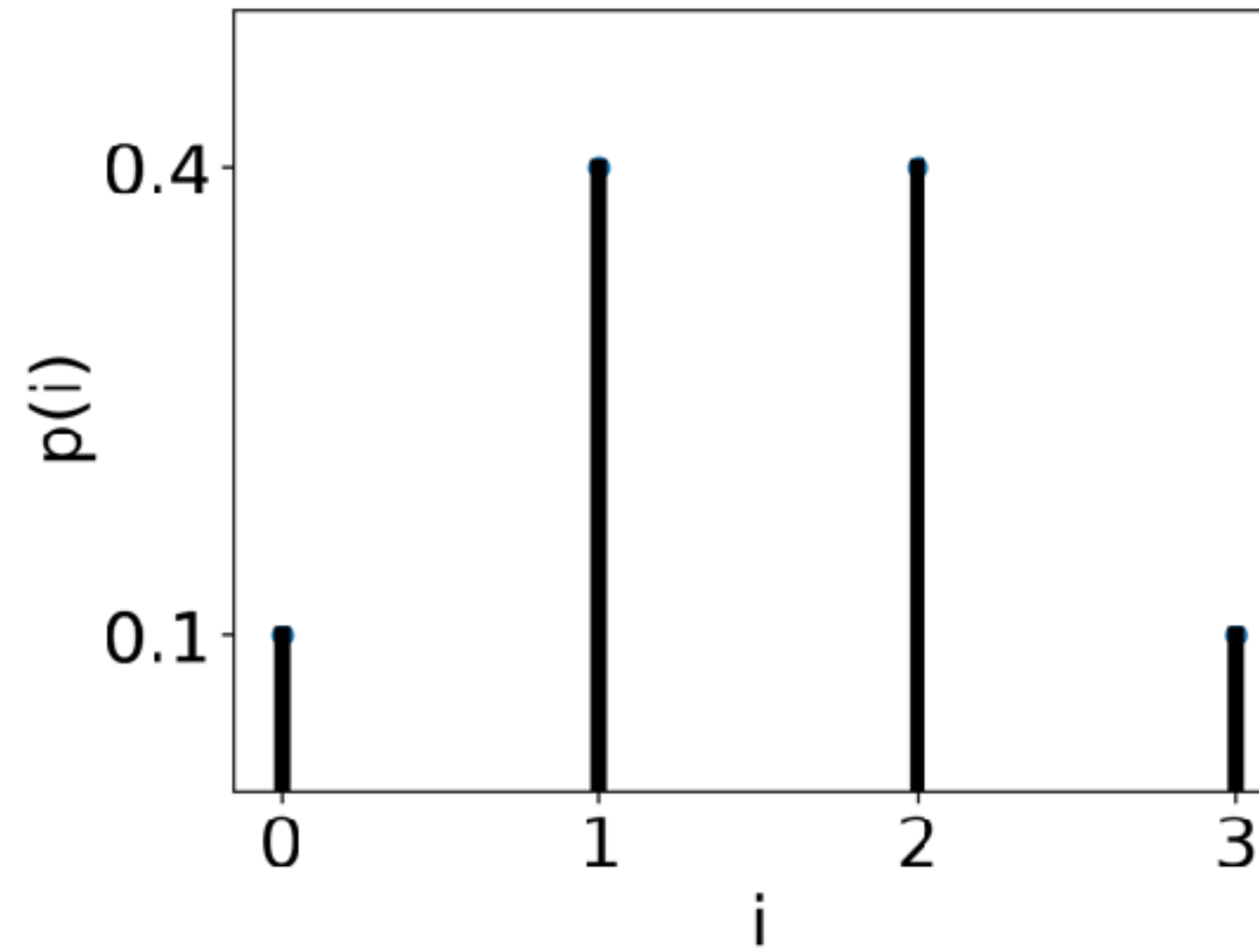
property of the distribution

Fair Die

$$\begin{aligned} E(X) &= \sum_{i=1}^6 P(i) \cdot i \\ &= \sum_{i=1}^6 \frac{1}{6} \cdot i \\ &= \frac{1 + 2 + \dots + 6}{6} \\ &= \frac{1}{6} \frac{(1 + 6) \cdot 6}{2} \\ &= \frac{7}{2} = 3.5 \quad \checkmark \end{aligned}$$



