ICC18 # Control System II

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→ **State Variable Analysis :**

We can use it for Analysis two or more systems.

→ **Slide bullet text :**

Matrices :

Matrix ⇒

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

rectangular array of mn elements n rows, m columns.

→ **Upper triangular matrix :** U has all its elements below the principal diagonal ~~signal~~ ~~to~~ ~~for~~ equal to zero; $U_{ij} = 0$ if $i > j \rightarrow 1 \leq i \leq n,$ $1 \leq j \leq m$.

$$U = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1m} \\ 0 & U_{22} & \cdots & U_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & U_{nm} \end{bmatrix}$$

→ Matrix transpose :

If the rows & columns of an $n \times m$ matrix A are interchanged, the resulting $m \times n$ matrix, denoted as A^T, is called the transpose of the matrix A, namely, if A is given by eqⁿ.

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$$

Some properties of the matrix transpose are

(i) $(A^T)^T = A$

(ii) $(kA)^T = kA^T$. k is a scalar

(iii) $(A+B)^T = A^T + B^T$

(iv) $(AB)^T = B^T A^T$

→ Determinant of a matrix :

Determinats are defined for square matrices only. The determinant of the $n \times n$ matrix A, written as $|A|$ is a scalar valued function of A. It is found through the use of minors & cofactors.

The minor m_{ij} of the element a_{ij} is the determinant of a matrix of order $(n-1) \times (n-1)$ obtained from A by remaining the row & the column ~~obkusee~~ containing a_{ij}.

The cofactor c_{ij} of the element a_{ij} is defined by the eq^n

$$a_{j} = (-1)^{i+j} m_{ij}$$

Determinants can be evoluated by the method of Laplace expansion. If A is nxn matrix, any arbitrary row k can be selected and |A| is the given by

$$|A| = \sum_{j=1}^{n} a_{kj} C_{kj}$$

Laplace expansion can be carried out wrt any arbitary column l.

$$|A| = \sum_{i=1}^{n} a_{ij} C_{il}$$

Laplace expansion reduces the evolution of an $n \times n$ det to evolution of a string of $(n-1) \times (n-1)$ dets cofactors.

* Properties:

(i) $\det AB = (\det A)(\det B)$

(ii) $\det A^T = \det A$

(iii) $\det kA = k^n \det A$ $\quad A \to n \times n$ matrix, k = scalar

(iv) det of any diag or A matrix is the product of its diagonal elements.

→ ~~Sig~~ Singular matrix: A square matrix is called singular if the assoc determinant is zero.

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→ **Non Singular Matrix :** A square matrix is a non singular if det is non zero.

→ **Adjoint Matrix :** of a square matrix A is found by replacing each element a_{ij} of matrix A by its cofactors C_{ij} & then transposing

$$\text{adj } A = A^{+}$$

$$= \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} = [C_{ji}]$$

$$A(\text{adj } A) = (\text{adj } A)A = |A| I$$

→ **Matrix inverse :** Inverse of a square matrix is written as A^{-1} and is defined by the relation

$$A^{-1}A = AA^{-1} = I$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

* properties :

$(A^{-1})^{-1} = A$

$(AB)^{-1} = B^{-1}A^{-1}$

$\det P^{-1}AP = \det A$

$(A^{T})^{-1} = (A^{-1})^{T}$

$\det A^{-1} = \frac{1}{\det A}$

Inverse of diag-matrix

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$$A^{-1} = \begin{bmatrix} 1/a_{11} & 0 & \cdots & 0 \\ 0 & 1/a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix}$$

$$\Rightarrow \text{diag}\left[\frac{1}{a_{11}}, \frac{1}{a_{22}}, \cdots \frac{1}{a_{nn}}\right]$$

→ **Block Diagonal Matrix:**

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & A_m \end{bmatrix}$$

$$= \text{diag}\,[A_1, A_2 \cdots A_m)$$

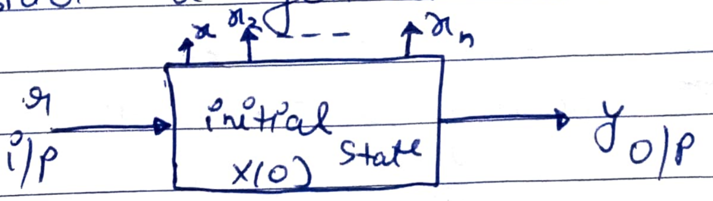
for this case

(i) $|A| = |A_1||A_2| \cdots |A_m|$

(ii) $A^{-1} = \text{diag}\,[A_1^{-1}\, A_2^{-1} \cdots A_m^{-1}]$ provided that A^{-1} exists

→ **State Variable representation:**

Consider a general SISO system



We'll denote the system state by n state variables

$\{x_1(t), x_2(t) ------ x_n(t)\}$, state vector

$$x(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

* **State Variable:** Minimal set of variable is static variable to define complete system.

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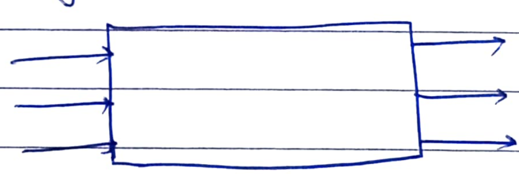
M/MO System



OR $\quad x(t) = Ax(t) + bu(t) \qquad x(t_0) \triangleq x^0$ state eq^n

$\quad\;\; y(t) = Cx(t) + du(t)$ o/p eq^n

$x(t) = n \times 1$ state vectors of n^{th} order dynamic sys

$u(t)$ = system I/P

$y(t)$ = defined o/p

d = Scalar represent direct coupling bet^n I/P + o/P (usually $d=0$)

$A = n \times n$ matrix
$b = n \times 1$ column matrix
$C = 1 \times n$ row matrix

