CHAPTE

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WHY ABSTRACT ALGEBRA'

Vhen we open a textbook of abstract algebra for the first time and peruse the table of contents, we ar truck by the unfamiliarity of almost every topic we see listed. Algebra is a subject we know well, bu ere it looks surprisingly different. What are these differences, and how fundamental are they?

First, there is a major difference in emphasis. In elementary algebra we learned the basic symbolisı nd methodology of algebra; we came to see how problems of the real world can be reduced to sets c quations and how these equations can be solved to yield numerical answers. This technique fc anslating complicated problems into symbols is the basis for all further work in mathematics and th xact sciences, and is one of the triumphs of the human mind. However, algebra is not only a technique, s also a branch of learning, a *discipline*, like calculus or physics or chemistry. It is a coherent and unifie ody of knowledge which may be studied systematically, starting from first principles and building up. S e first difference between the elementary and the more advanced course in algebra is that, wherea arlier we concentrated on technique, we will now develop that branch of mathematics called algebra i systematic way. Ideas and general principles will take precedence over problem solving. (By the way is does not mean that modern algebra has no applications—quite the opposite is true, as we will se oon.)

Algebra at the more advanced level is often described as *modern* or *abstract* algebra. In fact, bot f these descriptions are partly misleading. Some of the great discoveries in the upper reaches of present ay algebra (for example, the so-called Galois theory) were known many years before the American Civi Var ; and the broad aims of algebra today were clearly stated by Leibniz in the seventeenth century. Thus modern" algebra is not so very modern, after all! To what extent is it *abstract*? Well, abstraction is al elative; one person's abstraction is another person's bread and butter. The abstract tendency i athematics is a little like the situation of changing moral codes, or changing tastes in music: What shock ne generation becomes the norm in the next. This has been true throughout the history of mathematics.

For example, 1000 years ago negative numbers were considered to be an outrageous idea. After al was said, numbers are for counting: we may have one orange, or two oranges, or no oranges at all; bu ow can we have minus an orange? The *logisticians*, or professional calculators, of those days use egative numbers as an aid in their computations; they considered these numbers to be a useful fiction, fc you believe in them then every linear equation $ax + b = 0$ has a solution (namely $x = -b/a$, provided a). Even the great Diophantus once described the solution of $4x + 6 = 2$ as an *absurd* number. The idea c system of numeration which included negative numbers was far too abstract for many of the learne eads of the tenth century!

The history of the complex numbers (numbers which involve $\sqrt{-1}$) is very much the same. For hundreds of years, mathematicians refused to accept them because they couldn't find concrete examples or applications. (They are now a basic tool of physics.)

Set theory was considered to be highly abstract a few years ago, and so were other commonplaces of today. Many of the abstractions of modern algebra are already being used by scientists, engineers, and computer specialists in their everyday work. They will soon be common fare, respectably "concrete," and by then there will be new "abstractions."

Later in this chapter we will take a closer look at the particular brand of abstraction used in algebra. We will consider how it came about and why it is useful.

Algebra has evolved considerably, especially during the past 100 years. Its growth has been closely linked with the development of other branches of mathematics, and it has been deeply influenced by philosophical ideas on the nature of mathematics and the role of logic. To help us understand the nature and spirit of modern algebra, we should take a brief look at its origins.

ORIGINS

The order in which subjects follow each other in our mathematical education tends to repeat the historical stages in the evolution of mathematics. In this scheme, elementary algebra corresponds to the great classical age of algebra, which spans about 300 years from the sixteenth through the eighteenth centuries. It was during these years that the art of solving equations became highly developed and modern symbolism was invented.

The word "algebra"—*al jebr* in Arabic—was first used by Mohammed of Kharizm, who taught mathematics in Baghdad during the ninth century. The word may be roughly translated as "reunion," and describes his method for collecting the terms of an equation in order to solve it. It is an amusing fact that the word "algebra" was first used in Europe in quite another context. In Spain barbers were called *algebristas*, or bonesetters (they *reunited* broken bones), because medieval barbers did bonesetting and bloodletting as a sideline to their usual business.

The origin of the word clearly reflects the actual context of algebra at that time, for it was mainly concerned with ways of solving equations. In fact, Omar Khayyam, who is best remembered for his brilliant verses on wine, song, love, and friendship which are collected in the *Rubaiyat*—but who was also a great mathematician—explicitly defined algebra as the *science of solving equations*.