4 Sided- Die



$$E(X) = \sum_{i=1}^4 p_i \cdot i$$

$$= 0.1 \cdot 1 + 0.2 \cdot 2 + 0.3 \cdot 3 + 0.4 \cdot 4$$

$$= 3$$



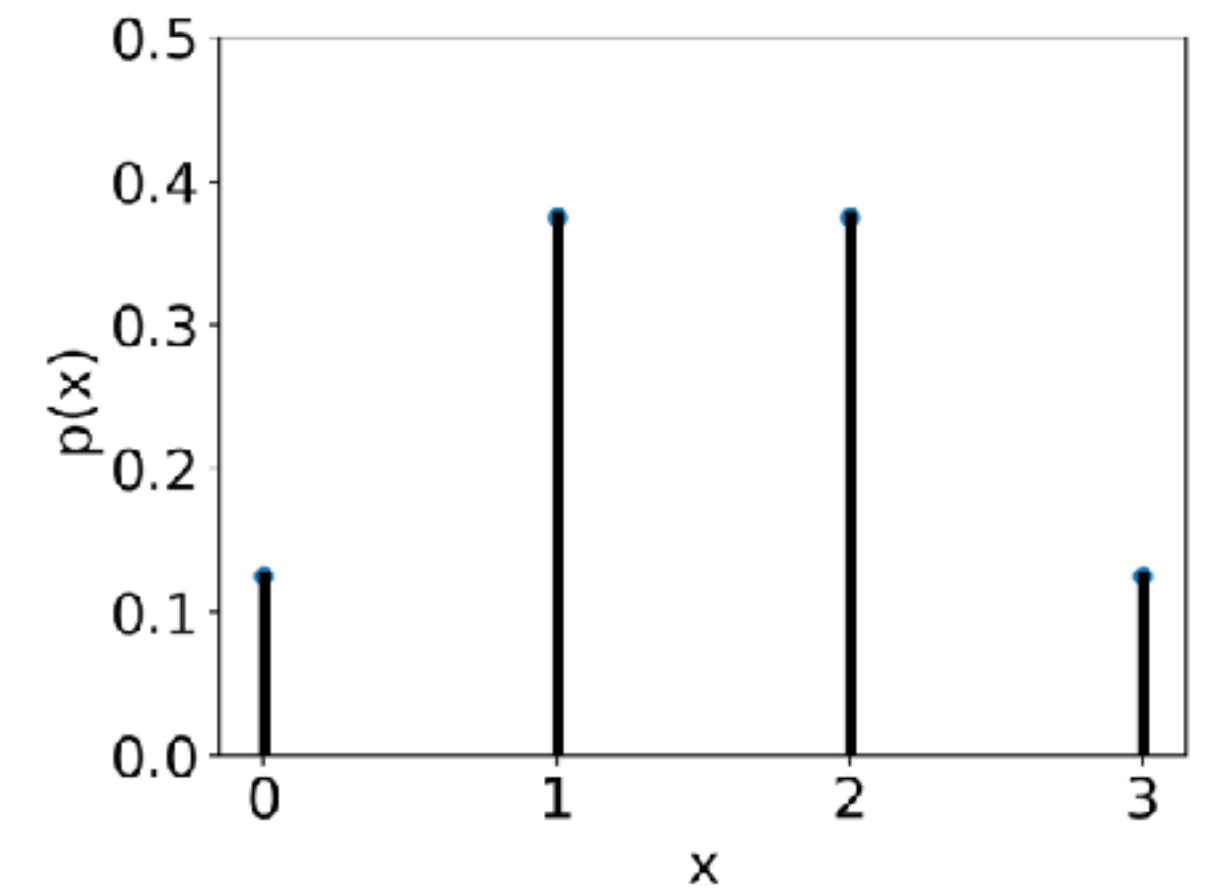
3 Coins

Toss a coin 3 times

X - # heads

$E(X) = ?$

| x | outcomes | $p(x)$ |
|-----|---------------|---------------|
| 0 | ttt | $\frac{1}{8}$ |
| 1 | tth,tht,htt | $\frac{3}{8}$ |
| 2 | thh, hth, hht | $\frac{3}{8}$ |
| 3 | hhh | $\frac{1}{8}$ |



$$\sum P(x) \cdot x = \frac{1}{8} \cdot 0 + \frac{3}{8} \cdot 1 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 3 = 1.5$$

