ESTIMATION OF STATES OF A FREE-FALLING BODY USING EXTENDED KALMAN FILTERING

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Abstract: In this paper, we look at an Extended Kalman Filter and use it to estimate the states of a free-falling body. We take a copper sphere and make it to fall from a height. It's states (height and velocity) depend on factors like air buoyancy and air resistance, thus making it a non-linear model. We simulate the results and the graphs show the Actual and Estimated positions and velocities of the copper sphere over time. We also plot the errors in our estimation and find them to be negligible.

1. INTRODUCTION

When we study physics in high school, we generally ignore the effects of air resistance and buoyancy on a free-falling body. This is done to make our task easy, convenient and to give insight into Newton's second law of motion. But in actual situations, air buoyancy and resistance come into play unless we are doing the experiment in vacuum. If we continue using our theoretical model of high school physics, we will get great error in our measurement. Also, in our experiment, there are various random factors that come into play like imperfections in modelling, measurement and other random factors. The model of a free-falling body is a non-linear model when air buoyancy and resistance are considered [1]. We have used an Extended Kalman Filter to estimate the states of the same.

An Extended Kalman Filter is an extension of the Kalman Filter for estimation when the model is non-linear. In the extended Kalman Filtering algorithm, we linearize the non-linear model by taking the Jacobian of the State Transition Function and then, applying the general Kalman Filtering Algorithm.

The paper is organized as follows. Section 2 discusses the construction of the free-falling object model. In Section 3, theory of the Extended Kalman Filtering is discussed. In Section 4, details of our particular experiment are given and how an Extended Kalman Filter is applied in our case. In section 5, the simulation results with analysis are presented followed by conclusions in Section 6.

2. FREE-FALLING OBJECT MODEL

When an object falls, 3 major forces act on the object. These are Gravitational Force, Force due to buoyancy and the force of air resistance with equations [1]: -

$$G = mg$$

$$F_f = \rho_l V_m g = \rho_l mg / \rho_m$$

$$F_a = \tfrac{1}{2}\rho_g C_d S v^2$$

m is the mass of the falling object, g is the acceleration of gravity, Vm is the volume of the falling objects, Cd is drag coefficient, l is the characteristic length of the cross section, S is the object by the air resistance the maximum cross-sectional area.

The net force on the object is given by : -

$$F = G - F_f - F_a$$

Now,

$\dot{h} = v$ (derivative of height is velocity)

$$\dot{v} = g - \frac{\rho_l}{\rho_m} g - \frac{\frac{1}{2}\rho_g C_d S v^2}{m}.$$

(acceleration)

Now, by Euler's Equation,

$$\frac{h(k+\Delta T)-h(k)}{\Delta T}=v(k)$$

$$h(k+1)=h(k)+\Delta T v(k)+w_1(k)$$

$$\frac{v(k+\Delta T)-v(k)}{\Delta T}=A-Bv^2(k)$$

Rearranging terms, we get

$$v(k+1)=A\Delta T+v(k)-B\Delta T v^2(k)+w_2(k)$$

Thus we have state equations for the states (height and velocity) where A and B are constants given by :-

$$A=g-\frac{\rho_l}{\rho_m}g,\ B=\frac{\frac{1}{2}\rho_g C_d S}{m}$$

Which are the model parameters and Δ T is the Sampling period.

Adding, process noises which are zero mean gaussian

$$w(k),\ w(k)=\begin{pmatrix} w_1(k) \\ w_2(k) \end{pmatrix}, k=1,2,\cdots,n$$

to the system equations, we get the final state equations:-

$$X(k+1)=\begin{pmatrix} h(k+1) \\ v(k+1) \end{pmatrix}=\begin{pmatrix} h(k)+\Delta T v(k)+w_1(k) \\ A\Delta T+v(k)-Bkv^2(k)+w_2(k) \end{pmatrix}$$
$$=f(X(k),w(k))$$

These state equations are non-linear.
Now, we have our measurement equation as:-

$$Z(k)=X(k)+v(k)=H(X(k))+v(k)$$

Here v is the zero mean gaussian noise and H is the observation matrix. Thus, we have our free-falling object model [1].

