The Maths Cook Book

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Part - I

Algebra

Progressions

INTRODUCTION

Sequence and Series is one of the most important chapter that is covered in the syllabus for IIT-JEE. The summation of series is one of the most important way to reduce large series into some value and it is widely used throughout the syllabus. The tricks and techniques discussed here are very useful and are widely used in other chapters as well. The A.M- G.M inequality which will be discussed here is also a very important result that is used widely.

So Let us Probe more into this Chapter...

Arithmetic Progression (A.P):

A sequence of numbers is said to be in A.P when difference $t_n - t_{n-1}$ is constant for all

$n \in N$. This constant $d = t_n - t_{n-1}$ is called common difference. If a the first term and d the common difference of the A.P then,

1) n^{th} term $t_n = a + (n-1)\, d$.

Proof

1st term of A.P = a = a+ (1-1) d.

2nd term of A.P = a+d = a+ (2-1) d.

3rd term of A.P = a+2d = a + (3-1) d.

4th term of A.P = a+3d = a + (4-1) d.

$\therefore$ n^{th} term of A.P = a + (n-1) d.

2) Sum of n terms

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$= \frac{n}{2}(a + (a + (n-1)d)))$$

$$= \frac{n}{2}(a + t_n)$$

Why?

Let S_n = a + (a+d) + (a+2d) + ------- + a+ (n-2) d + a+ (n-1) d --------- (1)

Or S_n = a+ (n-1) d + a+ (n-2) d + ---------- + (a+d) + a ---------- (2)

Now adding (1) and (2) we get

$$2S_n = (2a + (n-1)d) + (2a + (n-1)d) + - - - - - n \text{ times}$$

$$2S_n = n(2a + (n-1)d)$$

$$\text{Or } S_n = \frac{n}{2}(2a + (n-1)d)$$

Arithmetic Means (A.M):
The n number A_1, A_2, A_3---------, A_n are arithmetic means between a and b if a, A_1, A_2, -----A_n, b are in A.P.

Since a is first term and b is $(n+2)^{th}$ term.

$$\text{So, } b = a + ((n-2)-1)d$$

$$\Rightarrow d = \frac{b-a}{n+1}$$

$$\text{So, } A_i = a + id$$

$$= a + \frac{i(b-a)}{n+1}$$

<u>Note:</u>

1) $\sum_{i=1}^{n} A_i = \frac{n(a+b)}{2}$

Why?

$$A_i = a + \frac{i(b-a)}{n+1}$$

$$\text{So, } \sum_{i=1}^{n} A_i = a\sum_{i=1}^{n} 1 + \frac{b-a}{n+1}\sum_{i=1}^{n} i$$

$$= a(1+1+-----1) + \frac{b-a}{n+1}(1+2+----+(n-1)+n)$$

$$= na + \frac{b-a}{n+1}\left(\frac{n}{2}\right)(2.1+(n-1))$$

$$= na + \frac{n}{2}(b-a)$$

$$= n\left(\frac{a+b}{2}\right)$$

2) Arithmetic mean A of n numbers a_1, a_2, ---------- a_n is given by,

$$A = \frac{a_1 + a_2 + ----+a_n}{n}$$

Illustration 1:

For what value of n $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\ (a \neq b)$ is A.M between a and b?

A.M between a and b is $\frac{a+b}{2}$

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\frac{a+b}{2}$$

$$\Rightarrow 2a^{n+1}+2b^{n+1}=a^{n+1}+a^nb+b^na+b^{n+1}$$

$$\Rightarrow a^{n+1}+b^{n+1}-a^nb-b^na=0$$

$$\Rightarrow a^n(a-b)-b^n(a-b)=0$$

$$\Rightarrow (a-b)(a^n-b^n)=0$$

$$\Rightarrow a^n-b^n=0$$

$$\Rightarrow \left(\frac{a}{b}\right)^n=1$$

Or $n=0$

Properties of A.P:

1) If a fixed number is added (substracted) to each term of a given A.P. then resulting sequence is also A.P with same common difference as given A.P.

Dumb Question:

1) Why common difference has not changed?

Ans: Suppose the original A.P had $t_n = a+(n-1)d$, now let g be substracted from each term. So $t_n = (a-g)+(n-1)d$. Therefore only the first term of A.P has changed.

2) If each term of an A.P is multiplied (or divided) by fixed constant then resulting sequence is also an A.P with common difference multiplied (or divided) by same constant.

3) Sum and difference of corresponding terms of two A.P's will form an A.P.

Why?

Let first A.P be

$$a_1, a_1 + d_1, a_1 + 2d_1, ----, a_1 + (n-1)d_1$$

And second A.P be

$$a_2, a_2 + d_2, a_2 + 2d_2, -----, a_2 + (n-1)d_2$$

$$\begin{aligned} \text{So, } T_r &= t_{1r} + t_{2r} \\ &= (a_1 + (r-1)d_1 + (a_2 + (r-1)d_2) \\ &= (a_1 + a_2) + (r-1)(d_1 + d_2) \end{aligned}$$

So, resulting sequence is A.P with first term a_1+a_2 and common difference d_1+d_2.

4) If we want to pick terms of an A.P then convenient way of doing that is,

For three term's in A.P we choose a-d, a, a+d.

For four terms in A.P we choose a-3d, a-d, a+d, a+3d.

5) $t_n = S_n - S_{n-1}$

6) If a, b, c are in A.P $\Rightarrow$ 2b = a+c or b-a = c-b.

7) The sum of the terms of an A.P equidistant from beginning and end is constant and equal to sum of first and last term.

$t_k + t_{n-k+1}$ = Constant = $t_1 + t_n$.

Illustration 2:

Find the number of terms in the series $20, 19\frac{1}{3}, 18\frac{2}{3} -------$ of which the sum is 300.

Solution:

Here we observe that a=20, $d = -\frac{2}{3}$, and $S_n = 300$.

So, $\frac{n}{2}\left(2\times 20 + (n-1)\left(\frac{-2}{3}\right)\right) = 300$

$n^2 - 61n + 900 = 0$

$\Rightarrow n = 25 \text{ or } 36.$

Dumb Question:

1) Why do we get n as 25 and 36 both?

Ans: This clearly indicates that sum of terms from n = 26 to 36 is zero.

i.e. $T_{26} + T_{27} + - - - - - - + T_{36} = 0$

Geometric Progression:
A sequence is a G.P when its first term is non-zero and each of its succeeding term is r times the preceding term. The fixed term r is known as common ratio of G.P.

If a is first term of an G.P and r its common ratio then,

1) n^{th} term $t_n = ar^{n-1}$.

Why?

Let us observe the pattern

$t_1 = ar^{1-1}$

$t_2 = ar^{2-1}$

$t_3 = ar^{3-1}$

.

.

$t_n = ar^{n-1}$

So, clearly $t_n = ar^{n-1}$.

2) The sum of first n terms

$$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r - 1} & \text{if } r \neq 1 \\ na & \text{if } r = 1 \end{cases}$$

Why?

Case 1: Suppose r=1

So, $S_n = a + a + a + - - - + a$

$= na$

Case 2: If $r \neq 1$.

$S_n = a + ar + ar^2 + ar^3 + - - - - - ar^{n-2} + ar^{n-1} + ar^n$

$rS_n = ar + ar^2 + - - - - - + ar^{n-2} + ar^{n-1} + ar^n$

So, now $(1) - (2)$ gives

$S_n - rS_n = a - ar^n$

$S_n(1 - r) = a(1 - r^n)$

$\Rightarrow S_n = \dfrac{a(r^n - 1)}{(r - 1)}$

3) If $|r|<1$ and $n \to \infty$ then $\left(\lim_{n\to\infty} S_n\right) = \frac{a}{1-r}$

Why?

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\text{Now } \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a}{1-r} - \lim_{n\to\infty} \frac{ar^n}{1-r}$$

$$= \frac{a}{1-r} - 0 \qquad \left(\lim_{n\to\infty} r^n = 0\right)$$

$$= 0$$

Dumb Question:

1) Why $\lim_{n\to\infty} r^n = 0$?

Ans: Here $|r|<1$ so as we multiply r by itself so the value goes down and down. For example. Suppose $r = ½$, so $r^2=1/4$ is less than ½ and so the value keeps going down.

Illustration 3:

If a_1, a_2, a_3 are 3 consecutive terms of a G.P with common ratio k and $a_1<0$. Find values of k for which $a_3> 4a_2-3a_1$ is satisfied.

Solution:

Now since a_1, a_2, a_3 are in G.P

So $a_2 = a_1k$.

$a_3 = a_1k^2$.

$\therefore\ a_1k^2 > 4a_1k-3a_1$.

$\Rightarrow a_1 (k^2-4k+3) > 0$.

Or a_1 (k-1) (k-3) > 0.

Since $a_1<0$ so it means (k-1) (k-3) < 0.

$\Rightarrow 1<k<3$.

Geometric Means (G.M)

If G_1, G_2, G_3, G_4---------G_n are in G.M's between a and b then a, G_1, G_2, ---------- G_n, b are in G.P.

Now b is t_{n+2} so $b = ar^{n+1}$

Or $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

and Thus $G_i = a\left(\frac{b}{a}\right)^{\frac{i}{n+1}}$

Note:

1) The product of n G.M's between a and b is equal to n^{th} power of one G.M between a and b

i.e. G_1, G_2------- $G_n = \left(\sqrt{ab}\right)^n$.

Why?

$$G_i = a\left(\frac{b}{a}\right)^{\frac{i}{n+1}}$$

$$\therefore G_1, G_2, ---G_n = a^n \prod_{i=1}^{n} \left(\frac{b}{a}\right)^{\frac{i}{n+1}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{\sum i}{n+1}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}}$$

$$= (ab)^{\frac{n}{2}} = \left(\sqrt{ab}\right)^n$$

2) If a_1, a_2, ------- a_n are n non-zero numbers then their G.M is given by

$$G = (a_1, a_2 --- a_n)^{\frac{1}{n}}$$

Illustration 4:

If we insert odd number (2n+1) G.M's between 4 and 2916 then find the value of $(n+1)^{th}$ G.M?

Solution:

Now 4, G_1, G_2, G_3---------- G_{n+1}, ------- G_{2n}, G_{2n+1}, 2916 are in G.P. So G_{n+1} will be the middle mean of (2n+1) odd means and so it will be equidistant from 1st and last term.

So, 4, G_{n+1}, 2916 are also in G.P.

And thus,

$$G_{n+1} = (4.2916)^{\frac{1}{2}}$$

$$= (4 \times 9 \times 324)^{\frac{1}{2}}$$

$$= (4 \times 9 \times 4 \times 81)^{\frac{1}{2}}$$

$$= (2 \times 3 \times 2 \times 9)$$

$$= 108.$$

Properties of G.P:

1) If each term of a G.P is multiplied (or divided) by some non-zero quantity the resulting progression is G.P with some common ratio.

Why?

Suppose the G.P is with $t_n = ar^{n-1}$

Now this G.P is multiplied by some k ($\neq 0$).

So G.P will be $t_n^1 = (ak)r^{n-1}$

So, only the first term of G.P has changed and the common ratio remains unaffected.

2) If a_1, a_2, -------- and b_1, b_2, ---------- be two G.P's of common ratio r_1 and r_2 respectively, then a_1b_1, a_2b_2, ------ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}$ _ _ _ _ _ will also form G.P common ratio will be r_1r_2 and $\frac{r_1}{r_2}$ respectively.

Why?

Let the series a_1, a_2--------- have the nth term as $a_1r_1^{n-1}$ and the series b_1, b_2-------- have n^{th} term as $b_1r_2^{n-1}$.

So the series a_1b_1, a_2b_2 ------- will have n^{th} term as

$(a_1 r_1^{n-1})(b_1 r_2^{n-1})$
$= (a_1 b_1)(r_1 r_2)^{n-1}$

So the common ratio now becomes $r_1 r_2$.

3) If we have to take three terms in G.P we take them as $\frac{a}{r}, a, ar$ with common ratio r and four terms as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ with common ratio r^2.

4) If a, b, c are in G.P then $b^2 = ac$.

5) If a_1, a_2, a_3 -------- a_n is a G.P then $\log a_1, \log a_2$--------- $\log a_n$ is an A.P.

Why?

Now a_1, a_2, a_3 ------- a_n form a G.P.

So $a_i = a_1 r^{i-1}$ (Let)

$\Rightarrow \log a_i = \log a_1 + (i-1)\log r$

This is clearly term of an A.P.

Dumb Question:

1) How does $\log a_1 + (i-1) \log r$ represents term of an A.P?

Ans: Let $\log a_1 = A$

And $\log r = D$

So, $\log a_1 + (i-1) \log r = A + (i-1) D$.

So, A+ (i-1) D is term of an A.P with first term as A and common difference as D

Illustration 5:

If x = 1+ a+ a^2+a^3+ --------+∞ and y = 1+b+b^2+ b^3+ ------ +∞ Show that,

$$1+ab+a^2b^2+a^3b^3+------+\infty=\frac{xy}{x+y-1}$$

Where 0<a<1 and 0<b<1.

Solution:

Given x = 1+ a+ a^2+a^3+ --------+∞

$$\therefore\ x=\frac{1}{1-a}$$
$$\Rightarrow x-ax=1$$
$$\Rightarrow a=\frac{x-1}{x}----(1)$$

y = 1+b+b^2+ b^3 ------ +∞

$$\text{and } b=\frac{y-1}{y}----(2)$$

Since 0<a<1, 0<b<1

∴ 0<ab<1.

Now $$1+ab+a^2b^2+a^3b^3+------+\infty=\frac{1}{1-ab}$$

$$= \frac{1}{1-\left(\frac{x-1}{x}\right)\left(\frac{y-1}{y}\right)} \quad \{\text{from (1) and (2)}\}$$

$$= \frac{xy}{xy - xy + x + y - 1}$$

$$= \frac{xy}{x + y - 1}$$

Arithmetic Geometric Series (A.G.P):

Suppose a_1, a_2, a_3, ------- a_n be an A.P and b_1, b_2, b_3, ------- b_n is a G.P Then the sequence a_1b_1, a_2b_2, -------- a_nb_n is A.G.P.

A.G.P is of form $ab, (a+d)br, ------ (a+(n-1)d)br^{n-1}$

Where clearly $t_n = (a+(n-1)d)r^{n-1}$

1) Sum of n terms of an A.G.P

$$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)br^n}{1-r}$$

Why?

Let $S_n = ab + (a+d)br + ------ + [a+(n-1)d]br^{n-1}, \ d \neq 0, r \neq 1$

Now if we multiply the series by r then.

$$rS_n = abr + (a+d)br^2 + ------ + (a+(n-1)d)br^n$$

So on substraction,

$$S_n(1-r) = ab + bd(r + r^2 + ----- + r^{n-1}) - (a + (n-1)d)br^n$$

$$\text{Or } S_n(1-r) = ab + \frac{dbr(1-r^{n-1})}{1-r} - (a + (n-1)d)br^n$$

$$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{(a + (n-1)d)br^n}{1-r}$$

2) Sum of infinite series S_∞.

$$S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}, |r| < 1$$

Why?

If |r|<1 then $(n-1)r^n, r^{n-1} \to 0$ as $n \to \infty$

So, $S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$

Illustration 6:

If sum to infinity of the series $1 + 4x + 7x^2 + 10x^3 + ----- \text{is } \frac{35}{16}$ then find x.

Solution:

We know that $S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$

Here a=1, b=1, r=x, d = 3

$$\text{So, } \frac{35}{16} = \frac{1}{1-x} + \frac{3x}{(1-x)^2}$$

$$\Rightarrow \frac{35}{16} = \frac{(1+2x)}{(1-x)^2}$$

$$\Rightarrow 35x^2 - 102x + 19 = 0$$

$$\Rightarrow (7x-19)(5x-1) = 0$$

$$\text{Now } x \neq \frac{19}{7}$$

$$\text{So, } x = \frac{1}{5}$$

Dumb Question:

1) Why $x \neq \frac{19}{7}$?

Ans: For infinite series to be summable |x| needs to be less than 1 hence $x \neq \frac{19}{7}$

Harmonic Progression (H .P):

The sequence a_1, a_2, ------ a_n is said to be a H.P if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} --- \frac{1}{a_n}$ is an A.P.

The nth term of a H.P (t_n) is given by $t_n = \frac{1}{a+(n-1)d}$ where $a = \frac{1}{a_1}$ and $d = \frac{1}{a_2} - \frac{1}{a_1}$.

Harmonic Means (H .M):

If H_1, H_2, H_3-------- H_n be n H.M's between a and b then a, H_1, H_2, H_3-------- H_n, b is a H.P.

This means $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2} --- \frac{1}{H_n}, \frac{1}{b}$ is a A.P.

And hence $\frac{1}{H_i} = \frac{1}{a} + \frac{i(a-b)}{(n+1)ab}$

Note:

1) If a_1, a_2, a_3---------a_n are n non-zero numbers then H.M(H) of these number is given by

$$H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + --- + \frac{1}{a_n}}$$

2) If a, b, c are in H.P then $b = \frac{2ac}{a+c}$

Why?

a, b, c are in H.P so, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P

And hence $\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow \frac{2ac}{a+c} = b$

Illustration 7:

If the $(m+1)^{th}$, $(n+1)^{th}$ and $(r+1)^{th}$ term of an A.P are in G.P m, n, r are in H.P, Show that ratio of the common difference to the first term in the A.P is (-2/n).

Solution:

Let ‘a’ be the first term and ‘d’ be common difference of the A.P. Let x, y, z be the $(m+1)^{th}$, $(n+1)^{th}$ and $(r+1)^{th}$ term of the A.P then x = a+md, y = a+nd, z = a+rd. Since x, y, z are in G.P.

$\therefore$ $y^2 = xz$ i.e. $(a+nd)^2 = (a+rd)(a+md)$

$$\therefore \frac{d}{a} = \frac{r+m-2n}{n^2-rm}$$

Now m, n, r in H.P

$$\Rightarrow \frac{2}{n} = \frac{1}{m} + \frac{1}{r}$$

$$\Rightarrow \frac{2}{n} = \frac{m+r}{mr}$$

$$\text{Hence } \frac{d}{a} = \frac{2\left(\frac{r+m}{2} - n\right)}{n\left(n - \frac{rm}{n}\right)}$$

$$= \frac{2\left(\frac{mr}{n} - n\right)}{n\left(n - \frac{rm}{n}\right)}$$

$$= -\frac{2}{n}$$

Some Important Theorems:

If A, G, H are respectively AM, GM, HM between two positive unequal quantities then.

1) A>G>H

Why?

First of all let us Prove A>G.

The two numbers be x, y.

$A = \frac{x+y}{2}$ and $G = \sqrt{xy}$

So to prove $\frac{x+y}{2} > \sqrt{xy}$

Now $(\sqrt{x} - \sqrt{y})^2 > 0$ $(\because x, y \text{ are positive})$

$\Rightarrow x + y - 2\sqrt{xy} > 0$

$\Rightarrow \frac{x+y}{2} > \sqrt{xy}$

Hence A>G -------- (1)

Now Let us Prove G>H

Again $H = \frac{2xy}{x+y}$

$$\text{Also } (\sqrt{x}-\sqrt{y})^2 > 0$$
$$\Rightarrow x+y > 2\sqrt{xy}$$
$$\Rightarrow 1 > \frac{2\sqrt{xy}}{x+y}$$
$$\Rightarrow \sqrt{xy} > \frac{2xy}{x+y}$$
$$\Rightarrow G > H ------(2)$$

Combining (1) and (2) we get A>G>H.

Why G^2=AH?

Let x, y be two numbers.

So, $A = \frac{x+y}{2}, G = \sqrt{xy}, H = \frac{2xy}{x+y}$

Hence

$$AH = \left(\frac{x+y}{2}\right)\left(\frac{2xy}{x+y}\right)$$
$$= xy$$
$$= G^2$$

Illustration 8:

If a, b, c, d, be four distinct positive quantities in H.P then show that a+d>b+c.

Solution:

a, b, c, d are in H.P

Then A.M > H.M

For first three terms $\therefore \frac{a+c}{2} > b$

$\Rightarrow$ a+c>2b ------ (1)

And for last three terms

$\frac{b+d}{2} > c$

$\Rightarrow$ b+d > 2c ------ (2)

From (1) and (2)

a+c+b+d > 2b+2c

$\Rightarrow$ a+d > b+c.

Some Special Series:

1) $\sum n = 1+2+3+----+n = \frac{n(n+1)}{2}$.

Why?

This is an A.P with a =1 and d=1

So, $\sum n = \frac{n}{2}(2.1+(n-1)1)$

$= \frac{n(n+1)}{2}$

2) $\sum n^2 = 1^2+2^2+3^2+-----+n^2 = \frac{n(n+1)(2n+1)}{6}$

Proof:

$\because (x+1)^3 - x^3 = 3x^2 + 3x + 1$

Putting $x = 1,2,3,4,---n$ then

$2^3 - 1^3 = 3.1^2 + 3.1 + 1$

$3^3 - 2^3 = 3.2^2 + 3.2 + 1$

$4^3 - 3^3 = 3.3^2 + 3.3 + 1$

$5^3 - 4^3 = 3.4^2 + 3.4 + 1$

$-----$

$(n+1)^3 - n^3 = 3.n^2 + 3n + 1$

Adding all we get,

$(n+1)^3 - 1^3 = 3(1^2 + 2^2 + --- + n^2) + 3(1 + 2 + --- + n) + (1 + 1 --- n \text{ times})$

$\Rightarrow n^3 + 3n^2 + 3n = 3\sum n^2 = 3\sum n + n$

$\therefore 3\sum n^2 = n^3 + 3n^2 + 3n - 3\frac{n(n+1)}{2} - n$

$\Rightarrow \sum n^2 = \frac{n(n+1)(2n+1)}{6}$

3) $\sum n^3 = 1^3 + 2^3 + 3^3 --- + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Proof:

$\because (x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$

Putting $x = 1,2,3,4 ----- n$ then,

$2^4 - 1^4 = 4.1^3 + 6.1^2 + 4.1 + 1$

$3^4 - 2^4 = 4.2^3 + 6.2^2 + 4.2 + 1$

$4^4 - 3^4 = 4.3^3 + 6.3^2 + 4.3 + 1$

$------$

$(n+1)^4 - n^4 = 4.n^3 + 6.n^2 + 4.n + 1$

Adding all we get,

$$(x+1)^4 - 1^4 = 4(1^3 + 2^3 + --- + n^3) + 6(1^2 + 2^2 + ---- + n^2) + 4(1 + 2 + --- + n) + (1 + 1 + ------ + n \text{ times})$$

$$\Rightarrow n^4 + 4n^3 + 6n^2 + 4n = 4\sum n^3 + 6\sum n^2 + 4\sum n + n$$

$$= 4\sum n^3 + \frac{6.n(n+1)(2n+1)}{6} + \frac{4.n(n+1)}{2} + n$$

$$\Rightarrow 4\sum n^3 = n^4 + 4n^3 + 6n^2 + 4n - n(n+1)(2n+1) - 2n(n+1) - n$$

$$= n^2(n+1)^2$$

$$\therefore \sum n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Illustration 9:

If S_1, S_2, S_3, ------- S_n are the sums of infinite geometric series whose first term are 1, 2, 3, --------n and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4} ------ \frac{1}{n+1}$ respectively then find the value of $\sum_{r=1}^{2n-1} S_r^{\,2}$.

Solution:

The rth series has the first term a = r.

Common ratio = $\frac{1}{r+1}$, Sum = S_r.

$$\therefore S_r = \frac{r}{1-\frac{1}{r+1}} = \frac{r(r+1)}{r} = (r+1)$$

$$\text{So, } S_r^{\,2} = (r+1)^2$$

$$\therefore \sum_{r=1}^{2n-1} S_r^{\,2} = \sum_{r=1}^{2n-1} (r+1)^2 = 2^2 + 3^2 + ----+(2n)^2$$

$$= \{1^2 + 2^2 + 3^2 + ------+(2n)^2\} - 1$$

$$\frac{2n(2n+1)(4n+1)}{6} - 1$$

$$= \frac{1}{3} n(2n+1)(4n+1) - 1$$

Some Special Series

1)Method of Difference:

Suppose a_1, a_2, a_3--------- is a sequence such that sequence a_2-a_1, a_3-a_2, -------- is either an A.I or a G.P. Then n^{th} term of this sequence may be found as follows:

Let

$$S = a_1 + a_2 + ---------- a_{n-1} + a_n$$

$$S = a_1 + a_2 ------a_{n-2} + a_{n-1} + a_n$$

$$0 = a_1 + (a_2 - a_1) + ----+(a_n - a_{n-1}) - a_n$$

$$\Rightarrow a_n = a_1 + (a_2 - a_1) + ------+(a_n - a_{n-1})$$

So, we find value of a_n and hence S_n.

Dumb Question:

1) How do we ensure that value of a_n can be calculated?

Ans: The terms $a_2 - a_1, a_3 - a_2, -----a_n - a_{n-1}$ form either an A.P or a G.P. So

we can always sum them up.

2) Method of V_n:

Let T_1, T_2, T_3------- be terms of a sequence. If there exists a sequence V_1, V_2, V_3---- Satisfying $T_k = V_k - V_{k-1}, k \geq 1$ then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} V_k - V_{k-1}$$
$$= V_n - V_0$$

Illustration 10:

$$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + ---- \text{ to n terms } = \frac{6n}{n+1}$$. Then

find the sum to infinite terms of the series on the left hand side.

Solution:

$$t_n = \frac{3+(n-1).2}{1^2+2^2+3^2+---+n^2} = \frac{2n+1}{n(n+1)(2n+1)/6}$$
$$= \frac{6}{n(n+1)} = 6\left\{\frac{1}{n} - \frac{1}{(n+1)}\right\}$$
$$\therefore t_1 = 6\left(1 - \frac{1}{2}\right)$$

$$t_2 = 6\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$t_3 = 6\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$-------$$

$$t_n = 6\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Adding L.H.S = $6\left(1 - \frac{1}{n+1}\right) = \frac{6n}{(n+1)}$

Thus $S_n = 6\left(1 - \frac{1}{n+1}\right)$

$\therefore$ Sum of infinite terms

$$= \lim_{n\to\infty} S_n = \lim_{n\to\infty} 6\left(1 - \frac{1}{(n+1)}\right) = 6$$

PROBLEMS (EASY TYPE)

1) If the roots of the equation $x^3 - 12x^2 + 39x - 28 = 0$ are in A.P then what will be their common difference?

Solution:

Sum of three numbers a-d, a, a+d are in A.P.

= 3a = 12

$\therefore$ a = 4 is a root of cubic

$\therefore$ (x-4) (x^2-8x+7) = 0

$\therefore$ x = 1, 4, 7 or 7, 4, 1 and d = ±3.

2) The sum of the squares of three distinct real numbers which are in G.P is S^2 if their sum is αS, show that $\alpha^2 \in \left(\frac{1}{3},1\right) \cup (1,3)$.

Solution: Let the three numbers in G.P is $\frac{a}{r}, a, ar.$

$$\therefore \frac{a^2}{r^2} + a^2 + a^2r^2 = S^2$$

$$\Rightarrow \frac{a^2(1+r^2+r^4)}{r^2} = S^2 ---(1)$$

Again $\frac{a}{r} + a + ar = \alpha S$

$$\Rightarrow \frac{a(1+r+r^2)}{r} = \alpha S ----(2)$$

From (1) and (2) we get.

$$\frac{a^2(1+r^2+r^4)}{r^2} = \frac{a^2(1+r+r^2)^2}{r^2\alpha^2}$$

$$\Rightarrow \alpha^2 (1+r^2+r^4) = (1+r+r^2)^2$$

$$\Rightarrow \alpha^2 (1+r+r^2)(1-r+r^2) = (1+r+r^2)^2$$

$$\therefore 1+r+r^2 \neq 0$$

$$\therefore \alpha^2 (1-r+r^2) = (1+r+r^2)$$

$$r^2(\alpha^2-1) - (\alpha^2+1)r + (\alpha^2-1) = 0$$

$$\Rightarrow r^2 - \frac{(\alpha^2+1)}{(\alpha^2-1)}r + 1 = 0$$

Since r is real $\left(\frac{\alpha^2+1}{\alpha^2-1}\right)^2 - 4 > 0$ {the number in G.P are different}.

$$\Rightarrow \left(\frac{\alpha^2+1}{\alpha^2-1}-2\right)\left(\frac{\alpha^2+1}{\alpha^2-1}+2\right)>0$$

$$\Rightarrow \frac{(3-\alpha^2)(3\alpha^2-1)}{(\alpha^2-1)^2}>0$$

$$\Rightarrow \frac{(\alpha^2-3)(\alpha^2-1/3)}{(\alpha^2-1)^2}<0$$

Fig (1)

Critical points of α^2 are 1/3, 3, 1 from wavy curve method.

$$\therefore\ \alpha^2 \in \left(\frac{1}{3},1\right)\cup(1,3)$$

3) Find the sum

1.2.3.4.5+2.3.4.5.6+3.4.5.6.7+-------+ (n-4) (n-3) (n-2) (n-1) n

Solution:

The given sum = $\sum_{r=1}^{n-4} r(r+1)(r+2)(r+3)(r+4)$

$$=\sum_{r=1}^{n-4}\left[\frac{r(r+1)(r+2)(r+3)(r+4)(r+5)-r(r-1)(r+1)(r+2)(r+3)(r+4)}{6}\right]$$

$$=\frac{1}{6}\sum_{r=1}^{n-4}(V_r-V_{r-1}) \quad \text{where } V_r = r(r+1)(r+2)(r+3)(r+4)(r+5)$$

$$=\frac{1}{6}\left(V_{r-4}-V_0\right)$$

$$=\frac{1}{6}(n-4)(n-3)(n-2)(n-1)n(n+1)$$

3) Prove that natural number 7.7.7------7 consisting of seven is a composite number divisible by a multiple of 7.

Solution:

7.7.7------7 = 7×1.1.1----1

$$= 7\times\left\{10^{76}+10^{75}+10^{74}+------+10+1\right\}$$
$$= 7\times\frac{10^{77}-1}{10-1} = 7\times\frac{10^{77}-1}{10^{7}-1}\times\frac{10^{7}-1}{10-1}$$
$$= 7\times\left\{1+10^{7}+10^{14}+----+10^{70}\right\}\times\left\{1+10+10^{2}+---+10^{6}\right\}$$

= 7×integer×integer.

4) If a, b, c are in H.P and b, c, d are in G.P and c, d, e are in A.P show that

$$e = \frac{ab^2}{(2a-b)^2}$$

Solution:

Given a, b, c are in H.P

$$\therefore\ b = \frac{2ac}{a+c} ----(1)$$

b, c, d are in G.P $c^2 = bd----(2)$

And c, d, e are in A.P

$$d = \frac{c+e}{2} ----(3)$$

We want the relation in a, b, e

Then eliminate c, d from (1), (2) and (3)

From (1) ab+bc = 2ac

$$\Rightarrow c = \frac{ab}{2a-b} ----(4)$$

From (3) and (4)

$$d = \frac{1}{2}\left\{\frac{ab}{2a-b} + e\right\}$$

Substituting the values of c and d from (4) and (5) in (2) we get,

$$\frac{a^2b^2}{(2a-b)^2} = \frac{b}{2}\left[\frac{ab}{2a-b} + e\right]$$

$$\Rightarrow \frac{2a^2b}{(2a-b)^2} - \frac{ab}{(2a-b)} = e$$

$$\therefore e = \frac{2a^2b - 2a^2b + ab^2}{(2a-b)^2}$$

$$\text{Hence } e = \frac{ab^2}{(2a-b)^2}$$

5) Balls are arranged in rows to form an equilateral triangle the first row consist of one ball, the second row of two balls and so on. If 669 more balls are added then all the balls can be arranged in the shape of a square and each of the sides then contains 8 balls less than each side of the triangle then find the initial number of balls.

Solution:

S = 1+2+3+4+-------+n = $\sum n = \frac{n(n+1)}{2}$

S+669 = $(n-8)^2$ by given condition

Or $\frac{n(n+1)}{2} + 669 = n^2 - 16n + 64$

$n^2 - 33n - 1210 = 0$
$\Rightarrow (n-55)(n+22) = 0$
$\therefore n = 55$

$\therefore$ S = 1540.

Check: 1540+669 = 2209 = $(55-8)^2 = (47)^2$.

6) The equation $x^4 - 4x^3 + px^2 + qx + 1 = 0$ has four positive roots then find the value of (p, q).

Solution: Let the roots be $\alpha, \beta, \gamma, \delta$.

$\therefore \alpha + \beta + \gamma + \delta = 4$. and $\alpha.\beta.\gamma.\delta = 1$.

Or $\frac{\alpha+\beta+\gamma+\delta}{4} = 1, \quad (\alpha\beta\gamma\delta)^{\frac{1}{4}} = 1$

$\therefore$ A.M = G.M of the roots

$\therefore$ All the roots are equal and each equal to 1.

$\therefore x^4 - 4x^3 + px^2 + qx + 1 = (x-1)^4$
$= x^4 - 4x^3 + 6x^2 - 4x + 1$

Comparing we get p = 6, q = -4.

7) If a_1, a_2, a_3, --------- a_n are in A.P where $a_i \neq k\pi$ for all i, prove that

Cosec a_1.Cosec a_2 + Cosec a_2.Cosec a_3 + --------+Cosec a_{n-1}.Cosec a_n

$= \frac{\text{Cot}a_1 - \text{Cot}a_n}{\text{Sin}(a_2 - a_1)}$

Solution:

$$\text{L.H.S} = \frac{1}{\text{Sin}a_1.\text{Sin}a_2} + \frac{1}{\text{Sin}a_2.\text{Sin}a_3} + --- + \frac{1}{\text{Sin}a_{n-1}.\text{Sin}a_n}$$

$$t_r = \frac{\text{Sin}(a_{r+1} - a_r)}{\text{Sin}a_r.\text{Sin}a_{r+1}} \times \frac{1}{\text{Sin}(a_{r+1} - a_r)}$$

$$= \frac{\text{Sin}a_{r+1}.\text{Cos}a_r - \text{Cos}a_{r+1}.\text{Sin}a_r}{\text{Sin}a_r.\text{Sin}a_{r+1}} \times \frac{1}{\text{Sin}d}$$

Where common difference $= d$

$$= (\text{Cot}a_r - \text{Cot}a_{r+1}).\frac{1}{\text{Sin}d}$$

$$\therefore\ t_1 = (\text{Cot}a_1 - \text{Cot}a_2).\frac{1}{\text{Sin}d}$$

$$t_2 = (\text{Cot}a_2 - \text{Cot}a_3).\frac{1}{\text{Sin}d}$$

$$t_3 = (\text{Cot}a_3 - \text{Cot}a_4).\frac{1}{\text{Sin}d}$$

$$----------$$

$$t_{n-1} = (\text{Cot}a_{n-1} - \text{Cot}a_n).\frac{1}{\text{Sin}d}$$

Adding $\text{L.H.S} = (\text{Cot}a_1 - \text{Cot}a_n).\frac{1}{\text{Sin}d}$

$$= \frac{\text{Cot}a_1 - \text{Cot}a_n}{\text{Sin}d}$$

$$= \frac{(\text{Cot}a_1 - \text{Cot}a_n)}{\text{Sin}(a_2 - a_1)}$$

8) Prove that $\frac{S}{S-a} + \frac{S}{S-b} + \frac{S}{S-c} \geq \frac{9}{2}$ if S = a+b+c [a, b, c>0].

Solution:

We have to prove

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\geq\frac{9}{2(a+b+c)}$$

As A.M $\geq$ H.M

$$\Rightarrow\frac{(a+b)+(b+c)+(c+a)}{3}\geq\frac{3}{\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}}$$

$$\Rightarrow\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{9}{2(a+b+c)}$$

9) Two consecutive numbers from 1, 2, 3-------n are removed. Arithmetic mean of the remaining numbers is $\frac{105}{4}$. Find n and those removed number.

Solution:

Let p and (p+1) be removed numbers from 1, 2, 3, ------ n then sum of remaining numbers.

$$=\frac{n(n+1)}{2}-(2p+1)$$

From given condition

$$\frac{105}{4}=\frac{\frac{n(n+1)}{2}-(2p+1)}{(n-2)}$$

$$\Rightarrow 2n^2-103n-8p+206=0$$

Since n and p are integers so n must be even let n=2r.

We get $p=\frac{4r^2+103(1-r)}{4}$

Since p is an integer then (1-r) must be divisible by 4.

Let r = 1+4t we get

$n = 2+8t$ and $p = 16t^2-95t+1$.

Now $1 \leq p < n$.

$\Rightarrow 1 \leq 16t^2-95t+1 < 8t+2$.

$\Rightarrow t = 6$.

$\Rightarrow n = 50$ and $p = 7$.

Hence removed numbers are 7 and 8.

10) Find the sum of infinitely decreasing G.P whose third term, the triple product of the first term by the fourth term A.P with the common difference equal to 1/8.

Solution:

Let G.P be a, ar, ar^2------ Since the G.P is infinite and decreasing $-1 < r < 1$ and $r>0$, so $0 < r < 1$ and so $a>0$ according to the hypothesis.

ar^2, $3a\ ar^3$, ar are in A.P with common difference 1/8.

$$\Rightarrow 3a^2r^3 - ar^2 = \frac{1}{8} ----(1)$$

$$\text{and } ar = ar^2 + 2\times\frac{1}{8} ----(2)$$

$$\text{from (2) } a = \frac{1}{4(r-r^2)} = \frac{1}{4r(1-r)} ---(3)$$

from (1) and (3)

$$\frac{3r^3}{16r^2(1-r)^2} - \frac{r^2}{4r(1-r)} = \frac{1}{8}$$

$$\Rightarrow \frac{3r}{16(1-r)^2} - \frac{r}{4(1-r)} = \frac{1}{8}$$

$$\therefore 2r^2 + 3r - 2 = 0$$

$$\text{which gives } r = \frac{1}{2}, -2 \text{ but } 0 < r < 1$$

$$\therefore r = \frac{1}{2}$$

From (3) we get a = 1

Hence G.P is $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} ----$

$$\therefore S_\infty = \frac{1}{1-\frac{1}{2}}$$

$$= 2$$

PROBLEMS (MEDIUM TYPE)

1) A sequence a_1, a_2, a_3, ------- a_n of real numbers is such that $a_1 = 0$, $|a_2| = |a_1+1|$, $|a_3| = |a_2+1|$, ---------- $|a_n| = |a_{n-1}+1|$. Prove that arithmetic mean of a_1, a_2, -------- a_n can not be less than -1/2.

Solution:

Let us add one more number a_{n+1} to the given sequence. The number a_{n+1} is such that $|a_{n+1}| = |a_n+1|$.

Squaring all numbers we have,

$$a_1^2 = 0$$
$$a_2^2 = a_1^2 + 2a_1 + 1$$
$$a_3^2 = a_2^2 + 2a_2 + 1$$
$$----------$$
$$----------$$
$$a_n^2 = a_{n-1}^2 + 2a_{n-1} + 1$$
$$a_{n+1}^2 = a_n^2 + 2a_n + 1$$

Adding the above equalities we get,

$$a_1^2 + a_2^2 + ---- a_n^2 + a_{n+1}^2 = a_1^2 + a_2^2 + ----+a_n^2 + 2(a_1 + a_2 + --+a_n) + n$$
$$\Rightarrow 2(a_1 + a_2 + a_3 + --+a_n) = -n + a_{n+1}^2 \geq -n$$
$$\Rightarrow \frac{a_1 + a_2 + a_3 + ---+a_n}{n} \geq -\frac{1}{2}$$

Dumb Question:

1) Why $-n + a_{n+1}^2 \geq -n$?

Ans: a_{n+1} is given to be a real no in the question and hence a_{n+1}^2 is going to be a positive number.

There fore $-n + a_{n+1}^2 \geq -n$.

2) There are (4n+1) terms in a certain sequence of which the first (2n+1) terms form an A.P of common difference 2 and the last (2n+1) terms are in G.P of common ratio 1/2. If the middle terms of both A.P and G.P be the same what is the mid term of this sequence.

Solution: d=2, r=1/2

If there be odd number of terms then mid term = $\frac{1}{2}(\text{odd}+1)$.

T_{2n+1} is the mid term of sequence of (4n+1) odd terms a+2nd = a+4n ------ (1)

This mid term is the last term of A.P and first term of following G.P. Each of (2n+1) terms with this term being common to both.

T_{n+1} and t_{n+1} are mid terms of A.P and G.P

T_{n+1} = a+nd = a+2n.

$$t_{n+1} = AR^n = T_{2n+1}\left(\frac{1}{2}\right)^n = (a+4n)\left(\frac{1}{2}\right)^n \quad \text{by}(1)$$

By given condition $T_{n+1} = t_{n+1}$.

$$a + 2n = (a + 4n)\frac{1}{2^n}$$

$$\text{or } (2^n - 1)a = 4n - 2n.2^n$$

$$\therefore\ a = \frac{4n - n.2^{n+1}}{2^n - 1}$$

Hence mid term of sequence by (1) is

$$\begin{aligned} a + 4n &= \frac{4n - n.2^{n+1}}{2^n - 1} + 4n \\ &= \frac{-n.2^{n+1} + 2n.2^{n+1}}{2^n - 1} \\ &= \frac{n.2^{n+1}}{2^n - 1} \end{aligned}$$

Dumb Question:

1) Why middle term of the G.P is $(a + 4n)\left(\frac{1}{2}\right)^n$?

Ans: Well the first term of the G.P is the last term of the A.P

So, the first term = a+2(2n+1-1)

= a+4n.

Now the $(n+1)^{th}$ term of the G.P is

$$(a+4n)\left(\frac{1}{2}\right)^{n+1-1}$$

$$= (a+4n)\left(\frac{1}{2}\right)^{n}$$

3) If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$ then find $\sum_{r=1}^{n} \frac{1}{T_r}$.

Solution:

$$T_n = S_n - S_{n-1}$$

$$= \sum_{r=1}^{n} T_r - \sum_{r=1}^{n-1} T_r$$

$$= \frac{n(n+1)(n+2)(n+3)}{8} - \frac{(n-1)n(n+1)(n+2)}{8}$$

$$= \frac{n(n+1)(n+2)}{8}[(n+3)-(n-1)]$$

$$= \frac{n(n+1)(n+2)}{8}(4)$$

$$= \frac{n(n+1)(n+2)}{2}$$

$$\Rightarrow \frac{1}{T_n} = \frac{2}{n(n+1)(n+2)} --- (1)$$

$$\therefore V_n = \frac{2}{(n+1)(n+2)}$$

Then $V_{n-1} = \dfrac{2}{n(n+1)}$

$$\therefore V_n - V_{n-1} = \frac{2}{(n+1)(n+2)} - \frac{2}{n(n+1)}$$

$$= -\frac{4}{n(n+1)(n+2)}$$

$$= -2\left(\frac{1}{T_n}\right) ------ \text{from}(1)$$

$$\therefore \frac{1}{T_n} = -\frac{1}{2}\left(V_n - V_{n-1}\right)$$

Putting n = 1, 2, 3, ------n

$$\frac{1}{T_1} = -\frac{1}{2}(V_1 - V_0)$$

$$\frac{1}{T_2} = -\frac{1}{2}(V_2 - V_1)$$

$$\frac{1}{T_n} = -\frac{1}{2}(V_n - V_{n-1})$$

$$\therefore \frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}+-----+\frac{1}{T_n}=-\frac{1}{2}(V_n-V_0)$$

$$\Rightarrow \sum_{r=1}^{n}\frac{1}{T_r}=-\frac{1}{2}\left[\frac{2}{(n+1)(n+2)}-\frac{2}{2}\right]$$

$$=\frac{(n+1)(n+2)-2}{2(n+1)(n+2)}$$

$$=\frac{(n^2+3n)}{2(n+1)(n+2)}$$

4) Evaluate sum of n terms of the series $\frac{8}{5}+\frac{16}{65}+\frac{24}{325}+-----$

Solution:

$$S_n=\frac{8}{5}+\frac{16}{65}+\frac{24}{325}+-----\text{to n terms}$$

$$=\sum_{r=1}^{n}\left(\frac{8r}{4r^4+1}\right)$$

$$=\sum_{r=1}^{n}\frac{8r}{(2r^2-2r+1)(2r^2+2r+1)}$$

$$=2\sum_{r=1}^{n}\left\{\frac{1}{(2r^2-2r+1)}-\frac{1}{(2r^2+2r+1)}\right\}$$

$$=2\left\{\frac{1}{1}-\frac{1}{5}+\frac{1}{5}-\frac{1}{13}+\frac{1}{13}---+\frac{1}{(2n^2-2n+1)}-\frac{1}{(2n^2+2n+1)}\right\}$$

$$=2\left\{1-\frac{1}{(2n^2+2n+1)}\right\}$$

$$=\frac{4n^2+4n}{(2n^2+2n+1)}$$

5) Find the sum of $\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+-------+\frac{2^n x^{2^n-1}}{1+x^{2^n}}$

Solution:

If S be the sum then consider $S-\frac{1}{1+x}$

$$\text{Now } S-\frac{1}{1+x}=\frac{1}{1+x}-\frac{1}{1-x}+\left(\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+----+\frac{2^n x^{2^n-1}}{1+x^{2^n}}\right)$$

$$=-\frac{2x}{1-x^2}+\frac{2x}{1+x^2}+\left(\frac{4x^3}{1+x^4}+\frac{8x^4}{1+x^8}+---+\frac{2^n x^{2^n-1}}{1+x^{2^n}}\right)$$

$$=-\frac{4x^3}{1-x^4}+\frac{4x^3}{1+x^4}+\left(\frac{8x^4}{1+x^8}+---+\frac{2^n x^{2^n-1}}{1+x^{2^n}}\right)$$

$$----------$$

$$=\frac{-2^{n+1}x^{2^{n+1}-1}}{1-x^{2^{n+1}}}$$

$$S=\frac{1}{1-x}+\frac{2^{n+1}x^{2^{n+1}-1}}{x^{2^{n+1}}-1}$$

6) If 2a+b+3c = 1 and a>0, b>0, c>0 find the greatest value of $a^4b^2c^2$ and obtain the corresponding values of a, b, c.

Solution:

Consider the positive numbers $\frac{2a}{4},\frac{2a}{4},\frac{2a}{4},\frac{2a}{4},\frac{b}{2},\frac{b}{2},\frac{3c}{2},\frac{3c}{2}$. {As there is a^4, take

4 equal parts of 2a; as there is b^2 take 2 equal parts of b; as there is c^2 take 2 equal parts of 3c}.

For the numbers,

$$A = \frac{\frac{2a}{4}+\frac{2a}{4}+\frac{2a}{4}+\frac{2a}{4}+\frac{b}{2}+\frac{b}{2}+\frac{3c}{2}+\frac{3c}{2}}{4+2+2}$$

$$= \frac{2a+b+3c}{8} = \frac{1}{8} \quad (\because 2a+b+3c=1)$$

$$G = \left(\frac{2a}{4}.\frac{2a}{4}.\frac{2a}{4}.\frac{2a}{4}.\frac{b}{2}.\frac{b}{2}.\frac{3c}{2}.\frac{3c}{2}\right)^{\frac{1}{8}}$$

$$= \left(\frac{1}{2^4}.\frac{1}{2^2}.\frac{1}{2^2}.3^2.a^4.b^2.c^2\right)^{\frac{1}{8}}$$

$$\therefore A \geq G \Rightarrow \frac{1}{8} \geq \left(\frac{3^2}{2^8}.a^4.b^2.c^2\right)^{\frac{1}{8}}$$

$$\text{or } \frac{1}{8^8} \geq \frac{3^2}{2^8}.a^4.b^2.c^2$$

$$\text{or } \frac{2^8}{3^2.8^8} \geq a^4b^2c^2$$

$$\text{or } \frac{1}{9.4^8} \geq a^4b^2c^2$$

So the greatest value of $a^4b^2c^2 = \frac{1}{9.4^8}$

Now when the value is greatest the numbers themselves have to be equal.

So, $\frac{2a}{4} = \frac{b}{2} = \frac{3c}{2} = k$ (Let)

$$a = 2k$$

$$b = 2k$$

$$c = \frac{2k}{3}$$

$$2a + b + 3c = 1$$

$$\therefore\ 8k = 1$$

$$\text{or } k = \frac{1}{8}$$

Hence $a = \frac{1}{4}, b = \frac{1}{4}, c = \frac{1}{12}$

7) If $f(x) = x - 2x^2 + 3x^3 - - - + (-1)^{n-1}.n\,x^n$ where $x \in R$ then find $f^1\left(\frac{1}{2}\right)$.

Solution:

$$f(x) = 1 - 2^2 x + 3^2 x^2 - 4^2 x^3 + ------ + (-1)^{n-1}.n^2.x^{n-1}$$

$$\therefore f^1\left(\frac{1}{2}\right) = 1^2 - 2^2\left(\frac{1}{2}\right) + 3^2\left(\frac{1}{2^2}\right) - 4^2\left(\frac{1}{2^3}\right) + ------ + (-1)^{n-1}.n^2.\frac{1}{2^{n-1}}$$

$$\therefore -\frac{1}{2}f^1\left(\frac{1}{2}\right) = -1^2\frac{1}{2} + 2^2.\frac{1}{2^2} - 3^2.\frac{1}{2^3} + ------ + (-1)^{n-1}.n^2.\frac{1}{2^n}$$

Substracti ng

$$\frac{3}{2}f^1\left(\frac{1}{2}\right) = \left\{1 - 3.\frac{1}{2} + 5.\frac{1}{2^2} - 7.\frac{1}{2^3} + ---\text{to n terms}\right\} + (-1)^{n-1}.n^2.\frac{1}{2^n}$$

$$\text{Let } S_n = 1 - 3.\frac{1}{2} + 5.\frac{1}{2^2} - 7.\frac{1}{2^3} + ------ + (-1)^{n-1}.\frac{(2n-1)}{2^n - 1}$$

$$-\frac{1}{2}S_n = -1.\frac{1}{2} + 3.\frac{1}{2^2} - 5.\frac{1}{2^3} + ------ + (-1)^n.\frac{(2n-1)}{2^n}$$

Substracti ng –

$$\frac{3}{2}S_n = \left\{1 - 2.\frac{1}{2} + 2.\frac{1}{2^2} - 2.\frac{1}{2^3} + ----+\text{to n terms}\right\} - (-1)^n\frac{2n-1}{2^n}$$

$$\frac{3}{2}S_n = \left(1 - 1 + \frac{1}{2} - \frac{1}{2^2} + ---+\text{to n terms}\right) - (-1)^n.\frac{2n-1}{2^n}$$

$$= \frac{1}{2} - \frac{1}{2^2} + ------ + \text{to } (n-2) \text{ terms} - (-1)^n.\frac{2n-1}{2^n}$$

$$= \frac{\frac{1}{2}\left\{1 - \left(-\frac{1}{2}\right)^{n-2}\right\}}{1 - \left(-\frac{1}{2}\right)} - (-1)^n.\frac{2n-1}{2^n}$$

$$\therefore S_n = \frac{2}{9}\left\{1 - \left(-\frac{1}{2}\right)^{n-2}\right\} - (-1)^n.\frac{2n-1}{3.2^{n-1}}$$

$$\text{Thus } \frac{3}{2}f^1\left(\frac{1}{2}\right) = \frac{2}{9}\left\{1 - \left(-\frac{1}{2}\right)^{n-2}\right\} - (-1)^n.\frac{2n-1}{3.2^{n-1}} + (-1)^{n-1}.\frac{n^2}{2^n}$$

$$= \frac{2}{9}\left\{1-\left(-\frac{1}{2}\right)^{n-2}\right\} + (-1)^{n-1}.\frac{3n^2+4n-2}{3.2^n}.$$

$$\therefore f^1\left(\frac{1}{2}\right) = \frac{4}{27}\left\{1-\left(-\frac{1}{2}\right)^{n-2}\right\} + (-1)^{n-1}\frac{3n^2+4n-2}{9.2^{n-1}}$$

8) Let S_n, n=1, 2, 3, ------- be the sum of infinite geometric series whose first term is n and the common ratio is $\frac{1}{n+1}$ evaluate.

$$\lim_{n\to\infty} \frac{S_1S_n + S_2S_{n-1} + S_3S_{n-2} + ----+S_nS_1}{S_1^{\ 2} + S_2^{\ 2} + ----+S_n^{\ 2}}$$

Solution:

$$S_n = \frac{n}{1-\frac{1}{n+1}}$$

$$S_n = n+1$$

$$\therefore S_1S_n + S_2S_{n-1} + S_3S_{n-2} + ----+S_nS_1$$

$$= 2(n+1) + 3n + 4(n-1) + ----+(n+1).2$$

$$= \sum_{r=1}^{n}(r+1)(n-r+2)$$

$$= \sum_{r=1}^{n}(nr - r^2 + 2r + n - r + 2)$$

$$= \sum_{r=1}^{n}(n+1)r - r^2 + (n+2)$$

$$= (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 + (n+2)\sum_{r=1}^{n} 1$$

$$= (n+1)\sum n - \sum n^2 + n(n+2)$$

$$= \frac{(n+1)n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + n(n+2)$$

$$= \frac{n}{6}\left\{3(n^2+2n+1) - (2n^2+3n+1) + 6n + 12\right\}$$

$$= \frac{n}{6}\left\{n^2 + 9n + 14\right\} ------(1)$$

And

$$S_1^2 + S_2^2 + ------+S_n^2$$

$$= 2^2 + 3^2 + -------+(n+1)^2$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} - 1$$

$$= \frac{n(2n^2+9n+13)}{6} ----(2)$$

From (1) and (2) we get,

$$\frac{S_1S_n + S_2S_{n-1} + S_3S_{n-2} + ----+S_nS_1}{S_1^{\ 2} + S_2^{\ 2} + ----+S_n^{\ 2}} = \frac{n^2 + 9n + 14}{2n^2 + 9n + 13}$$

$$= \frac{1 + \frac{9}{n} + \frac{14}{n^2}}{2 + \frac{9}{n} + \frac{13}{n^2}}$$

$$\lim_{n\to\infty} \frac{S_1S_n + S_2S_{n-1} + S_3S_{n-2} + ----+S_nS_1}{S_1^{\ 2} + S_2^{\ 2} + ----+S_n^{\ 2}} = \lim_{n\to\infty} \frac{1 + \frac{9}{n} + \frac{14}{n^2}}{2 + \frac{9}{n} + \frac{13}{n^2}}$$

$$= \frac{1+0+0}{2+0+0} = \frac{1}{2}$$

Dumb Question:

1) $S_1S_n + S_2S_{n-1} + S_3S_{n-2} + ----+S_nS_1 = \sum_{r=1}^{n} (r+1)(n-r+2)$

Ans:

S_n is n+1

So, S_r = r+1.

$$\text{Now, } S_1S_n + S_2S_{n-1} + S_3S_{n-2} + ----+S_nS_1 = \sum_{r=1}^{n} S_rS_{n-r+1}$$

$$= \sum_{r=1}^{n} (r+1)(n-r+2)$$

9) Prove that the numbers of the sequence 121, 12321, 1234321, ----- are each a perfect square of odd integer.

Solution:

We have first term $T_1 = 121 = 10^2 + 2\times10 + 1$

Second Term $T_2 = 12321 = 10^4 + 2\times10^3 + 3\times10^2 + 2\times10 + 1$.

Third Term $T_3 = 1234321 = 10^6 + 2\times10^5 + 3\times10^4 + 4\times10^3 + 3\times10^2 + 2\times10 + 1$

---

n^{th} term T_n = 123---------- n (n+1) n ---------- 4321

$$= 1\times10^{2n} + 2\times10^{2n-1} + 3\times10^{2n-2} + ----+ n\times10^{n+1} + (n+1)\times10^n +$$
$$n\times10^{n-1} + (n-1)\times10^{n-2} + --------+ 3\times10^2 + 2\times10 + 1$$
$$= 10^{2n}\left(1 + 2\frac{1}{10} + 3\left(\frac{1}{10}\right)^2 + ----+ n\left(\frac{1}{10}\right)^{n-1}\right) + (1 + 2\times10 + 3\times10^2 + ----+$$
$$n\times10^{n-1} + (n+1)\times10^n)$$
$$= 10^{2n}S_1 + S_2 \quad (\text{Say}) ------(1)$$

$$\therefore S_1 = 1 + 2\left(\frac{1}{10}\right) + 3\left(\frac{1}{10}\right)^2 + ----+ (n-1)\left(\frac{1}{10}\right)^{n-2} + n\left(\frac{1}{10}\right)^{n-1}$$
$$\therefore \frac{1}{10}S_1 = 0 + \left(\frac{1}{10}\right) + 2\left(\frac{1}{10}\right)^2 + ------+ (n-1)\left(\frac{1}{10}\right)^{n-1} + n\left(\frac{1}{10}\right)^n$$

Substracting we get,

$$\frac{9}{10}S_1 = 1+\left(\frac{1}{10}\right)+\left(\frac{1}{10}\right)^2+-----+\left(\frac{1}{10}\right)^{n-1}-n\left(\frac{1}{10}\right)^n$$

$$= \frac{1\left\{1-\left(\frac{1}{10}\right)^n\right\}}{1-\frac{1}{10}} - n\left(\frac{1}{10}\right)^n$$

$$S_1 = \frac{100}{81}\left(1-\frac{1}{10^n}\right)-\frac{10n}{9.10^n}$$

$$S_1 = \frac{10^2}{81}\left(1-\frac{1}{10^n}\right)-\frac{90n}{81.10^n}------(2)$$

And $S_2 = 1+2(10)+3(10)^2+---+n(10)^{n-1}+(n+1)(10)^n$

$$\therefore 10S_2 = 10+2(10)^2+---+n(10)^n+(n+1)10^{n+1}$$

Substracting we get,

$$-9S_2 = 1+10+(10)^2+-----+(10)^n-(n+1)(10)^{n+1}$$

$$= \frac{10^{n+1}-1}{10-1}-(n+1)10^{n+1}$$

$$\therefore S_2 = \frac{1-10^{n+1}}{81}+\frac{(n+1)10^{n+1}}{9}----(3)$$

Substituting the values of S_1 and S_2 from (2) and (3) in (1) we get

$$T_n = 10^{2n}\left(\frac{10^2}{81}\right)\left(1-\frac{1}{10^n}\right)-\frac{90n.10^{2n}}{81.10^n}+\frac{1-10^{n+1}}{81}+\frac{(n+1)10^{n+1}}{9}$$

$$= \frac{1}{81}\left[10^{2n+2}-10^{n+2}+9n.10^{n+1}+1-10^{n+1}+9.(n+1).10^{n+1}\right]$$

$$= \frac{1}{81}\left[10^{2n+2}-10.10^{n+1}+1+8.10^{n+1}\right]$$

$$= \frac{1}{81}\left[10^{2n+2}-2.10^{n+1}+1\right]$$

$$= \left(\frac{10^{n+1}-1}{9}\right)^2$$

Since sum of digits of $10^{n+1}-1$ is divisible by 9.

$\therefore \dfrac{10^{n+1}-1}{9}$ is a positive integer.

Thus T_n is a perfect square.

10) Sum to n terms:

$$1+2^2-3^3+4+5^2-6^3+7+8^2-9^3+----$$

Solution:

Regrouping the terms the series becomes

$$(1+4+7+---)+(2^2+5^2+8^2+---)-(3^3+6^3+9^3+--)$$

The number of terms in each group will depend on n.

