

Hanan Madan

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Ques 1 Determine the analytic funcⁿ whose real part is $e^n(n \cos 2y - y \sin 2y)$

Solⁿ) $e^{2n} (x \cos 2y - y \sin 2y)$

$$\frac{u}{v} = \frac{e^n (x \cos 2y - y \sin 2y)}{e^{2n} (x \cos 2y - y \sin 2y)} = e^{-n} \cos 2y$$

$$= e^{2n} (x \cos 2y + 2y \sin 2y) = \phi_1(x, y)$$
$$\phi_1(z, 0) = e^{-2z}(0) = 0$$

By milne thomson method.

$$F(z) = \int \phi_1(z, 0) dz + \int \phi_2(z, 0) dz + C$$

$$= \int e^{2z} (1 + 2z) dz - 0 + C$$

$$= (1 + 2z) \left(\frac{e^{2z}}{2} \right) - z \left(\frac{e^{2z}}{1} \right) + C$$

$$= \frac{e^{2z}}{2} (1+2z-1) + C$$

$$\boxed{f(z) = e^{2z} + C}$$

Que 302 Find the regular function whose imaginary part is $\cos x \cosh y$.

$$\Rightarrow \cos x \cosh y \Rightarrow \cos x \left(\frac{e^y + e^{-y}}{2} \right)$$

$$\left. \begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial (\cos x \cosh y)}{\partial y} \\ &= \cos x \left(\frac{e^y - e^{-y}}{2} \right) \end{aligned} \right\} \rightarrow 0$$

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &= -\sin x \left(\frac{e^y + e^{-y}}{2} \right) \end{aligned} \right\} \begin{array}{l} x=z \\ y=0 \end{array} ; -\sin z$$

$$\frac{dw}{dz} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}$$

$$\frac{dw}{dz} = i \sin z$$

$$\int dw = \int -i \sin z dz$$

$$\boxed{w = i \cos z}$$

Que: 03 $f(z) = u + iv$ if

$$(i) u - v = e^x (\cos y - \sin y)$$

$$u = e^x \cos y$$

$$v = e^x \sin y$$

$$f(z) = u + iv$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x [\cos y + i \sin y]$$

by Euler's identity $\rightarrow e^{f(z)} = e^x e^{iy}$

$$f(z) = e^{(x+iy)}$$

$$\boxed{f(z) = e^z}$$

$$(ii) u + v = \frac{x}{x^2 + y^2}, \text{ when } f(1) = 1$$

$$f(z) = u + iv \quad \text{--- (i)}$$

$$i f(z) = iv - u \quad \text{--- (ii)}$$

(i) + (ii) gives $(1+i)f(z) = (u-v) + i(v+u)$
 $f(z) = u + iv$

V_s is given as $n/(x^2+y^2)$

$$\frac{dV}{du} = \frac{(1.08n^2 + y^2) - n(2y)}{(n^2 + y^2)^2}$$

$$\left. \begin{matrix} x \rightarrow 2 \\ y \rightarrow 0 \end{matrix} \right\} \frac{\partial V}{\partial n} = \frac{2^2 - 2 \cdot 2^2}{2^4} = -\frac{1}{2^2}$$

$$\frac{\partial V}{\partial y} = -2yn / (n^2 + y^2) = 0$$

$$\frac{dw}{dz} = 0 + (-1/z^2)$$

$$\int dw = -i \int \frac{1}{z^2} dz$$

$$w = i/z$$

$$f(z) = \frac{1}{z} \quad (1+i) f(2) = f(z)$$

$$f(z) = \left(\frac{i}{i+1} \right) \frac{1}{z}$$

Que: 04

$$\omega = \frac{1}{z}, \quad z = \frac{1}{\omega}$$

$$|z - 2i| = 2$$

$$|x + (y-2)i| = 2$$

$$x^2 + (y-2)^2 = 4$$

$$x^2 + y^2 - 2y = 0$$

$$z = \frac{1}{u+iv}$$

$$= \frac{u - i^2 v}{u^2 + v^2}$$

$$x + iy = \frac{u - i^2 v}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}$$

