# THE STATE OF STRESS NEAR THE SINGULARITIES CREATED BY A PLANE PUNCH

**N. P. ANDRIANOPOULOS** *Athens National Technical University*

**P. S. THEOCARIS** *Athens National Technical University*

An attempt is made to obtain photoelastically the stress distribution in the near vicinity of singular points, when a punch is indenting an elastic half-plane. The punch is to be assumed flat and rough so that friction is developed between the contact surfaces. By expanding the complex stress function into a Taylor series, an extrapolation law is obtained, which allows the calculation of stresses at the vicinity of the singular points by means of photoelastic measurements at positions remote from these points. The error limits of this technique are defined and, finally, a relation between the order of singularity and the parameters of the photoelastic pattern is established.

## 1 INTRODUCTION

The problem of determining the distribution of stresses created by a rigid punch pressed on the free boundary of an elastic half-plane constitutes a special case of the general problem of two bodies, when one of them is infinitely hard. Sadowski (**1**)† was the first to solve the problem of the punch on a half-plane when no friction between punch and plane existed. The problem of the punch in general has been solved by Muskhelishvili (**2**) by means of the complex potentials, which he had himself introduced. Galin (**3**) introduced a similar theoretical solution. Moreover, some numerical methods were developed (**4**), based on Galin's analysis, but by these methods the state of stress around singularities was not given special consideration.

Following Muskhelishvili's analysis, we assume that on the O$x$-boundary of an elastic half-plane a punch of length $2\alpha$ is applied and is pressed by a normal to the boundary load $P$ per unit length of the punch. Also, it is assumed that friction exists between punch and half-plane, which is parallel to the O$x$-axis. The punch can move only vertically and thus no slip occurs along the O$x$-axis. The presence of friction leads to an asymmetric distribution of stresses—that is, to a different order of singularity—around the two ends of the punch. More precisely, the singularity of either end of the punch depends on the direction of friction. The factor of friction, $T$, has the same direction along the contact segment AA′ (Fig. 2). So, it points 'outwards' (out of segment AA′) at one end of the punch and 'inwards' at the other end. The singularity at the first point, where the vector of friction points outwards, has the value of $p = 1/2 + \gamma$, while the singularity at the other point has the value of $q = 1/2 - \gamma$. The coefficient $\gamma$ depends on friction and will be defined later.

Each singularity works as a source of fringes in the photoelastic model, and therefore a high fringe density develops in the area. The difference in the photoelastic pattern between the problem of the punch and the corresponding first fundamental problem, by which the stress function possesses a logarithmic singularity while stresses do not become singular (**5**), is characteristic. Photoelastic patterns corresponding to the punch (Fig. 1*a*) and to uniformly distributed load (Fig. 1*b*), on the same length of the material with roughly the same applied load, show clearly the influence of the stress singularity on the photoelastic pattern. Thus, in areas remote from the end of the punch, or from the end of the distributed load, the fringe patterns are more or less similar.

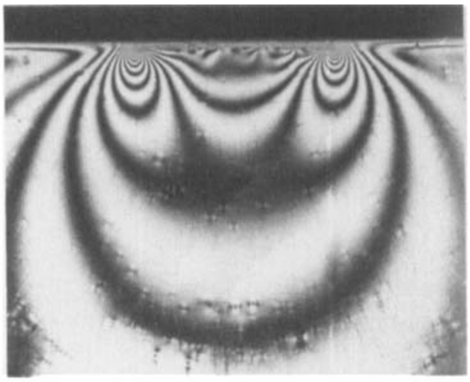
However, at points close to these ends the situation is absolutely different. In the case of the punch, fringes begin and end at the same point and their area decreases rapidly in a way that the evaluation of stress is not feasible. Certainly, other factors aggravate the situation, such as variation of the refractive index of the material, transformation of the plane stress state into a triaxial one, since a considerable change in the material thickness occurs near the singularities, material imperfections, etc. In fact, the method of caustics, as developed by Theocaris (**6**), gives satisfactory results in converting stress singularities into optical ones. Photoelastic measurements are ineffective in these areas for evaluating the state of stress at the near vicinity of the singularity, for the reasons already stated.

In order to counteract this experimental weakness, an extrapolation law is sought, by which one may reduce the values of photoelastic data at positions far away from the singular points to respective values at their near vicinity.