Thus, as we enter upon the threshold of the classical age of algebra, its central theme is clearly identified as that of solving equations. Methods of solving the linear equation $ax + b = 0$ and the quadratic $ax^2 + bx + c = 0$ were well known even before the Greeks. But nobody had yet found a general solution for *cubic* equations

$$x^3 + ax^2 + bx = c$$

or *quartic* (fourth-degree) equations

$$x^4 + ax^3 + bx^2 + cx = d$$

This great accomplishment was the triumph of sixteenth century algebra.

The setting is Italy and the time is the Renaissance—an age of high adventure and brilliant achievement, when the wide world was reawakening after the long austerity of the Middle Ages. America had just been discovered, classical knowledge had been brought to light, and prosperity had returned t ie great cities of Europe. It was a heady age when nothing seemed impossible and even the old barrier f birth and rank could be overcome. Courageous individuals set out for great adventures in the fa orners of the earth, while others, now confident once again of the power of the human mind, were boldl xploring the limits of knowledge in the sciences and the arts. The ideal was to be bold and many-facetec ɔ "know something of everything, and everything of at least one thing." The great traders were patrons c ie arts, the finest minds in science were adepts at political intrigue and high finance. The study of algebr /as reborn in this lively milieu.

Those men who brought algebra to a high level of perfection at the beginning of its classical age—al /pical products of the Italian Renaissanee —were as colorful and extraordinary a lot as have eve ppeared in a chapter of history. Arrogant and unscrupulous, brilliant, flamboyant, swaggering, an emarkable, they lived their lives as they did their work: with style and panache, in brilliant dashes an ıspired leaps of the imagination.

The spirit of scholarship was not exactly as it is today. These men, instead of publishing thei iscoveries, kept them as well-guarded secrets to be used against each other in problem-solvin ompetitions. Such contests were a popular attraction: heavy bets were made on the rival parties, an ieir reputations (as well as a substantial purse) depended on the outcome.

One of the most remarkable of these men was Girolamo Cardan. Cardan was born in 1501 as th .legitimate son of a famous jurist of the city of Pavia. A man of passionate contrasts, he was destined t ecome famous as a physician, astrologer, and mathematician—and notorious as a compulsive gamble: coundrel, and heretic. After he graduated in medicine, his efforts to build up a medical practice were s nsuccessful that he and his wife were forced to seek refuge in the poorhouse. With the help of friends h ecame a lecturer in mathematics, and, after he cured the child of a senator from Milan, his medica areer also picked up. He was finally admitted to the college of physicians and soon became its rector. A rilliant doctor, he gave the first clinical description of typhus fever, and as his fame spread he becam ie personal physician of many of the high and mighty of his day.

Cardan's early interest in mathematics was not without a practical side. As an inveterate gambler h /as fascinated by what he recognized to be the laws of chance. He wrote a gamblers' manual entitle *ook on Games of Chance*, which presents the first systematic computations of probabilities. He als eeded mathematics as a tool in casting horoscopes, for his fame as an astrologer was great and hi redictions were highly regarded and sought after. His most important achievement was the publication c book called *Ars Magna* (*The Great Art*), in which he presented systematically all the algebrai nowledge of his time. However, as already stated, much of this knowledge was the personal secret of it ractitioners, and had to be wheedled out of them by cunning and deceit. The most importar ccomplishment of the day, the general solution of the cubic equation which had been discovered b 'artaglia, was obtained in that fashion.

Tartaglia's life was as turbulent as any in those days. Born with the name of Niccolo Fontana abou 500, he was present at the occupation of Brescia by the French in 1512. He and his father fled with man thers into a cathedral for sanctuary, but in the heat of battle the soldiers massacred the hapless citizen ven in that holy place. The father was killed, and the boy, with a split skull and a deep saber cut acros is jaws and palate, was left for dead. At night his mother stole into the cathedral and managed to carr im off; miraculously he survived. The horror of what he had witnessed caused him to stammer for th est of his life, earning him the nickname *Tartaglia*, "the stammerer," which he eventually adopted.

Tartaglia received no formal schooling, for that was a privilege of rank and wealth. However, h ıught himself mathematics and became one of the most gifted mathematicians of his day. He translate 'uclid and Archimedes and may be said to have originated the science of ballistics, for he wrote eatise on gunnery which was a pioneering effort on the laws of falling bodies.

In 1535 Tartaglia found a way of solving any cubic equation of the form $x^3 + ax^2 = b$ (that is, withou n x term). When be announced his accomplishment (without giving any details, of course), he wa hallenged to an algebra contest by a certain Antonio Fior, a pupil of the celebrated professor c nathematics Scipio del Ferro. Scipio had already found a method for solving any cubic equation of th orm $x^3 + ax = b$ (that is, without an x^2 term), and had confided his secret to his pupil Fior. It was agree hat each contestant was to draw up 30 problems and hand the list to his opponent. Whoever solved th reater number of problems would receive a sum of money deposited with a lawyer. A few days befor he contest, Tartaglia found a way of extending his method so as to solve *any* cubic equation. In less than ours he solved all his opponent's problems, while his opponent failed to solve even one of thos roposed by Tartaglia.