If n=3m i.e. n is divisible by 3

Sum of the first group = 1+4+7+--------- to m terms;

$$= \frac{m}{2}\left[2.1+(m-1)3\right] = \frac{m(3m-1)}{2} = \frac{n}{6}(n-1) ----(1)$$

Sum of the second group;

$$= 2^2 + 5^2 + 8^2 + ----+\text{to m terms}$$

$$= \sum_{r=1}^{m}[2+(r-1)3]^2 = \sum_{r=1}^{m}(3r-1)^2 = \sum_{r=1}^{m}(9r^2-6r+1)$$

$$= 9\sum_{r=1}^{m} r^2 - 6\sum_{r=1}^{m} r + \sum_{r=1}^{m} 1$$

$$= 9\frac{m(m+1)(2m+1)}{6} - 6\frac{m(m+1)}{2} + m$$

$$= 3m(m+1)\left\{\frac{1}{2}(2m+1)-1\right\} + m$$

$$= \frac{3m(m+1)(2m-1)}{2} + m ----(2)$$

$$= \frac{n\left(\frac{n}{3}+1\right)\left(\frac{2n}{3}-1\right)}{2} + \frac{n}{3} = \frac{n(n+3)(2n-3)}{18} + \frac{n}{3}$$

Sum of the third group = $3^3 + 6^3 + 9^3 + ----\text{to m terms}$

$$= 3^3(1^3 + 2^3 + 3^3 + ---- + m^3)$$

$$= 27\left\{\frac{m(m+1)}{2}\right\}^2 ----(3)$$

$$= \frac{27}{4}\left\{\frac{n}{3}\left(\frac{n}{3}+1\right)\right\}^2$$

$$= \frac{27}{4}.\frac{1}{81}n^2(n+3)^2 = \frac{n^2(n+3)^2}{12}$$

$\therefore$ the sum

$$= \frac{n(n-1)}{6} + \frac{n(n+3)(2n-3)}{18} + \frac{n}{3} - \frac{n^2(n+3)^2}{12}$$

When n is divisible by 3.

If n=3m+1 i.e. When n is divided by 3. The remainder is 1. In this case the first group will have m+1 term while others will have m terms each.

$\therefore$ In (1) we shall get

$$\frac{(m+1)\{3(m+1)-1\}}{2} \text{ i.e. } \frac{(m+1)(3m+2)}{2}$$

$\therefore$ the sum

$$= \frac{(m+1)(3m+2)}{2} + \frac{3m(m+1)(2m-1)}{2} + m - 27\left\{\frac{m(m+1)}{2}\right\}^2$$

$$= \frac{\left(\frac{n-1}{3}+1\right)(n-1+2)}{2} + \frac{(n-1)\left(\frac{n-1}{3}+1\right)\left(2.\frac{n-1}{3}-1\right)}{2} +$$

$$\frac{n-1}{3} - \frac{27}{4}\left\{\frac{n-1}{3}\left(\frac{n-1}{3}+1\right)\right\}^2$$

$$= \frac{(n+2)(n+1)}{6} + \frac{(n-1)(n+2)(2n-5)}{18} + \frac{n-1}{3} - \frac{1}{12}\{(n-1)(n+2)\}^2$$

When n is of the form 3m+1.

If n=3m+2 i.e. When n is divided by 3, the remainder is 2. In this case the first and the second group will have (m+1) terms each while the third will have m terms.

$\therefore$ In (1) we shall get,

$$\frac{(m+1)\{3(m+1)-1\}}{2} \text{ i.e. } \frac{(m+1)(3m+2)}{2}$$

In (2) we shall get $\frac{3(m+1)(m+2)(2m+1)}{2} + (m+1)$

There fore the sum

$$= \frac{(m+1)(3m+2)}{2} + \frac{3(m+1)(m+2)(2m+1)}{2} + (m+1) - 27\left\{\frac{m(m+1)}{2}\right\}^2 \text{ (Using 3)}$$

$$= \frac{\left(\frac{n-2}{3}+1\right)n}{2} + \frac{3\left(\frac{n-2}{3}+1\right)\left(\frac{n-2}{3}+2\right)\left(2.\frac{n-2}{3}+1\right)}{2} + \left(\frac{n-2}{3}+1\right) - \frac{27}{4}\left\{\frac{n-2}{3}\left(\frac{n-2}{3}+1\right)\right\}^2$$

$$= \frac{(n+1)n}{6} + \frac{(n+1)(n+4)(2n-1)}{18} + \frac{n+1}{3} - \frac{1}{12}\{(n-2)(n+1)\}^2$$

Where n is of the form 3m+2.

KEYWORDS

- Arithmetic Progression (A.P.)
- Common Difference.
- Term.
- Arithmetic Mean (AM).
- Sequence.
- Geometric Progression (GP).
- Common Ratio.
- Geometric Mean (GM).
- Arithmetic Geometric Progression (AGP).
- Harmonic Progression (H.P.).
- Harmonic Mean (H.M.).

The Quadratic Equation

Quadratic equation is nothing but polynomial equation with degree 2.The graph of quadratic equation i.e. $ax^2 + bx + c$ represents a parabola.

It is often claimed that Babylonians (about 400 BC) were first to solve quadratic equation. However this is not true, Savasorda in 1145 came with complete solution of quadratic equation about 900 years from then we have many interesting things to calculate like intervals of values of x for which quadratic expression $ax^2 + bx + c$ is of

one sign and many more things.

So let us start solving "QUADRATIC EQUATION"

INTRODUCTION:

An equation of form $ax^2 + bx + c = 0$ where a, b, c e C is called quadratic equation.

- If a = 0 then one root is ∞ and other root is $-\frac{c}{b}$
- If a=b=0 then both roots are ∞.

In general roots of equation are given by

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

Where $D = b^2 - 4ac$ is known as Discriminant.

<u>Nature of roots:</u>

1) The quadratic equation has real and equal roots if D=0.
2) The quadratic equation has real and distinct roots if D>0
3) The quadratic equation has complex roots with non-zero imaginary parts if D<0.
4) p+iq is a root of quadratic equation if p-iq is a root of equation.

More:

a) In general if a polynomial equation with all real coefficients has a root p+iq then p-iq will also be root of equation.

b) So, any polynomial equation with all real coefficients will have non-real complex roots in conjugate pairs.

5) If a, b, c ∈Q and p+√q (q is not a perfect square) is an irrational root of quadratic equation then p-√q is also a root.

More:
a) In general if polynomial equation with all coefficients rational has an irrational root p+√q then p-√q will also be a root of equation.

b) So polynomial equation with all rational coefficients will have irrational roots in conjugate pair.

6) If quadratic equation is satisfied by more than two distinct numbers (real or imaginary) then it becomes an identity i.e. a=b=c=0

Illustration 1:
If a+b+c=0 then find the nature of the root of the equation
$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$?

Solution:

$$\begin{aligned} D &= 4(a^2 - bc)^2 - 4(c^2 - ab)(b^2 - ac) \\ &= 4\left[a^4 + b^2c^2 - 2a^2bc - (b^2c^2 - ac^3 - ab^3 + a^2bc)\right] \\ &= 4a\left[a^3 + b^3 + c^3 - 3abc\right] \\ &= 4a[a + b + c]\left[a^2 + b^2 + c^2 - ab - bc - ac\right] \\ &= 0 \qquad (\because a + b + c = 0) \end{aligned}$$

So the roots are real and equal.

Relation between roots and coefficients:

$ax^2 + bx + c = 0$ a ¹ 0, a, b, cÎC, If a,b are roots then a+b = $-\frac{b}{a}$, ab = $\frac{c}{a}$

Factorization is $ax^2 + bx + c$ = a(x-a) (x-b), If a and b are given then equation is

x2-(a+b) x+ab=0.

Illustration 2:
If one root of equation $x^2 + px + q$ is square of other then find the relation between p and q.
Solution: Let one root be a, so other root is a2
So, a+a2 = -p and a(a2) = q
Now (a+a2)3 = (-p)3

$$\alpha^3 + \alpha^6 + 3(\alpha(\alpha^2))(\alpha + \alpha^2) = -p^3$$

$$\text{or } q + q^2 + 3q(-p) = -p^3$$

$$p^3 + q(1 - 3p) + q^2 = 0$$

<u>Common roots</u>

If $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$
1) have a common root, then
$(a_2c_1 - a_1c_2)^2 = (b_1c_2 - c_1b_2)(a_1b_2 - b_1a_2)$ and common root is given by,

$$\alpha = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2} \text{ or } \alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - b_1a_2}$$

How?

Let a be a common root then

$$a_1\alpha^2 + b_1\alpha + c_1 = 0 \quad \text{And} \quad a_2\alpha^2 + b_2\alpha + c_2 = 0$$

Solving we get

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - a_1c_2} = \frac{1}{a_1b_2 - b_1a_2}$$

$$\text{So, } \alpha = \frac{b_1c_2 - b_2c_1}{c_1a_2 - a_1c_2}$$

$$\text{Also } \alpha = \frac{c_1a_2 - a_1c_2}{a_1b_2 - b_1a_2}$$

Eliminating a we get

$$\frac{b_1c_2 - b_2c_1}{c_1a_2 - a_1c_2} = \frac{c_1a_2 - a_1c_2}{a_1b_2 - b_1a_2}$$

Or $(c_1a_2 - a_1c_2)^2 = (a_1b_2 - b_1a_2)(b_1c_2 - b_2c_1)$

Have both the roots common if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Illustration 3:
Find the condition on a, b, c and d such that equations $2ax^3 + bx^2 + cx + d = 0$ and $2ax^2 + 3bx + 4c = 0$ have a common root.

Solution: Let a be a common root

So, $2a\alpha^3 + b\alpha^2 + c\alpha + d = 0$ ---------- (1)

And $2a\alpha^2 + 3b\alpha + 4c = 0$ ---------- (2)

Now (1) - (2) $\times \alpha$

$\Rightarrow -2b\alpha^2 - 3c\alpha + d = 0$

$\Rightarrow 2b\alpha^2 + 3c\alpha - d = 0$ ---------- (3)

Now from (2) and (3) we find the condition i.e.

$$((4c(2b)) - 2a(-d))^2 = (-2a.3c + 3b.2b)(4c.3c - (3b)(-d))$$
$$\Rightarrow (8bc + 2ad)^2 = (6b^2 - 6ac)(12c^2 + 3bd)$$
$$\Rightarrow 4(ad + 4bc)^2 = 6 \times 3(b^2 - ac)(4c^2 + bd)$$
$$\Rightarrow (4bc + ad)^2 = \frac{9}{2}(b^2 - ac)(4c^2 + bd)$$

Lagrange's Identity:

If a1, a2, a3, b1, b2, b3ÎR then,

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 = \begin{Bmatrix} (a_1b_2 - a_2b_1)^2 + \\ (a_2b_3 - a_3b_2)^2 + \\ (a_3b_1 - a_1b_3)^2 \end{Bmatrix}$$

Cauchy Swartz inequality:

If $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ then,

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \geq 0$$

Illustration 4:

If a1, a2, ----------an ÎR+ then show that

$$(a_1 + a_2 + ------ + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + ------ + \frac{1}{a_n}\right) \geq n^2$$

Solution:

Since a1, a2, ----------an ÎR+ so we can take root of each of the ai's

$$(a_1 + a_2 + ------ + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + ------ + \frac{1}{a_n}\right) =$$

$$\left((\sqrt{a_1})^2 + (\sqrt{a_2})^2 + ---- + (\sqrt{a_n})^2\right)\left(\left(\frac{1}{\sqrt{a_1}}\right)^2 + ---- + \left(\frac{1}{\sqrt{a_n}}\right)^2\right)$$

Now using the Cauchy Swartz inequality we get,

$$\left((\sqrt{a_1})^2 + (\sqrt{a_2})^2 + -- + (\sqrt{a_n})^2\right)\left(\left(\frac{1}{\sqrt{a_1}}\right)^2 + -- + \left(\frac{1}{\sqrt{a_n}}\right)^2\right) - \left(\begin{matrix} \sqrt{a_1}\,\frac{1}{\sqrt{a_1}} + \\ \sqrt{a_2}\,\frac{1}{\sqrt{a_2}} + -- + \\ \sqrt{a_n}\,\frac{1}{\sqrt{a_n}} \end{matrix}\right)^2 \geq 0$$

$$(a_1 + a_2 + --- + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + --- + \frac{1}{a_n}\right) - n^2 \geq 0$$

$$(a_1 + a_2 + --- + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + --- + \frac{1}{a_n}\right) \geq n^2$$

Equation of Higher Degree:

Consider the equation;

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} --- + a_1 x + a_0 = 0$$

Where $a_n, a_{n-1}, --- a_0 \in C$ and $a_0 \neq 0$

Let $\alpha_1, \alpha_2, --- \alpha_n$ be the roots of this equation then,

$$f(x) = a_n(x - \alpha_1)(x - \alpha_2) --- (x - \alpha_n)$$

Also,

$$\sum_{i=1}^{n} \alpha_i = \alpha_1 + \alpha_2 + \alpha_3 + --- \alpha_{n-1} + \alpha_n = (-1)\frac{a_{n-1}}{a_n}$$

$$\sum_{1 \leq i \leq j \leq n} \alpha_i \alpha_j = (-1)^2 \frac{a_{n-2}}{a_n}$$

$$\prod_{i=1}^{n} \alpha_i = \alpha_1 \alpha_2 \alpha_3 --- \alpha_n = (-1)^n \frac{a_0}{a_n}$$

Illustration 5:

If $f(x) = x^3 + bx^2 + cx + d$ and f (0), f (-1) are odd integers then prove that f(x) =0 cannot have all integral roots.

Solution:
f (0) =d, f (-1) = -1+b-c+d
As given in the equation, d is odd and also -1+b-c+d = odd
Or b-c = 1+odd-d
= even-odd
= odd ---------- (1)

So,d and b-c are odd
Now let a, b, g be integral roots

So, $\alpha\beta\gamma = (-1)^3 \frac{d}{1} = -d = \text{negative odd integer}$

abg are 3 integers whose product is odd.
So, a, b, g are odd.
Again ab+ga+bg = c
And a+b+g = -b
Now b and c will also be odd and so b-c will be even which contradicts (1).
So roots cannot be all integers.

<u>Wavy Curve Method:</u>

Let $f(x) = (x-a_1)^{k_1}(x-a_2)^{k_2} --- (x-a_n)^{k_n}$ where $k_1, k_2, k_3 --- k_n \in N$ and $a_1, a_2, a_3 --- a_n$ are fixed numbers satisfying the condition, $a_1 < a_2 < a_3 --- a_{n-1} < a_n$ First we mark the numbers $a_1, a_2, a_3 --- a_n$ on the real axis and the plus sign in interval of the right of the largest of these numbers i.e. on right of an. If kn is even, put plus sign on left of an otherwise put minus sign. Now in next interval the sign is put according to this rule: When passing through a_{n-1}

f (x) changes sign if is odd number otherwise it has same sign.
Similar rule is applied for the next interval and so on.
Union of all intervals with plus sign give solution of f (x)>0 and solution of f (x) <0 is union of all intervals in which we put the minus sign.

Illustration 6:

Solve the inequality

$(x+3)(3x-2)^5(7-x)^3(5x+8)^2 \geq 0$

Solution:

$(x+3)(3x-2)^5(7-x)^3(5x+8)^2 \geq 0$

$= (x+3)(3x-2)^5(x-7)^3(5x+8)^2 \leq 0$

$$= (x+3)\left(x-\frac{2}{3}\right)^5 (x-7)^3 \left(x+\frac{8}{5}\right)^2 \le 0$$

The critical points are $-3, -\frac{8}{5}, \frac{2}{3}, 7$

Fig (1)

$$\therefore x \in (-\infty, -3) \cup \left[\frac{2}{3}, 7\right]$$

<u>Sign of quadratic expression:</u>

The expression f(x) = y = $ax^2 + bx + c = 0$, a ¹ 0, a, b, cÎR can be written as

$$y = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$$

$$\text{Or } \left(x+\frac{b}{2a}\right)^2 = \frac{1}{a}\left[y - \frac{4ac-b^2}{4a}\right]$$

Which represents a parabola with vertex $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$ and axis $x = -\frac{b}{2a}$

Case 1: If $b^2 - 4ac < 0$:

1) If a>0

Min $y = \frac{4ac-b^2}{4a}$ at $x = -\frac{b}{2a}$ and

$y>0$ and $x \in R$.

Fig (2)

2) If $a<0$

Max $y = \frac{4ac-b^2}{4a}$ at $x = -\frac{b}{2a}$

$y<0$ and $x \in R$

Fig (3)

Case 2: If $b^2 - 4ac = 0$

1) If $a>0$

Min $y = 0$ at $x = -\frac{b}{2a}$

$y \geq 0$ and $x \in R$

Fig (4)

2) If $a<0$

Max $y = 0$ at $x = -\frac{b}{2a}$

$y \leq 0$ and $x \in R$

Fig (5)

Case 3: If $b^2 - 4ac > 0$

1) If $a>0$

$$y = \begin{cases} > 0 \text{ if } x < \alpha < \beta \text{ or } x > \beta > \alpha \\ = 0 \text{ if } x = \alpha, \beta \\ < 0 \text{ if } \alpha < x < \beta \end{cases}$$

$$\text{Min } y = \frac{4ac - b^2}{4a} \text{ at } x = -\frac{b}{2a}$$

2) If a<0

$$y = \begin{cases} < 0 \text{ if } x < \alpha < \beta \text{ or } x > \beta > \alpha \\ = 0 \text{ if } x = \alpha, \beta \\ > 0 \text{ if } \alpha < x < \beta \end{cases}$$

$$\text{Max } y = \frac{4ac - b^2}{4a} \text{ at } x = -\frac{b}{2a}$$

Illustration 7:

Find the condition so that function $\frac{(x-a)(x-b)}{(x-c)}$ will assume all real values?

Solution:

$$y = \frac{(x-a)(x-b)}{(x-c)}$$

Or $y\,(x\text{-}c) = x^2 - (a+b)x + ab$

Or $x^2 - (a+b+y)x + ab + cy = 0$

Now the determinant D of this equation must be greater than zero for all values of y

So, $(a+b+y)^2 - 4(ab+cy) \geq 0$

Or $y^2 + 2y(a+b-2c) + (a-b)^2 \geq 0$

Now in this equation A>0 so B2-4AC should be less than zero i.e. B2-4AC<0 for this condition to hold

Hence, $4(a+b-2c)^2 - 4(a-b)^2 < 0$

Or $(a+b-2c+a-b)(a+b-2c-(a-b)) < 0$

Or $(a-c)(b-c) < 0$

i.e. c lies between a and b.

So, a<c<b if a<b

Or b<c<a if b<a.

<u>Intervals of roots</u>

Let $f(x) = ax^2 + bx + c = 0$, a ¹ 0, a, b, cÎR and a>0 suppose k, k1, k2, p, qÎR and k1<k2 then

1) If D³0, f (k)>0 and $-\frac{b}{2a} > k$ then both roots of f (x) = 0 are greater than k why?

a>0 and D³0

\Shape of parabola is like this

Fig (8)

Now $-\frac{b}{2a} > k$, so k is on left hand side of the point $x = -\frac{b}{2a}$

Also f (k)>0, so it is possible only when k<a.
So,a, b>k.

1) If D³0, f (k)>0 and $-\frac{b}{2a} < k$ then both roots of f (x) =0 are less than k. Why?

As D³0 and a>0 so shape of parabola is like this

Fig (9)

Now f (k)>0 so k>b or k<a

And also $-\frac{b}{2a} < k$
So, k<a is ruled out
Therefore k>b
Hence, k>a,b.

2) If D>0 and f (k) <0 then k lies between root of f (x) = 0 why?

D>0 and a>0
So the shape of parabola is

Fig (10)

Now f (k) <0, so clearly a<k<b.

4) If D>0 and f (k1)´f (k2) <0, then exactly one root of equation f (x) = 0 lies in interval (k1, k2). When (k2>k1) Why?

Since f (k1)´f (k2) <0
So, f (k1) and f (k2) should be of opposite sign.

Fig (11)

1) Now let f(k1)<0 and f(k2)>0
So, a<k1<b and k2>b (because k2>k1)
And hence b lies in interval (k1, k2).

2) Second case is f (k1) >0 and f (k2) <0
So, a< k2<b and k1<a (because k2>k1)
Hence a lies in interval (k1, k2).

So exactly one root lies in interval (k1, k2).

5) If D³0, f (p)>0 and f (q)>0, q>p then both the roots of the equation f(x) = 0 will

lie between p and q, if $p < -\frac{b}{2a} < q$ why?

Suppose both roots lie in interval (p, q)

Now D³0, a>0 therefore shape of parabola is

Fig (12)

a, b lies in interval (p, q)

So, b<q and p<a

And hence p<a<b<q

Also $\alpha < -\frac{b}{2a} < \beta$

So, $p < \alpha < -\frac{b}{2a} < \beta < q$

Or $p < -\frac{b}{2a} < q$

6) If D³0, $-\frac{b}{2a} \geq 0$ and $\frac{c}{a}$ then both roots of the equation f(x) = 0 are positive. Why?

$-\frac{b}{2a} \geq 0 = 1, -\frac{b}{a} \geq 0$ or sum of roots is positive

Also $\frac{c}{a} \geq 0$ so, product of root is positive, and hence clearly both roots are positive.

Illustration 8:
For what values of 'a' exactly one root of the equation $2^a x^2 - 4^a x + 2^a - 1 = 0$ lies between 1 and 2.

Solution: Since exactly one root of given equation lie between 1 and 2,
So f (1)´f (2) <0
Here f(x) = $2^a x^2 - 4^a x + 2^a - 1 = 0$
So, f (1)´f (2) <0

$$\Rightarrow (2^a - 4^a + 2^a - 1)(2^a.2^2 - 4^a.2 + 2^a - 1) < 0$$

$$\Rightarrow -(4^a - 2.2^a + 1)(-2.4^a + 5.2^a - 1) < 0$$

$$\Rightarrow (2^a - 1)^2 (2.2^{2a} - 5.2^a + 1) < 0$$

$$\Rightarrow 2.2^{2a} - 5.2^a + 1 < 0$$

$$\Rightarrow \left(2^a - \left(\frac{5+\sqrt{17}}{4}\right)\right)\left(2^a - \left(\frac{5-\sqrt{17}}{4}\right)\right) < 0$$

Or $\frac{5-\sqrt{17}}{4} < 2^a < \frac{5+\sqrt{17}}{4}$

Or $\log_2\left(\frac{5-\sqrt{17}}{4}\right) < a < \log_2\left(\frac{5+\sqrt{17}}{4}\right)$

<u>Some important results using differentiability:</u>

Let f(x) = $a_0x^n + a_1x^{n-1} + - - - - + a_{n-1}x + a_n$

1) If f(x) = 0 has a real root a of multiplicity g (g>1) then f(x) = (x-a)gg (a) where g(a)¹0. Also f(x) = 0 has a as a real root with multiplicity (g-1)

2) If f(x) = 0 has n real roots then f (x) = 0 has (n-1) real roots.

Illustration 9:

A polynomial p(x) has $x^2 - 4x + 4$ as one of the factors, other roots of this polynomial lie in the range $(-3, \frac{5}{4})$. Prove that g(x) where g(x) = p1(x) has at least one positive root.

Solution:

Let $p(x) = (x^2 - 4x + 4)\ g(x)$

Or $p(x) = (x-2)^2\ g(x)$

Now p(x) has a root 2 with multiplicity 2.

So the derivative of p(x), p1(x) or g(x) must has 2 as root with multiplicity 2-1=1.

So g(x) has at least one positive root which is 2.

<u>Easy (Quadratic)</u>

Q-1: The roots of the quadratic equation $8x^2 - 10x + 3 = 0$ are a and b2 Where $\beta^2 > \frac{1}{2}$ then Show that $x^2 + x + 1 = 0$ has roots (a+ib)100 and (a-ib)100

Solution:

$f(x) = (2x-1)(4x-3) \qquad \therefore\ x = \frac{1}{2}, \frac{3}{4}$

$\therefore\ \alpha = \frac{1}{2}$ and $\beta^2 = \frac{3}{4}$ $(\because\ \beta^2 > \frac{1}{2} = \frac{2}{4})$

$$\alpha + i\beta = \frac{1}{2} + i\frac{\sqrt{3}}{2} = re^{i\theta} = e^{i\frac{\pi}{3}}$$

$$\because r = 1,\ \theta = \frac{\pi}{3}$$

$$\because \alpha - i\beta = e^{-i\frac{\pi}{3}} \quad \text{(conjugate)}$$

$$(\alpha + i\beta)^{100} = e^{i\frac{100\pi}{3}} = e^{i33}e^{i\frac{\pi}{3}} = -e^{i\frac{\pi}{3}}$$

$$\text{as } e^{i33\pi} = e^{i32\pi}.e^{i\pi} = 1(-1) = -1$$

$$(\alpha - i\beta)^{100} = -e^{-i\frac{\pi}{3}} \quad \text{(conjugate)}$$

$$\text{sum} = -\left(e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}\right) = -2\text{Cos}\frac{\pi}{3} = 1$$

Product = 1.

$\therefore$ Required equation is $x^2 + x + 1 = 0$

Q-2: If a, b are the roots of the equation $8x^2 - 3x + 27 = 0$ then prove that $\left(\frac{\alpha^2}{\beta}\right)^{\frac{1}{3}} + \left(\frac{\beta^2}{\alpha}\right)^{\frac{1}{3}} = \frac{1}{4}$

$$\alpha + \beta = \frac{3}{8},\ \alpha\beta = \frac{27}{8} = \left(\frac{3}{2}\right)^3$$

$$\therefore E = \frac{\alpha + \beta}{(\alpha\beta)^{\frac{1}{3}}} = \frac{\alpha + \beta}{\frac{3}{2}} = \frac{2}{3}\left(\frac{3}{8}\right) = \frac{1}{4}$$

Q-3: If $x = 2 + 2^{\frac{2}{3}} + 2^{\frac{1}{3}}$ then find value of expression $x^3 - 6x^2 + 6x$.

Solution:

$$(x - 2)^3 = 2^2 + 2 + 3.2(x - 2)$$

$$\text{or } x^3 - 6x^2 + 12x - 8 = 6 + 6x - 12$$

$$\therefore x^3 - 6x^2 + 6x = 2$$

Q-4: Solve the equation

$$2x^4 + x^3 - 11x^2 + x + 2 = 0 \qquad \text{-------- (1).}$$

Solution:

Since x=0 is not a solution of given equation. Dividing by x2 in both sides of (1) we get,

$$2\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 11 = 0 \qquad \text{----------(2)}$$

Putting $x + \frac{1}{x} = y$ in equation (2) then (2) reduces in the form,

$$2(y^2 - 2) + y - 11 = 0$$

$$\Rightarrow 2y^2 + y - 15 = 0$$

$$\Rightarrow y_1 = -3 \text{ and } y_2 = \frac{5}{2}$$

Consequently the original equation is equivalent to the collection of equations,

$$\begin{cases} x + \frac{1}{x} = -3 \\ x + \frac{1}{x} = \frac{5}{2} \end{cases}$$

We find that

$$x_1 = \frac{-3-\sqrt{5}}{2}, x_2 = \frac{-3+\sqrt{5}}{2}, x_3 = \frac{1}{2}, x_4 = 2$$

Q-5: If aÎz and the equation (x-a) (x-10) +1=0 has integral roots, then what is the value

Of ‘a’?

Solution:

As both x and a are integers and hence given equation implies that either x-a =1 and

x-10= -1 or x-a = -1 and x-10=1.

$\therefore$ x=9 and hence a=8 or x=11 and a=12.

Therefore possible values of a's are 8, 12.

Q-6: Find the values of 'a' for which inequality $\tan^2 x + (a+1)\tan x - (a-3) < 0$ is true for at least one $x \in \left(0, \frac{\pi}{2}\right)$.

Solution: The required condition will be satisfied if – The quadratic expression (quadratic in tanx).

1) f (x) = $\tan^2 x + (a+1)\tan x - (a-3) < 0$ has positive discriminant and ,

2) At least one root of f(x) =0 is positive as tanx>0, for all $x \in \left(0, \frac{\pi}{2}\right)$.

For (1) discriminant >0

$\Rightarrow (a+1)^2 + 4(a-3) > 0$

$\Rightarrow a > 2\sqrt{5} - 3$

or $a < -(2\sqrt{5} + 3)$ ----------(a)

For (2) we first find the condition, that both the roots of $t^2 + (a+1)t - (a-3) = 0$ (t=tanx) are non positive for which,

Sum of roots<0, product of roots ≥0

$\Rightarrow$ - (a+1) <0 and – (a-3) ≥0 $\Rightarrow$ -1 <a ≤3

Condition (2) will be fulfilled of a ≤-1or a>3 ----------(b)

Required value of a is given by intersection of (a) and (b)

Hence $a \in (-\infty, -(2\sqrt{5} + 3)) \cup (3, \infty)$

Q-7: Solve the equation- $(x)^2 = [x]^2 + 2x$ where (x) and [x] are the integer just less than or equal to x and just greater than or equal to x respectively.

Case 1: If x ∈I,

Then (x) = [x] -------- (1),

$\therefore \quad (x)^2 = [x]^2 + 2x$

$\Rightarrow [x]^2 = [x]^2 + 2x$

$\therefore x = 0.$

Case 2: if x ∉l

Then (x) = [x]-1

$\therefore (x)^2 = [x]^2 + 2x$

$$[x]^2 + 1 + 2[x] = [x]^2 + 2x$$

$$\Rightarrow x = [x] + \frac{1}{2}$$

$$= n + \frac{1}{2}, n \in I$$

Hence the solution of original equation is, x=0, $n + \frac{1}{2}$.$n \in I$.

Q-8: If the equation $ax^2 + 2bx + c = 0$ has real roots a, b, c being real numbers and if m and n are real number such that m2>n>0 then prove that the equation $ax^2 + 2mbx + nc = 0$ has real roots.

Solution: Since roots of the equation $ax^2 + 2bx + c = 0$ are real

$\therefore (2b)^2 - 4ac \geq 0$

$\therefore b^2 - ac \geq 0$ ---------- (1)

and discriminant of $ax^2 + 2mbx + nc = 0$

is $D = (2mb)^2 - 4anc$

$D = 4m^2b^2 - 4anc$ ----------(2)

From (1) $b^2 \geq ac$ ----------(3)

And given $m^2 > n$ ----------(4)

From (3) and (4)

$\therefore b^2m^2 \geq anc$

$\Rightarrow 4b^2m^2 - 4anc \geq 0$

$\Rightarrow D \geq 0$ (from (2))

Hence roots of equation $ax^2 + 2mbx + nc = 0$ are real.

Q-9 If graph of function $y = 16x^2+8(a+5)\,x-7a-5$ is strictly above the x-axis then show that -15<a<-2

Solution: Y has to be positive

∴16x2+8(a+5) x-(7a+5) = +ve

∴Since sign of first term is +ve therefore the expression will be +ve if D<0.

$64(a+5)^2 + 64(7a+5) < 0$

$a^2 + 17a + 30 < 0$ or

$(a+15)(a+2) < 0$

or $-15 < a < -2$

Q-10: For real roots what is the solution of the equation $2^{2x+2} - 6^x - 2 \times 3^{2x+2} = 0$

Solution: Put $6^x = 3^x.2^x$

Divide by 6^x and put $\left(\frac{3}{2}\right)^x = t$ where t is +ve being exponential function.

$$\therefore \frac{4}{t} - 1 - 18t = 0$$

or $18t^2 + t - 4 = 0$

$$\therefore t = \frac{4}{9}, -\frac{1}{2}$$

$$\left(\frac{3}{2}\right)^x = \left(\frac{2}{3}\right)^2 = \left(\frac{3}{2}\right)^{-2}$$

$$\therefore x = -2$$

The other value $t = -\frac{1}{2}$ is rejected as t is +ve.

Q-11: Obtain a quadratic expression in x and solve for it if

$$x = 1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \text{----to } \infty}}}}$$

Here

$$x = 1 + \cfrac{1}{3 + \cfrac{1}{2 + (x - 1)}}$$

Because,

$$\left[x - 1 = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + ----\text{to } \infty}}}} \right]$$

$$\text{Or } x = 1 + \cfrac{1}{3 + \cfrac{1}{1 + x}}$$

$$\text{Or } x = 1 + \cfrac{1}{\cfrac{4 + 3x}{1 + x}}$$

$$\text{Or } x = \frac{5 + 4x}{4 + 3x}$$

$$\text{Or } x(4 + 3x) = 5 + 4x$$

$$\text{Or } 3x^2 = 5$$

$$\therefore x = \pm\sqrt{\frac{5}{3}}$$

But clearly x is positive

$$\therefore x = +\sqrt{\frac{5}{3}}$$

Q-12: Solve for x $\text{If } (x-1)^3 + (x-2)^3 + (x-3)^3 + (x-4)^3 + (x-5)^3 = 0$

Solution: Here A.M of {x-1, x-5} =A.M of {x-2, x-4} =x-3, put x-3=y then,

x-1= y+2, x-2=y+1, x-4=y-1, x-5=y-2.

$\therefore$ The equation becomes,

$(y+2)^3 + (y+1)^3 + y^3 + (y-1)^3 + (y-2)^3 = 0$

Or $\left[(y+2)^3 + (y-2)^3\right] + \left[(y+1)^3 + (y-1)^3\right] + y^3 = 0$

Or $2(y^3 + 12y) + 2(y^3 + 3y) + y^3 = 0$

Or $5y^3 + 30y = 0$

Or $5y(y^2 + 6) = 0$

$\therefore y = 0, \pm\sqrt{-6} = 0, \pm\sqrt{6}i$

$\therefore x = 3, 3 \pm \sqrt{6}i.$

<u>Medium</u>

Q-1: Prove that $2x^4 + 1402 = y^4$ has no integral solution

Solution: Let $x^2 = u,\ y^2 = v$ then the equation is $2u^2 = v^2 - 1402$

As u, v are integers, >1402

But $37^2 = 1369$, $38^2 = 1444$

$\therefore$ Minimum possible value of v=38.

Also

$\therefore$ v must be even.

Let v=38+2k where k $\in$N

$\therefore 2u^2 = (38 + 2k)^2 - 1402$

$= 4k^2 + 152k + 42$

$\therefore\ u^2 = 2k^2 + 76k + 21$

So $2k^2 + 76k + 21$ is a perfect square

Hence D=0

i.e. $76^2 - 4.2.21 = 0$ which is incorrect.

$\therefore$ u, v cannot have positive integral solutions.

So, x, y cannot have integral solutions.

Q-2: If a, b are the roots of and also of $x^{2n} + p^n x^n + q^n = 0$ and if $\left(\frac{\alpha}{\beta}\right), \left(\frac{\beta}{\alpha}\right)$ are the roots of $x^n + 1 + (x+1)^n = 0$ then prove that n must be even integer

Solution: $x^2 + px + q = 0$

Sincea, b are the roots of $x^2 + px + q = 0$

Since α, β are the roots of $x^2 + px + q = 0$

$\therefore \alpha+\beta = -p, \alpha\beta=q$ ----------(1)

Also α, β are roots of $x^{2n} + p^n x^n + q^n = 0$

$\therefore \alpha^n + \beta^n = -p^n$ and $\alpha^n\beta^n = q^n$ ----------(2)

Now α/β is a root of $x^n + 1 + (x+1)^n = 0$

$$\Rightarrow \left(\frac{\alpha^n}{\beta^n}\right) + 1 + \left(\frac{\alpha}{\beta} + 1\right)^n = 0$$

$$\Rightarrow \frac{\alpha^n + \beta^n}{\beta^n} + \frac{(\alpha+\beta)^n}{\beta^n} = 0$$

$\Rightarrow -p^n + (-p)^n = 0$ {from (1) and (2)}

This is true only if n is an even integer.

Q-3: Show that the equation

$$\frac{A^2}{(x-a)} + \frac{B^2}{(x-b)} + ------ + \frac{K^2}{(x-k)} = x + 1$$ has no imaginary roots,

Where A, B, C---------K, a, b, c------k and l are real.

Solution: Assume a+ib is an imaginary root of the given equation then conjugate of this root a-ib is also root of this equation.

Putting x = a+ib and x = a-ib in the given equation then,

$$\frac{A^2}{(\alpha+i\beta-a)}+\frac{B^2}{(\alpha+i\beta-b)}+------+\frac{K^2}{(\alpha+i\beta-k)}=\alpha+i\beta+1 \qquad -----(1)$$

and

$$\frac{A^2}{(\alpha-i\beta-a)}+\frac{B^2}{(\alpha-i\beta-b)}+------+\frac{K^2}{(\alpha-i\beta-k)}=\alpha-i\beta+1 \qquad -----(2)$$

Substracti ng (1) from (2) we get

$$\Rightarrow 2i\beta\left[\frac{A^2}{(\alpha-a)^2+\beta^2}+\frac{B^2}{(\alpha-b)^2+\beta^2}+------+\frac{K^2}{(\alpha-k)^2+\beta^2}\right]=-2i\beta$$

$$\Rightarrow 2i\beta\left[1+\frac{A^2}{(\alpha-a)^2+\beta^2}+\frac{B^2}{(\alpha-b)^2+\beta^2}+------+\frac{K^2}{(\alpha-k)^2+\beta^2}\right]=0$$

The expression in bracket $\neq 0$

$\therefore 2ib = 0 \Rightarrow b = 0$ (because $i \neq 0$)

Hence all roots of the given equation are real.

Q-4: Solve in R: $(x+3)^5-(x-1)^5 \geq 244$

Solution:

The A.M of x+3, x-1 is $\frac{x+3+x-1}{2}$, i.e. (x+1)

Put x+1 = y then x+3 = y+2, x-1 = y-2

$\therefore$ The equation becomes $(y+2)^5-(y-2)^5 \geq 244$

Or $\left\{\begin{matrix} y^5+{}^5c_1y^4 2+{}^5c_2y^3 2^2+{}^5c_3y^2 2^3+{}^5c_4y2^4+{}^5c_5 2^5- \\ (y^5-{}^5c_1y^4 2+{}^5c_2y^3 2^2-{}^5c_3y^2 2^3+{}^5c_4y2^4-{}^5c_5 2^5) \end{matrix}\right\} \geq 244$

Or $2\left[{}^5c_1y^4 2+{}^5c_3y^2 2^3+{}^5c_5 2^5\right] \geq 244$

Or $10y^4+80y^2+32 \geq 122$

Or $y^4+8y^2-9 \geq 0$

Or $(y^2+9)(y^2-1) \geq 0$

$\Rightarrow y^2-1 \geq 0$ because $y^2+9 > 0$ for real y

The corresponding equation =0 has roots 1, -1.

The sign scheme of $y \in R$ is as follows.

Fig (13)

$\therefore y^2-1, \geq 0$ holds if $y \leq -1$ or $y \geq 1$

Now $y \leq -1 \Rightarrow x+1 \leq -1$ or $x \leq -2$

$y \geq 1 \Rightarrow x+1 \geq 1$ or $x \geq 0$

Hence $x \leq -2$ or $x \geq 0$

Therefore the solution set =.

The given equation can be written in the form,

$$|a^2x+1| + |a^3+a^2x| = (a^2x+1) - (a^3+a^2x)$$

$\therefore$ The given equation is equivalent to the system,

$$\begin{cases} a^2x+1 \geq 0 \\ a^3+a^2x \leq 0 \end{cases} \qquad \text{--------(1)}$$

The equation $|A|+|B| = A-B$

Holds true $\Leftrightarrow$ $A \geq 0$ and $B \leq 0$

Now consider following cases

Case 1: If $a=0$

Then system (1) gives equation have all $x \in R$ as their solutions.

Fig (17)

Case 2: If $a \neq 0$ then system (1) is equivalent to

$$\begin{cases} x \geq -a^{-2} \\ x \leq -a \end{cases} \qquad \text{--------(2)}$$

Q-5: let a, b, c be real, if $ax^2 + bx + c = 0$ has two real roots a and b where a< -1 and b >1 then show that $1+\frac{c}{a}+\left|\frac{b}{a}\right| < 0.$

Solution:

Let $f(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$ from graph $f(-1) < 0$ and $f(1) < 0$

$$1+\frac{c}{a}-\frac{b}{a} < 0 \text{ and } 1+\frac{c}{a}+\frac{b}{a} < 0 \Rightarrow 1+\frac{c}{a}+\left|\frac{b}{a}\right| < 0$$

Fig (14)

Q-6: If a<b<c<d prove that the equation (x-a) (x-c) + k (x-b) (x-d) = 0 has real roots for all k ∈R.

Solution:

Here the equation is $x^2 - (a+c)x + ac + k[x^2 - (b+d)x + bd] = 0$

Or $(1+k)x^2 - \{(a+c) + k(b+d)\}x + ac + kbd = 0$ ------(1)

Equation (1) will have real roots if,

$\{a + c + k(b + d)\}^2 - 4(1+k)(ac + kbd) \geq 0$

Or $\{(b + d)^2 - 4bd\}k^2 + \{2(a + c)(b + d) - 4ac - 4bd\}k + (a + c)^2 - 4ac \geq 0$

Or $(b - d)^2 k^2 + 2\{ab + ad + bc + cd - 2ac - 2bd\}k + (a - c)^2 \geq 0$ -------(2)

Now the discriminant of the equation corresponding (2) is,

$$D = 4\{ab + ad + bc + cd - 2ac - 2bd\}^2 - 4(b-d)^2(a-c)^2$$
$$= 4\{ab + ad + bc + cd - 2ac - 2bd + (b-d)(a-c)\} \times$$
$$\{ab + ad + bc + cd - 2ac - 2bd - (b-d)(a-c)\}$$
$$= 4\{ab + ad + bc + cd - 2ac - 2bd + ab - bc - ad + cd\} \times$$
$$\{ab + ad + bc + cd - 2ac - 2bd - ab + bc + ad - cd\}$$
$$= 4 \times 2(ab + cd - ac - bd) \times 2(ad + bc - ac - bd)$$
$$= 16\{a(b-c) - d(b-c)\}\{d(a-b) - c(a-b)\}$$
$$= 16(b-c)(a-d)(a-b)(d-c)$$
$$= 16(-ve)(-ve)(-ve)(+ve) < 0$$

Because a<b<c<d

Also (b-d)2>0 so (2) is true for all k ∈R.

Hence given equation have real roots.

<u>Hard</u>

Q-1: If p be the first of n arithmetic means between two numbers and q be the first of n harmonic means between the same two numbers, prove that q cannot lie between

$p\left(\frac{n+1}{n-1}\right)^2$ and p.

Solution: Let the numbers be x, y. Let the n AMs between them be A1, A2, -------- An

Then y = (n+2) th term = x + (n+1) d (d being the common difference)

$$\therefore d = \frac{y-x}{(n+1)}; \therefore A_1 = p = x + \frac{y-x}{(n+1)}$$

$$\therefore p = \frac{nx + y}{(n+1)} \qquad -----(1)$$

Similarly $\frac{1}{q} = \frac{n\frac{1}{x} + \frac{1}{y}}{(n+1)} = \frac{ny + x}{(n+1)xy}$ -----(2)

Equation (1) Þ y = (n+1) p-nx putting this in (2).

$$\frac{(n+1)x}{q}\{(n+1)p-nx\} = n\{(n+1)p-nx\} + x$$

$$\text{Or } -\frac{n(n+1)}{q}x^2 + \frac{(n+1)^2}{q}px = n(n+1)p + (1-n^2)x$$

$$\text{Or } \frac{n(n+1)}{q}x^2 - \left\{\frac{(n+1)^2 p}{q} + (n^2-1)\right\}x + n(n+1)p = 0$$

$$\text{Or } \frac{n}{q}x^2 - \left\{\frac{(n+1)p}{q} + (n-1)\right\}x + np = 0$$

$$\text{Or } nx^2 - \{(n+1)p + (n-1)q\}x + npq = 0$$

As x is real, $D \geq 0$

$$\therefore \{(n+1)p + (n-1)q\}^2 - 4n.npq \geq 0$$

$$\text{Or } (n-1)^2 q^2 + \{2(n^2-1) - 4n^2\}pq + (n+1)^2 p^2 \geq 0$$

$$\text{Or } q^2 - 2\frac{(n^2+1)}{(n-1)^2}pq + \left(\frac{n+1}{n-1}\right)^2 p^2 \geq 0 \qquad \text{-----(3)}$$

The corresponding equation in q is

$$q^2 - 2\frac{(n^2+1)}{(n-1)^2}pq + \left(\frac{n+1}{n-1}\right)^2 p^2 = 0$$

$$q = \frac{\left\{2\frac{n^2+1}{(n-1)^2}p \pm \sqrt{4\left\{\frac{n^2+1}{(n-1)^2}p\right\}^2 - 4\left(\frac{n+1}{n-1}\right)^2 p^2}\right\}}{2}$$

$$= \frac{n^2+1}{(n-1)^2}p \pm \frac{p}{(n-1)}\sqrt{\left(\frac{n^2+1}{n-1}\right)^2 - (n+1)^2}$$

$$= \frac{n^2+1}{(n-1)^2}p \pm \frac{p}{(n-1)^2}\sqrt{(n^2+1)^2 - (n^2-1)^2}$$

$$= \frac{p}{(n-1)^2}(n^2+1 \pm 2n)$$

$$= p\left(\frac{n+1}{n-1}\right)^2, p$$

The sign scheme for (3) is as shown

$\therefore$(3) $\Rightarrow$ q can not lie between $p\left(\frac{n+1}{n-1}\right)^2$ and p.

Subsequently:

a) If a>1 then system (2) has no solutions, and therefore the original equation has no solution.

b) If a=1, then system (2) has only unique solution i.e. x= -1 and the conditions of the original equation are not satisfied. Hence the original equation has no solution.

c) If 0<a<1 then -1<-a<0 and therefore the interval [-a-2, -a] contains no less than four integers provided the inequality -a-2£ -4 holds true. Now solve the system.

$$\begin{cases} 0 < a < 1 \\ -\frac{1}{a^2} \le -4 \end{cases} \Leftrightarrow \begin{cases} 0 < a < 1 \\ \frac{\left(a+\frac{1}{2}\right)\left(a-\frac{1}{2}\right)}{a^2} \le 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 0 < a < 1 \\ a \in \left[-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right] \end{cases}$$

Fig (16)

Thus $a \in \left(0, \frac{1}{2}\right]$, then the given equation has no less than four different integer solutions.

d) If -1<a<0 then 0<-a<1 and therefore the interval [-a-2, -a] contains no less than four

integer provided the inequality -a-2 £ -3 holds true. Now solve the system.

$$\begin{cases} -1 < a < 0 \\ -\dfrac{1}{a^2} \le -3 \end{cases} \Leftrightarrow \begin{cases} -1 < a < 0 \\ \dfrac{\left(a + \dfrac{1}{\sqrt{3}}\right)\left(a - \dfrac{1}{\sqrt{3}}\right)}{a^2} \le 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -1 < a < 0 \\ a \in \left[-\dfrac{1}{\sqrt{3}}, 0\right) \cup \left(0, \dfrac{1}{\sqrt{3}}\right] \end{cases}$$

Thus $a \in \left[-\dfrac{1}{\sqrt{3}}, 0\right)$, then the given equation has no less than four different integer solutions.

Q-2: Find all values of a for which the equation $a^3 + a^2|a + x| + |a^2x + 1| = 1$ has no less than four different integer solution.

Solution: The given equation can be written in the form,

$$|a^2x + 1| + |a^3 + a^2x| = (a^2x + 1) - (a^3 + a^2x)$$

∴The given equation is equivalent to the system,

$$\begin{cases} a^2x + 1 \ge 0 \\ a^3 + a^2x \le 0 \end{cases} \qquad \text{--------(1)}$$

The equation |A|+|B| = A-B
Holds true A ≥0 and B ≤0
Now consider following cases
Case 1: If a=0
Then system (1) gives equation have all xÎR as their solutions.

Fig (17)

Case 2: If a ¹0 then system (1) is equivalent to

$$\begin{cases} x \geq -a^{-2} \\ x \leq -a \end{cases} \qquad \text{---------(2)}$$

Now,

$$(-a^{-2}) - (-a) = -\frac{1}{a^2} + a$$

$$= \frac{a^3 - 1}{a^2}$$

$$= \frac{(a-1)(a^2 + a + 1)}{a^2}$$

$$= \frac{(a-1)\left\{\left(a + \frac{1}{2}\right)^2 + \frac{3}{4}\right\}}{a^2}$$

For a<1, a 0
-a-2+a<0

For a=1

-a-2+a=0

For a>1

-a-2+a>0

Fig (18)

e) If a= -1 then the interval [-1, 1] contains only three integer i.e. condition of the problem are not satisfied.

f) If a<-1 then -1<-a-2<0 and therefore the interval [-a-2, -a] contains no less than four integers. It is necessary that the inequality –a ³ 3 hold true thus for a £ -3 the given equation

has no less than four integer solutions.

Combining all the results we get the set of required values for a,

$$\therefore\ a \in (-\infty, -3) \cup \left(-\frac{1}{\sqrt{3}}, \frac{1}{2}\right] \text{ and } a \neq 0.$$

Q-3: Solve for x

$$3x^3 = \left\{x^2 + \sqrt{18}x + \sqrt{32}\right\}\left\{x^2 - \sqrt{18}x - \sqrt{32}\right\} - 4x^2$$

Solution:

Here, $3x^3 = \left\{x^2 + \sqrt{18}x + \sqrt{32}\right\}\left\{x^2 - \sqrt{18}x - \sqrt{32}\right\} - 4x^2$

$$= x^4 - \left(\sqrt{18}x + \sqrt{32}\right)^2 - 4x^2$$
$$= x^4 - 2(3x+4)^2 - 4x^2$$
$$\text{Or } 3x^3 + 4x^2 = x^4 - 2(3x+4)^2$$
$$\text{Or } x^4 - 2(3x+4)^2 - x^2(3x+4) = 0$$

put $3x + 4 = y$ then the equation becomes,

$$x^4 - yx^2 - 2y^2 = 0$$
$$\text{Or } x^4 - 2yx^2 + yx^2 - 2y^2 = 0$$
$$\text{or } x^2(x^2 - 2y) + y(x^2 - 2y) = 0$$
$$\text{or } (x^2 - 2y)(x^2 + y) = 0$$
$$\therefore x^2 - 2y = 0 \text{ or } x^2 + y = 0$$

if $x^2 - 2y = 0$ then $x^2 - 2(3x+4) = 0$

$$\text{or } x^2 - 6x - 8 = 0$$
$$\therefore x = \frac{6 \pm \sqrt{36+32}}{2} = \frac{6 \pm \sqrt{68}}{2} = 3 \pm \sqrt{17}$$

if $x^2 + y = 0$ then $x^2 + 3x + 4 = 0$

$$\therefore x = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{7}i}{2}$$

thus $x = 3 \pm \sqrt{17}, \frac{-3 \pm \sqrt{7}i}{2}$

KEYWORDS

- Quadratic
- Root
- Discreminant
- Polynomial
- Coefficient
- Common roots
- Lagrange's Identity
- Cauchy Swartz Inequality
- Multiplicity

The Binomial Theorm

Binomial Theoram is one chapter in which you see a question and then realise that it can easily be done in 2 or 3 ways. Definitely one of the most interesting chapters which uses concepts from calculus, trignometry, complex number prognessiong and wkhat not Binomial theoram finds use in number theory as well. So, let us prober deeper into this section.

Binomial theorm

Definition of Bionomial theoram

If n is a +ve integer and x_1 y are 2 complex number then .

$(x + y)^n \sum_{r=1}^{n} {}^nC_r x^{n-r} y^r$

$= {}^nC_oX^n - {}^nC_1 X^{n-1} y + {}^nC_2 X^{n-2} y^2 + ... + {}^nC_n y^n$

Similarly $(x - y)^n = {}^nCoX^n - {}^nC_1 X^{n-1} y + {}^nC_2 X^{n-2} y^2 + ... + (-1)^n {}^nC_n y^n$

The oefficients ${}^nC_o$, ${}^nC_1$, ...${}^nC_n$ are called Bionomial coeff.

Some important facts

(i) There are (n + 1) tevms in the expression .

(ii) lthe sum of in dices. Of x and y in each term of expansion is n .

(iii) The bionomial 3efficent of the terms equidistant from begining and the end are equal .
(iv) The r-th term from end in $(x + y)^n$ = (h + r + 2) th term from beginning .

General term.

In the expansion of $(x + y)^n$ the $(r + 1^{tn})$ term from beginning of expansion is
$Tr + 1 = {}^nC_r a^{n-r} b^b$

Illustration 1

Expand $\left[2a - \frac{3}{b}\right]^5$ by biobomial theoram .

<u>Ans</u>:- Using bionomial theoram we get

$$\left(2a-\frac{3}{b}\right)^5 5C_0(2a)^5 = 5C_1(2a)^4\left(\frac{3}{b}\right) + 5C_2(2a)^3\left(\frac{3}{b}\right)^2 - 5C_3(2a)^1\left(\frac{3}{b}\right)^3 + 5C_4(2a)^4 - 5C_5\left(\right.$$

$$= 32\,a^5 - \frac{240a^4}{b} + \frac{720a^3}{b^2} - \frac{1080a^2}{b^3} + \frac{810a^1}{b^4} - \frac{243}{b^5}$$

<u>Illustration 2</u>.

If the ratio of 7th term from beginning to the seventh term from the end in the expassion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$ is $\frac{1}{6}$ then what is n .

<u>Ans</u> T_7 from end of $(a + b)^n$ is same as T_7 from beginning of $(b + a)^n$

$$\frac{T_7 \text{ from beginning}}{T_7 \text{ from end}} \text{ of } (a + b)^n$$

or $$\frac{T_7 \text{ of } (a+b)^n}{T^1{}_7 \text{ of } (a+b)^n} = \frac{1}{6}$$

$$\therefore \frac{{}^nC_6\ a^{n-6}\ b^6}{{}^nC_6\ a^{n-6}\ a^6} = \frac{1}{6}$$

or $$\frac{a^{n-12}}{b^{n-12}} = \frac{1}{6}$$

or $(2^{1/3}\,3^{1/3})^{n-12} = \frac{1}{6}$

$\Rightarrow (6)^{\frac{n-12}{3}} = 6^{-1}$

or $\frac{n-12}{3} = -1$

So, n = 9

<u>Greates Binomial coefficient(or Middle terms)</u>

(1) If n is even, then there is only one middle term which $(h/2 + 1)^{th}$ term i.e.

$T_{n/2} + 1 = {}^{n}C_{n/2} X^{n/2}$

and ${}^{n}C_{n/2}$ is greatest bionomial coefficient

(2) If n is each them trere are two middle terms which are $\frac{n+1}{2}^{th}$ and $\frac{n+3}{2}^{th}$ terms i.e

$= \frac{T_{n+1}}{2} = \frac{{}^{n}C_{n-1}}{2} a^{(n+1)/2} b^{(n-1)/2}$

and $\frac{T_{n+3}}{2} = \frac{{}^{n}C_{n-1}}{2} a^{\left(\frac{n-1}{2}\right)} b^{\left(\frac{n-1}{2}\right)}$

And, $\frac{{}^{n}C_{n-1}}{2} = \frac{{}^{n}C_{n+1}}{2}$ are greatest bionoimial ccoefficients

<u>Numarically Greatest term of Bionmial expansion</u>

$(a + x)^n = Co\ a^n + C_1\ a^{n-1}\ x.....+ C_{n-1}ax^{n-1} + C_nx^n.$

The numerically greatest term will be T_{r+1} where r = ...

If $\dfrac{n+1}{1+\left|\dfrac{a}{x}\right|}$ itself is a natural number then $T_r = T_{r+1}$

both are numerically greatest terms.

<u>Why</u> ?

$$\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}}\right| \left|\frac{x}{a}\right| = \left|\frac{n-r+1}{r}\right| \left|\frac{x}{a}\right|$$

If $\left|\dfrac{n-r+1}{r}\right| \left|\dfrac{x}{a}\right| \geq 1$ for given a, x and n

then $\gamma \leq \dfrac{n+1}{1+\left|\dfrac{a}{x}\right|}$

So, when $\gamma = \left[\dfrac{n+1}{1+\left|\dfrac{a}{x}\right|}\right]$

<u>Illustration 3</u>

Show that middle term in the expansion of $(1 + x)^{2n}$ is img.....
where n isa +ve integer.

<u>Ans</u> This number of terms in expassion of $(1 + x)^{2n}$ is 2n + 1 (odd)

So, its middle term is $(n + 1)^{th}$ term.

Required Term = $T_{n+1} = {}^{2n}C_n\, x^n$

$= \dfrac{2n!}{n!\,n!} x^n$

$$= \frac{1.\ 2.\ 3.\ 4.....2n}{n!\ \ n!}\ x^n$$

$$= \frac{(1.\ 3.\ 5.....2n - 1)\ \ (2.\ 4.\ 6.....2n}{n!\ \ n!}\ x^n.$$

$$= \frac{(1.\ 3.\ 5.....(2n - 1))\ \ 2^n\ \ (1.\ 2.\ 3.....n)}{n!\ \ n!}\ x^n$$

$$= \frac{(1.\ 3.\ 5.....(2n - 1)\ )\ 2^n\ n!\ x^n}{n!\ \ n!}$$

$$= \frac{1.\ 3.\ 5.....(2n - 1)}{n!}\ 2^n\ x^n$$

<u>Illustration 4</u>

find the greatest term in the expansion of $\sqrt{3}\ \left(1 + \frac{1}{\sqrt{3}}\right)^{20}$

<u>Ans</u> Let us find $r = \frac{n+1}{1 + \left|\frac{a}{x}\right|}$

So, $r = \left[\frac{20\ +\ 1}{1\ +\ \sqrt{3}}\right]$

$= \left[\frac{21}{2}\ (\sqrt{3} - 1)\right]$

$= 7$

$\therefore T_{r+1} = T_8$ is greatest term

Now $T_8 = \sqrt{3}\,{}^{20}C_7 \left(\frac{1}{\sqrt{3}}\right)^7$

$= \frac{25840}{9}$

<u>Summation of series of Bionomial co-efficients</u>.

Series of Bionomia cofficients cn be umme by uing methods like by taaking prouct o expnion of two bionomial by differentiating bionomial expansion, integraating bionomial expansion by equating real and imaginory part of a eries etc.

(1) Sum of sevies by taking product o expnsions of two bionomials if we find the product of binomial coefficients in tnhe sevies then this method can be used .

<u>Illustration 5</u>

Prove that $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots\ldots\ldots C_n^2 = \frac{2n!}{n!\,n!}$

<u>Ans</u> : We know .

$(1 + x)^n = C_0 + C_1\ C_2\ x^2 \ldots\ldots + C_{n-1}x^{n-1} + C_n x^n$

Also. $(x + 1)^n = C_0\ x^n + C_1\ x^{n-1} + C_2\ x^{n-2} + \ldots\ldots\ldots\ldots + C_{n-1}x + C_n$

Now Let us multiply the two expansions and compare the coefficient of x^n on both side .

$(1 + x)^n\ (x - 1)^n = (C_0 + C_1\ x + C_2\ x^2 \ldots\ldots\ldots + C_n x + x^n)$ X

$(C_0\ x^n + C_1\ x^{n-1} + C_2\ x^{n-2} + \ldots\ldots\ldots\ldots + C_{n-1}x + C_n\ )$

Coefficient of x^n in $(1 + n)^n\ (1 + n)^n$

=> Coeff. of x^n in $(1 + x)^{2n}$

$= 2n_{Cn}$

Coeff. of x^n in ($C_0 + C_1 x + C_2 x^2 + ...C_n x^n$) X

$(C_0x^n + C_1x^n + ...+C_n)$

$= C_0^2 + C_1^2 + C_2^2 + ... + C_{n-1}^2 + C_n^2$

$C_0^2 + C_1^2 + C_2^2 + ... + C_{n-1}^2 + C_n^2$

So, Comparing the coefficients un the 2 sidse we get .

$C_0^2 + C_1^2 + C_2^2 + ... + C_{n-1}^2 + C_n^2 = 2n\ C_n$

$= \frac{2n!}{n!\,n!}$

(2) Sum of sevies by use of differentiation:

When numericals occur as product of bionomial Coefficients this method is used .

<u>Illustration 6</u>

Find thesum of the following sevies.

$C_0 + 2C_1 + 3C_2 + ... + (n + 1)\ C_n$?

<u>Ans</u> We know that

$(1 + x)^n = C_0 + C_1x + C_2 x^2 + ... + C_n x^n$

So, Multiplying both sides with x we get

$x(1 + x)^n = C_0 x + C_1 x^2 + C_2 x^3 + ... + C_n x^{n+1}$

Now differentiating both sides with respect x we get

$xn(1 + x)^{n-1} + (1 + x)^n = C_0 + 2C_1x + 3c_2 x^2 + ... + (n + 1)\ C_n x^n$

Now put x = 1

So, $n\,2^{n-1} + 2^n = C_0 + 2c_1 + 3C_2 + \ldots\ldots\ldots\ldots + (n+1)C_n$

So, $C_0 + 2c_1 + 3C_2 + \ldots\ldots\ldots\ldots + (n+1)C_n = (n+2)2^{n-1}$

<u>Dumb Question</u>:- Why did we multiplied binomial expansion with x initially ?