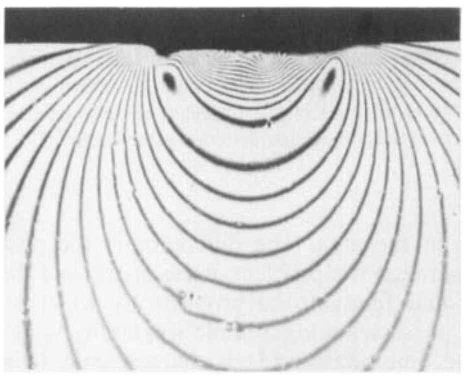
For this purpose, a technique, developed by Theocaris and Gdoutos (**7**) for a crack under uniaxial or biaxial tension, has been applied with satisfactory results. This technique is as follows. The given stress function is expanded into a Taylor series, of which only the first term is singular. Stresses can be expressed in terms of the stress function, which is truncated in this case to the first term of its Taylor expansion. The singular expressions of stress are thus obtained. It is easy to prove that the ratio of the exact value of a component of stress to its corresponding singular value approaches unity as the singularity is

*The MS. of this paper was received at the Institution on 24th October 1977 and accepted for publication on 13th July 1978.*

† *References are given in the Appendix.*



a



b

Fig. 1. Isochromatic fringes: (a) punch without friction; (b) uniformly distributed load ($2a = 12$ mm, $P = 90$ kp/cm; specimen: epoxy-resin with dimensions $150 \times 75 \times 10\ \text{mm}^3$)

approached. If one calculates this ratio for various distances from the singularity, then the exact value of one of the stresses far away from that point can be photoelastically determined and, by using the calculated ratio, one can estimate the singular stress near the singularity. The same technique is here extended to the case of singularities created at the ends of a rough punch in an elastic half-plane.

### 1.1 Notation

| | |
|---|---|
| $f$ | Stress-optical constant of material |
| $K$ | Coefficient of friction between punch and material |
| $\bar{K}$ | Reduced coefficient of friction $\left(=\dfrac{\kappa-1}{\kappa+1}K\right)$ |
| $N$ | Order of an isochromatic fringe |
| $n$ | $\tau_{max}/\bar{\tau}_{max}$ |
| $P$ | Normal load per unit length of punch |
| $(p,q)$ | Order of singularity at two ends of punch |
| $(R,\theta)$ | Polar co-ordinates |
| $T$ | Friction force |
| $t$ | Thickness of specimen |
| $v^-$ | Displacement along negative $y$-axis |
| $z$ | Complex co-ordinate |
| $\alpha$ | Half-length of punch |
| $\gamma$ | $\tan^{-1}\left(K\dfrac{\kappa-1}{\kappa+1}\right)/\pi$ |
| $\varepsilon$ | Relative error $\left(=\dfrac{\tau_{max}-\bar{\tau}_{max}}{\bar{\tau}_{max}}\right)$ |
| $\kappa$ | $\dfrac{\lambda+3\mu}{\lambda+\mu}$ |
| $(\lambda,\mu)$ | Lamé constants |
| $\rho$ | $R/2\alpha$ |
| $(\sigma_x,\sigma_\psi,\tau_{x\psi})$ | Stress components at point $z$ |
| $\tau_{max}$ | Exact (theoretical) value of maximum shear stress at point $z$ |
| $\bar{\tau}_{max}$ | Approximate (singular) value of $\tau_{max}$ |

## 2 THEORETICAL CONSIDERATIONS

Let a flat-ended rigid punch be pressed on the boundary $y = 0$, $|x| \leq \alpha$ of a half-plane $y \leq 0$ (Fig. 2). One assumes that the indenting force is $P$ per unit length of the punch.