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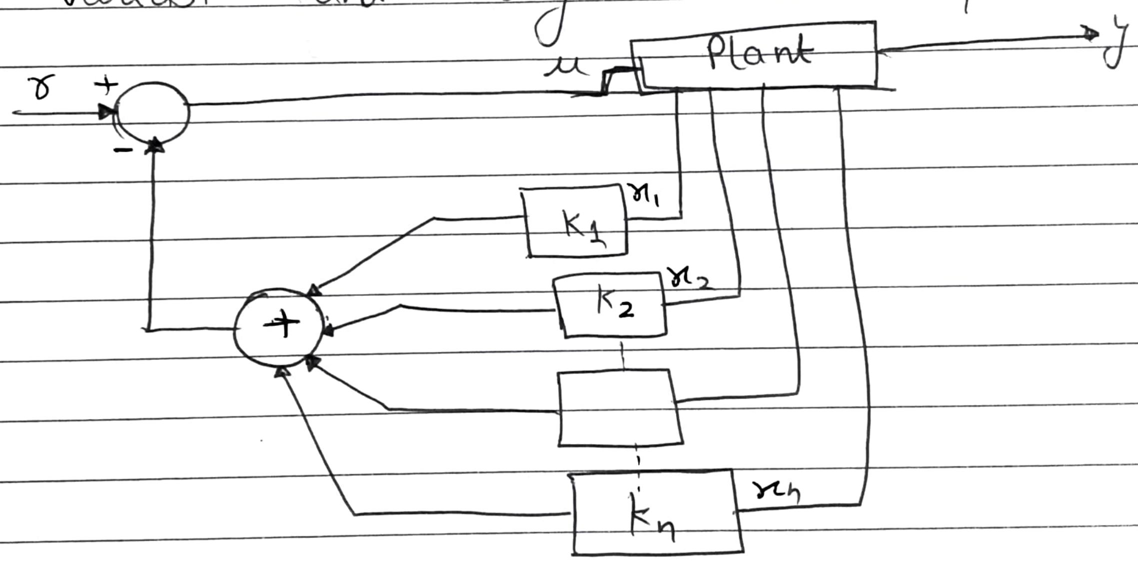
State variable representation :

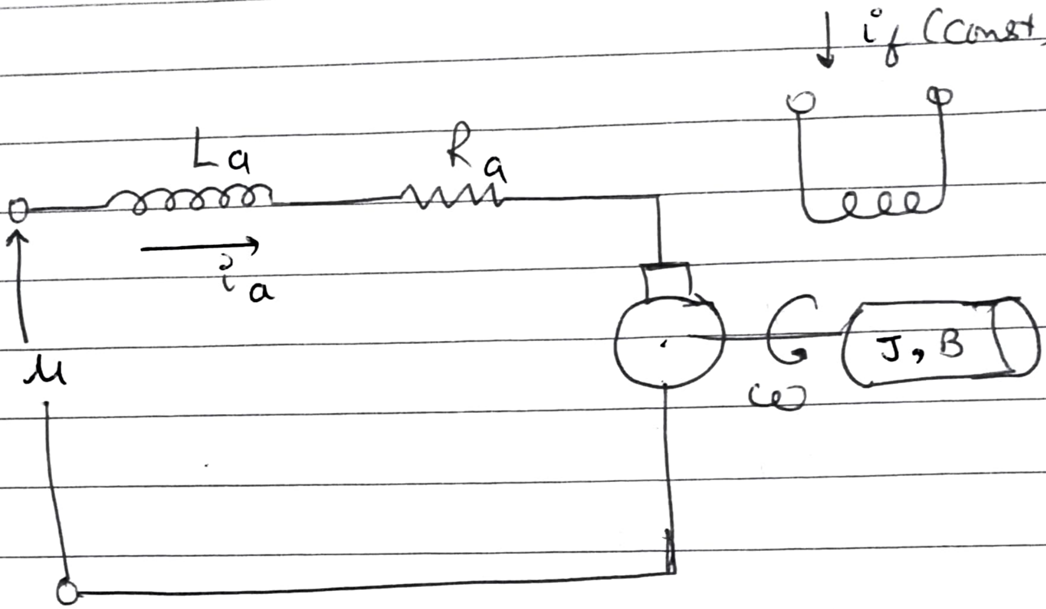
• State variable concepts :

The modelling process of linear systems involves setting up a chain of cause effect relationships, beginning from the i/p variables and ending at the o/p variables.



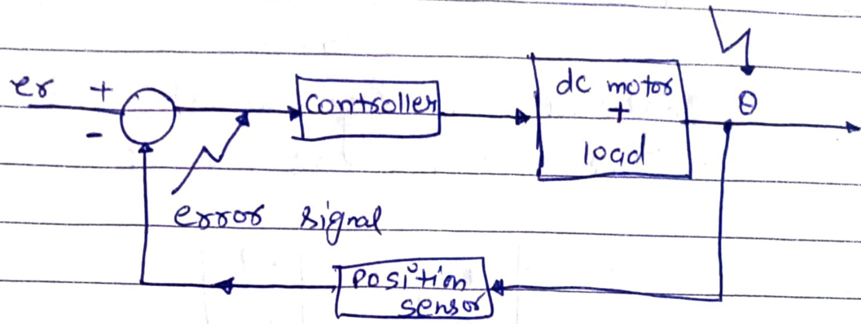
A state feedback control configuration

Two very ~~imp~~ usual applications of dc motors are in speed & position control systems.



Model of separately exited dc motor

Ex:



$$\frac{d\theta(t)}{dt} = \omega(t)$$

$$x_1(t) = \theta(t), \quad x_2(t) = \omega(t), \quad x_3(t) = i_a(t)$$

$$y(t) = \theta(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -B/J & K_T/J \\ 0 & -K_b/L_a & -R_a/L_a \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} u(t)$$

$$y(t) = x_1(t)$$

for the system parameters, the plant model for position system becomes

$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = Cx(t)$$

where,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -10 \end{bmatrix}; \quad b = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix};$$

$$C = [1 \; 0 \; 0]$$

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example :

for the system, we have taken angular velocity & $\omega(t)$ and armature current $i_a(t)$ as state variables :

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega \\ i_a \end{bmatrix}$$

We now define new state variable as

$$\bar{x}_1 = \omega, \quad \bar{x}_2 = -\omega + i_a$$

or

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We can express velocity $x_1(t)$ & armature current $x_2(t)$ in terms of variables $\bar{x}_1(t)$ & $\bar{x}_2(t)$:

$$x = P\bar{x}$$

with

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

We obtain the following state variable model for the system in terms of the transformed state vector $\bar{x}(t)$:

$$\dot{\bar{x}}(t) = \bar{A}\,\bar{x}(t) + \bar{B}\,u(t)$$

$$y(t) = \bar{C}\,\bar{x}(t)$$

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where

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & -10 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -11 & -11 \end{bmatrix}$$

$$\bar{b} = P^{-1}b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$\bar{c} = cP = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\boxed{\bar{x}_1(t_0) = x_1(t_0) \; ; \; \bar{x}_2(t_0) = -x_1(t_0) + x_2(t_0)}$$

Conversion of State Variable Models to T.F.

