Q1) Controlling matrix of the system

$M = [B \,\vdots\, AB \,\vdots\, A^2B]$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 31 \end{bmatrix}$$

so $M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$

$|M| = -1$ so rank of $M = 3$ means system is completely state controllable

1st Method

$$|sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{vmatrix}$$

$|sI - A| = s(s^2 + 6s + 5) - (-1)(0+1) + 0$

$|sI - A| = s^3 + 6s^2 + 5s + 1 = 0$

Comparing with

$s^n + a_1 s^{n-1} + \cdots \quad a_{n-1} s + a_n$

$a_1 = 6, \; a_2 = 5, \; a_3 = 1$

$T = MW$

$$M = [B \,|\, AB \,\vdots\, A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

$$W = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Desired characteristic eqn

$(s-\mu_1)(s-\mu_2)(s-\mu_3) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$

$(s+2-j4)(s+2+j4)(s+10) = ((s+2)^2+16)(s+10)$

$= (s^2+4s+20)(s+10)$

$= s^3 + 14s^2 + 60s + 200$

so $\alpha_1 = 14$, $\alpha_2 = 60$, $\alpha_3 = 200$

$K = [\alpha_3 - a_3 \,\vdots\, \alpha_2 - a_2 \,\vdots\, \alpha_1 - a_1] T^{-1}$ $\quad [T = MW]$

$K = [200-1 \,\vdots\, 60-5 \,\vdots\, 14-5]$ (state eqn is controllable canonical form so $T=1$)

2nd method

Defining desired state feedback gain matrix $K = [k_1 \; k_2 \; k_3]$

The desired characteristics eqn is $|sI - A + BK| = 0$

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \; k_2 \; k_3] = 0$$

$$\left| \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1+k_1 & 5+k_2 & s+6+k_3 \end{vmatrix} = 0$$

$$s^3 + (6+k_3)s^2 + (5+k_2)s + (1+k_1) = (s+2-j4)(s+2+j4)(s+10)$$

$$= (s^2+4s+20)(s+10)$$

$$= s^3 + 14s^2 + 60s + 200$$

$6 + k_3 = 14 \Rightarrow k_3 = 8$

$5 + k_2 = 60 \Rightarrow k_2 = 55$

$1 + k_1 = 200 \Rightarrow k_1 = 199$

$K = [199 \quad 55 \quad 8]$.

Method 2 Ackerman's Formula:

$K = [0 \ 0 \ 1] \ [B \,;\, AB \,;\, A^2B]^{-1} \ \phi(A)$

since char eq at A is $s^3 + 6s^2 + 5s + 1 = 0$

$(s+2-j4)(s+2+j4) \cdots \Rightarrow s^3 + 14s^2 + 60s + 200 = 0$

$\phi(A) = A^3 + 14A^2 + 60A + 200I$

$$A^3 = A \cdot A \cdot A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -5 & -6 \\ 6 & 29 & 31 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & -5 & -6 \\ 6 & 29 & 31 \\ -31 & -149 & -157 \end{bmatrix}$$

$\phi(A) = A^3 + 14A^2 + 60A + 200I$

$$= \begin{bmatrix} -1 & -5 & -6 \\ 6 & 29 & 31 \\ -31 & -149 & -157 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 14 \\ -14 & -70 & -84 \\ 84 & 406 & 434 \end{bmatrix} + \begin{bmatrix} 0 & 60 & 0 \\ 0 & 0 & 60 \\ -60 & -300 & -360 \end{bmatrix} + \begin{bmatrix} 200 \\ 0 \end{bmatrix}$$

$M = [B \; AB \; A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$

$|M| = -1$

$M^{-1} = \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

so $K = [0 \; 0 \; 1] \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$

$K = [1 \; 0 \; 0] \begin{bmatrix} 199 & 55 & 8 \\ -8 & 153 & 7 \\ -7 & -43 & 117 \end{bmatrix}$

$K = [199 \;\; 55 \;\; 8] A$

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