Priors for second-order unbiased Bayes estimators

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arXiv Code

Background: The Challenge of Bias in Bayes Estimation

- Bayes estimators **generally have $O(n^{-1})$ bias** with n sample size
- **Asymptotically Unbiased Priors** (Hartigan, 1965) yield **second-order unbiased** Bayes estimators, i.e. **bias** $= o(n^{-1})$
- **Limitation**: Hartigan's work **assumes i.i.d.** data, which excludes many models (e.g. Bayesian regression analysis)

Hartigan. (1965) The asymptotically unbiased prior distribution.

Purpose: Extend the theory of second-order unbiased Bayes estimators to **non-i.i.d. settings**

Setup & Assumptions

- $(X_1, \ldots, X_n) \sim f_n(x_1, \ldots, x_n \mid \theta)$
- $\theta = (\theta_1, \ldots, \theta_p) \in \Theta \subset \mathbb{R}^p$
- $\ell_n(\theta) = \log f_n(x_1, \ldots, x_n \mid \theta)$
- $\pi(\theta)$: prior density of θ
- $\hat{\theta}^B = \dfrac{\int_\Theta \theta \pi(\theta) \exp(\ell_n(\theta)) d\theta}{\int_\Theta \pi(\theta) \exp(\ell_n(\theta)) d\theta}$: Bayes est.

Regularity conditions

- $\hat{\theta}^{ML} - \theta = O_p(n^{-1/2})$, where $\hat{\theta}^{ML}$ is the MLE
- $\ell_n(\theta) = O_p(n)$
- $\ell_n(\theta)$ is three times continuously differentiable
- $\pi(\theta)$ is differentiable
- $H_n(\theta) = -\frac{1}{n}\frac{\partial^2}{\partial\theta\partial\theta^\top}\ell_n(\theta)$ is invertible
- $H_n(\hat{\theta}^{ML})$ is positive definite

Assumptions on the log likelihood

- $H_n(\theta)) = H(\theta) + O_p(n^{-1/2})$, and $H(\theta)$ is invertible
- $\frac{1}{n}\frac{\partial^3}{\partial\theta_r\partial\theta_s\partial\theta_t}\ell_n(\theta) = K_{rst}(\theta) + o_p(1)$
- $\mathbb{E}\left[\frac{1}{n}\left(\frac{\partial \ell_n(\theta)}{\partial\theta_r}\right)\left(\frac{\partial \ell_n(\theta)}{\partial\theta_s}\right)\right] = I(\theta) + o(1)$
- $\mathbb{E}\left[\frac{1}{n}\left(\frac{\partial^2}{\partial\theta_r\partial\theta_s}\ell_n(\theta)\right)\left(\frac{\partial}{\partial\theta_t}\ell_n(\theta)\right)\right] = J_{rs,t}(\theta) + o(1)$

Contribution 1: Characterization of asymptotically unbiased priors for non-i.i.d. models

The bias of the Bayes estimator

$$\mathbb{E}(\hat{\theta}^B - \theta) = \underbrace{\frac{1}{n}H^{-1}(\theta)\left(\frac{\partial}{\partial\theta}\log\pi(\theta) - \phi(\theta)\right)}_{O(n^{-1})\text{ bias}} + o\left(\frac{1}{n}\right),$$

$$\phi(\theta) := -\frac{1}{2}\sum_{r=1}^{p}\sum_{s=1}^{p} H^{rs}(\theta)A_{rs}(\theta), \qquad A_{rs}(\theta) = \begin{bmatrix} K_{1rs}(\theta) + 2J_{1r,s}(\theta) \\ \vdots \\ K_{prs}(\theta) + 2J_{pr,s}(\theta) \end{bmatrix} + \sum_{t=1}^{p}\sum_{u=1}^{p} H^{tu}(\theta)I_{su}(\theta)\begin{bmatrix} K_{rt1}(\theta) \\ \vdots \\ K_{rtp}(\theta) \end{bmatrix}$$

$$\frac{\partial}{\partial\theta}\log\pi(\theta) = \phi(\theta)$$

$\Rightarrow$ $\hat{\theta}^B$ is second-order unbiased

Contribution 2: Systematic construction of an asymptotically unbiased prior

Check the existence

$\exists\pi(\theta)$ s.t. $\frac{\partial}{\partial\theta}\log\pi(\theta) = \phi(\theta)$

$$\Leftrightarrow \quad \frac{\partial}{\partial\theta_u}\phi_t(\theta) = \frac{\partial}{\partial\theta_t}\phi_u(\theta)$$