For some time Tartaglia kept his method for solving cubic equations to himself, but in the end h uccumbed to Cardan's accomplished powers of persuasion. Influenced by Cardan's promise to help hi ecome artillery adviser to the Spanish army, he revealed the details of his method to Cardan under th romise of strict secrecy. A few years later, to Tartaglia's unbelieving amazement and indignation, Carda ublished Tartaglia's method in his book *Ars Magna*. Even though he gave Tartaglia full credit as th riginator of the method, there can be no doubt that he broke his solemn promise. A bitter dispute aros etween the mathematicians, from which Tartaglia was perhaps lucky to escape alive. He lost his positio s public lecturer at Brescia, and lived out his remaining years in obscurity.

The next great step in the progress of algebra was made by another member of the same circle. It wa udovico Ferrari who discovered the general method for solving quartic equations—equations of th orm

$$x^4 + ax^3 + bx^2 + cx = d$$

errari was Cardan's personal servant. As a boy in Cardan's service he learned Latin, Greek, an nathematics. He won fame after defeating Tartaglia in a contest in 1548, and received an appointment a upervisor of tax assessments in Mantua. This position brought him wealth and influence, but he was nc ble to dominate his own violent disposition. He quarreled with the regent of Mantua, lost his positio nd died at the age of 43. Tradition has it that he was poisoned by his sister.

As for Cardan, after a long career of brilliant and unscrupulous achievement, his luck finall bandoned him. Cardan's son poisoned his unfaithful wife and was executed in 1560. Ten years late Cardan was arrested for heresy because he published a horoscope of Christ's life. He spent severa nonths in jail and was released after renouncing his heresy privately, but lost his university position an he right to publish books. He was left with a small pension which had been granted to him, for som naccountable reason, by the Pope.

As this colorful time draws to a close, algebra emerges as a major branch of mathematics. It becam lear that methods can be found to solve many different types of equations. In particular, formulas ha een discovered which yielded the roots of all cubic and quartic equations. Now the challenge wa learly out to take the next step, namely, to find a formula for the roots of equations of degree 5 or highe in other words, equations with an x^5 term, or an x^6 term, or higher). During the next 200 years, there wa ardly a mathematician of distinction who did not try to solve this problem, but none succeeded. Progres vas made in new parts of algebra, and algebra was linked to geometry with the invention of analyti eometry. But the problem of solving equations of degree higher than 4 remained unsettled. It was, in th xpression of Lagrange, "a challenge to the human mind."

It was therefore a great surprise to all mathematicians when in 1824 the work of a young Norwegia rodigy named Niels Abel came to light. In his work, Abel showed that *there does not exist any formul*

ın the conventional sense we have in mind) for the roots of an algebraic equation whose degree is 5 c
reater. This sensational discovery brings to a close what is called the classical age of algebra
'hroughout this age algebra was conceived essentially as the science of solving equations, and now th
uter limits of this quest had apparently been reached. In the years ahead, algebra was to strike out in nev
irections.

'HE MODERN AGE

ıbout the time Niels Abel made his remarkable discovery, several mathematicians, workin
ıdependently in different parts of Europe, began raising questions about algebra which had never bee
onsidered before. Their researches in different branches of mathematics had led them to investigat
algebras" of a very unconventional kind—and in connection with these algebras they had to find answer
› questions which had nothing to do with solving equations. Their work had important applications, an
/as soon to compel mathematicians to greatly enlarge their conception of what algebra is about.

The new varieties of algebra arose as a perfectly natural development in connection with th
pplication of mathematics to practical problems. This is certainly true for the example we are about t
ɔok at first.

'he Algebra of Matrices

ı *matrix* is a rectangular array of numbers such as

$$\begin{pmatrix} 2 & 11 & -3 \\ 9 & 0.5 & 4 \end{pmatrix}$$

uch arrays come up naturally in many situations, for example, in the solution of simultaneous linea
quations. The above matrix, for instance, is the *matrix of coefficients* of the pair of equations

$$2x + 11y - 3z = 0$$
$$9x + 0.5y + 4z = 0$$

ince the solution of this pair of equations depends only on the coefficients, we may solve it by workin
n the matrix of coefficients alone and ignoring everything else.