3. THEORY OF EXTENDED KALMAN FILTERING

When the state equation is non-linear, we can use an Extended Kalman Filter to estimate the states. Extended Kalman Filter can take into account the non-linearity of the measurement equation too but in our case, as we make direct measurements, the measurement equation would be linear. Extended Kalman Filter involves linearizing the non-linear model by taking the Jacobian of the State Transition function, which is the function relating the next states to the previous ones in a process equation. So, if we have the state equations like [2]: -

$$\begin{aligned} \mathbf{x}_n &= \mathbf{f}(\mathbf{x}_{n-1})+\mathbf{v}_n, \\ \mathbf{y}_n &= \mathbf{h}(\mathbf{x}_n)+\mathbf{w}_n, \end{aligned}$$

Where f and h are the state transition function and observation function respectively. Assuming them to be differentiable, their Jacobian matrices which will linearize the model are given by [2]: -

$$\begin{aligned} \tilde{\mathbf{F}}_n &= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{m}_{n-1|n-1}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_D} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_D} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_D}{\partial x_1} & \frac{\partial f_D}{\partial x_2} & \cdots & \frac{\partial f_D}{\partial x_D} \end{bmatrix}\right|_{\mathbf{m}_{n-1|n-1}}, \\ \tilde{\mathbf{H}}_n &= \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{m}_{n|n-1}} = \left.\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_D} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_D} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_M}{\partial x_1} & \frac{\partial h_M}{\partial x_2} & \cdots & \frac{\partial h_M}{\partial x_D} \end{bmatrix}\right|_{\mathbf{m}_{n|n-1}}. \end{aligned}$$

The extended Kalman filtering prediction and update equations are given by [2]: -

$$\begin{aligned} \mathbf{m}_{n|n-1} &= \mathbf{f}(\mathbf{m}_{n-1|n-1}), \\ \mathbf{P}_{n|n-1} &= \tilde{\mathbf{F}}_n\,\mathbf{P}_{n-1|n-1}\,\tilde{\mathbf{F}}_n^T+\mathbf{Q}_n, \\ \mathbf{S}_n &= \tilde{\mathbf{H}}_n\,\mathbf{P}_{n|n-1}\,\tilde{\mathbf{H}}_n^T+\mathbf{R}_n, \\ \mathbf{K}_n &= \mathbf{P}_{n|n-1}\,\tilde{\mathbf{H}}_n^T\,\mathbf{S}_n^{-1}, \\ \mathbf{m}_{n|n} &= \mathbf{m}_{n|n-1}+\mathbf{K}_n\,(\mathbf{y}_n-\mathbf{h}(\mathbf{m}_{n|n-1})), \\ \mathbf{P}_{n|n} &= \mathbf{P}_{n|n-1}-\mathbf{K}_n\,\tilde{\mathbf{H}}_n\,\mathbf{P}_{n|n-1}, \end{aligned}$$

In the next section, we have applied the Extended Kalman Filtering Algorithm to our problem, i.e. Free fall model.

4. APPLICATION OF EXTENDED KALMAN FILTERING ALGORITHM TO FREE FALL MODEL

We assume that a copper sphere of mass about 4.66 kg with radius as 5 cm is dropped from 1000m height for our simulation. We can edit these values and the simulation will show the same pattern. Now, first we have considered the values of all the quantities model parameters A and B defined in section 2 depend upon. A and B were defined as: -

$$A=g-\frac{\rho_l}{\rho_m}g,\ B=\frac{\frac{1}{2}\rho_g C_d S}{m}$$

Where,

g = 9.81 m/s^2 (Acceleration due to gravity)

ρ_m = 8910 kg/m^3 (Density of Copper sphere)

