Final problem set for Physics 5300 Theoretical Mechanics

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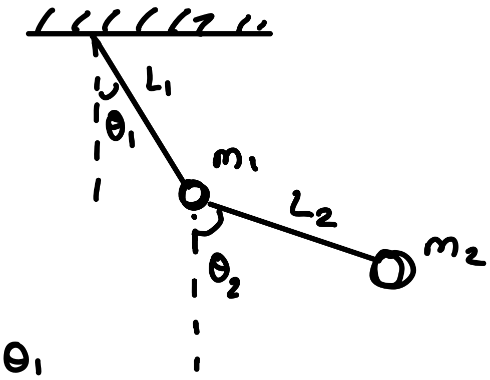
Due date: Friday, April 28, 2023 via Github

- To hand in the assignment, upload (or push) Jupyter notebooks to your github account (which you will have established in homework 11).
- Base your notebooks on the ones used in class (you can make your own, but please tell me in advance). Comments in the Python code are required.
- The minimum to do are the tasks in parts **a** and **b** of each problem. Your grade will improve based on how many of the other tasks you complete and their correctness (details in class).
- You are strongly encouraged to push versions of your notebooks in advance of the due date. You can have me check or help debug them at any time.

1. **Double (and more) pendulum**
   a. Make a notebook that solves the double pendulum using (11.37) and (11.38) in Taylor Section 11.4 (i.e., do not assume the small angle approximation). Implement the pendulum code as a Python class (as in our examples). **You can use either the Euler-Lagrange or Hamiltonian equations.** The basic output will be plots of the two angles as a function of time for given initial conditions.
   b. Show that the system is chaotic for initial conditions beyond the small angle approximation.
   c. Add Markdown/LaTeX documentation.
   d. Extend your simulation to three pendulums.
   e. Add widgets to control the simulation and/or animate it.

2. **Gravitational orbits in Cartesian coordinates**
   a. Make a notebook that solves the two-body problem for gravitational attraction between two bodies (e.g., earth and sun) in Cartesian coordinates. Implement the differential equation code as a Python class (as in our examples).
   b. Show that the problem reduces to the orbits considered in class if one of the bodies is very heavy and you are in its rest frame.
   c. Add Markdown/LaTeX documentation.
   d. Use the Leapfrog method to solve the differential equations and show that it conserves energy while using the SciPy ODE solvers do not.
   e. Extend the simulation to three bodies (i.e., add a planet) in the same plane and comment on the orbits you find.
   f. Add widgets to control the simulation and/or animate the orbits.

E-L: $\dfrac{\partial \mathcal{L}}{\partial q} = \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{q}}$



$\mathcal{L} = T - U$

$x_1 = L_1 \sin\theta_1 \Rightarrow \dot{x}_1 = L_1\dot{\theta}_1\cos\theta_1$

$y_1 = L_1\cos\theta_1 \Rightarrow \dot{y}_1 = -L_1\dot{\theta}_1\sin\theta_1$

$x_2 = x_1 + L_2\sin\theta_2$

$= L_1\sin\theta_1 + L_2\sin\theta_2 \Rightarrow \dot{x}_2 = L_1\dot{\theta}_1\cos\theta_1 + L_2\dot{\theta}_2\cos\theta_2$

$y_2 = y_1 + L_2\cos\theta_2$

$= L_1\cos\theta_1 + L_2\cos\theta_2 \Rightarrow \dot{y}_2 = -L_1\dot{\theta}_1\sin\theta_1 - L_2\dot{\theta}_2\sin\theta_2$

$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$

$= \frac{1}{2}m_1((L_1\dot{\theta}_1\cos\theta_1)^2 + (-L_1\dot{\theta}_1\sin\theta_1)^2)$
$+ \frac{1}{2}m_2((L_1\dot{\theta}_1\cos\theta_1 + L_2\dot{\theta}_2\cos\theta_2)^2 +$
$(-L_1\dot{\theta}_1\sin\theta_1 - L_2\dot{\theta}_2\sin\theta_2)^2)$

$= \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2$
$+ 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2$
$+ 2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2)$

$T = \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2))$

$$U = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g L_1 \cos\theta_1 - m_2 g (L_1 \cos\theta_1 + L_2 \cos\theta_2)$$
$$U = -(m_1+m_2) g L_1 \cos\theta_1 - m_2 g L_2 \cos\theta_2$$

$$\mathcal{L} = T - U$$
$$= \frac{1}{2} m_1 L_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 \left(L_1^2 \dot\theta_1^2 + L_2^2 \dot\theta_2^2 + 2L_1L_2 \dot\theta_1\dot\theta_2 \cos(\theta_1-\theta_2)\right)$$
$$+ (m_1+m_2) g L_1 \cos\theta_1 + m_2 g L_2 \cos\theta_2$$