The boundary conditions of the present problem have the form

$$\left.\begin{aligned} P(x) &= \text{const} \\ T(x) &= \mathrm{i}KP(x) \\ v^- &= \text{const} \end{aligned}\right\} \qquad -\alpha \leqq x \leqq \alpha$$

$$\left.\begin{aligned} P(x) &= 0 \\ T(x) &= 0 \\ v^- &= 0 \end{aligned}\right\} \qquad x < -\alpha, x > \alpha$$

The complex stress function, representing the stress state in the half-plane, is expressed by (**2**)

$$\Phi(z) = \frac{P(1+\mathrm{i}K)\,\mathrm{e}^{\mathrm{i}\pi\gamma}}{2\pi(\alpha+z)^{1/2+\gamma}\,(\alpha-z)^{(\frac{1}{2}-\gamma)}} \tag{1}$$

with

$$\gamma = \frac{\tan^{-1}\left(K\dfrac{\kappa-1}{\kappa+1}\right)}{\pi}, \qquad 0 \leq \gamma < \tfrac{1}{2}$$

$$\kappa = \frac{\lambda+3\mu}{\lambda+\mu}, \qquad 1 < \kappa < 3$$

where $K$ is the friction coefficient between punch and half-plane and $\lambda$, $\mu$ are the Lamé constants.

The positive values of $\gamma$ imply that $K$ is also positive, and thus the force of friction, $T$, is directed to the negative $x$-axis, because it is assumed that $T = \mathrm{i}KP$. However, one may assume that $T$ would be also directed to the positive $x$-axis and, consequently, $\gamma$ may vary in the range $-1/2 < \gamma < 1/2$. Negative values of $\gamma$ correspond to negative values of $K$.

Conventionally, one considers that negative $K$ implies that a friction force is directed towards the negative $x$-axis.

The stress function (1) exhibits singularities of order $(1/2 \pm \gamma)$ at the points $z = \mp\alpha$ respectively.

Stresses at each point $z$ of the half-plane are given by the well-known Kolosov–Muskhelishvili relations:

$$\sigma_x + \sigma_\psi = 2[\Phi(z) + \overline{\Phi(z)}] \tag{2}$$

$$\sigma_\psi - \sigma_x + 2\mathrm{i}\tau_{x\psi} = 2[(\bar{z}-z)\Phi'(z) - \Phi(z) - \bar{\Phi}(z)] \tag{3}$$

$$\tau_{max} = \tfrac{1}{2}|\sigma_\psi - \sigma_x + 2\mathrm{i}\tau_{x\psi}| \tag{4}$$

In order to expand $\Phi(z)$ into a Taylor series, we introduce the following notation:

$$\left.\begin{aligned} &\tfrac{1}{2}+\gamma = p, \quad \tfrac{1}{2}-\gamma = q, \quad (\alpha-z)^q = (z-\alpha)^q\,\mathrm{e}^{\mathrm{i}\pi q} \\ &z-\alpha = \zeta = R\,\mathrm{e}^{\mathrm{i}\theta}, \quad \frac{R}{2\alpha} = \rho \end{aligned}\right\} \tag{5}$$

when the stress function $\Phi(z)$ assumes the form

$$\Phi(z) = \frac{P(1 + \mathrm{i}K)\,\mathrm{e}^{\mathrm{i}\pi(\gamma - q)}}{2\pi\zeta^q(\zeta + 2\alpha)^p} \tag{6}$$

The binomial $(\zeta + 2\alpha)^{-p}$ is then expanded into a Taylor series with a centre located at the point $A(\alpha, 0)$ and a convergence radius $2\alpha$. This expansion has as a first term the quantity $1/(2\alpha)^p$.

Consequently, equation (6) gives

$$F_1(z) = \frac{P(1 + \mathrm{i}K)\,\mathrm{e}^{\mathrm{i}\pi(\gamma - q)}}{2\pi\zeta^q(2\alpha)^p}$$

which finally reduces to

$$F_1(z) = \frac{P(1 + \mathrm{i}K)\,\mathrm{e}^{-\mathrm{i}(\theta q + \pi(q - \gamma))}}{4\pi\alpha\rho^q} \tag{7}$$

Similarly, the singular part of the first term of the Taylor expansion of the expression $(\bar{z} - z)\Phi'(z)$, which appears in equation (3), becomes

$$F_2(z) = \frac{\mathrm{i}P \sin\theta(1 - 2\gamma)(1 + \mathrm{i}K)\,\mathrm{e}^{-\mathrm{i}(\theta(q+1) + \pi(q - \gamma))}}{4\pi\alpha\rho^q} \tag{8}$$

Now from equations (3) and (4) one obtains for the singular expression of maximum shear stress the expression

$$\bar{\tau}_{\max} = \frac{P}{4\pi\alpha\rho^q} \times [\sin\theta(1 - 2\gamma)(1 + \mathrm{i}K)\,\mathrm{e}^{-\mathrm{i}(\theta(q+1) + \pi(q - \gamma))} - 2K\,\mathrm{e}^{-\mathrm{i}(\theta q + \pi(q - \gamma))}] \tag{9}$$

From equations (3), (4) and (9) we can calculate numerically the ratio

$$n = \frac{\tau_{\max}}{\bar{\tau}_{\max}} \tag{10}$$

where $\tau_{\max}$ is the exact expression and $\bar{\tau}_{\max}$ the singular one for the maximum shear stress, for various distances, $\rho$, and angles, $\theta$.