We may consider the entries of a matrix to be arranged in *rows* and *columns;* the above matrix ha
vo rows which are

$$(2\ 11\ -3) \quad \text{and} \quad (9\ 0.5\ 4)$$

nd three columns which are

$$\begin{pmatrix} 2 \\ 9 \end{pmatrix} \begin{pmatrix} 11 \\ 0.5 \end{pmatrix} \text{ and } \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

: is a 2 × 3 matrix.

To simplify our discussion, we will consider only 2 × 2 matrices in the remainder of this section.

Matrices are added by adding corresponding entries:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}=\begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$$

he matrix

$$\mathbf{O}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

s called the *zero matrix* and behaves, under addition, like the number zero.

The multiplication of matrices is a little more difficult. First, let us recall that the *dot product* of tw ectors (a, b) and (a',b') is

$$(a,b)\cdot(a',b')=aa'+bb'$$

hat is, we multiply corresponding components and add. Now, suppose we want to multiply two matrice and **B**; we obtain the product **AB** as follows:

The entry in the first row and first column of **AB**, that is, in *this* position

$$\begin{pmatrix} x & \\ & \end{pmatrix}$$

s equal to the dot product of the first row of **A** by the first column of **B**. The entry in the first row an *econd* column of **AB**, in other words, *this* position

$$\begin{pmatrix} & x \\ & \end{pmatrix}$$

s equal to the dot product of the first row of **A** by the *second* column of **B**. And so on. For example,

$$\begin{pmatrix} \boxed{1 \quad 2} \\ 3 \quad 0 \end{pmatrix}\begin{pmatrix} \boxed{1} & 1 \\ \boxed{2} & 0 \end{pmatrix}=\begin{pmatrix} 5 & \\ & \end{pmatrix} \qquad \begin{pmatrix} \boxed{1 \quad 2} \\ 3 \quad 0 \end{pmatrix}\begin{pmatrix} 1 & \boxed{1} \\ 2 & \boxed{0} \end{pmatrix}=\begin{pmatrix} & 1 \\ & \end{pmatrix}$$

$$\begin{pmatrix} 1 \quad 2 \\ \boxed{3 \quad 0} \end{pmatrix}\begin{pmatrix} \boxed{1} & 1 \\ \boxed{2} & 0 \end{pmatrix}=\begin{pmatrix} & \\ 3 & \end{pmatrix} \qquad \begin{pmatrix} 1 \quad 2 \\ \boxed{3 \quad 0} \end{pmatrix}\begin{pmatrix} 1 & \boxed{1} \\ 2 & \boxed{0} \end{pmatrix}=\begin{pmatrix} & \\ & 3 \end{pmatrix}$$

o finally,

$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}=\begin{pmatrix} 5 & 1 \\ 3 & 3 \end{pmatrix}$$

The rules of algebra for matrices are very different from the rules of "conventional" algebra. Fc nstance, the commutative law of multplica-tion, **AB** = **BA**, is not true. Here is a simple example:

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{\mathbf{A}}\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{B}}=\underbrace{\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}}_{\mathbf{AB}}\neq\underbrace{\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}}_{\mathbf{BA}}=\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{B}}\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{\mathbf{A}}$$

If A is a real number and $A^2=0$, then necessarily $A=0$; but this is not true of matrices. For example,

$$\underbrace{\begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}}_{\mathbf{A}}\underbrace{\begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}}_{\mathbf{A}}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

at is, $A^2 = 0$ although $A \neq 0$.

In the algebra of numbers, if $AB = AC$ where $A \neq 0$, we may cancel A and conclude that $B = C$. I atrix algebra we cannot. For example,

$$\underbrace{\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}}_{\mathbf{A}}\underbrace{\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}}_{\mathbf{B}}=\begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}=\underbrace{\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}}_{\mathbf{A}}\underbrace{\begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}}_{\mathbf{C}}$$

at is, $\mathbf{AB} = \mathbf{AC}$, $\mathbf{A} \neq \mathbf{0}$, yet $\mathbf{B} \neq \mathbf{C}$.

The *identity* matrix

$$\mathbf{I}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

orresponds in matrix multiplication to the number 1; for we have $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for every 2×2 matrix A A is a number and $A^2 = 1$, we conclude that $A = \pm 1$ Matrices do not obey this rule. For example,

$$\underbrace{\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix}}_{\mathbf{A}}\underbrace{\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix}}_{\mathbf{A}}=\underbrace{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}_{\mathbf{I}}$$

at is, $\mathbf{A}^2 = \mathbf{I}$, and yet $\mathbf{A}$ is neither $\mathbf{I}$ nor $-\mathbf{I}$.