$\rho_l = \rho_g$ = 1.225 kg/m^3 (Air Density)

C_d = 0.2 (Drag Coefficient)

S = 0.0025π m^2 (Cross sectional area of sphere)

v_m =0.0001666π m^3 (Volume of sphere)

m = $\rho_m * v_m$ = 4.66 Kg (Mass of copper sphere)

Now, let's look at our State Equations for free fall constructed in Section 2.

$$X(k+1) = \begin{pmatrix} h(k+1) \\ v(k+1) \end{pmatrix} = \begin{pmatrix} h(k) + \Delta T v(k) + w_1(k) \\ A\Delta T + v(k) - Bkv^2(k) + w_2(k) \end{pmatrix}$$
$$= f(X(k), w(k))$$

We can write them as,

$$\begin{pmatrix} h(k+1) \\ v(k+1) \end{pmatrix} = \begin{pmatrix} 1 & -\vartriangle T \\ 0 & 1 - B\vartriangle Tv(k) \end{pmatrix} \begin{pmatrix} h(k) \\ v(k) \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \vartriangle T * A \end{pmatrix}$$

Now, we can take the Jacobian Matrix of the transition function as: -

$$Jacobian = \begin{pmatrix} 1 & -\vartriangle T \\ 0 & 1 - 2v(k)B\vartriangle T \end{pmatrix}$$

We have taken w_1 and w_2 as zero mean Gaussian noises with unity variances.

Thus, the process noise covariance matrix is an identity matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Now, take the measurement equation,

$$Z(k) = X(k) + v(k) = H(X(k)) + v(k)$$

which is linear and as we make direct measurements, H is an identity matrix.

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$v(k)$ is again zero mean Gaussian noise with unity variance, thus, measurement noise correlation matrix is,

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, as we drop the object from 1000m in our simulation, and because the initial velocity of a free-falling body is zero (because we dropped it and it was still before dropping), we get the initial value of our state matrix, i.e. our initial states as

$$\begin{pmatrix} h_i \\ v_i \end{pmatrix} = \begin{pmatrix} 1000 \\ 0 \end{pmatrix}$$

And the Process noise covariance matrix is initialized to identity matrix.

Thus, using all this information in the equations for Extended Kalman Filtering, i.e. using the Extended Kalman Filtering Algorithm, we have got some simulation results which are presented in the next section.

5. SIMULATION RESULTS WITH ANALYSIS

Using MATLAB, we plotted the actual and estimated position and velocity of the free-falling copper sphere over time.

