

Two Variables

Why 2

Outcomes often result from multiple factors

Rain temperature and humidity

Economy unemployment and inflation

Hiring experience and salary

Student # classes GPA

Human condition profession age cholesterol
location salary happiness
dinner plans ...

Two Fair Coins

$$U, V \sim B(1/2) \quad \perp\!\!\!\perp$$

Several ways to indicate distribution

Explicit $P(u,v) \stackrel{\text{def}}{=} P(U=u, V=v) = 1/4 \quad \forall \{u,v\} \in \{0,1\}$

1-d table

u	v	P(u,v)
0	0	1/4
0	1	1/4
1	0	1/4
1	1	1/4

2-d table

		v	
		0	1
u	0	1/4	1/4
	1	1/4	1/4

Use U, V, for several examples

Min - Max

$$U, V \sim B(1/2)$$

 \perp

$$X = \min(U, V)$$

$$Y = \max(U, V)$$

u	v	min	max	
0	0	0	0	} 1/4
0	1	0	1	
1	0	0	1	} 1/2
1	1	1	1	
				} 1/4

 $x = \min$ $y = \max$

	0	1
0	1/4	1/2
1	0	1/4

Product - Sum

$$X = U \cdot V$$

$$Y = U + V$$

		y		
		0	1	2
x	0	$\frac{1}{4}$	$\frac{1}{2}$	0
	1	0	0	$\frac{1}{4}$

3 Coins

$U_1, U_2, U_3 \sim B(1/2)$ $\perp\!\!\!\perp$

$X = U_1 + U_2$ # heads among first 2

$Y = U_2 + U_3$ # heads among last 2

U_1	U_2	U_3	X	Y
0	0	0	0	0
0	0	1	0	1
0	1	0	1	1
0	1	1	1	2
1	0	0	1	0
1	0	1	1	1
1	1	0	2	1
1	1	1	2	2

	0	1	2	
				Y
0	$\frac{1}{8}$	$\frac{1}{8}$	0	
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	
2	0	$\frac{1}{8}$	$\frac{1}{8}$	
				X

General B(p)

$$U \sim B(p), V \sim B(q) \quad \perp$$

$$X = \min(U, V)$$

$$Y = \max(U, V)$$

	v	
	$\bar{p}\bar{q}$	$\bar{p}q$
u	$p\bar{q}$	pq

		y = max	
		0	1
x = min	0	$\bar{p}\bar{q}$	$p\bar{q} + \bar{p}q$
	1	0	pq

General?

Joint Distribution

X, Y - random variables

Joint distribution: P : probability of every possible (x,y) pair

$$p(x,y) \stackrel{\text{def}}{=} P(X = x, Y = y)$$

$$\forall x,y \ p(x,y) \geq 0$$

$$\sum_{x,y} p(x,y) = 1$$

Joint Distribution Tells All

Joint distribution determines probabilities of all events

		0	1	y
x	0	0.1	0.2	
	1	0.3	0.4	

$$P(X \leq Y) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 1, Y = 1)$$

$$= P(0, 0) + P(0, 1) + P(1, 1)$$

$$= 0.1 + 0.2 + 0.4$$

$$= 0.7$$

Marginals

Marginal of X $P(x) \stackrel{\text{def}}{=} P_X(x) \stackrel{\text{def}}{=} P(X = x) = \sum_y p(x,y)$

Rule of total probability

Marginal of Y $P(y) \stackrel{\text{def}}{=} P_Y(y) \stackrel{\text{def}}{=} P(Y = y) = \sum_x p(x,y)$

		0	1	y	
x	0	0.1	0.2		← $P(X = 0) = .3$
	1	0.3	0.4		← $P(X = 1) = .7$

$$\begin{aligned}
 P(X = 0) &= P(X = 0, Y = 0) + P(X = 0, Y = 1) \\
 &= P(0,0) + P(0,1) = .1 + .2 = .3
 \end{aligned}$$

Conditionals

$$P(x \mid y) = \frac{p(x,y)}{p(y)}$$

$$P(y \mid x) = \frac{p(x, y)}{p(x)}$$

$$P(Y = 0 \mid X = 0) = \frac{P(X = 0, Y = 0)}{P(X = 0)} = \frac{0.1}{0.3} = \frac{1}{3}$$

$$P(Y = 1 \mid X = 0) = \frac{P(X = 0, Y = 1)}{P(X = 0)} = \frac{0.2}{0.3} = \frac{2}{3}$$

$$P(X = 0 \mid Y = 0) = \frac{P(X = 0, Y = 0)}{P(Y = 0)} = \frac{0.1}{0.4} = \frac{1}{4}$$

$$P(X = 1 \mid Y = 0) = 1 - P(X = 0, Y = 0) = 1 - \frac{1}{4} = \frac{3}{4}$$

S		0	1	y
0	0.1	0.2	$\leftarrow P(X = 0) = 0.3$	
1	0.3	0.4	$\uparrow P(Y = 0) = 0.4$	

Independence

X, Y independent

$$X \perp\!\!\!\perp Y$$

$$\forall x, y \quad p(y \mid x) = p(y)$$

$$p(x \mid y) = p(x)$$

$$p(x, y) = p(x) \cdot p(y) \quad \leftarrow \text{more robust}$$

		$P(y)$		y
		0.2	0.8	
$P(x)$	0.6	0.12	0.48	x
	0.4	0.08	0.32	

$\perp\!\!\!\perp$

		$P(y)$		y
		0.4	0.6	
$P(x)$	0.3	0.1	0.2	x
	0.7	0.3	0.4	

~~$\perp\!\!\!\perp$~~

Independence Checks

Independent \rightarrow rows proportional to each other

\rightarrow columns proportional to each other

$$X \sim B(1/2)$$

$$Y = X$$

		y	
		0	1
x	0	$1/2$	0
	1	0	$1/2$

\perp

$$Y = 1 - X$$

		y	
		0	1
x	0	0	$1/2$
	1	$1/2$	0

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