In the case of zero initial state (i.e $x^0 = 0$) the I/P & O/P behaviour of the system is determined entirely by the T.F.

$$\frac{Y(s)}{U(s)} = G(s) = c(sI - A)^{-1}b + d.$$

inverse of matrix $(sI - A)$ is

$$(sI - A)^{-1} = (sI - A)^{+} / |sI - A|$$

T.F. can be written as

$$G(s) = \frac{c(sI - A)^{+}b}{|sI - A|} + d$$

for

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$(sI - A) = \begin{bmatrix} s-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s-a_{nn} \end{bmatrix}$$

$$(s-a_{11})(s-a_{22}) \cdots (s-a_{nn}) = s^n + \alpha_1' s^{n-1} + \cdots \alpha_n'$$

$$|sI - A| = \Delta(s) = s^n + \alpha_1 s^{n-1} + \cdots \alpha_n$$

where α_i are constant scalars.

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* Invariance property :

It is recalled that the state variable model for a system is not unique, but depends on the choice of a set of state variables. A transformation, $x(t) = P\bar{x}(t)$; P is a non singular matrix.

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{b}u(t)$$
$$\bar{x}(t_0) = P^{-1}x(t_0)$$

$$y(t) = \bar{c}\bar{x}(t) + du(t)$$

$$\Rightarrow \bar{A} = P^{-1}AP, \quad \bar{b} = P^{-1}b, \quad \bar{c} = cP$$

(i) $|sI - \bar{A}| = |sI - P^{-1}AP| = |sP^{-1}P - P^{-1}AP|$

$$= |P^{-1}(sI-A)P| = |P^{-1}||sI-A||P| = |sI-A|$$

(ii) $\bar{G}(s) = \bar{c}(sI-\bar{A})^{-1}\bar{b} + d$

$$= cP(sI - P^{-1}AP)^{-1}P^{-1}b + d$$
$$= cP(sI - P - P^{-1}AP)^{-1}P^{-1}b + d$$
$$= cP[P^{-1}(sI-A)P]^{-1}P^{-1}b + d$$
$$= cPP^{-1}(sI-A)^{-1}PP^{-1}b + d$$

$$= c(sI-A)^{-1}b + d = G(s)$$

(iii) System o/p in response to initial state $\bar{x}(t_0)$ is

$$\bar{c}(sI-\bar{A})^{-1}\bar{x}(t_0) = cP(sI-P^{-1}AP)^{-1}P^{-1}x(t_0)$$

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$$= C(sI-A)^{-1} x(t_0)$$

Thus i/o behaviour $= C(SI-A)^{-1} x(t_0)$ is invarient under the transformation.

Ex:

$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = cx(t)$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -10 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} ; \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

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The characteristic polynomial of matrix A

$$|sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s+1 & -1 \\ 0 & 1 & s+10 \end{vmatrix}$$

$$= s(s^2 + 11s + 11)$$

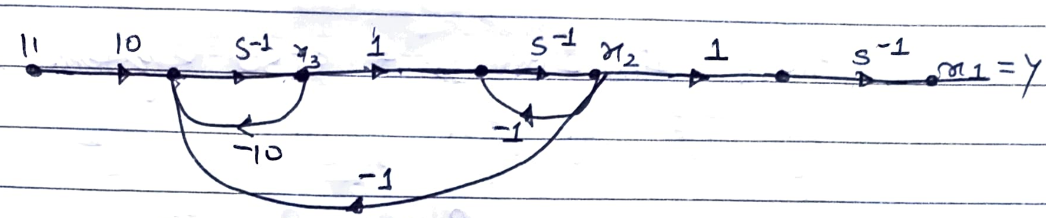
T.F :- $G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{c(sI-A)^{+} b}{|sI-A|}$

$$= \frac{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^2+11s+11 & s+10 & 1 \\ 0 & s(s+10) & s \\ 0 & -s & s(s+1) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}{s(s^2+11s+11)}$$

$$= \frac{10}{s(s^2+11s+11)}$$

$$\frac{Y(s)}{U(s)} = G(s) = \frac{10s^{-3}}{1-(-10s^{-1}-s^{-1}-s^{-2})+10s^{-3}}$$

$$= \frac{10}{s^3+11s^2+11s} = \frac{10}{s(s^2+11s+11)}$$



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Conversion of T.f to canonical state variable Models:

(i) System dynamics is determined experimentally using standard test signals like step, impulse, or sinusoidal signal.

In order to apply these techniques, experimentally obtained T.f descriptions must be realized into state variable models.

(ii) The realization of T.f into state variable models is needed even if the control system design is based on frequency-domain design methods. In these cases the need arises for the purpose of transient response simulation.

A rational function G(s) is realizable by a finite dimensional linear time-invariant state model if and only if G(s) is a proper rational function.
A proper rational function will have state model of the form:

$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = cx(t) + du(t)$$

A linear time-invariant SISO system is described by T.F of the form

$$G(s) = \frac{\beta_0 s^m + \beta_1 s^{m-1} + \cdots + \beta_m}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}, \quad m \le n$$

In the following we drive results for $m = n$. these results may be used for the case $m < n$ by setting appropriate B_i coeff $= 0$

∴ Our problem is to obtain a state variable model corresponding to the TF

$$G(s) = \frac{\beta_0 s^n + \beta_1 s^{n-1} + \cdots + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}$$

First companion form:

$$\frac{Z(s)}{U(s)} = \frac{1}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}$$

$$\left(s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n\right) Z(s) = U(s)$$

