Functions of Random Variables

X - income, Y = 0.3X - income tax

X - driving speed, Y= 2^x - speeding ticket

$X \sim f_X$ $\quad$ f_X known distribution

$Y = g(X)$ $\quad$ g known deterministic function

f_Y ?

What is the distribution of Y?

Power-law pdf's

$a > -1$

$\int_0^1 \underbrace{(a+1)x^a}_{\geq 0} dx = x^{a+1}\Big|_0^1 = 1$

$a < -1$ later

$$f(x) = \begin{cases} (a+1)x^a & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

pdf!

$F(x) = \int_0^x (a+1)u^a du = u^{a+1}\Big|_0^x = x^{a+1}$

$a \leq x \leq 1$

Use $f(x) = 3x^2$

$F(x) = x^3$

$0 \leq x \leq 1$

$g \nearrow$

$$X \sim f_X(x) = 3x^2$$
$$F_X(x) = x^3$$
$$0 \leq x \leq 1$$

$g \searrow$

$$Y = g(X) \triangleq X^{\frac{3}{2}}$$

$$F_Y(y) = P(Y \leq y) \qquad 0 \leq y \leq 1$$
$$= P(X^{\frac{3}{2}} \leq y)$$
$$= P(X \leq y^{\frac{2}{3}})$$
$$= F_X(y^{\frac{2}{3}})$$
$$= y^2$$

$$F_Y(0) = 0$$
$$F_Y(1) = 1$$ ✅

$$f_Y(y) = F_Y'(y) = 2y$$

$$Y = g(X) \triangleq X^{-3}$$

$$F_Y(y) = P(Y \leq y) \qquad y \geq 1$$
$$= P(X^{-3} \leq y)$$
$$= P(X \geq y^{-\frac{1}{3}})$$
$$= 1 - P(X \leq y^{-\frac{1}{3}})$$
$$= 1 - F_X(y^{-\frac{1}{3}})$$
$$= 1 - y^{-1}$$

$$F_Y(1) = 0$$
✅ $$F_Y(\infty) = 1$$

$$f_Y(y) = F_Y'(y) = y^{-2}$$

$g \nearrow$

$$\boxed{f_X(x) = 3x^2 \quad F_X(x) = x^3 \quad g(x) = x^{\frac{3}{2}} \quad h(y) = y^{\frac{2}{3}} \quad f_Y(y) = 2y}$$

$F_Y(y) \triangleq P(Y \leq y)$

$f_Y(y) = F'_Y(y)$

$= P(g(X) \leq y)$

$P(X^{\frac{3}{2}} \leq y)$

$= [F_X(h(y))]'$

$[(y^{\frac{2}{3}})^3]'$

$= P(X \leq g^{-1}(y))$

$P(X \leq y^{\frac{2}{3}})$

$= F'_X(h(y)) \cdot h'(y)$

$3(y^{\frac{2}{3}})^2 \cdot \frac{2}{3} y^{-\frac{1}{3}}$

$= F_X(g^{-1}(y))$

$F_X(y^{\frac{2}{3}})$

$= f_X(h(y)) \cdot h'(y)$

$2y$

$= F_X(h(y))$

$h(y) \overset{\text{def}}{=} g^{-1}(y)$

g ↘

$$F_Y(y) \triangleq P(Y \leq y)$$

$$= P(g(X) \leq y)$$

$$= P(X \geq g^{-1}(y))$$

$$= 1 - P(X \leq g^{-1}(y))$$

$$= 1 - F_X(g^{-1}(y))$$

$$= 1 - F_X(h(y))$$

$$f_Y(y) = F'_Y(y)$$

$$= [1 - [F_X(h(y))]'$$

$$= -F'_X(h(y)) \cdot h'(y)$$

$$= -f_X(h(y)) \cdot h'(y)$$

Combining

$g \nearrow$ $$f_Y(y) = f_X(h(y)) \cdot h'(y)$$

$g \searrow$ $$f_Y(y) = -\, f_X(h(y)) \cdot h'(y)$$

For both $$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$

Alternative formulation $$f_Y(y) = \frac{1}{|g'(x)|} f_X(x)\Big|_{y=g(x)}$$

Functions of Random Variables

Uniform Distributions