No more will be said about the algebra of matrices at this point, except that we must be aware, onc gain, that it is a new game whose rules are quite different from those we apply in conventional algebra.

oolean Algebra

n even more bizarre kind of algebra was developed in the mid-nineteenth century by an Englishma amed George Boole. This algebra—subsequently named boolean algebra after its inventor—has yriad of applications today. It is formally the same as the algebra of sets.

If S is a set, we may consider *union* and *intersection* to be operations on the subsets of 5. Let u gree provisionally to write

$$A + B \quad \text{for} \quad A \cup B$$

nd

$$A \cdot B \quad \text{for} \quad A \cap B$$

This convention is not unusual.) Then,

$$A + B = B + A \qquad A \cdot B = B \cdot A$$

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

$$A + \varnothing = A \qquad A \cdot \varnothing = \varnothing$$

nd so on.

These identities are analogous to the ones we use in elementary algebra. But the following identitie re also true, and they have no counterpart in conventional algebra:

$$A + (B \cdot C) = (A + B) \cdot (A + C)$$

$$A + A = A \qquad A \cdot A = A$$

$$(A + B) \cdot A = A \qquad (A \cdot B) + A = A$$

nd so on.

This unusual algebra has become a familiar tool for people who work with electrical network: omputer systems, codes, and so on. It is as different from the algebra of numbers as it is from the algebr f matrices.

Other exotic algebras arose in a variety of contexts, often in connection with scientific problem: here were "complex" and "hypercomplex" algebras, algebras of vectors and tensors, and many other: oday it is estimated that over 200 different kinds of algebraic systems have been studied, each of whic rose in connection with some application or specific need.

lgebraic Structures

s legions of new algebras began to occupy the attention of mathematicians, the awareness grew tha lgebra can no longer be conceived merely as the *science of solving equations*. It had to be viewed muc ore broadly as a branch of mathematics capable of revealing general principles which apply equally t *ll known and all possible algebras*.

What is it that all algebras have in common? What trait do they share which lets us refer to all c em as "algebras"? In the most general sense, every algebra consists of a *set* (a set of numbers, a set c atrices, a set of switching components, or any other kind of set) and certain *operations* on that set. A peration is simply a way of combining any two members of a set to produce a unique third member of th ame set.

Thus, we are led to the modern notion of algebraic structure. An *algebraic structure* is understood t e an arbitrary set, with one or more operations defined on it. And algebra, then, is defined to be *th tudy of algebraic structures*.

It is important that we be awakened to the full generality of the notion of algebraic structure. We mu: ake an effort to discard all our preconceived notions of what an algebra is, and look at this new notio f algebraic structure in its naked simplicity. *Any* set, with a rule (or rules) for combining its elements, i lready an algebraic structure. There does not need to be any connection with known mathematics. Fc xample, consider the set of all colors (pure colors as well as color combinations), and the operation c ixing any two colors to produce a new color. This may be conceived as an algebraic structure. It obey ertain rules, such as the commutative law (mixing red and blue is the same as mixing blue and red). In imilar vein, consider the set of all musical sounds with the operation of combining any two sounds t roduce a new (harmonious or disharmonious) combination.

As another example, imagine that the guests at a family reunion have made up a rule for picking th *losest common relative* of any two persons present at the reunion (and suppose that, for any two peopl t the reunion, their closest common relative is also present at the reunion). This too, is an algebrai tructure: we have a set (namely the set of persons at the reunion) and an operation on that set (namely th closest common relative" operation).

As the general notion of algebraic structure became more familiar (it was not fully accepted until th arly part of the twentieth century), it was bound to have a profound influence on what mathematician erceived algebra to *be*. In the end it became clear that the purpose of algebra is to study algebrai tructures, and nothing less than that. Ideally it should aim to be a general science of algebraic structure vhose results should have applications to particular cases, thereby making contact with the older parts c lgebra. Before we take a closer look at this program, we must briefly examine another aspect of moder nathematics, namely, the increasing use of the axiomatic method.

XIOMS

'he axiomatic method is beyond doubt the most remarkable invention of antiquity, and in a sense the mos uzzling. It appeared suddenly in Greek geometry in a highly developed form—already sophisticatec legant, and thoroughly modern in style. Nothing seems to have foreshadowed it and it was unknown t ncient mathematicians before the Greeks. It appears for the first time in the light of history in the grea extbook of early geometry, Euclid's *Elements*. Its origins—the first tentative experiments in forma eductive reasoning which must have preceded it—remain steeped in mystery.

