Assignment 2: Θ-method for Heat equation

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1 Introduction

The θ method is a semi-implicit forward time centered space scheme to solve the heat equation. The heat equation in one dimension is given as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

2 Θ Scheme

The scheme to solve the equation is as follows:

$$\partial_t^+ U_j^k = (\theta)\partial_x^+\partial_x^- U_j^{k+1} + (1-\theta)\partial_x^+\partial_x^- U_j^k$$

$$\left(\frac{U_j^{k+1} - U_j^k}{\Delta t}\right) = (\theta)\left(\frac{U_{j+1}^{k+1} - 2U_j^{k+1} + U_{j-1}^{k+1}}{\Delta x^2}\right) + (1-\theta)\left(\frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{\Delta x^2}\right)$$

Setting $\lambda_1 = \left(\frac{\Delta t}{\Delta x^2}\right)\theta$ and $\lambda_2 = \left(\frac{\Delta t}{\Delta x^2}\right)(\theta - 1)$, we can write it in a matrix-vector form,

$$\begin{bmatrix} 1+2\lambda_1 & -\lambda_1 & 0 & \dots & 0 \\ -\lambda_1 & 1+2\lambda_1 & -\lambda_1 & \ddots & 0 \\ 0 & -\lambda_1 & 1+2\lambda_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\lambda_1 & 1+2\lambda_1 \end{bmatrix} \begin{bmatrix} U_1^{k+1} \\ U_2^{k+1} \\ \vdots \\ \vdots \\ U_{J-1}^{k+1} \end{bmatrix} = \begin{bmatrix} 1+2\lambda_2 & -\lambda_2 & 0 & \dots & 0 \\ -\lambda_2 & 1+2\lambda_2 & -\lambda_2 & \ddots & 0 \\ 0 & -\lambda_2 & 1+2\lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\lambda_2 & 1+2\lambda_2 \end{bmatrix} \begin{bmatrix} U_1^k \\ U_2^k \\ \vdots \\ \vdots \\ U_{J-1}^k \end{bmatrix}$$

Taking the inverse, we can write it as a linear equation

$$U^{k+1} = \mathbf{A}(\mu, \theta)U^k$$

The eigenvalues of the matrix $\mathbf{A}$ dictate the stability of the scheme.

3 Stability Analysis using eigenvalues

It can be clearly seen in Figure1 that the method is stable for $\theta \geq 0.5$ and for $\theta < 0.5$ when $\mu < 0.5$. This is what we obtained from Von-Neumann Stability analysis of the method.

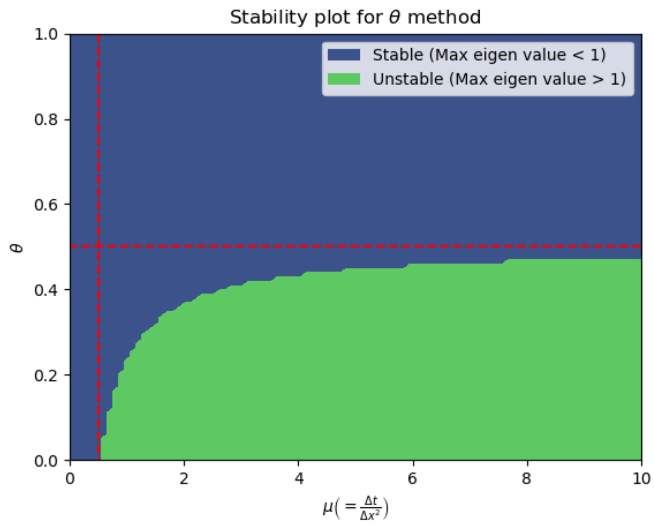

Figure 1: Stability plot fot θ method

4 Convergence study

The following set of plots show the solution for various values of μ and θ. The red line represents the exact solution and the blue line represents the numerical solution. The convergence plots are log log plotted against μ and Δt. The red line in this plot represents $\mu = 0.5$. The contour plot represents the solution in the x-t plane.