Define functions

Fix arbitrary $(c_1, \ldots, c_p) \in \Theta$ and define

$$\psi_t(\theta_t, \ldots, \theta_p) := \phi_t(c_1, \ldots, c_{t-1}, \theta_t, \ldots, \theta_p)$$

Integration

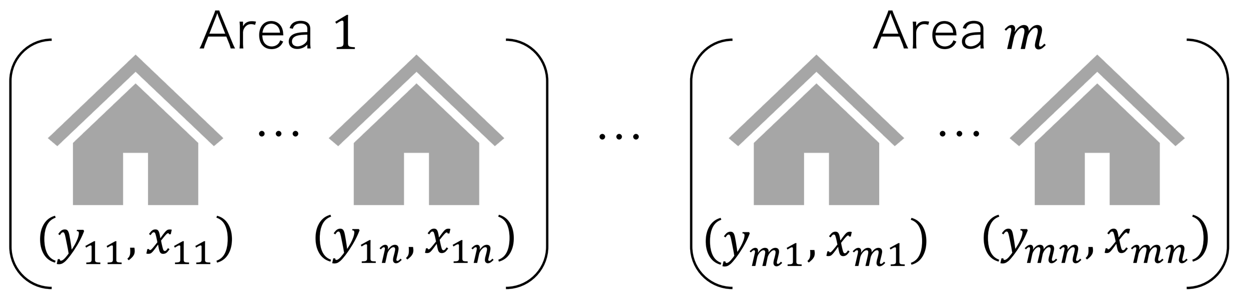
Take the prior as

$$\pi(\theta) \propto \exp\left(\sum_{t=1}^{p}\int_{c_t}^{\theta_t}\psi_t(z, \theta_{t+1}, \ldots, \theta_p)dz\right)$$

Contribution 3: New prior for the random effects model

Model

$$y_{ij} = x_{ij}^\top\beta + v_i + \epsilon_{ij} \qquad (i = 1, \ldots, m,\ j = 1, \ldots, n)$$



- $v_i \sim N(0, \tau^2)$, $\epsilon_{i1}, \ldots, \epsilon_{in} \overset{i.i.d.}{\sim} N(0, \sigma^2)$, $v_i \perp \epsilon_{ij}$
- $\theta = (\theta_1, \ldots, \theta_p, \theta_{p+1}, \theta_{p+2}) = (\beta, \tau^2, \sigma^2) \in \mathbb{R}^{p+2}$
- m : number of areas, $m \to \infty$
- n : number of units in an area, fixed
- Observe $y_{ij} \in \mathbb{R}$, $x_{ij} \in \mathbb{R}^p$, independently for each i