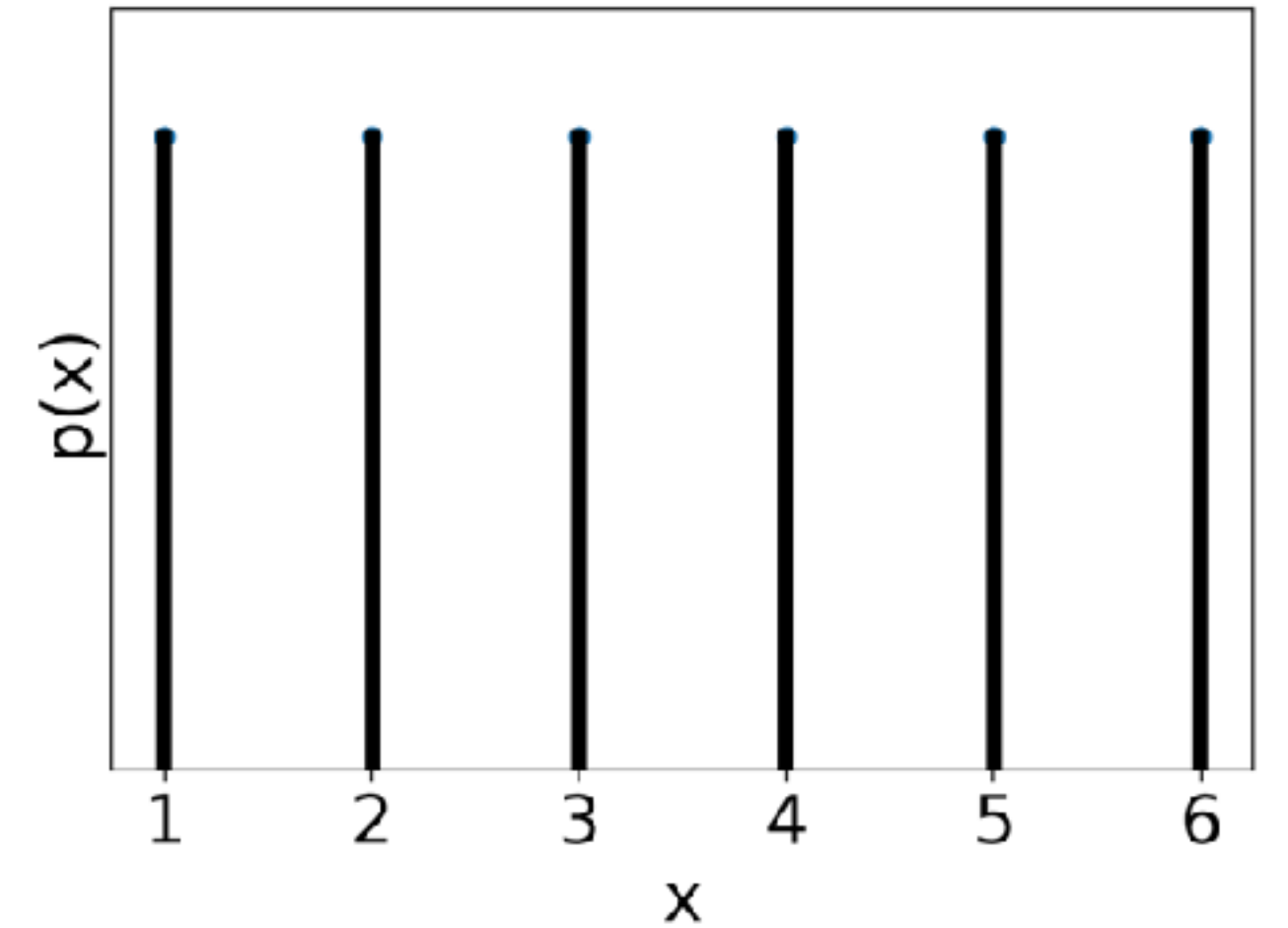
heads ranges from 0 to 3, on average 1.5



Uniform Variables

X uniform over Ω

$$p(x) = \frac{1}{|\Omega|}$$



$$E(X) = \sum_{x \in \Omega} p(x) \cdot x = \sum_{x \in \Omega} \frac{1}{|\Omega|} \cdot x = \frac{1}{|\Omega|} \sum_{x \in \Omega} x$$