4.1 Explicit ($\theta = 0$)

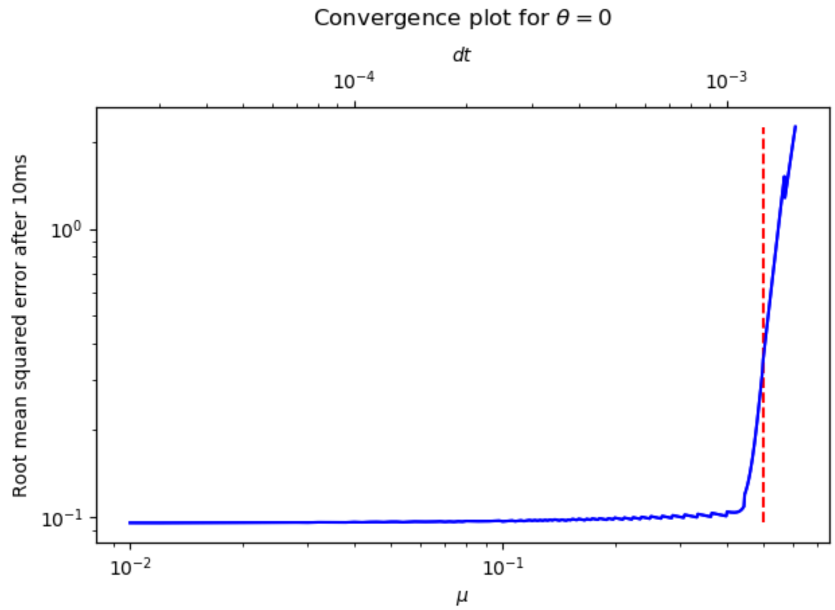

Figure 2: Convergence plot for $\theta = 0$

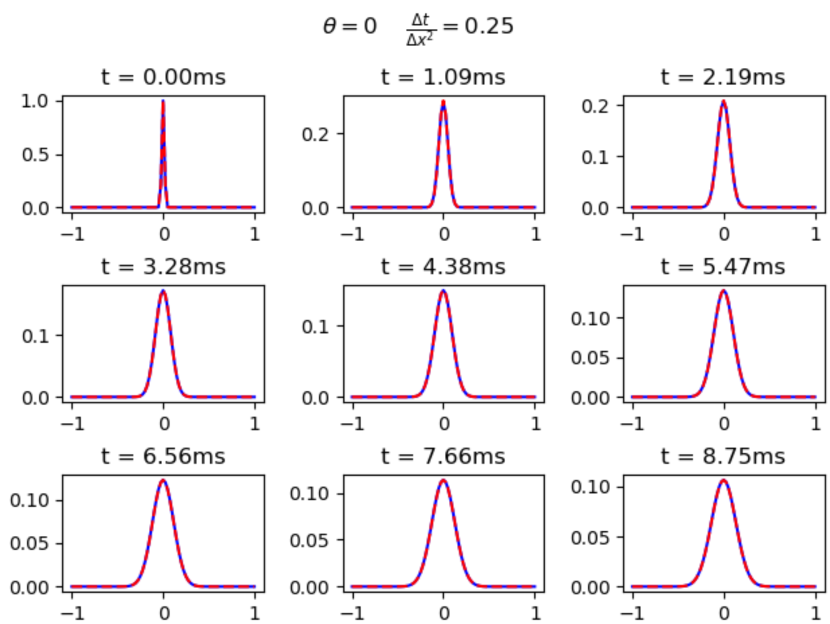

Figure 3: Solution for $\mu = 0.2$ and $\theta = 0$

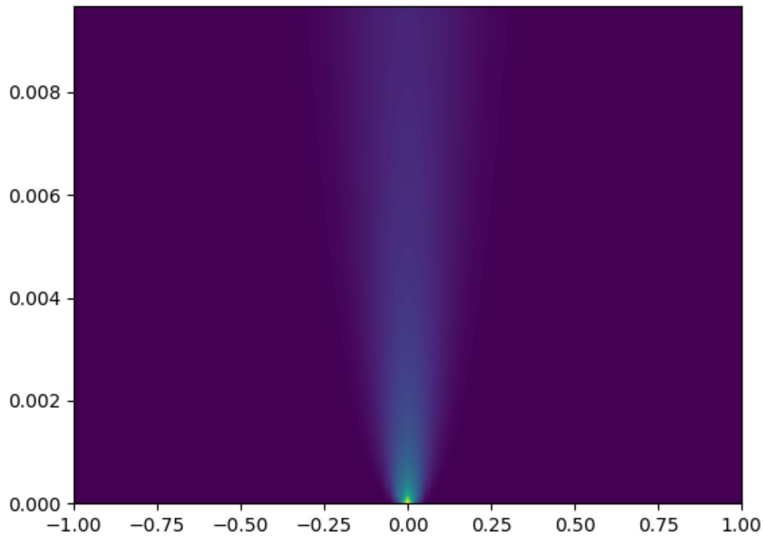

Figure 4: Solution for $\mu = 0.2$ and $\theta = 0$

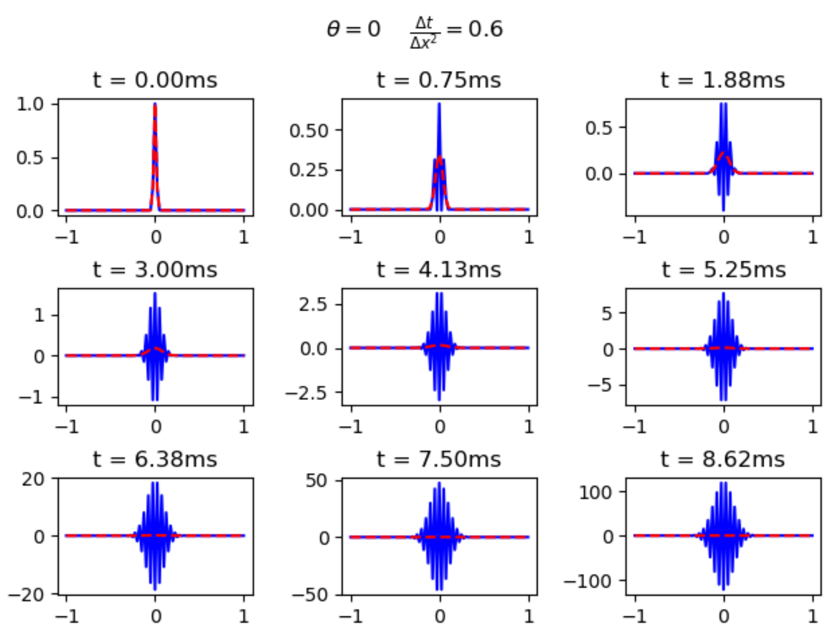

Figure 5: Solution for $\mu = 0.6$ and $\theta = 0$ (Blowup. Unstable)