E-L equations:

① $\dfrac{\partial\mathcal{L}}{\partial\theta_1} = \dfrac{d}{dt}\dfrac{\partial\mathcal{L}}{\partial\dot\theta_1}$, ② $\dfrac{\partial\mathcal{L}}{\partial\theta_2} = \dfrac{d}{dt}\dfrac{\partial\mathcal{L}}{\partial\dot\theta_2}$

① $-(m_1+m_2) g L_1 \sin\theta_1 - m_2 L_1 L_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2)$
$$= \frac{d}{dt}\left(m_1 L_1^2 \dot\theta_1 + m_2 L_1^2 \dot\theta_1 + m_2 L_1 L_2 \dot\theta_2 \cos(\theta_1-\theta_2)\right)$$

$\therefore -(m_1+m_2) g L_1 \sin\theta_1 - m_2 L_1 L_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2)$
$$= (m_1+m_2) L_1^2 \ddot\theta_1 + m_2 L_1 L_2 \ddot\theta_2 \cos(\theta_1-\theta_2)$$
$$- m_2 L_1 L_2 \dot\theta_2 \sin(\theta_1-\theta_2)(\dot\theta_1 - \dot\theta_2)$$

$\therefore -(m_1+m_2) g L_1 \sin\theta_1 - m_2 L_1 L_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2)$
$$= (m_1+m_2) L_1^2 \ddot\theta_1 + m_2 L_1 L_2 \ddot\theta_2 \cos(\theta_1-\theta_2)$$
$$- \dot\theta_1 m_2 L_1 L_2 \dot\theta_2 \sin(\theta_1-\theta_2)$$
$$+ m_2 L_1 L_2 \dot\theta_2^2 \sin(\theta_1-\theta_2)$$

$$\therefore \quad (m_1+m_2)gL_1\sin\theta_1 + (m_1+m_2)L_1^2\ddot{\theta}_1 + m_2L_1L_2\ddot{\theta}_2\cos(\theta_1-\theta_2)$$
$$+ m_2L_1L_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) = 0$$

$$\therefore \quad (m_1+m_2)g\sin\theta_1 + (m_1+m_2)L_1\ddot{\theta}_1 + m_2L_2\ddot{\theta}_2\cos(\theta_1-\theta_2)$$
$$+ m_2L_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) = 0$$

(2) $$m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2gL_2\sin\theta_2 = \frac{d}{dt}\Big(m_2L_2^2\dot{\theta}_2$$
$$+ 2L_1L_2\dot{\theta}_1\cos(\theta_1-\theta_2)\Big)$$

$$\therefore \quad m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2gL_2\sin\theta_2$$
$$= \frac{d}{dt}\Big(m_2L_2^2\dot{\theta}_2 + m_2L_1L_2\dot{\theta}_1\cos(\theta_1-\theta_2)\Big)$$

$$\therefore \quad m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2gL_2\sin\theta_2$$
$$= m_2L_2^2\ddot{\theta}_2 + m_2L_1L_2\ddot{\theta}_1\cos(\theta_1-\theta_2)$$
$$- m_2L_1L_2\dot{\theta}_1\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)$$

$$\therefore \quad m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2gL_2\sin\theta_2$$
$$= m_2L_2^2\ddot{\theta}_2 + m_2L_1L_2\ddot{\theta}_1\cos(\theta_1-\theta_2)$$
$$- m_2L_1L_2\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)$$

$$\therefore \quad m_2L_2^2\ddot{\theta}_2 + m_2gL_2\sin\theta_2 + m_2L_1L_2\ddot{\theta}_1\cos(\theta_1-\theta_2)$$
$$- m_2L_1L_2\dot{\theta}_1^2\sin(\theta_1-\theta_2) = 0$$

$$\therefore \quad m_2L_2\ddot{\theta}_2 + m_2g\sin\theta_2 + m_2L_1\ddot{\theta}_1\cos(\theta_1-\theta_2)$$
$$- m_2L_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) = 0$$

$$z_1 \equiv \dot{\theta}_1 \Rightarrow \dot{z}_1 = \ddot{\theta}_1 \;,\; z_2 = \dot{\theta}_2 \;,\; \dot{z}_2 = \ddot{\theta}_2$$