The corresponding differential eq^n is

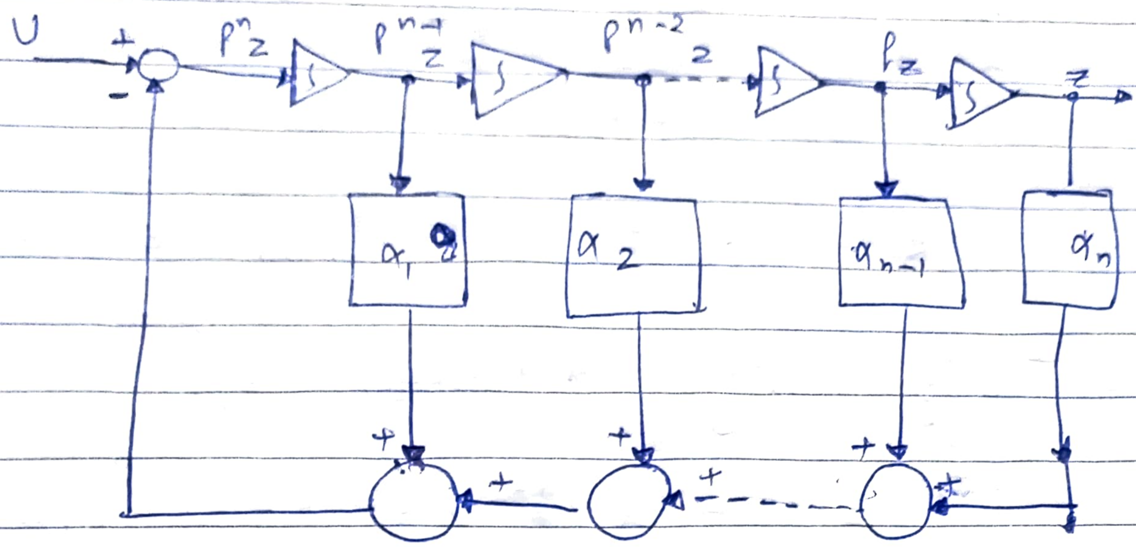
$$p^n z(t) + \alpha_1 p^{n-1} z(t) + \cdots + \alpha_n z(t) = u(t)$$

where

$$p^k z(t) \triangleq \frac{d^k z(t)}{dt^k}$$

Solving for highest derivative of $z(t)$,

$$p^n \dot{z}(t) = \alpha_1 p^{n-1} z(t) - \alpha_2 p^{n-2} z(t) - \cdots - \alpha_n z(t) + u(t$$

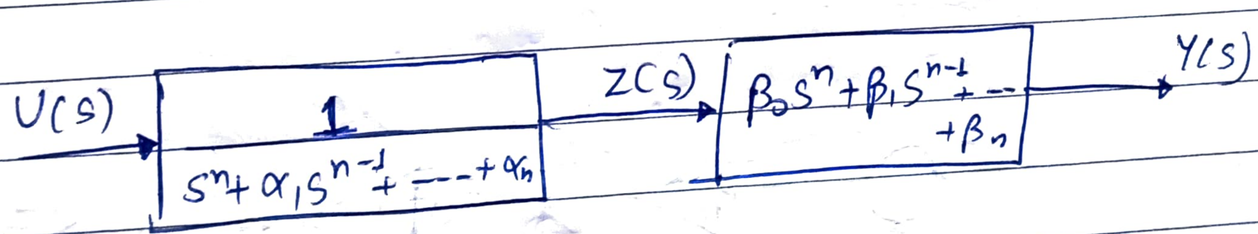


Realization of System

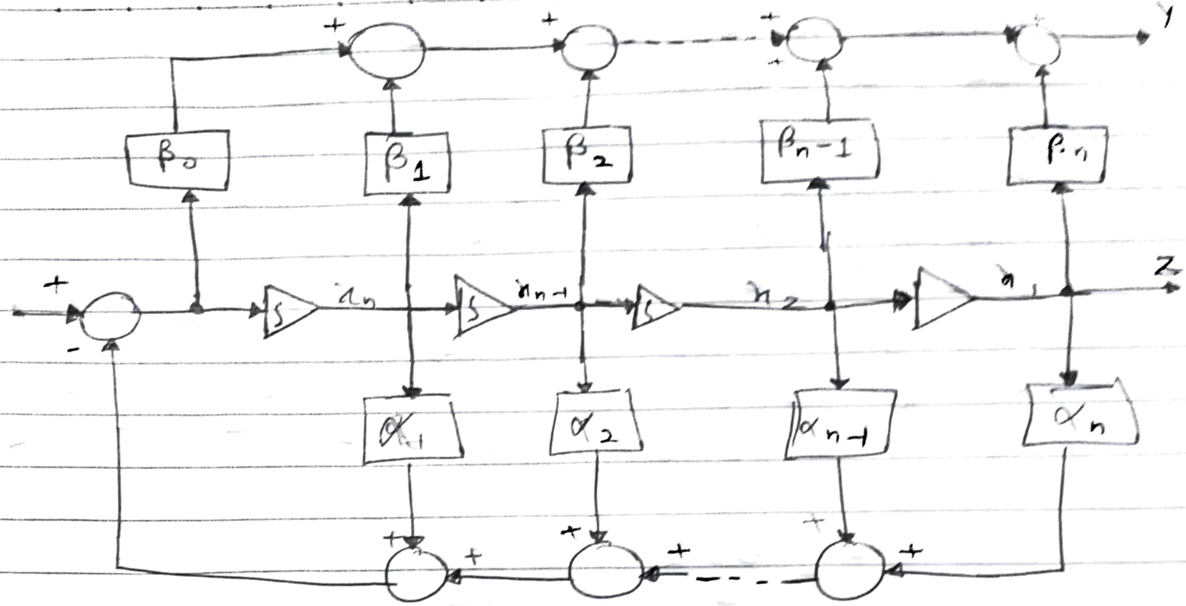
o/p → $Y(s) = (\beta_0 s^n + \beta_1 s^{n-1} + \dots \pm \beta_n) Z(s)$

