the number of groupings in G_2 is just

$$\binom{(n-k)+k-1}{(n-k)} = \binom{n-1}{n-k}. \qquad \blacksquare$$

❉

Now we turn to the case where the n elements are identical, and the k groups are indistinguishable; for example, placing those same $n = 15$ white marbles into $k = 5$ boxes, where now the boxes are indistinguishable from each other unless they contain different numbers of marbles. In this scenario, the stars and bars diagram

$$***|****|*|*****|**$$

describes the same outcome as the diagram

$$****|***|**|*****|*$$

which we might as well show by listing the boxes from smallest to largest:

$$*|**|***|****|*****$$

There are 5! such diagrams that all correspond to the same outcome: the set of numbers $\{1, 2, 3, 4, 5\}$. But not every outcome has 5! corresponding diagrams. For instance, the outcome represented by

$$***|***|***|***|***$$

has just this one diagram.

Counting the ways of putting indistinguishable elements into indistinguishable boxes is essentially the problem of expressing a positive integer n as the sum of k smaller positive integers, called a *partition* of n, where the k terms being added up are its *parts*.[1] If we write $p_k(n)$ for the number of ways of partitioning n into k parts, then, for example,

[1] Problem 15.8, regarding Bulgarian solitaire, was actually about such partitions.

$$\begin{aligned}
p_1(2) &= 1 & (2)\\
p_2(2) &= 1 & (1+1)\\
p_1(3) &= 1 & (3)\\
p_2(3) &= 1 & (1+2)\\
p_3(3) &= 1 & (1+1+1)\\
p_1(4) &= 1 & (4)\\
p_2(4) &= 2 & (1+3, 2+2)
\end{aligned}$$

$$p_3(4) = 1 \qquad (1+1+2)$$
$$p_4(4) = 1 \qquad (1+1+1+1).$$

There is no simple formula for $p_k(n)$. However, these numbers can be calculated recursively by breaking the problem down into smaller, similar problems until we reach base cases that are easy to solve.

Theorem 23.17.

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k),$$

with base cases

$$\begin{aligned} p_1(n) &= 1 \textit{ for all } n, \\ p_n(n) &= 1 \textit{ for all } n, \\ p_k(n) &= 0 \textit{ for } k > n. \end{aligned}$$

Proof. Let $P_{k,n}$ be the set of partitions of n into k parts. (For example, $P_{2,4} = \{1+3, 2+2\}$.) Let $A \subseteq P_{k,n}$ be the set of such partitions for which at least one part is equal to 1. Any member of A can be written as $\{1\} \cup A'$, where A' is a partition of $n-1$ into $k-1$ parts; that is, A' is a member of $P_{k-1,n-1}$. The function that removes one part of size 1 from such a partition is a bijection between A and $P_{k-1,n-1}$.

Let $B \subseteq P_{k,n}$ be the set of such partitions for which no part is equal to 1—that is, each of the k parts is greater than 1. For any member of B, we can subtract 1 from each part to get a partition of $n-k$ into k parts. The function that subtracts 1 from each part of such a partition is a bijection between B and $P_{k,n-k}$.

Then A and B are disjoint, and $P_{k,n}$ is their union. So

$$\begin{aligned} |P_{k,n}| &= |A| + |B| \\ &= |P_{k-1,n-1}| + |P_{k,n-k}|, \end{aligned}$$

or in other words,

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k).$$

As for the base cases, the only partition of n into 1 part is n itself, and the only partition of n into n parts is

$$n = \overbrace{1 + \ldots + 1}^{n \text{ times}}.$$

There are no partitions of n into more than n parts, since each part must be at least 1. ∎

For example, consider the partitions of 7 into 3 parts. There are three such partitions that contain one or more 1s, and each can be written as 1 plus a partition of 6 into 2 parts ($n-1=6$ and $k-1=2$):

$$7=5+1+1=(5+1)+1,$$

$$7=4+2+1=(4+2)+1,$$

$$7=3+3+1=(3+3)+1.$$

There is one such partition that does not contain any 1s:

$$7=3+2+2=(2+1+1)+3.$$

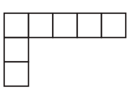
After subtracting 1 from each part, we are left with $2+1+1$, which is a partition of 4 into 3 parts ($n-k=4$ and $k=3$).

We can use Theorem 23.17 to check that the four partitions listed above are the only partitions of 7 into 3 parts:

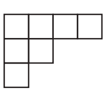
$$\begin{aligned} p_3(7) &= p_2(6)+p_3(4) \\ &= p_1(5)+p_2(4)+p_2(3)+p_3(1) \\ &= 1+p_1(3)+p_2(2)+p_1(2)+p_2(1)+0 \\ &= 1+1+1+1+0+0 \\ &= 4. \end{aligned}$$

This calculation is not too difficult when k and n are small, but with each step, each term splits into two, unless it is a base case. For larger values of k and n, the arithmetic becomes cumbersome—and may even call for redundant work. Problem 23.12 explores more efficient strategies for carrying out such calculations.