Furthermore, it is valid that

$$\tau_{\max} = \frac{Nf}{t} \tag{11}$$

where $N$ is the order of the isochromatic fringe, $f$ is the stress-optical constant of the material and $t$ is the thickness of the specimen.

Also, the singular expressions of $\bar{\tau}_{\max}$ for different distances $\rho_B$, $\rho_C$ from point $A(\alpha, 0)$ and for the same angle $\theta$ are connected through the relation

$$\frac{\bar{\tau}_{\max,B}}{\bar{\tau}_{\max,C}} = \left(\frac{\rho_C}{\rho_B}\right)^q \tag{12}$$

Finally, for the point B, which is supposed to be very close to the singular point A, one obtains

$$\bar{\tau}_{\max,B} = \frac{N_c f}{nt}\left(\frac{\rho_C}{\rho_B}\right)^q \tag{13}$$

where $N_C$ is the order of the fringe which passes through point C (Fig. 2).

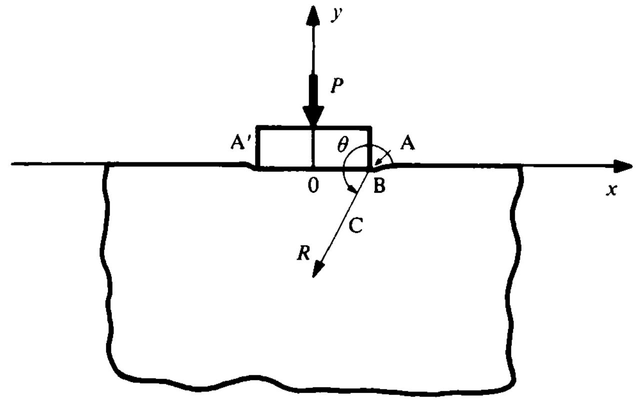



Fig. 2. Geometry of the problem

Plots of the variation of $\tau_{\max}/\bar{\tau}_{\max} = f(\rho)$ were traced for the values of the parameters $\rho$, $\theta$ and $q$, varying as follows: $0 < \rho \leq 0.1$, $\theta = 210°, 240°, 270°, 300°, 330°$ and $0.01 \leq q \leq 0.99$.

All these curves are practically straight lines intersecting at the ordinate equal to unity and they are omitted. The only interesting result is that the slopes of these straight lines are in the third quadrant greater than the corresponding lines in the fourth one, when the singularity $q$ of the point A is smaller than the corresponding singularity of the point $A'$ $(-\alpha, 0)$. This is to be expected since, when $p > q$, the third quadrant is more influenced than the fourth by the stronger singularity of the point $A'$ $(-\alpha, 0)$.

Figs 3*a* to *g* are similar to the previous plots but they correspond to variations of the radius $\rho$ lying between 0·1 and 1·0. The situation in these plots is more complicated because the only term of the Taylor expansion taken is not adequate to describe the stress situation at positions far away from the centre of expansion. However, the above remark made on the relative behaviour of curves in the third and fourth quadrants is still valid. An interesting remark for this second set of curves is that points with $\tau_{\max}/\bar{\tau}_{\max} = 1$ exist. Such points are: $(q, \theta, \rho) = (0{\cdot}1, 240°, 0{\cdot}75)$, $(0{\cdot}3, 300°, 0{\cdot}98)$, $(0{\cdot}5, 210°, 0{\cdot}49)$, etc.