4.2 Implicit ($\theta = 1$)

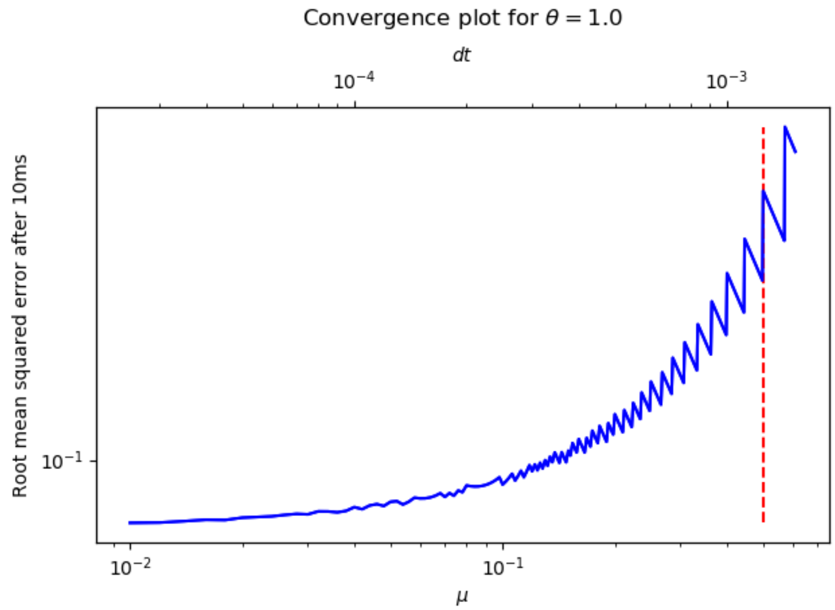

Figure 6: Convergence plot for $\theta = 1$

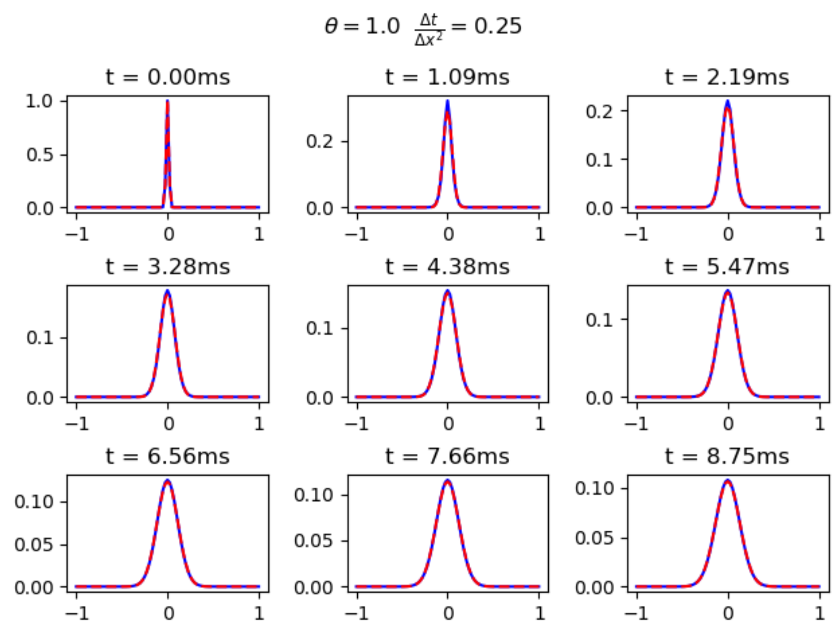

Figure 7: Solution for $\mu = 0.2$ and $\theta = 1$

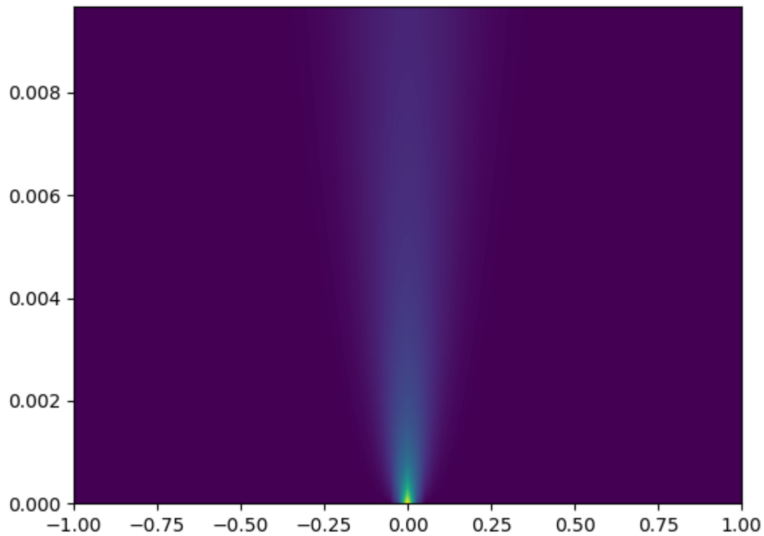

Figure 8: Solution for $\mu = 0.2$ and $\theta = 1$

