Solution of Nonlinear Equation

We are interested in finding the Solution of :

$$f(x) = 0$$

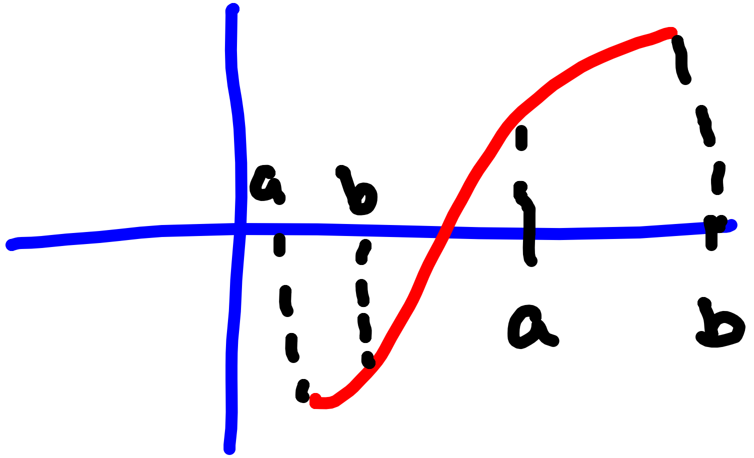
$$\left.\begin{aligned} x^2 + 2x + 5 &= 0 \\ x^3 + 1 &= 0 \end{aligned}\right\} \text{ You already know how to find root.}$$