Furthermore, the technique described here can provide another way of estimating the stress state near the singularity. For this purpose, one defines a maximum acceptable limit of error, $\varepsilon$:

$$\varepsilon = \frac{\tau_{\max} - \bar{\tau}_{\max}}{\bar{\tau}_{\max}} \tag{14}$$

and seeks, for each $\theta$, the maximum distance $\rho$ for which the value of $\tau_{\max}/\bar{\tau}_{\max}$ not exceeding the given limit of error $\varepsilon$, is given. In Figs 4*a* to *f* the relative errors, $\varepsilon$, varying between 3 per cent and 15 per cent around the singularity of the boundary are plotted. From these curves we can derive the areas where measurements of prescribed accuracy can be effected. It is worthwhile indicating that these areas increase with $q$, and for $q = 0{\cdot}99$ cover all the semicircle of radius $\rho = 1$.

Expression (9) can also provide the necessary relation between the order of singularity and the parameters of the photoelastic pattern.

Thus, we can seek the angle $\theta_m$ corresponding to the maximum value of distance $\rho$ for a given value of $q$. This corresponds to finding the roots of the equation $\mathrm{d}\rho/\mathrm{d}\theta =$

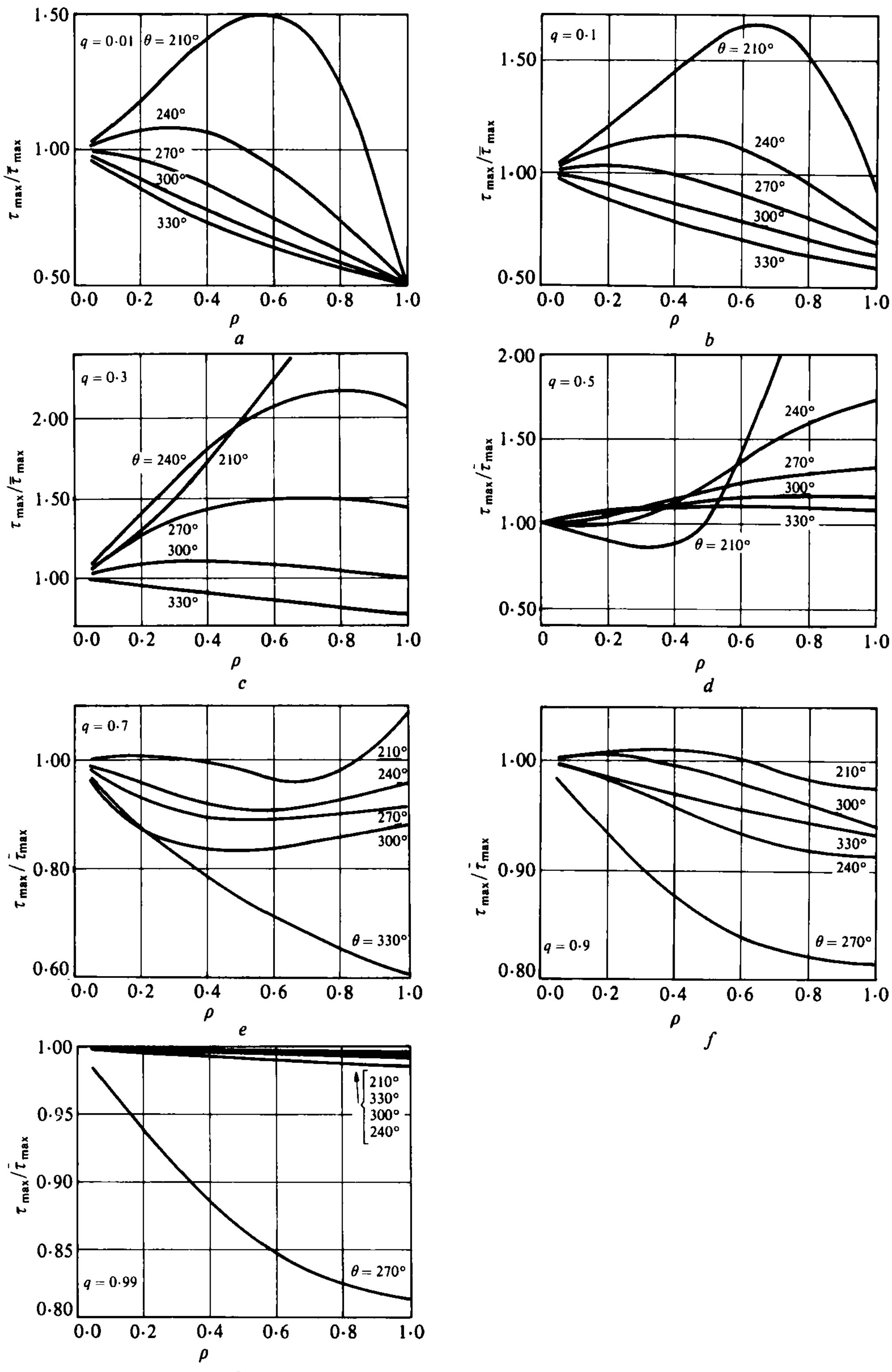



Fig. 3. The ratio $\tau_{max}/\bar{\tau}_{max}$ versus $\rho$ for various values of angle $\theta$. Fig. 3*a* corresponds to $q = 0{\cdot}01$, 3*b* to $q = 0{\cdot}1$, etc.

$f(\theta, q)$ resulting from equation (9). The expression for the function $f(\theta, q)$ is given by

$$f(\theta,q)=\frac{1}{2q}\left(\frac{P}{4\pi\alpha\bar{\tau}_{max}}\right)^{1/q}[(1-2\gamma)^2(1+K)^2\sin^2\theta$$
$$-4(1-2\gamma)K\sin\theta(\cos\theta+K\sin\theta)+4K^2]^{(1-2q)/(2q)}$$
$$\times[((1-2\gamma)^2(1+K^2)-4K^2(1-2\gamma))\sin 2\theta$$
$$-4(1-2\gamma)K\cos 2\theta] \qquad (15)$$

We can deduce from this relation that the equation has roots which do not depend on $P$, $\alpha$, $\bar{\tau}_{max}$. A solution of equation (15) leads to the curve of Fig. 5 which for a measured value of $\theta_m$, in which a photoelastic fringe shows the largest distance $\rho$ from the point A($\alpha$,0), defines the corresponding value of the singularity $q$. Certainly, this curve is mainly of theoretical interest, as values of $K$ outside the range $[-1, 1]$ are not of practical importance. This can be countered by observing that, for any material,