<u>Ans</u> Observe that the numerical with binomial coefficients were one more in value that the corresponding power of x, So we needed to multiply the expansion by x.

(3) Sum of series by use of integration

When numerical occur as the denominator of the binomial Coefficient we apply this method.

<u>Illustration 7</u>.

Find the sum of series $2C_o + 2^2 \frac{C_1}{2} + 2^3 \frac{C_2}{3} + 2 \frac{C_3}{4} + \ldots\ldots\ldots + 2^{n+1} \frac{(n)}{n+1}$

<u>And</u>

We know

$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots\ldots\ldots + C_n x^n$

Integrating both sides with respect to x we get

$$\int_0^2 (1+x)^n \, dx = \int_0^2 (C_0 + C_1 x + C_2 x^2 + \ldots\ldots\ldots\ldots + C_n x^{n)\,dx}$$

$$\Rightarrow \left(\frac{(1+x)\ n+1}{n+1}\right)^2 = 2C_0 + \frac{C_1(2^2)}{2} + \frac{C_1(2^3)}{3} + \ldots\ldots\ldots + \frac{C_n 2^{n+1}}{n+1}$$

$$2C_o + 2^2 \frac{C_1}{2} + 2^3 \frac{C_2}{3} + 2^4 \frac{C_3}{4} + \ldots\ldots\ldots + 2^{n+1} \frac{C_n}{n+1}$$

$$= \frac{3^{n+1}-1}{n+1}$$

(4) Sum of the series by equation real and imaginary parts.

Sometimes the use of complex nos make the calculation very easy

<u>Illustration 8</u>

If $(3 + 4x)^n = P_0 + P_1x + P_2x^2 + ... + P_nx^n$ then prove that $(P_o - P_2 + P_4...)^2 + (P_1 - P_3 + P_5 ....)^2 = 25^n$

<u>Ans</u> . Let us put x = i in the expansion

$(3 + 4x)^n = P_o + P_1x + P_2 x^2 + ... + P_nx^n$

So, we get

$(3 + 4i)^n = (P_o + P_2 + P_4 .....) + i (P_1 - P_3 + P_5 .....)$

Now equating the square of the modulus, we get

$(P_o + P_2 + P_4 .....)^2 + (P_1 - P_3 + P_5 .....)^2$

$= (3^2 + 4^2)^n$

$= 25^n$

<u>Dumb Question</u> Why did P_4 got a +ve sign in front of it ?

<u>Ans</u> P_4 is Coefficient of x^4 . So,when we put i. i^4 gives the sign and .

$i^4 = (i^2)^2 = (-1)^2 = 1$

<u>Binomial Series (for negative or fractional indices)</u>

If $n \in \theta$ and $| x | < 1$, then

$(1 + x)^n = 1 + nx +$

$$\frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!} + \ldots\ldots\ldots\ldots + \frac{n(n-1)\ldots\ldots(n-r-1)}{r!}x^r + \ldots\ldots$$

General term = T_{r+1} $\dfrac{n(n-1)\ldots\ldots\ldots(n-r+1)}{r!}$ x^2

<u>Illustration 9</u>

Find the coefficients of x^4 and x^{-4} in expansion of

$$\frac{1}{2+x-x^2}, |<x<2.$$

$$\frac{1}{2+x-x^2} = \frac{1}{(2-x)(1+x)}$$

$$= \frac{1}{3}\left[\frac{1}{1+x} + \frac{1}{2-x}\right]$$

$$= \frac{1}{3x}\left[\frac{1}{1+\frac{1}{x}}\right] + \frac{1}{3}\frac{1}{2[1-x/2]}$$

$$\frac{1}{3x}\left(1+\frac{1}{x}\right)^{-1} + \frac{1}{6}\left(1-\frac{x}{2}\right)^{-1}\left(\because \frac{1}{x} < 1, \frac{x}{2} < 1\right)$$

$$= \left(\frac{1}{3x}1 - \frac{1}{x} + \frac{1}{x^2} - \ldots\ldots\ldots\right) + \frac{1}{6}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \ldots\ldots\ldots\ldots\right)$$

Coefficient of $x^4 = 0 + \frac{1}{6}\left(\frac{1}{2}\right)^4 = \frac{1}{6}$

Coefficient of $x^{-4} = \frac{1}{3}(-1) + 0 = \frac{-1}{3}$

Binomial Theoram

If n is a positive integer then

$$(a_1 + a_2 + ...+ a_m)^n \sum \frac{n!}{n!n_2!.....n_m!} a_1^{n1} a_2^{n2}. a_3^{n3}.....a_m^{nm}$$

Where the summation is taken over all non-negative integers n_1, n_2 .....n_m such that $n_1 + n_2 + n_3 + ... + n_m = n$

and the number of terms in the expansion is

= number of non-negative integral sol^n of equation

$n_1 + n_2 + ...+n_m$

$= {}^{n+m-1}C_{m-1}$

Illustration 10 .

find the total number of terms in the kexpansion

of $(x + y + z + w)^n$ $n \in \mathbb{N}$

Ans. From Hultinomial Theorm

$$(x + y + z + w)^n = \sum \frac{n!}{n1!\ n2!\ n3!\ n4!} x_1^{n1}\ y^{n2}\ z^{n3}\ w^{n4}$$

where n_1, n_2, n_3, n_4 are non negative integers subject to the condition $n_1 + n_2 + n_3 + n_4 = n$

Hence no of distinct terms

= Coefficient of x^nin $(x^o + x^1 + x^2 + ...+x^n)^4$

= Coefficient of x^n in $\left(\frac{1 - x^{n+1}}{1 - x}\right)^4$

= Coefficient of x^n in $(1 - x^{n+1})^4 (1 - x)^{-1}$

= Coefficient of x^n in $(1 - x)^{-4}$ (since x^{n+1}

$= n + 3_{Cn}$

$$= \frac{(n+3)\ (n+2)\ (n+1)}{6}$$

<u>Dumb Question</u> Why is no of disfinft terms equal to coefficient

of x^n in $(x^o + x^1 + x^2 + \ldots\ldots\ldots\ldots\ldots\ldots + x^n)^4$?

<u>Ans</u> This is actually a problem of permutation and combipnation

But Let us discuss it here very brifly .

Equation is $n_1 + n_2 + n_3 + n_4 = n$

Now n_1 can vary from o to n

Similar is the case for n_2, n_3 and n_4.

So, $x^o + x^1 + x^2 + \ldots\ldots\ldots\ldots\ldots\ldots + x^n$ for n_1

and $(x^o + x^1 + x^2 + \ldots\ldots\ldots\ldots\ldots\ldots + x)^4$ for all n_1, $n_2 + n_3$ and n_4.

And then we try to find coeft of x^n as n_1 $n_2 + n_3 + n_4$ head to sum to n

<u>Easy</u>

(1)Prove that $C_o + C_1 + C_2 + \ldots\ldots\ldots C_n = 2^n$

<u>Ans</u> we know

$C_o + C_1 + C_2 + \ldots\ldots\ldots C_n x^n = (1 + x)^n$

Now put x = 1 or both sides

$C_o + C_1 + C_2 + \ldots\ldots\ldots C_n = 2^n$

(2) find the sum kof series $\sum_{r=0}^{10} {}^{20}C_r$?

<u>Ans</u> We know = $\sum_{r=0}^{20} {}^{20}C_r = 2^{20}$

Now L.H.S. mhas 21lterms out of which 10 pairs are equal.

because ${}^{n}C_r = {}^{n}C_{n-r}$

So, L.H.S. $2 \sum_{r=0}^{9} {}^{20}C_r - {}^{20}C_{10} = 2^{20}$

So, $2 \left(\sum_{r=0}^{9} {}^{20}C_r + {}^{20}C_{10} \right) = 2^{20} + {}^{20}C_{10}$

or $\sum_{r=0}^{10} {}^{20}C_r = \frac{1}{2} (2^{20} + {}^{20}C_{10})$

(3) Find the digit at unit place in the number .

$16^{1986} + 11^{1986} - 6^{1986}$. ?

<u>And</u> Let $E = 16^{1986} + 11^{1986} - 6^{1986}$

So, $E = (10 + 6)^n + (1 + 10)^n - 6^n$

where N = 1986

$= 10^N + {}^{N}C_1 \, 10^{N-1} \, 6 + \ldots\ldots\ldots + {}^{N}C_{N-1} \, 10. \, 6^{n-1} + 6^N$

$+ 1 + {}^{N}C_1 \, 10. + {}^{N}C_2 \, 10^2 + \ldots\ldots\ldots + {}^{N}C_N 10^N - 6^N$

= 1 + All inteqers being multiple of 10

So, The digit at unit place is 1.

(4) Find the value of x for which the sixth term of $\left[\sqrt{2^{\log(10-3^x)}}+\sqrt[5]{2^{(x-2)\log 3}}\right]^m$ is equal to 21 and binomial coefficient of second, third and fourth terms are first, third and fifth terms of an aritnmetic progression .

<u>Ans</u> The sixth term of the given binomial exponsion is

$${}^mC_5\left(\sqrt{2\log(10-3^x)}\right)^{m-5}\left(\sqrt[5]{2^{x-2\log 3}}\right)^5 \quad .....(1)$$

The other given lcondition is, $m_{C1} + m_{C3} = 2\ m_{C2}$

=> $m^2 - 9m + +14 = 0$ => $(m - 7)(m - 2) = 0$

or m = 7 Since m = 2 is not possible .

Put m = 7 in equation (1) to get x = 0, 2

<u>Dumb Question</u> Why m cannot be 2 ?

If m = 2 then there cannot be 6th term in the expansion and hence it is ruled out .

If n is any positive integer show that . $2^{3n+3} - 7n - 8$ is divisible by 49 ?

The given expression

$= 2^{3n+3} - 7n - 8$

$= 2^{3(n+1)} - 7n - 8$

$= 8^{n+1} - 7n - 8$

$=(1 + 2)^{n+1} - 7n - 8$

$= 8\ (1 + 7)^n - 7n - 8$

$= 8\ (1 + {}^nC_1\ 7 + {}^nC_2.\ 7^2 .....{}^nC_n\ 7^n) - 7_n - 8$

$= 8 + 56_n + 8\ ({}^nC_2\ 7^2 \ldots\ldots\ldots + {}^n C_n\ 7^n) - 7_n - 8$

$= 49_n\ 8\ ({}^n C_2\ 7^2 + \ldots\ldots\ldots\ldots\ldots + {}^n C_n\ 7^n)$

$= 49\ (n + 8\ ({}^n C_2 + \ldots\ldots\ldots + {}^n C_n\ 7^{n-2}))$

So, $2^{3n+3} - 7_n - 8$ is divisible by 49 .

(6) Find the coefficient of x^{50} in the polynoamial

$(1 + x) + 2(1 + x)^2 + 3(1 + x)^3 + \ldots\ldots\ldots + 1000\ (1 + x)^{1000}$?

<u>Ans</u> Let $P(x) = (1 + x) + 2(1 + x)^2 + 3(1 + x)^3 + \ldots\ldots\ldots + 1000\ (1 + x)^{1000}$

Now $(1 + x)\ p(x) - P(x)$

$= (1 + x)^2 + 2(1 + x)^3 + 3(1 + x)^4 + \ldots\ldots\ldots + 999\ (1 + x)^{1000} + 1000(1 + x)^{1000}$

$- (1 + x) - 2\ (1 + x)^2 - 3\ (1 + x)^3 \ldots\ldots\ldots\ldots\ldots\ldots - 1000\ (1 + x)^{1000}$

$= 1000\ (1 + x)^{1001} - (1 + x) - (1 + x)^2 - (1 + x)^3 - \ldots\ldots\ldots\ldots\ldots - (1 + x)^{1000}$

$1000\ (1 + x)^{1001} - [\ (1 + x) - (1 + x)^2 + \ldots\ldots\ldots + (1 + x)^{1000}]$

$$= 1000\ (1 + x)^{1001} - \left[\frac{(1+x)^{1001} - (1+x)}{(1+x) - 1}\right]$$

$$= 1000\ (1 + x)^{1001} - \left[\frac{(1+x)^{1001} - (1+x)}{x}\right]$$

$$= P(x) = \frac{100(1+x)^{1001}}{x} - \frac{(1+x)^{1001}}{x^2} + \frac{(1+x)}{x^2}$$

So, coefficient of x^{50} in $P(x)$

$= 1000\ X\ {}^{1000} C_{51} = {}^{1000}C_{52}$

$$= \frac{1000 \times 1001!}{51!\ 950!} - \frac{1001!}{52!\ 949!}$$

$$= \frac{(51050)\ 1001!}{52!\ 950!}$$

(7) If $(1 + x)^n = a_o + a_1x + a_2x^2 + \ldots\ldots\ldots + a_n x^n$ then find

value of $\left(1+\frac{a_1}{a_0}\right)\left(1+\frac{a_2}{a_1}\right)\ldots\ldots\left(1+\frac{an}{a_{n-1}}\right)$?

<u>Ans</u> Clearly $a_r = {}^n C_r$

So,

$$\Rightarrow 1 + \frac{ar}{a_{r-1}} = \frac{n+1}{r}$$

So, $\prod_{r=1}^{n}\left(1+\frac{a_r}{a_{r-1}}\right) = \prod_{r=1}^{n}\frac{(n+1)}{r} = \frac{(n+1)^n}{n!}$

Sum the following series

$C_o + 5c_1 + ac_2 + ac_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + (4n + 1)c_n$

<u>Ans</u> Let $S = C_o + 5c_1 + ac_2 + ac_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + (4n + 1)c_n \ldots\ldots\ldots\ldots(1)$

Sine $C_r = C_n - r$

So, $S\ C_n + 5C_{n-1} + {}^a C_n + \ldots\ldots\ldots\ldots + (4n = 1)C_o$

or $S = (4n + 1)\ C_o + (4n - 3)\ C_1 + \ldots\ldots\ldots\ldots\ldots\ldots + 5C_{n-1} + c_n \ldots\ldots(2)$ Now Adding (i) + (2) we get

$25 = (4n + 2)\ C_o + C_1 + \ldots\ldots\ldots\ldots C_n\ )$

$\Rightarrow S = (2n + 1)\ 2^n$

So, $C_o + 5C_1 + 9C_2 + ...+ (4n + 1) C_n = (2n + 1) 2^n$

(9) If is an odd natural lnumber then what is the value

Of $\sum_{r=1}^{n} \frac{(-1)^r}{{}^nC_r}$?

<u>Ans</u> Since n is odd, the number of terms will be n + 1 which is even,

Now since $\frac{n+1}{2}$ is an integer we make pairs of terms equidistant from the beginning and end .

So, $\sum_{r=1}^{n} \frac{(-1)^r}{{}^nC_r} = \sum_{r=1}^{\frac{(n+1)}{2}} \left[\frac{(-1)^{n-r}}{{}^nC_{n-r}} + \frac{(-1)^{rn}}{C_r} \right]$

$(-1)^r \sum_{r=0}^{\frac{(n+1)}{2}} \left[\frac{1}{{}^nC_r} + \frac{(-1)^{n-2r}}{{}^nC_r} \right]$

$(-1)^r \sum_{r=0}^{\frac{(n+1)}{2}} \left[\frac{1}{{}^nC_r} + \frac{-1}{{}^nC_r} \right] = 0$
f

Hence $\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r} = 0$

<u>Dumb Question :</u> Why $(-1)^{n-2r}$ is -1 for all v)

<u>Ans</u> Since n is odd so, n - 2r is odd as 2r is even .

Thus $(-1)^{odd} = -1$.

(10) find the sum of the series .

$$\sum_{r=0}^{n} (-1)^r \, {}^nC_r \left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\text{.....upto m terms}\right]$$

<u>Ans</u> $$\sum_{r=0}^{n} (-1)^r \, {}^nC_r \left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{3^r}{2^{2r}}+\text{.....upto m terms}\right]$$

$$= \sum_{r=0}^{n} (-1)^r \, {}^nC_r \left\{\left(\frac{1}{2^r}\right)^r + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + \text{..... +up to m terms}\right\}$$

$$= \sum_{r=0}^{n} (-1)^r \, {}^nC_r \left(\frac{1}{2}\right)^r + \sum_{r=0}^{n} (-1)^r \, {}^nC_r \left(\frac{3}{4}\right)^r \quad \text{+.....up to m terms}$$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n$$

$$= \frac{\left(\frac{1}{2}\right)^n\left[1-\left(\frac{1}{2^n}\right)^m\right]}{\left[1-\frac{1}{2n}\right]}$$

$$= \frac{(2^{mn}-1)}{2^{mn}(2^n-1)}$$

(11) Prove the following identity

$${}^nC_o + {}^{n+1}C_1 + {}^{n+2}C_2 + \text{.....} + {}^{n+r}C_r = {}^{n+r+1}C_r$$

<u>Ans</u> ${}^nC_o$ is coefficient of x^n in expansion of $(x + 1)^n$

${}^{n+1}C_1$ is coefficient of x^n in expansion of $(x + 1)^{n+1}$

.....

${}^{n+r}C_r$ is coefficient of $(x + 1)^{n+r}$

Thus L.H.S. is coefficient of x^n inexpansion of

$(x + 1)^n + (x + 1)^{n+1} + \ldots\ldots\ldots\ldots + (x + 1)^{n+r}$

Or coefficient of x^n in $(x + 1)^n \left[\dfrac{(x+1)^{r+1}-1}{(x+1)-1}\right]$

Or coefficient of x^n in $\dfrac{(x+1)^{n+r+1}-(x+1)^n}{x}$

Or coefficient of x^{n+1} in $(x + 1)^{n+r+1} - (x - 1)^n$

$= {}^{n+r+1}c_{n+1}$ (since $(x + 1)^n$ has no ten containing x^{n+1})

$= {}^{n+r+1}C_r$

<u>Medium</u>.

(1) Given that 4th term in expansion of $\left(2+\dfrac{3x}{8}\right)^{10}$ has the maximum numberical value, find the range of value of x for which this will be true .

<u>Ans</u> According to the question

$|t_4| \geq |t_3|, |t_4| \geq |t_5|$

Now, $t_{r+1} = {}^{10}C_r 2^{10-r} \left(\dfrac{3x}{8}\right)^r$

$t_4 = {}^{10}C_3\, 2^7 \left(\dfrac{3x}{8}\right)^3$

$t_3 = {}^{10}C_2\, 2^8 \left(\frac{3x}{8}\right)^2$

and $t_5 = {}^{10}C_2\, 2^6 \left(\frac{3x}{8}\right)^4$.

Now, $|t_4| \geq |t_3|$

$\Rightarrow {}^{10}C_3\, 2^7 \left(\frac{3}{8}\right)^3 |x|^3 \geq {}^{10}C_2 2^8 \left(\frac{3}{8}\right)^2 |x|^2$

$\Rightarrow |x| \geq 2$ .....(1)

and $|t_4| \geq |t_5|$

${}^{10}C_3\, 2^7 \left(\frac{3}{8}\right)^3 |x|^3 \geq {}^{10}C_4\, 2^6 \left(\frac{3}{8}\right)^2 |x|^2$

$\Rightarrow 1 - \frac{21}{64} |x| \geq 0$ .....(2)

Clearly $\left(5\sqrt{5}+11\right)^{2n+1}\left(5\sqrt{5}-11\right)^{2n+1}$

$= \frac{4}{\sqrt[5]{5}+11}$

$\therefore \sqrt[5]{5}-11$ is a positive proper fraction

and so $g = \left(5\sqrt{5}-11\right)^{2n+1}$ is also a positive proper fraction.

Thus, $0 < f < 1$ and $0 < g < 1$

Now, $[R] + f - g$

$= 2$

$$\left[{}^{2n+1}C_1 \left(5\sqrt{5}\right)^{2n} 11 + {}^{2n+1}C_3 \left(5\sqrt{5}\right)^{2n-2} 11^3 + \ldots\ldots + {}^{2n+1}C_{2n+1} \cdot 11^{2n+1} \right]$$

= 2 X inteqer = even integer.

$\therefore$ f - g must be an integer because [R] is ans itneger .

Now f - g = 0 i.e f = g .

$\therefore$ R. F{ [R] + f } = ([R] + f) g

$$= \left(5\sqrt{5}+11\right)^{2n+1} \left(5\sqrt{5}-11\right)^{2n+1}$$

$$= \left(5\sqrt{5}+11\right) \left(5\sqrt{5}-11\right)^{2n+1}$$

$$= \left(\left(5\sqrt{5}\right)^2 - 11^2\right)^{2n+1}$$

$= 4^{2n+1}$

<u>Dumb Question</u>.

Why f - g is zero ?

$\therefore \ |x| \le \dfrac{64}{21}$

Thus we get $|x| \ge 2$ and $|x| \le \dfrac{64}{21}$

So, $x \in \left[\dfrac{-64}{21}, -2\right] \cup \left[2, \dfrac{64}{21}\right]$

<u>Dumb Question</u>

Why we had an interval $\left[\frac{-64}{21}, -2\right]$ in answer ?

<u>Ans</u> . Well we have the situation

$|x| \geq 2$ and $\leq \frac{64}{21}$

So, $|x| \geq 2$ means

$x \geq 2$ or $x \leq -2$

Similariey $|x| \leq \frac{64}{21}$

and hence we get the interal $\left[\frac{-64}{21}, -2\right]$

(2) Let R = $\left(5\sqrt{5}+11\right)^{2n+1}$ and F = R - [R] where [] denotes the greatest integral funftion. Prove that RF = 4^{2n+1}.

<u>Ans</u> R = [R] + F = $\left(5\sqrt{5}+11\right)^{2n+1}$

$= {}^{2n+1}C_o \left(5\sqrt{5}\right)^{2n+1} + {}^{2n+1}C_1 \left(5\sqrt{5}\right)^{2n} . 11 + \ldots\ldots\ldots\ldots + {}^{2n+1}C_{2n+1}\, 11^{2n+1}$ .....(1)

Let g = $\left(5\sqrt{5}-11\right)^{2n+1}$

$= {}^{2n+1}C_o \left(5\sqrt{5}\right)^{2n+1} - {}^{2n+1}C_1 \left(5\sqrt{5}\right)^{2n} 11 + \ldots\ldots\ldots\ldots + {}^{2n+1}C_{2n+1}\, 11^{2n+1}$ .....(2)

F and g both are positive proper fraction. i.e.

$O < f < 1$, $O < g < 1$

So, F - g con not be any integer exept zero because where both f and g not kintegers how can there difference be.

(3) If $(1 + x + x^2)^n = a_o + a_1x + a_2x^2 + a_3 x^3 +.....+ a_{2n}x^{2n}$ then show that

$a_o + a_3 + a_6 +.....=a_1 + a_4 + a_7 +.....$

$= a_2 + a_5 + a_8 + ...$

<u>Ans</u> Putting x = 1, w, w^2 where w is a non ,real cine root of unity ?

So, $Z^n = a_o + a_1 + a_2 +.....(1)$

$O = a_0+ a_1 w + a_2 w^2+ .....(\because + w + w^2 = o) .....(2)$

$0 = a_0 + a_1 w^2 + a_2 w^4 .....(\because 1 + w + w^2 = 0) .....(3)$

Adding these .

$3^n\ 3 a_0 + a_1 (1 + w + w^2) + a_2 (1 + w + w^4) + a_3 (1 + w^3+ w^6) + ...$

$= 3(a_0 + a_3 + a_6 +.....)$

$=>a_0 + a_3 + a_6 + ...3^{n-1}$

from (1) + (2) X w^2 + (3) X w.....we get

$3^n + O X w^2 + O X W$

$= a_0 (1 + w^2 + w) + a_1 (1 +w^3+ w^3) + a_2 (1 + w^4 + w^5 + a_3 (1+ w^5 + w^7) + a_4 (1 + w^6 + w^9) +.....$

img.....$3^n = 3(a_1 + a_4 + a_7 + ...)$

So, $a_1 + a_4 + a_7 + ...3^{n-1}$

Again from (1) + (2) x w + (3) w^2 we get

$3^n = a_0 (1 + w + w^2) + a_1 (1 + w^2 + w^4) +a_2 (1 + w^3 + w^3) +.....$

$= 3(a_2 + a_5\ a_8 + ...)$

$\therefore a_2 + a_5 + a_8 + ... = 3^{n-1}$

<u>Dumb Question</u>.

Why $1 + w^4 + w^5$ is equal to zero and

$1 + w^6 + w^9$ is 3.?

$1 + w^4 + w^5 = 1 + w + w^2$ (using $w^3 = 1$)

$= 0$

Also $1 + w^6 + w^9 = 1 + (w^3)^2 + (w^3)^3$

$= 1 + 1 + 1$

$= 3$.

(4) Find the value of $C_0 - C_1$

$$\frac{(1+x)}{1+nx} + C_2\frac{(1+x)}{(1+nx)^2} - C_3\frac{(1+3x)}{(1+nx)^3} + ...$$

<u>Ans</u> . Let $C_o - C_1$ $\frac{(1+x)}{1+nx} + C_2\frac{(1+x)}{(1+nx)^2} - C_3\frac{(1+3x)}{(1+nx)^3} + ...$

$$= \underbrace{\left[C_0 - \frac{C_1}{1+nx} + \frac{C_2}{(1+nx)^2} - .....\right]}_{A} + \frac{x}{1+nx}\underbrace{\left[-C_0 + \frac{2C_2}{1+nx} + \frac{3C_3}{(1+nx)^2} + ...\right]}_{B}$$

So, $(1 - y)^n = C_0 - C_1\ y + C_2 y^2 - C_3\ y^3 + ...(-1)^n\ y^n$
defferentiating w.r.t. y we get

$-n\ (1 - y)^{n-1} = -\ C_1 + 2C_2\ y - 3C_3\ y^2 + ...$

Now $A = \left[1 - \frac{1}{1+nx}\right]^n = \left(\frac{nx}{1+nx}\right)^n$

and $B = -C_2 + 2C_2 \frac{1}{1+nx}$

$= - n \left[1- \frac{1}{1+nx}\right]^{n-1}$

$= - n \left(\frac{nx}{1+nx}\right)^{n-1}$

So, $C_0 - C_1 \frac{1+x}{1+nx} + C_2 \frac{(1+2x)}{(1+2x)^2} - \ldots\ldots\ldots$

$= A + \frac{x}{1+nx} B$

$= \left(\frac{x}{1+nx}\right)^{n} - \left(\frac{nx}{1+nx}\right) \left(\frac{nx}{1+nx}\right)^{n-1}$

$= 0$

(5) Prov that $\sum_{m=0}^{n} {}^{n}C_m \sin (mx) \cos ((n-m)x) = 2^{n-1} \sin <(nx)$

<u>Ans</u> $\sum_{m=0}^{n} {}^{n}C_m \sin (mx) \cos (n - m) x.$

$= {}^{n}C_0 (0.x) \cos nx + {}^{n}C_1 \sin x \text{ ccos } (n -1) x$

$= {}^{n}C_2 \sin 2x \cos (n - 2) x + \ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots = {}^{n}C_n \sin nx \cos (0 . x)$

$\Rightarrow 2 \sum_{m=0}^{n} {}^{n}C_m \sin (m x) \cos (n - m) x$

$[\ ^{n}C_0\ (0\ .\ x)\ \cos nx + {}^{n}\sin nx \cos 0\ .\ x]$

$+ [\ ^{n}C_1 \sin x \cos (n - 1)\ x + {}^{n}C_{n-1} \sin (n - 1)\ x\ .\ \cos x\ ]$

$.....+ [^{n}C_n \sin nx \cos(0\ .\ x) + {}^{n}C_0 \sin (0\ .\ x) \cos nx\ ]$

Using $^{n}C_r = {}^{n}C_{n-r}$ we get

$2 \sum_{m=0}^{n} {}^{n}C_m \sin (m\ x) \cos (n - m\ )\ x$

$= {}^{n}C_0\ [\ \sin (0\ .\ x) \cos (n\ x) + \sin (n\ x) \cos (0\ .\ x)]$

$+ {}^{n}C_1\ [\ \sin x \cos (n -1)\ x + \sin (n - 1)\ x \cos x\ ] + ...$

$.....+ {}^{n}C_n[\ \sin x \cos (0\ .\ x) + \cos (n\ x\ )\sin (0\ .\ x)]$

NOw using $\sin (A + B) = \sin A \cos B + \cos A \sin B$ we get

$(\ ^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 +.....+^{n}C_n\ ) \sin nx = 2^{n} \sin nx.$

So, $\sum_{m=0}^{n} {}^{n}C_m \sin (m\ x) \cos (n - m)\ x = 2^{n-1} \sin (nx)$

(6) If $a_n = \left(1+\frac{1}{n}\right)^n$ then prove that $2 < a_n < 3$ img..... $n \in \mathbb{N}$

Also show that

$n^{n-1} > (n + 1)^{n}\ ; \geq 3\ ,\ n \in \mathbb{N}$

<u>Ans</u> . We get $a_n = \left(1+\frac{1}{n}\right)^n$

$= 1 + n\ \frac{1}{n} + \frac{n(n - 1)}{2\ !}\ \frac{1}{n^2} + ...$

$$= 1 + 1 + \frac{(1-\frac{1}{n})}{2!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})}{3!} + \ldots\ldots\ldots\ldots$$

= 2 + Positive quantity

$\therefore a_n > 2$ . .....(1)

Alos, $a_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \ldots\ldots\ldots\ldots + \frac{1}{n!}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots\ldots\ldots \frac{1}{2^{n-1}}$

$< 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots\ldots\ldots + \frac{1}{2^{n-1}} + \ldots\ldots$

$< 1 + \frac{1}{1-\frac{1}{2}}$

< 3 .....(2)

From (1) and (2) we get

$2 < a_n < 3$

So, $a_n < 3$

Or $\left(1+\frac{1}{n}\right)^n < 3$

or $\left(1+\frac{1}{n}\right)^n < n$ ($\because$ n ≥ 3)

or.....img

So, $(1 +)^n < n^4$. n

or $(n + 1)^n < n^{n+1}$

<u>Dumb Question</u>

Why img.....

img.....

<u>Ans</u> We are dividing one by a smaller no because all natural no.s except 2 are bigger than 2 only

So, $\frac{1}{3!} < \frac{1}{2^2}$

as $\frac{1}{2.3} < \frac{1}{2.2}$

3 > 2 so, $\frac{1}{3!} < \frac{1}{2^2}$

For other factor also

So, $1 + 1 + \frac{1}{2!} + \frac{1}{3!} \ldots\ldots\ldots\ldots + \frac{1}{n!}$ is less than $1 + 1 + \frac{1}{2} + \frac{1}{2^2} . + \frac{1}{2^{n-1}}$

<u>Hard</u>

(1) Prove that $\sum_{r=1}^{k} (-1)^{r-1} {}^{3n}C_{2r-1} = 0$ where $k = \frac{3n}{2}$

is an even positive inteqer.

<u>Ans</u> Given n is an even positive inteqer.

Let $n = 2m \therefore R = 3m$ $m \in \mathbb{N}$

L.H.S. $\sum_{r=1}^{k} (-3)^{r-1} {}^{3n}C_{2r-1} = \sum_{r=1}^{3m} (-3)^{r-1} {}^{6m}C_{2r-1}$

$= {}^{6m}C_1 - 3\,{}^{6m}C_3 + 3^2\,{}^{6m}C_5 - \ldots\ldots + (-3)^{3m-1}\,{}^{6m}C_{6m} - 1$ .....(1)

$(1 + i\sqrt{3})^{6m} = {}^{6m}C_o + {}^{6m}C_1\,(i\sqrt{3}) + {}^{6m}C_2\,(i\sqrt{3})^2 + \ldots\ldots + {}^{6m}C_{6m-1}(i\sqrt{3})^{6m-1} + {}^{6m}C_{6m}(i\sqrt{3})^{6m}$img .....

or $2^{6m}\left[\cos\frac{\pi}{3} + i\ \sin\pi/3\right]^{6m} = 6m\,C_o + {}^{6m}C_1\,(i\sqrt{3}) + {}^{6m}C_2\,(i\sqrt{3})^2 + \ldots\ldots + {}^{6m}C_{6m-1}(i\sqrt{3})^{6m-1} + {}^{6m}C_{6m}(i\sqrt{3})^{6m}$

or $2^{6m}(\cos 2\pi + i \sin 2\pi m)$

$=({}^{6m}C_o - 3\,{}^{6m}C_2 + 3^2\,{}^{6m} - \ldots\ldots + (-3)^{3m}\,{}^{6m}C_{6m}) + i\sqrt{3}\,({}^{6m}C_o - 3\,{}^{6m}C_2 + 3^2\,{}^{6m} - \ldots\ldots + (-3)^{3m}\,{}^{6m}C_{6m})$

Companing imaqinary part on both sides we get .

$\sqrt{3}\,({}^{6m}C_1 - 3\,{}^{6m}C_3 + 3^2\,{}^{6m}C_5 \ldots\ldots\ldots\ldots + (-3)^{3m-1}\,{}^{6m}C_{6m-1}) = 0$

Or $({}^{6m}C_1 - 3\,{}^{6m}C_3 + 3^2\,{}^{6m}C_5 \ldots\ldots\ldots\ldots + (-3)^{3m-1}\,{}^{6m}C_{6m-1}) = 0$

$\Rightarrow \sum_{r=1}^{3m} (-3)^{r-1}\,{}^{6m}C_{2r-2} = 0$

Or $\sum_{r=1}^{k} (-33)^{r-1}\,{}^{3n}C_{2r-1} = 0$ (where n = 2m)

<u>Dumb Question</u>: How did $(\cos \pi/3 + i \sin \pi/3)^{6m}$ be came $\cos 2\pi m + i \sin 2\pi/3\, m$?

inmg.....

<u>Ans</u> We used Eulers Rule which is $((\cos\theta + \sin\theta)^m = \cos m\theta + \sin\theta$

So, $(\cos \pi/3 + i \sin \pi/3)^{6m} = \cos 6m\,\pi/3 + i \sin 6m\,\pi/3)$

$= \cos 2m\pi + i \sin 2m\pi$

(2) If n > 3 then prove that

$C_0\, ab - C_1(a - 1) + C_2\,(a - b)\,(b - 2) - \ldots\ldots\ldots\ldots + (-1)^n C_n = 0$

<u>Ans</u> Now, $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots + x$ put x = - 1

$0 = 1 - n + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \ldots\ldots\ldots ++ (-1)^n \ldots\ldots\ldots\ldots\ldots\ldots(1)$

Now replace n by n - 1

So, $0 = 1 - (n - 1) + \frac{n(n-1)(n-2)}{2!} - \frac{n(n-1)(n-2)(n-3)}{3!} + \ldots\ldots\ldots\ldots\ldots + (-1)^{n-1} \ldots\ldots\ldots\ldots(2)$

Multiplying (1) by a and (2) by n and odding we get.

$a - na + \frac{n(n-1)}{2!}a + \ldots\ldots\ldots\ldots\ldots + (-1)^n a + n - n(n-1) + \frac{n(n-1)(n-2)}{2!} - \ldots\ldots\ldots\ldots + (-1)^{n-1} n = 0$

$\Rightarrow a - n(a-1) + \frac{n(n-1)}{2!}(a-2) - \frac{n(n-1)(n-2)(n-3)}{3!} - \ldots\ldots\ldots\ldots + (-1)^n (a - n) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3)$

Now Replace n by n-1 and a by a-1 in (3) to get

$(a - 1) - (n - 1)(a - 2) + \frac{n(n-1)(n-2)}{2!}(a - 3) + \ldots\ldots\ldots + (-1)^{n-1}(a - n) = 0 \ldots\ldots(4)$

Multiplying (3) by b and (4) by n we get .

$ab - n(a - 1)^{b} + \frac{n(n-1)}{2!}(a - 2)b + \ldots\ldots\ldots\ldots + (-1)^{n}(a - n)b$

$+ n(a - 1) - n(n - 1)(n - 2) + \ldots\ldots\ldots\ldots + (-1)^{n-1}(a - n)n = 0$

On the other hand

$(\ldots\ldots(((x - 2)^2 - 2)^2 - \ldots\ldots\ldots\ldots - 2)^2$

$= ((\ldots\ldots((x - 2)^2 - 2)^2 - \ldots\ldots\ldots - 2)^2 - 2)^2$

k - 1tirmes

$= [\,(R_{k-1}x^3\ \theta x^2 + R_{k-1}\,x + 4) - 2\,]^2$

$= (R_{k-1}x^3\ \theta x^2 + R_{k-1}\,x + 2)^2$

$= R^2_{k-1}\,x^6 + 2R_{k-1}\,\theta_{k-1}x^5 + (\theta_{k-1}{}^2 + 2p_{k-1}R_{k-1})\,x^4$

$+ (4R_{k-1} + 2\theta_{k-1}P_{k-1})\,x^3$

$+ (P^{k-12}\,4\theta_{k-1})\,x^2 + 4P_{k-1}x + 4$.

$= [R^2_{k-1}x^3 + 2R_{k-1}\,\theta_{k-1}\,x^2 + (\theta^2_{k-1} + 2P_{k-1}\,R_{k-1})\,x + 4R_{k-1} + 2\theta_{k-1}P_{k-1})]\,x^2$

$+ (P^{k-12}\,4\theta_{k-1})\,x^2 + 4P_{k-1}x + 4$.

Whence $P_k = 4P_{k-1}$ and $\theta_k = P^2_{k-1} + 4\,\theta_{k-1}$

Since $(x - 2)^2\ x^2\ 4x + 4$

We have $P_1 = -4$.

So, $P_2 = -4^2$, $P_3 = -4^3$.....

and so, $P_k = -4^k$

NOw let us compute θ_k

$= P^2_{k-1} + 4\theta_{k-1}$

$\Rightarrow P^2_{k-1} + 4(P^2_{k-1} + 4\theta_{k-2})$

$P_{k-1}^2 + 4\,(P_{k-2}^2 + 4\,\theta_{k-2})$

$= P_{k-1}^2 + 4\,P_{k-2}^2 + 4^2\,P_{k-3}^2 + \ldots\ldots\ldots\ldots + 4^{k-2}P_1 + 4^{k-1}\theta_1$

$\Rightarrow ab - n\,(a-1)(b-1)\,\frac{n(n-1)}{2!}\,(a-2)(b-2)\ldots\ldots\ldots\ldots + (-1)^n$

Or. $C_0\,ab - C_1(a-1)\,(b-1) + C_2\,(a-2)(b-2)\ldots\ldots\ldots$

$+(-1)^n\,C_n\,(a-n)\,b-n) = 0$

(3) Determine coefficients in x^2 appeating parantheses have been removed and like terms have been callected is the expression

$(\ldots\ldots\ldots((x-2)^2 - 2)^2 \ldots\ldots\ldots -2\,)^2$

<u>Ans</u> Let us first of all determine constant term which is obtained from the expression.

$[\,((x-2)^2 - 2)^2 \ldots\ldots\ldots -2\,]^2$

k times

For kthis we put x = 0 i.e it is equal to

$(\ldots\ldots\ldots((-2)^2 - 2)^2 \ldots\ldots\ldots -2\,)^2$

k times

$(\ldots\ldots\ldots((4-2)^2 - 2)^2 \ldots\ldots\ldots -2\,)^2$

(R -1)times

$= (\ldots\ldots(4-2)^2 - \ldots\ldots\ldots\ldots -2)^2$

(R - 2) times

$=((4 - 2)^2 - 2)^2 = (4 - 2)^2 = 4$

Now let us denote ny P_k the coe fficient of x by $P_k$the coefficient of x^2 and the R_k the coefficient of x^3 the sum of terms involving x to the powers higher than 2 then we can write

$(.....((x - 2)^2 - 2)^2 -.....-2)^2= R_k x^3 + \theta_k x^2 +P_k x + 4$

Now by substituting img .....= 1, $P_1 = - 4$, $P_2 =-4^2$.....

into this expression we get,

$\theta_k= 4^{2k-2} + 4.4^{2k-4}+ 4^2\ 4^{2k-6}+.....4^{k-2}\ 4^2 + 4^{k-1}\ 1.$

$= 4^{2k-2} + 4.4^{2k-3}+ 4^2\ 4^{2k-4}+.....4^k\ ++4^{k-1}$

$= 4^{k-1}\ [\ 1 + 4 + 4^2 +.....+4^{k-1}]$

$$=4^{k-1}\ \frac{[4^k\ - 1]}{4\ -1}$$

$$= \frac{4^{2k-1}\ - 4^{k-1}}{3}$$

So, Coefficient of x^2 in the given expression is

$$\frac{4^{2k-1}\ - 4^{k-1}}{3}$$

<u>Dumb Question</u>. What is the significance of the subscript in R_k ,θ_k , P_k etc?

<u>Ans</u> One may note that the expression in a very symetric kind of fexpression where things are repeated k times The subscript k denotes that repetition olnly .

One should always remember that symetry isan important thing in matrematics and should be always used aas an important tool

Key words

(1) Binomial theoram

(2) Binomial Coefficient .

(3) General term .

(4) Multinormial theoram.

(5) Binomial series.

(6) Middle term.

Logarithms

Definition:-
We define log as, $a^y = x$ than $y = \log_a x$. in $\log_a x$. both x and a are positive ie. $x > 0$ and $a > 0$ and also $a \neq 1$.

Dumb Question:- Why a cannot be 1 ?
Ans:- Suppose a is 1 then let us attempt to y such that $y = \log_1 x$.
Now according to definition of log.
$1^y = x$.
But no matter what power we raise to 1 the answer will be. 1 only so we will never never be able to find y.
Hence a cannot be 1.

Some important formulae:- (Formulaes marked with * are important. This is not be printed)
1. $\log_a a = 1$.
2. $\log_{any} 1 = 0$.
3. $\log_c a = \log_b a.\log_c b$.
Why ?
Let $\log_b a = x$ and $\log_c b = y$
So, by definition, $a = b^x$ ..... (i)
$b = c^y$ ..... (ii)
Using (i) & (ii) $a = c^{xy}$
Now taking log on both sides.
$\log_c a = xy$
$\log_b a.\log_c b$.

Illustration - 1.
Find value of $\log_2 10.\log_{10} 2$?
Using formula 3 we get.
$\log_2 10.\log_{10} 2 = \log_{10} 10$
Now using formula 1 we get
$\log_{10} 10 = 1$
Hence $\log_2 10.\log_{10} 2 = 1$

4. $\log_a(m.n) = \log_a m + \log_a n$
Why ?
Let $\log_a m = x$
$\log_a n = y$
So, $m = a^x$ and $n = a^y$ [Using definition]
$m.n = a^x.a^y = a^{x+y}$
$\Rightarrow \log_a(mn) = \log_a a^{x+y}$

$= x + y$
$= \log_a m + \log_a n.$

5. $\log_a(m/n) = \log_a m - \log_a n$
<u>Why ?</u>
Let $\log_a m = x$
$\log_a n = y$
$\therefore m = a^x$ and $n = a^y$ [Using definition]
$m/n = a^x/a^y = a^{x-y}$
$\Rightarrow \log_a m/n = x - y$
$= \log_a m - \log_a n$

6. $\log_a m^n = n \log_a m.$
<u>why ?</u>
Let $\log_a m^n = y$
By definition $a^y = m^n$
or, $a^{y/x} = m$
Again using definition of log
$\log_a m = y/x$
or $n \log_a m = y$
so, $\log_a m^n = n \log_a m.$

<u>Illustration - 2</u>. If $\log_{12}27 = a$, then $\log_3 16 = ??$
$\log_3 16 = \log_3 2^4 = 4 \log_3 2$ ..... (i)
Now, $\log_{12}27 = \log_{12}3^3$
$= 3 \log_{12}3$
$= \dfrac{3}{\log_3 12}$
$= \dfrac{3}{\log_3 (4 \times 3)}$
$= \dfrac{3}{\log_3 4 + 1}$
$= \dfrac{3}{2\log_3 2 + 1}$
$=$
So, $\log_3 2 = \dfrac{2a}{3 - a}$
Now, $\log_3 = \dfrac{8a}{3 - a}$

*7. $\log_{a^q} n^p = \log_a n$

$\log_{a^q} n^p = p \log_a q^n$ - (Using formula 6.)

$= p \dfrac{1}{\log_n a^q}$ - (Using formula 3.)

$= \dfrac{p}{q \log_n a}$ - (Using formula 6.)

$= \dfrac{p}{q} \log_a n$ - (Using formula 3.)

*8. $a^{\log_a n} = n$

<u>Illustration - 3</u>.

Evaluate. $4^{\log_8 27}$

$4^{\log_8 27} = 4^{\log_{2^3} 3^3}$

$= 4^{3/3 \log_2 3}$ (Using formula 7)

$= 4^{\log_2 3}$

$= 2^{2\log_2 3}$

$= 2^{\log_2 9}$ (Using formula 6)

= 9 (using formula 8)

Hence $4^{\log_8 27} = 9$

$\log_p a > \log_p b$

=> $a \geq b$ if p is greater than 1 i.e. $p > 1$

or $b \leq a$ if p is positive and less than 1, $0 < p < 1$

<u>Remembering Tip</u>:- If base if > 1 then inequality remains same and if base is + ve but less than 1 then sign of inequality is reversed.

<u>Why ?</u>

Let $\log_p a = x$

$\log_p b = y$

So, $a = p^x$

and $b = p^y$

Now $x > y$ is given

So, if $p > 1$

then a $\geq$b
and if 0 < p < 1
then a $\leq$b

<u>Illustration - 4</u>.
If $\log_{0.2}(x - 1) < \log_{0.4}(x - 1)$ then x lies interval.
Now, $\log_{0.2}(x - 1) < \log_{(0.2)^2}(x - 1)$
=> $\log_{0.2}(x - 1) < \frac{1}{2}\log_{0.2}(x - 1)$
=> $\frac{1}{2}\log_{0.2}(x - 1) < 0$
or, $\log_{0.2}(x - 1) < 0 = \log_{0.2}1$
Since 0.2 is less than 1 so, x - 1 > 1 or x > 2

<u>log - flow Questions</u>
<u>Easy</u>
Q1. Find the value of $\log_8 128$?
<u>Ans</u>:- $\log_8 128 = \log_{2^3} 2^7$
$= \frac{7}{3}\log_2 2$
$= \frac{7}{3}$

Q2. What is the approx vaslue of $\log_1 9.903$?
<u>Ans</u>:- The value does not exist because the base of a logarithm cannot be 1.

Q3. Find solution of the equation $2^{3/\log_3 x} = \frac{1}{64}$?
<u>Ans</u>:- $\frac{1}{64} = 2^{-6}$
So, $\frac{3}{\log_3 x} = -6$
$-\frac{1}{2} = \log_3 x$

So, $x = \frac{1}{\sqrt{3}}$

Q4. Find equation $\log_e x + \log_e(1 + x) = 0$ free from log ?
<u>Ans</u>:- $\log_e x + \log_e(1 + x) = 0$
=> $\log_e(x(1 + x)) = 0$
So, $x(1 + x) = 1$
or, $x^2 + x + 1 = 0$

Q5. If $\log_2 x \times \log_2 \frac{x}{16} + 4 = 0$ then find the value of x.
<u>Ans</u>:- $\log_2 x \times (\log_2 x - \log_2 16) + 4 = 0$
=> $(\log_2 x)^2 - 4 \log_2 2 \log_2 x + 4 = 0$
=> $(\log_2 x)^2 - 4 \log_2 2 + 4 = 0$
=> $(\log_2 x - 2)^2 = 0$
=> $\log_2 x = 2$
or, $x = 4$.

Q6. If $2 \log_8 N = p$, $\log_2 2N = q$ and $q - p = 4$ then find the value of N ?
<u>Ans</u>:- $p = 2 \log_8 N$
$= \frac{2}{3} \log_2 N$
$q = \log_2 2N$
$= \log_2 2 + \log_2 N$
$= 1 + \log_2 N$
$q - p = 1 + \log_2 N - \frac{2}{3} \log_2 N$
$= 1 + \frac{1}{3} \log_2 N = 4$ (Given)
So, $\log_2 N = 9$
0r $N = 2^9 = 512$

Q7. If a, b, c are consecutive positive integrals and $\log(1 + ac) = 2K$ then find value of K ?
<u>Ans</u>:- Let $a = n$
$b = n + 1$
$c = n + 2$
So, $\log(1 + ac) = \log(1 + n(n + 2))$
$= \log(1 + n^2 + 2n)$
$= \log(n + 1)^2$

$= 2 \log(n + 1)$
$= 2 \log b$
$\therefore$ Value of K is logb.

Q8. Find value of x if $\log_3 x.\log_y 3.\log_2 y = 5$?
<u>Ans</u>:- $\log_3 x.\log_y 3.\log_2 y = \log_3 x.\log_2 3$
$= \log_2 x$
$= 5$ (Given)
So, $x = 2^5 = 32$.

Q9. The value of $e^{\ln(\ln 7)}$ IS 7, True or false.
<u>Ans</u>:- $e^{\ln(\ln 7)} = \ln 7^{\ln e}$
$= \ln 7$
So, it is false.

Q10. The value of $\frac{1}{\log_2 \pi} + \frac{1}{\log_6 \pi}$ is greater then 2. true or false ?
<u>Ans</u>:- $\frac{1}{\log_2 \pi} + \frac{1}{\log_6 \pi} = \log_\pi 2 + \log_\pi 6$
$= \log_\pi 12$
Now, $12 > \pi^2$
So, $\log_\pi 12 > \log_\pi \pi^2$
or $\log_\pi 12 > 2$
So, It is true.

<u>Logarithms - flow Questions</u>
<u>Medium</u>
Q1. If a, b, c are distinct positive numbers each different from 1 such that
$[\log_b a \log_c a - \log_a a] + [\log_a b \log_c b - \log_b b] + [\log_a c \log_b c - \log_c c] = 0$
then find the value of abc ?
<u>Ans</u>:- Let us change all logarithms to base $\alpha (\alpha > 0, \alpha \neq 1)$
So, the eq^n now becomes
$\sum \left[\frac{x}{y} \cdot \frac{x}{2} - 1 \right] = 0$ where $x = \log_\alpha a$ etc.

So, $\frac{x^2}{yz} + \frac{y^2}{xz} + \frac{z^2}{xy} = 3$
or, $x^3 + y^3 + z^3 - 3xyz = 0$
or, $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 0$

Since $x \neq y \neq 3$ we have, $x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2] \neq 0$

** Tip: Writing $x^2 + y^2 + z^2 - xy - yz - zx$ as $\frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2]$ which is non-negative for real x, y, z is an useful magnipulation is solving many questions.
So, we calculate $x + y + z = 0$ that is
$\log_a a + \log_a b + \log_a c = 0$
or, $\log_a abc = 0$
or, $abc = 1$

<u>Dumb Question</u>:- Why $x^2 + y^2 + z^2 - xy - yz - zx$ is not equal to 0 ?

<u>Ans</u>:- $x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2]$
Now since it is given in question that x, y, z are distinct positive numbers, this cannot be equal to zero.

Q2. If $\log_a(ab) = x$, then find value of $\log_b(ab)$?
<u>Ans</u>:- $\log_a(ab) = \log_a a + \log_a b = 1 + \log_a b = x$
So, $\log_a b = x - 1$
or $\log_b a = \frac{1}{x - 1}$
or, $\log_b a + \log_b b = \frac{1}{x - 1} + 1$
$\Rightarrow \log_b(ab) = \frac{1 + x - 1}{x - 1}$
$= \frac{x}{x - 1}$

Q3. Find the least value of the expression
$2 \log_{10} x - \log_x(0.01)$ for $x > 1$

Ans:- $2\log_{10}x = \log_x(0.01)$

$= 2\log_{10} x - \log_x 10^{-2}$

$= 2(\log_{10}x + \log_x 10)$

$= 2(\log_{10}x + \frac{1}{\log_x 10})$

$= 2\left[\left(\frac{1}{\sqrt{\log_{10}x}} - \sqrt{\log_{10}x}\right)^2 + 2\right]$

$= 2\left(\frac{1}{\sqrt{\log_{10}x}} - \sqrt{\log_{10}x}\right)^2 + 4$

So, the minimum value is 4

Dumb Question:- How does $\log_{10}x + \frac{1}{\log_{10}x}$ was equated to $\left(\frac{1}{\sqrt{\log_{10}x}} - \sqrt{\log_{10}x}\right)^2 + 2$?

Ans:- $\log_{10}x + \frac{1}{\log_{10}x} = \left(\sqrt{\log_{10}x}\right)^2 + \left(\frac{1}{\sqrt{\log_{10}x}}\right)^2 - 2\left(\sqrt{\log_{10}x}\right)\left(\frac{1}{\sqrt{\log_{10}x}}\right) + 2$

$= \left(\frac{1}{\sqrt{\log_{10}x}} - \sqrt{\log_{10}x}\right)^2 + 2$

Since x is given to be > 1. So, $\log_{10}x$ is a positive no.

Q4. Solve $\log_{2x+3}(6x^2 + 23x + 21) = 4 - \log_{3x+7}(4x^2 + 12x + 9)$

Ans:- $\log_{2x+3}(6x^2 + 23x + 21) = 4 - \log_{3x+7}(4x^2 + 12x + 9)$

$\Rightarrow \log_{2x+3}[(2x + 3)(3x + 7)] = 4 - \log_{3x+7}(2x + 3)^2$

$\Rightarrow 1 + \log_{2x+3}(3x + 7) = 4 - 2\log_{3x+7}(2x + 3)$

$$\log_{2x+3}(3x + 7) = 3 - \frac{2}{\log_{2x+3}(3x + 7)}$$

Let $\log_{2x+3}(3x + 7) = y$

$$y = 3 - \frac{2}{y}$$

$y^2 - 3y + 2 = 0$

=> $(y - 1)(y - 2) = 0$

So, $y = 1$, $y = 2$

So, $\log_{2x+3}(3x + 7) = 1$

=> $3x + 7 = 2x + 3$

=> $x = - 4$

But, this means that $2x + 3 = - 5$ is a -ve value and base cannot be -ve. So, $y = 1$ is ruled out.

So, Now let us test $y = 2$

$\log_{2x+3}(3x + 7) = 2$

=> $(2x + 3)^2 = 3x + 7$

$4x^2 + 12x + 9 - 3x - 7 = 0$

$4x^2 + 9x + 2 = 0$

=> $x = - 2, - \frac{1}{4}$

Following similar argument as above $x = -2$ is rule out, So, the final solution is $x = - \frac{1}{4}$

<u>Dumb Question</u>:- Why $x = -2$ sol^n was ruled out ?

<u>Ans</u>:- If $x = -2$ then $2x + 3 = -1$.

Again, the base cannot be a -ve no, so $x = -2$ is ruled out.

**Tip :- Never obtain a sol^n and believe it is a solution, always try and cross check or other way could be that you write conditions which sol^n should satisfy while solving the question itself definitely - a cleaner aproach.

Q5. Find the value of x satisfying the equation.

$$|x - 1|^{\log_3 x^2 + 2\log_x 9} = (x - 1)^7 \text{ ?}$$

Ans:- L.H.S. is a positive no being an exponential.
So, R.H.S. should also be +ve.
R.H.S. will be +ve if $x > 1$ and in that case $|x - 1| = x - 1$.
Hence, we can write given eq^n as

$$(x - 1)^{\log_3 x^2 + 2\log_x 9} = (x - 1)^7$$

This gives $\log_3 x^2 - 2 \log_x 9 = 7$
Now if $\log_3 x = y$, then eq^n reduces to

$$2y - \frac{4}{y} = 7$$

or, $2y^2 - 7y - 4 = 0$

or, $y = 4, -\frac{1}{2}$

or, $\log_3 x = 4, -\frac{1}{2}$

$\Rightarrow x = 3^4, 3^{-\frac{1}{2}}$

Since $x > 1$ so, $x = 3^{-\frac{1}{2}}$ is rejected.
Hence $x = 81$

Dumb Question:- Why would R.H.S. be positive only if $x > 1$?
Ans:- R.H.S. is $(x - 1)^7$
Now suppose if $x < 1$ then we have (-ve no) odd number.
Now let us explain this with an example say $(-2)^7 = -128$ which is -ve.
So, clearly R.H.S. will be +ve only if $x > 1$.

HARD
Q1. If n is a natural no such that $n = P_1^{a_1} P_2^{a_2} \ldots\ldots\ldots\ldots P_K^{a_K}$ and $P_1, P_2, \ldots P_K$ are distinct primes then show that $\log n > K\log 2$.
Ans:- Since n is a natural no and $P_1, P_2 \ldots\ldots P_K$ are prime nos, so, $a_1, a_2 \ldots\ldots a_K$ have to be natural nos.
So, $a_i \ldots\ldots 1$ for ie. $1, 2, \ldots\ldots K$
Now $\log n = a_1 \log P_1 + a_2 \log P_2 + \ldots\ldots + a_K \log P_K$
$\geq \log P_1 + \log P_2 + \ldots\ldots + \log P_n$
Now, since $P_1, P_2 \ldots\ldots P_K$ are distinct primes.
So, $P_i \geq 2$ for i.e. $1, 2, \ldots\ldots K$.

So, $\log P_i > \log 2$
Hence $\log n \geq \log 2 + \log 2 + \ldots\ldots\ldots\ldots\ldots\ldots + \log 2 = K \log 2$
Hence $\log n \geq K \log 2$

<u>Dumb Questions</u>:- Why $a_1, a_2, \ldots\ldots\ldots\ldots\ldots\ldots a_K$ are natural no.
<u>Ans</u>:- Suppose $a_1, \ldots\ldots\ldots\ldots a_K$ - ve no.
than we will have a denominator will not be able to cancel with numerator because the prime nos do not cancel each other. Also if the a_i will be an irrational no so again n cannot be natural no.

Q2. Prove that

$$2^{\left(\sqrt{\log_a \sqrt[4]{ab} + \log_b \sqrt[4]{ab}} - \sqrt{\log_a \sqrt[4]{b/a} + \log_b \sqrt[4]{a/b}}\right)\cdot\sqrt{\log_a b}} = \begin{cases} 2, & b \geq a > 1 \\ 2^{\log_a b}, & 0 < b < a \end{cases}$$

<u>Ans</u>:- Since

$$\sqrt{\log_a \sqrt[4]{ab} + \log_b \sqrt[4]{ab}} = \sqrt{\frac{1}{4}\log_a ab + \frac{1}{4}\log_b (ab)}$$

$$= \sqrt{\frac{1}{4}(1 + \log_a b + \log_b a + 1)}$$

$$= \sqrt{\frac{\log_a b + \frac{1}{\log_a b} + 2}{4}}$$

$$= \sqrt{\left(\frac{\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}} + 2}{2}\right)^2}$$

$$= \frac{1}{2}\left(\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}}\right)$$

& $\sqrt{\log_a \sqrt[4]{b/a} + \log_b \sqrt[4]{a/b}} = \sqrt{\frac{1}{4}\log_a (a/b) + \frac{1}{4}\log_b (a/b)}$

$$= \sqrt{\frac{1}{4}(\log_a b - 1 + \log_b a - 1)}$$

$$= \sqrt{\frac{1}{4}\left(\log_a b + \frac{1}{\log_a b} - 2\right)}$$

$$= \frac{\left|\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}}\right|}{2}$$

So, $\sqrt{\log_a \sqrt[4]{ab} + \log_b \sqrt[4]{ab}} - \sqrt{\log_a \sqrt[4]{b/a} + \log_b \sqrt[4]{a/b}} = R$ (say)

So, $R = \frac{1}{2}\left\{\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}} + \left|\sqrt{|\log_a b|} - \frac{1}{\sqrt{|\log_a b|}}\right|\right\}$

Case I: If $b \geq a \geq 1$

then $P = \frac{1}{2}\left\{\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}} + \sqrt{|\log_a b|} - \frac{1}{\sqrt{|\log_a b|}}\right\}$

$$= \frac{1}{\sqrt{|\log_a b|}}$$

So, $2^{P \cdot \sqrt{\log_a b}} = 2^1 = 2$

Case II: If $1 < b < a$

then $P = \frac{1}{2}\left\{\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}} + \sqrt{|\log_a b|} - \frac{1}{\sqrt{|\log_a b|}}\right\}$

$$= \frac{1}{\sqrt{|\log_a b|}}$$

$2^{P \cdot \sqrt{\log_a b}} = 2^{\log_a b}$

<u>Keywords</u>

1. Log
2. Logarithm
3. Exponential

Trigonometry

Basic Trigonometry

The word trignometry is devided from 2 greek words
(1) Trigonon and

(2) Meetron

Trignon means triangle and metron means measure
So, is science of measuning angle .A very interesting branch, it is used in almost all other branches of mathematic whether it be coordinatc goemetry or it be calculus.