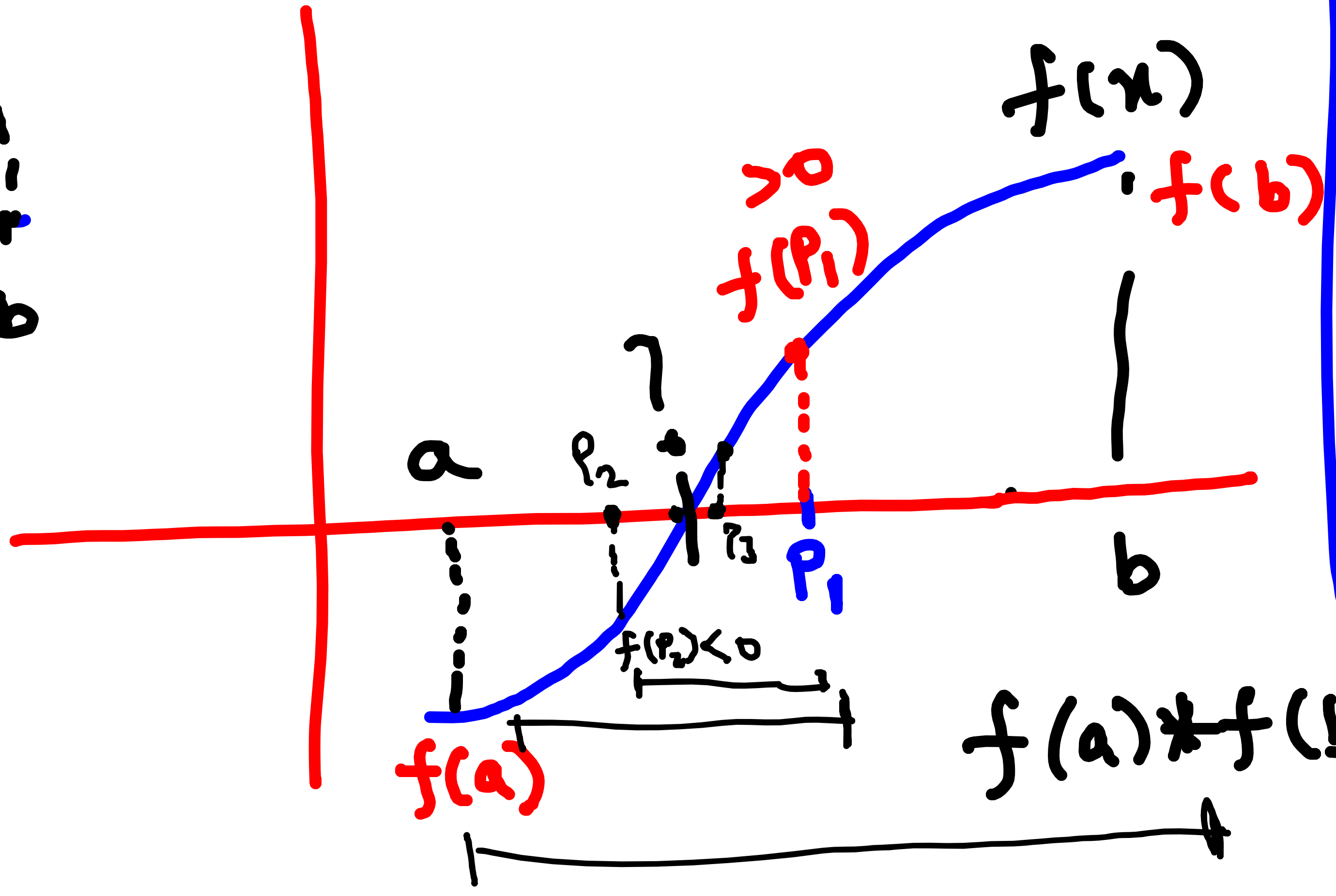
$$\sin x + x^3 + 1 = 0$$

But for this !!

1

Bisection Method





$f(a) * f(b) < 0$

$P_1 = \frac{a+b}{2}$

$P_2 = \frac{P_1 + a}{2}$

$P_3 = \frac{P_2 + P_1}{2}$

$P_4 = \frac{P_2 + P_3}{2}$

Q. Find root of $f(x) = x^3 + 4x^2 - 10 = 0$ in $[1, 2]$ by bisection Method.

Soln

$f(1) = -5$ | $f(2) = 14$

$\therefore$ root lies b/w $[1, 2]$

$P_1 = \frac{1+2}{2} = 1.5$; $f(1.5) = 2.375$ (+ve)

$P_2 = \frac{1.5+1}{2} = 1.25$; $f(1.25) = -1.796$ (-ve)

$P_3 = \frac{1.25+1.5}{2} = 1.375$; $f(1.375) = 0.1621$

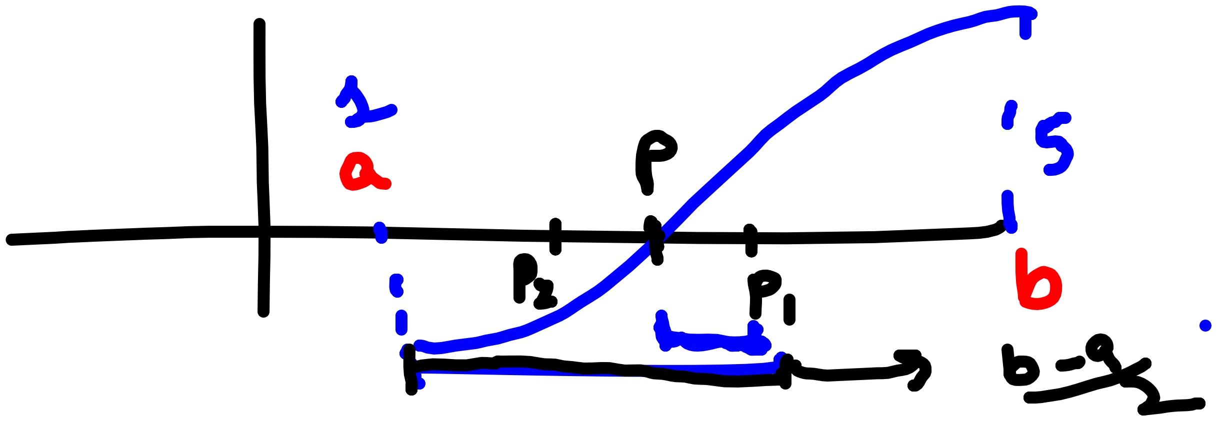
$P_4 = \frac{1.375+1.25}{2} = 1.3125$; $f(1.3125) =$

