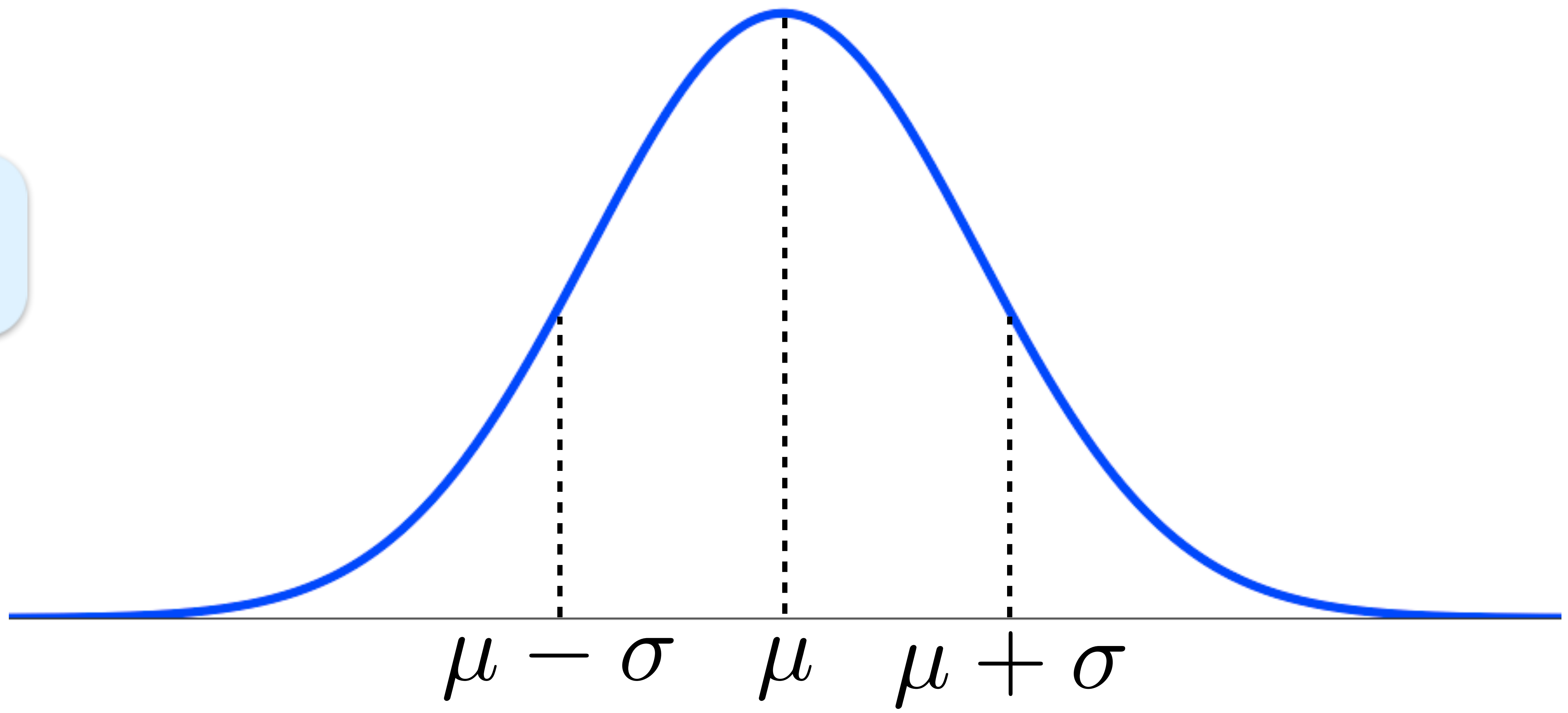
Normal (Gaussian) Distributions

Definition

Mean, Variance

$X \sim N(\mu, \sigma^2)$

$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$