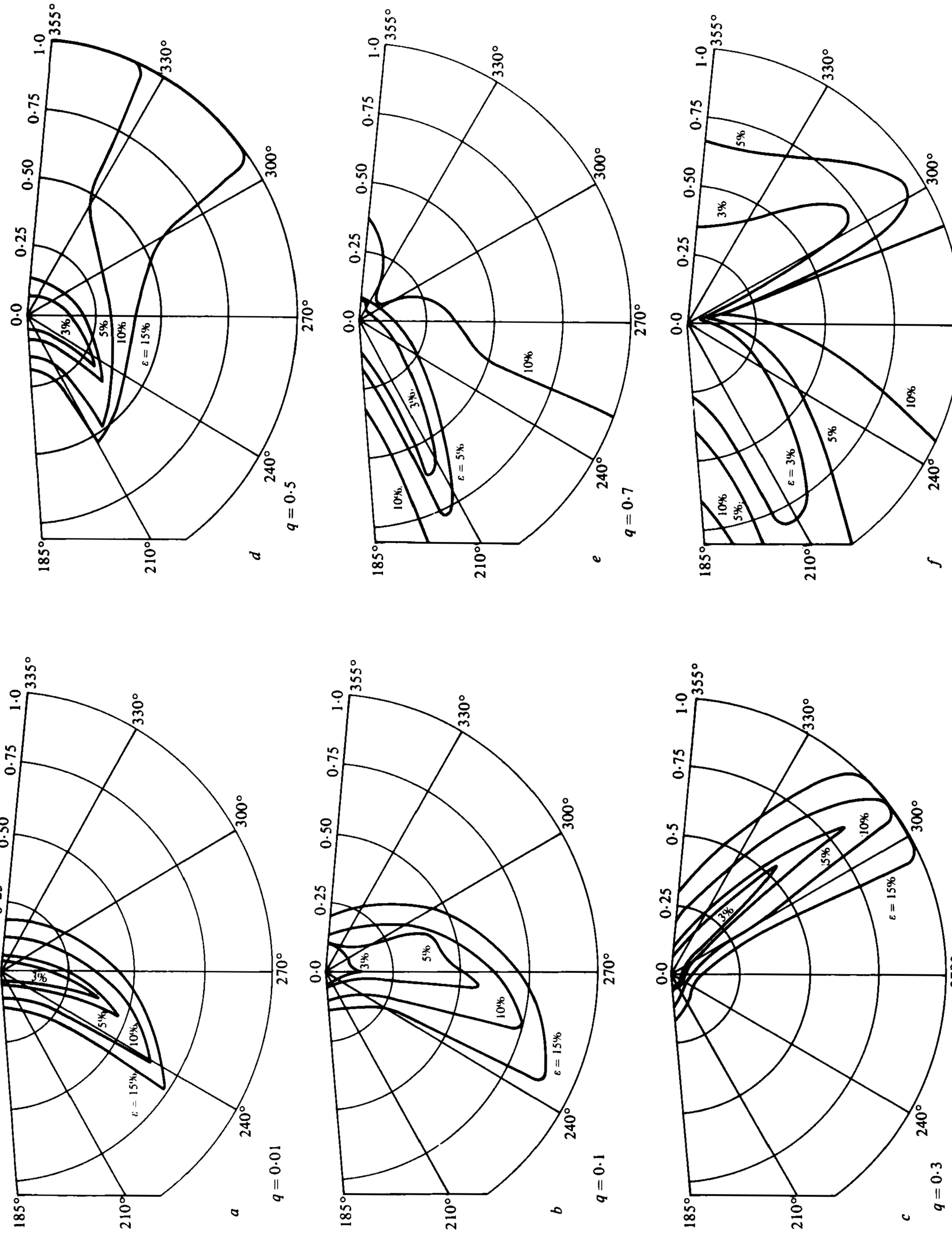



Fig. 4. Measurement areas around point A with pre-defined relative error, $\epsilon$ = 3 per cent, 5 per cent, 10 per cent and 15 per cent

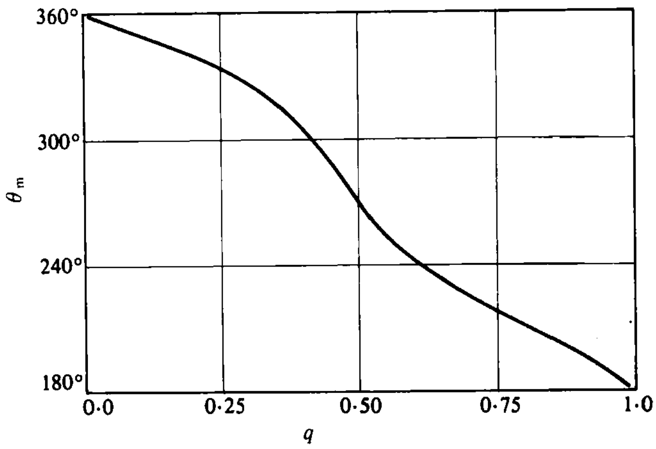



Fig. 5. Relation between order of singularity, $q$, and angle $\theta_m$ where the isochromatic fringe has its larger value of radius, $\rho$