For a given angle θ in a right angle these ate six fossible ratios of 2 sides and hence there are 6 trignometric functions. These are extremely useful in simplefying many cliculations in mathematics. So lets stady these in more details.

<u>Trignometric functions</u>:
Let O be centre if circle of radius r . Let a may po form an angle θ at centre of circle. Drop a $\perp$ pm.
So, in Δ OPM we obserre.

$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$

ton = $\frac{y}{x}(x \neq 0)$; Cosesθ = $\frac{x}{y}(y \neq 0)$
Further more about these 6 trignometric ratios is covered in this table.

<u>Illustration 1.</u>

Find the sol^r of the equation $e^{\ln \cos x} = 2$?
$e^{\ln \cos x} = \cos x \; {}^{\ln e} = \cos x$
$\cos x = 2$
Now the range of cos x is - 1 to 1 .
So, the equation has no solution .

<u>Trignometric function of allied angles</u>:
If θ is any angle then - θ,
$90^\circ \pm \theta,\ 180^\circ \theta,\ 270^\circ \pm \theta,\ 360^\circ \pm \theta$ etc are called allied angles of θ

<u>Remork</u>: (1) If we have allied angle $(n\pi \pm \theta)$ where n is any integer then ratio remains the some and sign (+ve and -ve) is given according to the quadrant in which $n\pi \pm \theta$ lies
(assuming $\in \theta$ Ist Quadrant).

(2) If we have allied angle $\left(p\frac{\pi}{2} \pm \theta\right)$ where P is an odd integer then ratio changes i.e 'sine' changs to 'cosine' ; 'cosine' changes t 'sine'; tangent and 'cosecant' changes to 'secant'changes to 'cotangent', 'secant' changes t 'cosecant' and ' cosecant' changes to 'secant' The sign (+ve or -ve) is given according t quadrant in which $\left(p\frac{\pi}{2} \pm \theta\right)$ lies(assumtg $\in \theta$ IstQuadrant.

<u>Illustration 2</u>:
Find the value of tan (1020^0) ?
$1020^0 = 10 \text{ X } 90^0 + 30^0$
If lies in the IV th Quadrant.
So, tan ($10 \text{ X } 90^0 + 30^0$) = - tan 30^0
$= -\frac{1}{\sqrt{3}}$

<u>Trignometric Ratios of Compound angles</u>:

(1) sin (A + B) = sin A cos B + cos A sin B
<u>Why</u>?

Draw < A o B = < A and < boc = < B .
And find any point P on oc.

Draw PM and PN $\perp$ to OA and oB respectively.
Through N draw NR parrallel to AO t meet MP in R and other N $\perp$ OA
< R P N = 90^0 - < P N R
= < R N O = < NO θ = < A

So, sin (A + B) = sin A o P = $\frac{MP}{OP} = \frac{MR + PR}{OP}$

$= \frac{\theta N}{OP} + \frac{PR}{OP} = \frac{\theta N}{ON} \cdot \frac{ON}{OP} + \frac{RP}{NP} \cdot \frac{NP}{OP}$

= sin A cos B + cos A sin B .

(2) os (A + B) = cos A cos B - sin A sin B
<u>Why</u> ?
cos(A + B) = sin($\pi/2$ - (A + B))
= sin(($\pi/2$ - A) - (-B))
= sin ($\pi/2$ - A) cos (- B) + cos ($\pi/2$ - A) sin(- 6)
= cos A cos B - sin A sin B .

3) sin (A + B) = sin A cos B - cos A sin B

4) cos (A + B) = cos A cos B + sin A sin B

5) tan (A + B) = $\frac{\tan A + \tan B}{1 - \tan A \tan B}$

6) $\tan(A + B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

7) $\cot(A + B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

8) $\cot(A + B) = \dfrac{\cot A \cot B + 1}{\cot A - \cot B}$

<u>Illustration 3:</u>
If $A + B = 45^0$, show that $(1 + \tan A)(1 + \tan B) = 2$.

Ans:- $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$1 = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ ($\because$ $A + B = 45^o$.so $\tan(A + B) = 1$)
So, $\tan A + \tan B + \tan A \tan B = 1$
or, $\tan A + \tan B + \tan A + \tan B + 1 = 2$
or, $(\tan A + 1)(\tan B + 1) = 2$

<u>Some more farmulae</u>:

1) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$
<u>Why</u> ?
Add $\sin(A + B) = \sin A \cos B + \cos A \sin B$
& $\sin(A + B) = \sin A \cos B - \cos A \sin B$

2) $2\cos A \sin B = \sin(A + B) - \sin(A - B)$

3) $2\cos A \cos B = \cos(A + B) - \cos(A - B)$

4) $2\sin A \sin B = \cos(A + B) - \cos(A - B)$

5) $\sin C + \sin D = 2 \sin\left(\dfrac{C + D}{2}\right) \cos\left(\dfrac{C - D}{2}\right)$
<u>Why</u> ?
Put $A + B = C$
and $A - B = D$
in $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$.

6) sin C - sin D = 2 cos $\left(\frac{C+D}{2}\right)$ sin $\left(\frac{C-D}{2}\right)$

7) cis C + cos D = 2 cos $\left(\frac{C+D}{2}\right)$ cos $\left(\frac{C-D}{2}\right)$

8) cos C - cos D = 2 sin $\left(\frac{C+D}{2}\right)$ sin $\left(\frac{C-D}{2}\right)$

<u>Illustration 4</u>:

Proove that sin $\alpha + \sin\left(\alpha+\frac{2\pi}{3}\right) + \sin\left(\alpha+\frac{4\pi}{3}\right) = 0$

Ans:- sin $\alpha + \sin\left(\alpha+\frac{2\pi}{3}\right) + \sin\left(\alpha+\frac{4\pi}{3}\right)$

= sin $\alpha + \sin\left(\pi-(\pi/3-\alpha)\right) + \sin\left(\pi+(\pi/3+\alpha)\right)$

= $\alpha + \sin\left(\pi-(\pi/3-\alpha)\right) - \sin\left(\pi+(\pi/3+\alpha)\right)$ [using sin($\pi-\theta$)= sin θ , sin ($\pi+\theta$) = - sin θ]

α+2 sin $\left(\frac{\pi/3-\alpha\ \pi/3-\alpha}{2}\right)$ cos $\left(\frac{\pi/3-\alpha\ \pi/3+\alpha}{2}\right)$

= sin $\alpha + 2\sin(-\alpha)\cos(\pi/3)$

= sin $\alpha + 2\sin(-\alpha)\ \frac{1}{2}$

= sin - sin α - sin α

= 0

<u>Trignometri Ratios multiples of an angle</u> :

1) sin 2θ = 2sin θcosθ= $\frac{2\tan\theta}{1+\tan^2\theta}$

<u>why</u> ? sin (A +B) = sin A cos B + cos A sin B

Put A = B = θ

So, sin 2 θ= 2sin θcosθ

= $\frac{2\sin\theta\ \cos\theta}{\sin^2\theta\ \cos^2\theta}$

= $\frac{2\tan\theta}{1+\tan^2\theta}$

2) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$= \cos^2\theta - 1$

$= \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

3) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

4) $\sin 3\theta = 3\sin\theta - 4\sin^2\theta$

<u>Why</u> ?

$\sin 3\theta = \sin(\theta + 2\theta)$

$\sin(\theta + 2\theta) = \sin\theta\cos^2\theta + \cos\theta\sin^2\theta$

$= \sin\theta(1 - 2\sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta(1 - \sin^2\theta)$

$= 3\sin\theta - 4\sin^3\theta$

5) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

6) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

<u>Illustration 5</u>:

$\sin x + \sin y = a$ and $\cos x + \cos y = b$, show that

$\sin(x + y) = \dfrac{2ab}{a^2+b^2}$ and $\tan(x - y)$ $\sqrt{\dfrac{4 - a^2 - b^2}{a^2+b^2}}$

Ans:- $\sin x + \sin y = a$

$\Rightarrow 2\sin\left(\dfrac{x+4}{2}\right)\cos\left(\dfrac{x-4}{2}\right) = a$ .....(1)

$\cos x + \cos y = b$

$\Rightarrow 2\cos\left(\dfrac{x+4}{2}\right)\cos\left(\dfrac{x-4}{2}\right) = b$ .....(2)

$(1)/(2) \Rightarrow \tan\left(\dfrac{x-4}{2}\right) = \dfrac{a}{b}$

$$\sin(x+y) = \frac{2\tan\left(\frac{x+4}{2}\right)}{1+\tan^2\left(\frac{x+4}{2}\right)}$$

$$= \frac{2\,{}^{a}/_{b}}{1+{}^{a^2}/_{b^2}} = \frac{2ab}{a^2+b^2}$$

Squaring (1), (2) and adding .

$$4\cos^2\left(\frac{x-4}{2}\right)\left\{\sin^2\left(\frac{x+4}{2}\right) + \cos^2\left(\frac{x+4}{2}\right)\right\} = a^2+b^2$$

$$\text{or } \cos^2\left(\frac{x-4}{2}\right) = \frac{a^2+b^2}{4}$$

$$\sin^2\left(\frac{x-4}{2}\right) = 1 - \left(\frac{a^2+b^2}{4}\right)$$

$$= \frac{4-a^2-b^2}{4}$$

$$\tan^2\left(x-{}^{4}/_{2}\right) = \frac{4-a^2-b^2}{4} \quad \frac{4}{a^2+b^2}$$

$$\tan\left(\frac{x-4}{2}\right) = \sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

<u>Illustration 6</u>:-

Find the value of $\sin 18^0$?

Ans:- Let $\theta\, 18^0$

So, $5\theta = 90^0$

$\Rightarrow 2\theta = 90^0 - 3\theta$

Or, $\sin 2\theta = \sin(90^0 - 3\theta)$

$\Rightarrow \sin 2\theta = \cos 3\theta$

$\Rightarrow 2\sin\theta\cos\theta = \cos\theta(4\cos^2\theta - 3)$

$\Rightarrow 2\sin\theta = 4\cos^2\theta - 3$ ($\because \cos\theta \neq 0$)

$= 1 - 4\sin^2\theta$

$\Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$

$$\text{So, } \sin\theta = \frac{-2 \pm \sqrt{4+16}}{24}$$

$$= \frac{-1 \pm \sqrt{5}}{4}$$

But since sin $\theta > 0$ we have sin $\theta = \frac{\sqrt{5}-1}{4}$

So, sin 18 ^{o} = $\frac{\sqrt{5}-1}{4}$

<u>Dumb Question</u>:- Why cos θ is not equal to 0 ?
Ans: cos 90^0 si O and cos O^o is 1
So, cos being a continous function.
cos θ i.e. cos 18^0 would have some value between o and 1.
<u>Some nice manipulations</u>:

1) $(\cos\theta \pm \sin\theta)^2 = 1 \pm \sin 2\theta$

Why ? $(\cos\theta \pm \sin\theta)^2 = \cos^2\theta \sin^2 \pm 2\sin\theta \cos\theta$
$= 1 \pm 2\sin\theta\cos\theta$
$= 1 \pm \sin^2\theta$

2) $\frac{1-\cos\theta}{\sin\theta} = \tan(\theta/2)$

Why ?

$$\frac{1-\cos\theta}{\sin\theta} = \frac{1 - (1 - \sin^2\theta)}{\sin\theta/2 \ \cos\theta/2}$$

$$= \frac{\sin^2\theta/2}{\sin\theta/2 \ \cos\theta/2} = \tan\theta/2$$

3) cos Asin A = $\sqrt{2} \sin(\pi/4 \pm A)$

Why !

cos A $\pm$ sin A = $\sqrt{2}\,(\sin\pi/4 \cos A \pm \cos\pi/4 \ \sin A)$
$= \sqrt{2}\,(\sin \pi/4 \ \cos A \pm \cos\pi/4 \ \sin A)$
$= \sqrt{2} \sin(\pi/4 \pm A)$

<u>Expressing sin</u> $\sin\pi/2 + \cos\pi/2$ <u>in terms of sin A</u>:

$\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = \pm\sqrt{1 + \sin A}$

$\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = \sin\frac{\pi}{2} + \cos\frac{\pi}{2}$

The ambiguity of sign is removed from following figure.

<u>Dumb Question</u>: Where does this diagram come from ?

Ans:- $\sin\frac{A}{2} + \cos\frac{A}{2} = \sqrt{2}\sin\left(\frac{A}{4} + \frac{\pi}{4}\right)$

So, if $\frac{A}{2} + \frac{A}{4}$ is in I st or IInd quadrent

then $\sin\frac{A}{2} + \cos\frac{A}{2}$ is + ve.

So, A/2 lie from 2_n $2_n\pi - \frac{\pi}{4} \neq 2_n\pi + \frac{3\pi}{4}$ for $\sin\frac{A}{2} + \cos\frac{A}{2}$ to be + ve

Similarily for $\sin\frac{A}{2} - \cos\frac{A}{2}$

<u>Illustration 7</u>:

If cos 25^0 + sin 25 ^{0} = P , then find value of cos 50^0 in trems of P ?

Ans:- $\cos 50^0 = \cos^2 25^0 - \sin^2 25^0$

$= (\cos 25^0 + \sin 25^0)(\cos 25^0 - \sin 25^0)$

$P(\cos 25^0 - \sin 25^0)$

Also, $(\cos 25^0 - \sin 25^0)^2 + (\cos 25^0 - \sin 25^0)^2 = 1 + 1$

$\cos 25^0 - \sin 25^0 = +\sqrt{2 - P^2}$

(+ve sign as cos 25^0 > sin 25^0)

$\cos 50^0 = P\sqrt{2 - P^2}$

<u>The qreatest and least valume of expression</u> (a sin θ + b cos θ)

$-\sqrt{a^2 + b^2} \leq a\sin\theta + b\cos\theta \leq \sqrt{a^2 + b^2}$

Why?

Let a = r cos α

b = r sin α so that r = $\sqrt{a^2 + b^2}$

So, a sin θ + b cos θ = r (sin θ cos α + cos θ sin α)

= r sin (θ + α)

Now sin (θ + α has minimum and maximum value as + 1 and - 1 esppectively.

So, - r rsin (θ + α) $\leq$ r

So, - $\sqrt{a^2 + b^2}$ $\leq$ a sin θ + b cos θ $\leq \sqrt{a^2 + b^2}$

Illustration 8:

Find the minimum and maximum value of

6 sin x cos x + 4cos^2 2x ?

Ans:- 6 sin x cos x + 4cos^2 2x = (2sin x cos x) + 4cos 2x

= 3 sin 2x + 4cos 2x .

- $\sqrt{3^2 + 4^2}$ 3sin 2x + 4cos2x $\leq \sqrt{3^2 + 4^2}$

=> Minimum value of 6sin x cos x + 4 cos 2x is - 5

and maximum value is 5.

Sum of sine and cosine senies when angles are in AP .

(1) sin α + sin $(\alpha+\beta)$ + $\sin(\alpha+2\beta)$.....+ $\sin(\alpha+(n-1)\beta)$ =

$$\frac{\sin\ (n.\beta/2)}{\sin\ (\beta/2)}\ \sin\left(\frac{2\alpha+(n-1)\beta}{2}\right)$$

Why ? $(\alpha+\beta)$ + $\sin(\alpha+2\beta)$+.....+$\sin(\alpha+(n-1)\beta)$

2sin α $\sin \beta/2$ = $\cos(\alpha-\beta/2)$ - $\cos(\alpha+\beta/2)$

2sin $(\alpha+\beta)$ $\sin\beta/2$ = $\cos(\alpha+2\beta/2)$ − $\cos(\alpha+3\beta/2)$

2sin

$(\alpha+(n-1)\beta)$ $\sin\beta/2$ = $\cos(\alpha+(n-3/2)\beta)$ − $\cos(\alpha+(n-1/2)\beta)$

By adding these n lines we have.

2sin $\beta/2$ s = $\cos(\alpha-\beta/2)$ − $\cos(\alpha+(n-1/2)\beta)$

= 2sin $\left(\frac{2\alpha+(n-1)\beta}{2}\right)$ $\sin(n\beta/2)$

$$S = \frac{\sin\left(\frac{2\alpha+(n-1)\beta}{2}\right)\ \sin(n\,\beta/2)}{\sin(\beta/2)}$$

(2) $\cos\alpha + \cos(\alpha+\beta) + \ldots\ldots\ldots + \cos(\alpha+(n-1)\beta)$

$$= \frac{\sin(n\,\beta/2)}{\sin\beta/2}\ \cos\left(\frac{2\alpha+(n-1)\beta}{2}\right)$$

<u>Illustration 9</u>:

Find sum of sin $(\alpha) - \sin(\alpha+\beta) + \sin(\alpha+2\beta)$-.....+ 0 n terms. Ans:- Now, sin $(\alpha+\beta+\pi) = -\sin(\alpha+\beta)$

sin$(\alpha + 2\beta + 2\pi) = -\sin(\alpha + 2\beta)$

sin$(\alpha + 3\beta + 3\pi) = -\sin(\alpha + 3\beta)$

.....

Hence the series is

sin $\alpha + \sin(\alpha+(\beta+\pi)) + \sin(\alpha + 2(\beta+\pi))$ .....

$$= \frac{\sin\left(\alpha+\frac{(n-1)}{2}(\beta+\pi)\right)\ \sin\left(n\left(\beta+\pi/2\right)\right)}{\sin(\beta+\pi/2)}$$

$$= \frac{\sin\left(\alpha+\frac{(n-1)}{2}(\beta+\pi)\right)\ \sin\left(\frac{n(\beta+\pi)}{2}\right)}{\cos\beta/2}$$

<u>Conditional Identities</u>.

If A + B + C = 180° then .

(i) sin 2A + sin 2B + sin 2c = 4sinA sin B sinC.

<u>Why</u> ?

A + B + C = 180°

So, A + B = 180° - C

sin(A + B) = sin C

cos (A + B) = - cos C

sin 2A + sin 2B + sin 2C = 2 sin (A + B) cos (A - B) + 2sin C cos C.

= 2sin C cos (A - B) + 2sin (- cos (A + B))

= 2sin C (cos (A - B) - cos (A + B))

= 2sin C (2sin A sin B)

= 4sin A sin B sin C .

(ii) cos 2A + cos 2B + cos 2C = - 1 - 4cos A cosB cosC

(iii) sinA + sinB + sinC = 4 cos $(A/2)\cos(B/2)\cos(C/2)$

(iv) cosA + cosB + cosC = 1 + 1 + 4sin $(A/2)\sin(B/2)\sin(C/2)$

(v) tanA + tanB + tanC =tanA tanB tanC

<u>Why</u> ?

$$\tan(A + B + C) = \frac{\tan(A + B) + \tan C}{1 - \tan C \tan(A+B)}$$

$$= \frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \tan C\left(\dfrac{\tan A + \tan B}{1 - \tan A \tan B}\right)}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan A \tan C}$$

So, since tan (A + B + C) = tan (180^0) = 0
So, tanA + tanB + tanC - tanA + tanB + tanC = 0
=> tanA + tanB + tanC = tanA + tanB + tanC.

(vi) cotA + cotB + cotA + cotC + cotB cotC = 1

(vii) $\tan A/2 \tan B/2 + \tan B/2 \tan C/2 + \tan C/2 \tan A/2$

(viii) $\cot A/2 + \cot B/2 + \cot C/2 = \cot(A/2)\cot(B/2)\cot(C/2)$

<u>Illustration 10</u>:
If A + B + C = 180 ^{o} then prove that
$\sin^2 A + \sin^2 C = 2 + 2 \cos A \cos B \cos C$.
Ans:- $\sin^2 A + \sin^2 B + \sin^2 C = 1/2\,(1 - \cos^2 A + 1 - \cos^2 B + 1 - \cos^2 C)$

$= \frac{1}{2}(3 - (\cos 2A + \cos 2B + \cos 2C))$

$= \frac{1}{2}[3 - (- 1 - 4\cos A \cos B \cos C)]$

$= \frac{1}{2}$ [4 + 4cosA cosB cosC]
$= \frac{1}{2}$ cosA cosB cosC
= 2 + 2 cosA cosB cosC

1) Question - If $\sin\theta + \operatorname{cosec}\theta = 2$ then what is the value of $\sin^2\theta + \operatorname{coses}^2\theta$.
Solution:- $\sin^2\theta + \operatorname{cosesc}^2\theta = (\sin\theta + \operatorname{cosec}\theta)^2 - 2\sin\theta \operatorname{cosec}\theta$
$= (1 + 1)^2 - 2.1 = 2$

2) Prove that $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0$
Solution:- L.H.S. $2 [(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\cos^2\theta(\sin^2\theta + \cos^2)] -[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\cos^2\theta] + 1$
$= 2 [1 - 3 \sin^2\theta\cos^2\theta] - 3 [1 - 2\sin^2\theta\cos^2\theta] + 1$
= 0

3) Prove that $\frac{3}{4} \leq \sin^2\theta + \cos^4\theta. \leq 1$ for all real θ

<u>Solution</u> $\sin^2\theta + \cos^4\theta - \cos^2\theta + 1$

$$= \left\{(\cos^2\theta)^2 - 2\,\frac{1}{2}\cos^2\theta + \frac{1}{4}\right\} + \frac{3}{4}$$

$$= \left(\cos^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} \qquad \left[\text{because } \left(\cos^2 - \frac{1}{2}\right)^2 \geq 0\right]$$

Also, $\sin^2\theta + \cos^2\theta = \cos^4\theta - \cos^2 + 1$

$= \cos^2\theta(\cos^2\theta - 1) + 1$
$= 1 - \sin^2\theta \cos^2\theta \leq 1$

[becaise $\sin^2\theta \cos^2\theta \geq 0$]

4) Evaluate $\sin 78^\circ - \sin 66^\circ - \sin 42^\circ + \sin 6^\circ$
$= (\sin 78^\circ - \sin 42^\circ) - (\sin 66^\circ - \sin 6^\circ)$

$= 2\cos 60^o \sin 18^o - 2\cos 36^o . \sin 30^o$

$= \sin 18^o - \cos 36^o$

$= \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right)$

$= -\frac{1}{2}$

(5) If a $\Delta ABC \angle C = 90^\circ$ then find the maximum value of sinA sinB .

Solution:- sinA sinB = $\frac{1}{2} \times 2$ sinA sinB

$= \frac{1}{2}$ [cos (A - B) - cos (A + B)]

$= \frac{1}{2}$ [cos (A - b) - cos 90°]

$= \frac{1}{2}$ cos (A - B) $\leq \frac{1}{2}$

So maximum valuie of sinA sinB = $\frac{1}{2}$

(6) Prove that $\sqrt{3}$ cosec 20° - sec 20° = 4

<u>Soluction</u>- L.H.S. = $\frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ} = \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ . \cos 20^\circ}$

$= \frac{4\left(\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{2\sin 20^\circ . \cos 20^\circ}$

$= 4. \frac{(\sin 60^\circ . \cos 20^\circ - \cos 60^\circ . \sin 20^\circ)}{\sin 40^\circ}$

$= 4. \frac{\sin (60^\circ - 20^\circ)}{\sin 40^\circ}$

$= 4.\dfrac{\sin 40^\circ}{\sin 40^\circ}$

= 4 = R.H.S.

(7) Find the value of, $\dfrac{\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}}{\sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}}$

When | tanA | < | and | A | is acure,

Solution:- $\dfrac{\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}}{\sqrt{1+\sin 2A} - \sqrt{1-\sin 2A}}$

=> $\dfrac{\sqrt{(\cos A + \sin A)^2} - \sqrt{(\cos A - \sin A)^2}}{\sqrt{(\cos A + \sin A)^2} - \sqrt{(\cos A - \sin A)^2}}$

=> $\dfrac{|\cos A + \sin A| + |\cos A - \sin A|}{|\cos A + \sin A| - |\cos A - \sin A|}$

=> $\dfrac{\cos A + \sin A + \cos A - \sin A}{\cos A + \sin A - \cos A - \sin A}$ $\because -\dfrac{\pi}{4} < A < \dfrac{\pi}{4}$ and in this interval casA > sinA }

=> $\dfrac{2\cos A}{2\sin A}$ => cotA

When |tanA| < 1 and |A| is acute.

Q. Find a and b such that for all x, a $\leq 3\cos x + 5\sin\left(x - \dfrac{\pi}{6}\right) \leq 6$

<u>Soluction</u>:- 3 cosx + 5 sin(x - $\dfrac{\pi}{6}$) = 3 cosx + 5 sinx cos $\dfrac{\pi}{6}$ - 5 cosx sin $\dfrac{\pi}{6}$

= (3 - 5/2)cosx + 5 $\dfrac{\sqrt{3}}{2}$ sinx

$$= \frac{1}{2}\cos x + \frac{5\sqrt{3}}{2}\sin x$$

$$a = -\sqrt{\frac{1}{4} + \frac{75}{4}} = -\sqrt{19}$$

and $b = \sqrt{19}$

(9) Find the maximum and minimum value of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$

Soluction:- $\cos^2\theta - 6\sin\theta.\cos\theta + 3\sin^2\theta + 2$

$= (1 - \sin^2\theta) - 3\sin 2\theta + 3\sin^2\theta + 2$

$= 2\sin^2\theta - 3\sin 2\theta + 3$

$= (1 - \cos 2\theta) - 3\sin 2\theta + 3$

$= 4 - (\cos 2\theta + 3\sin 2\theta)$ ..... (i)

as we have - $\sqrt{10} \le (\cos 2\theta + 3\sin 2\theta) \le \sqrt{10}$

$-\sqrt{10} \le -(\cos 2\theta + 3\sin 2\theta) \le \sqrt{10}$

or $4 - \sqrt{10} \le 4 - (\cos 2\theta + 3\sin 2\theta) \le 4 + \sqrt{10}$

(10) If a triangle ABC show that -

$$\frac{\sin A + \sin B + \sin C}{\sin A + \sin B - \sin C} = \cot\frac{A}{2}.\cot\frac{B}{2}$$

Solution:- Dr = sinA + sinB - sinC

$$= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right] \quad \because \sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$= 2\cos\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]$$

$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$

Similarly $N_r = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

L.H.S. = $\frac{N_r}{D_r} = \frac{4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}}{4 \sin\frac{A}{2} \sin\frac{B}{2} \cos\frac{C}{2}}$

$= \cot\frac{A}{2} \cot\frac{B}{2}$

(11) Prove that tanA + 2 tan2A + 4 tan 4A + 8 cot 8A = cot A

L.H.S. = tanA + 2 tan2A + 4 tan4A + $\frac{8\left(1 - \tan^2 4A\right)}{2 \tan 4A}$

= tanA + 2 tan2A + 4 cot4A

= tanA + 2 tan2A + 4 $\frac{\left(1 - \tan^2 4A\right)}{2 \tan 2A}$

= tanA + 2 cot2A

= tanA + $\frac{1 - \tan^2 4A}{\tan A}$

= cotA = R.H.S.

(12) If 2 cosA = x + 1/x , 2 cosB = y + 1/y show that 2 cos(A - B) = x/y + y/x

<u>Soluction</u>:- 2 cosA = x + 1/x since $4 \sin^2 A = 4 - 4 \cos^2 A$

$= 4 - (x + 1/x)^2$

$4 \sin^2 A = - [(x + 1/x)^2 - 4]$

$4 \sin^2 A = - i^2[(x - 1/x)^2]$

2 sinA = i(x - 1/x)

Similarily 2 cosB = y + 1/y

2 sinB = i(y - 1/y)

Now, 2 cos(A - B) = 2[cosA cosB + sinA sinB]

$$= \frac{2}{4}\left[\left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right) + i^2\left(x + \frac{1}{x}\right)\left(y - \frac{1}{y}\right)\right]$$

$$= \frac{1}{2}\left[\left\{xy + \frac{1}{xy} + \frac{y}{x} + \frac{x}{y}\right\} - \left\{xy + \frac{1}{xy} - \frac{y}{x} - \frac{x}{y}\right\}\right]$$

$$= \frac{1}{2}\left[2.\frac{y}{x} + 2.\frac{x}{y}\right] = \frac{x}{y} + \frac{y}{x}$$ = R.H.S.

Medium

Q-1 Determine the smallest positive value of x(in degrees) for which tan $(x + 100^0)$ = tan $(x + 50^0)$ tan x. tan $(x - 50^0)$

Soluction:- Rearranging expression-

$$\Rightarrow \frac{\tan(x + 100^0)}{\tan(x - 50^0)} = \tan(x + 50^0)\tan x$$

$$\Rightarrow \frac{\sin(x + 100^0).\cos(x - 50^0)}{\sin(2x + 50^0) - \sin(150^0)} = \frac{\sin(x + 50^0)\sin x}{\sin(x + 50^0)\cos x}$$

$$\Rightarrow \frac{\sin(2x + 50^0) + \sin(150^0)}{\sin(2x + 50^0) - \sin(150^0)} = \frac{\cos(x + 50^0) - \cos(2x + 50^0)}{\cos(50^0) + \cos(2x + 50^0)}$$

Applying componedo and dividendo, we get-

$$\frac{\sin(2x + 50^0)}{\sin 150^0} = \frac{-\cos 50^0}{\cos(2x + 50^0)}$$

=> $\cos 50^o + 2\sin(2x + 50^o) . \cos(2x + 50^o) = 0$

=> $\cos 50^o + \sin(4x + 100^o) = 0$

=> $\cos 50^o + \cos(4x + 10^o) = 0$

=> $\cos(2x + 30^o)\cos(2x - 20^o) = 0$

=> $x = 30^o , 55^o$

The smallest value of $x = 30^o$

Dump Question:- Why we rearranged tha $(x + 100^o)$ = tan $(x + 50^o)$tan x tan$(x -50^o)$

as $\frac{\tan(x + 100^0)}{\tan(x - 50^0)}$ = tan$(x + 50^o)$ tanx.

Ans:- Well the motiration behind doing this was the fact that the sum of $x + 100^o$ and $x - 50^o$ is $2x + 50^o$ and also the sum of $x + 50^o$ and x is $2x + 50^o$

And, what more the variable x is removed when we substract the one angle from another.

The usefulness of this obserration is clearly when we use compendo and dividerdo.

Question:- If $\sqrt{\frac{(a-b)}{(a+b)}}\cot\theta$, prove that $(a - b\cos 2\theta)(a - b\cos 2\phi)$ is independent of θ and ϕ.

Soluction:- Let us put $\tan\theta = t_1$ and $\tan\phi = t_2$

$\therefore t_1^2.t_2^2 = \frac{a-b}{a+b}$ ..... (i)

Also, $\cot 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-t_1^2}{1+t_1^2}$

$\cos 2\phi = \frac{1-\tan^2\phi}{1+\tan^2\phi} = \frac{1-t_1^2}{1+t_1^2}$

Now $a - b\cos 2\theta = a - b\left(\frac{1-t_1^2}{1+t_1^2}\right)$

$= \frac{(a-b)+(a+b)t_1^2}{1+t_1^2} = \frac{a+b}{(1+t_1^2)}\left[\frac{a-b}{a+b}+t_1^2\right]$

$= \frac{a+b}{(1+t_1^2)}\left[t_1^2t_2^2+t_1^2\right]$ [by (i)]

$= \frac{a+b}{(1+t_1^2)}(t_2^2+1)t_1^2$

Similarily $(a - b\cos 2\phi) = \frac{a+b}{(1+t_2^2)}(t_1^2+1)t_2^2$

Hence $a - b\cos 2\theta)(a - b\cos 2\phi) = (a+b)^2t_1^2t_2^2 = a^2 - b^2$

Which is independent of θ and ϕ.

Q. Show that $\tan\frac{\pi}{16} = \sqrt{4+2\sqrt{2}} - (\sqrt{2}+1)$

Soluction:- We know $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta}$ where $\theta = \frac{\pi}{16}$

Also $\sin 2\theta = \sqrt{\frac{1-\cos 4\theta}{2}} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}$

$$= \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}$$

and $\cos 2\theta = \sqrt{\frac{1 + \cos 4\theta}{2}} = \sqrt{\frac{1 - \cos\frac{\pi}{4}}{2}}$

$$= \sqrt{\frac{1 + \frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}}$$

$$\tan\frac{\pi}{6} = \frac{1 - \sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}}}{\sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}} = \frac{\sqrt{2\sqrt{2}} - \sqrt{\sqrt{2} + 1}}{\sqrt{\sqrt{2} - 1}}$$

$$= \frac{\sqrt{\sqrt{2} + 1}\left(\sqrt{2\sqrt{2}} - \sqrt{\sqrt{2} + 1}\right)}{\sqrt{\left(\sqrt{2} + 1\right)\left(\sqrt{2} - 1\right)}}$$

$$= \frac{\sqrt{\sqrt{2}\left(\sqrt{2} + 1\right)} - \sqrt{\left(\sqrt{2} + 1\right)^2}}{\sqrt{\left(\sqrt{2} + 1\right)\left(\sqrt{2} - 1\right)}}$$

$$= \frac{\sqrt{4 + 2\sqrt{2}} - \left(\sqrt{2} + 1\right)}{\sqrt{2} - 1} = \sqrt{4 + 2\sqrt{2}} - \left(\sqrt{2} + 1\right)$$

<u>Dumb Question</u>:- How should I approach when I look such a question for first time ?

Ans:- Look, here $\frac{\pi}{16}$ was the anle and so we observed that if some how $\pi \tan\left(\pi/16\right)$ could be expressed in tems of $\pi/4$ then our work will be done and so we procceded.

Q. In any triangle ABC prove that
$\sin^3 A \cos(B - C) + \sin^3 B \cos(C - A) + \sin^3 C \cos(A - B) = 3 \sin A \sin B \sin C$
Solution:- L.H.S. = $\sin^2 A \sin(B + C) \cos(B - C) + \sin^2 B \sin(C + A)$

$\cos(C - A) + \sin^2 C \sin(A + B) \cos(A - B)$

$= \frac{1}{2} \sin^2 A (\sin 2B + \sin 2C) + \frac{1}{2} \sin^2 B (\sin 2C + \sin 2A) + \frac{1}{2} \sin^2 C (\sin 2A + \sin 2B)$

$= \sin^2 A (\sin B \cos B + \sin C \cos C) + \sin^2 B (\sin C \cos C + \sin A \cos A) + \sin^2 C (\sin A \cos A + \sin B \cos B)$

$= \sin A \sin B (\sin A \cos B + \cos A \sin B) + \sin B \sin C (\sin B \cos C + \cos B \sin C) + \sin C \sin A (\sin A \cos C + \cos A \sin C)$

$= \sin A \sin B \sin(A + B) + \sin B \sin C \sin(B + C) + \sin C \sin A \sin(A + C)$

$= 3 \sin A \sin B \sin C =$ R.H.S.

<u>Dumb Question</u>:

How did $\sin A \sin B \sin(A + B) + \sin B \sin C \sin(B + C) + \sin C \sin A \sin(A + C)$ became equal to $3 \sin A \sin B \sin C$.

Ans:- Since $A + B + C = 180^0$ (Givien that A, B, C are angles of Δ)

So, $A + B = 180^0 - C$

and thus $\sin(A + B) = \sin(180^0 - C)$

$= \sin C$

So, $\sin A \sin B \sin(A + B) = \sin A \sin B \sin C$

Similarily $\sin B \sin C \sin(B + C) = \sin A \sin B \sin C$

and $\sin A \sin C \sin(A + c) = \sin A \sin B \sin C$

Q. Show that for all real θ, the expression, $a \sin^2\theta + b \sin\theta \cos\theta + c \cos^2\theta\theta$ lies between.

$\frac{1}{2}(a + c) - \frac{1}{2}\sqrt{b^2 + (a - c)^2}$ and $\frac{1}{2}(a + c) + \frac{1}{2}\sqrt{b^2 + (a - c)^2}$

Soluton:- Expression $= \frac{a}{2}(1 - \cos 2\theta) + \frac{b}{2} \sin 2\theta + \frac{c}{2}(1 + \cos 2\theta)$

$= \frac{a + c}{2} + (\frac{b}{2} \sin 2\theta - \frac{a - c}{2} \cos 2\theta)$

$$= \frac{a+c}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a-c}{2}\right)^2} \times \left\{ \frac{b/2}{\left(\frac{b}{2}\right)^2 + \left(\frac{a-c}{2}\right)^2} \sin 2\theta - \frac{\frac{a-c}{2}}{\left(\frac{b}{2}\right)^2 + \left(\frac{a-c}{2}\right)^2} \cos \right.$$

$$= \frac{a+c}{2} + \frac{1}{2}\sqrt{b^2 + (a-c)^2} \times \{\cos\alpha \sin 2\theta - \sin\alpha \cos 2\theta\}$$

$$\text{where } \cos\alpha = \frac{\frac{b}{2}}{\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a-c}{2}\right)^2}}$$

$$= \frac{a+c}{2} \; \frac{1}{2}\sqrt{b^2 + (a-c)^2} \sin(2\theta - \alpha)$$

But $-1 \leq \sin(2\theta - \alpha) \leq 1$

maximum value of expression-

$$= \frac{a+c}{2} + \frac{1}{2}\sqrt{b^2 + (a-c)^2}$$

and minimum value of the expression-

$$= \frac{a+c}{2} - \frac{1}{2}\sqrt{b^2 + (a-c)^2}$$

Q-6. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$ then show that $m^2 - n^2 = 4\sqrt{mn}$

Solution:- $(m + n) = 2\tan\theta$, $m - n = 2\sin\theta$

$m^2 - n^2 = 4\tan\theta . \sin\theta$

$4\sqrt{mn} = 4\sqrt{\tan^2\theta - \sin^2\theta}$

$= 4\sin\theta\sqrt{\sec^2\theta - 1}$

$= 4\sin\theta\tan\theta$

<u>Dumb Question</u>:-

Why $\sqrt{\sec^2\theta - 1}$ was equated to $\tan\theta$ and not $-\tan\theta$?

Ans:- Yes actually it could have been both, nothin is mentioned here in the question in the question so this ambiquity has arised.

Q-7. Evaluate $\sum_{r-1}^{n-1} \cos\left(\frac{r\pi}{n}\right)$

Solution:- Sum = $\frac{1}{2}\sum_{r=1}^{n-1}\left(1+\cos\frac{2r\pi}{n}\right)$

$$= \frac{1}{2}(n-1) + \frac{1}{2}\left\{\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots\ldots\ldots\ldots\ldots\ldots + \cos\frac{(2n-2)\pi}{n}\right\}$$

$$= \frac{1}{2}(n-1) + \frac{1}{2}\left\{\frac{\sin(n-1).\frac{2\pi}{2n}}{\sin\frac{2\pi}{n.2}} . \cos\left\{\frac{2\left(\frac{2\pi}{n}\right) + (n-2)\frac{2\pi}{n}}{2}\right\}\right\}$$

Using-

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots\ldots\ldots\ldots\ldots + \cos(\alpha + (n-1)\beta)$$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} . \cos\left\{\frac{2\alpha + (n-1)\beta}{2}\right\}$$

$$= \frac{1}{2}(n-1) + \frac{1}{2}\left\{\frac{\sin\frac{(n-1)\pi}{2} . \cos\pi}{\sin\left(\pi/n\right)}\right\}$$

$$= \frac{1}{2}(n-1) + \frac{1}{2}\left\{\frac{\left(\sin\frac{\pi}{2}\right)(-1)}{\sin\left(\pi/n\right)}\right\}$$

$$= \frac{1}{2}(n-1) - \frac{1}{2}$$

$$= \frac{n}{2} - 1$$

So, $\sum_{r=1}^{n-1} \cos^2\left(\frac{r\pi}{n}\right) = \frac{n}{2} - 1$

<u>Hard</u>

Q-1. If $m^2 + m^{12} + 2mm^1 \cos\theta = 1$

$n^2 + n^{12} + 2nn^1 \cos\theta = 1$ and $mn + m^1 n^1 + (mn^1 + m^1 n)$ $\cos\theta = 1$ then prove that $m^2 + n^2 = \text{cosec}^2\theta$

Solution:- $m^2 + m^{12} + 2mm^1 \cos\theta$

or $(m^2\cos^2 \cos^2\theta + m^2 \sin^2\theta) + m^{12} + 2mm^1 \cos\theta = 1$

or $m^2 \cos^2\theta + 2mm^1 \cos\theta + m^{12} = 1 - m^2 \sin^2\theta$

or $(m \cos\theta + m^1)^2 = 1 - m^2 \sin^2\theta$ .....(i)

Similaily, $n^2 + n^{12} + 2nn^1 \cos\theta = 1$

=> $n^2(n \cos\theta + n^1)^2 = 1 - n^2 \sin^2\theta$

.....(ii)

Finally, $mn + m^1n^1 + (mn^1 + m^1n \cos\theta = 0$

=> $(mn \cos^2\theta + mn \sin^2\theta) + m^1n^1 \cos\theta + m^1 n \cos\theta = 0$

=> $mn \cos^2\theta + m^1 n \cos\theta + m^1 n^1 + mn^1 \cos\theta = -mn \sin^2\theta$

=> $n \cos\theta(m \cos\theta + m^1) + n^1 (m^1 + m \cos\theta) = - m \sin^2\theta$

=> $(m \cos\theta + m^1)(n \cos\theta + n^1) = - mn \sin^2\theta$

=> $(m \cos\theta + m^1).(n \cos\theta + n^1)^2 = m^2n^2 \sin^4\theta$

=> $(1 - m^2 \sin^2\theta)(1 - n^2 \sin^2\theta = m^2n^2 \sin^4\theta$ (Using (i) and (ii))

=> $1 - (m^2 + n^2) \sin^2\theta + m^2n^2 \sin^4\theta = m^2n^2 \sin^4\theta$

=> $(m^2 + n^2) \sin^2\theta = 1$

=> $m^2 + n^2 = \text{cosec}^2\theta$

Q-2. Show that $\sin\frac{\pi}{7}.\sin\frac{9\pi}{7}.\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}$

Solution: Let $\theta = \frac{\pi}{7}$ or $\frac{3\pi}{7}$ or $\frac{5\pi}{7}$ (Note $\sin\frac{2\pi}{7} = \sin\frac{5\pi}{7}$)

$\therefore\ 7\theta = \pi, 3\pi, 5\pi$

$\therefore\ 4\theta = \pi - 3\theta, 3\pi - 3\theta, 5\pi - 3\theta$

$\therefore\ \sin4\theta = \sin(\pi - 3\theta), \sin(3\pi - 3\theta), \sin5\pi - 3\theta$

or $\sin4\theta = \sin3\theta$

or $2 \sin^2\theta \cos2\theta = 3 \sin\theta - 4 \sin^3\theta$

or $4\cos\theta.\cos 2\theta = 3 - 4\sin^2\theta$ ($\because \sin\theta \neq 0$)

or $4\cos\theta.(2\cos^2\theta - 1)$

This is a third degree equation in $\cos\theta$ and each of

$\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$ satisfied it.

$\therefore$ roots of (1) are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$

$\therefore \quad \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7}$ = sum of roots = $-\frac{-4}{8} = \frac{1}{2}$

..... (2)

$$\cos\frac{\pi}{7}.\cos\frac{3\pi}{7} + \cos\frac{3\pi}{7}.\cos\frac{5\pi}{7} + \cos\frac{5\pi}{7}.\cos\frac{\pi}{7} = \frac{-4}{8} = -\frac{1}{2}$$

..... (3)

$$\cos\frac{\pi}{7}.\cos\frac{3\pi}{7}.\cos\frac{5\pi}{7} = -\frac{1}{8}$$

..... (4)

Now $\left(\sin\frac{\pi}{7}.\sin\frac{2\pi}{7}.\sin\frac{3\pi}{7}\right)^2 = \left(\sin\frac{\pi}{7}.\sin\frac{5\pi}{7}.\sin\frac{3\pi}{7}\right)^2$

$$= \left(1 - \cos^2\frac{\pi}{7}\right)\left(1 - \cos^2\frac{3\pi}{7}\right)\left(1 - \cos^2\frac{5\pi}{7}\right)$$

$$= 1 - \left(\cos^2\frac{\pi}{7} + \cos^2\frac{3\pi}{7} + \cos^2\frac{5\pi}{7}\right) + \left(\cos^2\frac{\pi}{7}.\cos^2\frac{3\pi}{7} + \cos^2\right.$$

$$= 1 - \left\{\left(\cos^2\frac{\pi}{7} + \cos^2\frac{3\pi}{7} + \cos^2\frac{5\pi}{7}\right)^2 - 2\left(\cos^2\frac{\pi}{7}.\cos^2\frac{3\pi}{7} + \text{co}\right.\right.$$

$$+ \left\{\left(\cos\frac{\pi}{7}\cos\frac{3\pi}{7} + \cos\frac{3\pi}{7}\cos\frac{5\pi}{7} + \cos\frac{\pi}{7}\cos\frac{5\pi}{7}\right)^2\right.$$

$$- \left(\cos^2\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} + \cos^2\frac{3\pi}{7}\cos\frac{\pi}{7}\cos\frac{5\pi}{7} + \cos^2\frac{5\pi}{7}\text{ c}\right.$$

$$- \cos^2\frac{\pi}{7}.\cos^2\frac{3\pi}{7}.\cos^2\frac{5\pi}{7}$$

$$= 1 - \left\{\left(\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right)\right\} + \left\{\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{8}\right)\left(\frac{1}{2}\right)\right\} - \left(-\frac{1}{8}\right)^2$$

[Using (2), (3), (4)]

$$= 1 - \frac{5}{4} - \frac{3}{8} - \frac{1}{64}$$

$$= \frac{7}{64} = \left(\frac{\sqrt{7}}{8}\right)^2$$

Hence $\sin\frac{\pi}{7}\ \sin\frac{2\pi}{7}\ \sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}$

<u>Dumb Question</u>:- How was

$$\cos^2\frac{\pi}{7}\ \cos\frac{3\pi}{7}\ \cos\frac{5\pi}{7} + \cos^2\frac{3\pi}{7}\ \cos\frac{\pi}{7}\ \cos\frac{5\pi}{7} + \cos^2\frac{5\pi}{7}\ \cos$$

equated to $\left(\frac{-1}{8}\right)\left(\frac{1}{2}\right)$.

Ans:-

$$\sum\cos^2\frac{\pi}{7}\ \cos\frac{3\pi}{7}\ \cos\frac{5\pi}{7} = \cos\frac{\pi}{7}\ \cos\frac{3\pi}{7}\ \cos\frac{5\pi}{7}\left(\cos\frac{\pi}{7}\ \cos\frac{3\pi}{7}\right.$$

$$= \left(\frac{-1}{8}\right)\left(\frac{1}{2}\right)$$

Q-3. If A + B + C = π, sinA sinB sinC = P and cosA cosB cosC = 2 then otain the cubic equation whoose roots are tanA, tanB and tanC.

Solution:- The required cubic equation is

(x - tanA)(x - tanB)(x - tanC) = 0

or x^3 - (tanA + tanB + tanC) x^2 + x(tanA tanB + tanB tanC + tanC tanA)

- tanA tanB tanC = 0 .....(1)

Now tanA. tanB. tanC = $\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}\cdot\frac{\sin C}{\cos C} = \frac{p}{q}$

.....(2)

Also tan(A + B) = tan(π - C) = - tanC

or $\frac{\tan A + \tan B}{1 - \tan A.\tan B}$ = - tanC

or $\tan A + \tan B + \tan C = \tan A \tan B \tan C = \frac{p}{q}$
.....(3) [Using (2)]
Now $\sin^2 A + \sin^2 B + \sin^2 C$
$= \frac{1}{2}[3 - (\cos 2A + \cos 2B + \cos 2C)]$
$= \frac{1}{2}[3 - \{2\cos(A + B).\cos(A - B) + 2\cos^2 C - 1\}]$
$= \frac{1}{2}[4 - \{- 2\cos C \cos(A - B) + 2\cos^2 C\}]$
$= \frac{1}{2}[4 + 2\cos C\{\cos(A - B) + \cos(A + B)\}]$
$= 2(1 + \cos A \cos B \cos C) = 2(1 + q)$
$\therefore \sin^2 A \sin^2 B \sin^2 C (\operatorname{cosec}^2 B \operatorname{cosec}^2 C + \operatorname{cosec}^2 C \operatorname{cosec}^2 A + \operatorname{cosec}^2 A \operatorname{cosec}^2 B) = 2(1 + q)$
or $p^2\{(1 + \cot^2 B)(1 + \cot^2 C) + (1 + \cot^2 C)(1 + \cot^2 A) + (1 + \cot^2 A)(1 + \cot^2 B)\} = 2(1 + q)$
or $p^2\{3 + 2(\cot^2 A + \cot^2 B + \cot^2 C) + \cot^2 B \cot^2 C + \cot^2 C \cot^2 A + \cot^2 A \cot^2 B)\} = 2(1 + q)$
or $p^2[3 + 2\{\cot A + \cot B + \cot C)^2 - 2(\cot A \cot B + \cot B \cot C + \cot C \cot A)\} + \{(\cot B \cot C + \cot C \cot A + \cot A \cot B)^2 - 2 \cot A \cot B \times \cot C(\cot A + \mathrm{Cot}B + \cot C)\}] = 2(1 + q)$
or $p^2 [3 + 2\{(\cot A + \cot B + \cot C)^2 - 2$
$+ \{1 - 2.\frac{q}{p}(\cot A + \cot B + \cot C)\}] = 2(1 + q)$
.....(4)
[$\because \cot A. \cot B = 1$ if $A + B + C = \pi$]
Let $y = \tan A \tan B + \tan B \tan C + \tan C \tan A$
$= \tan A \tan B \tan C(\cot C + \cot A + \cot B)$
$= \frac{p}{q}(\cot A + \cot B + \cot C)$
$\therefore \cot A + \cot B + \cot C = \frac{yq}{p}$ .....(5)
$\therefore$ (4) gives
or $p^2\left[3 + 2\left\{\frac{y^2q^2}{p^2} - 2\right\} + 1 - \frac{2q}{p}.\frac{yq}{p}\right] = 2(1 + q)$

or $p^2\left(\frac{2q^2y^2}{p^2} - \frac{2q^2y^2}{p^2}\right)$ -2(1 + q)

or $q^2y^2 - q^2y - (1 - q) = 0$

$$y = \frac{q^2 \pm \sqrt{q^4 + 4q^2(1+q)}}{2q^2} = \frac{q^2 \pm q\sqrt{q^2 + 4q + 4}}{2q^2}$$

$$= \frac{q \pm (q+2)}{2q} = \frac{q+1}{q}, -\frac{1}{q}$$ .....

(6)

From (1), (2), (3), (4), (5) and (6),

$x^3 - \frac{p}{q}x^2 + \frac{q+1}{q}x - \frac{p}{q} = 0$ where $y = \frac{q+1}{q}$

or $x^3 \frac{p}{q}x^2 - \frac{1}{q}x - \frac{p}{q}\frac{p}{q} = 0$ when $y = -\frac{1}{q}$

∴ The required equation is

$qx^3 - px^2 + (q + 1)x - p = 0$

or $qx^3 - px^2 - x - p = 0$

<u>Dumb Question</u>:-

How does suddenly we started evaluating $\sin^2A + \sin^2B + \sin^2C$?

Ans:- The aim that we were trying to achievc at that point of time was to calculat the value of tanA tanB + tanB tanC + tan A tanC. Now when we try to evaluate this we realize that cotA + cotB + cotC needs to evaluated and thus we tried to make a equation is terms of cotA + cotB +cotC

So, all this factors were take in account and then only the $\sin^2A + \sin^2B + \sin^2C$ was eqaluated.

<u>Key words</u>:

(1) Trigonometry
(2) Quadrant
(3) Sin
(4) Cos
(5) Tan
(6) Cot
(7) Sec
(8) Cosec
(9) Allied angles.

(10) Compound angles.
(11) Perpendicular
(12) Hypotenuse
(13) Base

Solutions of Triangle

Every triangle has 3 angles and 3 sides. Given any 3 quantities out of 3 angles and 3 sides (at least one of which is a side) are given then remaining 3 can be found , which is termed as solution of the triangle. Many interesting relation among these quantities will be discussed in this section. Which formula is to be used where is a very important point here, as a use of wrong formula might lead to large calculations. So be prepared to see some very interesting stuff.

Terminology

Side AB , BC, CA of a ΔABC are denoted by c, a, b respectively and is semi perimeter of the triangle and Δ denotes area of the triangle.

Since Rule:

$$\frac{\text{Sin}A}{a} = \frac{\text{Sin}B}{b} = \frac{\text{Sin}C}{c}$$

How?

Fig (1)

Draw AD perpendicular to opposite side meeting it in point D.

In ΔABD,

$$\frac{AD}{AB} = SinB$$

$\Rightarrow$ AD = cSinB

In Δ ACD,

$$\frac{AD}{AC} = SinC$$

$\Rightarrow$ AD = bSinC

Equating 2 values of AD we get

cSinB = bSinc

So, $\frac{SinB}{b} = \frac{SinC}{c}$

Similarly drawing a perpendicular line from B upon CA we have

$$\frac{SinC}{c} = \frac{SinA}{a}$$

So, $\frac{SinA}{a} = \frac{SinB}{b} = \frac{SinC}{c}$

Illustration 1:

Given that $\angle$B=30, C=10 and b=5 find the angles of A and C of triangle.

Ans:

Fig (3)

By using sine rule,

$$\frac{\text{Sin}B}{b} = \frac{\text{Sin}C}{c}$$

<u>Cosine Rule:</u>

1) $\text{Cos } A = \dfrac{b^2 + c^2 - a^2}{2bc}$

2) $\text{Cos } B = \dfrac{c^2 + a^2 - b^2}{2ac}$

3) $\text{Cos } C = \dfrac{a^2 + b^2 - c^2}{2ab}$

How?

Fig (4)

Let ABC be a triangle and let perpendicular from A on BC meet it in point D

Now $AB^2 = AD^2 + BD^2$

$= (BC-CD)^2 + (AC^2 - DC^2)$

$= BC^2 + CD^2 - 2BC.CD + AC^2 - DC^2$

$= AC^2 + BC^2 - 2BC.CD$ ---------- (1)

Now $\frac{CD}{CA} = CosC$

$\Rightarrow CD = bCosC$

So, equation (1) is now

$c^2 = b^2 + a^2 - 2abCosC$

Or $CosC = \frac{b^2 + a^2 - c^2}{2ab}$

Similarly CosA, CosB can be found.

Illustration 2:

Show that the triangle is obtuse angled when sides of triangle 3x+4y, 4x+3y, 5x+5y units where x, y>0.

Ans:

Let a= 3x+4y

b= 4x+3y

c= 5x+5y

Since x, y>0 so c is largest side and angle C will be largest angle.

So,

$$\text{CosC} = \frac{b^2 + a^2 - c^2}{2ab}$$

$$= \frac{(3x + 4y)^2 + (4x + 3y)^2 - (5x + 5y)^2}{2(3x + 4y)(4x + 3y)}$$

$$= \frac{-2xy}{2(3x + 4y)(4x + 3y)}$$

$$= \frac{-xy}{(3x + 4y)(4x + 3y)}$$

Now since x, y>0

So, Cos C<0

And thus angle C is obtuse.

Projection Formula:

1) a = bCosC + cCosB

2) b = cCosA + aCosC

3) c = aCosB + bCosA

How?

Fig (5)

Let us draw a perpendicular line from C on Side AB

So, AB=BD+AD

In ΔBCD, $\frac{BD}{BC} = CosB$

So, BD= BC.CosB

In Δ ACD, $\frac{AD}{AC} = CosA$

So, AD= AC.CosA

So, AB=AC.CosA+BC.CosB

c= bCosA+aCosB

Illustration 3:

Drive Projection formula using the Cosine Rule

Ans:

$$aCosB + bCosA = a\left(\frac{a^2 + c^2 - b^2}{2ac}\right) + b\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$= \frac{1}{2c}\left[a^2 + c^2 - b^2 + b^2 + c^2 - a^2\right]$$

$$= \frac{2c^2}{2c}$$

$$= c$$

Napier Analogy:

$$1)\ \tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$$

$$2)\ \tan\left(\frac{C-A}{2}\right)=\left(\frac{c-a}{c+a}\right)\cot\left(\frac{B}{2}\right)$$

$$3)\ \tan\left(\frac{A-B}{2}\right)=\left(\frac{a-b}{a+b}\right)\cot\left(\frac{c}{2}\right)$$

How?

$$\frac{b-c}{b+c}=\frac{k\mathrm{Sin}B-k\mathrm{Sin}C}{k\mathrm{Sin}B+k\mathrm{Sin}C}$$

$$=\frac{2\mathrm{Sin}\left(\frac{B-C}{2}\right)\mathrm{Cos}\left(\frac{B+C}{2}\right)}{2\mathrm{Sin}\left(\frac{B+C}{2}\right)\mathrm{Cos}\left(\frac{B-C}{2}\right)}$$

$$=\tan\left(\frac{B-C}{2}\right)\cot\left(\frac{B+C}{2}\right)$$

$$=\tan\left(\frac{B-C}{2}\right)\cot\left(\frac{\pi-A}{2}\right)$$

$$=\tan\left(\frac{B-C}{2}\right)\tan\left(\frac{A}{2}\right)$$

So, $\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$

Illustration 4:

If in a triangle ABC, a = 6, b = 3, and Cos (A-B) = 4/5 then find the angle C.

Ans:

Cos (A-B) = 4/5

$$\text{So, } \frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A+B}{2}\right)} = \frac{4}{5}$$

Componendo and dividendo:

$$\frac{2\tan^2\left(\frac{A-B}{2}\right)}{2} = \frac{5-4}{5+4} = \frac{1}{9}$$

$$\text{Or } \tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$$

$$\text{But } \tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\left(\frac{c}{2}\right)$$

$$\frac{1}{3} = \left(\frac{6-3}{6+3}\right)\cot\left(\frac{C}{2}\right)$$

$$\text{Or } \cot\left(\frac{C}{2}\right) = 1$$

$$\text{or } C = 90$$

Trigonometric ratios of half angles:

$$1)\ \text{Sin}\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$2)\ \text{Sin}\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$3)\ \text{Sin}\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

How?

$$\text{Cos}A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$1 - 2\text{Sin}^2\left(\frac{A}{2}\right) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right) = 2\text{Sin}^2\left(\frac{A}{2}\right)$$

$$\frac{2bc - b^2 - c^2 + a^2}{4bc} = \text{Sin}^2\left(\frac{A}{2}\right)$$

$$\frac{a^2 - (b - c)^2}{4bc} = \text{Sin}^2\left(\frac{A}{2}\right)$$

$$\frac{(a - b + c)(a + b - c)}{4bc} = \text{Sin}^2\left(\frac{A}{2}\right)$$

$$\text{Now } S = \frac{a + b + c}{2}$$

$$\text{So, Sin}^2\left(\frac{A}{2}\right) = \frac{(2s - b - b)(2s - c - c)}{4bc}$$

$$= \frac{(s - b)(s - c)}{bc}$$

$$\text{So, Sin}\left(\frac{A}{2}\right) = \pm\sqrt{\frac{(s - b)(s - c)}{bc}}$$

Since A is angle of triangle, so 0<A<180 and thus A/2 is acute.

So, Sin A/2 has to be +ve and thus $\sqrt{\frac{(s - b)(s - c)}{bc}}$

4) $\text{Cos}\left(\frac{A}{2}\right) = \sqrt{\frac{s(s - a)}{bc}}$

5) $\text{Cos}\left(\frac{B}{2}\right) = \sqrt{\frac{s(s - b)}{ac}}$

6) $\text{Cos}\left(\frac{C}{2}\right) = \sqrt{\frac{s(s-c)}{ab}}$

7) $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

8) $\tan\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

9) $\tan\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Area of Triangle ABC:

1) $\Delta = \frac{1}{2}bc\text{Sin}A = \frac{1}{2}ca\text{Sin}B = \frac{1}{2}ab\text{Sin}C$

How?

Fig (6)

Let there be a triangle ABC and draw a perpendicular AD on the side BC from A

So, now $\Delta = ½$ AD.BC.

In triangle ABD

$$\frac{AD}{AB} = \text{Sin}B$$

$$AD = AB.\text{Sin}B$$

$$\text{So, } \Delta = \frac{1}{2}(AB.\text{Sin}B)(BC)$$

$$= \frac{1}{2}ac\text{Sin}B$$

2) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ Why?