Theorem: Suppose that $f \in C[a,b]$ and $f(a) * f(b) < 0$. The bisection method generates a sequence $\{p_n\}$ approximating a zero of f (say p) with

$$|p_n - p| \le \frac{b-a}{2^n}, \quad \text{when } n \ge 1$$

$n=1$

$$|p_1 - p| \le \frac{b-a}{2}$$



Q. To find the number of iterations necessary to get desired accuracy upto say ϵ

$$|P_n - P| \le \frac{b-a}{2^n}$$ (from previous theorem)

$$|P_n - P| < \epsilon$$ (desired accuracy)

$$\Rightarrow \left|\frac{b-a}{2^n}\right| < \epsilon$$

$$2^{-n} < \frac{\epsilon}{|b-a|}$$

$\Rightarrow$ taking log,

$$-n \log 2 < \log \epsilon - \log(b-a)$$

$$\boxed{n > \frac{\log(b-a) - \log \epsilon}{\log 2}}$$

Fixed point iteration

Definition: The number p is a fixed pt for a given function g if $g(p) = p$

You are given a root finding problem $f(x) = 0$.

Consider a function $g(x) = x - f(x)$

$$g(p) = p - f(p)$$

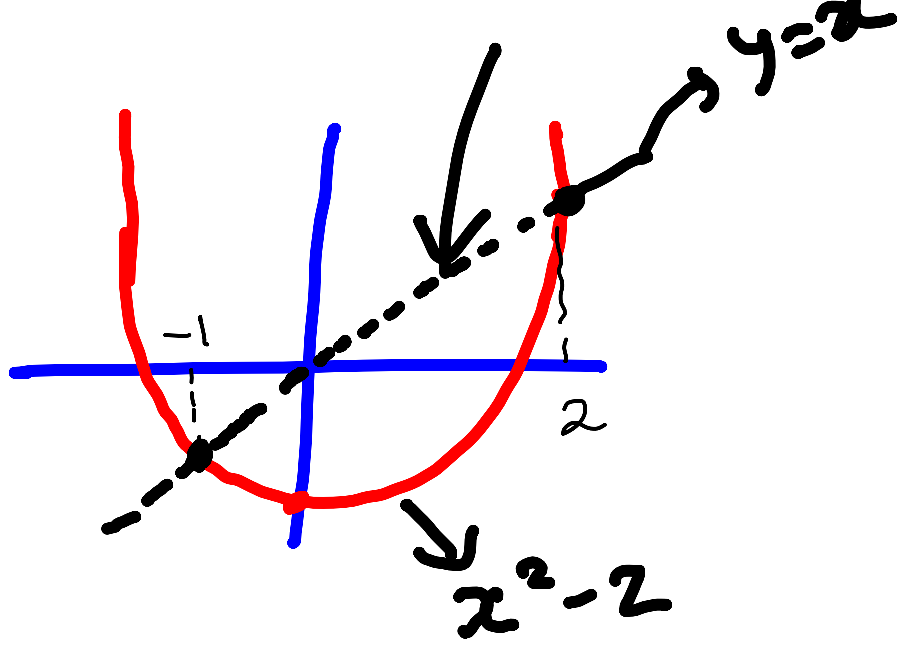
Suppose p is fixed pt of g

$$p = \cancel{p} - f(p) \Rightarrow f(p) = 0$$

$\Rightarrow$ p is root of f

Q. Determine fixed pts of function

$$g(x) = x^2 - 2$$



$$x = x^2 - 2$$

$$\boxed{p = -1, \quad p = 2}$$

These are two fixed pt of $g(x)$

Check!

