MATHS

ASSIGNMENT -3

MANAN
2018UIC3087
ICE - 2

Non Linear Partial Differential Equation:

I) $f'(p,q)=0$

The complete solution is given by

$$z = ax + by + c \quad —①$$

where a, b are connected by the relation

$f(a,b) = 0 \quad —②$

from ①, we get $p = \frac{dz}{dx} = a$

$q = \frac{\partial z}{\partial y} = b$ From ② we get b in terms of a.

let $b = \phi(a)$; here

$$z = ax + \phi(a)y + c$$

Q1) solve $pq + q + p = 0$

$$z = ax + by + c$$

where

$$ab + a + b = 0 \quad \text{or} \quad b = \frac{-a}{1+a}$$

hence $z = ax - \frac{ay}{1+a} + c$

Q2) solve $x^2p^2 + y^2q^2 = z^2$

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sol. $\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$ —— (1)

Now let $\frac{\partial x}{x} = \partial X$, $\frac{\partial y}{y} = \partial Y$, $\frac{\partial z}{z} = \partial Z$

$X = \log x$ $Y = \log y$ $Z = \log z$

Hence, $\frac{\partial Z}{\partial X} = \frac{x}{z}\frac{\partial z}{\partial x}$ &

$\frac{\partial Z}{\partial Y} = \frac{y}{z}\frac{\partial z}{\partial y}$ —— (2)

From (1) & (2), we get

$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$ or $P^2 + Q^2 = 1$ —— (3)

where, $P = \frac{\partial Z}{\partial X}$, $Q = \frac{\partial Z}{\partial Y}$

The complete sol^n of (3) is given by
$z = ax + by + c$ where $a^2 + b^2 = 1$

or $b = \sqrt{1-a^2}$

$\log z = a\log x + \sqrt{1-a^2}\log y + c$
$\log z = \log x^a + \log y^{\sqrt{1-a^2}} + \log K$
$\log z = \log\{K x^a y^{\sqrt{1-a^2}}$

$$\boxed{z = K x^a y^{\sqrt{1-a^2}}}$$

Q3. $p^2 + q^2 = 1$

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sol Let $f(p, q) = p^2 + q^2 - 1 = 0$

$z = ax + by + c$

$a^2 + b^2 = 1 \qquad b = \sqrt{1-a^2}$

$$z = ax + \sqrt{1-a^2}\, y + c$$

Q4. $p^2 - q^2 = 4$

solⁿ $f(p, q) \Rightarrow p^2 - q^2 - 4 = 0$

$z = ax + by + c \qquad a^2 - b^2 = -4$

$b = \sqrt{-4-a^2} \qquad z = ax + \sqrt{4+a^2}\, i y + c$

Q5. $q = 3p^2$

solⁿ $f(p, q) \Rightarrow q = 3p^2 \qquad q - 3p^2 = 0$

$z = ax + by + c \qquad a - 3b^2 = 0 \qquad b = \sqrt{\frac{a}{3}}$

$$\therefore \quad z = ax + \sqrt{\frac{a}{3}}\, y + c$$

II $f(z, p, q) = 0$ — (I)

Let $z = \phi(x + ay) = \phi(t)$, where $t = x + ay$

Now, $p = \frac{\partial z}{\partial x} = \phi'(x + ay) = \frac{dz}{dt}$

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& $q = \frac{\partial z}{\partial y} = \phi'(x+ay).a = \frac{a\,dz}{dt}$

substituting p & q in (1).

$f\left(z, \frac{dz}{dt}, \frac{a\,dz}{dt}\right) = 0$

Q1. $z^2(p^2+q^2+1) = 1$

sol^n Let $t = x+ay$ so that $p = \frac{dz}{dt}$

now the given eq^n reduced to,

$$z^2\left[\left(\frac{dz}{dt}\right)^2 + a^2\left(\frac{dz}{dt}\right)^2 + 1\right] = 1$$

$$z^2(1+a^2)\left(\frac{dz}{dt}\right)^2 = 1-z^2$$

$$\int \sqrt{1+a^2}\,\frac{dz}{dt} = \int \pm \frac{\sqrt{1-z^2}}{z}$$

$$\sqrt{1-z^2} = \pm \frac{t}{\sqrt{1+a^2}} + c$$

$$\sqrt{1-z^2} = \pm \frac{t}{\sqrt{1+a^2}} + c$$

Q2. $z^2(p^2x^2+q^2y^2) = 1$

$$z^2\left[\left(x\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right] = 1 \quad \text{—— (1)}$$