Solution:

$$= \frac{1}{2}ab\text{Sin}C$$

$$= \frac{1}{2}ab\left(2\text{Sin}\frac{C}{2}.\text{Cos}\frac{C}{2}\right)$$

$$= ab\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

<u>m-n theorem</u>

If in a triangle ABC, point D divides BC in the ratio m: n and $\angle ADC = Q$ then,

Fig (7)

1) (m+n) CotQ = mCotα-nCotβ

2) (m+n) CotQ= nCotβ-mCotC

Illustration 5:

Find the value of y in

$$\frac{b-c}{a}\cos^2\left(\frac{A}{2}\right)+\frac{c-a}{b}\cos^2\left(\frac{B}{2}\right)+\frac{a-b}{c}\cos^2\left(\frac{C}{2}\right)=y\ ?$$

Ans:

$$\frac{b-c}{a}\cos^2\left(\frac{A}{2}\right)=\frac{b-c}{a}\frac{s(s-a)}{bc}=\frac{s}{abc}(b-c)(s-a)$$

$$\text{L.H.S}=\frac{s}{abc}\left(\sum(s-a)(b-c)\right)$$

$$=\frac{s}{abc}\left(s\sum(b-c)-\sum a(b-c)\right)$$

$$=0$$

So, y = 0.

CIRCUMCIRCLE:

The Circle passing through points A, B, C of a triangle ABC, its radius is denoted by R.

$$R = \frac{a}{2\text{Sin}A} = \frac{b}{2\text{Sin}B} = \frac{c}{2\text{Sin}C} = \frac{abc}{4\Delta}$$

How?

Bisect the 2 side BC and AC in D and E respectively and draw DO and EO perpendicular to BC and CA

Fig (8)

So, O is center of circumcircle.

Now $\Delta BOD \cong \Delta DOC$ (By RHS congruency rule)

So, $\angle BOD = \angle DOC$
$= \frac{1}{2} \angle BOC$
$= \angle BAC$
$= \angle A$

Now, in ΔBOD, BD = BO Sin (∠BOD)

$$\therefore \frac{a}{2} = R\sin A$$

$$\therefore R = \frac{a}{2\sin A}$$

$$\text{Now, } \Delta = \frac{1}{2} bc\sin A$$

$$\therefore R = \frac{a}{2\left(\frac{2\Delta}{bc}\right)} = \frac{abc}{4\Delta}$$

<u>Illustration 6:</u>

If in the ΔABC, O is circumcenter and R is the circumradius and R_1, R_2, R_3, are circumradii of the triangle OBC, OCA, OAB respectively then prove that

$$\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}$$

Ans:

Fig (9)

Clearly in Δ OBC, ∠BOC = 2A, OB = OC = R, BC = a

$$\therefore 2R_1 = \frac{a}{\text{Sin}2A} \quad \{\text{By using sine rule in } \Delta \text{ BOC}\}$$

$$\text{Similarly } \therefore 2R_2 = \frac{b}{\text{Sin}2B},\ 2R_3 = \frac{c}{\text{Sin}2C}$$

$$\text{So, } \frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = 2\,(\text{Sin}2A + \text{Sin}2B + \text{Sin}2C)$$

$$= 2\,(4\ \text{Sin}A.\text{Sin}B.\ \text{Sin}C)$$

$$= 8\left(\frac{a}{2R}\right)\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right)$$

$$= \frac{abc}{R^3}$$

<u>Incircle:</u>

The Circle which touches all the sides of ΔABC internally its radius is denoted by r.

$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$$

$$= 4R\text{Sin}\left(\frac{A}{2}\right).\text{Sin}\left(\frac{B}{2}\right).\text{Sin}\left(\frac{C}{2}\right)$$

How?

Bisect the ∠B and ∠C by line BI and CI meeting in I

So now I is in center of the circle. Draw ID, IE and IF ⊥ to 3 sides.

So, ID = IE = IF = r

Fig (10)

Now we have

Area of ΔIBC = ½ ID.BC= ½ r.a

Area of ΔIAC = ½ IE.AC= ½ r.b

Area of ΔIAB = ½ IF.AB= ½ r.c

Area of ΔABC= area of ΔIBC + area of ΔIAC + area of ΔIAB.

$$\text{So, } \Delta = \frac{1}{2}r.a + \frac{1}{2}r.b + \frac{1}{2}r.c$$

$$\Delta = r\left(\frac{a+b+c}{2}\right)$$

$$\Delta = r.s$$

$$r = \frac{\Delta}{s}$$

$$= \sqrt{\frac{(s(s-a)\,(s-b)\,(s-c)}{s}}$$

$$= (s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= (s-a)\tan\left(\frac{A}{2}\right)$$

Now what about $r = 4R\mathrm{Sin}\left(\frac{A}{2}\right).\mathrm{Sin}\left(\frac{B}{2}\right).\mathrm{Sin}\left(\frac{C}{2}\right)$

$$a = BD + CD$$

$$= r\cot\left(\frac{B}{2}\right) + r\cot\left(\frac{C}{2}\right)$$

$$= \frac{r\mathrm{Sin}\left(\frac{B+C}{2}\right)}{\mathrm{Sin}\left(\frac{B}{2}\right)\mathrm{Sin}\left(\frac{C}{2}\right)}$$

Also, $a = 2R\mathrm{Sin}A$

$$r = \frac{2R\mathrm{Sin}A.\mathrm{Sin}\left(\frac{B}{2}\right).\mathrm{Sin}\left(\frac{C}{2}\right)}{\mathrm{Sin}\left(\frac{B+C}{2}\right)}$$

$$= \frac{2R\left(2\mathrm{Sin}\left(\frac{A}{2}\right).\mathrm{Cos}\left(\frac{A}{2}\right).\mathrm{Sin}\left(\frac{B}{2}\right).\mathrm{Sin}\left(\frac{C}{2}\right)\right)}{\mathrm{Cos}\left(\frac{A}{2}\right)}$$

$$= 4R\mathrm{Sin}\left(\frac{A}{2}\right).\mathrm{Sin}\left(\frac{B}{2}\right).\mathrm{Sin}\left(\frac{C}{2}\right)$$

Illustration7:

A, B, C are the angles of a triangle, prove that:

$$\text{Cos } A + \text{Cos } B + \text{Cos } C = 1 + \frac{r}{R}$$

Ans:

$$\text{Cos}A + \text{Cos}B + \text{Cos}C = 2\text{Cos}\left(\frac{A+B}{2}\right)\text{Cos}\left(\frac{A-B}{2}\right) + \text{Cos}C$$

$$= 2\text{Sin}\frac{C}{2}.\text{Cos}\left(\frac{A-B}{2}\right) + 1 - 2\text{Sin}^2\frac{C}{2}$$

$$= 1 + 2\text{Sin}\frac{C}{2}\left[\text{Cos}\left(\frac{A-B}{2}\right) - \text{Sin}\frac{C}{2}\right]$$

$$= 1 + 2\text{Sin}\frac{C}{2}\left(\text{Cos}\left(\frac{A-B}{2}\right) - \text{Cos}\left(\frac{A+B}{2}\right)\right)$$

$$= 1 + 2\text{Sin}\frac{C}{2}\left(2\text{Sin}\frac{A}{2}.\text{Sin}\frac{B}{2}\right)$$

$$= 1 + 4R\text{Sin}\left(\frac{A}{2}\right).\text{Sin}\left(\frac{B}{2}\right).\text{Sin}\left(\frac{C}{2}\right)$$

$$= 1 + \frac{r}{R} \qquad \left(r = 4R\text{Sin}\left(\frac{A}{2}\right)\text{Sin}\left(\frac{B}{2}\right)\text{Sin}\left(\frac{C}{2}\right)\right)$$

$$\text{Cos } A + \text{Cos } B + \text{Cos } C = 1 + \frac{r}{R}$$

Excircles or Escribed circles:

The circle which touch BC and the two sides AB and AC produced is called escribed circle. Opposite the angle A and its radius is denoted by r_1. Similarly radii of circle opposite the angle B and C are denoted by r_2 and r_3 respectively.

1) $r_1 = \frac{\Delta}{s-a} = \text{stan}\left(\frac{A}{2}\right) = 4R\text{Sin}\left(\frac{A}{2}\right)\text{Cos}\left(\frac{B}{2}\right)\text{Cos}\left(\frac{C}{2}\right)$

How?

Fig (11)

Produce AB and AC to L and M. Bisect ∠CBL and ∠BCM by lines BI and CI and, let these lines meet in I.

Draw ID, IE, IF ⊥ to 3 sides respectively.

I is center of the escribed circle and so ID=IF=IE=r_1.

Now area of ΔABC + area of ΔIBC = area of ΔIAB + area of ΔIAC

So, $\Delta + \frac{1}{2}(ID)(BC) = \frac{1}{2}(IF)(AB) + \frac{1}{2}(IE)(AC)$

$\Rightarrow \Delta + \frac{1}{2} r_1 a = \frac{1}{2} r_1 c + \frac{1}{2} r_1 b$

So, $\Delta = \frac{1}{2} r_1 (b+c-a)$

$= \frac{1}{2} r_1 ((a+b+c)-2a)$

$= \frac{1}{2} r_1 (2s-2a)$

$= r_1 (s-a)$

$$\Rightarrow r_1 = \frac{\Delta}{s-a}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{(s-a)}$$

$$= s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= s\tan\left(\frac{A}{2}\right)$$

Now what about $r_1 = 4R\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$

$$4R\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) = 4R\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-b)}{ab}}$$

$$= \frac{4R\, s(s-b)\,(s-c)\,(s-a)}{abc\,(s-a)}$$

$$= \frac{1}{\Delta} \times \frac{\Delta^2}{s-a}$$

$$= \frac{\Delta}{s-a}$$

$$= r_1$$

2) $r_2 = \frac{\Delta}{s-b} = s\tan\left(\frac{B}{2}\right) = 4R\cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$

3) $r_3 = \frac{\Delta}{s-c} = s\tan\left(\frac{C}{2}\right) = 4R\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

Illustration 8:

Show that $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$

$$\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=\frac{(b-c)(s-a)}{\Delta}+\frac{(c-a)(s-b)}{\Delta}+\frac{(a-b)(s-c)}{\Delta}$$

$$=\frac{s\,(b-c+c-a+a-b)-[ab-ac+bc-ba+ac-bc]}{\Delta}$$

$$=\frac{0}{\Delta}$$

$$=0$$

So, $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$

<u>Orthocenter and Pedal Δ:</u>

Let ABC be any triangle and let AD, BE and CF be the altitudes of ΔABC. Then the ΔDEF formed by joining point D, E and F the feet of ⊥ is called the pedal Δ of ΔABC.

Fig (12)

Now

1) AH = 2RCosA, BH = 2RCosB, CH = 2CosC

How?

$$AH = AE\ SecDAC$$
$$= (C\ CosA)\ Sec\ (90^0 - C)$$
$$= C\ Cosec\ C\ CosA$$
$$= \frac{C}{SinC} CosA$$
$$= 2RCosA$$

2) HD = 2R CosB CosC, HE = 2R CosA CosC, HF = 2R CosA CosB

Why?

Fig (14)

$$HD = BD \tan \angle HBD$$

$$= BD \tan (90^0 - c)$$

$$= AB \, CosB.CotC$$

$$= \frac{C}{SinC} CosB.CosC$$

$$= 2RCosB.CosC$$

Illustration 9:

Prove that centroid, orthocenter and circumcenter of a Δ are collinear and centroid divides the line joining orthocenter and circumcenter in ratio 2:1.

Ans:

Fig (15)

Let O and P be circumcenter and orthocenter respectively. Draw OD and PK ⊥ to BC.

Let AD and OP meet in G.

Now the ΔAGP and ΔDGO are equiangular which is clearly due to the fact that OD and PK are parallel lines.

Now OD = OB Cos ∠BOD

= OB CosA

= RCosA

Also AP = 2RCosA

So, by similar triangles,

$$\frac{AG}{GD} = \frac{AP}{OD} = 2$$

So, point G is centroid of the Δ

Again by the same proposition

$$\frac{OG}{GP} = \frac{OD}{AP} = \frac{1}{2}$$

So, centroid divides line joining circumcenter to orthocenter in ratio 1:2.

Bisectors of theangles:

If AD is the angle bisector of ∠A then AD $= \frac{2bc}{b+c} \text{Cos}\left(\frac{A}{2}\right)$

$$BE = \frac{2ac}{a+c} \text{Cos}\left(\frac{B}{2}\right)$$

$$CF = \frac{2ab}{a+b} \text{Cos}\left(\frac{C}{2}\right)$$

How?

Fig (16)

Now area of ΔABD + area of ΔADC = area of ΔABC

⇒ ½ AB.AD Sin (A/2) + ½ AC.AD Sin (A/2) = ½ AB.AC SinA

⇒ AD (AB Sin (A/2) + AC Sin (A/2)) = AB.AC.2Sin (A/2) Cos (A/2)

⇒ AD (b+c) = 2bc Cos (A/2)

$$\Rightarrow AD = \frac{2bc}{b+c}\operatorname{Cos}\left(\frac{A}{2}\right)$$

Also, $x = \frac{ac}{b+c}$, $y = \frac{ab}{b+c}$

How?

Now in ΔABD

$$\frac{AD}{\operatorname{Sin}B} = \frac{BD}{\operatorname{Sin}\left(\frac{A}{2}\right)}$$

$$\frac{2bc\operatorname{Cos}\left(\frac{A}{2}\right)}{(\operatorname{Sin}B)(b+c)} = \frac{x}{\operatorname{Sin}\left(\frac{A}{2}\right)}$$

$$x = \frac{\left(2\operatorname{Sin}\left(\frac{A}{2}\right)\operatorname{Cos}\left(\frac{A}{2}\right)\right)bc}{(\operatorname{Sin}B)(b+c)}$$

$$= \frac{\operatorname{Sin}Abc}{\operatorname{Sin}B(b+c)}$$

Now by using sine rule.

$$x = \frac{(\text{Sin}A)ac}{\text{Sin}A(b + c)}$$

$$= \frac{ac}{b + c}$$

Also $y = a - x$

$$= \frac{ab}{b + c}$$

<u>Medians:</u>

If AD is a median then

$$AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

similarly,

$$BE = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$$

$$CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

How?

Fig (17)

In ΔADC use cosine law

So $AD^2 = AC^2+DC^2-2AC.DC.CosC$

$$\Rightarrow AD^2 = b^2 + \left(\frac{a}{2}\right)^2 - 2b\left(\frac{a}{2}\right)CosC$$

$$= b^2 + \left(\frac{a^2}{4}\right) - abCosC$$

Also, $c^2 = b^2 + a^2 - 2abCosC$

$$2AD^2 = c^2 + b^2 - \left(\frac{a^2}{2}\right)$$

Or, $AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

<u>Illustration 10</u>

In a ΔABC prove that $CosA.CosC = \frac{2(c^2 - a^2)}{3ac}$ where AD is the median through A and AD⊥AC

Ans:

Fig (18)

In ΔADC

$AD^2 = DC^2 - AC^2$

$$= \left(\frac{a}{2}\right)^2 - b^2$$

$$= \left(\frac{a^2}{4}\right) - b^2$$

But

$$AD^2 = \left(\frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}\right)^2$$

$$\Rightarrow \frac{a^2}{4} - b^2 = \frac{1}{4}\left(2b^2 + 2c^2 - a^2\right)$$

or $a^2 - 4b^2 = 2b^2 + 2c^2 - a^2$

or $a^2 - c^2 = 3b^2$ ----------(1)

Now, $\text{Cos}C = \frac{AC}{DC} = \frac{b}{(a/2)} = \frac{2b}{a}$

Also, $\text{Cos}A = \frac{b^2 + c^2 - a^2}{2bc}$

So, $\text{Cos}A.\text{Cos}C = \frac{b^2 + c^2 - a^2}{2bc} \times \frac{2b}{a}$

$$= \frac{b^2 + c^2 - a^2}{ac}$$

$$= \frac{a^2 - c^2 + 3(c^2 - a^2)}{3ac}$$ (from equation (1))

So, $\text{CosA.CosC} = \dfrac{2(c^2 - a^2)}{3ac}$

<u>Solution of triangles</u>

When any 3 of the 6 elements (except all 3 angles) of a Δ given, the triangle is completely known. This process is called solution of triangle.

Case 1: When the sides a, b, c are given then how to find?

$$\text{CosA} = \frac{b^2 + c^2 - a^2}{2ab}$$

Similarly B and C can be obtained.

Case 2: When two side's b, c and included angle A are given then how to find?

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\text{Cos}\frac{A}{2}, \text{ we get } \frac{B-C}{2}$$

Also $\dfrac{B+C}{2} = 90^0 - \dfrac{A}{2}$ So, B and C can be evaluated.

The third side is given by a $= \dfrac{\text{bSinA}}{\text{SinB}}$

Case 3: When 2 sides' b and c and angle B (opposite to side b) are given

1) If angle B is acute.

   a. No triangle possible if b<cSinB

   b. One right angled Δ, right angled at C if b=cSinB

   c. two Δ's are possible if cSinB<b<c

How?

Now $\quad \text{CosB} = \dfrac{a^2 + c^2 - b^2}{2ac}$

$$\text{Or } a^2 - (2c\text{CosB})a + (c^2 - b^2) = 0$$

So, for real values of a

$$(2c\text{CosB})^2 - 4(1)(c^2 - b^2) \geq 0$$

$$\Rightarrow \quad 4b^2 - 4c^2\text{Sin}^2\text{B} \geq 0$$

$\therefore$ If b<cSinB no triangle is possible.

Now if b=cSinB

Then $\dfrac{b}{\text{SinB}} = \dfrac{c}{1}$

So, a right angle triangle with angle c= 90^0 is possible

Now, $a = C\text{CosB} \pm \sqrt{b^2 - c^2\text{Sin}^2\text{B}}$

$C\text{CosB} + \sqrt{b^2 - c^2\text{Sin}^2\text{B}}$ Is positive (B is acute)

But if $C\text{CosB} - \sqrt{b^2 - c^2\text{Sin}^2\text{B}}$ is also be positive

Then $C\text{CosB} > \sqrt{b^2 - c^2\text{Sin}^2\text{B}}$

Or $c^2 > b^2$

Or c>b,

So, if CSinB<b<c

Then 2Δ's are possible.

2) If ∠B is obtuse: only one Δ is possible if b>c

How?

$$\text{CosB}=\frac{a^2+c^2-b^2}{2ac}$$

Or $a^2-(2c\text{CosB})a+(c^2-b^2)=0$

$a=\text{CCosB}\pm\sqrt{b^2-c^2\text{Sin}^2\text{B}}$

Since B is obtuse CosB<0

So, a cannot be $\text{CCosB}-\sqrt{b^2-c^2\text{Sin}^2\text{B}}$ as it is less than zero.

So, only possibility is $a=\text{CCosB}+\sqrt{b^2-c^2\text{Sin}^2\text{B}}$ but this will be

Positive only if $\text{CCosB}<\sqrt{b^2-c^2\text{Sin}^2\text{B}}$

Or $c^2<b^2$

Or c<b so, one Δ is possible if b>c.

Q-3: If p, q are perpendiculars from the angular points A and B of the ΔABC drawn to any line

Through the vertex C then prove that

$$a^2b^2\text{Sin}^2\text{C}=a^2p^2+b^2q^2-2abpq\text{CosC}$$

Solution:

Let $\angle ACE = \alpha$ clearly from fig we get

Fig (21)

$$\frac{p}{AC} = \operatorname{Sin}\alpha,\ \frac{q}{BC} = \operatorname{Sin}(\alpha + C)$$

$$\Rightarrow \frac{p}{b} = \operatorname{Sin}\alpha,\ \frac{q}{a} = \operatorname{Sin}\alpha \operatorname{Cos}C + \operatorname{Cos}\alpha \operatorname{Sin}C$$

$$\therefore \frac{q}{a} = \frac{p}{b}\operatorname{Cos}C + \operatorname{Cos}\alpha \times \operatorname{Sinc}$$

$$\text{for } \left(\frac{q}{a} - \frac{p}{b}\operatorname{Cos}C\right)^2$$

$$= \operatorname{Cos}^2\alpha . \operatorname{Sin}^2 C$$

$$= \left(1 - \frac{p^2}{b^2}\right)(1 - \operatorname{Cos}^2 C)$$

$$\text{or } \frac{q^2}{a^2} + \frac{p^2}{b^2}\operatorname{Cos}^2 C - \frac{2pq}{ab}\operatorname{Cos}C = 1 - \frac{p^2}{b^2} - \left(1 - \frac{p^2}{b^2}\right)\operatorname{Cos}^2 C$$

$$\text{or } \frac{q^2}{a^2} + \frac{p^2}{b^2} - \frac{2pq}{ab}\operatorname{Cos}C = \operatorname{Sin}^2 C$$

$$\text{or } a^2b^2\operatorname{Sin}^2 C = a^2p^2 + b^2q^2 - 2abpq\operatorname{Cos}C$$

Q-4: Find the expression for area of cyclic quadrilateral.

Solution: A quadrilateral is cyclic quadrilateral if its vertices lie on a circle.

Let ABCD be a cyclic quadrilateral such that AB=a, BC=b, CD=c and DA=d.

Then $\angle B+\angle D = 180$ and $\angle A+\angle C = 180$

Let 2s = a+b+c+d be the perimeter of the quadrilateral.

Fig (22)

Now, Δ = area of cyclic quadrilateral ABCD

= area of ΔABC + area of ΔACD

$$= \frac{1}{2}\text{abSinB} + \frac{1}{2}\text{cdSinD}$$

$$= \frac{1}{2}\text{abSinB} + \frac{1}{2}\text{cdSin}(\pi - \text{B})$$

$$= \frac{1}{2}\text{abSinB} + \frac{1}{2}\text{cdSinB}$$

$$= \frac{1}{2}(\text{ab} + \text{cd})\text{SinB} \qquad \text{------(1)}$$

Using Cosine formula in a triangle ABC and ACD we have

$$AC^2 = AB^2 + BC^2 - 2AB.BC.CosB$$

$$AC^2 = a^2 + b^2 - 2abCosB \qquad -------(2)$$

and

$$AC^2 = AD^2 + CD^2 - 2AD.CD.CosD$$

$$AC^2 = d^2 + c^2 - 2cdCos(\pi - B) \qquad ------(3)$$

From (2) and (3) we have

$$a^2 + b^2 - 2abCosB = d^2 + c^2 + 2cdCosB$$

$$\Rightarrow 2(ab + cd)CosB = a^2 + b^2 - c^2 - d^2 \qquad ---(4)$$

$$\Rightarrow 4(ab + cd)^2 Cos^2B = (a^2 + b^2 - c^2 - d^2)^2$$

$$\Rightarrow 4(ab + cd)^2 (1 - Sin^2B) = (a^2 + b^2 - c^2 - d^2)^2$$

$$\Rightarrow 4(ab + cd)^2 Sin^2B = 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$$

$$\Rightarrow 4(ab + cd)^2 Sin^2B = \{2(ab + cd) + (a^2 + b^2 - c^2 - d^2)\}.\{2(ab + cd) - (a^2 + b^2 - c^2 - d^2)\}$$

$$\Rightarrow 4(ab + cd)^2 Sin^2B = \{(a + b)^2 - (c - d)^2\}\{(c + d)^2 - (a - b)^2\}$$

$$\Rightarrow 4(ab + cd)^2 Sin^2B = (a + b + c - d)(a + b - c + d)(c + d - a + b)(c + d + a - b)$$

$$\Rightarrow 4(ab + cd)^2 Sin^2B = (2s - 2d)(2s - 2c)(2s - 2b)(2s - 2a)$$

$$\Rightarrow 16\Delta^2 = 16(s - a)(s - b)(s - c)(s - d) \qquad \{using\ (1)\}$$

$$\Rightarrow \Delta = \sqrt{(s - a)(s - b)(s - c)(s - d)} \qquad -------(5)$$

$\therefore$ from (1) (4) and (5) are of cyclic quadrilate ral

$$\Delta = \frac{1}{2}(ab + cd)SinB$$

$$\Delta = \sqrt{(s - a)(s - b)(s - c)(s - d)}$$

and $CosB = \dfrac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$

Q-5: Find the distance between the circumcenter and incenter.

Solution:

Let O be the circumcenter and I be the incenter of ΔABC. Let OF be perpendicular to AB and IE be perpendicular to AC and ∠OAF = 90-C.

∴∠OAI = ∠IAF-∠OAF

$$= \frac{A}{2} - (90 - C) = \frac{A}{2} + C - \frac{A + B + C}{2}$$
$$= \frac{C - B}{2}$$

Fig (23)

Also,

$$AI = \frac{IE}{\text{Sin}\,\frac{A}{2}} = \frac{r}{\text{Sin}\,\frac{A}{2}}$$
$$= 4R\text{Sin}\,\frac{B}{2}\text{Sin}\,\frac{C}{2}$$
$$\therefore OI^2 = OA^2 + AI^2 - 2OA.AI.\text{Cos}OAI$$

$$= R^2 + 16R^2 \text{Sin}^2 \frac{B}{2} \text{Sin}^2 \frac{C}{2} - 8R^2 \text{Sin} \frac{B}{2} \times \text{Sin} \frac{C}{2} \times \text{Cos} \frac{C-B}{2}$$

$$\therefore \frac{OI^2}{R^2} = 1 + 16\text{Sin}^2 \frac{B}{2} \text{Sin}^2 \frac{C}{2} - 8\text{Sin} \frac{B}{2} \times \text{Sin} \frac{C}{2} \left(\text{Cos} \frac{B}{2} \text{Cos} \frac{C}{2} + \text{Sin} \frac{B}{2} \text{Sin} \frac{C}{2} \right)$$

$$= 1 - 8\text{Sin} \frac{B}{2} \text{Sin} \frac{C}{2} \left(\text{Cos} \frac{B}{2} \text{Cos} \frac{C}{2} - \text{Sin} \frac{B}{2} \text{Sin} \frac{C}{2} \right)$$

$$= 1 - 8\text{Sin} \frac{B}{2} \text{Sin} \frac{C}{2} \text{Sin} \frac{A}{2}$$

$$\therefore OI = R\sqrt{1 - 8\text{Sin} \frac{A}{2} \text{Sin} \frac{B}{2} \text{Sin} \frac{C}{2}}$$

Q-6: Prove that $r_1^2 + r_2^2 + r_3^2 + r^2 = 16R^2 - a^2 - b^2 - c^2$ Where r = inradius, R=circum radius r_1, r_2, r_3 are exradii.

Solution:

L.H.S = $r_1^2 + r_2^2 + r_3^2 + r^2$

$$= 16R^2 \text{Sin}^2 \frac{A}{2} \text{Sin}^2 \frac{B}{2} \text{Sin}^2 \frac{C}{2} + 16R^2 \text{Sin}^2 \frac{A}{2} \text{Cos}^2 \frac{B}{2} \text{Cos}^2 \frac{C}{2} +$$

$$16R^2 \text{Cos}^2 \frac{A}{2} \text{Sin}^2 \frac{B}{2} \text{Cos}^2 \frac{C}{2} + 16R^2 \text{Cos}^2 \frac{A}{2} \text{Cos}^2 \frac{B}{2} \text{Sin}^2 \frac{C}{2}$$

$$= 16R^2 \text{Sin}^2 \frac{A}{2} \left[\text{Sin}^2 \frac{B}{2} \text{Sin}^2 \frac{C}{2} + \text{Cos}^2 \frac{B}{2} \text{Cos}^2 \frac{C}{2} \right] + 16R^2 \text{Cos}^2 \frac{A}{2}$$

$$\left[\text{Sin}^2 \frac{B}{2} \text{Cos}^2 \frac{C}{2} + \text{Cos}^2 \frac{B}{2} \text{Sin}^2 \frac{C}{2} \right]$$

$$= 4R^2 \text{Sin}^2 \frac{A}{2}\left[\left(2\text{Sin}^2 \frac{B}{2}\right)\left(2\text{Sin}^2 \frac{C}{2}\right) + \left(2\text{Cos}^2 \frac{B}{2}\right)\left(2\text{Cos}^2 \frac{C}{2}\right)\right] +$$

$$4R^2 \text{Cos}^2 \frac{A}{2}\left[\left(2\text{Sin}^2 \frac{B}{2}\right)\left(2\text{Cos}^2 \frac{C}{2}\right) + \left(2\text{Cos}^2 \frac{B}{2}\right)\left(2\text{Sin}^2 \frac{C}{2}\right)\right]$$

$$= 4R^2 \text{Sin}^2 \frac{A}{2}[(1 - \text{Cos}B)(1 - \text{Cos}C) + (1 + \text{Cos}B)(1 + \text{Cos}C)] +$$

$$4R^2 \text{Cos}^2 \frac{A}{2}[(1 - \text{Cos}B)(1 + \text{Cos}C) + (1 + \text{Cos}B)(1 - \text{Cos}C)]$$

$$= 4R^2 \text{Sin}^2 \frac{A}{2}[2 + 2\text{Cos}B\text{Cos}C] + 4R^2 \text{Cos}^2 \frac{A}{2}[2 - 2\text{Cos}B\text{Cos}C]$$

$$= 8R^2 + 8R^2 \text{Cos}B\text{Cos}C\left(\text{Sin}^2 \frac{A}{2} - \text{Cos}^2 \frac{A}{2}\right)$$

$$= 8R^2 - 8R^2 \text{Cos}A.\text{Cos}B.\text{Cos}C$$

$$= 8R^2 + 4R^2[\text{Cos}\ (A + B) + \text{Cos}(A - B)].\text{Cos}(A + B)$$

$$= 8R^2 + 4R^2[\text{Cos}^2(A + B) + \text{Cos}(A - B).\text{Cos}(A + B)]$$

$$= 8R^2 + 4R^2[\text{Cos}^2 C + \text{Cos}^2 A - \text{Sin}^2 B]$$

$$= 8R^2 + 4R^2[1 - \text{Sin}^2 C + 1 - \text{Sin}^2 A - \text{Sin}^2 B]$$

$$= 16R^2 - 4R^2 \text{Sin}^2 A - 4R^2 \text{Sin}^2 B - 4R^2 \text{Sin}^2 C$$

$$= 16R^2 - a^2 - b^2 - c^2$$

= R.H.S.

Q-7: Solve $b\text{Cos}^2 \frac{C}{2} + c\text{Cos}^2 \frac{B}{2}$ in terms of K where K is perimeter of Δ ABC.

Solution:

Here, $b\text{Cos}^2 \frac{C}{2} + c\text{Cos}^2 \frac{B}{2}$

$$\Rightarrow \frac{b}{2}(1+\text{Cos}C)+\frac{c}{2}(1+\text{Cos}B)$$

$$\Rightarrow \frac{b+c}{2}+\frac{1}{2}(b\text{Cos}C+c\text{Cos}B) \qquad \text{(using projection formula)}$$

$$\Rightarrow \frac{(b+c)}{2}+\frac{1}{2}a$$

$$\Rightarrow \frac{a+b+c}{2}$$

$$\Rightarrow \frac{K}{2} \qquad \text{(where } K=a+b+c \text{ is given)}$$

Q-8: Find the sides and angles of the pedal triangle.

Solution:

Fig (19)

Since the angle PDC and PEC are right angles, the points P, E, C and D lie on a circle.

$\therefore \angle PDE = \angle PCE = 90\text{-}A$

Similarly P, D, B and F lie on a circle and therefore

$\angle PDF = \angle PBF = 90\text{-}A$

Hence $\angle FDE = 180-2A$

Similarly $\angle PEF = 180-2B$

$\angle EFD = 180-2C$

Also from triangle AEF we have

$$\frac{EF}{SinA} = \frac{AE}{Sin\angle AFE} = \frac{ABCosA}{Cos\angle PFE} = \frac{CCosA}{Cos\angle PAE} = \frac{CCosA}{SinC}$$

$$\therefore EF = \frac{c}{SinC} SinACosA = aCosA$$

Similarly $DF = bCosB$ and $DE = cCosC$

Q-9: Prove that in a ΔABC

$$\left(Cot\frac{A}{2} + Cot\frac{B}{2}\right)\left(aSin^2\frac{B}{2} + bSin^2\frac{A}{2}\right) = CCot\frac{c}{2}$$

Solution:

L.H.S = $\left(\frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta}\right)\left(\frac{a}{2}(1-CosB) + \frac{b}{2}(1-CosA)\right)$

$$= \frac{s}{\Delta}(s - a + s - b)\frac{1}{2}\{a + b - (a\text{Cos}B + b\text{Cos}A)\}$$

$$= \frac{s}{2\Delta}(2s - a - b)(a + b - c)$$

$$= \frac{s}{2\Delta}(a + b + c - a - b)(a + b + c - 2c)$$

$$= \frac{s}{2\Delta}(c)(2s - 2c)$$

$$= \frac{cs(s - c)}{\Delta}$$

$$= C\text{Cot}\frac{c}{2}$$

Q-10: Find the radii of the inscribed and the circumscribed circle of a regular polygon of n side

with each side and also find the area of the regular polygon.

Solution:

Fig (20)

Let AB, BC and CD be three successive sides of the polygon and O be the center of both

the incircle and the circumcircle of the

polygon.

$$\angle BOC = \frac{2\pi}{n}$$

$$\angle BOL = \frac{1}{2}\left(\frac{2\pi}{n}\right) = \frac{\pi}{n}$$

If a be a side of the polygon, we have a=BC=2BL=2RSinBOL =

$$2R\sin\frac{\pi}{n}$$

$$r = \frac{a}{2}\operatorname{Cot}\frac{\pi}{n}$$

Now the area of the regular polygon = n times the area of the ΔOBC

$$= n\left(\frac{1}{2}\times OL\times BC\right) = n\times\frac{1}{2}\times\frac{a}{2}\operatorname{Cot}\frac{\pi}{n}\times a$$

$$= \frac{na^2}{4}\operatorname{Cot}\frac{\pi}{n}$$

Q-11: If a_1b and A are given in a triangle and c_1, c_2 are the possible values of the third side,

prove that

$$c^2{}_1 + c^2{}_2 - 2c_1c_2\operatorname{Cos}2A = 4a^2\operatorname{Cos}^2 A$$

Solution:

$$\text{Cos}A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= c^2 - 2bc\text{Cos}A + b^2 - a^2 = 0 \qquad \text{which is quadratic in 'c'.}$$

$$\therefore c_1 + c_2 = 2b\text{Cos}A \qquad \text{------(1)}$$

$$c_1c_2 = b^2 - a^2 \qquad \text{-------(2)}$$

$$\therefore c_1^2 + c_2^2 - 2c_1c_2\text{Cos}2A$$

$$= (c_1 + c_2)^2 - 2c_1c_2 - 2c_1c_2\text{Cos}2A \qquad \text{(by using equation 1 and 2)}$$

$$= (c_1 + c_2)^2 - 2c_1c_2(1 + \text{Cos}2A)$$

$$= 4b^2\text{Cos}^2A - 2(b^2 - a^2)2\text{Cos}^2A$$

$$= 4a^2\text{Cos}^2A$$

$$\therefore c^2{}_1 + c^2{}_2 - 2c_1c_2\text{Cos}2A = 4a^2\text{Cos}^2A$$

Q-12: Prove that in a triangle the sum of exradii exceeds the inradius by twice the diameter of

the circumcircle.or prove that $r_1+r_2+r_3 = r+4R$.

Solution: Let the exradii be r_1, r_2, r_3 and inradius = r, circum radius = R.

Then we have to prove that $r_1+r_2+r_3 = r+4R$.

Now,

$$r_1 + r_2 + r_3 - r = \frac{\Delta}{(s-a)} + \frac{\Delta}{(s-b)} + \frac{\Delta}{(s-c)} - \frac{\Delta}{s}$$

$$= \Delta\left\{\left(\frac{1}{s-a} - \frac{1}{s}\right) + \left(\frac{1}{s-b} + \frac{1}{s-c}\right)\right\}$$

$$= \Delta\left\{\frac{a}{s(s-a)} + \frac{2s-b-c}{(s-b)(s-c)}\right\}$$

$$= \Delta\left\{\frac{a}{s(s-a)} + \frac{a}{(s-b)(s-a)}\right\} \qquad \text{for } 2s = a+b+c$$

$$= \Delta \times a \times \frac{s^2 - s(b+c) + bc + s^2 - as}{s(s-a)(s-b)(s-c)}$$

$$= \Delta \times a \times \frac{2s^2 - s(a+b+c) + bc}{\Delta^2}$$

$$= \frac{a}{\Delta}(2s^2 - 2s^2 + bc)$$

$$= \frac{abc}{\Delta}$$

$$= 4R \qquad \left\{R = \frac{abc}{4\Delta}\right\}$$

<u>Medium</u>

Q-1:

In a ΔABC the angles A, B, C are in A.P show that

$$2\text{Cos}\frac{A-C}{2} = \frac{a+c}{\sqrt{a^2 - ac + c^2}}$$

Solution:

Here, A, B, C are in A.P So $B=A-\alpha$; $C=A-2\alpha$

Also

$$B = \frac{A + C}{2} = \frac{180^0 - B}{2} = 90^0 - \frac{B}{2}$$

$$\therefore \frac{3B}{2} = 90^0 \quad \therefore B = 60^0$$

$$\therefore A - \alpha = 60^0 \text{ ie } A = 60^0 + \alpha$$

$$\text{And } C = A - 2\alpha = 60^0 + \alpha - 2\alpha = 60^0 - \alpha$$

$$\therefore A - C = 2\alpha$$

$$= \frac{a}{\text{Sin}(60^0 + \alpha)} = \frac{b}{\text{Sin}60^0} = \frac{c}{\text{Sin}(60^0 - \alpha)}$$

$$= \frac{a + c}{\text{Sin}(60^0 + \alpha) + \text{Sin}(60^0 - \alpha)} = \frac{a + c}{\sqrt{3}\text{Cos}\alpha}$$

$$\therefore \frac{b}{\frac{\sqrt{3}}{2}} = \frac{a+c}{\sqrt{3}\text{Cos}\alpha} \qquad \therefore 2\text{Cos}\alpha = \frac{a+c}{b}$$

$$\text{Now } \frac{a+c}{\sqrt{a^2 - ac + c^2}} = \frac{a+c}{b\sqrt{\left(\frac{a}{b}\right)^2 - \frac{a}{b}\frac{c}{b} + \left(\frac{c}{b}\right)^2}}$$

$$= 2\text{Cos}\alpha \frac{1}{\left[\left\{\frac{\text{Sin}(60^0+\alpha)}{\text{Sin}60^0}\right\}^2 - \frac{\text{Sin}(60^0+\alpha)}{\text{Sin}60^0} \times \frac{\text{Sin}(60^0-\alpha)}{\text{Sin}60^0} + \left\{\frac{\text{Sin}(60^0-\alpha)}{\text{Sin}60^0}\right\}^2\right]^{\frac{1}{2}}}$$

$$= \frac{2\text{Cos}\alpha}{\left\{\left(\text{Cos}\alpha + \frac{1}{\sqrt{3}}\text{Sin}\,\alpha\right)^2 - \left(\text{Cos}\alpha + \frac{1}{\sqrt{3}}\text{Sin}\,\alpha\right) \times \left(\text{Cos}\alpha - \frac{1}{\sqrt{3}}\text{Sin}\,\alpha\right) + \left(\text{Cos}\alpha - \frac{1}{\sqrt{3}}\text{Sin}\,\alpha\right)^2\right\}^{\frac{1}{2}}}$$

$$= \frac{2\text{Cos}\alpha}{\sqrt{2\left(\text{Cos}^2\alpha + \frac{1}{3}\text{Sin}^2\alpha\right) - \left(\text{Cos}^2\alpha - \frac{1}{3}\text{Sin}^2\alpha\right)}}$$

$$= \frac{2\text{Cos}\alpha}{\sqrt{\text{Cos}^2\alpha + \text{Sin}^2\alpha}}$$

$$= 2\text{Cos}\alpha$$

$$= 2\text{Cos}\frac{A-C}{2}$$

Q-2: If in a ΔABC, CosA.CosB+SinA.SinB.SinC = 1, Show that a: b: c = 1:1:$\sqrt{2}$

Solution: Given relation yields,

$$\text{SinC} = \frac{1 - \text{CosA.CosB}}{\text{SinA.SinB}} \le 1$$

$$\Rightarrow \frac{1 - \text{CosA.CosB}}{\text{SinA.SinB}} - 1 \le 0$$

$$\Rightarrow 1 - \text{CosA.CosB} - \text{SinA.SinB} \le 0$$

$$\Rightarrow 1 - \text{Cos}(A - B) \le 0$$

$$\Rightarrow \text{Cos}(A - B) \ge 1$$

But $\text{Cos}(A - B)$ can not be greater than 1

$$\Rightarrow \text{Cos}(A - B) = 1$$

$$\Rightarrow A - B = 0$$

$$\Rightarrow A = B$$

also

$$\text{SinC} = \frac{1 - \text{Cos}^2 A}{\text{Sin}^2 A} = \frac{\text{Sin}^2 A}{\text{Sin}^2 A} = 1$$

$$\Rightarrow C = 90^0$$

$$\Rightarrow A = B = 45^0$$

Using sine law,

$$\frac{\text{SinA}}{a} = \frac{\text{SinB}}{b} = \frac{\text{SinC}}{c}$$

$$\Rightarrow \frac{\text{Sin}45^0}{a} = \frac{\text{Sin}45^0}{b} = \frac{\text{Sin}90^0}{c}$$

$$\Rightarrow \frac{1}{\sqrt{2}a} = \frac{1}{\sqrt{2}b} = \frac{1}{c}$$

$$\Rightarrow a : b : c = 1 : 1 : \sqrt{2}$$

Hard

Q-1: The internal bisectors of the angles of the ΔABC meet the sides BC, CA and AB at P, Q, and R respectively. Show that the area of the ΔPQR is equal to $\frac{2abc\Delta}{(a+b)(b+c)(c+a)}$

Solution:

Let the bisector of the angles meet at I. then I is the incenter. Let ID⊥BC, then ID = inradius = r.

Clearly $\angle BIP = \angle IBA + \angle IAB = \frac{B}{2} + \frac{A}{2}$

Let ∠DIP=Q, PI=x, QI=y and RI=z then

Q=∠BIP-∠BID = $\frac{A}{2} + \frac{B}{2} - \left(90 - \frac{B}{2}\right) = \frac{A}{2} + B - 90$

Fig (24)

Now from ΔDIP, $\frac{r}{x} = \text{Cos}Q$

$$\therefore x = \frac{r}{\text{Cos}Q} = \frac{r}{\text{Cos}\left(\frac{A}{2} + B - 90\right)} = \frac{r}{\text{Sin}\left(\frac{A}{2} + B\right)}$$

$$= \frac{r}{\left(\frac{b+c}{a}\right)\text{Sin}\frac{A}{2}}$$

$$\because a\text{Sin}\left(\frac{A}{2} + B\right) = (b+c)\text{Sin}\frac{A}{2}$$

$$\therefore \qquad x = \frac{ar}{(b+c)\text{Sin}\,\frac{A}{2}}$$

Similarly $\quad y = \dfrac{br}{(c+a)\text{Sin}\,\frac{B}{2}}$

and $\quad z = \dfrac{cr}{(a+b)\text{Sin}\,\frac{C}{2}}$

Now $\text{ar}(\Delta r(\Delta P = \frac{1}{2}\text{PI.IQ.Sin}\angle\text{PIQ}$

$$= \frac{1}{2}\text{xy.Sin}(\angle\text{PIC} + \angle\text{QIC})$$

$$= \frac{1}{2}\text{xySin}\left\{\left(\frac{A}{2}+\frac{C}{2}\right)+\left(\frac{C}{2}+\frac{B}{2}\right)\right\}$$

$$= \frac{1}{2}\frac{ar}{(b+c)\text{Sin}\,\frac{A}{2}} \times \frac{br}{(c+a)\text{Sin}\,\frac{B}{2}} \times \text{Sin}\left(\frac{A+B+C}{2}+\frac{C}{2}\right)$$

$$= \frac{abr^2}{2(b+c)(c+a)\text{Sin}\,\frac{A}{2}\text{Sin}\,\frac{B}{2}}\text{Sin}\left(90+\frac{C}{2}\right)$$

$$= \frac{abr^2\text{Cos}\left(\frac{C}{2}\right)}{2(b+c)(c+a)\text{Sin}\,\frac{A}{2}\text{Sin}\,\frac{B}{2}}$$

Similarly, $\quad \text{ar}(\Delta\text{QIR}) = \dfrac{bcr^2\text{Cos}\left(\frac{A}{2}\right)}{2(c+a)(a+b)\text{Sin}\,\frac{B}{2}\text{Sin}\,\frac{C}{2}}$

$$\text{ar}(\Delta\text{RIP}) = \frac{car^2\text{Cos}\left(\frac{B}{2}\right)}{2(a+b)(b+c)\text{Sin}\,\frac{C}{2}\text{Sin}\,\frac{A}{2}}$$

Now $ar(\Delta PQR) = ar(\Delta PIQ) + ar(\Delta QIR) + ar(\Delta RIP)$

$$= \frac{abcr^2}{2(a+b)(b+c)(c+a)\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\text{Sin}\frac{C}{2}} \times \left\{ \begin{array}{l} \frac{a+b}{c}\text{Sin}\frac{C}{2}\text{Cos}\frac{C}{2} + \frac{b+c}{a}\text{Sin}\frac{A}{2}\text{Cos}\frac{A}{2} + \\ \frac{c+a}{b}\text{Sin}\frac{B}{2}\text{Cos}\frac{B}{2} \end{array} \right\}$$

$$\frac{abcr^2}{2(a+b)(b+c)(c+a)\frac{r}{4R}} \times \frac{1}{2}\left\{ \frac{a+b}{c}\text{Sin}C + \frac{b+c}{a}\text{Sin}A + \frac{c+a}{b}\text{Sin}B \right\}$$

$$\left[\because r = 4R\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\text{Sin}\frac{C}{2} \right]$$

$$= \frac{abcRr}{(a+b)(b+c)(c+a)} \times \left\{ (a+b)\frac{1}{2R} + (b+c)\frac{1}{2R} + (c+a)\frac{1}{2R} \right\}$$

$$= \frac{abcr}{(a+b)(b+c)(c+a)}(a+b+c)$$

$$= \frac{abcr}{(a+b)(b+c)(c+a)}2s$$

$$= \frac{2abc\Delta}{(a+b)(b+c)(c+a)} \qquad \left(\because r = \frac{\Delta}{s}\right)$$

Q-2: Show that the ΔABC is equilateral if its circumradius is double of the inradius.

Solution:

Here R=2r (given) .We know that,

$$r = \frac{\Delta}{s} = \frac{\Delta^2}{s\Delta} = \frac{s(s-a)(s-b)(s-c)}{s\Delta}$$

$$= \frac{1}{\Delta}(s-a)(s-b)(s-c)$$

$$= \frac{1}{\Delta}\sqrt{(s-b)(s-c)} \times \sqrt{(s-c)(s-a)} \times \sqrt{(s-a)(s-b)}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= 4R\text{Sin}\frac{A}{2}.\text{Sin}\frac{B}{2}.\text{Sin}\frac{C}{2} \quad \left\{\because R = \frac{abc}{4\Delta}\right\}$$

$$r = 8r\text{Sin}\frac{A}{2}.\text{Sin}\frac{B}{2}.\text{Sin}\frac{C}{2}$$

$$\text{or}\ \frac{1}{8} = \text{Sin}\frac{A}{2}.\text{Sin}\frac{B}{2}.\text{Sin}\frac{C}{2} \qquad \text{----------}(1)$$

Now

$$\text{Sin}\frac{A}{2}.\text{Sin}\frac{B}{2}.\text{Sin}\frac{C}{2} = \frac{1}{2}\text{Sin}\frac{A}{2}\left\{\text{Cos}\frac{B-C}{2} - \text{Cos}\frac{B+C}{2}\right\}$$

$$= \frac{1}{2}\text{Sin}\frac{A}{2}\left\{\text{Cos}\frac{B-C}{2} - \text{Sin}\frac{A}{2}\right\}$$

$$\le \frac{1}{2}\text{Sin}\frac{A}{2}\left(1 - \text{Sin}\frac{A}{2}\right) \quad \left\{\because \max\ \text{Cos}\frac{B-C}{2} = 1\right\}$$

equality holds when $B = C$

$$\text{But}\ \frac{1}{2}\text{Sin}\frac{A}{2}\left(1 - \text{Sin}\frac{A}{2}\right) = \frac{1}{2}\left[\text{Sin}\frac{A}{2} - \text{Sin}^2\frac{A}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} - \left(\text{Sin}^2\frac{A}{2} - \text{Sin}\frac{A}{2} + \frac{1}{4}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} - \left(\text{Sin}\frac{A}{2} - \frac{1}{2}\right)^2\right]$$

$$\leq \frac{1}{2}, \frac{1}{4} \qquad \left\{\because \left(\text{Sin}\,\frac{A}{2} - \frac{1}{2}\right)^2 \geq 0\right\}$$

equality holds when $\text{Sin}\,\frac{A}{2} = \frac{1}{2}$

$\because \text{Sin}\,\frac{A}{2}.\text{Sin}\,\frac{B}{2}.\text{Sin}\,\frac{C}{2} \leq \frac{1}{8}$

equality holds when $B = C$ and $\text{Sin}\,\frac{A}{2} = \frac{1}{2}$

From (1) equality must hold

So $B = C$ and $\text{Sin}\,\frac{A}{2} = \frac{1}{2}$

i.e. $\frac{A}{2} = 30^0 \qquad \therefore A = 60^0$

$\therefore B + C = 180^0 - A = 120^0$

But $B = C$ So $B = C = 60^0$

Hence $A = B = C = 60^0$

So, triangle is equilateral.

The Coordinate Geometry

Circle

Most of the things that we see around us are circular. Sun, moon on full moon day bangle mery-go around, which you loved so much when you where a child; all happened to circles. Ever wondered how a circle can be represented mathematically; well no!!, then we will tell you in this chapter. Also we will take about tangent normulas, chords which we all have hear about. So let us prove deep in to circles.

<u>Definition</u>: locus of a set of points equidistant from a fixed point

Equation ofcircle -

$(x - h)^2 + (y - k)^2 = r^2$

$x^2 h_2 - 2hx + y^2 - 2ky + k^2 - r^2 = 0$

$x^2 + y^2 + 2gx + 2fy + C = 0$

Center $\rightarrow (-g,-f)$. radius $\rightarrow \sqrt{g^2+f^2-c}$

<u>Genreal second degree equation-</u>

$ax^2 + by^2 + 2hxy + 2yx + 2fy + C = 0$

this equation refresent circle when,

$a = b, h = 0$, $g^2 + f^2 \geq C$

<u>Equation of circle in different forms -</u>

(1) <u>Centre(h_1K) radius a</u> :-

$(x - h)^2 + (y - k)^2 = a^2$

<u>standard form (when center is origin)</u>:-

$x^2 + y^2 = a^2$

(2).<u>center (h,k)and pass through origin-</u>

$x^2 + y^2 - 2hx - 2ky = 0$

Why :-

here $(\pm h, \pm h)$

$\therefore (x - h)^2 + (y - k)^2 = r^2 = h^2 + k^2$

$x^2 + y^2 - 2hx - 22ky = 0$

<u>Center(h,k) and touches the axis of</u> x-y

$x^2+y^2- 2hx - 2ky + h^2= 0$
$(x - h)^2 + (y - k)^2 = k^2$
or, $x^2+ y^2 - 2hx - l2ky + h^2 = 0$
(4)<u>Center(h,k) and touches the axis of y-</u>
$x^2 + y^2-2hx -2ky + k^2 = 0$
Why :-
From fig it is clear that radius will be h .

$(x - h)^2 + (y - k)^2 = h^2$
or, $x^2+ y^2- 2hx - 2ky +k^2 = 0$
<u>Circle which touches both the axis:-</u>
Center will be(h, h)and radius will be h. But since center would be in any of the four quadrants its coordinates can be taken as $(\pm h, \pm h)$ radius h.
$\therefore x^2 + y^2 \pm 2hx \pm 2hx + h^2 = 0$

<u>Illustration -1-</u>. Find the equation of circle passing through (-2, 3) and touching both the axes.
<u>Solution</u>- As the circle toucher both theaxed and lies in 2nd quadrant, lits centre is $C \equiv (-r,r)$ Where r is the radius , Distance of center from (- 2, 3) = radius .
$\Rightarrow \sqrt{(r-2)^2 + (3-r)^2} = r$
$\Rightarrow \quad nr = 5 \pm 2\sqrt{3}$
the circle are :- $(x + r)^2 + (y - r)^2 = r^2$
$x^2+y^2+2(5 \pm 2\sqrt{3})x - 2(5 \pm 2\sqrt{3})y + (5 \pm 2\sqrt{3})^2 = 0$
$X^2 + y^2+ 2agx + 2fy +c = 0$ ----equation of circle.

x intercept $\therefore$
Why :-
let it cut the axis x ie y = 0

in points $(x_2,0)$ and $(x_2, 0)$
$\therefore$ x_1, x_2 are the roots of $x^2 + 2yx + c = 0$
$\therefore$ $x_1, x_2 = -2y$, $x_1 . x_2 = c$
Intercelt = $x_2 - x_1$

$= [(x_2 + x_1)^2 - 4x_1x_2]^{1/2} = \sqrt{4g^2 - 4c} = 2\sqrt{g^2 - c}$

similarily .y intercept = $2\sqrt{f^2 - c}$
(6) Circle whose diameter is the line joining two point A (s_1, y_1) <u>and B (x_2, y_2)</u>-

<u>Diametric form</u>:-
$(xx - x_1)(x - x_2) + (y - y_1(y - y_2) = 0$

Why :-
Angle in a semicircle is a right angle $PA \perp PB$
$m_1m_2 = -1$

$$\frac{y - y_1}{(x - x_1)} \cdot \frac{y - y_2}{(x - x_2)} = -1$$

or $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
(7) <u>Parametric form</u>-

$$\left.\begin{aligned} x &= a\cos\theta \\ y &= a\sin\theta \end{aligned}\right]$$ – general point of a circle if centre is (0,0) $\theta \in (-\pi, \pi)$ isparameter (radius)

$x = h + a\cos\theta$

$y = k + a\sin\theta$

<u>Illustration -2</u>- Find the equation a circle which touches the .y axis at (0, 4) and cuts an intercept of length .6 units on x axis .

<u>Solution</u>- The equation of circle toching x = 0 at (0,4) can be taken as

$(x - 0)^2 + (y - 4^2) + kx = 0$

$x^2 + y^2 + kx - 8y + 16 = 0$

the circle cuts x -axis point $(x_1, 0)$ .8 $(x_2, 0)$given by, $x^2 + kx + 16 = 0$

Xintercept difference of root of this quadratic equation $6 = | x_2 - x_1|$

$36 = (x_2 + x_1)^2 - 4x_1 .x_2$

$36 = k^2 - 4 (16)$

$k = \pm 10$

Hence the required circle is ,

$x^2 + y^2 \pm 10 x - 8y + 16 = 0$

<u>Some natations in a circle</u>-

1) $s = x^2 + y^2 + 2gx + 2fy + c$

2) $s_1 = x x_1^2 + y_1^2 + 2yx_1 + 2fy_1 + c$

3) $T = xx_1 + yy_1 + g(x + x_1) + f (y + y_1) + c$

<u>Standrad form</u>-

1)$s = x^2 + y^2 - a^2$

2)$s = {}_1x_1^2 + y_1^2 - a^2$

3) $T = xx_1 + yy_1 - a2$

* If $s_1 > 0 \rightarrow$ point lies out side the circle
* If $s_1 < 0 \rightarrow$ point lies in side the circle
* If s_1 0 $\rightarrow$ point lies upon the circle

<u>Why</u> :-

Let equation of circle be $X^2 + y^2 + 2yx + 2fy + c = 0$

having centre C≡(-y, -f) and radius $r = \sqrt{g^2 + f^2 - c}$

Let P (x_1, y_1) be any point then :-

P lies outside the circle if :-

PC > r

=> $x_1^2 + y_1^2 + 2yx_1 + 2fy_1 + c > 0$

P lies on the circle if :-

PC = r

=> $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

P lies inside the circle if :-

PC < r

=> $x_1^2 + y_1^2 + 2gx_1\ 2fy_1 + c < 0$

<u>Dumb question</u> :-

How does PC > r leads to -

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$

<u>Ans</u>- $= \sqrt{(x_1 - (-g))^2 + (y_1 - (-f))^2}$

$= \sqrt{(x_1 + y)^2 + (y_1 + f)^2}$

and r = $\sqrt{g^2 + f^2 - c}$

Now PC > r

=> $PC^2 > r^2$

=> $(x_1 + y)^2 + (y_1 + f)^2 > y^2 + f^2 - c$

=> $x_1^2 + y_1^2\ 2gx, + 2fy_1 + c > 0$

(1) A line L and a circle intersed in two point A and B .

=> d < r

=> Perpendi cular distance of line L from the centre of circle is less than the radius, and the length of te chors AB is :-

$AB = 2\sqrt{r^2 - d^2}$

(2) A line L and +a circle touch each other at a point P.

=> d = r

=> Perpendicular distance of L from the centre of circle = radius.

(3) A line L and a circle may not intersect at all
=> .d > r
=> Perpendicular distance of line from the centre of circle is greater than the radius .
(4) A line y = mx + c touches circle $x^2 + y^2 = a^2$
If :- perpendicular distance of line from centre of the circle
= radius of the circle

$$\Rightarrow \frac{|c|}{\sqrt{1+m^2}} = a$$

$$\Rightarrow c = \pm a\sqrt{1+m^2}$$

<u>Illustration</u>- For what value of m, will line y = mx does not intersect the circle $x^2+y^2 + 20 X +20y + 20 = 0$
<u>Solution</u>- IF the line y = mx does not intersect the circle ; the perpendicular distance of the line from the centre of the circle must be greater than its radius .

Centre of circle ≡(-10, -10) ; radius $r = 6\sqrt{5}$
distance of line mx - y = 0 from (-10, -10)
=> |m(-10)-(-10)|

$$\sqrt{m^2+1} > 6\sqrt{5}$$

=>(2m + 1) (m + 2) < 0
=> -2 < m < - 1/2
<u>Intersection of line with circle-</u>
Let the line be y = mx + d and circle is $x^2+ y^2 +2gx + 2fy + c$
thes x. Coordinate of their point of intersection are given by, $(1 + m^2)x2+ (2g + 2fm +2dm) x + d^2+ 2fd + = o$
Why :-
When the two curves intersect, both the curves will be simultaneously satisfied.
So y = mx + d can be replaced in
$x^2y^2 + 2gx + 2fy + c =0$
=> $x^2 + (mx + d)^2 + 2gx + 2f (mx + d) + c =0$
=> $(1 + m)^2 + (2g + 2fm + 2dm) x + d^2 + 2fd + c = 0$
<u>if.</u> (i) $B^2 - 4AC = 0$ then line touches the circle.
(ii) $B^2 - 4AC = > 0$ then the line intersect circle at 2 different point.
(iii) $B^2 - 4AC = < 0$ then no real intersecti takes place.

<u>Illustration 4</u>- Find the point on the circle $x^2 + y^2$
= 4 whose distance from the line 4x + 3y = 12 is 4/5 .
<u>Soluction-</u> Let A,B be the point on x^2/u> + y^2 = 4 luing ar a distance 4/5 from 4x + 3y = 12

=> AB will be parallel to 4x + 3y = 12

distance between the two line is $\frac{|c-|2|}{\sqrt{9+16}} = \frac{4}{5}$

=> C = 16, 8

=> the equation of AB is :- 4x + 3y = 8 4x + 3y = 16

the point A,B can be formed by sliving for point of intersection of x^2 $y^2 = 4$ with AB.