$E(X)$ is the arithmetic average of elements in Ω



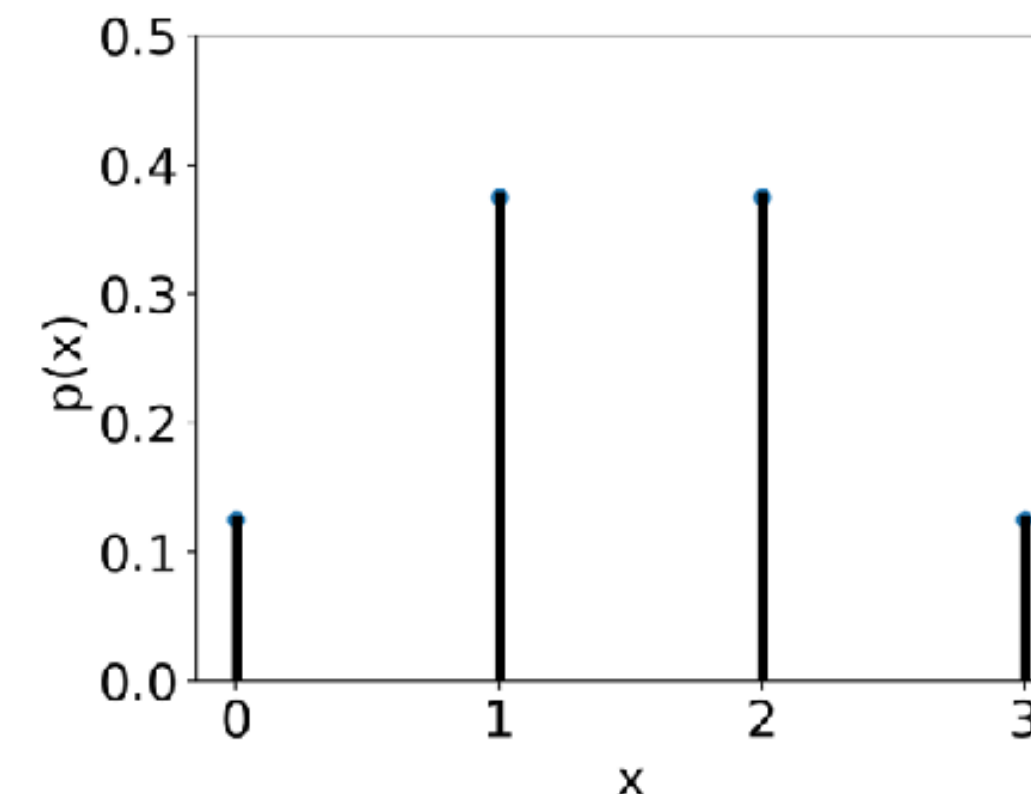
$$E(X) = \frac{1+2+\dots+6}{6} = 3.5$$

Symmetry

A distribution p is symmetric around a if for all $x > 0$, $p(a+x) = p(a-x)$

If p is symmetric around a , then $E(X) = a$

| x | outcomes | P(x) |
|---|---------------|---------------|
| 0 | ttt | $\frac{1}{8}$ |
| 1 | tth,tht,htt | $\frac{3}{8}$ |
| 2 | thh, hth, hht | $\frac{3}{8}$ |
| 3 | hhh | $\frac{1}{8}$ |