Bell Curve

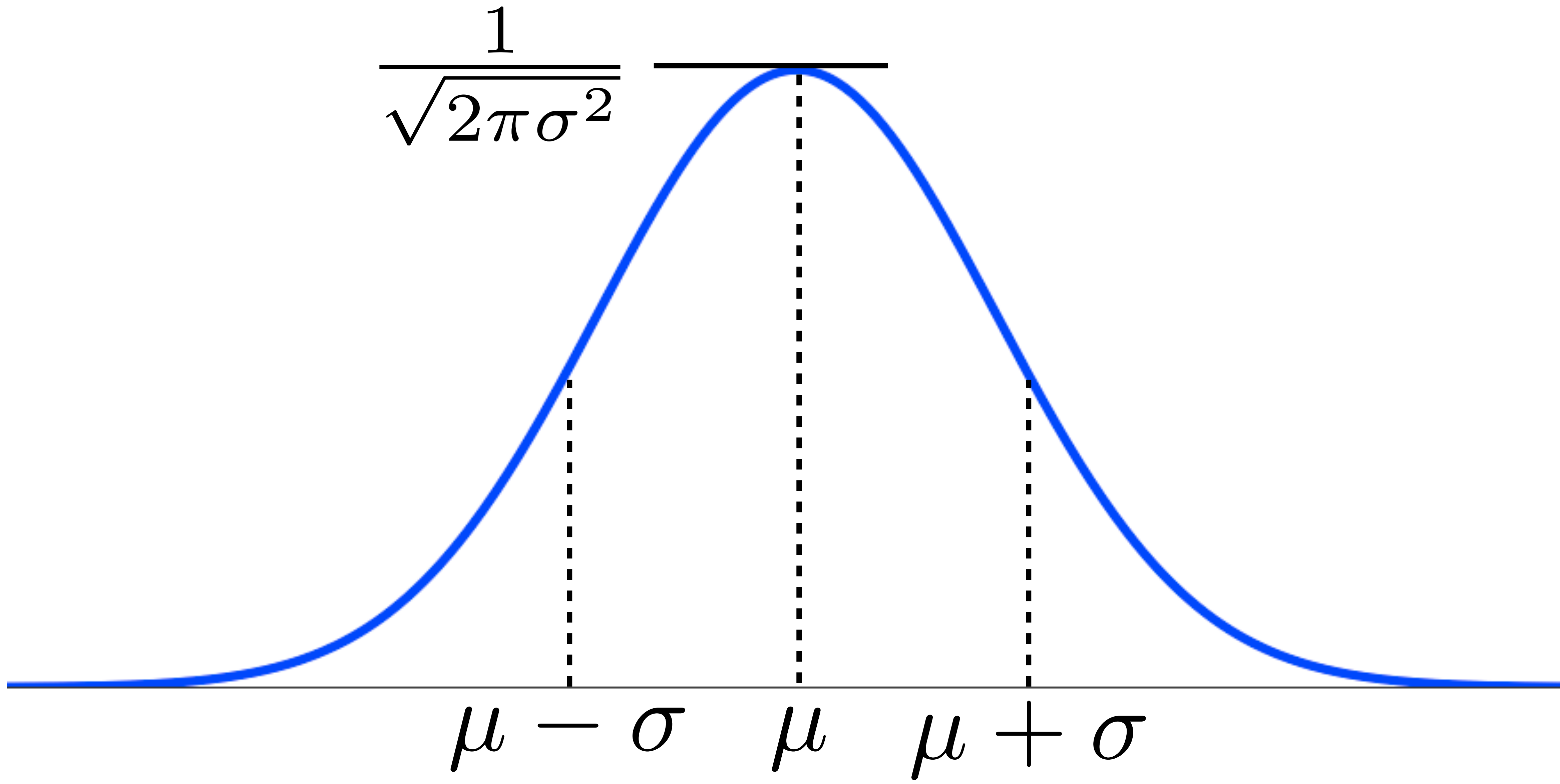
Very common

Occurs whenever adding many independent factors

height, weight, rainfall, salaries, ….

