Independence

Informal and formal definition

Examples

Motivation

$P(F \mid E) > P(F)$

$P(2 \mid \text{Even}) = \frac{1}{3} > \frac{1}{6} = P(2)$

E ↗ probability of F

$P(F \mid E) < P(F)$

$P(2 \mid \text{Odd}) = 0 < \frac{1}{6} = P(2)$

E ↘ probability of F

$P(F \mid E) = P(F)$

$P(\text{Even} \mid \leq 4) = \frac{1}{2} = P(\text{Even})$

E neither ↗ nor ↘ probability of F

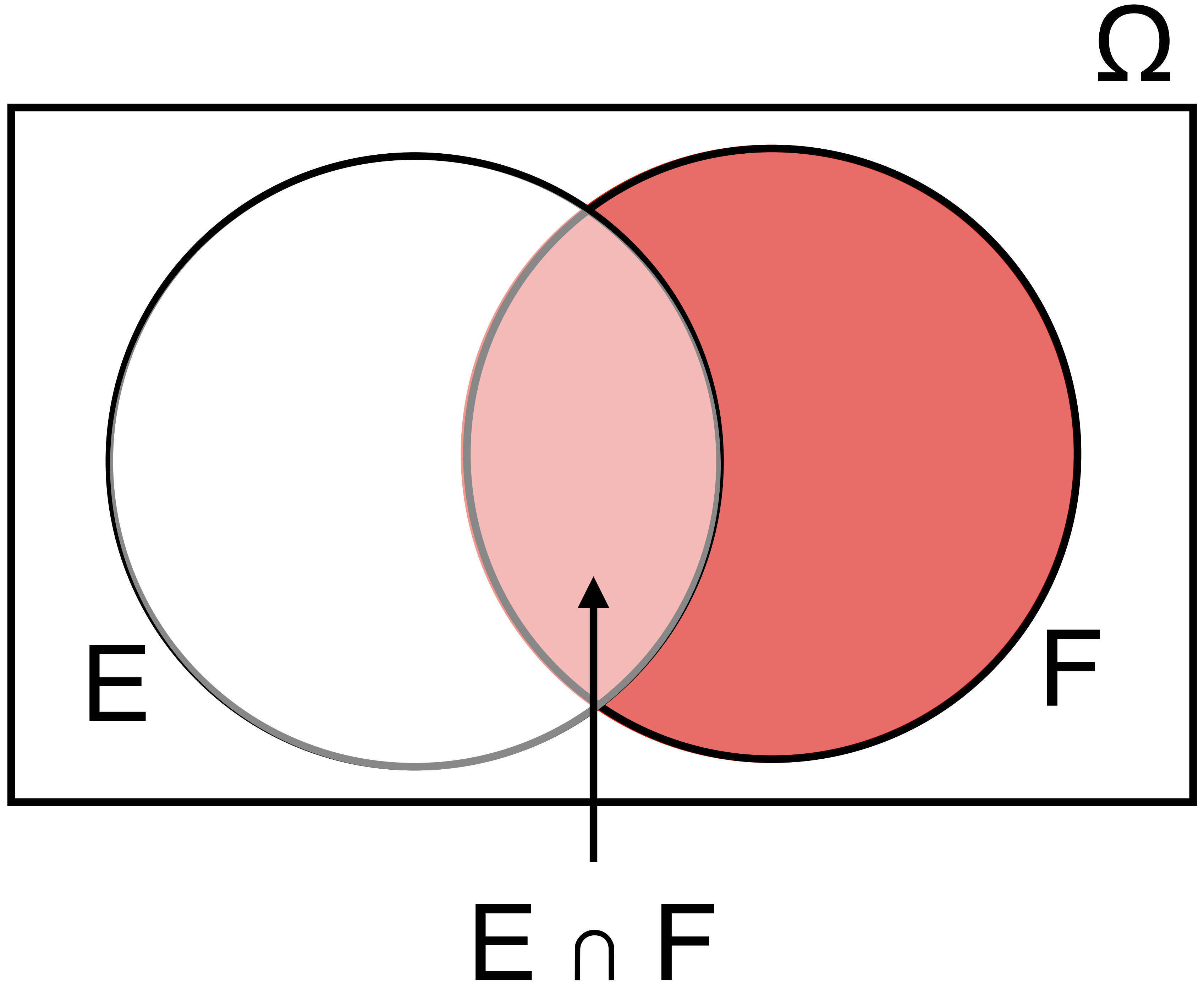
Whether or not E occurs, does not change P(F)