$Z(s) \Rightarrow$

$$\frac{Z(s)}{U(s)} = \frac{1}{s^n + \alpha_1 s^{n-1} + \dots + \alpha_n}$$



$y(t) = \beta_0 p^n z(t) + \beta_1 p^{n-1} z(t) + \dots \beta_n z(t)$



$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\vdots$

$\dot{x}_{n-1} = x_n$

$\dot{x}_n = -\alpha_n x_1 - \alpha_{n-1} x_2 + \cdots \alpha_1 x_n + U$

$y = (\beta_n - \alpha_n \beta_0) x_1 + (\beta_{n-1} - \alpha_{n-1} \beta_0) x_2 + \cdots$

$\cdots + (\beta_1 - \alpha \beta_0) x_n + \beta_0 u$

Que: Consider the following state-space representation of a system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

$$Y = [1 0 0] \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}$$

If the transforming matrix P matrix is given as below

$$= \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

obtain the new A, B, C & D matrices after transformation

⇒ i.e.,

$$\dot{Z} = P^{-1} APZ + P^{-1} Bu$$
$$Y = CPZ$$

Sol^n

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} +$$

$$+\begin{bmatrix} 3 & 25 & 0.1 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

Simplifying gives:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} u$$

The o/p eq:

$$Y = CPZ$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Que : obtain the TF of the system whose governing eqn are as given below :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \begin{matrix} = -4x_1 - x_2 + 3u \\ = -2x_1 - 3x_2 + 5u \end{matrix}$$

$$y = x_1 + 2x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} U.$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

TF $G(s) = C[SI - A]^{-1} B$

Sol If $G(s) = C[sI - A]^{-1} B$

$$G(s) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s+4 & 1 \\ -2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s+3/s^2+7s+14 & -1/s^2+7s+14 \\ 2/s^2+7s+14 & s+4/s^2+7s+14 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \frac{13s + 56}{s^2 + 7s + 14}$$

$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = Cx(t) + du(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{bmatrix} ; \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = [\beta_n - \alpha_n \beta_0, \ \beta_{n-1} - \alpha_{n-1}\beta_0, \cdots, \beta_1 - \alpha_1 \beta_0]$$
$$d = \beta_0$$

If the direct path through β_0 is absent, then the scalar d is zero & the row matrix C contains only the β_i coefficients

Second companion form

In this companion form in which the coefficients appear in a column of the A matrix.

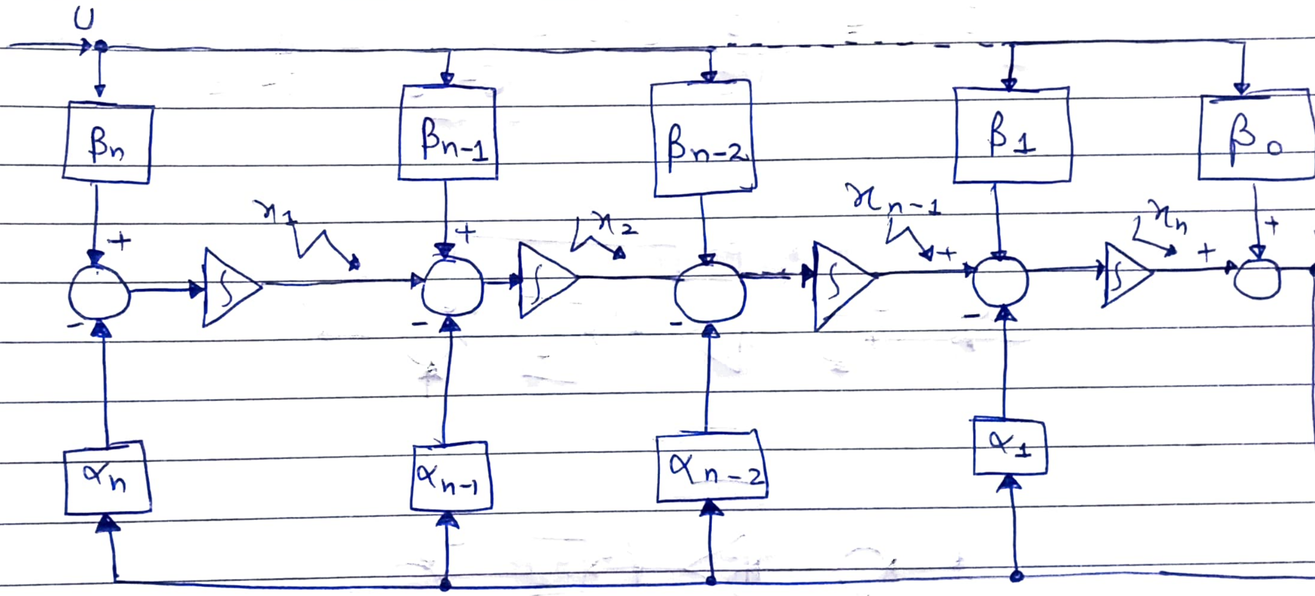
$$(s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n) Y(s) = (\beta_0 s^n + \beta_1 s^{n-1} + \cdots + \beta_n) U(s)$$

$$s^n [Y(s) - \beta_0 U(s)] + s^{n-1} [\alpha_1 Y(s) - \beta_1 U(s)] + \cdots + [\alpha_n Y(s) - \beta_n U(s)] = 0$$

on dividing by s^n & solving for $Y(s)$

$$y(s) = \beta_0 U(s) + \frac{1}{s}\left[\beta_1 U(s) - \alpha_1 y(s)\right] + \cdots + \frac{1}{s^n}\left[\beta_n U(s) - \alpha_n y(s)\right]$$

*Note that $1/s^n$ is the T.F of a chain of n integrators. Realization of $\frac{1}{s^n}\left[\beta_n U(s) - \alpha_n y(s)\right]$ requires a chain of n integrators with $[\beta_n u - \alpha_n y]$ to the first integrator in the chain from left-to-right



$$\dot{x}_n = x_{n-1} - \alpha_1 (x_n + \beta_0 u) + \beta_1 u$$
$$\dot{x}_{n-1} = x_{n-2} - \alpha_2 (x_n + \beta_0 u) + \beta_2 u$$
$$\vdots$$
$$\dot{x}_2 = x_1 - \alpha_{n-1} (x_n + \beta_0 u) + \beta_{n-1} u$$
$$\dot{x}_1 = -\alpha_n (x_n + \beta_0 u) + \beta_n u$$