Approximates binomial distribution

Observations

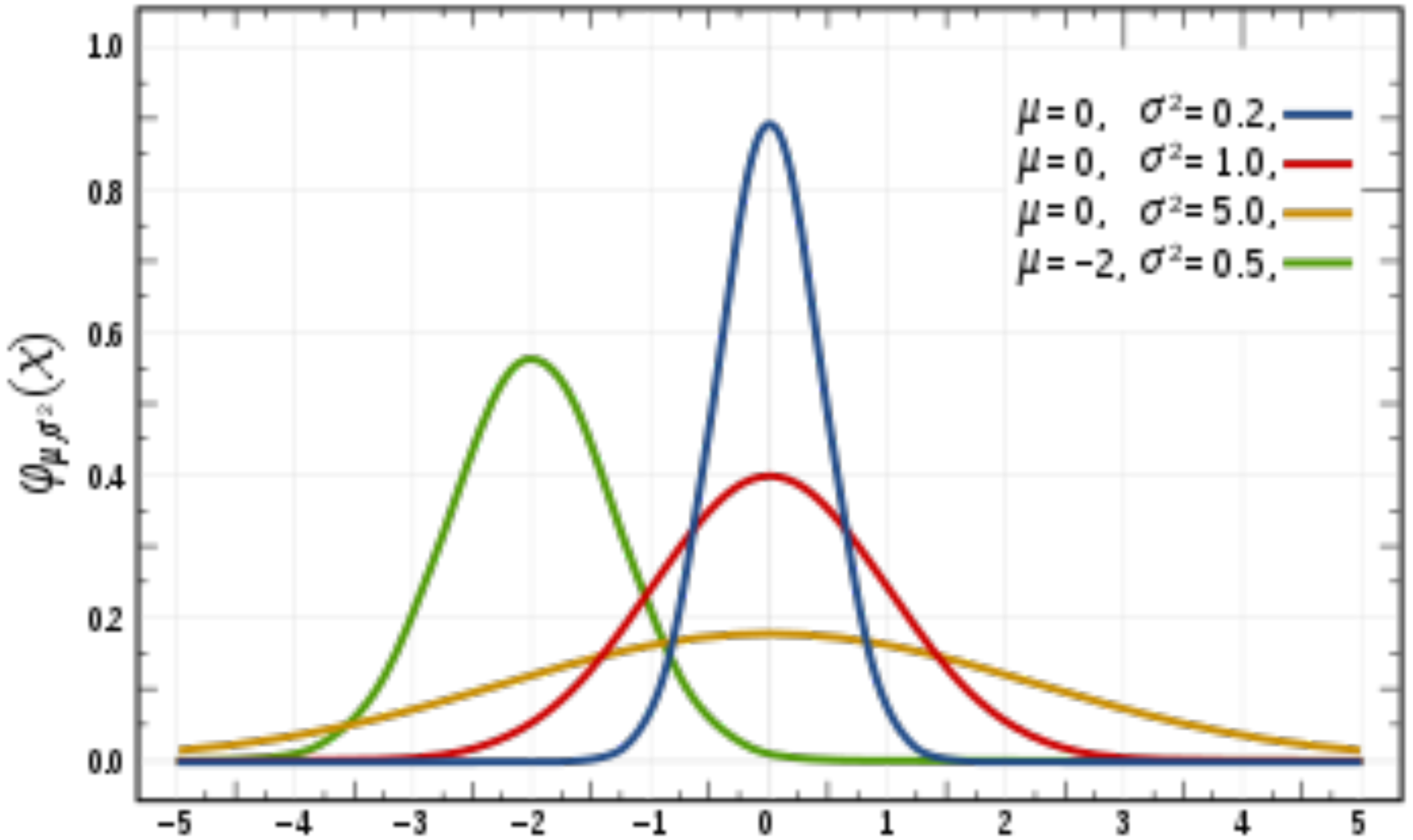


Symmetric around mean μ

μ most likely

$$f(\mu) = \frac{1}{\sqrt{2\pi\sigma^2}}$$

As σ grows, distribution gets more flat



Linear Transformations

Linear transformations of normal distributions are normal

$X \sim N(\mu, \sigma)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$Y = aX + b$

As for all r.v. $\mu_Y = a\mu_X + b$ $\sigma_Y = a\sigma_X$

Show normal

Variable transformation

$$f_Y(y) = \frac{1}{(ax+b)'} f_X(x)\Big|_{y=ax+b} \longleftrightarrow x = \frac{y-b}{a}$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\frac{y-b}{a}-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\underbrace{(a\sigma)}_{\sigma_Y}^2}} e^{-\frac{(y-\overbrace{(a\mu+b)}^{\mu_Y})^2}{2\underbrace{(a\sigma)}_{\sigma_Y}^2}}$$