$g(-1) = -1$

$g(2) = 2$

Theorem: (i) Existence of fixed pt: Proof

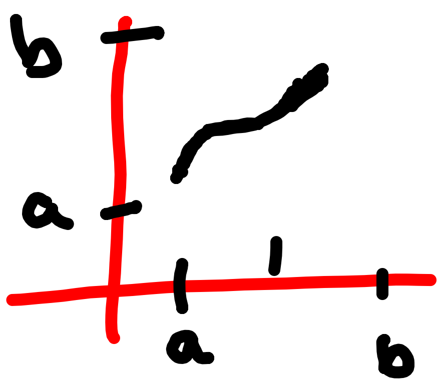
If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$
then g has at least one fixed point in $[a,b]$

(ii) Uniqueness:

If $g'(x)$ exists on (a,b) and
a 'tve' const $K<1$ exists with

$|g'(x)| \leq K$, $\forall x \in (a,b)$

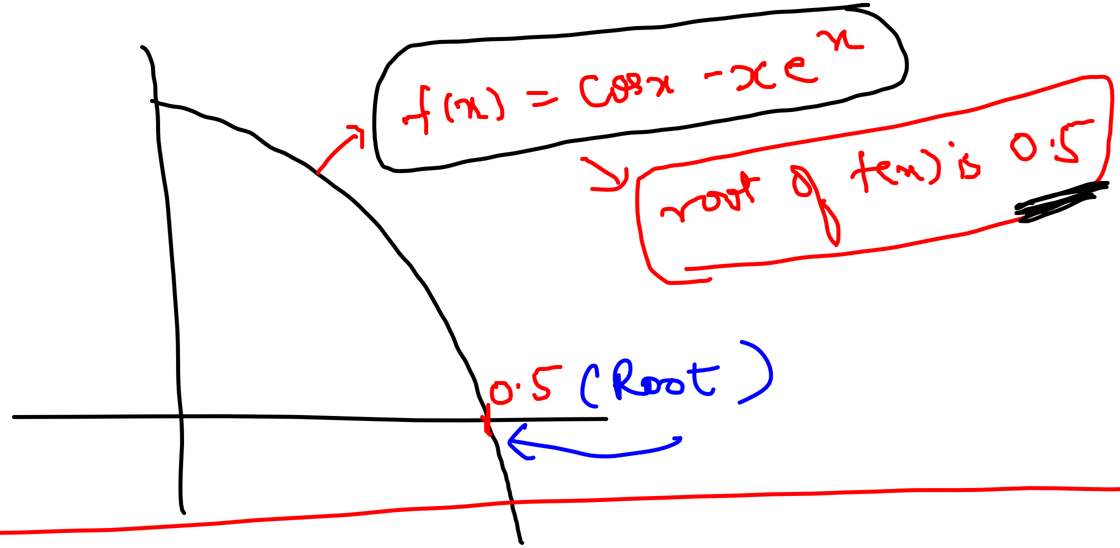
then $\exists$ exactly one fixed pt in $[a,b]$

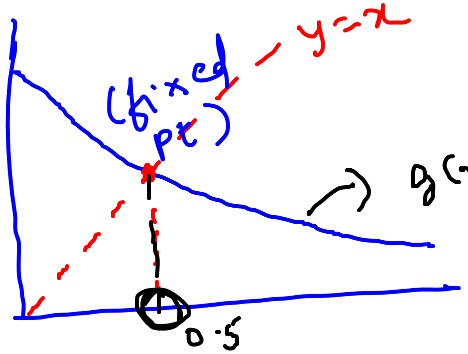


Recall Fixed point Method

$$f(x) = y = \cos x - xe^{x}$$

Graph





$\Rightarrow g(x) = e^{-x} \cos x$

Graph

Desmos

$g(x) = ?$

$f(x) = \boxed{\cos x - xe^{x} = 0}$

$\Rightarrow \cos x = x e^{x}$

$\Rightarrow e^{-x} \cos x = x$

$f(x)\ \boxed{= 0 = x - e^{-x} \cos x}$

$g(x)$

$g(x) = x - f(x)$

$\boxed{f(x) = x - g(x)}$

$\Rightarrow \boxed{g(x) = e^{-x} \cos x}$

we need to find fixed pt of g(x)