O/P eq^n : $y = x_n + \beta_0 u$

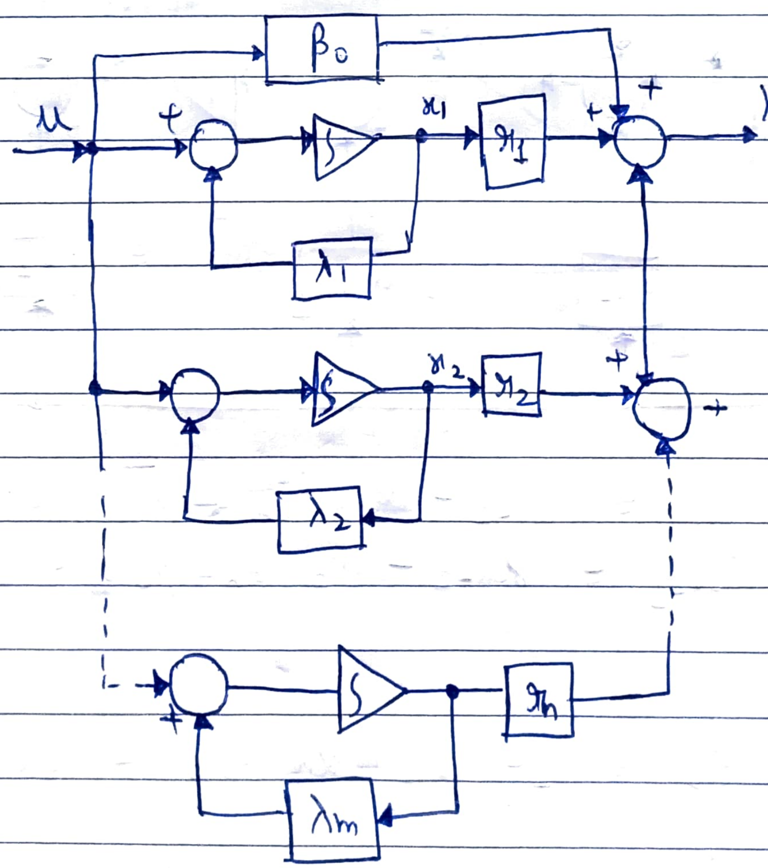
Jordan Canonical Form:

$$G(S) = \frac{\beta_0 S^n + \beta_1 S^{n-1} + \cdots + \beta_n}{S^n + \alpha_1 S^{n-1} + \cdots + \alpha_n}$$

By long division, $G(S)$ can be written as

$$G(S) = \beta_0 + \frac{\beta_1' S^{n-1} + \beta_2' S^{n-2} + \cdots + \beta_n'}{S^n + \alpha_1 S^{n-1} + \cdots + \alpha_n}$$

$$G(S) = \frac{Y(S)}{U(S)} = \beta_0 + \frac{r_1}{S-\lambda_1} + \frac{r_2}{S-\lambda_2} + \cdots + \frac{r_n}{S-\lambda_n}$$



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$$\dot{x}(t) = Ax(t) + bu(t)$$
$$y(t) = cx(t) + du(t)$$

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} ; \quad b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} ; \quad d = \beta_0$$

$$\dot{x}_i(t) = \lambda_i x_i(t) + u(t)$$

$$i = 1, 2, \cdots n$$

$$G(s) = d + \frac{p + jq}{s - (\sigma + j\omega)} + \frac{p - jq}{s - (\sigma - j\omega)} + \frac{r}{s - \lambda}$$

$$\dot{x} = Ax + bU$$
$$y = Cx + dU$$

$$A = \begin{bmatrix} \sigma + j\omega & 0 & 0 \\ 0 & \sigma - j\omega & 0 \\ 0 & 0 & \lambda \end{bmatrix} ; \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} p + jq & p - jq & r \end{bmatrix}$$

Introducing an equivalence transformation:

$$x = P\bar{x}$$

$$P = \begin{bmatrix} 1/2 & -j1/2 & 0 \\ 1/2 & j1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{b}u(t)$$
$$y(t) = \bar{c}\bar{x}(t) + du(t)$$

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 1 & 1 & 0 \\ j & -j & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma + j\omega & 0 & 0 \\ 0 & \sigma - j\omega & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 1/2 & -j1/2 & 0 \\ 1/2 & j1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma & \omega & 0 \\ -\omega & \sigma & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\bar{b} = P^{-1}b = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{c} = cP = \begin{bmatrix} p & q & r \end{bmatrix}$$

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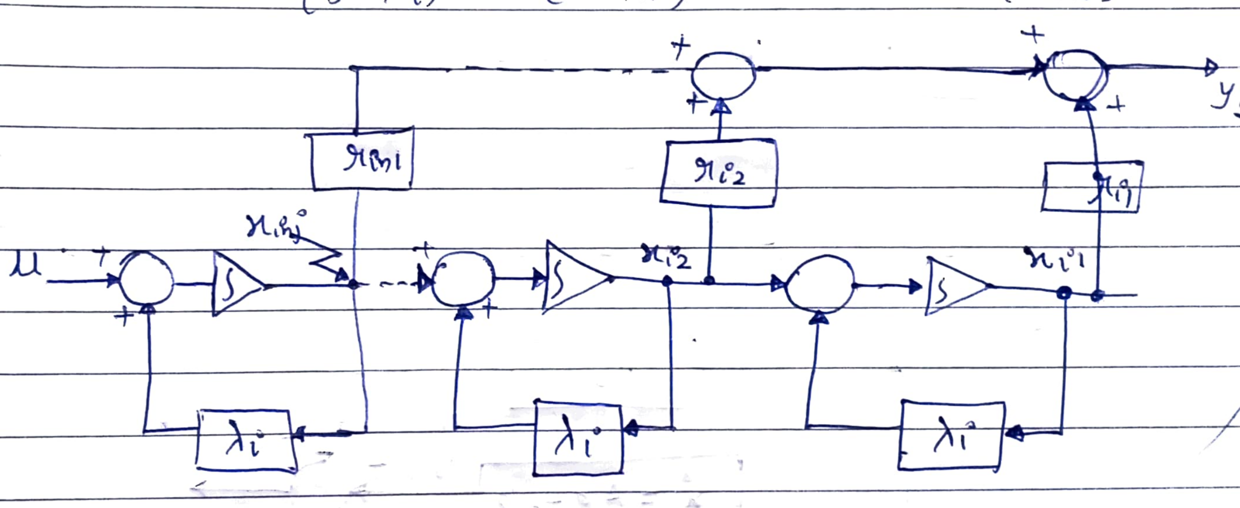
when the T.F. $G(s)$ has repeated poles, the partial fraction expansion will not be as simple