AB ≡ (4x + 3y - 8 = 0)

=> $x^2 + \left(\frac{8-4x}{3}\right)^2 . = 4$

=> 25 x^2 - 64x + 28 = 0

=> x = 2, 14/25

y = 0, 48/25

AB ≡ (4x + 3y - 16 = 0)

=> $x^2 + \left(\frac{16-4x}{3}\right)^2 = 4$ => 25 x^2 - 128 x + 220 = 0

=> D < 0 => no real roots

Hence these are two pointr on circle at distance 4/5 from liine

A ≡ (2,0) . & B ≡ (14/25, 48/25)

<u>Alter native method -</u>

let P ≡ $P \equiv (2\cos\theta, 2\sin\theta)$ be the point on the circle $x^2 + y^2 = 4$

distance . 4/5 from given line. the distsnce from line = 4/5

$\Rightarrow \frac{.(4(2\cos\theta) + 3(2\sin\theta) - 12)}{5} = 4/5$

Solve for θ to get the point .

<u>Equation of tangent in general form is</u> :-

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

<u>equation of tangent on standard form</u> :-

$xx_1 + yy_1\ a^2 = 0$

<u>Why</u> :-

Slope of tangent = - $\frac{x_1 + g}{y_1 + f}$

equation of tangent :-

$y - y_1 - \frac{x_1 + g}{y_1 + f} (x - x_1)$

$(y - y_1) (y_1 + f) = - (x_1 + y) (x - x_1)$

on solving we get,

$xx_1 + yy_1 + g (x + x_1) + f (y + y_1) + = 0$

Equation of tangen $\rightarrow$ T = 0 >

<u>Dumb question</u>- Why slope of tangent $= \frac{y_1 - (- f)}{x_1 - (- g)} = \frac{y_1 + f}{x_1 + g}$?

<u>Ans</u> - The slope of line Joining the centre to point of contact is

$= \frac{y_1 - (- f)}{x_1 - (- g)} = \frac{y_1 + f}{x_1 + g}$

Now tangent os perpendicular to this line -

$\therefore$ slope of tangent is - $\left(\frac{x_1 + f}{y_1 + f}\right)$

<u>Note</u>:- Golden rule to write equation of tangent is to replace. $x^2 \rightarrow xx_1$, $y^2 \rightarrow yy$

1

$2x \rightarrow x + x_1$, $2y = y + y_1$ in equation of circle where (x_1, y_1) is of contact.

$T \equiv (a \cos\theta) x + (a \sin\theta) y - a^2$

$x \cos\theta + y \sin\theta = a$

Equation of tangent.

<u>Lngth of tangent</u>:-

length $= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{s_1}$

<u>Why</u> :-

let equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

then center is $c \equiv (-g, -f)$ and radius = f

length of tangent = PA

$= \sqrt{PC^2 - AC^2}$

$= \sqrt{(x_1 + y)^2 + (y_1 + f)^2 - r^2}$

on solving we get ,

length of tangent $= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$= \sqrt{s_1}$

length $= \sqrt{s_1}$

<u>Condition of or line y = mx + c to be a ltangent to $x^2 + y^2 = a^2$</u> -

<u>Condition</u>:-

$c^2 = a^2 (1 + m^2)$

$\therefore$ <u>Equation of tangent</u>:-

$y = mx \pm a\sqrt{1+m^2}$

Why -?
putting $y = mx + c$ in $x^2 + y^2 = a^2$

$x^2 + (mx + c)^2 = a^2$

$(1 + m^2)\, x^2 + 2mxc + c^2 - a^2 = 0$

$4m^2 - c^2 - 4\,(c^2 - a^2)\,(1 + m^2) = 0$

$c^2 = a^2(1 + m^2)$

Similarily when circle equation is -
$(x - h)^2 + (y - k)^2 = a^2$
equation of tangent with slope m
=> $(y - k) = m\,(x - h) \pm a\sqrt{1+m^2}$

<u>Dumb question</u> :- Why D is taken zero ?

<u>Ans</u>- Line is touching circle.
It means on ksolving line and circle -
We will get only are solution -
It means quadratic of x will lhave repeated roots if means D = 0

<u>Illustration</u>- Find the equation of two tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from point (0, 1)
<u>Solution</u>- let m be the slope of the tangent .
For true lengths here will be two values of m which are requited.
As the regent pases through (0, 1) its equation will be .
$(y - 1) = m(x - 0)$
$mx - y + 1 = 0$
Now the centre of circle ($x^2 + y^2 - 2x + 4y = 0$) is (1,-2) and radius r

=÷

So using the condition of tangencu ÷ distance of centre (1, - 2) from line = radius (r)

$$\Rightarrow m = 2, -\frac{|m(1) - (-2) + 1|}{\sqrt{m^2 + 1}} = \sqrt{5}$$ 1/2

∴ equation of tangents are :-

2x - y = 1 = 0 and x = 2y - 2 = 0

<u>Illustration</u>:- Find the equation mcircle passing through (-4, 3)an touching the line x + y = 2 and x - y = 2.

<u>Solution .</u>- let (h, k) be the centre of the circle the ditance of the centre from th given line and the given point must be equation to radius :-

$$\Rightarrow \frac{|h+k-2|}{\sqrt{2}} = \frac{|h-k-2|}{\sqrt{2}} = \sqrt{(h+4)^2+(k-3)^2}$$

<>br /> Consider $\frac{|h+k-2|}{\sqrt{2}} = \frac{|h-k-2|}{\sqrt{2}}$

=> h + k - 2 = ±(h - k - 2)

<u>Case(I)</u>- h + k - 2 = h - k -2 => k = 0

$$\frac{|h-2|}{\sqrt{2}} = \sqrt{(h+4)^2+9}$$

=> $(h - 2)^2 = 2(h + 4)^2 + 18$

=> $h^2 + 20h + 46 = 0$

=> $k = 10 \pm 3\sqrt{6}$

radius = $\frac{|h+k-2|}{\sqrt{2}} = \left|\frac{-12 \pm 3\sqrt{6}}{\sqrt{2}}\right|$

Circle is :

$$(x+10\mp 3\sqrt{6})^2+(y-0)^2 = \frac{(-12\pm 3\sqrt{6})^2}{2}$$

$$\Rightarrow x^2 + y^2 + 2(10 \pm 3\sqrt{6})x + (10 \pm 3\sqrt{6})^2 - \frac{(-12\pm 3\sqrt{6})^2}{2} = 0$$

$$\Rightarrow x^2 + y^2 + 2(10 \pm 3\sqrt{6})x + 55 \pm 24\sqrt{6} = 0$$

Case(II) - h + k - 2 = - (h - k - 2)

=> $k^2 = 72 + 2 (k - 3)^2$

=> $k^2 = 12k + 90 = 0$
The equation has no real roots. Hence no circle is possible for h = 2 .
Hence only two circle are possible (k = 0)
$\Rightarrow x^2 + y^2 + 2(10 \pm 3\sqrt{6})x + 55 \pm 24\sqrt{6} = 0$

<u>Pair of tangents -</u>
$T^2 = ss_1$

<u>Equation of normal</u>-
The normal to a curve at any P of a curve is the straight line which passes through P and is perpendicular to the tangent at P .
The equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point $l(x_1,y_1)$ is :-
$y(x_1 + g) - x(y_1 + f) + fx_1 + gy_1 = 0$

<u>Why</u> :-
normal will be OP
slope OP = $\frac{y_1 + f}{x_1 + y}$

equation of normal -

$$y - y_1 = \frac{y_1 + f}{x_1 + y}(x - x_1)$$

on sloving we get,

$y(x_1 + y) - x(y_1 + f) + fx_1 - gy_1 = 0$

With respect to circle S = 0

Equation of chord of contact:-

T = 0

<u>Why</u> :-

Equation of circle S = 0

Whose, $S = x^2 + y^2 - a^2$

Equation of AP => $xx_2 + yy_2 - a^2 = 0$

Equation of BP => $xx_3 + yy_3 - a^2 = 0$

Both passes through P .

$\therefore x_1x_2 + y_1y_2 - a^2 = 0$ .....(1)

$x_1x_3 + y_1y_3 - a^2 = 0$ .....(2)

Now consider

$xx_1 + yy_1 - a^2 = 0$ .....(3)

from (1), (2) and (3) it is lvery that clear that A and B lies on (3)

$\therefore$ equation of AB is $\rightarrow xx_1 + yy_1 - a^2 = 0$

=> T = 0

<u>Equation of chord lhaving mid point (x_1, y_1)</u>

Only one such chord is possible

Equation of chord

$T - s_1 = 0$

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Slop of CP = - $\left(\frac{y_1 + 2}{x_1 + y}\right)$

Equation of CP =>

$(y - y_1) = -\left(\frac{y_1 + g}{x_1 + f}\right)(x - x_1)$

on solving we get,

$T = S_1$

$T - S_1 = 0$

<u>Illustration</u>- Find the co-ordinates of the point from which tangen are drawn to the circle $x^2 + y^2 - 6x - 4y + 3 = 0$ such that mid point of its chord of contact is (1, 1).

<u>Ans</u>- $S = x^2 + y^2 - 6x - 4y + 3$

$T_{AB} = xx_1 + yy_1 - 3(x + x_1) - 2(y_1 + y_1) + 3$

$(x_1 - 3)x + (y_1 - 2)y - 3x_1 - 2y_1 + 3 = 0$.....(1)

Equation of AB $\rightarrow T - S_1 = 0$

on solving

$2x + y = 3$ .....(2)

Comparing (1) and (2)

$$\frac{x_1 - 3}{2} = \frac{y_1 - 3}{1} = \frac{3x_1 + 2y_1 - 3}{3}$$

on solving, .

$x_1 = -1$, $y_1 = 0$

In previous illastration why .

$$\frac{x_1 - 3}{2} = \frac{y_1 - 3}{1} = \frac{3x_1 + 2y_1 - 3}{3}$$

Why not $x_1 - 3 = 2$

y $x_1 - 3 = 2$?

<u>Solution</u> - On comparing two equation -

$a_1 x + b_1 y + c_1 = 0$

$a_2 x + b_2 y + c_2 = 0$

If both the above equation are of some line, then we get :-

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

<u>Two circle touching each other</u> -

(a) <u>External tiuch</u>-

$c_1 c_2 = r_1 r_2$

Point P divides the line Joining c_1 & c_2 internally in th ratio

r_1; r_2

$$\frac{c_1 p}{c_2 p} = \frac{r_1}{r_2}$$

(b) <u>Internal touch</u>:-

$c_1 c_2 = |r_1 - r_2|$

point P divider the line Joining c_1 & c_2 externally in the ratio $r_1 : r_2$.

$$\frac{c_1 p}{c_2 p} = \frac{r_1}{r_2}$$

Conclition :-

$|(r_1 - r_2)| < c_1\ c_2 < r_1 - r_2$

(a)<u>One is completely inside other</u>-

<u>Conclition</u> :-

$c_1c_2 < r_1 - r_2$

<u>Illustration</u>- Examine if the two circle $x^2 + y^2 - 8y - 4 = 0$ & $x^2 + y^2 - 2x - 4y = 0$ touch each other find the point contact if they touch.

<u>Solution</u>- For $x^2 + y^2 - 2x - 4y = 0$ centre $c_1 \equiv (1, 2)$

& $x^2 + y^2 - 8x - 4 = 0$ centre $c_2 \equiv (0, 4)$

using $r = \sqrt{g^2+f^2-c}$ $r_1 = \sqrt{5}$ & $r_2 = 2\sqrt{5}$

Now $c_1c_2 = \sqrt{(0-1)^2 + (4-2)^2} = \sqrt{5}$

=> $r_2 - r_1 = 2\sqrt{5} - \sqrt{5} = \sqrt{5}$

For point of contact -

Let P(x, y) be the point of contact,

Pdivixles c_1c_2 externally in ratio of $\sqrt{5} : 2\sqrt{5}$ = 1 : 2
using section formula, we get,

$$x = \frac{1(0) - 2(1)}{1-2} = 2$$

$$y = \frac{1(4) - 2(2)}{1-2} = 0$$

P(x, y)≡(2, 0) is the point of contact .

15 and touching the circle $x^2 + y^2$ = at point (6, 8).

<u>solution</u>- <u>Case</u>: It the required circle touches $x^2 + y^2$
at (6, -8) centerally .

then P(6, -8) divider OA in the ratio 2 : 3 internally .
Let centre of the circle be (h, k). Now using section formula,

$$\Rightarrow \frac{2h-3(0)}{2-3} = 6$$

$$\Rightarrow \frac{2\,k-3\,(0)}{2-3} = -\,8$$

=> h = 15, & k = -20
=> $(x - 15)^2 + (y + 20)^2 = 225$ is the required circle .
<u>Case-2</u> If the required circle touches $x^2 + y^2 = 100$ at (6, -8)internally,
then P(6, - 8) divides oa in the ratio 2 : 3 exterternally .

$$\Rightarrow \frac{2h-3(0)}{2-3} = 6$$

$$\Rightarrow \frac{2k-3(0)}{2-3} = -8$$

=> h = - 3 and k = 4

=> $(x + 3d)^2 + (y - 4)^2 = 225$ is the required circle .

(1) If two circle neither intersect nor touch eache other then they have 4 common tangents.

DCT means direct common tangent

TCT means Transverse common tangent .

(2) If two circle touch each other extenally then there are 3 common tangents .

(3) If two circle intersect each other at 2 point then there are 2 common tangents.

tangent is possible .

(5) If two circle are concentric then there is no common tangent between them .

<u>Director circle</u> - The locus of meeting point of perpendicular two is also a circle concertric with the given circle is called director circle of the given circle .

Equation of given circle -

$(x - h)^2 + (y - k)^2 = r^2$

Equation of director circle-

$(x - h)^2 + (y - k)^2 = 2r^2$

Question- The line $x\cos\theta + y\sin\theta = a$ and $y\cos\theta - \sin\theta\, x = a$ meet at point (h, k) find the locus of point (h, k) for various values of θ.

Ans- Note that the line $x\cos\theta + y\sin\theta = a$ and $x\cos\theta + y\sin\theta = a$ are perpendicular to each othe and both are tangent to circle $x^2 + y^2 = a^2$.
So, the locus of point of intersection of the two kof tangents ie, (h, k) is director circle with equation -
(1)Equatio of family of circle passing through point of intersection of a circle (s = 0) and a line (L =0) is given by -

Illustration- Tangents PA and PB are drawn from the point P to circle $x^2 + y^2 = a^2$. find the equation of circum circle of ΔPAB and the area of ΔPAB.
sloution - AB is the chord of contact for point P.
Equation of AB is :-
$hx + ky = a^2$

The circum circle of ΔPAB passes through the intersection of circle.

$x^2 + y^2 - a^2 = 0$ and the line $hx + ky - a^2 = 0$.
using $S + KL = 0$ (family of circle)
we can write equation of circle:-
$(x^2 + y^2 - a^2 + k(hx + kg - a^2) = 0$ where k is paramete.
As this circle passes trough p(h, k);
$\Rightarrow h^2 + k^2 - a^2 + k(h^2 + k^2 - a^2) = 0$
$\Rightarrow k = -1$
The circle kis = $1/2 \times PM \times AB (PM \perp AB)$
PM = distance of p from AB

$$= \frac{|h^2 + k^2 - a^2|}{\sqrt{h^2 + k^2}}$$

PA = length of tangent from P = $\sqrt{h^2 + k^2 - a^2}$

area = $\frac{1}{2} PM \left[2\sqrt{PA^2 + PM^2}\right] = PM\sqrt{PA^2 - PM^2}$

area $\frac{a(h^2 + k^2 - a^2)^{3/2}}{h^2 + k^2}$

Note taht $h^2 + k^2 - a^2 > 0$
$\because$ (h, k) lies outside the circle .
(2) Equation of family of circle passing through points of intersection of two given circle -

$$S + \lambda L = 0$$

$$S + \lambda (s - s^1) = 0$$

$$S - \frac{\lambda}{\lambda + 1} S^1 = 0$$

$S + KS^1 = 0$

(3) Equation of family of circle through .
two fixed given points .

S and L can be find out as fallows →

$S = (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

(diametric form)

$$L \equiv \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0$$

$S + \lambda L = 0$

<u>Illustration</u>- Find the value of t so that the point (1, 1), (2, -1), (3, -2). and (12, t) are concyclic .

We will find the equatiion of the circle passing through A, B & C and then find t so that D lies on that circle .

Any circle passing through A, B can be taken as :-

$$(x - 1)(x - 2) + (y - 1)(y + 1) + K\begin{vmatrix} x & y & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow x^2 + y^2 - 3x + 1 + k(2x + y - 3) = 0$

$c \equiv (3, -2)$ lies on this circle .

$\Rightarrow 9 + 4 - 9 + 1 + k(6 - 2 - 3) = 0$

$\Rightarrow k = -5$

=> circle through A; B & C is :-

$x^2 + y^2 - 3x + 1 - 5(2x + y - 3) = 0$

$x^2 + y^2 - 13x - 5y + 16 = 0$

Point $D \equiv (12, t)$ .will lie on this circle if :-

$\Rightarrow 144 + t^2 - 156 - 5t + 16 = 0$

$\Rightarrow t^2 - 5t + 4 = 0$

$\Rightarrow t = 1, 4$

=> for t = 1, 4 the point are concyclic.

(4) Equation of family of circles through .
one point an a fixed given line-

$S = (x - x_1)^2 + (y - y_1)^2 = 0$

$(x - x_1)^2 + (y - y_1)^2 + \lambda L = 0$

<u>Illustration</u>- Find the equation of a circle touching the line $x + 2y = 1$ at the point (3, - 1) and passing through the point (2, 1).
the point (3, 1) can be talen as :-
$(x - 3)^2 + (y + 1)^2 + k(x + 2y - 1) = 0$ (using family of circle)
As the circle passes through (2, 1)
$(2 - 3)^2 + (1 + 1)^2 + k(2 + 2) = 0$
=> $k = - 5/3$
=> the required circle is :-
$3 (x^2 + y^2 - 23x - 4y + 35 = 0$

<u>Orthognal cut of two circle</u>-
If two circle are cutting each other orthogonally then the tangents at the point of intersection is perpendicular to each other :-

$(O_1o_2)^2 = r_1^2 + r_2^2$

$(g_1 - g_2)^2 + (f_1 - f_2)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_1$

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$

Illustration - The center of circle S .line on the line $2x - 2y + 9 = 0$ and S cuts at right anges the circle $x^2 + y^2 = 4$. Show that S passes through two fixed point and fond heir coordinates

Contre lies on $2x - 2y + 9 = 0$

$\Rightarrow - 2g + 2f + 9 = 0$ .....(1)

So cuts $x^2 + y^2 - 4 = 0$ orthogonally -

$\Rightarrow 2g (0) + 2f (0) = c - 4$

$\Rightarrow c = 4$ .....(2)

Using (1) and (2) the equation of S be come :-

$x^2 + y^2 + (2f + 9) x ++2fy + 4 = 0$

$\Rightarrow (x^2 + y^2 + 9x + 4) + f(2x + 2y) = 0$

We can compare this equation with the equation the family of circle through the point of interectin of a circle and a line (S + FL = 0, where f is a parameter).

Hence the circle S always pass through two fixed point A and B .

Which are the point of intersection of $x^2 + y^2 + 9x + 4 = 0$ and $2x + 2y = 0$

Solving these equation we get,

$x^2 + xy^2 + 9x + 4 = 0$

$\Rightarrow x = - 4, -1/2 \Rightarrow y = 4, 1/2$

$\Rightarrow A \equiv (- 4, 4)$ & $B \equiv (- 1/2, 1/2)$

Radical axes- Locus of a point which moves such mthat mthe tangents drawn from tjhis point oto the two given circle will be of equal in lingth .

$l_1 = l_2$

$$\frac{5}{5} = \frac{8}{2l} = \frac{-12}{-2m}$$

$$\text{or } 1 = \frac{4}{l} = \frac{6}{m}$$

$S_1 = S_1^1$
equation of radical axes $\rightarrow S_1 = S_1^1 = 0$

<u>Illustration</u>- Find the equation of radical axes of the circle $\rightarrow$
$S = x^2 + y^2 + 4x - 6y + 3 = 0$
$S^1 = 3x^2 + 3y^2 - 12x + 9y + 1 = 0$

<u>Ans</u>- $S_1 = S_1^1$

$\Rightarrow x_1^2 + y_1^2 + 4x_1 - 6y_1 + 3 = x_1^2 + y_1^2 - 4x_1 + 3y_1 + 1/3$

$\Rightarrow 24\, x_1 - 27y_1 + 8 = 0$
<u>Important Results</u>:-
(1)When circle are touching each other, the radical axes is the common tangent beween them.

axes is common chord between them :-

<u>Radical centre</u>- If there are thre circle (whose centre are not clooinear.) then there will be three radical axes. All these three radical axes are concurrent . And the point of concurrency is called radical centre.
* Radical axes are perpendical to kthe line Joining centre of the circles .

$L_{23} \rightarrow S_2 - S_3 = 0$

$L_{12} \rightarrow S_1 - S_2 = 0$

adding above three -
$L_{13} = -(L_{23} + L_{12})$
pair of lines.

<u>Co axal System of circle</u>-
A system of circle is said to be co axial when they have common radical axis ie, when radicaal axis of each pair of circles of system is same .

<u>Illustration</u>-
For what values of l and m the circle $5(x^2 + y^2) + ly - m = 0$ belongs to the coaxal system determined by pthe circle .
$x^2 + y^2 + 2x + 4y - 6 = 0$ and $2^2 + y^2) - x = 0$?
<u>Ans</u>- If the radical axis for each pair of the three given circle is the same them the result is established.
Let the circle be .
$S_1 = x^2 + y^2 + 2x + 4y - 6 = 0$
$S_2 = x^2 + y^2 - 1/2\ x = 0$
$S_3 = x^2 + y_2 + l/2y - m/5 = 0$
the equation of radical axis of circle $S_1 = 0$, $S_2 = 0$ is $S_1 - S_2 = 0$
ie, $x^2 + y^2 + 2x + 4y - 6\ (x^2 + y^2 - 1/2x) = 0$
or, $5/2x + 4y - 6 = 0$
or, $5x + 8y - 12 = 0$ .....(1)
the equation of the radical axis of circle $S_2 = 0$, $S_3 = 0$ is $S_2 - S_3 = 0$
ie, $x^2 + y^2 - 1/2x - (x^2 + y^2 + l/5y - m/5) = 0$
or, $5x + 2ly - 2m = 0$ .....(2)
(1) and (2) must be identical, so comparing them,

$$\frac{5}{5} = \frac{8}{2l} = \frac{-12}{-2m}$$

$$\text{or } 1 = \frac{4}{l} = \frac{6}{m}$$

$\therefore l = 4, m = 6$

(Q |) Find the equation of a circle which touches the line $x + y = 5$ at the point A(-2, 7) and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally .

(A |)

Since the circle is touching the line $x + y - 5 = 0$ at

(-2, 7), its equation can be written as

$(x + 2)^2 + (y - 7)^2 + K(x + y - 5) = 0$ (whose k b vatiable

$\Rightarrow x^2 + y^2 + x(K + 4) + y(K - 14) + 53 - 5K = 0$

It is orthogonal to the circle

$x^2 + y^2 + 4x - 6y + 9 = 0$

$\Rightarrow 2g_1 g_2 + 2f_1 f_2 = c_1 + c_2$

$\Rightarrow 2(K + 4) - 3(K - 14) = 53 - 5K + 9$

$4K = 12$

$K = 3$

$\therefore$ eqn of circle,

$x^2 + y^2 + 7x - 11y + 38 = 0$

DUNB QUESTION

(Q) Why in this question we used family of circles and not did it directly by taking an an arbitrary circle and applying the given information ?

(A) Whenever it is given that kthe the circle is fouduing a line L and the point of contact P is also given, then always as e the following pointly.

$(x - x_1)^2 + (y - y_1)^2 + K L = 0$ K is variable .

It makes the solution very easy as only one variable is left and we have used 2 conditions namely

(a) Line L is tangent to circle.

(b) Point of coutact lis P .
So only are mose in formation is needed to define the circle .
∵To define a circle we any three paramets
(Q2) Find the equation of circle passing through (- 2_1 3) and touching both the axes.
(A2)

As the circle touches both the axes and lies in second quadeant, its contre is ;
C≡(- r_1 r) where r is the radius.
Distance of contre from (- 2, 3) = eadius.

$$(5 \pm 2\sqrt{3})x - 2(5 \pm 2\sqrt{3})y$$
$$+(5 \pm 2\sqrt{3})^2 = 0$$

The circles we $(x + r)^2 + (y - r)^2 = r^2$

(Q3) If ; i =1,2,3,4 are four distinct points on a circle, show that m_1 m_2 m_3 m_4 = 1
Let the eqn of circle be
$x^2 + y^2 + 2gx + 2fy + c = 0$

Now since is on the circle,

$$\therefore \quad mi^2 + \frac{1}{mi}2 + \frac{2f}{mi} + 2gmi + c = 0$$

=> $mi^4 + 2g\, mi^3 + cmi^2 + 2f\, mi + 1 = 0$
The roots of this on are (m_1,m_2, m_3, m_4)

∴ Product of the roots m_1,m_2, m_3, m_4 = = 1
Hence prooved .
Dumb question

(Q) Why are the roots of the above equation m_1, m_2, m_3, m_4 and how is $m_1, m_2, m_3, m_4 = 18$

(A) Since $\left(m_1, \frac{1}{m_1}\right), \left(m2, \frac{1}{m_2}\right), \left(m3, \frac{1}{m_3}\right), \left(m4, \frac{1}{m_4}\right)$ individually satiefy the equation of circle, hence they all must be the solution of m : . And hence it is equation of degree 4. For a fore degree eqn $ax^4 + bx^3 + cx^2 + dx + e = 0$.

product of roots = e/a .

$\therefore m_1, m_2, m_3, m_4 = 1$.

(Q4) For what value of m, will the line y = mx does not intersect the circle

$x^2 + y^2 + 20x + 20y + 20 = 0$

(A4) If the line y = m, does not ointersect the circle, the perpendicular distance of the line from the contre must be greater than its radius.

Contre of the circle $\equiv$(- 10, - 10); radius 6r = 61.

distance of the line mx - y = 0 from (- 10, - 10) .

$$= \frac{|m(-10) - (-10)|}{\sqrt{m^2 + 1}}$$

$$= \frac{|10 - 10m|}{\sqrt{m^2 + 1}} > 6\sqrt{5}$$

$= (2m + 1)(m + 2) < 0$

$\Rightarrow -2 < m < -1/2$

(Q5) Secants are drawn from origin to the circle $(x - h)^2 + (y - k)^2 = r^2$. Find the locus of the mid - point of the portion of secants intercept inside the circle.

(A5)

Let C$\equiv(h_1\ k)$ be the centre of the given circle . and P$\equiv(x_1, y_1)$ be the mid point of the postion AB of the secant O AB.

$\Rightarrow CP \perp AB$

Slope COP) $\times$slope(CP) = - 1

$$\left(\frac{y_1 - 0}{x_1 - 0}\right) \times \left(\frac{y_1 - k}{x_1 - h}\right) = -1$$

=> $x_1^2 + y_1^2 - hx_1 - ky_1 = 0$

the locus of the poine P : $x^2 + y^2 - hx - ky = 0$

A circle has radius equal to 3 units and its centre lies the line y = x - 1 . Find the equation of the circle if it passes through (7, 3) .

<u>Ans</u> - Let the centre of the circle be (α, β)

It lies of the line y = x - 1

=> $\beta = \alpha - 1$. Hence the centre is $(\alpha, \alpha - 1)$

=> The equation of the circle is

$$(x - \alpha)^2 + (y - \alpha + 1)^2 = 9$$

It passes through (7, 3)

$$\Rightarrow (7 - \alpha)^2 + (4 - \alpha)^2 = 9$$

$$\Rightarrow \alpha^2 - 11\alpha + 28 = 0$$

$$\Rightarrow (\alpha - 4)(\alpha - 7) = 0$$

$$\Rightarrow \alpha = 4, 7$$

Hence the required equation are,

$x^2 + y^2 - 8x - 6y + 16 = 0$

and $x^2 + y^2 - 14x - 12y + 76 = 0$

<u>Que</u> - Prove that the circle $x^2 + y^2 - 6x - 4y + 9 = 0$ bisects the circumference of lthe circle.$x^2 + y^2 - 8x - 6y + 23 = 0$

<u>Ans</u> The given circles are

$S_1 \equiv x^2 + y^2 - 6x - 4y + 9 = 0$ .....(1)

and $S_2 x^2 + y^2 - 8x - 6x + 23 = 0$ .....(2)

Equation of the common chord of circle (1) and (2) which is also the radical axis of the circle S_1 and $_2$ is,

$S_1 - S_2 = 0$ or $2x + 2y - 14 = 0$

or $x + y - 7 = 0$ .....(3)

Centre if the circle S_2 is (4, 3). clearly, line (3) passes through h point (4, 3) and hence line (3) is equation of diameter of circle (2)

Hance circle (1) bisects the circumfrance of circle(2)

extesnally. Find the angle θ beween the direct common tangents it given that $r_1 > r_2$.

<u>Ans</u> - let us draw the diagram.

Cleatly $\theta = 2\alpha$

Now, $\sin\alpha = \dfrac{r_1 - r_2}{r_1 + r_2}$

$$\therefore \alpha = \sin^{-1}\frac{r_1 - r_2}{r_1 + r_2}$$

$$\therefore \theta = 2\alpha$$

$$= 2\sin^{-1}\left(\frac{r_1 - r_2}{r_1 + r_2}\right)$$

Q. Find th number of common tangents that can be drawn to circle $x^2 + y^2 - 4x - 6y - 3 = 0$
and $x^2 + y^2 + 2x + 2y + 1 = 0$?
<u>Ans</u>. The circle $x^2 + y^2 - 4x - 6y - 3 = 0$ has
Centre C_1 (2, 3) and radius $r_1 = 4$.
And the circle $x^2 + y^2 + 2x + 2y + 1 = 0$ has
Centre C_2 (-1, -1) and radius $r_2 = 1$

Now $C_1C_2 = \sqrt{(2 - (-1))^2 + (3 - (-1))^2}$
$= 5$
Also $r_1 + r_2 = 4 + 1 = 5$.
So, the two circle touch each other externally and thus 3 common tangents are possible.

Q. If two circle cut the third circle orthogonally, prove that their common chord will pass through centre of third circle.
<u>Ans</u>. Let us take equation of two circle as .

$$x^2 + y^2 + 2\lambda_1 x + a = 0 \quad .....(1)$$
$$x^2 + y^2 + 2\lambda_2 x + a = 0 \quad .....(2)$$

Let third circle be $x^2 + y^2 + 2fy + c = 0$ .....(3)
The circle (1) and (3) cut orthergonally .
So, $2\lambda_1 g = a + c$. .....(4)
$2\lambda_2 g = a + c$. .....(5)
from (4) and (5)
$2g(\lambda_1 - \lambda_2) = 0$
$\lambda_1 \neq \lambda_2$, as then circles would be same
So, g = 0
$\therefore$ centre of third is (0, -f)
The common chord of (1) and (2) has equation $S_1 - S_2$
i.e $x^2 + y^2 + 2\lambda_1 x + a - (x^2 + y^2 + 2\lambda_2 x + a) = 0$
=> $2(\lambda_1 - \lambda_2)x = 0$
or x = 0
So centre (0, - f) satisfies equation x = 0.
<u>Dumb Question</u>. Why the equation of circle were taken to be $x^2 + y^2 + 2\lambda i\, x + a = 0$? Where did the 'y' term went ?
<u>Ans</u>- Such equation of the circle can be taken if we chose the axis suitably .
The axis can be chosen as the nline joining the centre of two circles as x - axis and the point midway between the centre as origin.
And then we can obtain the equation as used in the question

(Q 1) Let AB be a chord of the circle $x^2 + y^2 = r^2$ substens aright angle at the centre θ. Show that the locu of the coutroid of the triangle PAB as P mo ves on the circle is a circle.

ΔODB is isosceles with OD = OB = x(say)
We may assume AB is parallel to and below x - axis .

$\therefore x^2 + x^2 = r^2 \Rightarrow x = r/\sqrt{2}$

$\therefore$ B is $\left(\frac{r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right)$ and t is $\left(\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right)$

Let P be $(r\cos\theta, r\sin\theta)$ and centroid of ΔPAB be G

$\therefore x_1 = \frac{r\cos\theta + r1\sqrt{2} - r\sqrt{2}}{3}, \quad y_1 = \frac{r\sin\theta - r1\sqrt{2} - r1\sqrt{2}}{3}$

=> 3 $x_1 = r\cos\theta, \quad 3y_1 = r(\sin\theta - \sqrt{2})$

Eliminating θ, we get

$$\left(\frac{3x_1}{r}\right)^2 + \left(\frac{3y_1}{r} + \sqrt{2}\right)^2 = 1$$

or $x_{12} + \left(y_1 + \frac{\sqrt{2}r}{3}\right)^2 = \frac{r^2}{9}$ $\therefore$ Locus of (x_1, y_1) is circle

(Q Dumb question)

Why was $x^2 = r^2/2$ implied

x = r 1 $\sqrt{2}$ and correspondingly

A $\equiv \left(\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right)$, B $\left(\frac{r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right)$.

(A) We have assumed OD as '+' ve y axis and OB '+' ve axis .

$\therefore$ x - co-ordinate of B is positive

$\therefore$ x = $\frac{+r}{\sqrt{2}}$ and its y co-ordinate (iv)quadrent is below y - axis axis

$\therefore$ B $\equiv \left(\frac{r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right)$

For A, x - co-ordinate on left of 0.

$\therefore$ x = $\frac{-r}{\sqrt{2}}$, and its y - co-cordinate is '-' ve (i.e. (iii) quadrant)

$\therefore A \equiv \left(\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right)$.

(Q 2) The centre of circle S on use line 2x - 2y + 9 = 0 and cuts at right angles the circle $x^2 + y^2 = 4$ Show that S passes through two fixed point and find their co-ordinates.

(A 2) Let the circle S be : $x^2 + y^2 + 2gx + 2fy + c =$

centre lies on 2x - 2y + 9 = 0.

=> -2g + 2f + 9 = 0 .....(1)

S cuts $x^2 + y^2 - 4 = 0$ orthogonelly .

2g (0) + 2f (0) = c - 4 .

c = 4 - .....(ii)

Using (i) and (ii) the equation of S becomes:

$x^2 + y^2 + (2f + a) x + 2fy + 4 = 0$

$(x^2 + y^2 + 9x + 4) + \& (2x + 2y) = 0$

=> The circle S always passes through 2 fixed point A and B which are point of intersection of $x^2 + y^2 + ax + 4 = 0$ and 2x + 2y = 0 (as the above eqn can be compared to family circle S + KL = 0, where K is parameter).

Solving these eqn we get

$x^2 + x^2 + 9x + 4 = 0$

x = - 4, -1/2

y = 4 1/2 $\therefore$ A ≡(-4, 4), B ≡(-1/2, 1/2)

Q3: The circle of $X^2+Y^2-4X-4Y+4=0$ is inscribed in atraingle has two sides long

the co-ordinates axes . the locus ofthe circumcenter of the traingle is

x + y - xy + k

find the value of k.

The given circle is $(x-2)^2+(y-2)^2=4$

=>centre=(2,2) and radius=2.
Let oab be the traingle in which the circle is inscribed. as $\triangle$OAB is right angled, the circumcentre is mid point of AB. Let p=(x_1,y_1) be the circumcentre.
A=(2x,0) & B(0,2y_1) THe eqn of AB is:,

$$\frac{x}{2x_1} + \frac{y}{2y_1} = 1$$

As $\triangle$AOB touches the circle, distance of AB=radius.

$$\frac{\left[\frac{2}{2x_1} + \frac{2}{2y1} - 1\right]}{\sqrt{\frac{1}{4x_1} + \frac{1}{4y_1^2}}} = 2 \ldots\ldots\ldots\ldots(1)$$

as the center (2,2) lies on the origin side of the line

$$\Rightarrow \frac{x}{2x_1} + \frac{2}{2y_1} - 1$$

$$\Rightarrow -\left(\frac{2}{2x_1} + \frac{2}{2y_1} - 1\right) = 2\sqrt{\frac{1}{4x_1^2} + \frac{1}{4y_1^2}}$$

$$\Rightarrow x_1 + y_1 - x_1y_1 + \sqrt{x_1^2 + y_1^2} = 0$$

k=1
dumb question
(q) why is $\left|\frac{x}{2x_1} + \frac{y}{2y_1} - 1\right|$ taken - $\left(\frac{x}{2x_1} + \frac{y}{2y_1} - 1\right)$ lies on the origin side of the line

$$\frac{x}{2x_1} + \frac{y}{2y_1} - 1 = 0$$

the expression $\frac{x}{2x_1} + \frac{y}{2y_1} - 1 = 0$ has the same sign as the constant term (-1) in the eqn-

$\frac{x}{2x_1} + \frac{y}{2y_1} - 1$ is negative

Q4: Two circles each of radius 5 units touch each other at (1,2).if the equation of their common tangent is 4x+3y=10, find the equation of the circles.

Equation of common tangent is 4x+3y=10. the 2 circles touch each other at (1,2)

. EQuation of family of circles touching a given line 4x+3y=10 at a given point (1,2) is

. $(x-1)^2+(y-2)^2+k(4x+3y-10)=0$,

$x2+y^2+(4k-2)x+(3k-4)y+5-10k=0$

$\frac{3k-4}{2}$ and radius $^2= g^2+f^2-c=(2k-1)^2+(\pm a\sqrt{1+m^2})^2-(5-10)$

as the radius of the reqd. circle is 5

$(2k-1)^2+\frac{3k-4}{2}-(5-10k)=5^2$

$k=\frac{20}{25} \Rightarrow k=\pm\frac{2}{\sqrt{5}}$

Put the values of k in 1 to get the Equation of reqd. circles

The reqd circles are

$\sqrt{5}(x2+y2)+(8-2\sqrt{5})x+(6-4\sqrt{5})y+5\sqrt{5-20}$

and $\sqrt{5}(x2+y2)+(8+2\sqrt{5})x-(6+4\sqrt{5})y+5\sqrt{5}+2$

P is a variabe point on the line y=4.Tangent are drawn t the circle $x^2+y^2=4$ from p to touch it at A and B. the parallelogram. PAQB is completed. Prove that the locus of the point $(y+4)(x^2+y^2)=2y^2$

p lies on (y=4), then AB is chord of contact with respect to circle

$x^2+y^2=4$ whose equation is
. $hx + 4y = 4$.....(1)
solving with circle we get

$$x^2 + \left(\frac{4 - hx}{4}\right)^2 = 4$$

or $x^2(16+h^2)-8hx-48=0$
above gives abscissas of the point A And b,

$$\therefore x_1+x_2= \frac{8h}{16+h^2}$$

Also the points A and B lie on(1)
$h(x_1+x_2)+ 4(y_1+y_2)=y+4$

$$\therefore 4(y_1+y_2)= 8 - \frac{8h^2}{16+h^2}$$

$$\therefore y_1+y_2= \frac{32}{16+h^2}$$

Now if the point q be α, β, then the figure PAQ being a paralelogram its diagnal bisect

$$\therefore x_1x_2=h+\alpha = \frac{y_1 - (-f)}{x_1 - (-g)} = \frac{y_1 + f}{x_1 + g} \text{.....(2)}$$

$$y_1+y_2=4+\beta = \frac{32}{16+h^2}$$

Now we have to eliminate the variable between(2) and (3) to find the locus of Q i.e, α, β

dividing $\frac{h+\alpha}{4+\beta} = \frac{h}{4}$ $\therefore 4\alpha = h\beta$

or $h= \frac{4\alpha}{\beta}$ put in (3) and we get

$$(4+\beta)\left(16 + \frac{16\alpha^2}{\beta}\right)=32$$

$\therefore$ locus is $(y+4(x^2+y^2)=2y^2$

Q11. find all common tangents to s1=x2+y2-2x-6y+9=0
S2=x2+y2+6x-2y-1

The centres and radii of the circles are $c_1(1,3)$ and

$r_1 = r_1 = \sqrt{1+9-9} = 1$

$c_2(-3,1)$ and $r_2 = -\sqrt{1+9-9} = 3$

<u>Transverse common tangents</u> are tangents drawn from the point p which divides c1c2 internally
in the ratio of radii 1:3.

<u>Direct common tanent</u> are tangents drawn from the point q which divides c1c2 externally in th ratio 1:3

co-ordinate of p are:

$$\left(\frac{1(-3)+3.1}{1+3}, \frac{1.1+3.3}{1+3}\right) = (0, \frac{5}{2})$$

co-ordinate of q are:

$$\left(\frac{1(-3)+3(1)}{1-3}, \frac{1.1+3.3}{1-3}\right) = (3,4)$$

Transverse tangents are the tangents through any line through (0,5/2) is

(y-5/2)=mx or mx-y+5/2=0

apply the usual condition of tangency to any of the circle,

$$\frac{m.1-3+5/2}{\sqrt{m^2+1}} = 1$$

$\Rightarrow$ m=-3/4,∞

=> eqn of transverse tangents

3x+4y-10=0,x=0

direct tangents are tangents drawn from the p and q(3,4). let the eqn be (y-4)=m(x-3)

proceeding in same way,

$$\frac{|3-4-m(1-3)|}{\sqrt{1+m^2}} = 1 \quad \Rightarrow m=0, \frac{4}{3}$$

hence eqns of direct common tangents are

y=4,4x-3y=0

(Dumb question)

How does P and Q divide c1c2 externally and internally in the ratio r1:r2 and why in first case m is $=\infty$ (for transverse tangents) why?

In $\triangle C_2MQ$ and C_1NQ

$\angle C_2mq = \angle c_1NQ = 90$

$\angle MQC_2 = \angle NQC_1$(vertically opposite)

$\therefore \Delta C_2M\theta = \Delta C_1N\theta$

Hence, $\frac{C_2\theta}{C_1\theta} = \frac{C_2M}{C_1N} = \frac{r_2}{r_1}$

Now,($m=\infty$),

$$\left(m-\frac{1}{2}\right)^2 = m^2+1$$

Eqn was $\Rightarrow -M-\frac{3}{4}=0$

But since there are 2 values of m as there have to be 2 transverse tangents(isolated is not possible)

hence eqn is (o)m^2-m-3/4=0

Now product of roots $=\infty$

$\therefore$ if one is real and finite other root is infinite

Find the interval of values of a for which t line y+x=0 bisects 2 chord drawn from point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle

$$2x^2 + 2y^2 - \left(1+\sqrt{2}a\right) x - \left(1-\sqrt{2}a\right) y = 0$$

let m,n= $=\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$

Equation of the circle reduces

$x^2+y^2-mx-ny=0$

let p(t,-t) be a point on the line y+x=0

Equatin of the chord passing through (t,-t) as mid -point

$$\frac{m}{2}(x+t) + \frac{-n}{2(y-t)} = t^2 + t^2 - mt + nt$$

since chord (1) also passes through (m,n) it shows satisfy the equation of chord i.e

$$xt - yt + -\frac{m}{2}(x+t) + \frac{-x}{2}(n-t) = t^2 + t^2 - mt + 4t^2 + m^2 + n^2 = 3t(m-n)$$

on substituting the values of m& n, we get

$$4t^2 - 3\sqrt{2}at + \left(1+2a^2\right) 12 = 0$$

Now if there exists 2 chords passing through (m,n) and are bisected by the line y+x=0 then the equation of(2) should have real and distinct roots. D>0 => $18a^2-16(1+2a^2)/2>0$

$a^2-4>0 \Rightarrow (a+2)(a-2)>0$

$a \in (-\infty, -2) \cup (2, \infty)$

hence values of a are $a \in (-\infty, -2) \cup (2, \infty)$

Q.3: the circle x2+y2 =1 cuts the x-axis at p and Q. another circle with center at Q and veriable redius intercept the first circle at R above x-axis and the line segment PQ at s. find the maximum of the triangle QSR.

Equation of circle I is $x^2 + y^2 = 1$. it acts x-axis at point p(1,0) and Q (-1,0) Let the radius of the variable circle be r. center of the variable circle is Q(-1,0). Let the radius of the variable circle is

$(x+1)^2+y^2=r^2$.....(2)

solving circle (1) and variable circle we get coordinate of r as

$$\left(\frac{r^2-2}{2}, \frac{r}{2}\sqrt{4-r^2}\right)$$

Area of the $\triangle$ $QSt = \frac{1}{2} \times QS \times RS = \frac{1}{2} \times r \times \frac{r}{2}\sqrt{4}$

to maximise the area of the traingle to maximise the square,

let $A(r) = \frac{1}{16} r^2 \left(4-r^2\right) = \frac{4r^4-r^6}{16}$

$$A'(r) = \frac{16r^3-6r^5}{16}$$

For A(r) to be maximum or minimum A'r=0

$$r = \sqrt{\frac{8}{3}}$$

see yourself A'' $r = \sqrt{\frac{8}{3}}$ <0.

Area is maximum for $r = \sqrt{\frac{8}{3}}$

Maximum area of $\triangle QRS = \frac{1}{2}\times\frac{1}{2}\times\frac{8}{3}\times\sqrt{\frac{4}{3}} = \frac{4}{3\sqrt{3}}$ 57unit

(Q.4) y = mx is a chord of the corcle of radius and whose diameter is along the axis of x. find the equation of the circle whose diameter is the chord and hence find the locus of center for all value of m.

Ans. The circle whose chordf is y = mx and center lies on x-axis will touch y-axis at origin. The equation of such circle is given by $(x-a)^2 +y^2=a^2 \Rightarrow x^2+y^2-2ax=0$.....(1)

furher family of circles passing through the intersection of circle (1) and the line y=mx is

$x^2+y^2-x(2a+km)+ky=0$

center of the circle is =(a + km/2.- k/2)

we require that the member of this family whose diameter is y=mx.

$-k/2=am+km^2/2$

=> $k=-2ma/(1+m^2)$ => center of the requird circle lies on y=mx

Put the value of k in (1) to get the eqn of circle

$$x^2+y^2-x\left(2a-\frac{2aM^2}{1+M^2}\right)-\frac{2amy}{1+M^2}=0$$

$(1+m^2(x^2+y^2)-2a(x+my)=0$.....(2)

let the coordinate of the point whose locus is reqd. be (x1,y1)

=>(x1,y1) is the centre of the circle (2)

$$(x_1,y_1)=\left(\frac{a}{1+m^2},\frac{ma}{1+m^2}\right)$$

$$x_1=\frac{a}{1+m^2}\text{------}(3)\quad \text{and } y_1=\frac{ma}{1+m^2}\text{--------}(4)$$

on squaring and adding (3) and (4) we get:

$$x_1^2 + y_1^2 = \frac{a^2}{1+m^2} \quad 1 + m^2 = \frac{a^2}{x_1^2+y_1^2}$$

substituting the value of $(1+m^2$in (3) to get:)

$x1^2+y1^2=ax1$

required locus: x,sup>2+y^2=ax Tangents PA and PB are drawn from point (h,k) to the circle $x^2+y^2=a^2$

find the equation of circumcircle of △PAB and the area of △PAB

AB is the chord of contact for point p. Equation of ab is $HX+ky=a^2$

the circumcircle of △PAB passes through the intersection of circle. $x^2+y^2-a^2=0$ and the line hx+ky-a=0

using s+kl=0, we can write the wequation of circle as: $(x^2+k^2-a^2)+k(h^2+k^2-a^2)=0$

=.k=-1

The circle is $x^2+y^2-hx-ky=0$

Area of △PAB= 1/2 PM X AB (PM is⊥ to AB)

PM= distance of p from AB

$$= \frac{\left|h^2+K^2-a^2\right|}{\sqrt{h^2+k^2}}$$

PA= length of tangent from p

$$\sqrt{h^2 + k^2 - a^2}$$

AREA= $\frac{1}{2}PM\left[2\sqrt{PA^2-PM^2}\right] = PM\sqrt{PA^2-PM^2}$

AREA= $\frac{\left|h^2+k^2-a^2\right|}{\sqrt{h^2K^2}} \frac{a\left(\sqrt{h^2+K^2-a^2}\right)}{\sqrt{h^2+K^2}}$

$$A=\frac{a\left(h^2+K^2-a^2\right)^{3/2}}{h^2+K^2}$$

Note that $h^2 + k^2 > a^2$ ∴ (h, k) lies outside the circle. 1 center

2 circle

3 Radius

4 Diameter

5 Semicircle

6 Chord

7 Tangent

8 Normal

9 Chord of contact

10. External touch

11. Internal touch

12. point of contact
13. Direct tangent
14. Transverse Tangent
15. Concentric

Parabola

Parabola is a curve of infinite extent and is not often obsened in real life as circles or shaight line are . However, the parabplaic curres are very important and have some vert good propertvies like the light raus reflected from parabolic surface passes through one foxed point. This property of parabola is luidely used in making lenses and in optics.
The brridges also take parabolic shape have been centuries lod. So, now let us start studying this interesting curre in more detail.
Parabola is locus of point which moves such that its distance from fixed point (focus) is always equat to its distance from fixed line (directrix) i.e. eccentricity, e = 1 .

<u>Note</u>: The distance from fixed point to the distance from fixed line is calle

<u>General second degree equation</u>: eccentncity.
The equation ax^2 by + 2hxy + 2gx + 2fy + c = 0

(1) Second degrel thrms make a perfict square. or $n^2 = ab$.
<u>Why ?</u>
Let the fixed point be (α, β). and fixed line ax + by + 1 = 0 Let variable point be (h,k).
Now,according to the definition of parabola.
the distance of (h,K) from ((α, β)) is same as distance from line ax + by + 1 = 0.

$$\sqrt{(h-\alpha)^2 + (\alpha+\beta)^2} = \left|\frac{ah + b\alpha + 1}{\sqrt{a^2+b^2}}\right|$$

$$(h-\alpha)^2 + (\alpha+\beta)^2 = \frac{(ah + b\alpha + 1)^2}{a^2+b^2}$$

So locus of $(h-\alpha)$ is $(x-\alpha)^2 + (y-\beta)^2 = \dfrac{(ax+by+1)^2}{a^2+b^2}$
On simplification, we get
$(a^2 + b^2)$ $(x-\alpha)^2 + ((y-\beta)^2)$=$(ax + by + 1)^2$
The expression reduces to
$b^2 x^2 + a^2 y^2$ - 2abxy + ...linear terms .... + conjtant=$(bx - ay)^2$ + ...linear terms + constant = 0.
So, second deqree terms make perfect squarc.

<u>Dumb Question</u>: Why is h^2= ab same as second term making a

perfect squarc ?

Ans The second degree terms in equation are

$ax^2 + by^2 + 2hxy$.

Now if $h^2 = ab$

then $h = \pm\sqrt{ab}$

$$\Rightarrow ax^2+by^2+2hxy = ax^2+by^2 \pm 2\sqrt{ab}$$

$$= (\sqrt{a}x \pm \sqrt{b}y)^2$$

So, the two thing are same .

(2) $$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$$

Illustration 1

Ide ntify the locus of point which move such that its distance from given point and line is equal ?

(i) (-3,7) is the point and x + 2y - 8 = 0 is given line .

Ans Now according to the question, let the point whose locus is to be determined be (x,y).

$$\therefore \sqrt{(x-(-3))^2 + (y-7)^2} = \left|\frac{x + 2y\ 8}{\sqrt{1^2+2^2}}\right| .$$

or, $$(x+3)^2 + (y-7)^2 = \frac{(x + 2y - 8)^2}{5} .$$

=>. 5 (x^2 + 9 + bx + y^2 + 49 -14y)

= $x^2 4y^2$ + 64 -16 x - 32y +4xy

= $4x^2 + y^2 - 4xy + 46x - 38y + 226 = 0$

Considening 2^0 term, $4x^2\ y^2 - 4xy = (2x - y)^2$ which is a perfect square.

Now consider

$$\Delta = \begin{vmatrix} 4 & -2 & 23 \\ -2 & 1 & -19 \\ 23 & -19 & 226 \end{vmatrix}$$

$$= -425 \neq 0$$

So, the locus is a parabola

(ii) (-6,7) is point and x + 2y - 8 = 0 is given line .

Ans. Note taht (- 6,7) satisfies the given line .
So, the locus is stratight line itself .
Standared Equation of parabola.

Important properties
(i) Vertex 0 (0,0)
(ii) Focus s(9,0)
(iii) Foot of directrix (-9,0)
(iv) Directrix x + a = 0
(v) Equation of catus rectum x = a
and length of Latus retum = 4a .
(vi) Axis y= 0.
(vii) Extremiofies of latus retum (9_1 2a) & (9_1-2a)

Note : Two parabolas are said to be equal if their length of latus rectii are equal
Types of parabola

x + a = 0 $y^2 = 4ax$, a > 0 $Y^2 = - 4ax$, a > 0

s

<u>Transformation of parabola</u>.

(a) $(y - R)^2 = 4a (x - h)$, $a > 0$

The vcetex will be (h , k), openitng of parabola will be on = ve side of axis, axis will be || to x axis and dire ctrox well be || to y axis.

(b) $(y - R^2) = 4a (x-h)$, $a < 0$

Same as above except, the opening of parabolanwill be on -ve side of x - axis .

s

(c) $(x - h^2) = 4a (y - yR)$, $a > 0$
The vertex will be at (h,K) opening of parabola will be on +ve side of yaxis , axis will be || t y - axid and directrix || to x - axis

(d) $(x - h^2) = 4a (y-K)$, $a < 0$
Same ad above except, lthe opening of parabola will be on -ve side of y-axis .

Illustration 2.
Find the vertex, axis, focus, and latus rectum of parabola
$4y^2 + 12x - x - 20y + 67 = 0$
Ans The equation can be written as
$y^2 - 5y = - 3x - 67/4$.
i.e. $(y-5/2^2) = - 3x - 64/2 + 25/4$.
$= - 3(x + 7/2)$
So, this a transformed parabola whose vertex is (-7/2, 5/2), the axis is $y = 5/2$.
The length of latus rectum = $|4a| = 3$
and $a = -3/4$.
So, focis is (-7/2 - 3/4 1 5/52)
= (-17/4 1 5/2).

Natations :
For standard parabola ($y^2 = 4ax$)
1) $S \equiv y^2 - 4ax$
2) $S_1 \equiv y_1^2 - 4ax_1$
3) $T \equiv yy_1 - 2a(x + x_1)$
4) $F \equiv (at, 2at)$

Position of a point (x_1,y_1) w. r. t $y^2 = 4ax$
If $S_1 > 0$ => Outside parabola.
$S_1 < 0$ => Inside parabola.
S_1 0 => On parabola.

Why ?
Suppose a point is outside parabola .

s So, PL > PM.
=> PL^2 PM^2
or $y_1^2 > y_2^2$
or $y_1^2 > 4ax_1$(∵ M lics oh parabola).
So, $S_1 > 0$ => Point is outside parabola .
Similarily when point is inside the parabola .
$S_1 < 0$.

<u>Dumb Question</u>:- What dose inside and outside the parabola mean in a curre like parabola which is not closed ?

<u>Ans</u> :- the wird "outside" refers to the region from where tangent can be drawn . On other hand, the region from where tangent cannot be drawn is fefference as "in side" the parabola.

<u>Par ame tric form</u>.
$x = at^2$, $y = 2at$ where t is a parameter represents the parametric form.
$(at^2, 2at)$ is general point on parabola $y^2 = 4ax$.

<u>Illustration 2</u>.
Find the equations of the parabola if the extremeties of its latus ractum are (3,5) and (3,7).
<u>Ans</u>.
Now the length of latus ractuin is
$$\sqrt{(3-3)^2+(7-5)^2} = 2.$$
So, 4a = 2 or a = 1/2 .
Now middle point of catus ractum is
(3+3/2, 5+7/2) = (3,6) which is focus of parabola.
So the two cases as shown in figure below are possible.

Since the vertex is at a distance a away from the parabola the vertices are (7/2, 6) and (5/2,6).
Now lBy transformation of parabolas the two parabolas possible are
$(y-6)^2 = 4(1/2)(x-5/2)$
& $(y-6)^2 = 4(1/2)(x-7/2)$.
So, equation of parabolas are
$(y-6)^2 = 2(x-5/2)$
& $(y-6)^2 = 2(x-7/2)$
<u>Dumb Question</u> . Why two aprabolas are possible ?

<u>Ans</u> In this question only lasus ractum is given and two parabola are possible having same latus ractum on either side as shown in figure

<u>Tangents to a parabola</u> :
Let us find equation of tangents to parabola $y^2 = 4ax$

(i)<u>Equation of tangent of slope m</u> : -
$y = mx + a/m$
And, the point of contact is $p = (a/m^2, 2a/m)$

<u>Why</u> ?
Let line $y = mx + c$. be tangent to parabola $y^2 = 4ax$.
Saluing the two curres tom gether.
We get, $(mx + c)^2 = 4ax$.
Since the line is tangent, this equation will have repeated roots.
So, D = O.
Now, $(mx)^2 + (2mc - 4a)x + c^2 = 0$
is equation.
$D = 4(mc-2a)^2 - 4m^2c^2 = 0$
=, $(mc - 2a)^2 - m^2c^2 = 0$
=, $(mc - 2a - mc)(mc - 2a + mc) = 0$

=| - 4a (mc - a) = 0
=, c = a/m

(ii) Equation of tangent at (x_1,y_1)
$yy_1 = 2a (x+x_1)$

<u>Why</u>?
Now $y^2 = 4ax$.
so, $dy/dx = 2a/y$
$\therefore \left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ = is slope of tangent at (x_1,y_1)
So, $(\because y_1^2 = 4ax_1)$
Put m - 2a / y_1 in equation y = mx + a/m
=> $y = 2a/y_1\, x + ay_1/2a$
=> $yy_1 = 2ax + 4ax_1/2$ $(\because y_1^2 = 4ax_1)$
$\therefore yy_1 = 2a(x+x_1)$

(iii) Equation of tangent at $(at^2, 2at)$
$ty = x + at^2$

<u>Why</u> ?
Put $x_1 = at^2$| $y_1 = 2at$ in
Equation $yy_1 = 2a (x + x1)$
=> $(2at)\, ly = 2a (x + at)^2$
= $ty = x + at^2$

<u>Illustation 4</u>.
Prove that line $x + my + am^2 = 0$ touches the parabola $y^2 = 4ax$. Also find or dinates of point of contact .

<u>Ans</u>:- Solving, parabola $y^2 = 4ax$ with line
$x + my + am^2 = 0$, we get
y^2 4a $(-my - am^2)$
or. $y^2 + 4amy + 4a^2m^2 = 0$
or. $(y + 2am)^2 = 0$.
Since above is a perfect square, therefore both the values of y are equal. Hence given line is tangent to parabola and ordinates of the point to contact is - 2am (from y - 2am = 0)
Ptting, value of m in equation of the line we get $x = am^2$.
Hence point of contact is $(am^2, - 2am)$

Illustration 5

Find equation of straught lines touching both $x^2 + y^2 = 2a^2$ and $y^2 = 8a$

Ans:- the parabola is $y^2 = 8ax$

or $y^2 = 4\,(2a)\,x$.

So, equation of tangent is

$y = mx + 2a/m$

or. $m^2x - lmy + 2a = 0$

Since this is tan gent to $x^2 + y^2 = 2a^2$, So the length of perpendicular from (0,0) must be equal to the radius ie $\sqrt{2}a$

$(\because m^2 + 2^1 0)$

or $\dfrac{4a^2}{m^4 m^2} = 2a^2$

=> $m^4 + m^2 - 2 = 0$

=> $(m^2+2)(m^2-1) = 0$

=> $m^2 - 1 = 0$ $(\because m^2 + 2 \neq 0)$

=> $m = \pm 1$

So, the equired tangent is

$x \pm y + 2a = 0.$

Equation of pair of tangents.

The equation of pair of tangent from (x_1, y_1) is

$t^2 = ss_1$

or. $(yy_1 - 2a(x + x_1))^2 = (y^2 - 4ax)(y_1^2 - 4ax^1)$

Why ?

Let $p(x_1 1\ y_1)$ be point from which tangents is drawin to $y^2 = 4ax$

Let $m(h_1R)$ be any point on either of tangent of straight line joining p and M is

$$y - y_1 = \left(\frac{K - y_1}{h - x_1}\right)(x - x_1) \quad \text{i.e.}$$

$$y = \left(\frac{K - y_1}{h - x_1}\right) x + \left(\frac{hy_1 - Kx_1}{h - x_1}\right) \text{--(2)}$$

As this is tangent to given parabola, it should be of the form.
y = mx + a/m .-------(3)
Comparing (2) and (3) we get

$$m = \frac{K - y_1}{h - x_1} \text{-----(4)}$$

$$\text{and } \frac{a}{m} = \frac{hy_1 - Kx_1}{h - x_1} \text{----(5)}$$

Eliminating M from equation(4) and (5) we get .
=> $a(h - x_1)^2 = (K - y_1)(hy_1 - K x_1)$
Putting $(x_1 y_1)$ in place of $(h_1 K)$ gibes equation of locus of M. as
$a(x - x_1)_2 = (y - y_1)(xy_1 - yx_1)$
Which on rearranging gives .

$$\{yy_1 - 2a(x + x_1)\}^2 = (y^2 - 4ax)(y_1^2 - 4ax)$$

<u>Illustration 6</u>.
Find the equation of tangents drawn to $y^2 + 12x = 0$from point (3,8).