Symmetric around 1.5

$E(X) = 1.5$

Properties

$E(X)$ Despite notation Not random Number Property of distribution

$$E(X) = 1.5$$

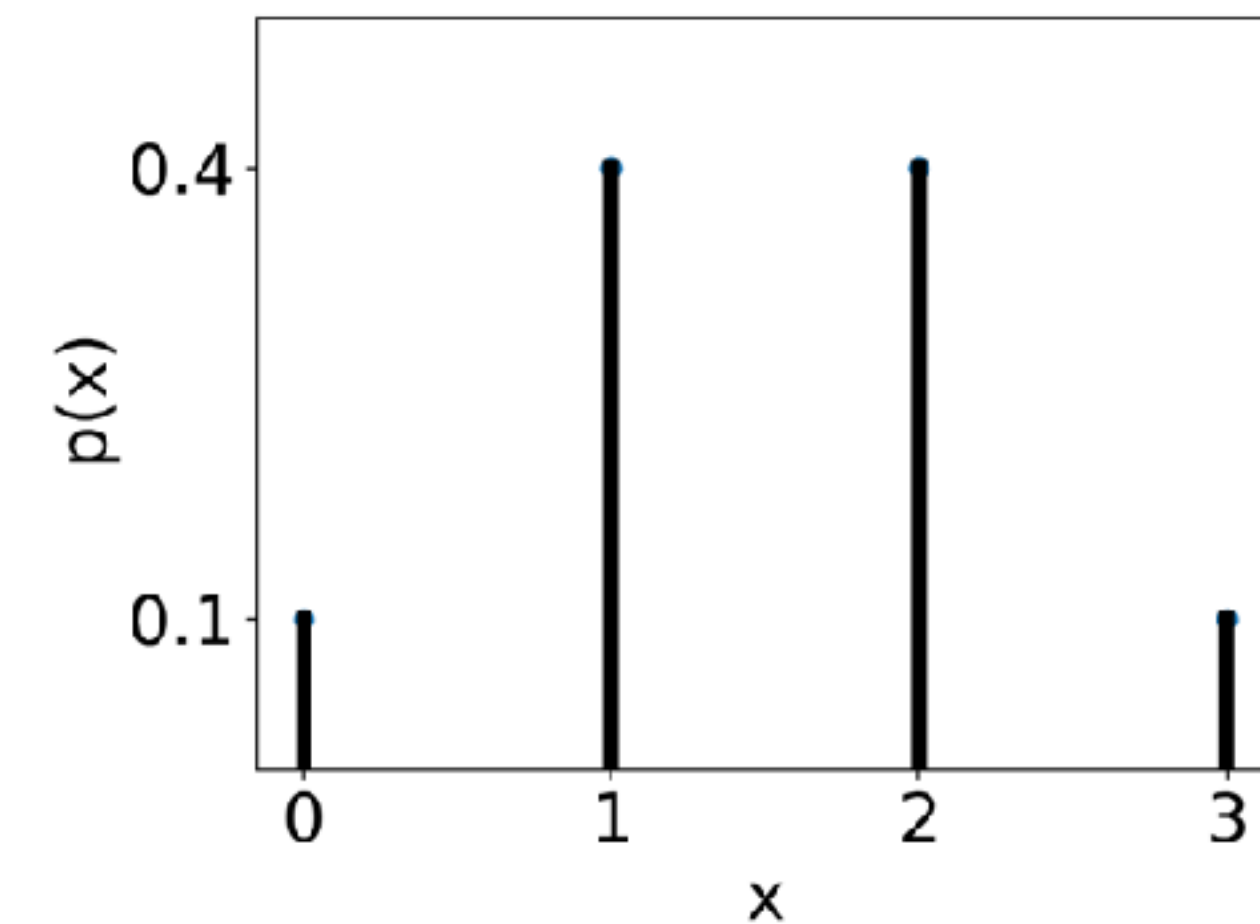
$$x_{\min} \leq E(X) \leq x_{\max}$$

$$= \text{iff } X = c$$

$$0 \leq E(X) \leq 3$$

X is a constant, namely $X=c \rightarrow E(X)=c$

$$E(E(X)) = E(X)$$



Is Expectation Expected?

$\mu = EX$ - expectation of X

Do we expect to see it?

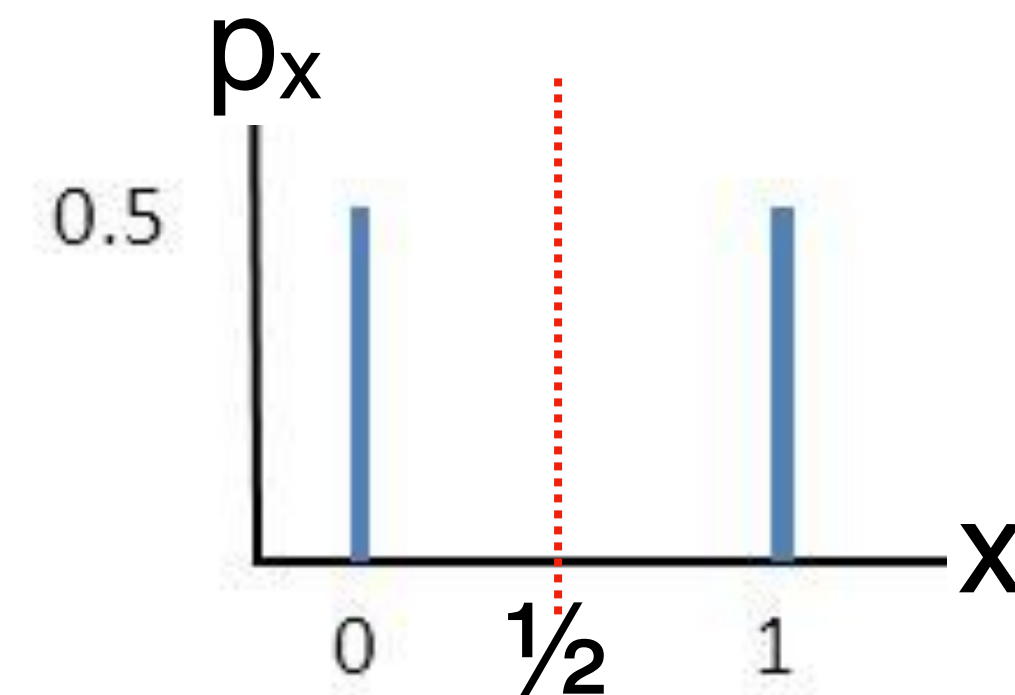
Is p_μ high?