$$\therefore (m_1+m_2) g\sin\theta_1 + (m_1+m_2) L_1 \dot{z}_1 + m_2 L_2 \dot{z}_2 \cos(\theta_1-\theta_2) + m_2 L_2 z_2^2 \sin(\theta_1-\theta_2) = 0 \qquad (1)\ a$$

$$m_2 L_2 \dot{z}_2 + m_2 g\sin\theta_2 + m_2 L_1 \dot{z}_1 \cos(\theta_1-\theta_2) - m_2 L_1 z_1^2 \sin(\theta_1-\theta_2) = 0 \qquad (2)\ b$$

$$\therefore \dot{z}_2 = \frac{-m_2 g\sin\theta_2 - m_2 L_1 \dot{z}_1 \cos(\theta_1-\theta_2) + m_2 L_1 z_1^2 \sin(\theta_1-\theta_2)}{m_2 L_2}$$

Substituting in a :

$$(m_1+m_2) g\sin\theta_1 + (m_1+m_2) L_1 \dot{z}_1 + \cos(\theta_1-\theta_2) \cdot \left(-m_2 g\sin\theta_2 - m_2 L_1 \dot{z}_1 \cos(\theta_1-\theta_2) + m_2 L_1 z_1^2 \sin(\theta_1-\theta_2)\right) + m_2 L_2 z_2^2 \sin(\theta_1-\theta_2) = 0$$

$$\therefore (m_1+m_2) L_1 \dot{z}_1 - m_2 L_1 \dot{z}_1 \cos^2(\theta_1-\theta_2) = -(m_1+m_2) g\sin\theta_1 - m_2 \sin(\theta_1-\theta_2)\left[L_2 z_2^2 + L_1 z_1^2 \cos(\theta_1-\theta_2)\right] + m_2 g\sin\theta_2 \cos(\theta_1-\theta_2)$$

$$\therefore m_1 L_1 \dot{z}_1 + m_2 L_1 \dot{z}_1 \sin^2(\theta_1-\theta_2) = -(m_1+m_2) g\sin\theta_1 - m_2 \sin(\theta_1-\theta_2)\left[L_2 z_2^2 + L_1 z_1^2 \cos(\theta_1-\theta_2)\right] + m_2 g\sin\theta_2 \cos(\theta_1-\theta_2)$$

$$\therefore \dot{z}_1 = \frac{\left(m_2 g\sin\theta_2 \cos(\theta_1-\theta_2) - (m_1+m_2) g\sin\theta_1 - m_2 \sin(\theta_1-\theta_2)\left[L_2 z_2^2 + L_1 z_1^2 \cos(\theta_1-\theta_2)\right]\right)}{L_1\left(m_1 + m_2 \sin^2(\theta_1-\theta_2)\right)}$$

From b,

$$\dot{z}_1 = -\frac{m_2L_2\dot{z}_2 - m_2 g\sin\theta_2 + m_2L_1z_1^2\sin(\theta_1-\theta_2)}{m_2L_1\cos(\theta_1-\theta_2)}$$

Plugging into a,

$$m_2L_2\dot{z}_2\cos(\theta_1-\theta_2) = -(m_1+m_2)g\sin\theta_1 - m_2L_2z_2^2\sin(\theta_1-\theta_2)$$
$$-(m_1+m_2)L_1\left(-\frac{m_2L_2\dot{z}_2 - m_2 g\sin\theta_2 + m_2L_1z_1^2\sin(\theta_1-\theta_2)}{m_2L_1\cos(\theta_1-\theta_2)}\right)$$

$$\therefore \left(m_2L_2\cos(\theta_1-\theta_2) - \frac{(m_1+m_2)L_2}{\cos(\theta_1-\theta_2)}\right)\dot{z}_2 =$$

$$-(m_1+m_2)g\sin\theta_1 - m_2L_2z_2^2\sin(\theta_1-\theta_2)$$
$$-(m_1+m_2)\left(\frac{-g\sin\theta_2 + L_1z_1^2\sin(\theta_1-\theta_2)}{\cos(\theta_1-\theta_2)}\right)$$