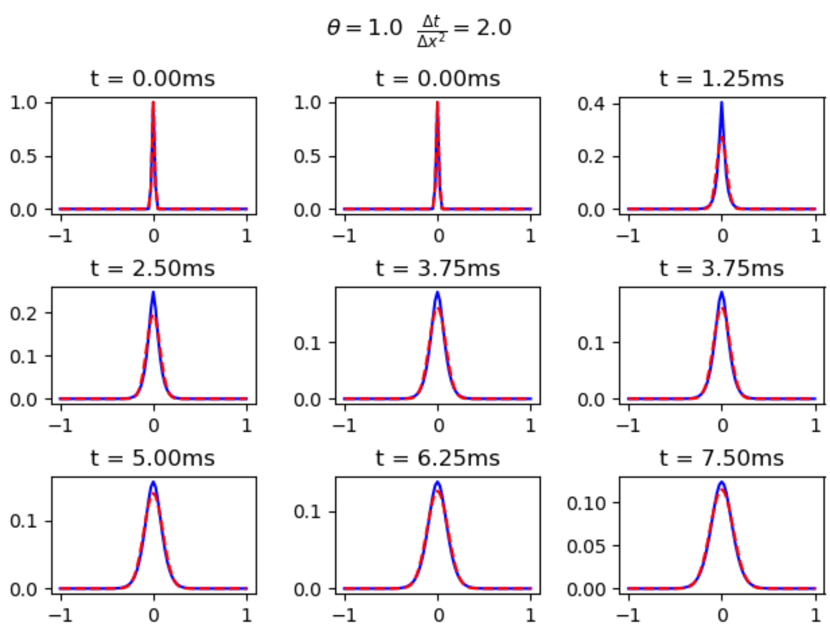

Figure 9: Solution for $\mu = 2.0$ and $\theta = 1$

4.3 CN ($\theta = 0.5$)

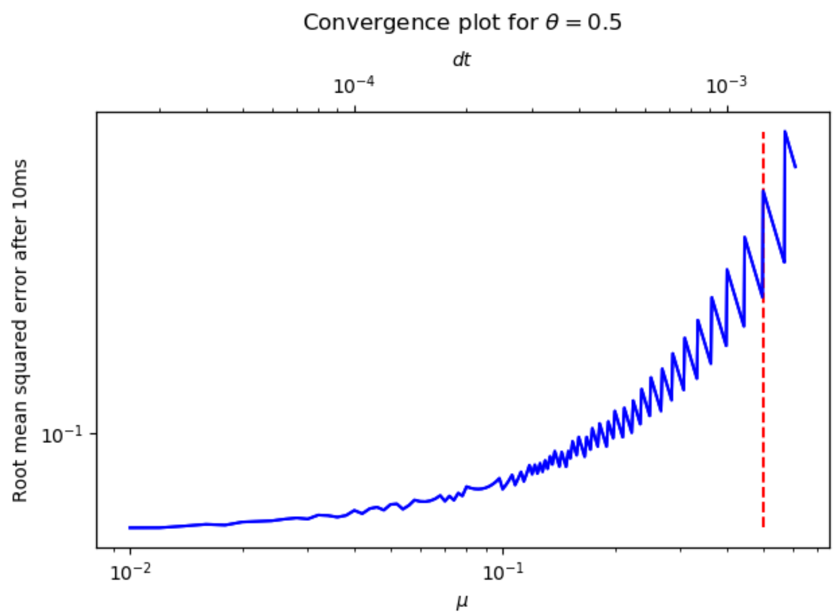

Figure 10: Convergence plot for $\theta = 0.5$

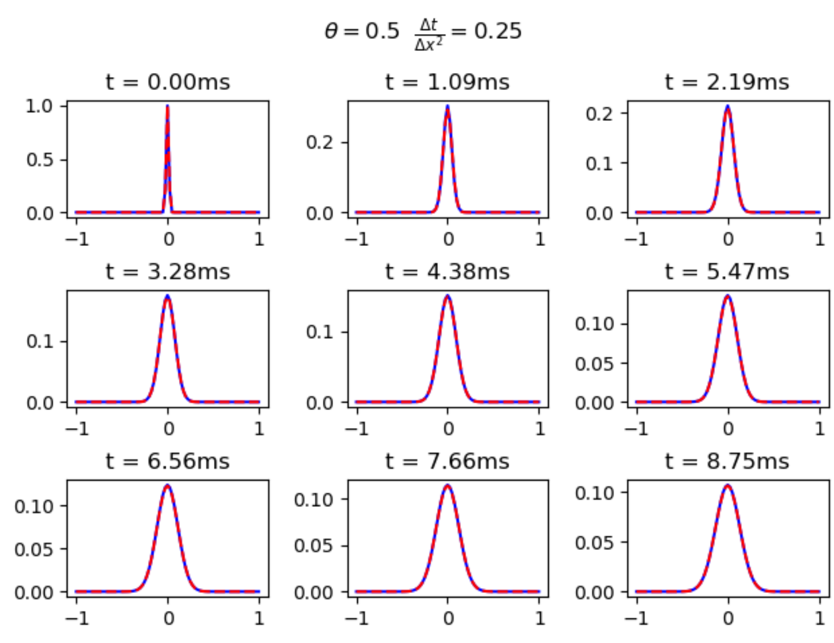

Figure 11: Solution for $\mu = 0.2$ and $\theta = 0.5$