Euclid's *Elements* embodies the axiomatic method in its purest form. This amazing book contain 65 geometric propositions, some fairly simple, some of astounding complexity. What is reall emarkable, though, is that the 465 propositions, forming the largest body of scientific knowledge in th ncient world, are derived logically from only 10 premises which would pass as trivial observations c ommon sense. Typical of the premises are the following:

Things equal to the same thing are equal to each other.
The whole is greater than the part.
A straight line can be drawn through any two points.
All right angles are equal.

o great was the impression made by Euclid's *Elements* on following generations that it became th nodel of correct mathematical form and remains so to this day.

It would be wrong to believe there was no notion of demonstrative mathematics before the time c uclid. There is evidence that the earliest geometers of the ancient Middle East used reasoning t iscover geometric principles. They found proofs and must have hit upon many of the same proofs we fin n Euclid. The difference is that Egyptian and Babylonian mathematicians considered logica emonstration to be an auxiliary process, like the preliminary sketch made by artists—a private menta rocess which guided them to a result but did not deserve to be recorded. Such an attitude shows littl nderstanding of the true nature of geometry and does not contain the seeds of the axiomatic method.

It is also known today that many—maybe most—of the geometric theorems in Euclid's *Element* ame from more ancient times, and were probably borrowed by Euclid from Egyptian and Babylonia ources. However, this does not detract from the greatness of his work. Important as are the contents c he *Elements*, what has proved far more important for posterity is the formal manner in which Eucli resented these contents. The heart of the matter was the way he *organized* geometric facts—arrange nem into a logical sequence where each theorem builds on preceding theorems and then forms the logica asis for other theorems.

(We must carefully note that the axiomatic method is not a way of discovering facts but of organizin hem. New facts in mathematics are found, as often as not, by inspired guesses or experienced intuitior 'o be accepted, however, they should be supported by proof in an axiomatic system.)

Euclid's *Elements* has stood throughout the ages as the model of organized, rational thought carrie › its ultimate perfection. Mathematicians and philosophers in every generation have tried to imitate it ucid perfection and flawless simplicity. Descartes and Leibniz dreamed of organizing all huma nowledge into an axiomatic system, and Spinoza created a deductive system of ethics patterned afte ;uclid's geometry. While many of these dreams have proved to be impractical, the method popularized b ;uclid has become the prototype of modern mathematical form. Since the middle of the nineteenth century e axiomatic method has been accepted as the only correct way of organizing mathematical knowledge.

To perceive why the axiomatic method is truly central to mathematics, we must keep one thing i nind: mathematics by its nature is essentially *abstract*. For example, in geometry straight lines are nc tretched threads, but a concept obtained by disregarding all the properties of stretched threads except tha f extending in one direction. Similarly, the concept of a geometric figure is the result of idealizing fror ll the properties of actual objects and retaining only their spatial relationships. Now, since the objects c nathematics are *abstractions*, it stands to reason that we must acquire knowledge about them by logic an ot by observation or experiment (for how can one experiment with an abstract thought?).

This remark applies very aptly to modern algebra. The notion of algebraic structure is obtained b dealizing from all particular, concrete systems of algebra. We choose to ignore the properties of th ctual objects in a system of algebra (they may be numbers, or matrices, or whatever—we disregard wha ey *are*), and we turn our attention simply to the way they combine under the given operations. In fac ıst as we disregard what the objects in a system *are*, we also disregard what the operations *do* to then Ve retain only the equations and inequalities which hold in the system, for only these are relevant t lgebra. Everything else may be discarded. Finally, equations and inequalities may be deduced from on nother logically, just as spatial relationships are deduced from each other in geometry.

'HE AXIOMATICS OF ALGEBRA

;et us remember that in the mid-nineteenth century, when eccentric new algebras seemed to show up a very turn in mathematical research, it was finally understood that sacrosanct laws such as the identitie $b = ba$ and $a(bc) = (ab)c$ are not inviolable—for there are algebras in which they do not hold. B arying or deleting some of these identities, or by replacing them by new ones, an enormous variety c ew systems can be created.

Most importantly, mathematicians slowly discovered that all the algebraic laws which hold in an ystem can be derived from a few simple, basic ones. This is a genuinely remarkable fact, for it parallel e discovery made by Euclid that a few very simple geometric postulates are sufficient to prove all th eorems of geometry. As it turns out, then, we have the same phenomenon in algebra: a few simpl lgebraic equations offer themselves naturally as axioms, and from them all other facts may be proved.

These basic algebraic laws are familiar to most high school students today. We list them here fc eference. We assume that A is any set and there is an operation on A which we designate with the symbc

$$a * b = b * a \qquad (1$$

f Equation (1) is true for any two elements a and b in A, we say that the operation * is *commutative*. Wha means, of course, is that the value of $a * b$ (or $b * a$) is independent of the order in which a and b ar aken.