Fixed point theorm

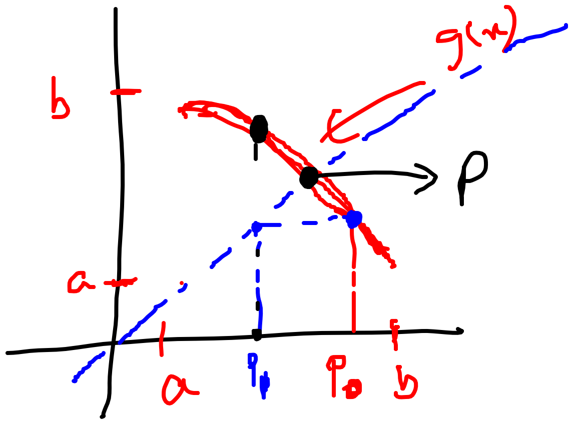
Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$ for all $x \in [a,b]$.

Suppose, in addition $g'(x)$ exists on (a,b) & $\exists$ a constant $0 < K < 1$ with $|g'(x)| \le K$ for all $x \in (a,b)$

Then for any number p_0 in $[a,b]$ the sequence defined by

$$p_{n+1} = g(p_n), \quad n = 0, 1, \ldots$$

converges to the unique fixed pt p in $[a,b]$



$p_1 = g(p_0)$

$g(p_1)$ - - - p

Q Find the root of the equation

$$\boxed{2x = \cos x + 3}$$ by fixed pt method

1.5

Ans

Soln

$$\boxed{2x - \cos x - 3 = f(x) = 0}$$

$[1 \quad 2]$

$$2x = \cos x + 3$$

$$x = \frac{\cos x + 3}{2}$$

$$x - \underbrace{\left(\frac{\cos x + 3}{2}\right)}_{g(x)} = 0$$

$$g(x) = \frac{\cos x + 3}{2} \quad (1.5)$$

Check

① g is cont. on [1, 2]

② $g(x) \in [1, 2]$

③ $|g'(x)| = \left|\frac{\sin x}{2}\right| < 1$?!.

$P_0 = 1.5$

$g(1.5) = 1.535$

$g(1.535) = 1.517$

$g(1.517) = 1.526$

$g(1.526) = 1.522$

$= 1.524$

Q Find root of $x^3 + x^2 - 1 = 0$ on the interval $[0, 1]$.

$(x+1)^{-1/2}$

Soln:

$x^2(x+1) - 1 = 0$

① $x = \frac{1}{\sqrt{x+1}} \longrightarrow g_1(x)$

0.5

① $g_1(x)$ is cont $[0,1]$
② $g_1(x) \in [0,1]$
③ $g_1'(x) = \left|\frac{1}{(x+1)^{3/2}}\right| < 1 \quad \forall x \in (0,1)$

All condition is satisfied convergence guaranted

Let $\boxed{P_0 = 0.5}$

$g_1(0.5) = 0.816$
$g_1(0.816) = 0.742$
$g_1(0.742) = 0.757$
$g_1(0.757) = 0.754$

② $x = (1 - x^2)^{1/3} \leftarrow g_2(x)$

① $g_2(x)$ is cont
② $g_2(x) \in [0,1]$
③ $g_2'(x) = \left|\frac{2x}{3(1-x^2)^{2/3}}\right| \not< 1 \quad \forall x$