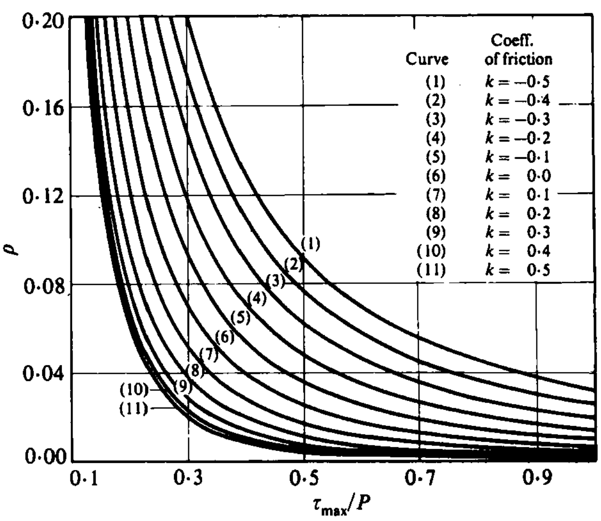



Fig. 6. Variation of $\rho$ with $\bar{\tau}_{max}/P$ for various values of coefficient of friction

the following relation holds:

$$\left.\begin{array}{l} \dfrac{\tan^{-1}(-0{\cdot}5\,K)}{\pi} < \gamma < \dfrac{\tan^{-1}(0{\cdot}5\,K)}{\pi} \\ \text{or} \\ -0{\cdot}148 \leq \gamma \leq 0{\cdot}148 \end{array}\right\} \qquad (16)$$