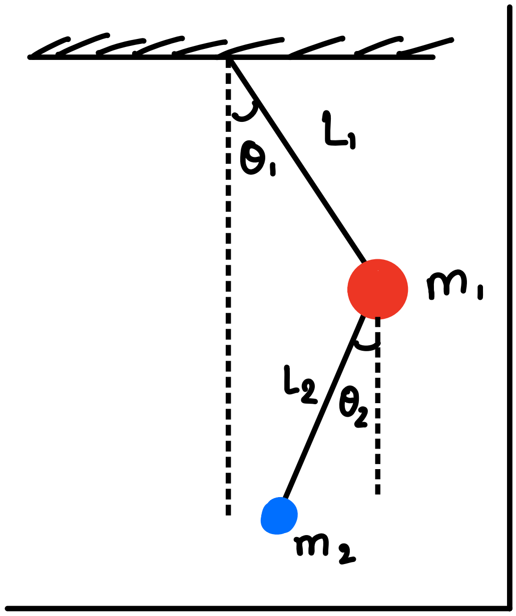
$$\therefore \left(\frac{-m_1 - m_2\sin^2(\theta_1-\theta_2)}{\cos(\theta_1-\theta_2)}\right)L_2\dot{z}_2 =$$

$$\left(\frac{-(m_1+m_2)g\sin\theta_1 - m_2L_2z_2^2\sin(\theta_1-\theta_2)\big)\cos(\theta_1-\theta_2)}{\cos(\theta_1-\theta_2)}\right)$$
$$-(m_1+m_2)\left(\frac{-g\sin\theta_2 + L_1z_1^2\sin(\theta_1-\theta_2)}{\cos(\theta_1-\theta_2)}\right)$$

$$\therefore \dot{z}_2 = \left(\frac{-(m_1+m_2)g\sin\theta_1 - m_2L_2z_2^2\sin(\theta_1-\theta_2)\big)\cos(\theta_1-\theta_2)}{-(m_1+m_2\sin^2(\theta_1-\theta_2))L_2}\right)$$
$$-(m_1+m_2)\left(\frac{-g\sin\theta_2 + L_1z_1^2\sin(\theta_1-\theta_2)}{-(m_1+m_2\sin^2(\theta_1-\theta_2))L_2}\right)$$

$$\therefore \dot{z}_2 = \frac{(m_1+m_2)\left[g\sin\theta_1\cos(\theta_1-\theta_2) - g\sin\theta_2 + L_1 z_1^2 \sin(\theta_1-\theta_2)\right]}{(m_1+m_2\sin^2(\theta_1-\theta_2))L_2}$$

$$+ \frac{(m_2 L_2 z_2^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2))}{(m_1+m_2\sin^2(\theta_1-\theta_2))L_2}$$



Gravitational Orbits

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \qquad \frac{\partial \mathcal{L}}{\partial y_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}_1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} \qquad \frac{\partial \mathcal{L}}{\partial y_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}_2}$$

$$U = \frac{-Gm_1m_2}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} = \frac{-Gm_1m_2}{r_{12}}$$

$$T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$$

$$= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$\mathcal{L} = \frac{1}{2}\left(m_1(\dot{x}_1^2 + \dot{y}_1^2) + m_2(\dot{x}_2^2 + \dot{y}_2^2)\right) + \frac{G m_1 m_2}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -\frac{1}{2} \frac{G m_1 m_2}{((x_2-x_1)^2 + (y_2-y_1)^2)^{3/2}} \cdot -2(x_2-x_1)$$

$$= \frac{G m_1 m_2 (x_2 - x_1)}{((x_2-x_1)^2 + (y_2-y_1)^2)^{3/2}}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \frac{d}{dt}(m_1 \dot{x}_1) = m_1 \ddot{x}_1$$

$$m_1 \ddot{x}_1 = \frac{G m_1 m_2 (x_2 - x_1)}{((x_2-x_1)^2 + (y_2-y_1)^2)^{3/2}}$$

$$\ddot{x}_1 = \frac{G m_2 (x_2 - x_1)}{((x_2-x_1)^2 + (y_2-y_1)^2)^{3/2}} = \frac{G m_2 (x_2 - x_1)}{r_{12}^3}$$

Similarly, $\ddot{y}_1 = \dfrac{G m_2 (y_2 - y_1)}{((x_2-x_1)^2 + (y_2-y_1))^{3/2}}$

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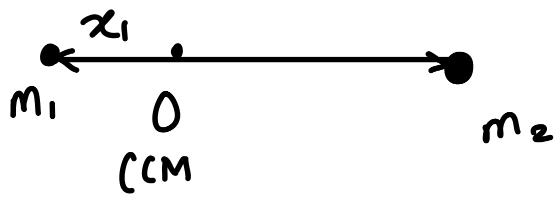

$$0 = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$m_1 x_1 = -m_2 x_2$$

$$x_2 = -\left(\frac{m_1}{m_2}\right) x_1$$

$$\dot{x}_2 = -\left(\frac{m_1}{m_2}\right) \dot{x}_1$$