$\boxed{x = 0.9}$

last condition does not hold true

∴ We cannot garuntee convergence

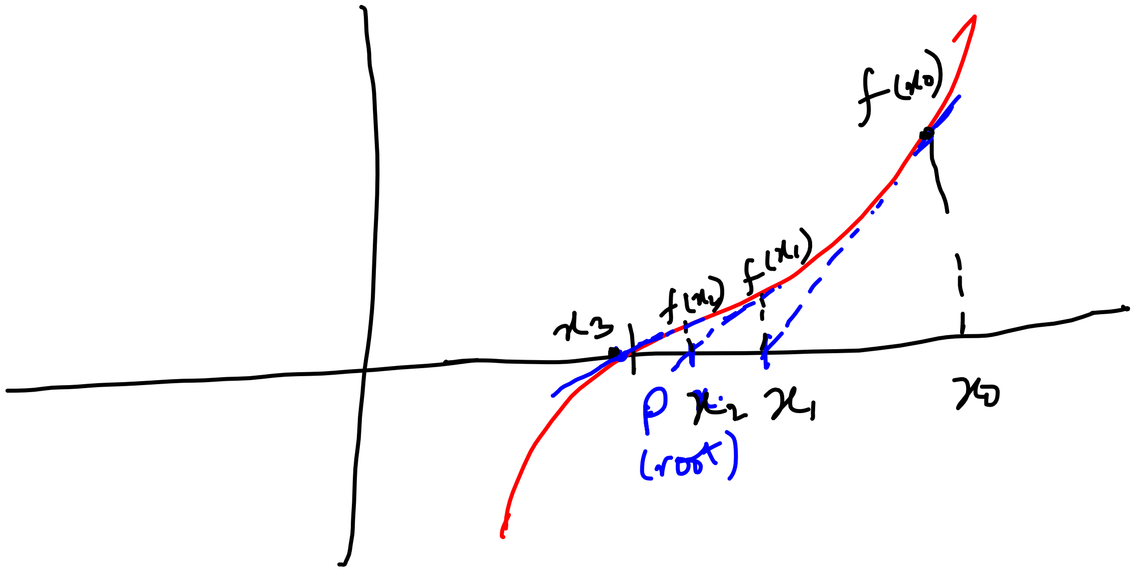
③ $g_3(x)$: $\sqrt{1-x^3} = x$

Check all condition for $g_3(x)$

Newton Raphson Method

Iteration formula for NR method

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$



graphically

Start with initial value x_0 near to the root p

$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\vdots$

until desired accuracy is reached!

Q Perform four iteration of Newton Raphson Method to approximate value of $(17)^{1/3}$ Starting with Initial approximation $x_0 = 2$.

Sol:

$$f(x) = x^3 - 17$$

$$f'(x) = 3x^2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$n=0$;

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

2 Ans

$(2(Ans)^3 + 17) / (3 * Ans^2)$

$$x_{n+1} = x_n - \frac{x_n^3 - 17}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + 17}{3x_n^2}$$

$$\boxed{x_{n+1} = \frac{2x_n^3 + 17}{3x_n^2}}$$

1

$$x_1 = 2 - \frac{f(2)}{f'(2)}$$

$$x_1 = 2.75$$

2

$$x_2 = 2.75 - \frac{f(2.75)}{f'(2.75)}$$

$$x_2 = 2.582$$

$$x_3 = 2.5715$$

$$x_4 = 2.5712$$

Q. Find a root of equation $x \sin x + \cos x = 0$ starting with IV $x_0 = \pi$. Perform 4 iteration.

$f(x) = x \sin x + \cos x$

$f'(x) = x \cos x$

$$\boxed{x_0 = \pi}$$

Soln:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n \sin(x_n) + \cos(x_n)}{x_n \cos(x_n)}$$

$$= \frac{x_n^2 \cos(x_n) - x_n \sin x_n - \cos x_n}{x_n \cos x_n}$$

$$x_1 = 2.82328$$

$$x_2 = 2.79860$$

$$x_3 = 2.79838$$

$$\boxed{x_4 = 2.798386}$$

Check it is root or not in graph

Q. Find a real root of eqn $x = e^{-x}$ using Newton Raphson Method.

Perform 4 iterations.