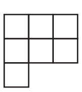
A lovely symmetry exists between partitioning an integer into k parts and partitioning that integer in such a way that the largest part is k. Remarkably, the number of partitions of these two kinds is the same.



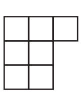
$7 = 5 + 1 + 1$



$7 = 4 + 2 + 1$



$7 = 3 + 3 + 1$



$7 = 3 + 2 + 2$

Figure 23.3. Young diagrams representing each of the 4 partitions of 7 into 3 parts.

Theorem 23.18. *The number of partitions of n into k parts is equal to the number of partitions of n for which the largest part is equal to k.*

Proof. Consider a diagram like those in Figure 23.3, called *Young diagrams*, consisting of n boxes arranged in rows, where the rows are arranged in nonincreasing order from top to bottom, and each row is left-justified.

Every partition of n into k parts corresponds to exactly one Young diagram of n boxes in k rows, and vice versa. To map a partition to a diagram, order the parts from largest to smallest, then construct a diagram for which the number of boxes in the i^{th} row is equal to the i^{th} part. To map from diagram to partition, include one part equal to the length of each row.

But every partition of n for which the largest part is equal to k also corresponds to exactly one such diagram, with n boxes and k rows, and vice versa. To map a partition to a diagram, order the parts from largest to smallest, and then construct a diagram for which the number of boxes in the i^{th} *column* is equal to the i^{th} part, with the columns top-justified. This results in the desired diagram of n boxes, with its rows in nonincreasing order from top to bottom, and with k rows (since the leftmost column represents the largest part, which is k). To map from diagram to partition, include one part equal to the height of each column.

So the number of Young diagrams containing n boxes in k rows counts both the number of partitions of n into k parts and the number of partitions of n for which the largest part is equal to k. Therefore, these counts are equal. ■

❈

This chapter has covered several variations on the problem of selecting subgroups from a collection of elements. In the case of distinct elements, we specified the size of the subgroup (k) and the size of the original collection (n), and the problems varied mainly in two ways: either the outcome is ordered, or not; and either replacement is allowed, or not. Figure 23.4 shows the four cases, with $n = 5$ and $k = 3$. The selection is ordered in the first two cases and unordered in the last two. Replacement is made in the first and third cases, so some of the chosen marbles can be the same color, but not in the second and fourth cases, so the colors of the chosen marbles must be different.

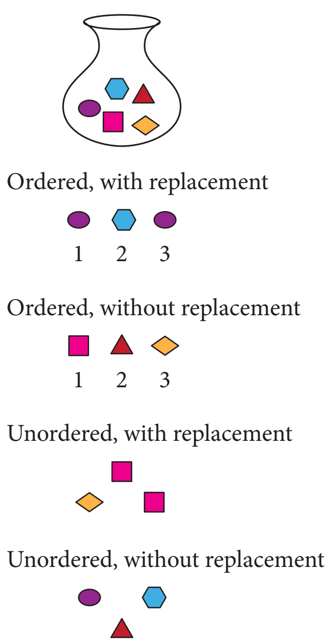


Figure 23.4. Various kinds of selections from a jar containing 5 different marbles.

For some of these variations it made sense to also analyze the case where multiple subgroups are chosen. In the case of identical elements, we specified the number of subgroups (k) and the size of the original collection (n), and the problems varied according to whether the subgroups are ordered or not. We have consistently used the metaphor of picking marbles from a jar, but the same techniques are applicable to a wide range of concrete situations:

- Distinct elements, ordered, with replacement—though we didn't use this description at the time, this scenario is discussed in Chapter 22: how many 10-character passwords are there, using an alphabet of 26 characters?
- Permutations—distinct elements, ordered, without replacement: In a tournament with 8 teams, how many different rankings of the teams are possible (on the assumption that ties cannot occur)? How many possibilities are there for just first, second, and third place? (The first question is about permutations of the entire set, and the second question about subset permutations.)

- Combinations—distinct elements, unordered, without replacement: if 10 students run for student council and 3 can be elected, how many outcomes are possible?
- Combinations with replacement—distinct elements, unordered, with replacement: how many different orders of 12 pizzas can be purchased from a shop that sells 3 types of pizza?
- Identical elements, ordered—simply another perspective on distinct elements, unordered, with replacement: if a shop has 12 identical plain pizzas and each pizza must be topped with exactly 1 of 3 possible toppings, how many ways are there to choose how many pizzas receive each topping?
- Partitions—identical elements, unordered: How many ways are there to write 25 as a sum of 9 positive integers? How many ways are there to divide a $1000 prize among first-, second-, and third-place winners, with first place getting at least as much as second place and second place at least as much as third place?