Let $X = \log x$, so that $x\frac{dz}{dx} = \frac{\partial z}{\partial X}$

$$z^2\left[\left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{dz}{dy}\right)^2\right] = 1$$

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$z^2[p^2+q^2] = 1$

$\Rightarrow$ Now let $t = x + ay$

$p = \frac{dz}{dx} = \frac{dz}{dt}$, $q = \frac{\partial z}{\partial y} = \frac{a\,dz}{dt}$

$(1+a^2) z^2 \left(\frac{dz}{dt}\right)^2 = 1$

$\sqrt{1+a^2}\; z \left(\frac{dz}{dt}\right) = \pm 1$

$\sqrt{1+a^2}\; \frac{z^2}{2} = \pm t + c$

$\frac{z^2}{2}\sqrt{1+a^2} = \pm (x+ay) + c$

$\Rightarrow z^2\sqrt{1+a^2} = \pm 2(\log x + ay) + b$

where $b = 2c$.

Q4. $pz = 1 + q$

soln $f(p, q, z) \Rightarrow pz - 1 - q = 0$

$X = x + ay$

$p = \frac{\partial z}{\partial X}$ & $q = a\frac{\partial z}{\partial X}$

$\left(\frac{dz}{dX}\right) z - 1 - \frac{a\,dz}{dX} = 0 \Rightarrow \frac{dz}{dt} = \frac{1}{z-a}$

$\int (z-a)\, dz = \int dX$

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$$\frac{z^2}{2} - az = X + c$$

$$z(z-2a) = 2(x+ay) + 2c$$

$$z(z-2a) = 2(x+ay) + b$$

Q5. $p^2 = zq$

$f(p, z, q) \Rightarrow p^2 - zq = 0$

put $X = x + ay$.

$p = \frac{\partial z}{\partial x}$ & $q = a\frac{\partial z}{\partial x}$

$\left(\frac{\partial z}{\partial x}\right)^2 = za\frac{\partial z}{\partial x}$ $\qquad \frac{dz}{dx} = za$

similarly,

$\int \frac{dz}{z} = \int a\,dx$ $\qquad \log z = ax + c$

$z = e^{ax} + c \Rightarrow z = c_2 e^{ax}$

III $f_1(x, p) = f_2(y, q)$

Let $f_1(x, p) = a$ & $f_2(y, q) = a$ — (1)

Now solve these eq^n for p & q and put them in

$$dz = p\,dx + q\,dy \quad \text{— (2)}$$

Integration of (2) gives the solution

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Q1. Solve $p - x^2 = q + y^2$

Sol^n Let $p - x^2 = q + y^2 = a$

then
$$p = x^2 + a \quad \& \quad q = a - y^2$$
$$dz = (x^2+a)\,dx + (a-y^2)\,dy$$
$$z = \frac{x^3}{3} + ax + ay - \frac{y^3}{3} + c$$
$$z = \frac{1}{3}(x^3 - y^3) + a(x+y) + c$$

Q2. $p^2 - q^2 = x - y$

sol^n Let $p^2 - x = q^2 - y = a$

then $p = (x+a)^{1/2}$ & $q = (y+a)^{1/2}$
$$dz = (x+a)^{1/2}\,dx + (y+a)^{1/2}\,dy$$
$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + c$$

Q3. $z^2(p^2+q^2) = x^2+y^2$
$$\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2+y^2$$

Let $z\,dz = dZ$ so that $Z = \frac{z^2}{2}$

Now $\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} = z\frac{\partial Z}{\partial x} = P$ let

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& $\frac{\partial z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = z\frac{\partial z}{\partial y} = Q$ let

hence, $P^2 + Q^2 = x^2 + y^2$

$P^2 - x^2 = y^2 - Q^2 = a^2$ let

then,

$P = \sqrt{x^2+a^2}$ & $Q = \sqrt{y^2-a^2}$

$dZ = \sqrt{ax^2+a^2}\,dx + \sqrt{y^2-a^2}\,dy$

$Z = \frac{x}{2}\sqrt{x^2+a^2} \pm \frac{a^2}{2}\log\left(y+\sqrt{y^2-a^2}\right) + c$

$+ \frac{y}{2}\sqrt{y^2-a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$

$z^2 = x\sqrt{x^2+a^2} + y\sqrt{y^2-a^2} + a^2\log\left(\frac{x+\sqrt{x^2+a^2}}{y-\sqrt{y^2-a^2}}\right)$
$+ b$