H.W

Ans: For $x_0 = 1$

$x_1 = 0.6839397$

$x_2 = 0.5774545$

$x_3 = 0.5672297$

$x_4 = 0.5671433$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) \to 0}$$

Newton raphson method converges slowly when $f(x)$ has multiple roots

Let $f(x) = (x-1)^2$

It has multiple roots $\underline{\underline{1, 1}}$

$f'(x) = 2(x-1)$

$\Rightarrow \quad f(1) = 0$

$f'(1) = 0$

$\therefore$ f has root 1 with multiplicity 2

Multiple Roots

Definition : A solution p of $\boxed{f(x) = 0}$ is a zero of multiplicity m of f if for $x \neq p$, we can write $f(x) = (x-p)^m \, q(x)$ where $\lim_{x \to p} q(x) \neq 0$.

$\boxed{\text{OR}}$

The function $f \in C^m[a,b]$ has a zero of multiplicity m at p in (a,b) iff

$$f(p) = f'(p) = f''(p) \; - - - = f^{(m-1)}(p) = 0$$

$$f(x) = (x-1)^2$$

$$f'(x) = x-1$$

$$f''(x) = 1$$

$f(1) = 0$

$f'(1) = 0$

Q. $f(x) = e^x - x - 1$

Show that f has a zero of multiplicity 2 at $x = 0$.

Soln:

$$f(0) = e^0 - 0 - 1 = 0$$

$$f'(x) = e^x - 1 \quad \Rightarrow \quad f'(0) = e^0 - 1 = 0$$

$$f''(x) = e^x \quad \Rightarrow \quad f''(0) = 1 \neq 0$$

HW: Apply NR Method to this to find root

Take $\boxed{x_0 = 1}$

Modified NR Method

Q. To find root of $f(x) = 0$, where f has multiple roots

Let p is a zero of f with multiplicity m

Let us consider a function $\mu(x) = \frac{f(x)}{f'(x)}$

Since f has root p with multiplicity m

$$f(x) = (x-p)^m q(x), \quad q(p) \neq 0$$

$$\Rightarrow f'(x) = m(x-p)^{m-1} q(x) + (x-p)^m q'(x)$$

$$\mu(x) = \frac{(x-p)^{\cancel{m}} q(x)}{\cancel{(x-p)^{m-1}} \left[m\, q(x) + (x-p)\, q'(x)\right]}$$

$$\mu(x) = (x-p) \left[\frac{q(x)}{m\, q(x) + (x-p)\, q'(x)}\right] \quad \left\{ \frac{q(p)}{m\, q(p)} \neq 0 \text{ at } x = p \right.$$

$\mu(x)$ has simple root at p

$\therefore$ we can apply normal NR method to $\mu(x)$

$\Rightarrow$ $H(x)$ has single zero at $x = P$

$$x_{n+1} = x_n - \frac{H(x_n)}{H'(x_n)}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)\, f'(x_n)}{\left(f'(x_n)\right)^2 - f(x_n)\, f''(x_n)}}$$

Modified N R method

Q. H.W. Find root of $f(x) = x^3 - x^2 - x + 1 = 0$ using modified Newton Raphson Method. Take $x_0 = 0.8$

System of Non-linear Equations

$f(x) = 0$

We now extend NR method for single eqn $f(x)=0$ to the system of nonlinear equations

$$f(x, y) = 0$$
$$g(x, y) = 0$$

Iteration scheme

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - J_k^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix}$$

where $J_k = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$ at (x_k, y_k)