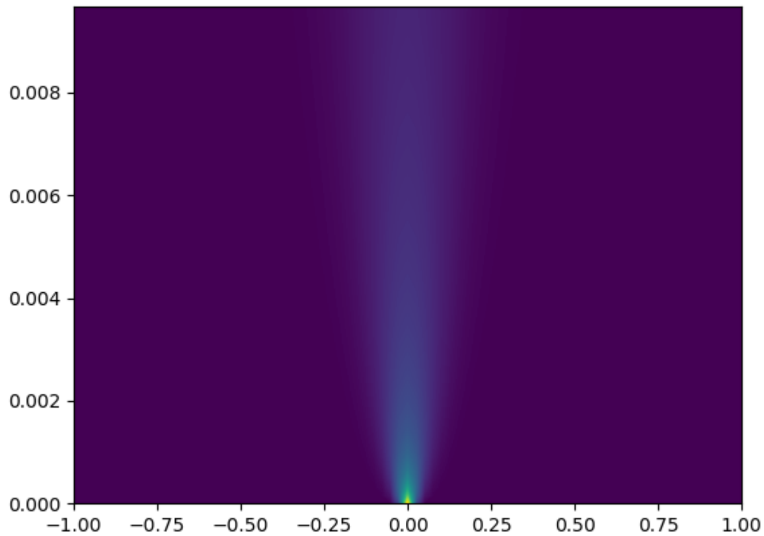

Figure 12: Solution for $\mu = 0.2$ and $\theta = 0.5$

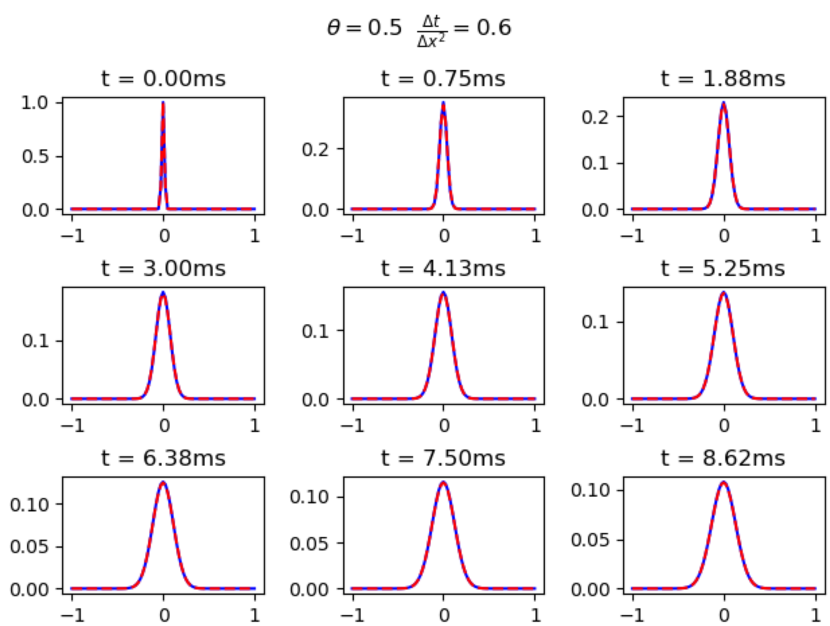

Figure 13: Solution for $\mu = 0.6$ and $\theta = 0.5$

4.4 $\theta = 0.6$

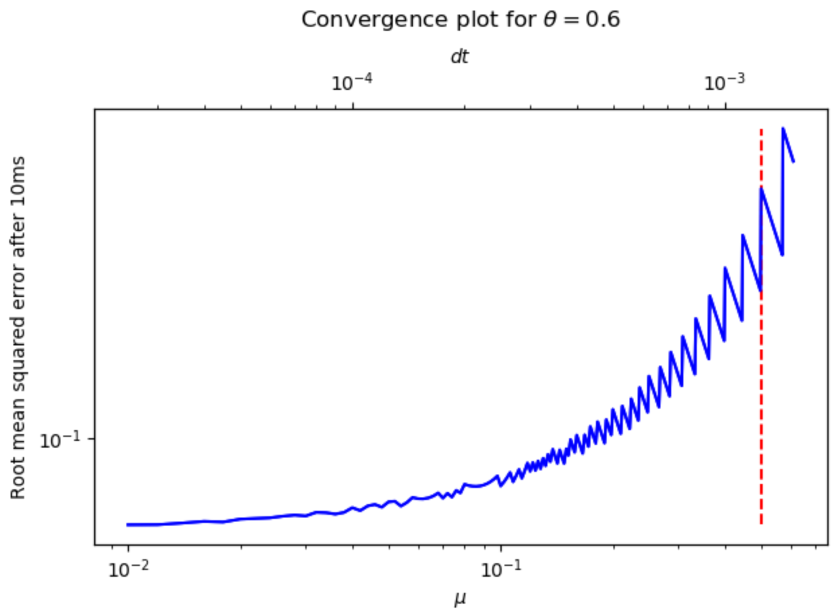

Figure 14: Convergence plot for $\theta = 0.6$

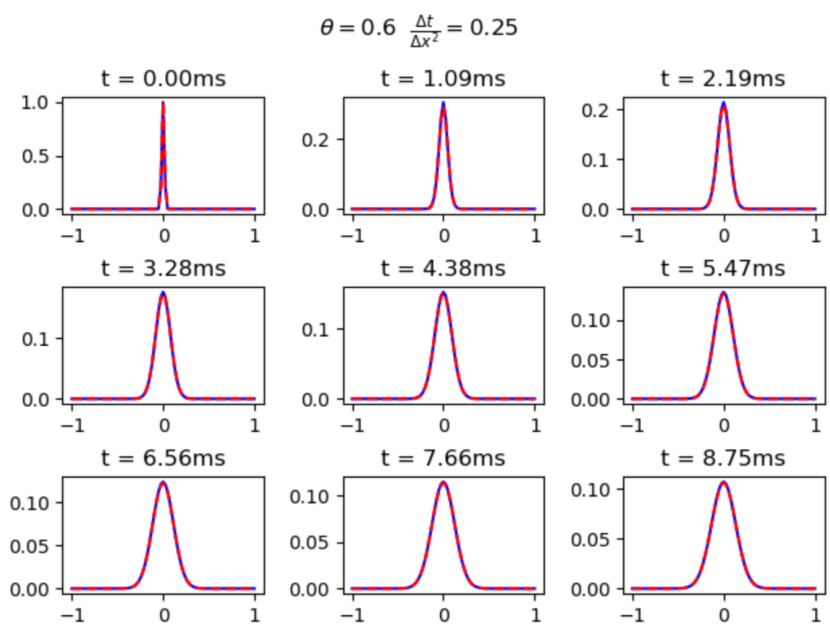