Chapter Summary

- The number of ways of ordering k elements chosen from a set of n elements with *replacement*, or from n types of elements, is n^k.
- The number of ways of ordering k elements chosen from a set of n elements without replacement is ${}_nP_k = \frac{n!}{(n-k)!}$. Such an ordering is called a *permutation*.
- The number of ways of choosing k elements from a set of n elements without replacement is ${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$. Such a selection is called a *combination*.
- A *counting argument* demonstrates that two quantities are the same by showing that they are different ways of counting the same set.
- Suppose a set of size n is partitioned into m disjoint subsets of sizes $k_1, k_2, \ldots, k_m$. If the order of the subsets matters, the number of ways to do so is

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}.$$

  If the order of the subsets does not matter, divide the above by the number of ways of ordering the subsets of each size. If each size k occurs with multiplicity $\mu(k)$, this is

$$\frac{n!}{\prod_{k \in K} \left(\mu(k)!(k!)^{\mu(k)}\right)}.$$

- The number of ways of choosing k elements from n elements with replacement, or from n types of elements, is equal to $\binom{k+n-1}{k}$. Such a selection is called a *combination with replacement*.The number of such selections is also called *n multichoose k*.

- A *stars and bars diagram* suggests a correspondence between a combination with replacement of k elements chosen from n types of elements, and a combination without replacement of k positions selected as stars (or $n-1$ as bars) out of $k+n-1$ positions.
- The number of ways of grouping n identical elements into k distinguishable groups is $\binom{n+k-1}{n}$, or if each group must contain at least one element, $\binom{n-1}{n-k}$.
- A grouping of n identical elements into k indistinguishable groups is called a *partition* of n into k *parts*. The number of such partitions can be defined recursively as $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$.
- The value $p_k(n)$ also counts the number of partitions of n with largest part equal to k.

Problems

23.1. Suppose a jar contains r red balls and b blue balls, each with a unique identifier on it. How many ways are there to choose a set of two balls of the same color? Of different colors? Show that the sum of these two numbers is the number of ways of choosing two balls from the total, ignoring color.

23.2. Consider a distributed computer network, comprising n computers.

(a) If one computer is designated as the central computer, and every other computer is connected to the central computer, how many connections are needed?

(b) If instead, each computer is connected to every other computer, how many connections are needed?

23.3. What proportion of all 5-character strings, composed from the 26 letters of the English alphabet, use 5 distinct letters?

23.4. Suppose a computer program takes a sequence of letters without spaces (for example, a URL), and finds where the spaces could go, by testing all possible ways of breaking the sequence into segments and then checking the segments against a dictionary to determine whether each is a word.

(a) Given a sequence that is 20 characters long, how many ways are there to break the sequence into 3 segments?

(b) Find a formula that describes the number of ways to break a sequence of n characters into any number of segments. *Hint:* Write this as a summation over the possible values for the number of segments, ranging from 1 to n.

23.5. How many different undirected graphs can be made from a set of n labeled vertices? (Consider two graphs to be different even if they are isomorphic, but have different vertex labels.)

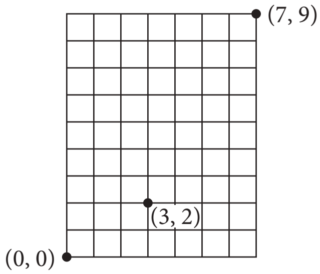


Figure 23.5. Paths within a grid.

23.6. Consider a grid that is 7 units wide and 9 units tall Figure 23.5). A *monotonic path* is one that begins at point $(0, 0)$ (the bottom left corner) and traverses to $(7, 9)$ (the top right corner), using only upward and rightward moves.

(a) How many different paths are possible?
(b) How many such paths go through the point $(3, 2)$?

23.7. In this problem we'll explore the connection between permutations of multisets, and combinations in which a set is split into several subsets.

(a) How many permutations are there of the letters ABBCCC?
(b) Suppose there are 6 marbles, labeled with the integers 1 through 6; and 3 boxes, labeled with the letters A, B, and C. In how many ways is it possible to arrange the marbles with 1 marble in box A, 2 marbles in box B, and 3 marbles in box C?
(c) Describe a bijection between the permutations of ABBCCC from part (a) and the arrangements of the marbles 1 through 6 in boxes A, B, and C from part (b). *Hint:* See Figure 23.6.

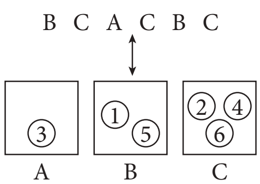


Figure 23.6. We can create a bijection between permutations of the letters and combinations of the marbles. The above image suggests a natural way to do so.