<u>Ans</u> Now, $y^2 = - 12x$
or 4a = - 12
=> a = - 3
On . Comparing with standard form of tangent to parabola should be
y = mx + a/m or y = mx -3/m
Since tangent passes through (3,8) we ha
8 = 3m -3/m
or. $3mm^2 - 8m - 3 = 0$
or (m - 3) (3m+1) = 0
=> m = 3, -1/3
Hence there $_L$ two tangents tnrough point (3,8)which are
3x - y -1 = 0
& x + 3y -27 = 0

<u>Dumb Question</u> Why tangent was taken in the form y = mx + a/m .?

<u>Ans</u> Whenever we have to take tangent from a point lying outside the parabola it is preferabee to take equation in this form as it is left

second by satistying given opoint in the equation.

Normal to a parabola.
(1) Equation of normal at point (t)
$y = -tx + 2at + at^3$

Why?
Slope of normal at $(at^2, 2at)$ $= \dfrac{-1}{\text{slope of tangent at } (at^2, 2t)} = \dfrac{-1}{1/t} = -t$.
$\therefore$ Equation of normal at $(at^2, 2at)$ is $y - 2at = -t(x - at^2)$
or $y = -tx + 2at + at^3$

(2) Equation of normal slope m
Putting $m = -t$ in above equation
$y = mx - 2am - am^3$ (3) Equation of normal at (x_1, y_1)
$y - y_1 = -y_1/2a\,(x - x_1)$

Some important point .
(1) Maximum 3 normals can be drawn from anyn piont (h_1K) to parabola $y^2 = 4ax$.

Why?
Equation of normal is
$y = -tx + 2at + at^3$
So, 3 different value of t are possible when we put (h_1 K) in given

eqquation
i.r $K = - t(n) + 2at + at^3$
Hence, 3different normals are possing .
(2) Three distirnt normals can be drawn from point (h_1 K) to parabola $y_2$4qx iff 4 > 2a .

<u>Illustration 7</u>.
Three normal to $y^2 = 4x$ pass through point (15,12) Show that one the normal is given by y =x - -3 and find equation of the others.

<u>Ans</u>:- $y^2 = 4x$.
=> 4a =4 ie a = 1.
Any normal to parabola is
$y = mx - 2am - am^3$
putting a= 1 , we get
$y = mx - 2am - am^3$
As it passses through (15,12) we get
$12 = 15m - 2m - m^3$
or $m^3 - 13m + 12 = 0$
or $(m - 1) (m + 4) (m - 3) = 0$
$\therefore m = 1, 3; - 4.$
Hence three naormals are
y = x - 3
y = -4x + 72
y = 3x - 33.
<u>Dumb Question</u>. How was $m^3 - 13m + 12 = 0$
factorized into $(m - 1) (m + 4)(m - 3) = 0$. ?

<u>Ans</u>:- It was given in the question kthat it is to lbe shown that y = x - 3 is nirmal .
So, m = 1 should be factor of $m^3 - 13m + 12 = 0$ which it is.
Hence $m^3 - 13 + 12 = 0$ is written as
$(m -1) (m^2 + m - 12) = 0$
or $(m -1) (m + 4) (m - 3) = 0$
<u>Chord of contact</u> .

Equation of chord of contact: -
T≡ 0
or $yy_1 - 2a (x + x) = 0$.

<u>Why</u> ?
Let point $p(x_1, y_1)$ be
A (p_1q) and B (h_1K) on the parabola.
Equation of PB is lKy - 2a(x + h) = 0
Both these stangents pass through $p(x + x_1)$
$\therefore 9y_1 = 2a (x_1 + p)$
and Ky_1= 2a (x_1 + h)
Now consiedr equation $yy_1 - 2a (x + x_1) = 0$
This equation is of first degree in x & y and it represcnts straight lline passing through
A (p_1q) and B (h_1K)
Hence, equation yy_1- 2a $(x + x_1) = 0$
represents equation of line A B which is called chord of contact of point ($x_1$1y $_1$)

<u>Illustration 8</u>.
From point (- 1, 2) tangent line are drawn to parabola $y^2 = 4ax$. find equation of hord of contact . Also find area of triangle formed by hoed of contac and the tangents.

<u>Ans</u>. Chord of lcontact of 0(-1,2) is
$yy_1 = 2a (x + x_1)$
or y = x - 1
Solving with parabola y = ^{2} 4x we get the points

p $(3+2\sqrt{2},\ 2+2\sqrt{2})$

θ $(3-2\sqrt{2},\ 2-2\sqrt{2})$

$\therefore\ PCl=\sqrt{(3+2\sqrt{2})-(3-2\sqrt{2})^2+(2+2\sqrt{2-(2-2\sqrt{2})}}$

$=\sqrt{(4\sqrt{2})^2+(4\sqrt{2})^2}$

$=\sqrt{64}\ =8$

The ⊥ distance of point 0(-1, 2)from line y = x -1 (Chord of contact) is

$\left|\dfrac{-1-2-1}{\sqrt{1+1}}\right| = 2\sqrt{2}$.

∴ the are a of Δ is

$\dfrac{1}{2}$ (PCl). $2\sqrt{2}$

$= 8\sqrt{2}$.

<u>Chord of mid point (x_1,y_1)</u>

$T \equiv S_1$

or $yy_1 - 2a\,(x+x_1) = y_1^2 - 4ax_1$

Note: Only one such chord is possible .

Why?
Any line through point (x_1, y_1) is
$(y - y_1) = m(x - x_1$ ----(1)
Now wehave to derermine value of m_1 its slope.
If it meets parabolain $P(at_1, 2at_1)$and
$\theta(at_2^2, 2at_2)$ then its equation is
$y(t_1 + t_2) - 2x - 2at_1t_2 = 0$ ----(2)
So, by companing slopes in (1) and (2) we get
Since (x_1,y_1) is mid point of $P\ \theta$.

$$y_1 = \frac{2at_1 + 2at_2}{2}$$

$$\text{or } t_1 + t2_2 = \frac{y_1}{a} \qquad \therefore\ m = \frac{2}{t_1 + t_2} = \frac{2a}{y_1}$$

$$m = \frac{2a}{y_1} \text{ in (1) we get}$$

$$y - y_1 = \frac{2a}{y_1}\ (x - x_1)$$

Putting
On reamanging, we get
$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$
or $T = S_1$

Eauation of chord.

(i) Equation of chord joing $(x_1),y_1)$ and (x_2, y_2) is
$y(y_1 + y_2) = 4ax + y_1y_2$

(ii)Equation of chard joing $(at_1^2, 2at_2)$ and$(at_1^2, 2at_2)$ is
$y(t_1 + t_2) = 2 (x + at_1 t_2)$

Note that both these results can be obtained by finding the middle point of the line joinign the two points and then using chord with mid point equation.

Illustration 9.
Show that locus of mid point of any focal chord is $y^2 = 2ax - 2a^2$

Ans Let the mid point be (h_1K)

So, the equation of this kchord is $T = S_1$
i.e yK - 2a (x+h) = K^2 - 4ah.
Now this chord is focal chord, si it must pass through (9,0).
=> (9,0) must satisty
yk -2a (x+h) = K^2 - 4aK
or . 0 - 2a(a + h) = k^2 - 4ah
or k^2 - 2ah + $2a^2$ = 0
∴ Locus is $y^2$2ax + $2a^2$ = 0
or y^2 = 2ax - $2a^2$.

<u>Diamcter</u>
Thelocus of middle point of a system of || chords of a parabola is called a diamcter .
If y = mx + c represents a system of | | chord of parabola y^2 = 4ax, then likne y = 2a/m is equation of diamcter and this will meet parabola at $\left(\frac{a}{m_2}, \frac{2a}{m}\right)$
<u>Why?</u>
The chords y = mx + c where c varies infersed the parabola y^2 = 4ax at point y_1 and y_1
Now, y' and y" are roots of equation

$$y^2 - \frac{4a}{m} y + \frac{4ac}{m} = 0$$

Which is obtained by solving y = mx + c
With y^2= 4ax .
∴ Equation is $y^2 - \frac{4a}{m} y + \frac{4ac}{m} = 0$ Let the middle point of the chords be (h,k)

$$K = \frac{y' + y''}{2}$$

$$= \frac{4a/m}{2} \text{ (from } \theta)$$

So, $= 2a/m$
∴ The equation of diametcr is y = 2a/m
<u>llustration 10</u>.
The general equation to a system of parrallel chords in parabola $y^2 = \frac{25}{7} x$ is 4x - y + K = 0 . What is equation of corresponding

diamcter ?

Ans Equation of parabola is $y^2 = \frac{25}{7}x$

$\therefore \quad 4a = \frac{25}{7}$

or $2a = \frac{25}{14.}$

And the equation of system of | | chords
is 4x = y + K = y .
∴ m = 4 .

$$y = \frac{2a}{m}$$
$$= \frac{25}{14\times4}$$
$$= \frac{25}{56}$$

So, diameetcr
or 56y = 25.

Some Important and useful results
1. Focal distance of a point $(at^2,2at)$ is $a + at^2$
Why?

Now focal distance = Pf = PB = $a + at^2$
= a +x,sub>1 (if (s_1, y_1) is point)

2. Foot of perperdicilar drawn from focus of parabola to any tangent will lie on tangent at vertex.
Why?
Equation of tangent is y = mx + a/m ---(1)

Equation of normal is $\frac{y - 0}{x - a} = \frac{-1}{m}$

or my + x = a ----(2)

Solving (1) & (2) we get

x = 0, y =a/m .

(O,a/m) is the point and clearly it lies or y - axis

3. The circle descvibed on any focal chord as diamcter touches the directrix of the parabola .

OR.

Any two tangents drawn from point on the directrix to parabola are perpendicular to each other and the chord of contact is focal chord .

4.) If any focal chord meets parabola at $t_1$and t_2.

then $t_1t_2 = - 1$.

<u>Why</u> ?

Slope of AC = slope of AB

So, $\frac{2at_1}{a-at_1^2} = \frac{2at_2}{a-at_2^2}$

$\Rightarrow t_1(1 - t_2^2) = t_2(1 - t_1^2)$

$t_1 - t_1t_2(t_2) = t_2 - t_1t_2(t_1)$

$(t_1 - t_2) + t_1t_2(t_1 - t_2) = 0$

$(t_1 - t_2)(1 + t_1t_2) = 0$

$\Rightarrow t_1t_2 = -1.$

<u>Dumb Question</u> Why $t_1 - t_2 \neq 0$?

<u>Ans</u> If $t_1 - t_2 = 0 \Rightarrow t_1 = t_2$

So, this will mean that point B and C are same point which is not true .

Hence $t_1 - t_2 \neq 0$

5). If any chord joining t_1 and t_2 subtends right angle at vertex that $t_1t_2 = -4$.

<u>Why</u>?

Slope of VA $\times$ slope of VB = - 1 .

$\frac{2at_1}{at_1^2} \times \frac{2at_2}{at_2^2} = -1$

$t_1t_2 = -4$.

6). Normal at t meets parabola at point (- t -2/t)
<u>Why</u> ?
Normal at t is ly = - tx + 2at + at^3
It meets parabola at t^1
$\therefore$ t^1 will also satisfy the equation

$2ar^1 = - t (at'^2) + 2at + at^3$

or $t(t'^2 - t^2) + 2 (t'-t) = 0$

or $(t'-t) (t(t' + t) + 2) = 0$

=> t' = t or t' = - t -2/t

But t' $\neq$ t as lpoint is different . So, t' = - t-2/t

7) If two normals at point t_1 and t_2 meet again on parabola then t_1t_2 = 2 .
<u>Why</u> ?

Now $t_3 = -t_1 \frac{-2}{t_1}$ (from previous result) and also

$$t_3 = -t_2 \frac{-2}{t_2}$$

So, $-t_1 \frac{-2}{t_2} = -t_2 \frac{-2}{t_2}$

=> $t_1 t_2 = 2$

8) Point of intersection of tangents at t_1 and t_2 is $(at_1 t_2, a(t_1 = t_2))$.

9) Point of intersection of normal drawn at t_1 and t_2 is
$[2a + a(t_1^2 + t_2^2 + t_1 t_2), -at_1 t_2(t_1 + t_2)]$.

(10) Reflection property of parabola. the tangent (pT) and normal (PN) ofparabola $y^2=4ax$ at p are the internal and external bisectors

of∠ SPM and Bp is to axis of parabola and ∠BPN=∠SPN
diagram 25

<u>Illustration</u>:
A ray of light coming along line y=b from the positive direction of x-axis strikes a concave mirror whose intersection with x-y plane is a parabola y^2= 4ax. Find equation of reflected ray and show it passes through focus of parabola
. both a and b are positive.
Ans:- given parabola is y^2=4ax.
Equation of tangent at
diagram26 $P(\frac{b^2}{4A}, b)$ is $yb=2a(x+\frac{b^2}{4a})$

slope of tangent= $\frac{2a}{b}$

hence, slope of normal $= \frac{-b}{2a} = \tan(180^\circ - \alpha)$

$\therefore \tan\alpha = \frac{b}{2a}$

slope of reflected ray
=tan(180-2α)
=-tan 2α

$$= \frac{-2\tan\alpha}{1-\tan^2\alpha} = \frac{-2.\frac{b}{2a}}{1-(\frac{b}{4a})^2} = \frac{-4ab}{4a^2-b^2}$$

=(y-b)($4a^2$-b^2)=-(4ax-b^2)b
which clearly passes through focus s(a,0)

<u>Easy</u> :-
E_1 show that the line xcos α+ysinα=p
touches the parabola y^2
=4ax if p cosα+a $\sin^2\alpha$=0
and that the point of contact is (a$\tan^2\alpha$,-2atanα)

<u>Solution</u>: The given line is
xcos α+ysinα=p
or y=-xcotα+pcosecα
$\therefore$ m=-cotα and c=p cosecα
since the given line touches the parabola

$\therefore$ c=$\frac{a}{m}$
or cm=a
or(pcosecα)(-cotα=a
and point of contact is

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

$$i.e, \left(\frac{a}{\cot 2\alpha}, -\frac{2a}{\cot \alpha}\right)$$

$$or\ (a\tan^2\alpha, -2a\tan\alpha)$$

E-2 show that normal to the parabola y_2=8x at the point(2,t) meets it again t (18,-12). find also the length of the normal chord.
solution-

s comparing the given parabola(i.e, y^2=8x) with =4ax
∴ 4a=8
∴ a=2
since normal at (x1,y1) to the
parabola y^2=4ax is

y-y1= $= \frac{y_1}{2a}(x - x_1)$

Here x1=2 and y1=4
∴ equation of normal is,

y-4= $= \frac{4}{4}(x - z)$

= y-4=-x+z
=y+x-6=0.....(1)
diagram solving (1) and y2 =8x
y2=8(6-y)
=y^2+8y-48=0
(y+12)(y-4)=0
∴ y=-12 and x=2
hence point of intersection of normal and parabola are(18,-12) and(2,4) therefore
normal meets the parabola at(18,-12)

and length of normal chord is distance between their points
$= pQ = \sqrt{(18-2)2+(-12-4)4} = 16\sqrt{2}$

<u>E-3</u> IF Three distinct and real normals can be drawn to $y^2=8x$ from the point (a,0) then show that a>4.

ans:- Equation of normal in terms of m is $y=mx-4m-2m^3=0$
it passes through (a,0) then $am-4m-2m^3=0$
m(a-4-2m2)=0

$= m=0, m2= \frac{a-4}{2}$

for three distinct normal , (a-4)>0
=a>4

E-4 if y+b=m,(x+a) and y+b=m(x+a) are two tangents to the parabola $y^2$4ax, then show that m1m2=-1
which lies on the directrix x+a=0. hence th etwo tangents intersect on directrix which we know is the locus of perpendicular tangents .
hence m1.m2=-1

E-5 show that the parametric representation $(2+t_2,2t+1)$ represents a parabola with vertex at (2,1).

Ans:- $x=2+t^2$, y=2t+1.
Eliminating t= we get $(y-1)^2=4(x-2)$
i.e, a parabola with vertex at (2,1).

Ans: Equation of the given parabola can be written as,
$9x^2+12x+4+18y-18=0$
i.e, $(3x+2)^2=18(y-1)$

$i.e, (x+\frac{2}{3})^2 = -2(y-1)$.....(1)

Equation of tangent to the above parabola can be written as-

$(x+\frac{2}{3}) = m(y-1) - \frac{1}{2m} \quad [\because 4a=-2, \therefore a=-\frac{1}{2}]$.....(2)

IF the tangent passes through (0,1) then we have,
$0.m^2 -4m-3=0$

gives,m=- $m = -\frac{3}{4}, \infty$

hence equation of the required lines are,

$$(x+\frac{2}{3})=-\frac{3}{4}(y-1)$$

$$(x+\frac{2}{3})=\infty(y-1)$$

i.e, 12x+9y-1=0
and y-1=0

E-8: the tangents to the parabola y^2 =4ax at p(at^2,2at) and q(at_2^2,2at^2) intersect at r. prove that the area of the triangle

PQR is $\frac{1}{2}A^2\left|(T_1-T_2)\right|$

Ans:

equation of tangents at p(at^2,2at1) and Q(at_2^2,2at_2)
are, $t_1y=x+at1^2$.....(1)
and $t_2y=x+at_2^2$
since point of intersection of (1) and (2) is r(at_1t_2) (at_1t_2,a(t_1+t_2)

E-6 the normals with slopes m1, m2, and m3 are drawn from apoint p not on the axis of the parabola y^2
=4ax ifv m1 and m2=α results in the locus of p being a part of parabola, find the value of α
Ans:

Any normal of the parabola $y^2=4x$ with slope m is
$y=mz-2m-m^3$
thus, m1.m2.m3=-k
α m_3=-k($\therefore$ m1m2=α
$$= m_3 = -\frac{k}{\alpha}$$
$\therefore$ m_3 is aroot of(1) then,
$$-\frac{k^3}{\alpha^3}+(2-h)(-\frac{k}{\alpha})+k=0$$
=k3+(2-h)kα^2-kα^3=0
$\therefore$ locus of p(h,k) is,
y^3+(2-x)yα^2-yα^3=0
(p does not lie on the axis of the parabola)
=y^2=α^2x-2α^2+α^3
it is a part of the parabola y^2=4x
then α^2=4
and -2α^2+α^3=0
=α-2=0
$\therefore$ α=2

E-7: find the equation of aline which touches the parabola
$9x^2+12x+18y-14=0$
and passes through the point (0,1).

<u>AREA OF TRAINGLE</u>:
$$PQR = \frac{1}{2} i \begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix} .$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \frac{1}{2} i \begin{vmatrix} at_1^2 & 2at_1 & 1 \\ a\left(t_2^2 - t_1^2\right) & a(t_2 - t_1) & 0 \\ at_1(t_2 - t_1) & a(t_2 - t_1) & 0 \end{vmatrix} 1$$

Expanding with respect to first row-

$$= \frac{1}{2} i \begin{vmatrix} a\left(t_2^2 - t_2^2\right) & 2a(t_2 - t_1) \\ a(t_2 - t_1) & a(t_2 - t_1) \end{vmatrix} 1$$

$$= \frac{1}{2} a^2 (t_2 - t_1)^2 i \begin{vmatrix} t_2 + t_1 & 2 \\ t_1 & 1 \end{vmatrix} 1$$

$$= \frac{1}{2} a^2 (t_2 - t_1)^2 \left|(t_2 - t_1)\right|$$

$$= \frac{1}{2} a^2 \left|(t_2 - t_1)^3\right|$$

.

E-9: Find the length of the normal chord to the parabola $y^2=4x$ which substends a right angle
at the vertex Ans:- a=1 for parabola PQ being normal chord.

$$\therefore \; t_2 = -t_1 - \frac{2}{t_1} \quad \ldots\ldots\ldots\ldots \; (1)$$

PQ substend a right angle at vertex,

$\therefore t_1 t_2 = -4$

$\text{or, } t_1\left(-t_1 - \frac{2}{t_1}\right) = -4$

$\text{or, } -t_1^2 = -4$

$\text{or , } t_1 = \sqrt{2}$

$\therefore t_2 = \frac{-4}{t} = -2\sqrt{2}$

$\therefore \text{ P is } \left(at_1^2, 2at_1\right) = \left(2, 2\sqrt{2}\right)$

$\text{Q } \left(at_1^2, 2at_1\right) = \left(8, -4\sqrt{2}\right)$

$\therefore PQ^2 = (36 + 36 \times 2)$

$\therefore PQ = 6\sqrt{3}$

E - 10. An equilateral triangle SAB is inscribed in the parabola y2 = 4ax Having its focus at 'S'.
It chord AB lies towards the left of S. Then find the side length of this triangle.
<u>Ans</u>:

let $A=(at_1^2,2at_1)$,$B=(at_2^2-2at_1)$
we have

$$M_{as} = \tan\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \frac{2at_1}{at_1^2 - a} = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow t_1^2 + 2\sqrt{3}t_1 = -1 = 0$$

$$\Rightarrow t_i = -\sqrt{3} \pm 2$$

clearly $t_1 = -\sqrt{3} - 2$ is rejected

This $t_1 = \left(2 - \sqrt{3}\right)$. Hence AB $= 4at_1 = 4a\left(2 - \sqrt{3}\right)$.

Q Prove that 9x2 - 24xy + 16y2 - 20x - 15 y - 60 = 0 represents a parabola. Also find its focus and directrix.

<u>Ans</u>: Here h^2-ab=$(-12)^2$-9x16=144-144=0 also $\Delta \neq 0$ $\therefore$ the equation represents a parabola Now, the equation is$(3x-4y)^2$=5(4x+3y+12) clearly , the lines 3x-4y=0 and 4x+3y+12=0 are perpendicular to each other.so, let $\frac{3x-4y}{\sqrt{3^2+(-4)^2}} = y, \quad = \frac{4x+3y+12}{\sqrt{4^2+3^2}} = x$ .....(1)

the equation of the parabola becomes -

$$Y^2 = X = 4.\frac{1}{4}.X$$

$\therefore$ Here $a = \frac{1}{4}$ in the standerd equation.

$\therefore$ the focus $= (a,0)_{X,Y} = \left(\frac{1}{4},0\right)_{X,Y}$

Now if $X = \frac{1}{4}, Y = 0$

then we havefrom the equations of tranformation in(1)

$$\frac{3x-4y}{5}=0, \frac{4x+3y+12}{5}=\frac{1}{4}$$

$$\Rightarrow 3x-4y=0, 4x+3y=-\frac{43}{4}$$

$$\Rightarrow y=\frac{3x}{4}, y=\frac{1}{3}\left(-\frac{46}{3}-4x\right)$$

$$\therefore \quad \frac{3x}{4}=\frac{1}{3}\left(-\frac{43}{3}-4x\right)$$

$$\text{or } 9x=-43-16x$$

$$\therefore x=\frac{-43}{25}$$

$$\text{and } y=\frac{3x}{4}=\frac{3}{4}.\frac{-43}{25}=\frac{-129}{100}$$

$$\therefore \text{focus}=\left(-\frac{43}{25}, -\frac{129}{100}\right)$$

The equation of directrix is,

$$x+a=0, i.e, X+\frac{1}{4}=0$$

$$\text{or } \frac{4x+3y+12}{5}+\frac{1}{4}=0$$

$$\text{or, } 4x+3y=-\frac{5}{4}-12=-\frac{53}{4}$$

$\therefore$

The directrix is $4x+3y+\frac{53}{4}=0$

<u>Dump Question</u>: The $\frac{3x-4y}{5}$ was written as Y and $\frac{4x+3y+12}{5}$ was
written as y.how?

<u>Ans</u>: The lines 3x-4y=0 and 4x+3y+12=0 are $\perp$ to each other as clear from
their slopes. so, the rotation of axis and transformation of axes was performed
to get the desired easy form of parabola.

Q.2:- Three normals from a point to the parabola y2 = 4ax meet the

axis of the parabola in points whose abscissa
are in A.P. Find the locus of the point. Ans : The equation of any normal to
the parabola is Y = mx - 2am -am3 It passes through the point (h,k) if am^3
+ m (2a - h) + k = 0 ??????. (1) the normal cuts the axis of the parabola viz
, y = 0 at point where x = 2a + am2 hence the abscissa of the point in which
the normal through (h,k) meet the axis of the parabola are. $X_1 = 2a + am1^2$, $x_2 = 2a + am_2^3$, $x_3 = 2a + am_3{}^2$ Since X_1 ,x_2, x_3 are in A.P. $(2a + am2^1) +(2a + am2^3) = 2 (2a + am2^2)$
$= m_1^2+m_2^2=2m_2^2$.....(2)
also from (1) $m_1+m_2+m_3=0$.....(3)

$$m_2m_3 + m_3m_1 + m_1m_2 = \frac{2a - h}{a} \quad \ldots\ldots (4)$$

$$\text{and } m_1.m_2.m_3 = -\frac{k}{a} \quad \ldots\ldots(5)$$

from (3) $(m_1+m_2)^2=m_2^2==m_1^2+m_3^2+2m_1m_3=m_2^2$

$$\Rightarrow 2m_2^2 - 2\frac{k}{am_2} = m_2^2 \Rightarrow m_2^2 = \frac{2k}{am_2}$$

$$\Rightarrow am_2^3 = 2k$$

Since m_2 is a root of (1), $am_2{}^3+m_2(2a-h)+k=0$
$= 2 k+m_2 (2a-h) + k = 0$
$27 ak^2 = 2 (h - 2a)^3$
Hence the locus of(h,k) is $27ay^2=2(x-2a)^3$ M-3- prove that the length of the intercept on the normal at the point $p(at^2,2at)$ of a parabola $y^2=4ax$ made by the circle on the line joining the focus and point p as diameter is a $\sqrt{1+t^2}$.
ANS: let the normal at $p(at^2,2at)$ cut the circle in k and the axis of parabola at g then pk is required intercept.
$SP=PM=a+at^2$
Since angle in a semicircle being right angle .$\angle$ SPR=90.
and normal at $p(at^2,2at)$ is
$y = - tx+2at+at^3$
$= tx + y - 2at - at^3=0$
$\therefore$ SK is the perpendicular distance from s(a,0) to the normal(1)

$$Sk = \frac{|at + 0 - 2at - at^3|}{\sqrt{x^2+1}}$$

$$\therefore a|t|\sqrt{1+t^2}$$

$\therefore$ in Δ SPK

$(PK)^2 = (SP)^2 - (SK)^2$

$= a^2(1+t^2)^2 - a^2t^2(1+t^2)$

$= a^2(1+t^2)$

$$\therefore Pk = a\sqrt{1+t^2}$$

<u>Dumb Question</u>:- why is SP=PM in the above question? Ans: the point p lies on the parabola . so the distance of p from directrix is same as the distance from the focus. and hence SP=PM. Q-4:- Three normals are drawn from the point (0,0) to the currve $y^2=x$. show that c must be greater than 1/2. one normal is always the x-axis . Find c for which the other two normals are perpendicular to each other.

<u>Ans</u>: Equation to normal to the parabola

$y^2 = x$ is $y = mx - \frac{m}{2} - \frac{m^3}{4}$ is passing through (c,o)

$$0 = cm - \frac{m}{2} - \frac{m^3}{4}$$

$$\Rightarrow m = 0 \text{ or } c - \frac{1}{2} = \frac{m^3}{4}$$

$$\left(c - \frac{1}{2}\right) \geq 0 \quad \left(\text{since } \frac{m^2}{4} \geq 0\right)$$

$$\Rightarrow c \geq \frac{1}{2} \text{ if } c = \frac{1}{2} \text{ then } m = 0 \text{ and}$$

Then only one normal will be there i.e , x axis

Now the slope of other two normal are

$$m_2 = 2\sqrt{c - \frac{1}{2}} \text{ and } m_3 = -2\sqrt{c - \frac{1}{2}}$$

For these normals to be $\perp$ to each other

we need $m_2.m_3=-1$

so,

$$\left(2\sqrt{c-\frac{1}{2}}\right)\left(-2\sqrt{c-\frac{1}{2}}\right)=-1$$

$$\Rightarrow \quad -4\left(c-\frac{1}{2}\right)=-1$$

$$\text{or } c=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

M-3:- Find the equation to the common tangents to the circle $x^2+y^2=2a^2$
and the parabola $y^2=8ax$.

<u>Ans</u> :-

The Equation of any tangent to the parabola $y^2=8ax$ i.e,
will touch the circle $x^2+y^2=2a^2$ if

radius = $\sqrt{2}\,a$

= length of the perpendicular from the centre(0,0) to the line

$$\text{or, } \sqrt{2}\,a=\left|\frac{\frac{2a}{m}}{\sqrt{1+m^2}}\right|$$

$$\text{or , } 2a^2\left(1+m^2\right)=\frac{4a^2}{m^2}$$

or, $m^2(1+m^2)=2$
or, $m^4+m^2-2=0$
or,$(m^2+2)(m^2-1)=0$
But $m^2+2=0$ gives non real values of m.
$\therefore$ $m^2-1=0$;
$\therefore$ $m=\pm$
Putting m $\pm$ in (1) we get the equation of common tangents are
$\therefore$ the equations of common tangents are.

$y = \pm x \pm 2a$
or $y=x + 2a$ and $y=-(x+2a)$

<u>Dumb question</u> : why $y= \pm x \pm 2a$ gave only four equations of straight line
$y=x+2a$ & $y= -(x+2a)$ where as four lines are possible?

<u>Ans</u>:- Note that the equation of tangent in $y= y = mx + \frac{2a}{m}$
So, it is not possible m with x be +ve and m with 2a be -ve . hence
$y=x-2a$ & $y=-x+2a$ are impossible solutions.

<u>Q.7</u>: Prove that the circle circumscribing the triangle formed by tangents to a parabola passes through the focus

<u>Ans</u>: Let $P(at_1^2, 2at_1)$,$q(at_2^2,2at_2)$ and $r(at_3^2,2at_3)$ be three points on the parabola $y^2=4ax$

The equation of tangents at $P't_11$ and $Q't2'$ are
$Y\ t_11 = x + at_1^2$
$Y\ t_2 = x + at_2^2$
Solving these, $x = at_1t_2^2$ and $y = a(t_1 + t_2)$
The point of intersection C of the tangents at P and Q is
$\{ at_1\ t_2 , a(t_1 + t_2)\}$.
similarly other points of intersection of tangents are
$B = \{at_3t_1, a(t_3+t_1)\}$,$A=\{at_2t_3,a(t_2+t_3)\}$
let the equation of circumcircle of the ΔABC be
$x^2+y^2+2gx+2fy+c=0$----------(1)
(1) will pass through the focus (a,0) if
$a^2 + 2ga + c=0$.....(2)
A,B,C are points on (1) .so-

$a^2 t_1^2 t_2^2 + a^2 (t_1+t_2)^2 + 2g.at_1t_2 + 2.f.a(t_1+t_2) + c = 0$.....(3)

$a^2 t_2^2 t_3^2 + a^2 (t_2+t_3)^2 + 2g.at_2t_3 + 2.f.a(t_2+t_3) + c = 0$.....(4)

$a^2 t_3^2 t_1^2 + a^2 (t_3+t_1)^2 + 2g.at_3t_1 + 2.f.a(t_3+t_1) + c = 0$.....(5)

(3)-(4)

$a^2 t_2^2(t_1^2 - t_3^2 + a^2 (t_1-t_3)(t_1+2t_2 + t_3 + 2g.at_2(t_1 - t_3) + 2.f.a(t_1-t_1) + c = 0$

or, $a^2 t_2^2 (t_1+t_3+a^2(t_1+2t_2 + t_3 2g.at_2t_2 + 2.f.a = 0$.....(6)

similarly (4)-(5)
=> $a^2[t_2^2 (t_1 + t_3) - {}_3^2(t_2 + t_1) + t_2 - t_3] + 2g.a(t_2 - t_3) = 0$

or,$a [t_2t_3(t_2 - t_3) + t_1(t_2^2 - t_3^2) + t_2 - t_3)] + 2g(t_2 - t_3) = 0$

or,$a[t_2t_3 + (t_1t_2) + t_1t_3 + 1] + 2g = 0$
$\therefore 2g = -a(1 + t_1t_2 + (t_2t_3) + t_3t_1)$
From t_3x(6) - t_2 X(7),we get
$2f = -a(t_1 + t_2 + t_3 - t_1t_2t_3)$
putting values of 2g, 2f in (3) we get
$\therefore$ the equation of the circumcircle is-

$$x^2 + y^2 - a\left(1 + \sum t_1t_2\right)x - a\left(\sum t_1 - t_1t_2t_3\right)y + a^2 \sum t_1t_2 = 0$$

it passes through(a,0) because.-

$$a^2 - a^2\left(1 + \sum t_1t_2\right) + a^2 \sum t_1t_2 = 0$$

Hence the problem.-
<u>Dumb Question</u>: How does the fact that the circle
$x^2+y^2+2gx+2fy+c=0$ passes through(a,0) leads to the condition -
$a^2+2ga+c=0$-

<u>Ans</u>:- Since the circle passes through (a,0) the equation of circle must satisfy the point(a,0)-
so, $a^2+0^2+2g(a)+2f(0)+c=0$-
0r $a^2+2ga+c=0$ is obtained.-
let c1 and c2 be respectively the parabola $x^2=y-1$ and $y^{2=x}$ p be any point on c1 and q be any point on c2. Let p1 and Q1 be reflections of P and Q-
respectively with respect to the line=y
prove that p1 lies on c2, Q1 lies on c1 and PQ> min[pp1,QQ1]-

Hence or otherwise determine points p_0 and $Q_{0\ on}$ the parabolas c1 and c2 respectively. such that p_0Q_0 (Pq)-for all pairs of points (p,Q) with p on c1 and Q on c2.-

<u>Ans</u>:

Let co-ordinates of p and q are p(t, t^2+1) and Q (s^2 + 1, s) which lies on x^2=y-1

and y^2=x-1 respectively.-

∴ p1 and Q1 be reflections of P and Q respectively with respect to the line y=x then-

$P_1 \equiv (t^2+1, t)$ and $Q_1 \equiv (\delta, \delta^2+1)$

we have, $(PQ_1)^2=(t-s)^2+(t^2-s^2)^2$

$(p_1Q)^2$

=>PQ1=PIQ

Also $PP_1 \parallel QQ_1$

Thus PP1QQ1 is an isosceles trapezium

we have Pq> min {PP1,QQ1}

Let us take min {PP1,QQ1}=PP_1

then $(PQ)^2=(pp_1)^2$

=>$(t^2+1-t)^2+(t-t^2-1)^2=2(t^2-t+1)^2$=f(t)say

we have f'(t)=$4(t^2-t+1)(2t-1)$

NOw $f'(t)=0, t=\frac{1}{2}$

Also $f''(t)<0$ for $t<\frac{1}{2}$ and $f'(t)>0$ for $t>\frac{1}{2}$

Hence f(t) is least when t=1/2 point p_0 on c1 is $\left(\frac{1}{2},\frac{5}{4}\right)$ and p1(which we take as Q_0

and c2 are $\left(\frac{5}{4},\frac{1}{2}\right)$.Note that PQ $P_0Q_0 \le PQ$ for all pairs of(P,Q) with p on C1 and Q on c2.
Hence proved.

<u>Dumb Question</u>:- f' (t) is$4(t^2-t+1)(2t-1)$, but the only solution is t=1/2. what about the factor t^2-t+1?

<u>Ans</u>:- Note that $t^2-t+1=\left(t-\frac{1}{2}\right)^2+\frac{3}{4}$.
Now, this a positive quantity if t is + ve.
The coordinate plane is a real plane Where the points can take only real values and hence t has to real only. So, t2 - t + 1 cannot be zero.

$$=-\frac{at^2}{\sqrt{1+t^2}}=k$$

=k (from the equation)
from (1) the equation of the axis of the parabola in x,y coordinate becomes-

$$yt+x=-2at\left(-\frac{at^2}{k}\right)$$

$$\text{or } yt+x-\frac{2a^2t^3}{k}=0 \quad \ldots\ldots\ldots\ldots (8)$$

The given parabola is $x^2=-8k(y-2k)$.....(9)
solving (8) and (9) we get

$$x^2=-8k.\frac{1}{t}\left(-x+\frac{2a^2t^3}{k}\right)+16k^2$$

here D= $64k^2-64t^2(a^2t^2-k^2$

$$=64\left[k^2-a^2t^4+t^2k^2\right]$$

$$=64\left[\left(1+t^2\right).\frac{a^2t^4}{1+t^2}-a^2t^4\right]=0$$

$\therefore$ the axis given by(8) touches the given parabola.

<u>Note</u>:- if we take $k=\frac{at^2}{\sqrt{1+t^2}}$, the points of intersection of the axis and the given parabola will be imaginary.

Que:- A parabola drawn touching the axis of x at the origin and having its vertex at a given distance k from the x - axis . Prove that the axis of parabola is a tangent to the parabola x2 + 8k(y - 2k) = 0.

Ans : Let the equation of the parabola be Y2 = 4ax

Any tangent to it at the point (at^2, 2at) is
$Yt = X + at^2$.....(1) The normal at the point(at^2, 2at) is
$Y + tX = 2at + at^3$.....(2)
Take the equations of transformation -

$$\frac{tY-x-at^2}{\sqrt{1+t^2}}=y \qquad \ldots\ldots\ldots\ldots \quad (3)$$

$$\text{and } \frac{Y+tx-2at-at^3}{\sqrt{1+t^2}}=x \qquad \ldots\ldots\ldots\ldots \quad (4)$$

∴ in xy coordinates p=(0,0) and PT is the axis which is tangent to the parabola at the origin.
Now,

$$(3)\ \ tY-x-at^2=y\sqrt{1+t^2} \qquad \ldots\ldots\ (5)$$

$$(4)Y+tx-2at-at^3=x\sqrt{1+t^2} \qquad \ldots\ldots\ (6)$$

$$(5)xt + (6) \Rightarrow (t^2+1)y - 2at^2 - 2at = yt\sqrt{1+t^2} + x\sqrt{1+t^2}$$

∴ the axis of the parabola(y=0) becomes-

$$-2at^3 - 2at = (yt + x)\sqrt{1+t^2}$$

$$\text{or, } yt + x = \frac{-2at(1+t^2)}{\sqrt{1+t^2}} = -2at\sqrt{1+t^2} \qquad \ldots\ldots\ldots\ (7)$$

the distance of the vertex v(0,0) in the x,y coordinates from pt -
keywords:-

1. Parabola

2. focus
3. Directrix
4. Eccentricity
5. Vertex
6. Axis.
7. Latus rectum.
8. Diameter
9. Focal distance
10. Focal Chord.

Hyperbola

Hyperbola is a very important lonic section. It has a wide use in further engineering courses. Hyperbola is a very special curve which is very rarely seen in day to day life. The new concepts such as touching the curve at infinity fills us with a great excitement to read this chapter, so enjoy the curve named hyperbola by getting into the chapter and feel the touch of the line to the line to the curve at infinity.

<u>DEFINATION</u>:

The locus of point which moves in a plane such that its distance from a fixed point called focus is e times (e > 1) its distance from a fixed line called directrix.

EQUATION OF HYPERBOLA

The standard equation of hyperbola is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(e^2-1)$

Why ?

Let s be the focus & ZM the directix of the hyperbola. Draw SZ ⊥ ZM & Divided SZ internally & externally in the ratio e:1 (e>1) & let A & A' be their internal & external points of division, then

SA = eAZ & .....(1)

SA'= eA'Z .....(2)

Points A & A' will lie on the hyperbola. Let AA' = ZA & take C, the mid point of AA' of AA' as origin.

i.e. CA = CA' = a

Let P be any point on the hyperbola.

Then adding (1)& (2) we get

SA + SA' = e(AZ + A'Z)

$\Rightarrow$(CS - CA) + (CS + CA') = eAA'

$\Rightarrow$2cs + CA' - CA = e(za)

But CA' = CA, Hence

2CS = 2ae

$\Rightarrow$CS = ae

Thus focus S is (ae, 0)

Now subtrating (1) from (2) use get

SA'- SA = C(A'Z - AZ)

AA' = e[(CA' + CZ) - (CA - CZ)]

AA' = e[2CZ + CA' - CA]

But CA'= CA, then

2a = e(2cz)

$\Rightarrow$CZ = a/e

Thus directrix is x = a/e

Now draw PM $\perp$MZ

We know by the definaton of hyperbola, that

= SP = ePM

= $(sp)^2 = e^2(pm)^2$

= $(x- ae)^2 + y^2 = e^2\left(x-\frac{a}{e}\right)^2$

= $(x - ae)^2 + y^2 = (ex - a)^2$

= $x^2 + a^2e^2 - 2aex + y^2 = e^2x^2 + a^2 - 2aex$

= $x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$

= $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

= $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

where $b^2 = a^2(e^2 - 1)$

<u>ILLUSTRATION : 1</u> .

Find the equation of hyperbola whose foci are (2, 4) & (10, 4) &

eccentricity is 2.

Ans: We know that the center of hyperbola is the mid point of two foci i.e. coordinates of centers are (6, 4). We also know that the distance between two foci is 2ae.

i.e $(10 - 2)^2 + (4 - 4)^2 = (zae)^2$

$\Rightarrow 2ac = 8$

$\Rightarrow 4a = 8$

$\Rightarrow a = 2$

$b^2 = a^2(e^2 - 1)$, Hence

$\Rightarrow a = 2$

$b^2 = 4(4 - 1) = 12$

The required equation of hyperbola is

$$\frac{(x - 6)^2}{4} - \frac{(y - 4)^2}{12} = 1$$

<u>SOME TERM RELATED TO HYPERBOLA</u>

(i) **<u>CENTER</u>** : This is the mid point of line joining the two foci. In standard form C is (0,0).

(ii) **<u>ECCENTRICITY</u>** : This is the ratio of the distances of any point on hyperbola from focus to the directrix. Here we have

& $b^2 = a^2(e^2 - 1)$

$$\Rightarrow e = \sqrt{1 + \frac{b^2}{a^2}}$$

(iii) **<u>FOCI</u>**: Foci consists of two points on the transvers axis of hyperbola whose mid point is the center of hyperbola. Coordinates: (Iae, 0)

(iv) **<u>DIRECTRICES</u>** : There are the two lines perpendicula to the transverse axis on oppsite sieds of centre. In standard form directrices are

$$x \pm \frac{a}{e} = 0$$

(v) **<u>AXES</u>** : In standard form points A(a,0) & A'(-a, 0) are called the vertices of hyperbola line AA' is called transverse axis &

perpendicular to this is called conjugate axis.

(vi) **DOUBLE ORDINATES** : This is the length of the chord of hyperbola with end points as a, a', which is perpendicular to the transverse axis.
If the abscissa of Q is h then

$$\frac{h^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow y = \pm \frac{b}{a}\sqrt{h^2 - a^2}$$

Hence coordinates of Q & Q' are $\left(h, \frac{b}{a}\sqrt{h^2 - a^2}\right)$ & $\left(h, -\frac{b}{a}\sqrt{h^2 - a^2}\right)$.

(vii) **LATUS RECTUM:** The double coordinates which pass through foci are called Latus rectums. The abscissa of Latus rectums are Iae. Hence the coordinates are

$\left(ae, \frac{b^2}{a}\right), \left(ae, -b^2/a\right), \left(-ae, b^2/a\right), \left(-ae, -b^2/a\right)$, & the length of Lotus rectum is $\frac{2b^2}{a}$.

(viii) **FOCAL CHORD** : A chord of hyperbola that passes through focus is called the focal chord of hyperbola.

(ix) **PARAMETRIC EQUATION OF HYPERBOLA**: The parametric equations of given hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

$x = a \sec\theta$

$y = b \tan\theta$

why ?

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola with centre C & transverse axis AA', Then the circle drawm with centre C & segment AA' as

diameter is called auxiliary circle. Equation is:

$x^2 + y^2 = a^2$

Let P be any point on hyperbola & Q on circle Draw PN perpendicular to axis & NQ be the tangent ot circle, then join CQ.

Let $\angle QCN = \phi =$ eccentric angle of P

since Q = (a cos ϕ, a sin ϕ)

$x = CN = a \sec \phi$

But P(x, y) lies on hyperbola, then

$$\frac{a^2 \sec^2 \phi}{a^2} - \frac{y^2}{b^2} = 1$$

$\Rightarrow y = b \tan \theta$

Hence the parametric equations of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are x= sec ϕ , y = tan ϕ

<u>ILLUSTRATION : 2</u>. Find the centre, foci, vertices, eccentricity, length, of axes, Latus Rectum & directrices of the hyperbola $3x^2 - 2y^2 + 12x + 4y + 4 = 0$

Ans: $3x^2 - 2y^2 + 12x + 4y + 4$

$= 3(x^2 + 4x + 4 - 4) - 2(y^2 - 2y + 1 - 1) + 4$

$\Rightarrow 3(x+2)^2 - 12 - 2(y - 1)^2 + 2 + 4 = 0$

$\Rightarrow 3(x + 2)^2 - 2(y - 1)^2 = 6$

$$\Rightarrow \frac{(x+2)^2}{2} - 2\frac{(y - 1)^2}{3} = 1$$

Centre = (- 2, 1)

Here $a^2 = 2$ $\quad \therefore \; a = \sqrt{2}$

$b^2 = 3$ $\quad \therefore \; b = \sqrt{3}$

Hence $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{3}{2}} = \sqrt{\frac{5}{2}}$

foci = (± ae, 0) = $\left(\pm\sqrt{5}, 0\right)$

vertics = (± ae,0) = $\left(\pm\sqrt{2}, 0\right)$

eceentricity = $\sqrt{\frac{5}{2}}$

length of axex = 2a & 2b

$= 2\sqrt{2} \;\&\; 2\sqrt{3}$

Latus Rectum = $\frac{2b^2}{a} = \frac{2 \times 3}{\sqrt{2}} = 3\sqrt{2}$

Directrices = $x \pm \frac{a}{e} = 0$

$\Rightarrow x \pm \frac{\sqrt{2} \times \sqrt{2}}{\sqrt{5}} = 0$

$\Rightarrow x \pm \frac{2}{\sqrt{5}} = 0$

<u>FOCAL DISTANCES OF A POINT</u>:

Another Defination of hyperbola is that the difference of focal distances of any point on hyperbola is constant & equal to the length of transverse axis of the hyperbola.

Why ?

Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We know that

Distance from focus = (Distance from Directrix)

Hence Distance of point $P(x_1, y_2)$ from S,(ae,0) is

$SP = ePM = e\left(x_1 - \frac{a}{e}\right) = ex_1 - a$

Similarly S'P = e(PM') = $e\left(x_1 - \frac{a}{e}\right)$ = $ex_1 + a$
Hence S'P - SP = 2a
= Transverse axis
Hence Hyperbola is the Locus of a point which moves in a plane such that the difference of its distances from two fixed points i.e. foci is constant.

POINT AND HYPERBOLA

The point (x_1, y_2) lies outside, on ,or inside the hyperbola
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ accordingly as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ < , = or > 0

Why ?
Let P = (x_1, y_2) & Q = $(x_1\ y_1)$

Draw QL perpendicular to x axis then
QL > PL
$\Rightarrow y_1 > y_2$

$\Rightarrow \frac{y_1^2}{b^2} > \frac{y^2_2}{b^2}$
$\Rightarrow \frac{-y_1^2}{b^2} < \frac{-y^2_2}{b^2}$

Adding $\frac{x_1^2}{a^2}$ on both sides

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < \frac{x_1^2}{a^2} - \frac{y_2^2}{b^2}$$

But $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

Hence $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$

$\Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$

When point lies outside. Similarly we can prove that when point lies on of inside the hyperbola Then $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 = \text{or} > 0$

LINE AND A HYPERBOLA

The line y = mx + c will cut the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in two points, one point or will not cut accordingly as $c^2 >, = \text{or} < a^2m^2 - b^2$

Why ?

Let the line be y = mx + c ..... (1)

& the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ..... (2)

Eliminating y form (1) & (2) we get

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

$\Rightarrow x^2(a^2m^2 - b^2) + 2mca^2x + a^2(b^2 + c^2) = 0$

This is a quadratic is x, hence

Discriminant = $b^2 - 4ac$

$= 4m^2c^2a^4 - 4(a^2m^2 - b^2)(a^2)(b^2 + c^2)$

$= c^2 + b^2 - a^2m^2$

Line will cut in two points if D > 0

$\Rightarrow c^2 + b^2 - a^2m^2 > 0 \Rightarrow c^2 > a^2m^2 - b^2$

Line will touch the parabola if D = 0

i.e. $c^2 + b^2 - a^2m^2 = 0 \Rightarrow c^2 = a^2m^2 - b^2$

Line will not be touch or be touch a chord of parabola
if $D < 0$ i.e. $c^2 + b^2 - a^2m^2 < 0$
$\Rightarrow c^2 < a^2m^2 - b^2$

<u>ILLUSTRATION : 3</u>.
For what values of K will the line $y = 3x + K$ be a chord to the hyperbola $\frac{x^2}{9} - \frac{y^2}{45} = 1$

Ans: For this hyperbola we have
$a^2 = 9$ & $b^2 = 45$
From the equation of given line
$m = 3$ & $c = k$
Hence we know that for a line to be a chord to hyperbola $c^2 > a^2m^2 - b^2$
i.e. $K^2 > 9 \times 9 - 45$
$\Rightarrow K^2 > 36$
$\Rightarrow K^2 - 36 > 0$
$\Rightarrow (K - 6)(K + 6) > 0$
Hence $K \in (-\infty, -6) \cup (6, \infty)$

<u>EQUATIONS OF TANGENT</u>

(a) **<u>POINT FORM</u>**:

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(x_1 y_1)$ is
$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

i.e. $T = 0$ where $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

Why ?
The equation of hyperbola is:
$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$$

Differentiating w.r.t.x. we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}$$

$$\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{b^2x_1}{a^2y_1}$$

Equation of tangent when it passes through (x_1y_1) is

$$(y - y_1) = \frac{b^2x_1}{a^2y_1}(x - x_1)$$

$\Rightarrow a^2y_1y - a^2y_1^2 = b^2x_1x - b^2x_1^2$

Dividing whole equation by a^2b^2 we get

$$\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$$

$$\Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

But $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ as (x_1, y_1) lies on hyperbola

$\Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ is the requurired equation

(b) **<u>PARAMETRIC FORM</u>**:

The equation of tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec\phi, b \tan\phi)$ is

$$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1$$

Why ?

We have to paranetric equations of hyperbola as

$x = a \sec\phi$ & $y = b \tan\phi$

Differentiating both equations we get $dx = a \sec\phi \tan\phi \, d\phi$ & $dy = b$

$\sec^2\phi \, d\phi$

Hence dividing these equations we get,

$$\frac{dx}{dy} = \frac{b}{a} \operatorname{cosec} \phi$$

Hence the equations of tangent is $(y - b \tan\phi) =$

$$\frac{b}{a} \sin\phi \, (x - a \sec\phi)$$

$\Rightarrow ay \sin\phi \cos\phi - ab \sin^2\phi = \cos\phi \, bx - ab$

$\Rightarrow say \sin\phi \cos\phi - bx \cos\phi = - ab \cos^2\phi$

$$\Rightarrow \frac{y \sin\phi \cos\phi}{b} - \frac{x}{a} \cos\phi = \cos^2\phi$$

$$\Rightarrow \frac{y}{b} \tan\phi - \frac{x}{a} \sec\phi = - 1$$

$$\Rightarrow \frac{x}{a} \sec\phi - \frac{y}{b} \tan\phi = 1$$ is required equation.

<u>SLOPE FORM</u>:

The equations of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$ & coordinates of points of contact are

$$\left(\pm \frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2 m^2 - b^2}} \right)$$

Why ?

Let the line with slope m be

$y = mx + c$, tangent to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Eliminating y from these two equations we get

$(a^2m^2 - b^2)x^2 + 2 mca^2x + a^2(c^2 + b^2) = 0$

This is a quadratic equations in x hence for only one solution D should be zero.

$\therefore \; D = b^2 - 4ac = 4m^2c^2a^4 - 4(a^2m^2 - b^2(a^2b^2 + c^2) = 0$

$\Rightarrow c^2 = a^2m^2 - b^2$

$\Rightarrow c = \pm\sqrt{a^2m^2-b^2}$

Hence the equation of tangent is:

$y = mx \pm\sqrt{a^2m^2-b^2}$ ..... (1)

Now, we also know that the equation of tangent at (x_1, y_1) is

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ ..... (2)

Comparing (1) & (2) as they are the same equation we get

$\frac{x_1}{ma^2} = \frac{y_1}{b^2} = \pm 1 \frac{1}{\sqrt{a^2m^2-b^2}}$

$\Rightarrow x_1 = \frac{\pm a^2 m}{\sqrt{a^2m^2-b^2}}$ &

$y_1 = \frac{\pm b^2}{\sqrt{a^2m^2-b^2}}$

Hence the coordinates of point of contact are

$\left(\frac{\pm a^2m}{\sqrt{a^2m^2-b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2-b^2}}\right)$.

<u>ILLUSTRATION : 4</u>.

Find the equations of the tangent to the hyperbola $\frac{x^2}{2} - \frac{y^2}{9} = 1$ which is perpendicular to the line 3y + x =1

Let the slope of line be m then $m \times \frac{-1}{3} = -1$

$\Rightarrow m = 3$

& $a^2 = 2$ & $b^2 = 9$

Hence equations of tangents are:

$y = mx \pm\sqrt{a^2m^2-b^2}$

i.e. $y = 3x + \sqrt{2 \times 9 - 9}$

& $y = 3x - \sqrt{2 \times 9 - 9}$

$\Rightarrow$ equations are

$y = 3x + 3$

& $y = 3x - 3$

EQUATIONS OF NORMAL

(a) **POINT FORM**:

Equations of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1,y_1) is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

Why ?

The equations of hyperbola is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

Differentiating w.r.t.x we get

$$\frac{zx}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \left.-\frac{dx}{dy}\right|_{(y_1,y_2)} = -\frac{a^2y_1}{b^2x_1}$$

Hence the equation of normal is

$$(y - y_1) = \frac{-a^2y_1}{b^2x_1}(x - x_1)$$

$$\Rightarrow b^2x_1y - b^2x_1y_1 = -a^2y_1x + a^2x_1y_1$$

Dividing whole equation by x_1y_1

$$\frac{b^2y}{y_1} - b^2 = -\frac{a^2x}{x_1} + a^2$$

$$\Rightarrow \frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$ is the required equation

(b) **PARAMETRIC FORM**:

The equation of normal at $(a\sec\phi, \tan\phi)$ to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$xa\cos\phi + by\cot\phi = a^2 + b^2$$

Why ?

We know that the parametric equations of hyperbola are

$x = a \sec\phi \quad y = \tan\phi$

Differentiating them we get

$dx = a \sec\phi \tan\phi \, d\phi \quad \& \quad dy = b\sec^2\phi \, d\phi$

Dividing them we get

$$\frac{-dx}{dy} = \frac{-a \sin\phi}{b}$$

Hence the equation of normal is

$$(y - b \tan\phi) = \frac{-a}{b} \sin\phi \, (x - a \sec\phi)$$

$\Rightarrow by - b^2 \tan\phi = - ax \sin\phi + a^2 t an\phi$

$\Rightarrow ax \sin\phi + by = (a^2 + b^2)\tan\phi$

Dividing whole equation by $\tan\phi$ we get

$$\frac{ax \sin\phi}{\tan\phi} + \frac{by}{\tan\phi} = a^2 + b^2$$

$\Rightarrow as \cos\phi + by \cot\phi = a^2 + b^2$ is required equation.

(c) **<u>SLOPE FORM</u>**:

The equations of the mormals of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by $\frac{ax \sin\phi}{\tan\phi} + \frac{by}{\tan\phi}$

Why ?

We know that the equation of normal at (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

since m is the slope, then

$$m = \frac{-a^2y_1}{b^2x_1}$$

$$\Rightarrow y_1 = \frac{-b^2mx_1}{a^2}$$

& (x_1, y_1) lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \quad \frac{x_1^2}{a^2} - \left(\frac{b^2 m x_1}{a^2}\right)^2 \times \frac{1}{b^2} = 1$

$\Rightarrow \quad x_1 = \pm \frac{a^2}{\sqrt{a^2 - b^2 m^2}}$

$y_1 = \pm \left(\pm \frac{mb^2}{\sqrt{a^2 - b^2 m^2}}\right)$

Hence the equation of normal in term of slope is

$y - \left(\pm \frac{mb^2}{\sqrt{a^2 - b^2 m^2}}\right) = m\left[x - \left(\frac{\pm a^2}{\sqrt{a^2 - b^2 m^2}}\right)\right]$

$\Rightarrow y = mx \pm \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2 m^2}}$ is the required equation of normal in slope form.

<u>Illustration : 5</u>.

Find the equation of normal to $\frac{x^2}{2} - \frac{y^2}{4} = 1$ at the poimt whose eccentric angle is $\pi/4$.

Ans: We know the equation of normal in terms of eccentric angle is

$ax \cos\phi + by \cot\phi = a^2 + b^2$

Here $\phi = 45^0$, $a = \sqrt{2}$, $b = 2$

$\therefore \quad ax \cos 45^0 + by \cot 45^2 = a^2 + b^2$

$\Rightarrow \sqrt{2}.x.\frac{1}{\sqrt{2}} + 2.y.1 = 2 + 4$

$\therefore \quad x + 2y = 6$ is the required equation.

<u>EQUATION OF A CHORD BISECTED AT A GIVEN POINT</u>:

The equation of a chord of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, bisected at $(x_1,$

y_1) is $T = S_1$ where $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$ & $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

i.e. $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$

Why ?

Let QR be the chord whose mid point is P. Since Q & R lies on hyperbola.

$\therefore \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = 1$ ..... (i) & also

$\therefore \frac{x_3^2}{a^2} - \frac{y_3^2}{b^2} = 1$ ..... (ii)

Subtracting (ii) from (i) we get

$$\frac{x_2^2 - x_3^2}{a^2} = \frac{1}{b^2}(y_2^2 - y_3^2)$$

$$\Rightarrow \frac{(x_2 + x_3)(x_2 - x_3)}{a^2} = \frac{(y_2 - y_3)(y_2 + y_3)}{b^2}$$

$$\Rightarrow \frac{y_2 - y_3}{x_2 - x_3} = \frac{x_2 + x_3}{y_2 + y_3} \times \frac{b^2}{a^2}$$

$$\Rightarrow \frac{y_2 - y_3}{x_2 - x_3} = \left(\frac{x_2 + x_3}{2}\right)\left(\frac{2}{y_2 + y_3}\right) \times \frac{b^2}{a^2}$$

$$\Rightarrow \frac{y_2 - y_3}{x_2 - x_3} = \frac{x_1 b^2}{y_1 a^2}$$

Hence slope of chord joining Q & R is $\frac{x_1 b^2}{y_1 a^2}$

$\therefore$ The equation of chord QR is

$$\frac{y - y_1}{x - x_1} = \frac{x_1 b^2}{y_1 a^2}$$

$\Rightarrow yy_1a^2 - y_1^2a^2 = xx_1b^2 - x_1^2b^2$

Dividing whole equation by a^2b^2 we get

$$\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

$$\Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

$$\Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

$\Rightarrow T = S_1$

is the required equation of chord with given mid point.

PAIR OF TANGENTS

The combined equation of pair of tangents drawn from a point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$$

i.e. $SS_1 = T^2$

where $S = \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$; $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

Why ?

Let R(h, k) be any point on pair of tangents PQ or PT from any external point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Equation of PR is

$$\frac{y - y_1}{x - x_1} = \frac{k - y_1}{h - x_1}$$

$$\Rightarrow y = \left(\frac{k - y_1}{h - x_1}\right)x + \left(\frac{hy_1 - kx_1}{h - x_1}\right)$$

But this line is a tangent to the hyperbola so it must be of the form.