Figure 15: Solution for $\mu = 0.2$ and $\theta = 0.6$

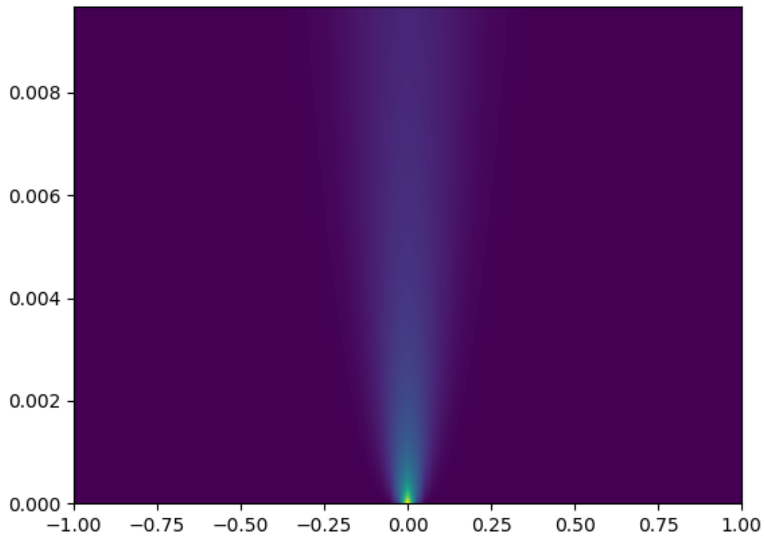

Figure 16: Solution for $\mu = 0.2$ and $\theta = 0.6$

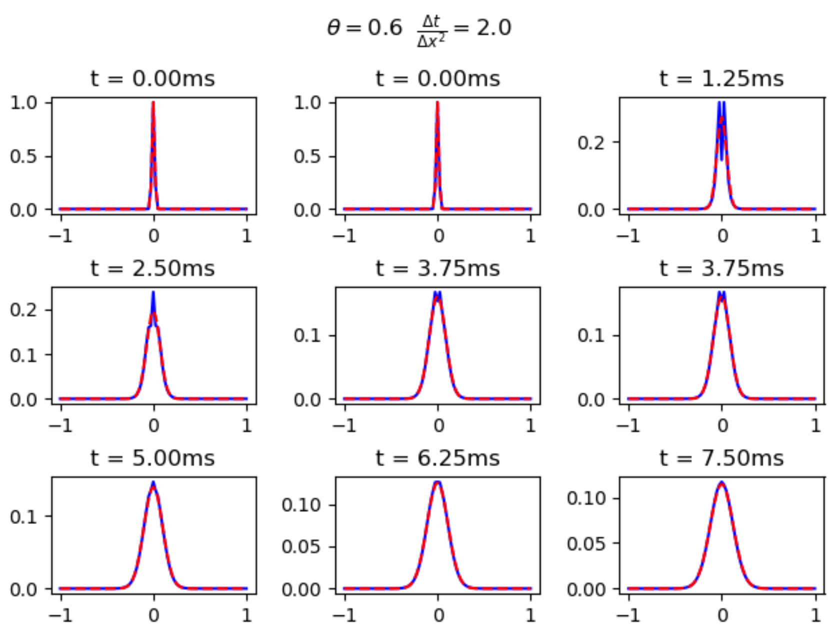

Figure 17: Solution for $\mu = 2.0$ and $\theta = 0.6$

4.5 $\theta = 0.4$

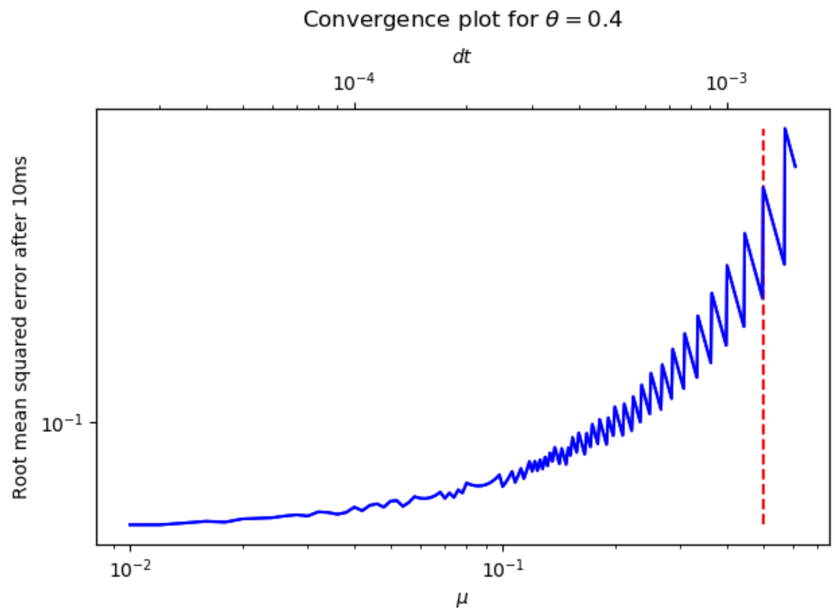