motivation → intuitive definition → formal

Independence - Intuitive

Events E and F are independent, denoted $E \perp\!\!\!\perp F$, if the occurrence of one does not affect the other's probability

$P(F \mid E) = P(F)$



Visually

$P(F) = \dfrac{P(F)}{P(\Omega)}$ F as a fraction of Ω

=

$P(F \mid E) \triangleq \dfrac{P(E \cap F)}{P(E)}$ $E \cap F$ as a fraction of E

Two issues

Asymmetric

Undefined if $P(E)=0$

Independence - Formal

Informally

$P(F) = P(F \mid E) \triangleq \dfrac{P(E \cap F)}{P(E)}$

Asymmetric

Undefined if $P(E)=0$

Formally

E and F are independent if $P(E \cap F) = P(E) \cdot P(F)$

Otherwise, dependent

Symmetric ✅

Applies when $P = 0$ ✅

Implies

intuitive def. $P(F|E) = P(F)$

$P(E|F) = P(E)$

$P(F \mid \overline{E}) = P(F)$

$P(E \mid \overline{F}) = P(E)$

Non-Surprising Independence

Two coins

H_1 First coin heads $P(H_1)=½$

H_2 Second coin heads $P(H_2)=½$

$H_1 \cap H_2$ Both coins heads $P(H_1 \cap H_2)=¼$

$P(H_1 \cap H_2) = ¼ = P(H_1) \cdot P(H_2)$ $H_1 \perp\!\!\!\perp H_2$

Not surprising as two separate coins

Can have $\perp\!\!\!\perp$ even for one experiment

Single Die

Three events

Event	Set	Probability
Prime	$\{ 2, 3, 5 \}$	$\frac{1}{2}$
Odd	$\{ 1, 3, 5 \}$	$\frac{1}{2}$
Square	$\{ 1, 4 \}$	$\frac{1}{3}$

Which pairs are $\perp\!\!\!\perp$ and $\not\perp\!\!\!\perp$

Intersection	Set	Prob	Product	=?	Independence
Prime $\cap$ Odd	$\{ 3, 5 \}$	$\frac{1}{3}$	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$	$\neq$	dependent
Prime $\cap$ Square	$\varnothing$	0	$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$	$\neq$	dependent
Odd $\cap$ Square	$\{ 1 \}$	$\frac{1}{6}$	$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$	$=$	independent

Three Coins

Three events

Event	Description	Set	Probability
H_1	first coin heads	$\{h{*}{*}\}$	½
H_2	second coin heads	$\{{*}h{*}\}$	½
HH	exactly 2 heads in a row	{hht, thh}	¼

Which pairs are $\perp\!\!\!\perp$ and $\not\perp\!\!\!\perp$

Intersection	Set	Prob	=?	Product	Independence
$H_1 \cap H_2$	$\{hh{*}\}$	¼	=	½ · ½ = ¼	independent
$H_2 \cap HH$	{hht, thh}	¼	≠	½ · ¼ = ⅛	dependent
$H_1 \cap HH$	{hht}	⅛	=	½ · ¼ = ⅛	independent

Independence of Ω and $\varnothing$

$\forall A$ $P(\Omega \cap A) = P(A) = P(\Omega) \cdot P(A)$ $\Omega \perp\!\!\!\perp$ of any event

A occurring doesn't modify likelihood of Ω

$\forall A$ $P(\varnothing \cap A) = P(\varnothing) = P(\varnothing) \cdot P(A)$ $\varnothing \perp\!\!\!\perp$ of any event

A occurring doesn't modify likelihood of $\varnothing$

Independence

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Examples

Total Probability