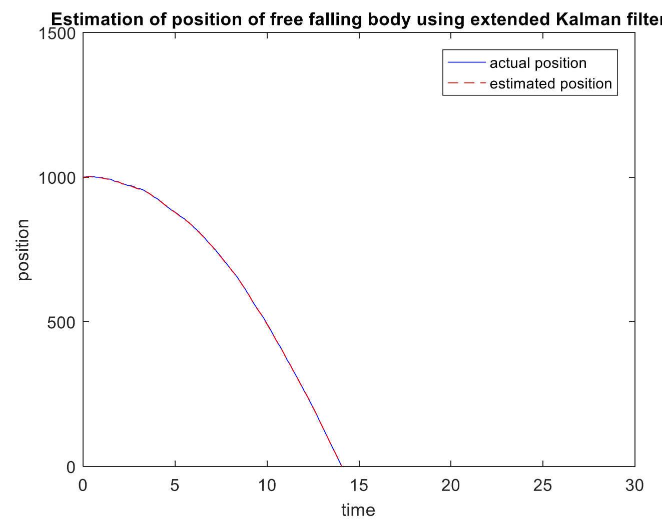

Fig.1. Estimation of position of free falling body using Extended Kalman Filter

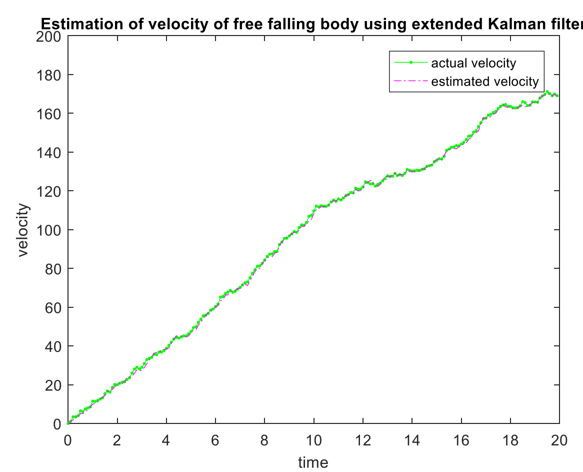

Fig.2. Estimation of velocity of free falling body using Extended Kalman Filter

From the above two graphs, we can observe that the Extended Kalman Filter excellently estimates the position and velocity of the free-falling copper sphere.

Now, we plot the errors in estimation of the position and velocity of the copper sphere with time.

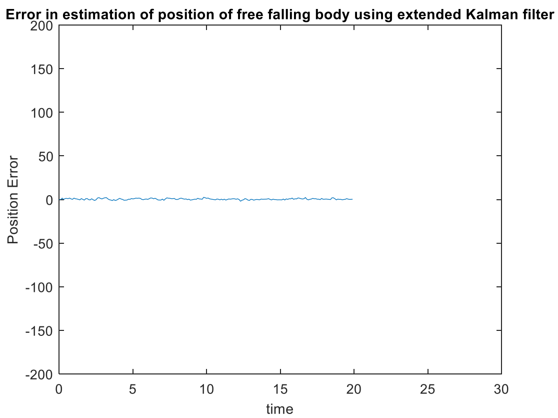

Fig.3. Errors in estimation of position using Extended Kalman Filter

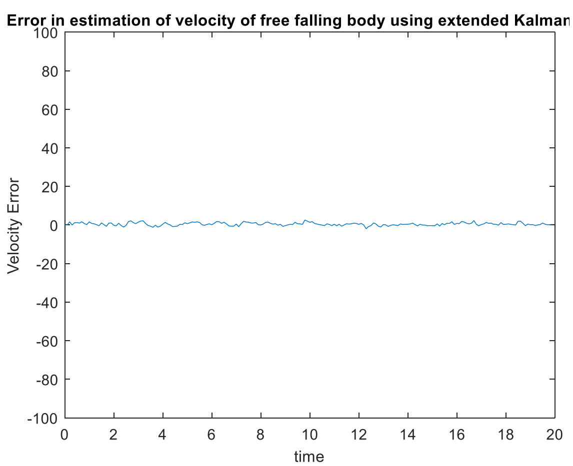

Fig.4. Errors in estimation of velocity using Extended Kalman Filter

Thus, we see that the errors in estimation are close to zero or negligible which justifies the same thing, that the Extended Kalman filter is a very good tool to estimate the states of a non-linear model.

6. CONCLUSIONS

In this paper, we saw how an Extended Kalman Filter can estimate the states of a free-falling body when the effects of air buoyancy and air resistances are considered. First, we constructed a free-falling object model and got some model equations. After that, we discussed in brief the theory of an Extended Kalman Filter and then, applied it to our free fall problem. We saw from our simulation results that an Extended Kalman Filter is an excellent tool to estimates the states of a non-linear model. And thus, it has numerous applications in military, space, communications and various other fields.

7. REFERENCES

[1] Weifu, Ding. "The State Estimation for Falling Objects Based on Unscented Kalman Filter." *Proceedings of the 2012 International Conference on Cybernetics and Informatics*. Springer New York, 2014.

[2] Kovvali, Narayan V. S. K, M. K. Banavar and A. Spanias. *An Introduction to Kalman Filtering with MATLAB Examples* (1st ed.) 2014;2013;*12*.