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad (2$$

f Equation (2) is true for any three elements a, b, and c in A, we say the operation * is *associative*. Remember that an operation is a rule for combining any *two* elements, so if we want to combine *thre* lements, we can do so in different ways. If we want to combine a, b, and c *without changing thei rder*, we may either combine a with the result of combining b and c, which produces $a *(b * c)$; or w nay first combine a with b, and then combine the result with c, producing $(a * b)* c$. The associative law sserts that these two possible ways of combining three elements (without changing their order) yield th ame result.

There exists an element e in A such that

$$e * a = a \quad \text{and} \quad a * e = a \quad \text{for every } a \text{ in A} \qquad (3$$

f such an element e exists in A, we call it an *identity element* for the operation *. An identity element i ometimes called a "neutral" element, for it may be combined with any element a without altering a. Fc xample, 0 is an identity element for addition, and 1 is an identity element for multiplication.

For every element a in A, there is an element a^{-1} ("a inverse") in A such that

$$a * a^{-1} = e \quad \text{and} \quad a^{-1} * a = e \qquad (4$$

f statement (4) is true in a system of algebra, we say that every element has an inverse with respect to th peration *. The meaning of the inverse should be clear: the combination of any element with its invers roduces the neutral element (one might roughly say that the inverse of a "neutralizes" a). For example, i ! is a set of numbers and the operation is addition, then the inverse of any number a is $(-a)$; if th peration is multiplication, the inverse of any $a \neq 0$ is $1/a$.

Let us assume now that the same set A has a second operation, symbolized by $\perp$, as well as th peration * :

$$a * (b \perp c) = (a * b) \perp (a * c) \qquad (5$$

f Equation (5) holds for any three elements a, b, and c in A, we say that * is *distributive* over $\perp$. If ther re two operations in a system, they must interact in some way; otherwise there would be no need t onsider them together. The distributive law is the most common way (but not the only possible one) fc wo operations to be related to one another.

There are other "basic" laws besides the five we have just seen, but these are the most commo nes. The most important algebraic systems have axioms chosen from among them. For example, when nathematician nowadays speaks of a *ring*, the mathematician is referring to a set A with two operations sually symbolized by + and ·, having the following axioms:

> *Addition is commutative and associative, it has a neutral element commonly symbolized by* 0, *and every element a has an inverse –a with respect to addition. Multiplication is associative, has a neutral element* 1, *and is distributive over addition.*

Aatrix algebra is a particular example of a ring, and all the laws of matrix algebra may be proved fror ne preceding axioms. However, there are many other examples of rings: rings of numbers, rings c unctions, rings of code "words," rings of switching components, and a great many more. Every algebrai aw which can be proved in a ring (from the preceding axioms) is true in every *example* of a ring. In othe vords, instead of proving the same formula repeatedly—once for numbers, once for matrices, once fc witching components, and so on—it is sufficient nowadays to prove only that the formula holds in rings nd then of necessity it will be true in all the hundreds of different concrete examples of rings.

By varying the possible choices of axioms, we can keep creating new axiomatic systems of algebr ndlessly. We may well ask: is it legitimate to study *any* axiomatic system, with *any* choice of axioms egardless of usefulness, relevance, or applicability? There are "radicals" in mathematics who claim th

eedom for mathematicians to study any system they wish, without the need to justify it. However, th
ractice in established mathematics is more conservative: particular axiomatic systems are investigate
n account of their relevance to new and traditional problems and other parts of mathematics, or becaus
ney correspond to particular applications.

In practice, how is a particular choice of algebraic axioms made? Very simply: when mathematician
ook at different parts of algebra and notice that a common pattern of proofs keeps recurring, an
ssentially the same assumptions need to be made each time, they find it natural to single out this choice c
ssumptions as the axioms for a new system. All the important new systems of algebra were created i
his fashion.

BSTRACTION REVISITED

nother important aspect of axiomatic mathematics is this: when we capture mathematical facts in a
xiomatic system, we never try to reproduce the facts in full, but only that side of them which is importan
r relevant in a particular context. This process of *selecting what is relevant* and disregarding everythin
lse is the very essence of abstraction.

This kind of abstraction is so natural to us as human beings that we practice it all the time withou
eing aware of doing so. Like the Bourgeois Gentleman in Molière's play who was amazed to learn tha
e spoke in prose, some of us may be surprised to discover how much we think in abstractions. Natur
resents us with a myriad of interwoven facts and sensations, and we are challenged at every instant t
ingle out those which are immediately relevant and discard the rest. In order to make our surrounding
omprehensible, we must continually pick out certain data and separate them from everything else.