Figure 18: Convergence plot for $\theta = 0.4$

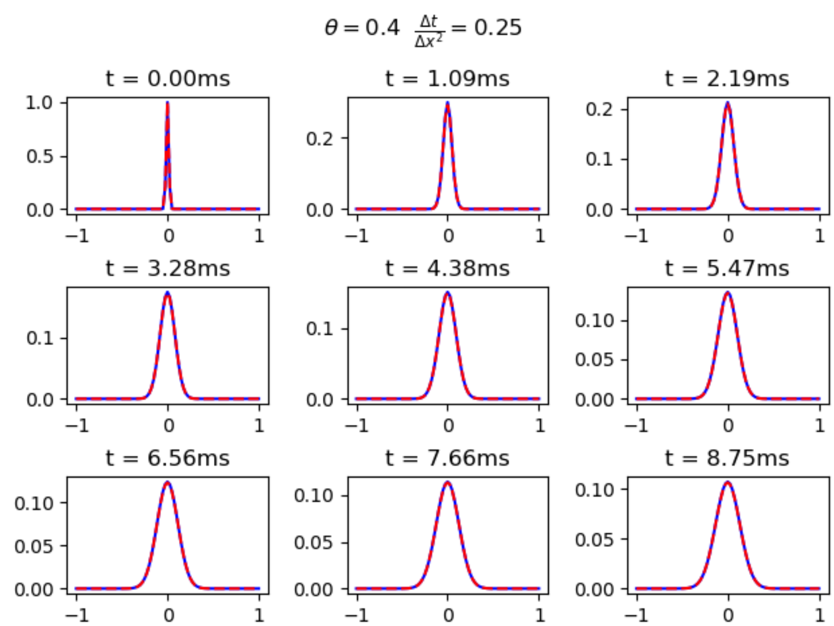

Figure 19: Solution for $\mu = 0.2$ and $\theta = 0.4$

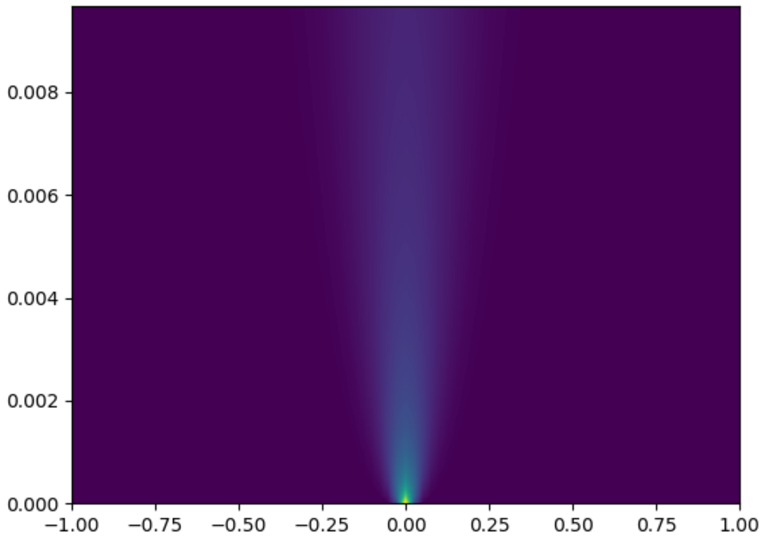

Figure 20: Solution for $\mu = 0.2$ and $\theta = 0.4$

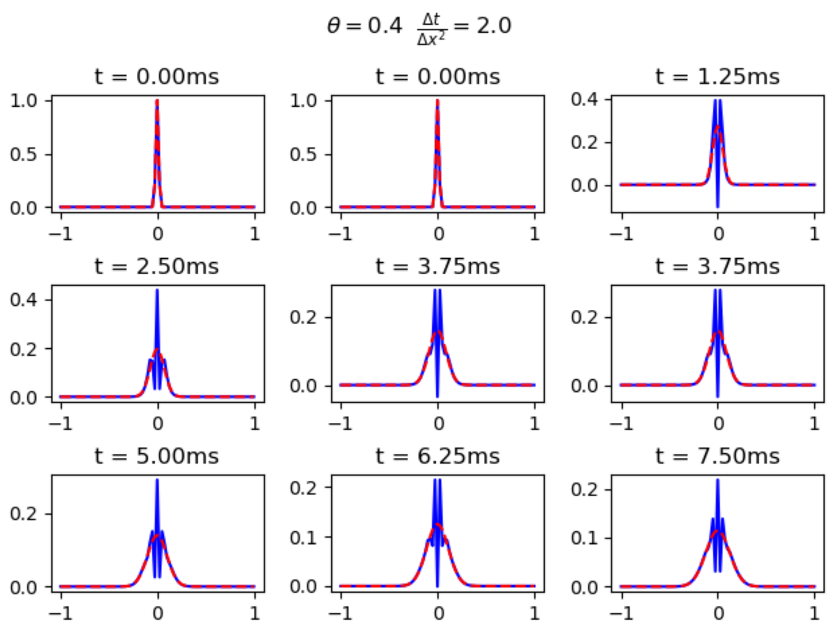

Figure 21: Solution for $\mu = 2.0$ and $\theta = 0.4$ (Unstable)

5 Conclusion

We have successfully checked the convergence and stability of the θ method for the heat equation for various values of μ and θ. We have shown that the sceme is stable for large time steps when $\theta \geq 0.5$ and is conditionally stable otherwise.