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MATHS

ASSIGNMENT - 4

HEAT EQUATION

UMANG
2018 UIC 3127
ICE-2

Ques 1. A rod of length 'l' with insulated sides is initially at a temperature $u_0(x)$. Its ends are suddenly cooled to 0°C and are kept at that temperature. Prove that the temp. fn $u(x,t)$ is given by $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\frac{c^2\pi^2 n^2}{l^2}t}$

where b_n is determined from the equation

$$b_n = \frac{2}{l}\int_0^l u_0(x) \sin\frac{n\pi x}{l}\, dx.$$

sol.

let $u(x,t) = (A\cos kx + B\sin kx)e^{-k^2c^2t}$ be the general soulution of the heat eq^n

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} \quad \text{---} \quad (1)$$

since the ends $(0=x, x=l)$ are cooled to 0°c and kept at that temp throughout, we have

$$u(0,t) = u(l,t) = 0 \quad \text{for all } t$$

$u(x,0) = u_0(x)$ is the initial condition —(2)

From (1) & (2),

$$Ae^{-k^2c^2t} = 0. \Rightarrow A = 0$$

and $B\sin kl e^{-k^2c^2t} = 0 \Rightarrow \sin kl = 0$

Date..................

or $kl = n\pi$ or $k = \frac{n\pi}{l}$

Hence, $u(x,t) = b_n \sin \frac{n\pi x}{l} e^{\frac{-n^2\pi^2 ct}{l^2}}$

on replacing B by B_n

The most general solⁿ is obtained by adding all such solutions for $n = 1, 2, 3 \ldots$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{\frac{-n^2\pi^2 c^2 t}{l^2}} \quad \text{--- (3)}$$

$t = 0$ in (3) then

$$u(x,0) = \sum b_n \sin \frac{n\pi x}{l} = u_0(x)$$

$$b_n = \frac{2}{l} \int_0^l u_0(x) \sin \frac{n\pi x}{l} \, dx$$

Ques 2. The initial temp of an insulated infinite rod is given by $u(x,0) = (-1)^n U$ between $x = nc$ and $x = (n+1)c$, where $n \in I$. Show that for $t = 0$

$$u(x,t) = \frac{4U}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin\left\{(2m+1)\frac{\pi x}{a}\right\} e^{-\frac{d^2(2m+1)^2\pi^2 t^2}{c^2}}$$

Sol Initial temp is alternatively U and -U over equal distances on the infinite rod. Hence the final rod at $t = \infty$ will be the average of anytime t will be an odd periodic function of distance, with period 2c. It will satisfy the conditions :-

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i) $u = 0$ at $x = 0$ & ii) $u = 0$ at $x = c$

Let the general solution of the heat eq^n $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ be $u(x,t) = A \cos kx + B \sin kx \, e^{-k^2 \alpha^2 t}$ — (2)

Applying conditions (1) & (2) we have,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{c} e^{-\frac{n^2\pi^2 \alpha^2 t}{c^2}}$$

This is an odd fn of x with period $2c$.

since $u(x,0) = U$ for $0 < x < c$ — (3)

gives.

$$U = \sum B_n \sin \frac{n\pi x}{c}$$

$$\therefore \quad B_n = \frac{2}{c} \int_0^c U \sin \frac{n\pi x}{c} dx.$$

$$= 2U \left[\frac{-c}{n\pi} \frac{\cos n\pi x}{c} \right]_0^c$$

$$= \frac{2U}{n\pi} (1 - \cos n\pi) = \frac{4U}{n\pi} \quad (\text{when } n \text{ is odd})$$

$$= 0 \quad (\text{when } n \text{ is even})$$

Let $n = 2m + 1$, So that

$$B_n = \frac{4U}{(2m+1)\pi} \quad \text{— (4)}$$

Now putting the value of B_n from (4) in (3) we get

$$u(x,t) = \frac{4U}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin \frac{(2m+1)\pi x}{c} e^{-\frac{(2m+1)^2 \alpha^2 \pi^2 t}{c}}$$