16

Partial Differential Equations

16.1. INTRODUCTION

A differential equation which involves partial derivatives is called a *partial differential equation*.

For example,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \quad \text{...(1)}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{...(2)}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial z}\right)^3 \quad \text{...(3)}$$

are partial differential equations.

The *order* of a partial differential equation is the order of the highest ordered partial derivative in the equation. The *degree* of a partial differential equation is the degree of the highest ordered partial derivative occurring in the equation.

Thus, equation (1) is of first order, equations (2) and (3) are of second order. The degree of all the above equations is one.

If z is a function of two independent variables x and y, then we shall use the following notation for the partial derivatives of z :

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t.$$

16.5. LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

A differential equation involving first order partial derivatives p and q only is called a *partial differential equation of the first order*. If p and q both occur in the first degree only and are not multiplied together, then it is called a *linear partial differential equation of the first order*.

16.6. LAGRANGE'S LINEAR EQUATION

The partial differential equation of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z is the standard form of a linear partial differential equation of the first order and is called *Lagrange's Linear Equation*.

Now Lagrange's linear equation $\qquad Pp + Qq = R \qquad$ ...(1)

is obtained by eliminating an arbitrary function ϕ from $\qquad \phi(u, v) = 0 \qquad$ ...(2)

where u, v are some definite functions of x, y, z.

Differentiating (2) partially w.r.t. x and y

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right) = 0$$

and

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) = 0$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$, we get

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right)\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) - \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right)\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) = 0$$

or

$$\left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

which is the same as (1) with

$$P = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}, \quad Q = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}, \quad R = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}.$$

To determine u, v from P, Q, R, suppose $u = a$ and $v = b$, where a, b are constants, so that

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = dv = 0$$

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}}$$

or

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad ...(3)$$

The solutions of these equations are $u = a$ and $v = b$. Thus determining u, v from the simultaneous equations (3), we have the solution of the partial differential equation.

$$Pp + Qq = R \quad \text{as} \quad \phi(u, v) = 0 \quad \text{or} \quad u = f(v).$$

Note. Equations (3) are called **Lagrange's auxiliary equations** or **subsidiary equations.**

16.7. WORKING METHOD

To solve the equation $\quad Pp + Qq = R$

(*i*) form the auxiliary equations $\quad \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

(*ii*) solve the auxiliary equations by the method of grouping or the method of multipliers or both to get two independent solutions $u = a$ and $v = b$, where a, b are arbitrary constants.

(*iii*) then $\phi(u, v) = 0$ or $u = f(v)$ is the general solution of the equation $Pp + Qq = R$.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following differential equations :*

(*i*) $\frac{y^2 z}{x} p + xzq = y^2$ $\qquad$ (*ii*) $pz - qz = z^2 + (x + y)^2$

(*iii*) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy.$

Sol. (*i*) The given equation can be written as $\quad y^2zp + x^2zq = xy^2$

Comparing with $Pp + Qq = R$, we have $\quad P = y^2z, \quad Q = x^2z, \quad R = xy^2$

$\therefore$ The auxiliary equations are $\quad \frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{xy^2}$

Taking the first two members, we have $\quad x^2dx = y^2dy$

which on integration gives $\quad x^3 - y^3 = a \qquad ...(1)$

Again taking the first and third members, we have $\quad xdx = zdz$

which on integration gives $\quad x^2 - z^2 = b \qquad ...(2)$

From (1) and (2), the general solution is $\quad \phi(x^3 - y^3, x^2 - z^2) = 0.$

(*ii*) Here the auxiliary equations are $\quad \frac{dx}{z}=\frac{dy}{-z}=\frac{dz}{z^2+(x+y)^2}$

Taking the first two members, we have $\quad dx + dy = 0$

which on integration gives $\quad x + y = a \qquad ...(1)$

Again taking the first and third members, we have

$$dx = \frac{zdz}{z^2 + a^2}, \text{ since } x + y = a$$

or $$\frac{2z\,dz}{z^2+a^2} = 2dx$$

which on integration gives $\log(z^2+a^2) = 2x + b$ or $\log[z^2+(x+y)^2] - 2x = b$ ...(2)