$$G(s) = \beta_0 + \frac{\beta_1' s^{n-1} + \beta_2' s^{n-2} + \dots + \beta_n'}{(s-\lambda_1)^{n_1}(s-\lambda_2)^{n_2} \dots (s-\lambda_m)^{n_m}}$$

$$G(s) = \beta_0 + H_1(s) + \dots + H_m(s) = \frac{Y(s)}{U(s)}$$

$$H_i(s) = \frac{r_{i1}}{(s-\lambda_i)^{n_i}} + \frac{r_{i2}}{(s-\lambda_i)^{n_i-1}} + \dots + \frac{r_{in_i}}{(s-\lambda_i)} = \frac{Y_i(s)}{U(s)}$$



$$\dot{x}_{i1} = \lambda_i x_{i1} + x_{i2}$$
$$\dot{x}_{i2} = \lambda_i x_{i2} + x_{i3}$$
$$\vdots$$
$$\dot{x}_{in_i} = \lambda_i x_{in_i} + u$$

$$y_i = r_{i1} x_{i1} + r_{i2} x_{i2} + \dots + r_{in_i} x_{in_i}$$

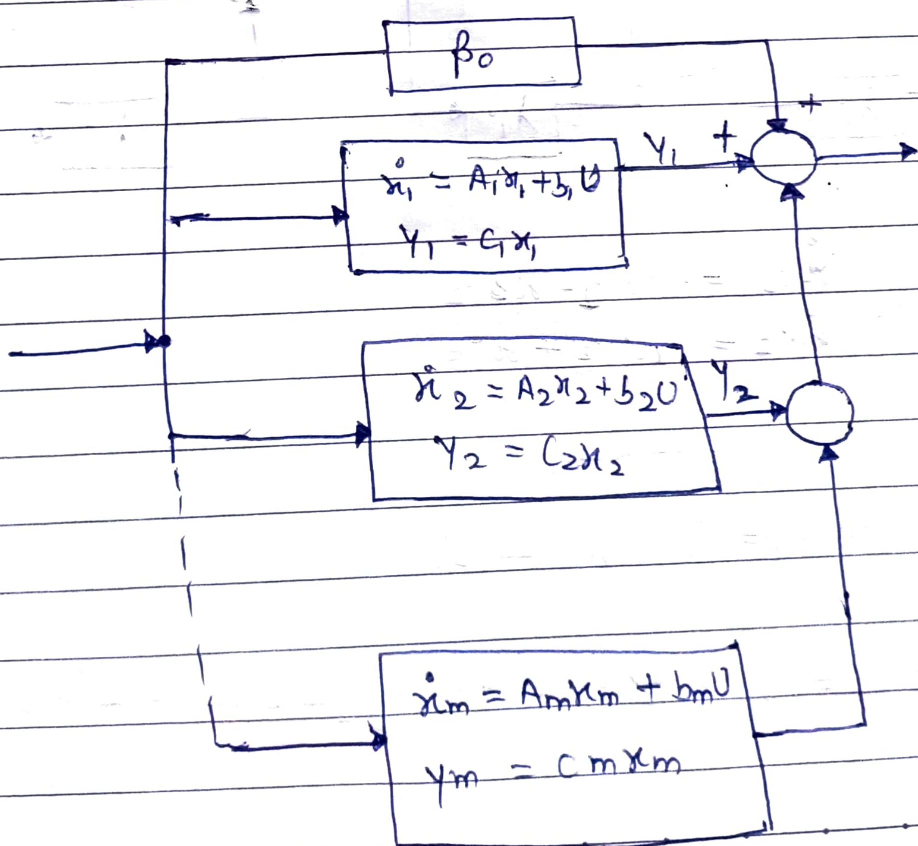
If the state vector for the subsystem is defined by

$$\dot{x}_i = A_i x_i + b_i U$$
$$y_i = c_i x_i$$

$$A_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix} ; \quad b_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_i = \begin{bmatrix} r_{i1} & r_{i2} & \cdots & r_{in_i} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$



$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} ; \quad C = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix} ; \quad d = \beta_0$$

ex 3

$$G(s) = \frac{s+3}{s^3 + 9s^2 + 24s + 20} = \frac{Y(s)}{U(s)}$$

first companion form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -20 \\ 0 & 0 & -24 \\ -20 & -24 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Y = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Comments : The T.F, has no cancellation if & only if the system is completely state controllable & completely observable.
This means that the canceled T.F does not carry along all ~~the~~ the information characterizing the dynamic system.

Alternative form of the condition for complete observability :

$$\dot{x} = Ax$$
$$y = Cx$$

$$P^{-1}AP = D$$
D is diagonal matrix

$$x = Pz$$

The eq^n
$$\dot{z} = P^{-1}APz = Dz$$
$$y = CPz$$

Hence
$$y(t) = CPe^{Dt}z(0)$$

The system described by eq is said to be completely o/p controllable if it is possible to construct an unconstrained control vector $u(t)$ that will transfer any given initial o/p $y(t_0)$ in any final o/p $y(t_1)$ in a finite time in $t_0 \leq t \leq t_1$

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The system described by eq is completely o/p controllable if & only if the $m \times (n+1)r$ matrix

$$[CB \,\vdots\, CAB \,\vdots\, CA^2B \,\vdots\, --- \,|\, CA^{n-1}B \,\vdots\, D]$$

is of rank m.

~~Untso~~ Uncontrollable System: An Uncontrollable system has a subsystem that is physically disconnected from the I/P.

O/P controllability:

In the practical design of a control system, we may want to control the o/p rather than the state of system.

Consider the system described by

$$\dot{x} = Ax + BU$$
$$y = Cx + DU$$

X = state vector (n)
U = control vector (r)
y = o/p vector (m)
A = $n \times n$ matrix
B = $n \times r$ "
C = $m \times n$ "
D = $m \times r$ "

Condition for complete State Controllability in the s plane :

- The condition for complete state controllability can be stated in terms of transfer functions or transfer matrices.