23.8. Use counting arguments to prove the following:

(a)

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(b)

$$\sum_{k=1}^{n} \binom{n}{k} \cdot k = n2^{n-1}.$$

Hint: Consider the number of ways to choose a committee of at least 1 person from a group of size n, with 1 committee member designated as its leader.

23.9. In the word game Scrabble, players end their turns by picking letter tiles to replace the ones they have just used. Suppose a player picks 4 letter tiles from the 7 tiles that remain: A, A, B, C, D, E, and F.

(a) How many different selections of letters are possible?
(b) How many different "words" (sequences of letters, not necessarily valid English words) can be made from 4 of these 7 letters?

23.10. Suppose that 27 students get together to play soccer.

(a) There are three different fields for the students to practice on. How many different ways are there to assign the 27 players to the 3 fields in teams of 9?
(b) How many ways are there to assign the 27 players to 3 teams of 9, without regard for which team is on which field?
(c) How many ways are there to assign the 27 players to 3 teams of 9, and for each team to choose 1 of its players as captain?

(d) One of the teams plays 10 games against teams from other schools, ending the season with a 7-3 record. How many different sequences of wins and losses could have led to this outcome?

23.11. Consider a standard deck of playing cards (as described on page 242).

(a) How many hands (sets) of 5 cards can be dealt from such a deck (which contains 52 distinct cards)?

(b) In poker, such a 5-card hand is called a *full house* if it contains two cards of one value and three of another (for example, two 5s and three 9s). How many 5-card hands are a full house?

(c) A much more common poker hand is *two pair*: two cards of one value, two cards of another value, and one card of a third value (for example two 5s, two 9s, and one K). How many 5-card hands are a two pair?

23.12. On page 254 we noted that calculating a single value $p_k(n)$ by the recursive definition may require many steps, since each term splits into two terms that must also be calculated recursively, and may call for redundant calculations.[2]

[2] Problem 23.12 describes two forms of *dynamic programming*, an algorithmic strategy that is useful for problems that can be defined in terms of overlapping subproblems. The first method, which proceeds from the bottom up and computes all values smaller than the target, is called *tabulation*. The second method, which proceeds from the top down and computes only the necessary smaller values, is called *memoization*.

(a) If we want to find not just one $p_k(n)$ but all $p_k(n)$ up to some maximum values of k and n, there is a more efficient way, which requires just a single addition to calculate each one. This is achieved by starting with the base cases and then working up from smaller cases to larger ones, tabulating the results along the way, such that the values $p_{k-1}(n-1)$ and $p_k(n-k)$ are already present in the table when we need to calculate $p_k(n)$.

Complete the following table with the values of $p_k(n)$ for $1 \leq k, n \leq 10$, proceeding from top to bottom and within each row from left to right. The base cases are already filled in.

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	0	1								
3	0	0	1							
4	0	0	0	1						
5	0	0	0	0	1					
6	0	0	0	0	0	1				
7	0	0	0	0	0	0	1			
8	0	0	0	0	0	0	0	1		
9	0	0	0	0	0	0	0	0	1	
10	0	0	0	0	0	0	0	0	0	1

(b) To find just a single value $p_k(n)$, it is still useful to record intermediate results in a table, though it is not necessary to fill in every cell. Find $p_5(10)$ as follows: Start by circling its corresponding cell. Then circle the two cells whose values are used to compute $p_5(10)$. For each newly circled cell, circle the two cells that it depends on, and repeat until reaching the base cases. Finally, proceed from top to bottom and left to right, filling in only the circled cells, until reaching $p_5(10)$.

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	0	1								
3	0	0	1							
4	0	0	0	1						
5	0	0	0	0	1					

23.13. How many partitions of n have all their parts equal to 1 or 2?

23.14. Three children are given 10 identical marbles, which they must divide among themselves.

(a) How many ways can the 10 marbles be divided among the 3 children?

(b) How many ways can the 10 marbles be divided among the 3 children, if each child must receive at least 1 marble?

(c) How many ways can the 10 marbles be divided, if each child must receive at least 1 marble, and the oldest child takes the largest share (or one of the two largest, if the two largest are equal), the second-oldest child takes the next-largest share (or one of the two next-largest, if the two are equal), and the youngest child takes the last remaining share? *Hint:* Use one of the strategies described in Problem 23.12.

(d) How many ways can the 10 marbles be divided, if a child may receive no marbles, and the oldest child takes the largest share (or one of the two largest, if the two largest are equal), the second-oldest child takes the next-largest share (or one of the two next-largest, if the two are equal), and the youngest child takes the last remaining share? *Hint:* View this as a division of the 10 pieces into 3, 2, or 1 piles, and use one of the strategies described in Problem 23.12.

23.15. Are there more ways to split 12 people up into 4 groups of 3 each, or into 3 groups of 4 each?