Not necessarily

We may never see it!

$$X \in \{0, 1\}$$

$$p_0 = p_1 = 0.5$$



$$EX = 0 \cdot p_0 + 1 \cdot p_1 = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

Symmetric around $\frac{1}{2}$

$\frac{1}{2}$ will never happen!

Many samples \rightarrow average = $\frac{1}{2}$

EX - average of large sample

Not necessarily likely

May not be observed at all

Infinite Expectation

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

Basel problem

Euler → famous

$$\frac{6}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} = 1$$

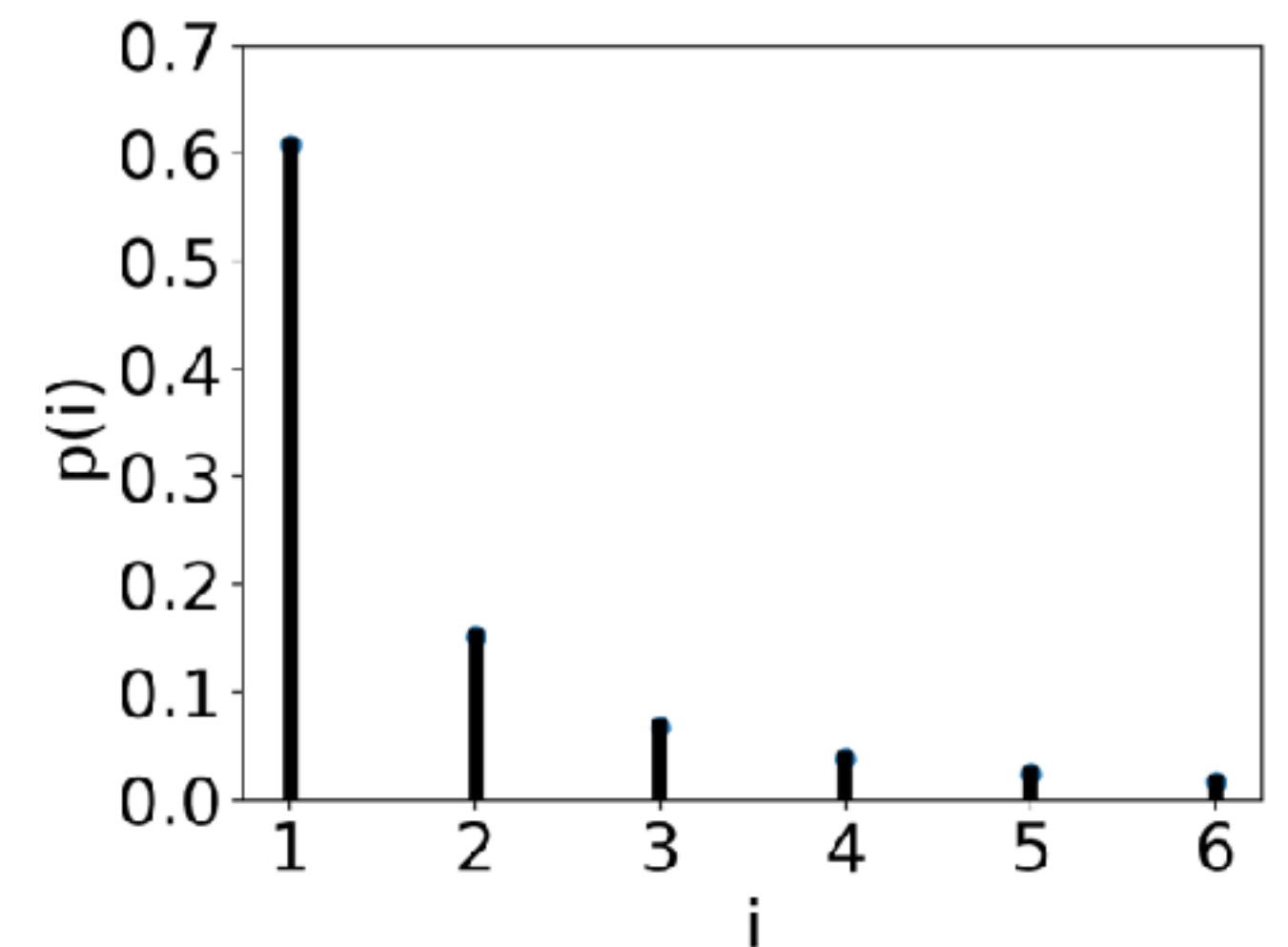
$$p_i = \frac{6}{\pi^2} \cdot \frac{1}{i^2}$$

probability distribution over \mathbb{P}

$$E(X) = \sum_{i=1}^{\infty} i \cdot p_i = \frac{6}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