- It can be proved that a necessary and sufficient condition for complete state controllability is that no cancellation occur in the T.F. or T. matrices. If cancellation occurs, the system cannot be controlled in the dirn of the canceled mode.

- The same conclusion can be obtained by writing the T.F. in form of a state eqn. A state space representation is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

since

$$[B \mid AB] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Complete Stability of Continous-Time Systems :

Consider the system described by eqn. the o/p vector $y(t)$ is

$$y(t) = Ce^{At}x(0)$$

Referring to eq

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t) A_k$$

where n is the degree of the characteristic polynomial.

$$y(t) = \sum_{k=0}^{n-1} \alpha_k(t) CA^k x(0)$$

or

$$y(t) = \alpha_0(t) Cx(0) + \alpha_1(t) CAx(0) + \cdots \alpha_{n-1}(t) CA^{n-1} x(0)$$

Complete observability of continuous-time systems

The o/p vector $y(t)$ is

$$y(t) = Ce^{At}x(0)$$

referring to eqn

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t) A^k$$

where n is the degree of the characteristic polynomial.

$$y(t) = \sum_{k=0}^{n-1} \alpha_k(t) CA^k x(0)$$

or

$$y(t) = a_0(t) Cx(0) + \alpha_1(t) CAx(0) + \cdots$$
$$\cdots + \alpha_{n-1}(t) CA^{n-1} x(0)$$

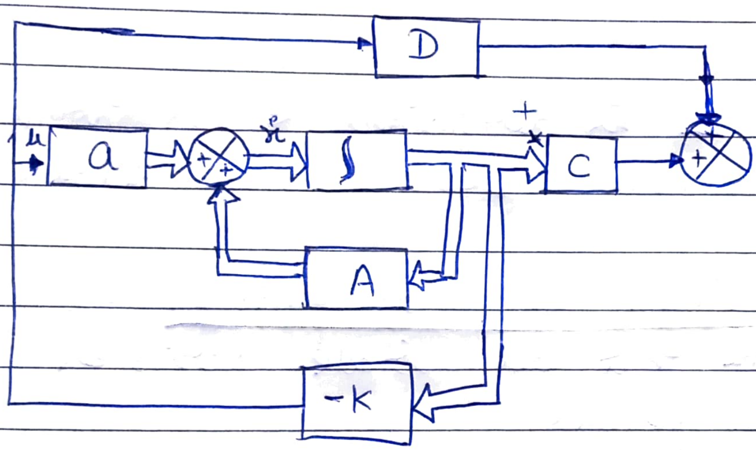
31/08/2020

$$\dot{x}(t) = (A-BK)x(t)$$

The solⁿ of this eqⁿ is given by

$$x(t) = e^{(A-BK)t}x(0)$$

where $x(0)$ is the initial state caused by external disturbances. The Stability & transient response characteristics are determined by the eigenvalues of matrix $A-BK$. If matrix

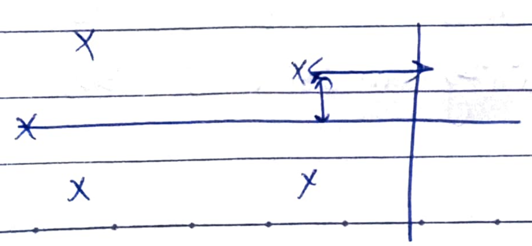


$$\dot{x} = Ax + BU$$
$$y = Cx + DU$$

+ Ch. eqⁿ $\quad q(s) = 0 \qquad G(s) = \frac{num}{den}$

S.O. roots $= (s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})(s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2})$

$M_p, t_p, t_s,$ ~~t_r~~, $e_{ss} \longrightarrow M_p, e_{ss}$



n order

n - 2 → non dominant

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→ Pole - Placement or pole - assignment technique:

1. All state variables are measurable and are available for feedback.
2. If the system considered is completely state controllable, then poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.

3. Single-i/p, single-o/p systems.

#

1. The control signal u is determined by an instantaneous state. such a scheme is called state feedback.

2. The matrix K is called the state feedback gain matrix. We assume that all state variables are available for feedback.

3. This closed-loop system has no input.

Necessary & sufficient condn:

- System be completely state controllable.

- the system is not completely state controllable. Then rank of the controllability matrix is less than n.

$$\text{rank}\,[B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B] = q < n$$

$$P = [r_1 \,\vdots\, r_2 \,\vdots\, \cdots \,\vdots\, r_v \,\vdots\, r_{v+1} \,\vdots\, r_{v+2} \,\vdots\, \cdots v_n]$$

is of rank n. Then it can be shown that

$$\hat{A} = P^{-1}AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \qquad \hat{B} = P^{-1}B = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}$$

Now define

$$\hat{K} = KP = [K_1 \,\vdots\, K_2]$$

then we have :

$$|sI - A + BK| = |P^{-1}(sI - A + BK)P|$$

$\dot{x} = Ax + BU$

$\dot{x} = Ax - BKx$

$\dot{x} = (ABK)x$

$$= |sI - P^{-1}AP + P^{-1}BKP|$$

$$= \left| sI - \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} [K_1 \,\vdots\, K_2] \right|$$

$$= |sI_q - A_{11} + B_{11}K_1| \cdot |sI_{n-q} - A_{22}| = 0$$

the characteristic eq^n for the system with state feedback. Therefore, it must be to the eq^n

$$a_1 + \delta_1 - \alpha_1$$
$$a_2 + \delta_2 - \alpha_2$$
$$\vdots$$
$$a_n + \delta_n = \alpha_n$$

Solving the proceeding eqn

$$K - [\delta_n \, \delta_{n-1} \, \cdots \, \delta_1] T^{-1}$$

$$= [\alpha_n - a_n \,\vdots\, \alpha_{n-1} - \alpha_{n-1} \,\vdots\, \cdots \,\vdots\, \alpha_2 - a_2 \,\vdots\, \alpha_1 - a_1] T^{-1}$$

if the system is completely state controllable, all eigenvalues can be arbitarily placed by choosing matrix K.