because

$$1 < \kappa < 3 \quad \text{and} \quad -1 < K < 1$$

Furthermore, equation (9) gives

$$\rho = \left(\frac{P}{4\pi\alpha\bar{\tau}_{max}}\right)^{1/q} [(1-2\gamma)(1+\bar{K}^2)\sin^2\theta$$
$$- 4(1-2\gamma)\bar{K}\sin\theta(\cos\theta + \bar{K}\sin\theta) + 4\bar{K}^2]^{1/(2q)} \qquad (17)$$

where

$$\bar{K} = \frac{\kappa - 1}{\kappa + 1} K$$

The curves of Fig. 6 represent the above relation, $\rho = f(\bar{\tau}_{max}/p)$, for parametric values of $\bar{K}$ in the range $[-0{\cdot}5, 0{\cdot}5]$ and $\theta = 270°$. From these curves, if $P$ is given, it is possible by means of a photoelastic experiment to determine the coefficient of friction between the two materials, provided that for one fringe the value of $\bar{\tau}_{max}$ is calculated and the corresponding distance $\rho$ along the $y$-axis is measured.

## 3 CONCLUSIONS

The distribution of stresses that are created when a flat and rough punch indents the free boundary of a half-plane was studied in the area near the singular points.

The given stress function was expanded to a Taylor series, whose first term only is singular. Near the singular points the stress function was truncated to its first term.

From this singular form of the stress function, the singular forms of the stresses were obtained. The ratio of the exact to the singular expression of the maximum shear stress was numerically calculated around the centre of expansion, which provided an estimate of the value of $\tau_{max}$ near the singularity, by photoelastically measuring its value at remote points. The divergence of the singular expression from the exact one was studied and curves of relative error were obtained.

A relation between the order of singularity and the parameters of the photoelastic pattern was established. The relation provided a means of estimating photoelastically the order of the singularity. Thus, as we can see in Fig. 1*a*, any fringe around both ends of the punch has its larger distance, $\rho$, in the direction of negative $y$-axis—that is, $\theta_m = 270°$—and according to Fig. 5 the order of singularity is $q = 0{\cdot}5$ (zero friction). This was expected, since the experiment of Fig. 1*a* was executed under no friction between punch and half-plane (lubricated contact area).

Finally, a set of curves relating the maximum shear stress to the coefficient of friction between the two materials was established in the more realistic case when this coefficient does not exceed unity. By means of a photoelastic experiment, one can calculate the value of $\bar{\tau}_{max}$ and measure the distance $\rho$ along the negative $y$-axis of a given isochromatic fringe. Thus, given the normal load, $P$, the pair $(\rho, \bar{\tau}_{max}/P)$ defines a point in the nomograph of Fig. 6, that is, the corresponding value of the reduced coefficient of friction, $\bar{K}$, and by means of the relation between $K$ and $\bar{K}$, the value of coefficient of friction, $K$.

It is evident that the same technique can be extended to other cases where the stress function contains one or more singular points.

## APPENDIX

### REFERENCES


(1) SADOWSKI, M. A. 'Zweidimensionale Problem der Elastizitätstheorie', *Zeitschr. f. angew. Math. Mech.* 1928 **8**, 107–121.

(2) MUSKHELISHVILI, N. I. *Some basic problems of the mathematical theory of elasticity* 1963 (Noordhoff).

(3) GALIN, L. A. *Contact problems in the theory of elasticity* 1961 (North Carolina State College Translation). Editor: I. Sneddon.

(4) CONWAY, H. D. *et al.* 'Normal and shearing contact stresses in indented strips and slabs', *Int. J. Engng Sci.* 1966 **4**, 343–359.

(5) THEOCARIS, P. S. and RAZEM, C. I., 'Deformed boundaries determined by the method of caustics', *J. Strain Analysis* 1977 **12**, 223–232.

(6) THEOCARIS, P. S. 'Local yielding around a crack tip in Plexiglas', *J. appl. Mech.* 1970 **37**, 409–415.

(7) THEOCARIS, P. S. and GDOUTOS, E. E. 'A photoelastic determination of $K_I$ stress intensity fractors', *Engng Fr. Mech.* 1975 **7**, 331–339.