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \left(J_k^{-1}\right)_{x_0, y_0} \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix}$$

Ex: Solve

$$\left.\begin{aligned} x^2 + xy + y^2 &= 7 \\ x^3 + y^3 &= 9 \end{aligned}\right\}$$

Take initial approximation as

$$x_0 = 1.5$$
$$y_0 = 0.5$$

here,

$$f(x,y) = x^2 + xy + y^2 - 7$$
$$g(x,y) = x^3 + y^3 - 9$$

Solution:

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

$$J_k = \begin{bmatrix} 2x_k + y_k & x_k + 2y_k \\ 3x_k^2 & 3y_k^2 \end{bmatrix} ; \quad J_k^{-1} = \frac{1}{D_k} \begin{bmatrix} 3y_k^2 & -(x_k + 2y_k) \\ -3x_k^2 & 2x_k + y_k \end{bmatrix}$$

where $\boxed{D_k = 6x_k y_k^2 + 3y_k^3 - 3x_k^3 - 6x_k^2 y_k}$

$$x_1 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - J_0^{-1} \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix}$$

$$J_0^{-1} = \frac{1}{-14.25} \begin{bmatrix} 0.75 & -2.5 \\ -6.75 & 3.5 \end{bmatrix}$$

$$\boxed{D_0 = -14.25}$$

$$f(1.5, 0.5) = -3.75$$

$$g(1.5, 0.5) = -5.50$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} - \frac{1}{-14.25} \begin{bmatrix} 0.75 & -2.5 \\ -6.75 & 3.5 \end{bmatrix} \begin{bmatrix} -3.75 \\ -5.50 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2.2675 \\ 0.9254 \end{bmatrix}$$

$$f(2.2675, 0.9254) = 1.0963$$

$$g(2.2675, 0.9254) = 3.4510$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.2675 \\ 0.9254 \end{bmatrix} - J_1^{-1} \begin{bmatrix} 1.0963 \\ 3.4510 \end{bmatrix}$$

$$J_1^{-1} = \frac{1}{-40.4951} \begin{bmatrix} 2.5691 & -4.1183 \\ -15.4247 & 5.4604 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.0373 \\ 0.9645 \end{bmatrix}$$

Perform one more iteration

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \ ?$$

Q - Solve

$$10x + \sin(x+y) = 1$$
$$8y - \cos^2(z-y) = 1$$
$$12z + \sin z = 1$$

I.C

$x_0 = \frac{1}{10}$

$y_0 = \frac{1}{4}$

$z_0 = \frac{1}{2}$

✓ $f(x,y,z) = 10x + \sin(x+y) - 1$

✓ $g(x,y,z) = 8y - \cos^2(z-y) - 1$

✓ $h(x,y,z) = 12z + \sin z - 1$

$$J_k = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} = \begin{bmatrix} 10+\cos(x+y) & \cos(x+y) & 0 \\ 0 & 8-\sin(2(z-y)) & \boxed{} \\ 0 & 0 & 12+\cos z \end{bmatrix}$$

$\sin 2(z-y)$

$$J_0 = \begin{bmatrix} 10.939373 & 0.939373 & 0 \\ 0 & 8.32719 & -0.32719 \\ 0 & 0 & 12.99653 \end{bmatrix}$$

$$J_0^{-1} = \begin{bmatrix} 0.0914 & -0.011 & -0.0002 \\ 0 & 0.12 & 0.003 \\ 0 & 0 & 0.0769 \end{bmatrix}$$ ✓

$$f(x_0, y_0, z_0) = 0.342898$$
$$g(x_0, y_0, z_0) = 0.027522$$
$$h(x_0, y_0, z_0) = 0.083237$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} - J_0^{-1} \begin{bmatrix} f(x_0, y_0, z_0) \\ g(x_0, y_0, z_0) \\ h(x_0, y_0, z_0) \end{bmatrix}$$

$=$