Many samples

Average will go to ∞

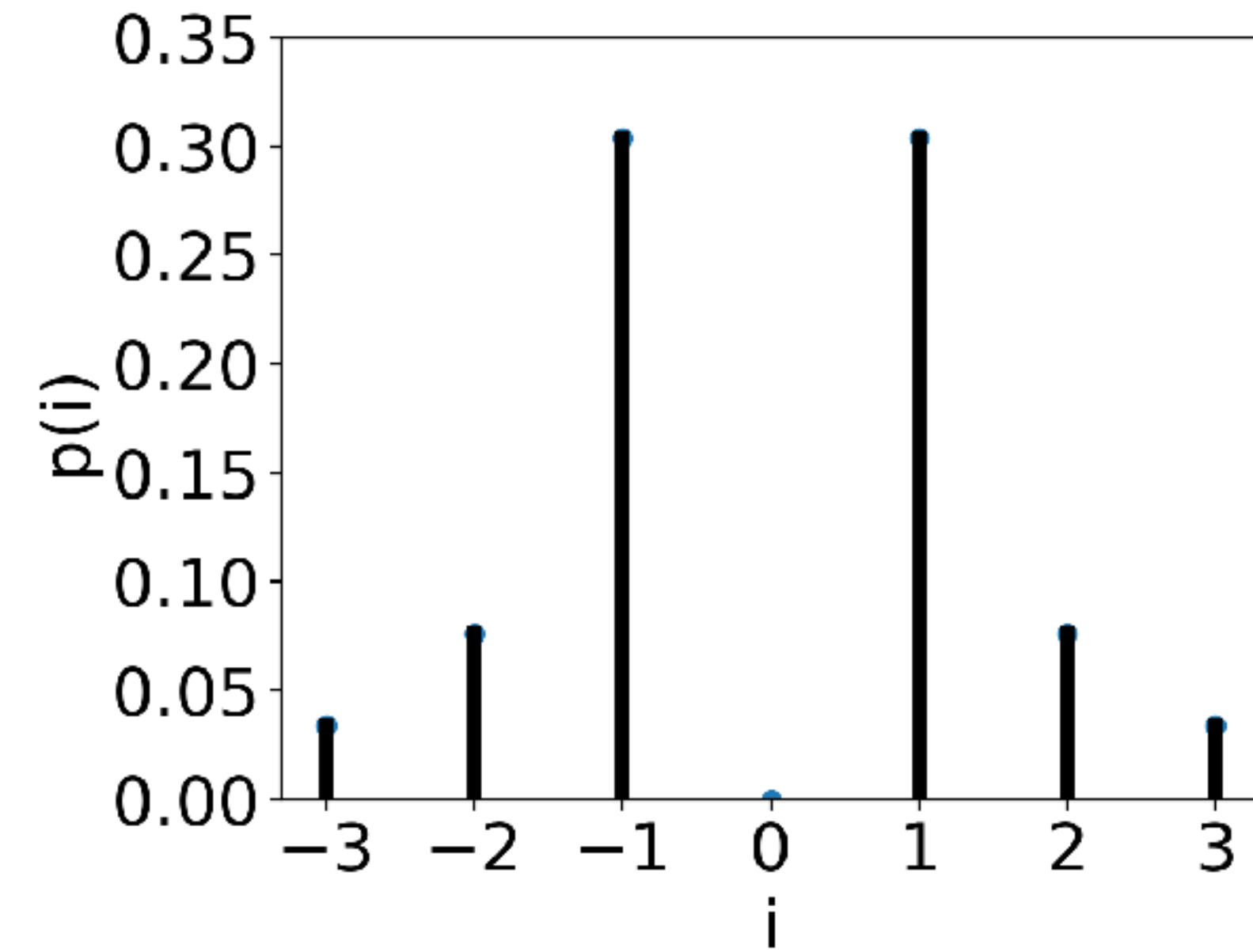


Undefined Expectation

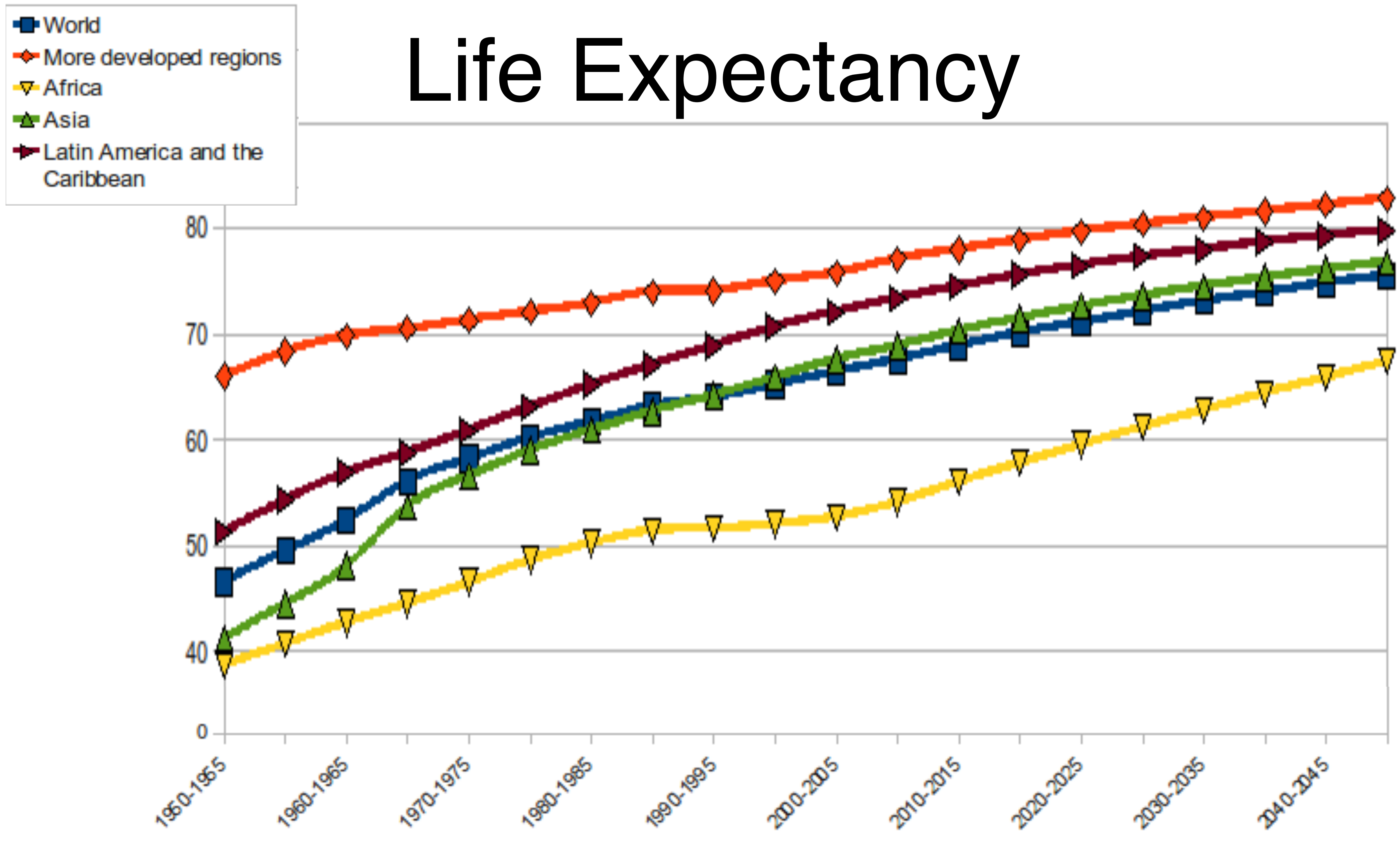
$$p_i = \frac{3}{\pi^2} \cdot \frac{1}{i^2} \quad \text{for } i \neq 0$$

$$E(X) = \infty - \infty$$

Undefined



Life Expectancy



1967 Mr Average

2017 Mr Average

Expectation

Average of many samples



Chest: 38in

Waist 34in

Weight: 11st 8lbs



Expectations of Functions
of Random Variables

ctancy

Collar: 16



Chest: 43in

Waist 37in

Weight: 13st 3lbs

Height: 5ft 10in

Shoe: 9

Life expectancy
81 years

1967 Mr Average

2017 Mr Average

Expectation

Collar: 14.5

Chest: 38in

Waist 34in

Height: 11st 8lbs

Collar: 16

Chest: 43in

Waist 37in

Weight: 13st 3lbs

Height: 5ft 10in

Shoe: 9

Life expectancy
81 years

$$E(X) = \sum_{x \in \Omega} p(x) \cdot x$$

Expectations of Functions
of Random Variables



ctancy