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} - 2 \frac{v}{u^2 + v^2} = 0$$

$$\frac{u^2 + v^2 - 2v(u^2 + v^2)}{(u^2 + v^2)^2} = 0$$

$$u^2 + v^2 - 2vu^2 - 2v^3 = 0$$

$$(u^2 + v^2) + 2v(u^2 + v^2) = 0$$

$$(u^2 + v^2)(1 + 2v) = 0$$

$$1 + 2v = 0$$

$$v = -\frac{1}{2}$$

Que: 05 Show that the transformation $w = \frac{2z+3}{z-4}$

maps the circle $x^2 + y^2 - 4x = 0$ into the straight line $4u + 3 = 0$

\Rightarrow

$$w = \frac{2z+3}{z-4}$$

$$4u + 3 = 0$$

\rightarrow it is a bilinear transformation.
Inverse transformation

~~$w = \frac{2z+3}{z-4}$~~

$$z = \frac{4w+3}{w-2}$$

The eqⁿ of circle $\rightarrow x^2 + y^2 - 4x = 0$

i.e. $|z|^2 - 4\operatorname{Re} z = 0$

$$z\bar{z} - 2(z + \bar{z}) = 0$$

$\therefore w = u + iv$

$$\frac{4w+3}{w-2} \cdot \frac{\bar{4w}+3}{\bar{w}-2} - 2\left(\frac{4w+3}{w-2} + \frac{\bar{4w}+3}{\bar{w}-2}\right) = 0$$

$$(4\omega+3) \cdot (4\bar{\omega}+3) - 2\{(4\omega+3)(\bar{\omega}-2) + (4\bar{\omega}+3)(\omega-2)\} = 0$$

$$\text{i.e. } 2(\omega+\bar{\omega})+3=0$$

$$\text{i.e. } 4u+3=0$$

h.p.

Que: 06 find the bilinear transformation which maps the points $z=1, i, -1$ into the points $w=i, 0, -i$

Solⁿ $w = \frac{az+b}{cz+d} \quad (i)$

put $z=1$ & $w=i$

$$i = \frac{a+b}{c+d}$$

$$ic + id = a + b \quad (ii)$$

put $z=i$ & $w=0$

$$0 = \frac{ai+b}{c(i)+d} \Rightarrow ai+b=0$$

$$b = -ai \quad (iii)$$

put $z = -1$ & $w = -i$

$$-i = \frac{a(-1) + b}{c(-1) + d}$$

$$ic - id = -a + b \quad \text{--- (v)}$$

add eq (2) & (iv)

$$ic + id = a + b$$

$$ic - id = -a + b$$

we, get $2ic = 2b$

$$c = \frac{b}{i} = -\frac{ai}{i} = -a \rightarrow (v)$$

$$\therefore w = \frac{a \cdot z - ai}{-a^2 - ai}$$

$$\therefore w = \frac{-a(-2+i)}{-a(2+i)}$$

$\Rightarrow \frac{z-2}{z+1}$ is the bilinear transformation.

Que 307

$$\int \frac{3z^2 + z}{z^2 - 1}$$

$$|z-1|=1$$

$$z^2 - 1 = 0$$

$$z = + - 1$$

$$|z-1|=1$$

$$z=1$$

$r=1$ includes the point $z=1$

$$\oint \frac{3z^2 + z}{z^2 - 1} dz = \int \frac{\left(\frac{3z^2 + z}{z+1} \right)}{z-1} dz$$

$$= 2\pi i \left(\frac{3z^2 + z}{z+1} \right)_{z=1}$$

$$= \underline{\underline{4\pi i}}$$

Que : 08

~~$(z+1)^2$~~ $= 0$
 ~~$z = -1$~~

$$\frac{z^2 - 2z}{(z+1)^2(z^2+4)}$$

$$z = -1, -1$$

$$z = \pm 2i$$

Residue at -1

$$\frac{d}{dz} [(z - (-1))]^2 f(z)$$

$$\frac{d}{dz} \frac{(z^2 - 2z)}{(z^2 + 4)}$$

$$\frac{(z+1)^2 \cdot (z^2 - 2z)}{(z+1)^2(z^2+4)}$$

$$2z + 8z - 2z^2 - 8 - 2z + 4z^2$$

$$= \frac{2z^2 + 8z - 8}{(z^2 + 4)^2}$$

$$= \frac{2 - 8 - 8}{25} = -\frac{14}{25}$$

Residue at $z = 2i$

$$\lim_{z \rightarrow 2i} (z - 2i) f(z) = \frac{(z - 2i)}{(z + 1)^2} \cdot \frac{z^2 - 2z}{(z - 2i)(z + 2i)}$$

$$= \frac{z^2 - 2z}{(z^2 + 1 + 2z)(z + 2i)}$$

$$= \frac{7 + 4(1 + i)}{-1(3i + 4)4} = \frac{1 + i}{3i + 4}$$

$$\boxed{= \frac{7 + i}{25}}$$

Ans

Residue at $z = -2i$

$$= \frac{7 - i}{25}$$