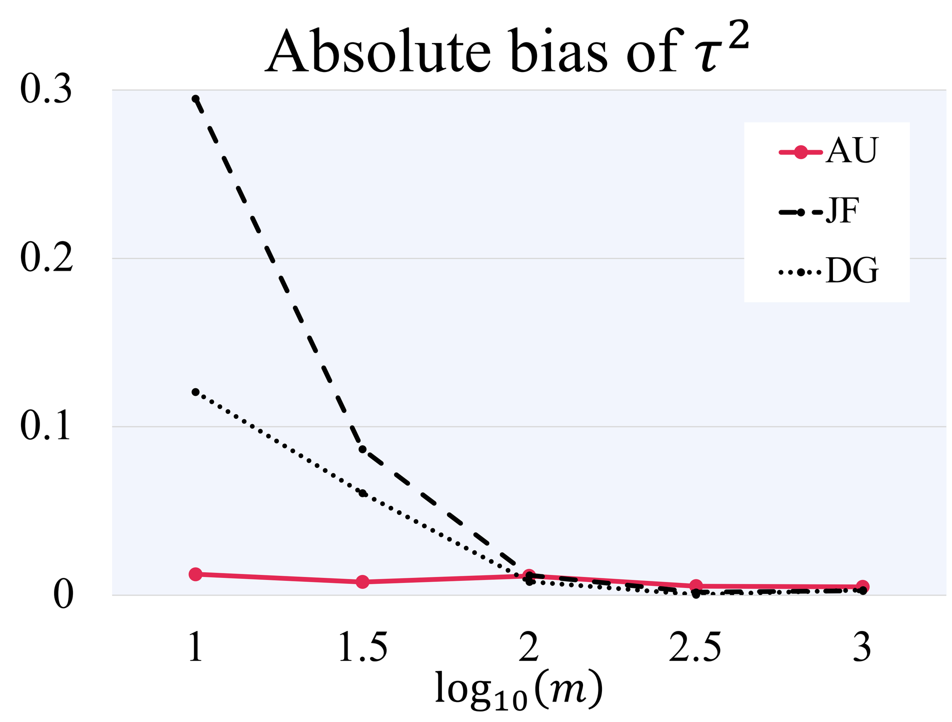
Asymptotically unbiased prior

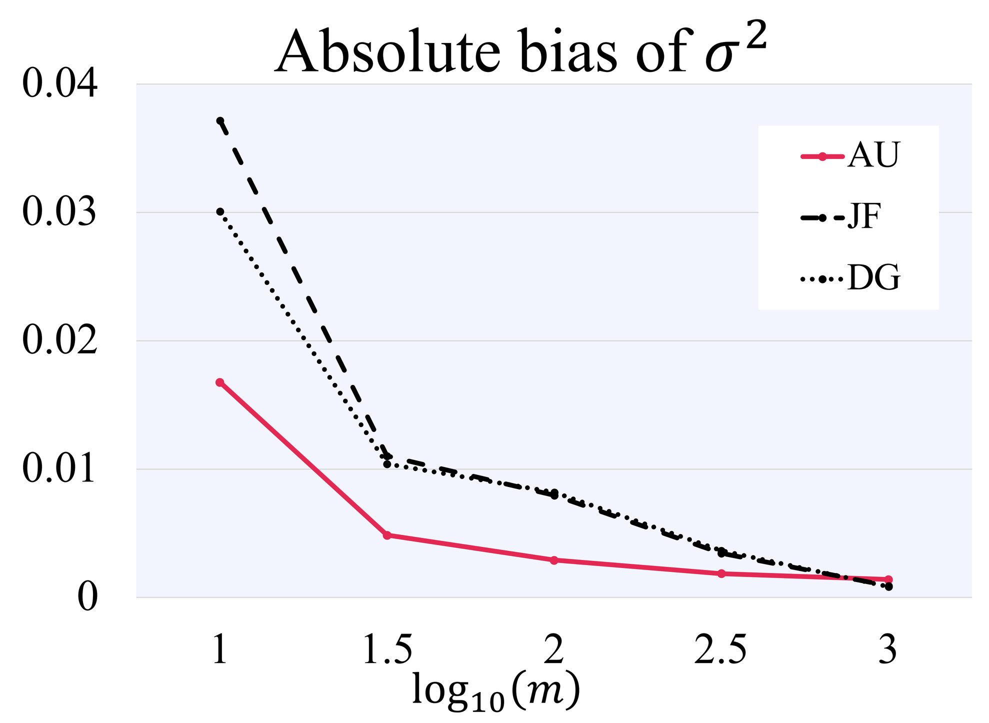
$$\pi(\theta) \propto \left[\sigma^2(\sigma^2 + n\tau^2)\right]^{-2}$$

Simulation

AU	Asymptotically unbiased prior	$\pi(\theta) \propto \left[\sigma^2(\sigma^2 + n\tau^2)\right]^{-2}$
JF	Jeffreys' prior for (τ^2, σ^2)	$\pi(\theta) \propto \left[\sigma^2(\sigma^2 + n\tau^2)\right]^{-1}$
DG	Prior of Datta & Ghosh (1991)	$\pi(\beta) \propto 1$, $\tau^2 \sim IG(a_\tau, b_\tau)$, $\sigma^2 \sim IG(a_\sigma, b_\sigma)$

Datta & Ghosh. (1991). Bayesian prediction in linear models: Applications to small area estimation.





* The bias of β remains small and stable across priors and sample sizes