$Y \sim N(a\mu + b, (a\sigma)^2)$

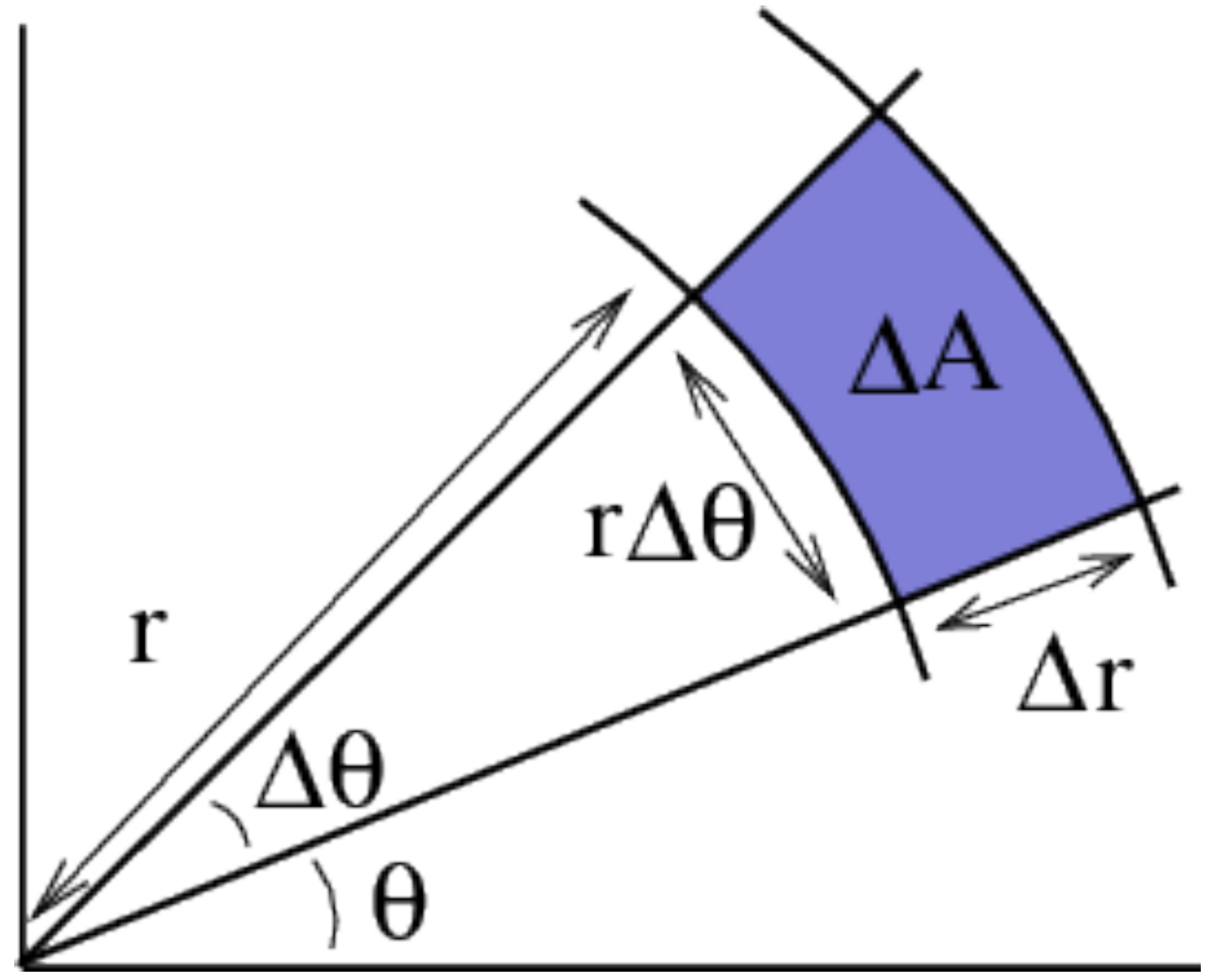
Standard Normal Distribution

Without loss of generality consider X ~ N(0,1)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

Helpful integral $\int xe^{-\frac{x^2}{2}} dx = -e^{-\frac{x^2}{2}}$

ΣWill It ADD?



$$I = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} dy\right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{-(x^2+y^2)}{2}} dx\ dy$$