For natural scientists, this process is the very core and essence of what they do. Nature is not mad
p of forces, velocities, and moments of inertia. Nature is a whole—nature simply *is*! The physicis
solates certain aspects of nature from the rest and finds the laws which govern these *abstractions*.

It is the same with mathematics. For example, the system of the integers (whole numbers), as know
y our intuition, is a complex reality with many facets. The mathematician separates these facets from on
nother and studies them individually. From one point of view the set of the integers, with addition an
nultiplication, forms a *ring* (that is, it satisfies the axioms stated previously). From another point of view
t is an ordered set, and satisfies special axioms of ordering. On a different level, the positive integer
orm the basis of "recursion theory," which singles out the particular way positive integers may b
onstructed, beginning with 1 and adding 1 each time.

It therefore happens that the traditional subdivision of mathematics into subject matters has bee
adically altered. No longer are the integers one subject, complex numbers another, matrices another, an
o on; instead, particular *aspects* of these systems are isolated, put in axiomatic form, and studie
bstractly without reference to any specific objects. The other side of the coin is that each aspect i
hared by many of the traditional systems: for example, algebraically the integers form a ring, and so d
he complex numbers, matrices, and many other kinds of objects.

There is nothing intrinsically new about this process of divorcing properties from the actual object
aving the properties; as we have seen, it is precisely what geometry has done for more than 2000 years
omehow, it took longer for this process to take hold in algebra.

The movement toward axiomatics and abstraction in modern algebra began about the 1830s and wa
ompleted 100 years later. The movement was tentative at first, not quite conscious of its aims, but i
ained momentum as it converged with similar trends in other parts of mathematics. The thinking of man
reat mathematicians played a decisive role, but none left a deeper or longer lasting impression than
ery young Frenchman by the name of Évariste Galois.

The story of Evariste Galois is probably the most fantastic and tragic in the history of mathematics. ensitive and prodigiously gifted young man, he was killed in a duel at the age of 20, ending a life whic ı its brief span had offered him nothing but tragedy and frustration. When he was only a youth his fathe ommited suicide, and Galois was left to fend for himself in the labyrinthine world of French universit fe and student politics. He was twice refused admittance to the Ecole Polytechnique, the mos restigious scientific establishment of its day, probably because his answers to the entrance examinatio vere too original and unorthodox. When he presented an early version of his important discoveries i lgebra to the great academician Cauchy, this gentleman did not read the young student's paper, but lost i ater, Galois gave his results to Fourier in the hope of winning the mathematics prize of the Academy c ciences. But Fourier died, and that paper, too, was lost. Another paper submitted to Poisson wa ventually returned because Poisson did not have the interest to read it through.

Galois finally gained admittance to the École Normale, another focal point of research i ıathematics, but he was soon expelled for writing an essay which attacked the king. He was jailed twic or political agitation in the student world of Paris. In the midst of such a turbulent life, it is hard t elieve that Galois found time to create his colossally original theories on algebra.

What Galois did was to tie in the problem of finding the roots of equations with new discoveries o roups of permutations. He explained exactly *which* equations of degree 5 or higher have solutions of th aditional kind—and which others do not. Along the way, he introduced some amazingly original an owerful concepts, which form the framework of much algebraic thinking to this day. Although Galois di ot work explicitly in axiomatic algebra (which was unknown in his day), the abstract notion of algebrai tructure is clearly prefigured in his work.

In 1832, when Galois was only 20 years old, he was challenged to a duel. What argument led to th hallenge is not clear: some say the issue was political, while others maintain the duel was fought over ckle lady's wavering love. The truth may never be known, but the turbulent, brilliant, and idealisti alois died of his wounds. Fortunately for mathematics, the night before the duel he wrote down his mai ıathematical results and entrusted them to a friend. This time, they weren't lost—but they were onl ublished 15 years after his death. The mathematical world was not ready for them before then!

Algebra today is organized axiomatically, and as such it is abstract. Mathematicians study algebrai tructures from a general point of view, compare different structures, and find relationships between then his abstraction and generalization might appear to be hopelessly impractical—but it is not! The genera pproach in algebra has produced powerful new methods for "algebraizing" different parts c ıathematics and science, formulating problems which could never have been formulated before, an nding entirely new kinds of solutions.

Such excursions into pure mathematical fancy have an odd way of running ahead of physical science roviding a theoretical framework to account for facts even before those facts are fully known. Thi attern is so characteristic that many mathematicians see themselves as pioneers in a world c *ossibilities* rather than facts. Mathematicians study *structure* independently of content, and their scienc s a voyage of exploration through all the kinds of structure and order which the human mind is capable c iscerning.