Q4. $\sqrt{p} + \sqrt{q} = 2x$

$\sqrt{p} - 2x = -\sqrt{q}$ let $\sqrt{p} - 2x = a = -\sqrt{q}$

$\sqrt{P} = a + 2x \Rightarrow P = (a+2x)^2, \quad q = a^2$

using P & Q in

$dz = p\,dx + q\,dy.$

$dz = (a+2x)^2\,dx + a^2\,dy$

Integrating $z = \frac{(a+2x)^3}{6} + a^2x + b$

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Q5. $q = xyp^2$

sol^n $xp^2 = q/y$ let $xp^2 = a$ & $q/y = a$

$p = \sqrt{a/x}$ & $q = ay$.

using p & q in $dz = pdx = qdy$

$dz = \sqrt{a/x}\,dx + ay\,dy$

integrating $z = 2\sqrt{a}\sqrt{x} + \frac{ay^2}{2} + b$.

$2z = 4\sqrt{ax} + ay^2 + 2b$.

squaring,

$16ax - (2z - ay^2 - 2b)^2 = 0$

IV $z = px + qy + f(p, q)$.

solution is given by $z = ax + by + f(a,b)$

it can be seen, ——— (1)

$p = \frac{\partial z}{\partial x} = a$ & $q = \frac{\partial z}{\partial y} = b$

Then complete sol^n of (1) consists of 2 family of planes.

Q1. $z = px + qy + p^2 + q^2$

sol^n since, it is in standard form, the

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complete soulition is obtained by writing a & b for p & q.

$$\therefore \quad z = ax + by + a^2 + b^2$$

Q2. $4xyz = pq + 2px^2y + 2qxy^2$

Let $x = \sqrt{X}$ & $y = \sqrt{Y}$ then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{dX}{\partial x} = 2\sqrt{X}\frac{\partial z}{\partial X}$$

$$q = 2\sqrt{Y}\frac{\partial z}{\partial Y}$$

$$z = X\frac{\partial z}{\partial X} + Y\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial y}$$

$$z = XP + YQ + PQ.$$

$$z = aX + bY + ab.$$

$$z = ax^2 + bx^2 + ab$$

Q3. $z = px + qy + c\sqrt{1+p^2+q^2}$

differentiate wrt. a & b,

$$\frac{\partial f}{\partial a} = -x - \frac{ca}{\sqrt{1+a^2+b^2}}$$

$$\frac{\partial f}{\partial b} = -y - \frac{cb}{\sqrt{1+a^2+b^2}}$$

$$-x - \frac{ca}{\sqrt{1+a^2+b^2}} = 0 \quad \& \quad -y - \frac{cb}{\sqrt{1+a^2+b^2}} = 0$$

$$\frac{x^2}{c^2a^2} = \frac{1}{1+a^2+b^2} \quad \& \quad \frac{y^2}{c^2b^2} = \frac{1}{1+a^2+b^2}$$

Hence, $\dfrac{c^2a^2}{x^2} = \dfrac{c^2b^2}{y^2} \Rightarrow \quad a\cancel{b} = \dfrac{bx}{y}$

$$\frac{c^2b^2x^2}{x^2y^2} = 1 + \frac{b^2x^2}{y^2} + b^2$$

$$c^2b^2 = y^2 + b^2x^2 + b^2y^2$$

$$(c^2 - x^2 - y^2)\,b^2 = y^2$$

$b = \dfrac{y}{\sqrt{c^2-x^2-y^2}}$ similarly $a = \dfrac{x}{\sqrt{c^2-x^2-y^2}}$

$$z = \frac{x^2}{\sqrt{c^2-x^2-y^2}} + \frac{y^2}{\sqrt{c^2-x^2-y^2}} + c\sqrt{1+\frac{x^2+y^2}{\sqrt{c^2-x^2-y^2}}}$$

$$\Rightarrow \quad \underline{z^2(c^2-x^2-y^2) = (x^2+y^2+c^2)^2}$$

Q4. $(px + qy - z)^2 = 1 + p^2 + q^2$

$$z = px + qy \pm \sqrt{1+p^2+q^2}$$

hence complete integral is

$$\underline{z = ax + by \pm \sqrt{1+a^2+b^2}}$$

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Q5.) $(p+q)(z - xp - yq) = 1$

$$z = xp + yq + \frac{1}{p+q}$$

putting $p = a$ & $q = b$

$$z = ax + by + \frac{1}{a+b}$$

x ———— x

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