$$= \int_{0}^{\infty} \int_{0}^{2\pi} e^{\frac{-r^2}{2}} r\ d\theta\ dr$$

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$dx\ dy = r\ dr\ d\theta$

$$= \int_{0}^{\infty} r\ e^{\frac{-r^2}{2}} \int_{0}^{2\pi} d\theta\ dr$$

$$= 2\pi \int_{0}^{\infty} r\ e^{\frac{-r^2}{2}} dr$$

$$\int re^{\frac{-r^2}{2}} dr = -e^{\frac{-r^2}{2}}$$

$$= -2\pi e^{\frac{-r^2}{2}} \Big|_{0}^{\infty} = 2\pi$$

$$I = \sqrt{2\pi}$$

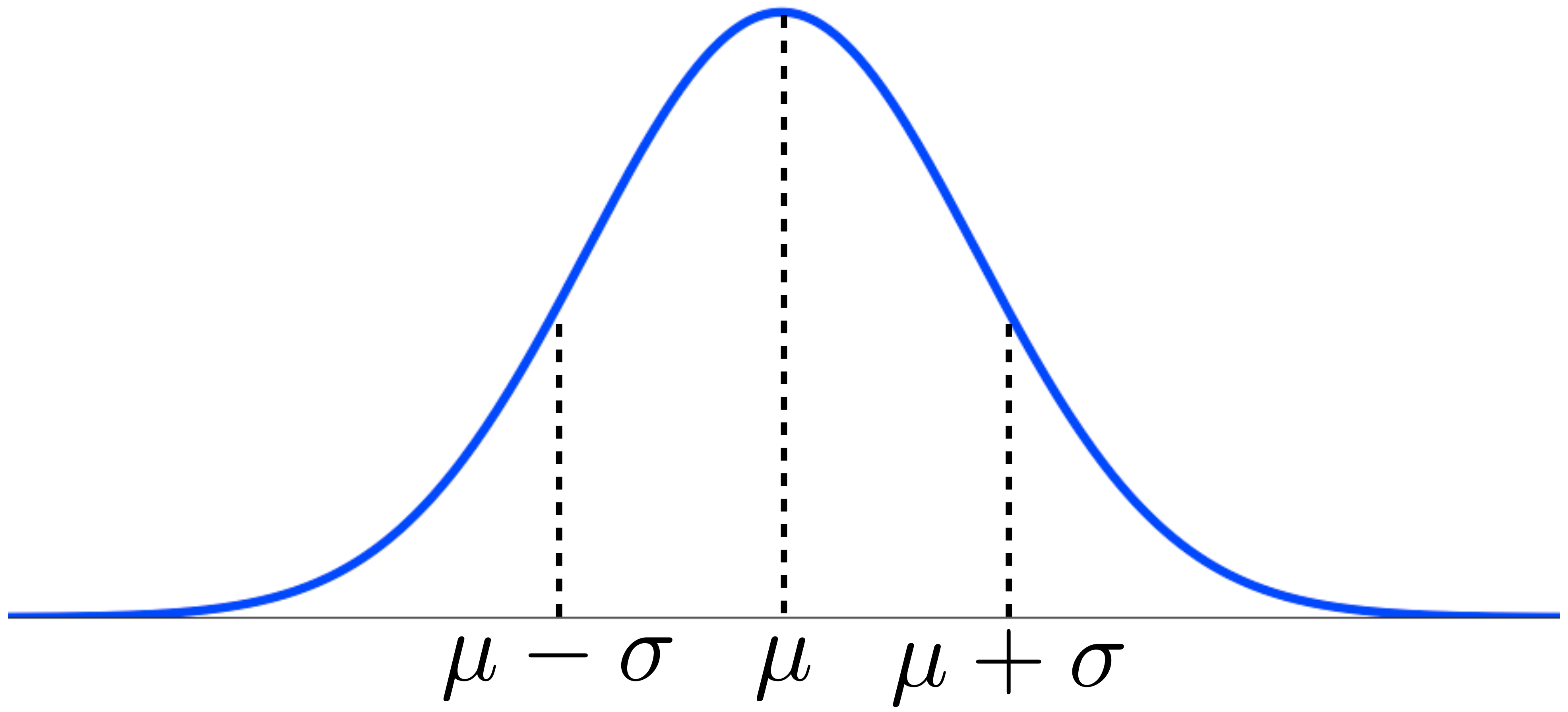
$$\int \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = 1$$

YES IT ADDS!

Expectation

Symmetry

$$E(X) = 0$$



Calculation

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{\frac{-x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} [-e^{\frac{-x^2}{2}} |_{-\infty}^{\infty} = 0$$

Variance

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \int x^2 e^{\frac{-x^2}{2}} dx$$

$$u = x \qquad dv = x\ e^{\frac{-x^2}{2}} dx$$

$$du = dx \qquad v = -e^{\frac{-x^2}{2}}$$

$$\int u\ dv = uv - \int v\ du$$

$$= \frac{-1}{\sqrt{2\pi}} x e^{\frac{-x^2}{2}} \Big|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx = 0 + 1 = 1$$

$$V(X) = E(X^2) - (EX)^2 = 1 - 0 = 1$$

Normal (Gaussian) Distributions

$$\mathbb{N}(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

$$EX = \mu$$

$$V = \sigma^2$$

$$\sigma = \sigma$$

Very common in nature

Next Probabilities