From (1) and (2), the general solution is

$$\phi[x+y, \log(x^2+y^2+z^2+2xy) - 2x] = 0.$$

(*iii*) Here the auxiliary equations are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \quad ...(1)$$

$$\therefore \quad \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)}$$

or $$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

which on integration gives $\log(x-y) = \log(y-z) + \log a$

or $$\log\left(\frac{x-y}{y-z}\right) = \log a \quad \text{or} \quad \frac{x-y}{y-z} = a \quad ...(2)$$

Using x, y, z as multipliers,

$$\text{each fraction of (1)} = \frac{x\,dx + y\,dy + z\,dz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \quad ...(3)$$

Also, each fraction of (1) $= \dfrac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$ ...(4)

From (3) and (4), $\dfrac{x\,dx + y\,dy + z\,dz}{x+y+z} = dx + dy + dz$

$$xdx + ydy + zdz = (x+y+z)\,d(x+y+z)$$

Integrating $$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} + c$$

$$x^2+y^2+z^2 = (x+y+z)^2 + 2c$$

$$2xy + 2yz + 2zx + 2c = 0$$

$$xy + yz + zx = b, \quad \text{where } b = -c \quad ...(5)$$

From (2) and (5), the general solution is

$$\phi\left(\frac{x-y}{y-z}, xy+yz+zx\right) = 0.$$

Example 2. *Solve the following differential equations :*

(*i*) $(mz - ny)p + (nx - lz)q = ly - mx$ (*ii*) $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$.

Sol. (*i*) Here the auxiliary equations are $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx-lz} = \dfrac{dz}{ly-mx}$

Using x, y, z as multipliers, we get

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\therefore \quad xdx + ydy + zdz = 0$$

which on integration gives $\quad x^2 + y^2 + z^2 = a \quad$ ...(1)

Again using l, m, n as multipliers, we get

$$\text{each fraction} = \frac{ldx + mdy + ndz}{0}$$

$$\therefore \quad ldx + mdy + ndz = 0$$

which on integration gives $\quad lx + my + nz = b \quad$ ...(2)

From (1) and (2), the general solution is $x^2 + y^2 + z^2 = f(lx + my + nz)$.

(*ii*) Here the auxiliary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers, we get

$$\text{each fraction} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\therefore \quad \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

which on integration gives $\quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a \quad$ ...(1)

Again using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

which on integration gives $\quad \log x + \log y + \log z = \log b$

or $\quad \log (xyz) = \log b \quad \text{or} \quad xyz = b \quad$ ...(2)

From (1) and (2), the general solution is $\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0.$

Example 3. *Solve the following differential equations :*

(*i*) $(x^2 - y^2 - z^2)p + 2xyq = 2xz$ $\qquad$ (*ii*) $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx.$

Sol. (*i*) Here the auxiliary equations are

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Taking the last two members, we have

$$\frac{dy}{y} = \frac{dz}{z}$$

which on integration gives $\log y = \log z + \log a$

or $\log \frac{y}{z} = \log a$ or $\frac{y}{z} = a$ ...(1)

Using x, y, z as multipliers, we get

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}$$

$$\therefore \quad \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)} = \frac{dz}{2xz}$$

or $$\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

which on integration gives $\log (x^2 + y^2 + z^2) = \log z + \log b$

or $\log \left(\frac{x^2 + y^2 + z^2}{z}\right) = \log b$ or $\frac{x^2 + y^2 + z^2}{z} = b$ ...(2)

From (1) and (2), the general solution is

$$\frac{x^2 + y^2 + z^2}{z} = f\left(\frac{y}{z}\right) \quad \text{or} \quad x^2 + y^2 + z^2 = z\, f\left(\frac{y}{z}\right).$$

(*ii*) Here the auxiliary equations are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}$$

Taking x, y, z as multipliers, we have

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$ ...(1)

which on integration gives $x^2 + y^2 + z^2 = a$

Again, taking the last two members, we have

$$\frac{dy}{y + z} = \frac{dz}{y - z}$$

or $(y - z)dy = (y + z)dz$

or $y\,dy - (z\,dy + y\,dz) - z\,dz = 0$

or $y\,dy - d(yz) - z\,dz = 0$ ...(2)

which on integration gives $y^2 - 2yz - z^2 = b$

From (1) and (2), the general solution is $\phi(x^2 + y^2 + z^2, y^2 - 2yz - z^2) = 0$.