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

Here $c = \frac{hy_1 - kx_1}{h - x_1}$ & $m = \frac{k - y_1}{h - x_1}$

$$\therefore \left(\frac{hy_1 - kx_1}{h - x_1}\right)^2 = a^2\left(\frac{k - y_1}{h - x_1}\right)^2 - b^2$$

$$\Rightarrow (hy_1 - kx_1)^2 = a^2(k - y_1)^2 - b^2(h - x_1)^2$$

Hence locus of (h, k) is

$$(xy_1 - yx_1)^2 = a^2(y - y_1)^2 - b^2(x - x_1)^2$$

$$\Rightarrow (xy_1 -- yx_1)^2 = - (b^2x^2 - a^2y^2) - (b^2x_1^2 - a^2y_1^2) - 2(a^2yy_1 - b^2xx_1)$$

Dividing whole equation by a^2b^2 we get

$$\left(\frac{xy_1 - yx_1}{ab}\right)^2 = -\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) + 2\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)$$

$$\Rightarrow - \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) - \left(\frac{xy_1 - yx_1}{ab}\right)^2 = - 2\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)$$

Adding $1 + \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)^2$ on both sides we get

$$\Rightarrow - \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) - \left(\frac{xy_1 - yx_1}{ab}\right)^2 + 1 + \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)^2 = 1 + \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)^2 - 2\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right)$$

$$\Rightarrow \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$$

i.e. $SS_1 = T^2$

<u>Illustration : 6</u>.

Find the equation of pair of tangents drawm from a point P(1, 1) to the hyperbola $\frac{x^2}{2} - \frac{y^2}{1} = 1$.

Ans: Here $a^2 = 2$ $b^2 = 1$

$S = \frac{x^2}{2} - \frac{y^2}{1} - 1$

$S_1 = \frac{1}{2} - \frac{1}{1} - 1 = -\frac{3}{2}$

$T = \frac{x.1}{2} - \frac{y.1}{1} - 1 = \frac{x}{2} - y - 1$

Hence equation is $SS_1 = T^2$

$\Rightarrow \left(\frac{x^2}{2} - y^2 - 1\right)\left(-\frac{3}{2}\right) = \left(\frac{x}{2} - y - 1\right)^2$

$\Rightarrow -3(x^2 - 2y^2 - 2) = (x - 2y - 2)^2$

$\Rightarrow 4x^2 - 2y^2 - 4x + 8y - 4xy - 2 = 0$

$\Rightarrow 2x^2 - y^2 - 2x + 4y - 2xy - 1 = 0$ is the equation of pair of tangents.

<u>CHORD OF CONTACT</u>:

If the tangent from a point P(x', y') to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ touch the equation of chord of contact QR is

Why ?

Let $Q = (x_1, y_1)$ & $R = (x_2, y_2)$

We know that PQ & Pr are tangents, hence equation of PQ is

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ ..... (i)

equation of PR is

$\frac{xx_2}{a^2} - \frac{yy_2}{b^2} = 1$ ..... (ii)

Since both (i) & (ii) passes through P(x', y')

$\therefore \quad \frac{x'x_1}{a^2} - \frac{y'y_1}{b^2} = 1$ ..... (iii)

& $\frac{x'x_2}{a^2} - \frac{y'y_2}{b^2} = 1$ ..... (iv)

By looking carefully at (iii) & (iv) we can say that (x_1, y_1) & (x_2, y_2) lies on

$$\frac{x'x}{a^2} - \frac{y'y}{b^2} = 1 \quad \text{i.e.} \quad T = 0.$$

<u>POLE AND POLAR</u>:

Let $P(x_1, y_1)$ be any point inside or outside the hyperbola the if any straight line drawn through P intersects the hyperbola at A & B. Then the locus of points of intersection of tangents to the hyperbola at A & B is called the polar of the given point P with respect to hyperbola & the point P is called the pole of the polar.

Why ?

Let P be (x_1, y_1) & hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If tangents to the hyperbola at A & B meets at Q(h, k), then AB is the chord ofcontact with respect to Q(h,k).

$\therefore$ Equation of AB is

$$\frac{hx}{a^2} - \frac{ky}{b^2}$$

But $P(x_1, y_1)$ lies on it, hence

$$\frac{hx_1}{a^2} - \frac{ky_1}{b^2} = 1$$

Hence locus of Q(h, k) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$$

This is required equation of polar with (x_1, y_1) as its pole.

<u>Illustration : 7</u>.

Find the pole of given line y = mx + c with respect to hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Ans: Let the pole of line be (α, β) then the polar with respect to hyperbola is

$$\frac{x\alpha}{a^2} - \frac{y\beta}{b^2} = 1$$

But we have given the polar to be

$$y = mx + c$$

Hence by comparing the coefficients we get

$$\frac{-\alpha}{b^2\alpha x} = \frac{\alpha}{ma^2} = -\frac{1}{c}$$

$$\Rightarrow \alpha = -\frac{ma^2}{c} \quad \& \quad \beta = \frac{\alpha b^2}{ma^2}$$

Hence pole is $\left(-\frac{ma^2}{c}, \frac{\alpha b^2}{ma^2}\right)$

<u>DEAMETER</u>:

The locus of the middle points of any set of parallel chords of a hyperbola is called a diameter & the point where diameter intersects the hyperbola iscalled the vertex of diameter & equation of diameter is $y = \frac{b^2x}{a^2m}$.

Why ?

Let y = mx + c be a set of parallel chords with c as a variable.

By solving y = mx + c , we get

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$$

$\Rightarrow (a^2m^2 - b^2)x^2 + 2mca^2x + a^2(b^2 + c^2) = 0$

This equation has roots x_1 & x_2

$\therefore \quad x_1 + x_2 = \dfrac{-2mca^2}{a^2m^2 - b^2}$ ..... (i)

But (h, k) is the middle point of QR then,

$h = \dfrac{x_1 + x_2}{2}$ ..... (ii)

From (i) & (ii)

$$h = \frac{-mca^2}{a^2m^2 - b^2}$$

$\Rightarrow c = \dfrac{b^2h - a^2m^2h}{ma^2}$

We also know that (h, k) lieson y = mx + c

$\therefore \quad k = mh + c$

$\Rightarrow k = mh + \dfrac{b^2h}{ma^2} - mh$

$\Rightarrow k = \dfrac{b^2h}{ma^2}$.

<u>CONJUGATE DIAMETERS</u>

Two diameters are said to beconjugate when each bisects all chords parallel to the others.

Why ?

Let one set of parallell chords be

$y = mx + c$ ..... (i)

then its diameter is $y = \frac{b^2x}{ma^2}$

Let other set of parallel chords be

$y = m_1x + c$

then its diameter is $y = \frac{b^2x}{m_1a^2}$ ..... (ii)

But slope of equation (i) & (ii) are same, therefore

$\frac{b^2x}{m_1a^2} = m$

$\Rightarrow mm_1 = \frac{b^2}{a^2}$

Therefore product of the slopes of two conjugate diameter is $\frac{b^2}{a^2}$.

<u>Illustration : 8</u>.

Find the condition for the lines $2Ax^2 + 2Hxy + By^2 = 0$ to be conjugate diameter of $\frac{x^2}{5} - \frac{y^2}{3} = 1$.

Ans: Let the slope of lines whose equation is $2Ax^2 + 2Hxy + By^2 = 0$ be m_1 & m_2 then we know that

$m_1m_2 = \frac{2A}{B}$ ..... (i)

We also know that if m_1 & m_2 are the slopes of two conjugate diameters then

$m_1m_2 = \frac{b^2}{a^2}$ ..... (ii)

Hence by (i) & (ii)

$\frac{2A}{B} = \frac{b^2}{a^2} = \frac{3}{5}$

$\Rightarrow 10A = 3B$ is the required condition.

<u>DIRECTOR CIRCLE</u>:

The locus of the point of intersection of the tangent to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which are perpendicular to each other is called director circle & the equation of director is

$x^2 + y^2 = a^2 - b^2 \quad (a > b)$

Why ?

The equation of tangent to hyperbola with slope m is

$y = mx + \sqrt{a^2m^2 - b^2}$ ..... (i)

& the perpendicular tangent with slope $-\frac{1}{m}$ is

$y = -\frac{1}{m}x + \sqrt{\frac{a^2}{m^2} - b^2}$ ..... (ii)

$\Rightarrow x + my = \sqrt{\frac{a^2 - m^2b^2}{m}}$ ..... (iii)

Squaring & adding equations (i) & (iii) we get

$(y - mx)^2 + (x + my)^2 = a^2m^2 - b^2 + a^2 - b^2m^2$

$\Rightarrow (1 + m^2)(x^2 + y^2) = (1 + m^2)(a^2 - b^2)$

$\Rightarrow (1 + m^2)[x^2 + y^2 - (a^2 - b^2)] = 0$

As $1 + m^2 \neq 0$, therefore

$x^2 + y^2 = a^2 - b^2$ is the equation of director circle of hyperbola.

<u>Illustration : 9</u>.

Find the radius of the director circle of the hyperbola $\frac{x^2}{4} = \frac{y^2}{2} = 1$.

Ans: We know the equation of director circle of is

$x^2 + y^2 = a^2 - b^2$

Here $a^2 = 4$ & $b^2 = 2$

Therefore $x^2 + y^2 = 4 - 2 = 2$

$\Rightarrow x^2 + y^2 = (\sqrt{2})^2 = r^2$

Here $r = \sqrt{2}$, Hence the radiusof director circle is $\sqrt{2}$ units.

<u>ASYMPTOTES</u>

An asymptote of a hyperbola or any curve is a straight line which touches the curve at infinity at two points. The equation of

asymptotes is $y = \pm \frac{bx}{a}$

Why ?

Let $y = mx + c$ be an asymptote to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Hence solving bot we get

$(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(b^2 + c^2) = 0$

Now thishas two solutions for x & the asymptote touches the hyperbola at ∞ therefore both the roots must be ∞. By looking at the above equation the only way that ∞ can be a solution is to be zero. Therefore

$a^2m^2 - b^2 = 0$ & $2a^2mc = 0$

$\Rightarrow m = \pm \frac{b}{a}$ & $c = 0$

Substituting m & c in $y = mx + c$

We get

$y = \pm \frac{b}{a} x$ is the required equation of asymptotes.

<u>Illustration : 10</u>.

Find the angle between asymptotes of hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

Ans: THe equation of asymptotes is

$y = \pm \frac{bx}{a} = \pm \frac{2}{3}$

Now we have to find angle between two lines

$y = \frac{2x}{3}$ & $y = -\frac{2x}{3}$

If m_1 & m_2 are slopes than angle between them is

$$\theta = \pm \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$$

$$= \pm \tan^{-1}\left(\frac{\frac{2}{3} - \left(-2/3\right)}{1 + \frac{2}{3} \times \frac{-2}{3}}\right)$$

$$= \pm \tan^{-1}\left[\frac{4/3}{5/9}\right]$$

$$\theta = \pm \tan^{-1}\left[\frac{12}{5}\right]$$

Hence the acute angle between the asymptotes is $\theta_1 = \tan^{-1}\left[\frac{12}{5}\right]$
& the obstuse angle is

$$\theta_2 = \pi - \tan^{-1}\left[\frac{12}{5}\right]$$

CONJUGATE HYPERBOLA

The hyperbola whose transverse & conjugate axes are respectively the conjugate & transverse axes of the standard hyperbolais called the conjugate hyperbola of the standard given hyperbola. The conjugate hyperbola of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

SOME RESULTS OF CONJUGATE HYPERBOLA

(a) Length of Transverse axis = 2b
(b) Length of conjugate axis = 2a
(c) Foci = (0, ±be)
(d) Equation of Directrices = y = ± b/e
(e) Eccentricity = $e = \sqrt{1 + \frac{a^2}{b^2}}$
(f) Length of Latus Rectum = $\frac{2a^2}{b}$
(g) Parametric co-ordinates = $(a \tan\theta, b \sec\theta)$
(h) Equation of transverse axis = x = 0
(i) Equation of conjugate axis = y = 0

<u>Illustration : 11</u>.

Find the length of transverse axis, conjugate axis eccentricity, coordinates of foci, vertices, length of latus rectum & equation of directrices of the hyperbola $9x^2 - 4y^2 = -36$.

Ans: The hyperbola is

$9x^2 - 4y^2 = -36$

$\Rightarrow \frac{x^2}{9} - \frac{y^2}{4} = -1$ i.e of the form

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ where a = 2 & b = 3

Length of transverse axis = 2b = 6units

Length of conjugation axis = 2a = 4 units

Eccentricity = $\sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{4}{9}} = \frac{\sqrt{13}}{3}$

Vertices = (0, ± b) = (0, ± 3)

Length of Latus Rectum = $\frac{2a^2}{6} = \frac{8}{3}$ units

Equation of directrices = y = ± b/e

$= y = \pm \frac{3}{\sqrt{3}/3}$

$= y = \pm \frac{9}{\sqrt{13}}$

$= y = \pm \frac{9\sqrt{13}}{13}$

RECTANGULAR HYPERBOLA DEFINITION

(1) A hyperbola whose asymptotes include aright angle is called a rectangular Hyperbola. THe general form of the equation of hyperbola is $x^2 - y^2 = a^2 = b^2$.

Why ?
The asysmptotes of a hyperbola is

$$y = \pm \frac{bx}{a}$$

If these are at right angles than

$$m_1 m_2 = -1$$

$$\Rightarrow \frac{b}{a} \times \frac{-b}{a} = -1$$

$$\Rightarrow b^2 = a^2$$

Hence the equation of ractangular hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$

$$\Rightarrow x^2 - y^2 = a^2$$

(2) If the length of transverse & conjugate axes of any hyperbola are equal, it is called as rectangular hyperbola.

Why ?

IF a = b, then $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ become

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \Rightarrow x^2 - y^2 = a^2$$

RECTANGULAR HYPERBOLA ($xy = c^2$)

The equation of Rectangular Hyperbola is $x^2 - y^2 = a^2$
i.e. $(x + y)(x - y) = a^2$
We know that $x + y = 0$ & $x - y = 0$ are at 45^0 & 135^0 to the x axis.
Now if we can rotate the axes through $\theta = -45^0$ without changing the origin.
So, we can replace (x, y) by

$(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$ i.e.

$$\left(\frac{x+y}{\sqrt{2}}, \frac{-x+y}{\sqrt{2}}\right)$$

$\therefore$ The equation $x^2 - y^2 = a^2$ becomes

$$\left(\frac{x+y}{\sqrt{2}}\right)^2 - \left(\frac{y-x}{\sqrt{2}}\right)^2 = a^2$$

$$\Rightarrow \frac{4xy}{2} = a^2$$

$$\Rightarrow xy = a^2/2$$

Let $a^2/2 = c^2$ = any positive constant

Therefore $xy = c^2$ is the another form of Rectangular hyperbola.

<u>RESULTS OF RECTANGULAR HYPERBOLA</u>

(a) The parametric coordinates are $x = ct$ & $y = c/t$.

(b) Equation of chord joining t_1 & t_2 is
$x + yt_1t_2 = c(t_1 + t_2)$

(c) Equation of tangent at t is $\frac{x}{t} + yt = x$.

(d) Point of intersection of tangents at 't_1' & 't_2' is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.

(e) Equation of normal at 't' is
$xt^3 - yt - ct^4 + c = 0.$

(f) Points of intersection of normals at 't$_1$' & 't$_2$' is

$$\left(\frac{c\left[t_1t_2\left(t_1^2 + t_2^2 + t_1t_2\right) - 1\right]}{t_1t_2\left(t_1 + t_2\right)}, \frac{c\left[t_1^2t_2^3 + \left(t_1^2 + t_2^2 + t_1t_1\right)\right]}{t_1t_2\left(t_1 + t_2\right)}\right)$$

<u>Illustration : 12</u>.

Find the eccentricity of Rectangular Hyperbola.

Ans: For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; $c = \sqrt{1 + \frac{a^2}{b^2}}$

Therefore for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we have $b = a$

Hence eccentricity of Rectangular hyperbola is $\sqrt{2}$.

<u>Easy</u>

1. For what value of k does $\frac{x^2}{3 + \alpha} - \frac{y^2}{6 - \alpha} = 1$ represents a hyperbola ?

Ans: We know that the standard form of hyperbola can be $\frac{x^2}{a^2} - \frac{y^2}{b^2} =$ 1 or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

<u>CASE - I</u>

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

As a^2 & b^2 are +ve quantities greater than zero simultaneously, therefore

$3 + \alpha > 0$ & $6 - \alpha > 0$

$\Rightarrow \alpha > -3$ & $\alpha < 6$

Hence common solution is

$-3 < \alpha < 6$

CASE II

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ or $\frac{x^2}{(-a^2)} - \frac{y^2}{(-b^2)} = 1$

Here also $-a^2$ & $-b^2$ are -ve quantities.

Therefore

$3 + \alpha < 0$ & also $6 - \alpha < 0$

$\Rightarrow \alpha < -3$ & $\alpha > 6$

There is no common solution.

Hence the only solution is

$-3 < \alpha < 6$ or $\alpha \in (-3, 6)$

Q.2. Find the value of c if the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ & $x^2 - y^2 = c^2$ cut at right angles ?

Ans: The curves cut at right angle means that the tangents at the point of intersection of two curves are at right angles. If m_1 is the slope of the tangent to ellipse & m_2 is the slope of the tangent to hyperbola, then

$m_1 = -\frac{b^2x}{a^2y} = -\left.\frac{4x_1}{9y_1}\right|_{x_1y_1} = -\frac{4x_1}{9y_1}$ ..... (i)

$m_2 = \left.\frac{x_1}{y_1}\right|_{x_1y_1} = \frac{x_1}{y_1}$ ..... (ii)

using (i) & (ii)

$m_1m_2 = -1$

$\therefore \frac{4x_1^2}{9y_1^2} = 1$

$\Rightarrow \frac{x_1^2}{9} = \frac{y_1^2}{4}$

But $\frac{x_1^2}{9} + \frac{y_1^2}{4} = 1$ as (x_1y_1) lies on hyperbola

$\therefore \frac{x_1^2}{9} = \frac{y_1^2}{4} = \frac{1}{2}$

$\Rightarrow x_1^2 = 9/2$ ..... (iii)

$y_1^2 = 4/2$ ..... (iv)
Bur $x_1^2 - y_1^2 = c^2$ ..... (v)

Using (iii) & (iv) in (v) we get

$$c^2 = \frac{9-4}{2} = \frac{5}{2} \quad \Rightarrow c = \pm\sqrt{\frac{5}{2}}$$

Q.3. Prove that if normal to the hyperbola $xy = c^2$ at any point t_1 meets the curve again at t_2, then, $t_1^3 t_2 = -1$.

Ans: Equation of normal at point t_1 is

$$yt_1 - xt_1^3 = c - ct_1^4$$

Now if this normal again meets at 't_2', then $(ct_2, \frac{c}{t_2})$ must satisfy the normal

$$\therefore \frac{c}{t_2} t_1 - ct_2t_1^3 = c - ct_1^4$$

$$\Rightarrow ct_1^3[t_2 - t_1] = \frac{c}{t_2}[t_1 - t_2]$$

$$\Rightarrow ct_1^3t_2[t_2 - t_1] + c(t_2 - t_1) = 0$$

$$\Rightarrow c(t_2 - t_1)(t_1^3 + 1) = 0$$

As $t_2 \neq t_1$ & $c > 0$

$$t_1^3t_2 + 1 = 0$$

$$\Rightarrow t_1^3t_2 = -1$$

Q.4. If $(a \sec\theta, b \tan\theta)$ & $(a \sec\phi, b \tan\phi)$ are the end points of a focal chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then prove that $\tan\frac{\phi}{2} \tan\frac{\phi}{2} = \frac{1-e}{1+e}$.

Ans: The equation of chord joining the points $(a \sec\theta, b \tan\theta)$ & $(a \sec\phi, b \tan\phi)$ is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

It passes through focus (ae, 0), then

$$\frac{ae}{a}\cos\left(\frac{\theta-\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow \frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)} = e$$

Applying componendo & dividendo we get

$$\frac{e-1}{e+1} = \frac{\cos\left(\frac{\theta+\phi}{2}\right) - \cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right) + \cos\left(\frac{\theta+\phi}{2}\right)}$$

$$\Rightarrow \frac{e-1}{e+1} = \frac{-2\sin\left(\theta/2\right)\sin\left(\phi/2\right)}{2\sin\left(\theta/2\right)\cos\left(\phi/2\right)}$$

$$\Rightarrow \tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right) = \frac{e-1}{e+1}$$

Q.5. Find the locus of poles of normal chords of the rectyangular hyperbola $xy = c^2$.

Ans: Equation of normal at any t is

$xt^3 - yt = c(t^4 - 1)$ ..... (i)

Lert its ple be $P(x_1, y_1)$ then the equation of polar is

$xy_1 + x_1y = 2c^2$..... (ii)

Comparing the coefficients of equations (i) & (ii) as they represent same equation

$$\frac{t^3}{y_1} = \frac{-t}{x_1} = \frac{c(t^4-1)}{2c^2}$$

$\Rightarrow t^2 = -\frac{y_1}{x_1}$ ..... (iii)

& $\frac{t^2}{x_1^2} = \frac{c^2(t^4-1)^2}{4c^4}$ ..... (iv)

From (iii) & (iv)

$$-\frac{y_1}{x_1} = \frac{x_1^2}{4c^2}\left(\frac{y_1^2}{x_1^2} - 1\right)^2$$

$$\Rightarrow -\frac{y_1}{x_1^3} = \frac{1}{4c^2}\frac{\left(y_1^2 - x_1^2\right)^2}{x_1^4}$$

$\Rightarrow (x_1^2 - y_1^2)^2 = -4c^2x_1y_1$

Hence locus of (x_1, y_1) is

$(x - y)^2 + 4c^2xy = 0$

Q.6. Find the co-ordinates of the foci & equation of the directrices of rectangular hyperbola $xy = c^2$?

Ans: When the hyperbola $x^2 - y^2 = a^2$ converts into $xy = c^2$ by rotation by -45^0 then $a^2 = 2c^2$.

So, for $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$

Coordinates of foci are $(\pm ae, 0) = (\pm \sqrt{2}\, a, 0)$

Also directrices are $x = \pm a/e = \pm a/\sqrt{2}$

Now we replace $\left(\frac{x+y}{\sqrt{2}}, \frac{y-x}{\sqrt{2}}\right)$ for (x, y) & $2c^2 = a^2$.

$\therefore$ Foci $= \frac{x+y}{\sqrt{2}} = \pm a\sqrt{2}$ & $\frac{y-x}{\sqrt{2}} = 0$

$\Rightarrow x = y = \pm 2a$

So Foci are $(\pm 2a, \pm 2a)$

i.e. $(\pm \sqrt{2}\, c, \pm \sqrt{2}\, c)$

Also directrices are

$\frac{x+y}{\sqrt{2}} = \pm a/\sqrt{2}$

$\Rightarrow x + y = \pm a$

$\Rightarrow x + y = \pm \sqrt{2}\, c \qquad (\because a^2 = 2c^2)$

are the directrices of the rectangular hyperbola.

Q.7. If the tangent at P of the hyperbola $xy = c^2$ meets the asymptotes ate L & M & C is the centre of hyperbola. Prove that PL = PM = CP.

Ans:

Let P(ct, $\frac{c}{t}$) be any point, then equation of tangent at P is

$x + t^2y = 2ct$

It meets the asymptotes i.e. $x = 0$ & $y = 0$ at L, M respectively.

$\therefore$ L = (2ct, 0)

& M = (0, $2c/t$)

Clearly mid point of LM is $\left(\frac{2ct+0}{2}, \frac{0+2c/t}{2}\right)$

= (ct, $\frac{c}{t}$) = P

Therefore PL = PM = $\frac{ML}{2}$ ..... (i)

& CP = $\sqrt{c^2t^2 + c^2/t^2}$

But ML = $\sqrt{4c^2t^2 + 4c^2/t^2}$

ML = $2\sqrt{c^2t^2 + c^2/t^2}$

ML = 2CP

$\Rightarrow$ CP = $\frac{ML}{2}$ ..... (ii)

By (i) & (ii)

PL = PM = CP

Q.8. Find the equation of hyperbola whose asymptotes are $2x + y + 2 = 0$ & $x + y + 3 = 0$ & which passes through (0, 1), also find the equation of conjugate Hyperbola.

Ans: We know that the combined equation of asymptotes & hyperbola differ by a constant.
The combined equation of assymptotes is

$(2x + y + 2)(x + y + 3) = 0$

$\Rightarrow 2x^2 + y^2 + 3xy + 8x + 5y + 6 = 0$

Then let the equation of hyperbola be

$2x^2 + y^2 + 3xy + 8x + 5y + 6 + \lambda = 0$

As it passes through (0, 1) we get

$0 + 1 + 0 + 0 + 5 + 6 + \lambda = 0$

$\Rightarrow \lambda = -12$

Hence the equation of hyperbola is

$2x^2 + y^2 + 3xy + 8x + 5y - 6 = 0$

We also know that

$\Rightarrow$The equation of conjugate hyperbola = 2(combined equation of assymptotes) - (equation of hyperbola)

$\Rightarrow$equation of conjugate hyperbola is

$2(2x^2 + y^2 + 3xy + 8x + 5y + 6) - (2x^2 + y^2 + 3xy + 8x + 5y - 6)$

$\Rightarrow$equation of conjugate hyperbola is

$2x^2 + y^2 + 3xy + 8x + 5y + 18 = 0$

Q.9. If the polars of (x_1, y_1) & (x_2, y_2) with respect to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are at right angles, then show that $\frac{x_1x_2}{y_1y_2} + \frac{a^4}{b^4} = 0$.

Ans: Equation of polar of (x_1, y_1) & (x_2, y_2) with respect to hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ & $\frac{xx_2}{a^2} - \frac{yy_2}{b^2} = 1$

Therefore the slopes are

$\frac{b^2x_1}{a^2y_1}$ & $\frac{b^2x_2}{a^2y_2}$

Since the polars are at right angles

$$\frac{b^2x_1}{a^2y_1} \times \frac{b^2x_2}{a^2y_2} = -1$$

$$\Rightarrow \frac{x_1x_2}{y_1y_2} = -\frac{a^4}{b^4}$$

$$\Rightarrow \Rightarrow \frac{x_1x_2}{y_1y_2} + \frac{a^4}{b^4} = 0$$

Q.10. Prove that product of perpendiculars from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is $\frac{a^2b^2}{a^2+b^2}$.

Ans: The asymptotes are

$y = \frac{bx}{a}$ & $y = -\frac{bx}{a}$

Let the point be P (x_1, y_1)

Perpendicular distance from $y = \frac{bx}{a}$ is

$$L_1 = \left|\frac{y_1 - \frac{bx_1}{a}}{\sqrt{1 + \frac{b^2}{a^2}}}\right|$$

Perpendicular distance from $y = -\frac{bx}{a}$ is

$$L_2 = \left|\frac{y_1 + \frac{bx_1}{a}}{\sqrt{1 + \frac{b^2}{a^2}}}\right|$$

$$L_1L_2 = \left|\frac{y_1^2 - \frac{b^2x_1^2}{a^2}}{\left(\sqrt{a^2+b^2}\right)^2}\right| a^2$$

$$= \frac{\left|\frac{b^2y_1^2}{b^2} - \frac{b^2x_1^2}{a^2}\right| a}{\left(\sqrt{a^2+b^2}\right)^2}$$

$$= \frac{a^2b^2\left|\frac{y_1^2}{b^2} - \frac{x_1^2}{a^2}\right|}{a^2+b^2}$$

But $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ (x_1, y_1 lies on hyperbola)

$$L_1L_2 = \frac{a^2b^2|-1|}{a^2 + b^2}$$

$$L_1L_2 = \frac{a^2b^2}{a^2 + b^2}$$

MEDIUM

Q.1. Show that the locus of the centre of circle which touches two given circles externally is a hyperbola.

Ans: Let C_1, C_2 be the centres of the two given circles & p & q be their radii. Let ♉ be the radius of the circle touching them externally & c be its centre, then,

$CC_1 = p + r$

$CC_2 = q +$ ♉

We can see that

$CC_1 - CC_2 = p - q = \text{constant}$

Here we observe that the difference of distances of the centre from two fixed points is constant. Hence it satisfies the property of hyperbola the locus of centre is a hyperbola.

Q.12. Find the locus of mid points mid of the chord of $x^2 + y^2 = 16$ which are tangets to the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

Ans: The equation of chord of circle with (h, k) as mid point is

$T = S_1$

i.e. $hx + ky - 16 = h^2 + k^2 - 16$

$\Rightarrow hx + ky = h^2 + k^2$ ..... (i)

If (i) is the tangent to hyperbola therefore it must be of the form

$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

here a = 2 b = 3

$\therefore \frac{x}{2}\sec\theta - \frac{y}{3}\tan\theta = 1$ ..... (ii)

Equating coefficients of (i) & (ii)

$$\frac{2h}{\sec\theta} = \frac{3k}{-\tan\theta} = \frac{h^2 + k^2}{1}$$

$$\Rightarrow \sec\theta = \frac{2h}{h^2 + k^2}$$

$$\& \ \tan\theta = -\frac{3k}{h^2 + k^2}$$

Using $\sec^2\theta - \tan^2\theta = 1$ we get

$$\left(\frac{2h}{h^2 + k^2}\right)^2 - \left(\frac{-3k}{h^2 + k^2}\right)^2 = 1$$

$\Rightarrow 4h^2 - 9k^2 = (h^2 + k^2)^2$

$\therefore$ Locus of (h, k) is $(x^2 + y^2)^2 = 4x^2 - 9y^2$

Q.13. A circle with cintre $(3\alpha, 2\beta)$ & of variable radius cuts the rectangular hyperbola $x^2 - y^2 = 9a^2$ at the points A, B, C, D. Find the locus of centroid of triangle ABC ?

Ans: The equation of circle is

$(x - 3\alpha)^2 + (y - 3\beta)^2 = \gamma^2$ ..... (i)

& hyperbola is

$x^2 - y^2 = 9a^2$ ..... (ii)

Eliminating y from (i) & (ii) we get

$4x^4 - 24\alpha x^3 +$ ..... $= 0$

It is a equation of power four, having roots as x_1, x_2, x_3, x_4.

Let (h, k) be the centroid of PQR

Then,

$$h = \frac{x_1 + x_2 + x_3}{3}$$

& $k = \dfrac{y_1 + y_2 + y_3}{3}$

We have $x_1 + x_2 + x_3 + x_4 = 6\alpha$ ..... (iii)

& similarly $y_1 + y_2 + y_3 + y_4 = 6\beta$

Therefore using (iii) & (iv) we get

$h = \dfrac{6\alpha - x_4}{3}$

& $k = \dfrac{6\beta - y_4}{3}$

But (x_4, y_4) lies on hyperbola

$\therefore\ x_4^2 - y_4^2 = 9a^2$

$\therefore\ 9(h - 2\alpha)^2 - 9(k - 2\beta)^2 = 9a^2$

$\Rightarrow (h - 2\alpha)^2 - (k - 2\beta)^2 = a^2$

Hence the Locus of (h, k) is

$(x - 2\alpha)^2 - (y - 2\beta)^2 = a^2$

Q.14. A normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ meets the axes in M & N & lines MP & NP are drawn perpendicular to the axes meeting at P. Find the Locus of P.

Ans:

Equation of normal at Q is $ax\cos\phi + by\cot\phi = a^2 + b^2$. The normal meets the x-axis at M

$M = \left(\dfrac{a^2 + b^2}{a}\sec\phi\ ,\ 0\right)$

& N = $\left(0, \frac{a^2+b^2}{b}\tan\phi\right)$

Equating of line MP is

$$x = \frac{(a^2+b^2)\sec\phi}{a} \quad \text{..... (i)}$$

& equating of line NP is

$$y = \frac{(a^2+b^2)\tan\phi}{b} \quad \text{..... (ii)}$$

From (i) & (ii)

$\sec^2\phi - \tan^2\phi = 1$

$\Rightarrow \left(\frac{ax}{a^2+b^2}\right)^2 - \left(\frac{by}{a^2+b^2}\right)^2 = 1$

$\Rightarrow a^2x^2 - b^2y^2 = (a^2+b^2)^2$

is the required Locus of P.

Q.15. A point P moves such that the tangents from it to the hyperbola $4x^2 - 9y^2 = 36$ are mutually perpendicular. Find the Locus of P.

Ans: We can write the combined equation of both thetangents. Let P $= (\alpha, \beta$

Then, $SS_1 = T^2$

$S< = 4x^2 - 9y^2 - 36$

$T = 4x\alpha - 9y\beta - 36$

$S_1 = 4\alpha^2 - 9\beta^2 - 36$

Using $SS_1 = T^2$

$(4x^2 - 9y^2 - 36)(4\alpha^2 - 9\beta^2 - 36) = (4x\alpha - 9y\beta - 36)^2$

This is an equation of pair of lines, if these are perpendicular then the coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore$ Coefficient of x^2

$= -144 - 36\beta^2$ ..... (i)

Coefficient of y^2

$= 324 - 36\alpha^2$ ..... (ii)

$\therefore$ by (i) & (ii)

$-36\beta^2 - 36\alpha^2 = -180$

$\Rightarrow \alpha^2 + \beta^2 = 5$

Therefore the Locus of (α, β) is $x^2 + y^2 = 5$

Which is the equation of director circle. Hence we can say that the Locus of point of intersection of two perpendicular tangents is the

director circle.

HARD

Q.16. The normals at three points P, Q, R on a rectangular hyperbola intersects at a point T on the curve. Prove that the centre of hyperbola is the centroid of triangle PQR.

Ans: Equation of normal at any point $(ct, \frac{c}{t})$ on $xy = c^2$ is

$xt^3 - yt - ct^4 + c = 0$

It will pass through T. So, $ht^3 - kt - ct^4 + c = 0$ ..... (i)

Let $T = (h, k)$ & $k = \frac{c}{p}$ ..... (ii)

Curve P is the parameter

By equations (i) & (ii) we get

$cqt^3 - \frac{c}{q}t - ct^4 + c = 0$

$\Rightarrow qt^3 - \frac{t}{q} - t^4 + 1 = 0$

$\Rightarrow q^2t^3 - t - qt^4 + q = 0$

$\Rightarrow q(qt^3 + 1) - t(qt^3 + 1) = 0$

$\Rightarrow (q - t)(qt^3 + 1) = 0$

but $q \neq t$

$\therefore$ $qt^3 + 1 = 0$ ..... (iii)

This is a cubic equation in t representing the points P, Q, R where normals are drawn.

Let the roots be t_1, t_2, t_3 such that

$P = (ct_1, \frac{c}{t_1}$ $q = (ct_2, \frac{c}{t_2})$ $R = (ct_3, \frac{c}{t_3})$

From equation (iii) we get

$t_1 + t_2 + t_3 = 0$ ..... (iv)

& $t_1t_2 + t_2t_3 + t_1t_3 = 0$ ..... (v)

Now the centroid of Triangle PQR is

$x = c(t_1 + t_2 + t_3)$

$= c(0)$ (From equation (iv)

$= 0$

$$y = \frac{c}{t_1} + \frac{c}{t_2} + \frac{c}{t_3}$$

$$= c\left[\frac{t_1t_2 + t_2t_3 + t_1t_3}{t_1t_2t_3}\right]$$

$$= \frac{c[0]}{-1}$$

$= 0$ (From equation (v))

Hence the centroid of Δ PQR is (0, 0) which is the centre of the given hyperbola.

<u>Dumb Question</u>: Initially we have considered the point 't' & Later use considered t_1, t_2, t_3 so this t is t_1 or t_2 or t_3 ?

Ans: Initially we have written that consider any point 't' which can be either t_1 or t_2 or t_3. If it is t_1, other two are t_2, t_3 & if its is t_2, other two are t_1, t_3 & so on. These three point forms a symmet4ry, therefore t is general, either t_1, t_2 or t_3.

Q.17. The chord of contact of the tangents through P to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ subtends a right angle at the centre. Prove that the Locus of P is the ellipse ?

Ans: Let the point P be (h, k), then the equation of chord of contact is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1$$

Now we know that the equations of line passing through origin & the points of intersection of chord of contact with hyperbola is obtained by homogenizing the hyoerbola.

Therefore

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Can be written as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = (1)^2 = \left(\frac{hx}{a^2} - \frac{ky}{b^2}\right)^2$$

because $\frac{hx}{a^2} - \frac{ky}{b^2} = 1$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{h^2x^2}{a^4} + \frac{k^2y^2}{b^4} - \frac{2hkxy}{a^2b^2}$$

This is a homogeneous equation representing pair of staright lines, if these subtend a right angle at the centre than

Coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \frac{1}{a^2} - \frac{h^2}{a^4} - \frac{1}{b^2} - \frac{k^2}{b^4} = 0$$

$\Rightarrow$Multiplying by a^4b^4 we get

$a^2b^2 - h^2b^4 - a^4b^2 - k^2a^4 = 0$

$\Rightarrow b^4h^2 + a^4k^2 = a^2b^2(b^2 - a^2)$

Hence the Locus of (h, k) is

$b^2x^2 + a^2y^2 = a^2b^2(b^2 - a^2)$

$$\Rightarrow \frac{y^2}{a^2(b^2 - a^2)} + \frac{y^2}{b^2(b^2 - a^2)} = 1$$

Hence the Locus is an ellipse.

<u>Dumb Question</u>: Why we have homogenised the hyperbola ?

Ans: Let us consider a general curve

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

& any general line $lx + my + n = 0$

By making the curve homogeneous means

By usig given line make the curve homogeneous of degree 2.

Consider $lx + my + n = 0$, then

$$1 = \frac{lx + my}{-n}$$

Now consider curve

$ax^2 + 2hxy + by^2 + 2gx.1 + 2fy.1 + c.(1)^2 = 0$

$$\Rightarrow ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx + my}{-n}\right) + 2fy\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$$

Hence this is a curve passing through origin & also homogeneous & because it is obtained by the help of curve & the line it represents the pair ofstaright lines passing through origin & points of intersection of line with the curve.

i.e. of OA & OB.

KEY WORDS

- Focus.
- Vertex.
- Directrix.
- Eccentricity.
- Latus Rectum.
- Absicca.
- Ordinate.
- Centre.
- Axes.
- Double Ordinate.
- Focal Chord.
- Parametric Equation.
- Eccentric Angle.
- Focal Distance.
- Conjugate Hyperbola.
- Tangent.
- Normal.
- Slope.
- Pair of Tangents.
- Chord of Cantact.
- Pole.
- Polar.
- Diameter.
- Conjugate Diameter.
- Director Circle.
- Asymtotes.
- Rectangular Hyperbola.

- Locus.
- Transverse Axis.
- Conjugate Axis.

3D-Geometry

Three dimensional geometry developed accordance to Einsteins field equations. It is useful in several branches of science like it is useful in Electromagnetism. It is used in computer alogorothms to construct 3D models that can be interactively experinced in virtual reality fashion. These models are used for single view metrology. 3-D Geometry as carrier of information about time by Einstein. 3-D Geometry is extensively used in quantum & black hole theory.

Section Formula:

(1) **Integral division**: If R(x, y, z) is point dividing join of $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_3)$ in ratio of m : n.

Then, $x = \dfrac{mx_2 + nx_1}{m+n}$, $y = \dfrac{my_2 + ny_1}{m+n}$, $z = \dfrac{mz_2 + nz_1}{m+n}$

(2) **External division**: Coordinates of point R which divides join of $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ externally in ratio m : n are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

Illustration: Show that plane ax + by + cz + d = 0 divides line joining (x_1, y_1, z_1) & (x_2, y_2, z_2) in ratio of $\left(\dfrac{-ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}\right)$

Ans: Let plane ax + by + cz + d = 0 divides line joining (x_1, y_1, z_1) & (x_2, y_2, z_2) in ratio K : 1

∴ Coordinates of P

$$\left(\frac{Kx_2 + x_1}{K+1}, \frac{Ky_2 + y_1}{K+1}, \frac{Kz_2 + z_1}{K+1}\right)$$

must satisfy eq. of plane.

$ax + by + cz + d = 0$

[<u>Dumb Question</u>: Why coordinates of P satisfy eq. of plane ?
Ans: Point P lies in the plane so, it satisfy eq. of plane.]

$$a\left(\frac{Kx_2 + x_1}{K+1}\right) + b\left(\frac{Ky_2 + y_1}{K+1}\right) + c\left(\frac{Kz_2 + z_1}{K+1}\right) + d = 0$$

$\Rightarrow a(Kx_2 + x_1) + b(Ky_2 + y_1) + c(Kz_2 + z_1) + d(K + 1) = 0$

$\Rightarrow K(ax_2 + by_2 + cz_2 + d) + (ax_1 + by_1 + cz_1 + d) = 0$

$\Rightarrow K = -\dfrac{(ax_1 + by_1 + cz_1 + d)}{(ax_2 + by_2 + cz_2 + d)}$

<u>Direction Cosines</u>:

Let $\overrightarrow{OP}$ is a vector α, β, γ inclination with x, y & z-axis respectively. Then $\cos\alpha$, $\cos\beta$ & $\cos\gamma$ are direction cosines of $\overrightarrow{OP}$. They denoted by

α, β, $\gamma \rightarrow$ direction angles.

& lies $0 \le \alpha, \beta, \gamma \le \pi$

<u>Note</u>: (i) Direction cosines of x-axis are (1, 0, 0)
Direction cosines of y-axis are (0, 1, 0)
Direction cosines of z-axis are (0, 0, 1)

(ii) Suppose OP be any line through origin O which has direction l, m, n
$(r\cos\alpha, r\cos\beta, r\cos\gamma)$ where OP = r

$\Rightarrow$ coordinates of P are ($r\cos\alpha$, $r\cos\beta$, $r\cos\gamma$)
or $x = lr$, $y = mr$, $z = nr$

(iii) $l^2 + m^2 + n^2 = 1$

Proof: $|\overrightarrow{OP}| = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$|\vec{r}|^2 = x^2 + y^2 + z^2 = l^2|\vec{r}|^2 + m^2|\vec{r}|^2 + n^2|\vec{r}|^2$

$\Rightarrow l^2 + m^2 + n^2 = 1$

(iv) $\hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$

<u>Direction ratios</u>: Suppose l, m, & n are direction cosines of vector $\vec{r}$ & a, b, c are no.s such that a, b, c are proportional to l, m, n. These a, b, c are c/d direction ratios.

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$$

Suppose a, b, c are direction ratios of vector $\vec{r}$ having direction cosines l, m, n. Then,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \lambda \quad \Rightarrow l = \lambda a,\ m = \lambda b,\ n = \lambda c$$

$\therefore\ l^2 + m^2 + n^2 = 1$

$$a^2\lambda^2 + b^2\lambda^2 + c^2\lambda^2 = 1 \quad \Rightarrow \lambda = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

<u>Note</u>: (i) If $\vec{r} = a\hat{z} + b\hat{j} + c\hat{k}$ having direction cosines l, m, n. Then, $l = \frac{a}{|\vec{r}|}$, $m = \frac{b}{|\vec{r}|}$, $n = \frac{c}{|\vec{r}|}$

(ii) Direction ratios of line joining two given points (x_1, y_1, z_1) & (x_2, y_2, z_2) is $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

(iii) If direction ratio's of $\vec{r}$ are a, b, c

$$\vec{r} = \frac{|\vec{r}|}{\sqrt{a^2 + b^2 + c^2}}\left(a\hat{z} + b\hat{j} + c\hat{k}\right)$$

(iv) Projection of segment joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line with direction cosines l, m, n, is:

$$(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$$

<u>Illustration</u>: If a line makes angles α, β & γ with coordinate axes, prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Ans: Line is making α, β & γ with coordinate axes.
Then, direction cosines are $l = \cos\alpha$, $m = \cos\beta$ & $n = \cos\gamma$.
But $l^2 + m^2 + n^2 = 1$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \cos^2\gamma = 1$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

<u>Angle b/w two vectors in terms of direction cosines & direction ratios</u>:

(i) Suppose $\vec{r_1}$ & $\vec{r_2}$ are two vectors having d.c's l_1, m_1, n_1 & l_2, m_2, n_2 respectively. Then,

$$\hat{r}_1 = l_1\hat{z} + n_1\hat{j} + n_1\hat{k} \quad \& \quad \hat{r}_2 = l_2\hat{z} + n_2\hat{j} + n_2\hat{k}$$

<u>Dumb Question</u>: How $\hat{r}_1 = l_1\hat{z} + n_1\hat{j} + n_1\hat{k}$?

Ans: $\left|\hat{r}_1\right| = \sqrt{l_1^2 + n_1^2 + n_1^2}$

But $l_1^2 + m_1^2 + n_1^2 = 1$

$$\left|\hat{r}_1\right| = \sqrt{1} = 1$$

So, $\hat{r}_1 = l_1\hat{z} + n_1\hat{j} + n_1\hat{k}$

$$\cos\theta = \frac{\hat{r}_1 \cdot \hat{r}_2}{\left|\hat{r}_1\right|\left|\hat{r}_2\right|} = \frac{\left(l_1\hat{z} + n_1\hat{j} + n_1\hat{k}\right)\left(l_2\hat{z} + n_2\hat{j} + n_2\hat{k}\right)}{\sqrt{l_1^2 + n_1^2 + n_1^2}\sqrt{l_2^2 + n_2^2 + n_2^2}}$$

$$\boxed{\cos\theta = l_1l_2 + m_1m_2 + n_1n_2}$$

(ii) IF a_1, b_1, c_1 and a_2, b_2, c_2 are d.r.s of $\vec{r_1}$ & $\vec{r_2}$. Then

$$\hat{r}_1 = l_1\hat{z} + n_1\hat{j} + n_1\hat{k} \quad \& \quad \hat{r}_2 = l_2\hat{z} + n_2\hat{j} + n_2\hat{k}$$

$$\cos\theta = \frac{\vec{r_1}\cdot\vec{r_2}}{|\vec{r_1}|\,|\vec{r_2}|} = \frac{\left(a_1\hat{z} + b_1\hat{j} + c_1\hat{k}\right)\cdot\left(a_2\hat{z} + b_2\hat{j} + c_2\hat{k}\right)}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

<u>Note</u>: (i) If two lines are $\perp$ then,

$\cos\theta = 0$ or $l_1l_2 + m_1m_2 + c_1c_2 = 0$

or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) If two lines are || then

$\cos\theta = 1$ or $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

<u>Illustration</u>: Find angle b/w lines whose direction cosines are

$\left(-\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2}\right)$ & $\left(-\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$.

Ans: Let θ be angle

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

$$= -\frac{\sqrt{3}}{4} \times -\frac{\sqrt{3}}{4} + \frac{1}{4} \times \frac{1}{4} + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{3}{16} + \frac{1}{16} - \frac{12}{16}$$

$$= -\frac{1}{2}$$

$$\cos\theta = -\frac{1}{2} \Rightarrow \theta = 120^0$$

<u>Straight line</u>:

Vectorial eq. of line:

Let a line passing through a point of P.V. $\vec{a}$ & || to given line EF($\vec{b}$).
Then, eq. of line $\vec{r} = \vec{a} + \lambda\vec{b}$

Proof: Let P be any point on line AP & its P.V. is $\vec{r}$
Then, $\vec{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ (By Δ law)
$\vec{r} = \vec{a} + \lambda\vec{b}$

Dumb Question: Why $\overrightarrow{AP} = \lambda\vec{b}$?

Ans: Since $\overrightarrow{AP}$ is || to EF of P.V. $\vec{b}$
So, $\overrightarrow{AP} = \lambda\vec{b}$

Note: Eq. of line through origin & || to $\vec{b}$ is $\vec{r} = \lambda\vec{b}$

Cartesion eq. of straight line:

Passing through point & parallel to given vector .
Let line is passing through A(x_1, y_1, z_1) & || to EF whose d.r.'s are a, b, c.

d.r.'s of AP = $(x - x_1, y - y_1, z - z_1)$
d.r.'s of EF = (a, b, c)

Since EF || AP, then,

$$\boxed{\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}} \rightarrow$$ b Eq. of line in parametric form.

<u>Illustration</u>: Find eq. of line || to $2\hat{z} + \hat{j} + 3\hat{k}$ & passing through pt.(1, 2, 3) ?

Ans: A(1, 2, 3) $\Rightarrow \hat{z} + 2\hat{j} + 3\hat{k} = \vec{a}$

$\therefore$ eq. of line passing through $\vec{a}$ & || to $\vec{b}$

$$\vec{r} = \vec{a} + \lambda\vec{b} = (\hat{z} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{z} + \hat{j} + 3\hat{k})$$

<u>Vector eq. of line passing through two given points</u>:

Vector eq. of line passing through two points whose P.V. S $\vec{a}$ & $\vec{b}$ is:

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

Proof: $\therefore$ $\overrightarrow{AP}$ is collinear with $\overrightarrow{AB}$

$\therefore$ $\overrightarrow{AP} = \lambda\,\overrightarrow{AB}$

$\Rightarrow$ $\overrightarrow{OP} - \overrightarrow{OA} = \lambda\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$

$\Rightarrow$ $\vec{r} - \vec{a} = \lambda\left(\vec{b} - \vec{a}\right)$

$\Rightarrow$ $\boxed{\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)}$

<u>Cartesian form</u>:

Eq. of line passing through (x_1, y_1, z_1) & (x_2, y^2, z_2)

D.R.'s of AB = $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$
D.R.'s of AP = $(x - x_1, y - y_1, z - z_1)$
Since AB || AP

$$\boxed{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}}$$

<u>Illustration</u>: Find cartesian eq. of line are $6x + 2 = 3y - 1 = 2z + 2$.
Find its direction ratios.

Ans: $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ is cartesion eq. of line

$6x + 2 = 3y - 1 = 2z + 2$

$6(x + 1/3) = 3(y - 1/3) = 2(z + 1)$

$$\frac{(x + 1/3)}{1/6} = \frac{(y - 1/3)}{1/3} = \frac{(z + 1)}{1/2}$$

on comparing $a = 1/6$, $b = 1/3$, $c = 1/2$

<u>Angle b/w Two lines</u>:

$\vec{r} = \vec{a} + \lambda\vec{b}$

$\vec{r} = \vec{a'} + \mu\vec{b'}$

Angle b/w two lines $\cos\theta = \frac{\vec{b} \cdot \vec{b}}{|\vec{b}|\,|\vec{b}|}$

If $\vec{b} = a_1\hat{z} + b_1\hat{j} + c_1\hat{k}$

$= \vec{b'} = a_2\hat{z} + b_2\hat{j} + c_2\hat{k}$

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

<u>Condition for perpendicularity</u>:

$$\vec{b} \cdot \vec{b'} = 0 \quad \Rightarrow a_1a_2 + b_1b_2 + c_1c_2$$

<u>Condition for parallelism</u>:

<u>Dumb Question</u>: Why angle between pair of lines

$$\vec{r} = 3\hat{z} + 2\hat{j} + 4\hat{k} + \lambda\left(\hat{z} - 2\hat{j} + 3\hat{k}\right)$$

$$\vec{r} = 5\hat{z} + 2\hat{j} + \mu\left(3\hat{z} + 2\hat{j} + 5\hat{k}\right)$$

Ans: $\vec{r} = \vec{a_1} + \lambda\vec{b}$ & $\vec{r} = \vec{a_2} + \mu\vec{b}$

$$\cos\theta = \frac{\vec{b_1} \cdot \vec{b_2}}{\left|\vec{b_1}\right|\ \left|\vec{b_2}\right|} = \frac{\left(\hat{z} - 2\hat{j} + 3\hat{k}\right)\cdot\left(3\hat{z} + 2\hat{j} + 5\hat{k}\right)}{\sqrt{1^2 + 2^2 + 3^2}\ \sqrt{3^2 + 2^2 + 5^2}}$$

$$\cos\theta = \frac{3 - 4 + 15}{\sqrt{4}\ \sqrt{38}} = \frac{7}{\sqrt{19 \times 7}} \quad \Rightarrow \theta = \cos^{-1}\left(\frac{7}{\sqrt{35}}\right)$$

<u>Perpendicular distance of point froma line</u>:

(a) **<u>Castesian form</u>**:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Suppose 'L' is foot of line of $\perp$.
Since L lies on line AB so,
coordinate of L is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$$

i.e. $L(x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$
direction ratios of PL are:

$$(x_1 + \lambda a - \alpha, y_1 + \lambda b - \beta, z_1 + \lambda c - \gamma)$$

also direction ratios of AB are (a, b, c)
Since PL $\perp$ AB

$$\Rightarrow a(x_1 + \lambda a - \alpha) + b(\lambda b - \beta) + c(z_1 + \lambda c - \gamma) = 0$$

$$\Rightarrow \lambda = \frac{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)}{a^2 + b^2 + c^2}$$

<u>Vector Form</u>:

Since L lies on line AB
P.V. of L = P.V. of line AB

$$= \vec{a} + \lambda \vec{b}$$

dr's of $\overrightarrow{PL} = \vec{a} + \lambda\vec{b} - \vec{\alpha} = (\vec{a} - \vec{\alpha}) + \lambda\vec{b}$

Since $\overrightarrow{PL} \perp \vec{b}$

$$\Rightarrow (\vec{a} + \vec{\alpha} + \lambda\vec{b}).\vec{b} = 0$$

$$\Rightarrow \lambda = \frac{-(\vec{a} - \vec{\alpha}).\vec{b}}{|\vec{b}|^2}$$

$\therefore$ P.V. of L is $\vec{a} + \lambda \vec{b}$

$$= \vec{a} - \left(\frac{(\vec{a} - \vec{\alpha}).\vec{b}}{|\vec{b}|^2} \right) \vec{b}$$

<u>Reflection of point in straight line</u>:

<u>Castesian Form</u>:

From above, we get coordinate of L(foot of $\perp$)
But L is mid point of PQ

$\therefore \frac{\alpha + \alpha'}{2} = x_1 + a\lambda$, $\frac{\beta + \beta'}{2} = y_1 + b\lambda$, $\frac{\gamma + \gamma'}{2} = z_1 + c\lambda$

$\Rightarrow \alpha' = 2(x_1 + a\lambda) - \alpha$, $\beta' = 2(y_1 + b\lambda) - \beta$, $\gamma' = 2(z_1 + c\lambda) - \gamma$

<u>Vector Form</u>:

From above, we get

P.V. of L, $\vec{a} + \lambda \vec{b}$

$$= \vec{a} - \left(\frac{(\vec{a} - \vec{\alpha}).\vec{b}}{|\vec{b}|^2} \right) \vec{b}$$

$$\frac{\vec{\alpha} + \vec{\alpha'}}{2} = \vec{a} - \left(\frac{(\vec{a} - \vec{\alpha}).\vec{b}}{|\vec{b}|^2} \right) \vec{b}$$

$$\Rightarrow \alpha' = 2a - 2\left(\frac{2(\vec{a} - \vec{\alpha}).\vec{b}}{|\vec{b}|^2} \right) \vec{b} - \vec{\alpha}$$

Let P.V. of Q is

Since L is mid point of PQ

$$\frac{\vec{\alpha} + \vec{\alpha'}}{2} = \vec{a} - \left(\frac{(\vec{a} - \vec{\alpha}).\vec{b}}{|\vec{b}|^2} \right) \vec{b}$$

$$\Rightarrow \alpha' = 2\vec{a} - \left(\frac{2(\vec{a} - \vec{\alpha}).\vec{b}}{|\vec{b}|^2} \right) \vec{b} - \vec{\alpha}$$

<u>Illustration</u>: Find reflection of point P(2, 3, 1) in line

$$\frac{x - 2}{3} = \frac{y - 1}{2} = \frac{z + 3}{4}$$

Ans:

Since L lies on line AB

$\therefore$ coordinate of L $(3\lambda + 2, 2\lambda + 1, 4\lambda - 3)$

DR's of PL are

$= (3\lambda + 2 - 2, 2\lambda + 1 - 3, 4\lambda - 3 - 1)$

$= (3\lambda, 2\lambda - 2, 4\lambda - 4)$

DR's of AB are (3, 2, 4)

Since PL $\perp$ AB

$3(3\lambda) + 2(2\lambda - 2) + 4(4\lambda - 4) = 0$

$9\lambda + 4\lambda - 4 + 16\lambda - 16 = 0$

$29\lambda = 20 \quad \Rightarrow = \frac{20}{29}$

Since L is mid point of PQ

So, $\frac{2+\alpha}{2} = 3\lambda + 2$, $\frac{2+\beta}{2} = 2\lambda + 1$, $\frac{2+\gamma}{2} = 4\lambda - 3$

$\alpha = (6\lambda + 4 - 2)$, $\beta = (4\lambda + 2 - 3)$, $\gamma = 8\lambda - 6 - 1$

$\alpha = 6\lambda + 2$, $\beta = 4\lambda - 1$, $\gamma = 8\lambda - 7$

where $\lambda = \frac{20}{29}$

<u>Skew lines</u>: Those lines which do not lies in same plane.

<u>Shortest distence b/w two skew lines</u>:

The line which is $\perp$ to both line l_1 & l_2 are c/d line of shortest distance.

<u>Vector Form</u>:

Let l_1 & l_2 are:

$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ & $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ respectively.

Since $\overrightarrow{PQ}$ is $\perp$ to both l_1 & l_2 which are parallel to $\vec{b_1}$ & $\vec{b_2}$

$\therefore$ $\overrightarrow{PQ}$ is || to $\vec{b_1} \times \vec{b_2}$

Let $\hat{n}$ be unit vector along $\overrightarrow{PQ}$, then $\hat{n} = \pm \dfrac{\vec{b_1} \times \vec{b_2}}{|\vec{b_1} \times \vec{b_2}|}$

$\therefore$ PQ = Projection of $\overrightarrow{AB}$ on $\overrightarrow{PQ}$

$\Rightarrow \overrightarrow{PQ} = \overrightarrow{AB} . \hat{n}$

$$= \pm \left(\vec{a_2} . \vec{a_1}\right) . \frac{\vec{b_1} \times \vec{b_2}}{|\vec{b_1} \times \vec{b_2}|} = \pm \frac{\left(\vec{b_1} \times \vec{b_2}\right).\left(\vec{a_2} \times \vec{a_1}\right)}{|\vec{b_1} \times \vec{b_2}|}$$

<u>Dumb Question</u>: How PQ = Projection $\overrightarrow{AB}$ on $\overrightarrow{PQ}$?

Ans: Form fig., it is clear that PQ is projection of $\overrightarrow{AB}$ on $\overrightarrow{PQ}$.

$$PQ = \left| \frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

<u>Cartesian form</u>: Two skew lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \quad \& \quad \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

<u>Shortest distance</u> :

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}}$$

<u>Condition for lines to intersect</u>:

Two lines intersects if shortest distance = 0

$$\left| \frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right| = 0 \Rightarrow (\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1}) = 0$$

or $$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

<u>Shortest distance b/w parallel line</u>:

Let l_1 & l_2 are

$\vec{r} = \vec{a} + \lambda\vec{b}$

$\vec{r} = \vec{a_2} + \mu\vec{b}$ respectively

& BM is shortest distance b/w l_1 & l_2

$\sin\theta = \frac{BM}{AB} \Rightarrow BM = AB \sin\theta = |\vec{AB}| \sin\theta$

$|\vec{AB} \times \vec{b}| = |\vec{AB}|\,|\vec{b}| \sin(\pi - \theta)$

$= |\vec{AB}|\,|\vec{b}| \sin\theta = (|\vec{AB}| \sin\theta)\,|\vec{b}| =$

<u>Dumb Question</u>: Why we have taken $\sin(\pi - \theta)$?

Ans:

Since direction of vector $\vec{b}$ is opposite to || lines. So, we have taken $(\pi - \theta)$ instead of θ

$$\therefore BM = \frac{|\vec{AB} \times \vec{b}|}{|\vec{b}|} = \frac{|(\vec{a_2} - \vec{a_1}) \times \vec{b}|}{|\vec{b}|}$$

$$\boxed{d = \frac{\left|\left(\vec{a_2} - \vec{a_1}\right) \times \vec{b}\right|}{\left|\vec{b}\right|}}$$

<u>Illustration</u>: Find shortest distance b/w lines:

$$\vec{r} = \left(a\hat{z} - \hat{j}\right) + \lambda\left(\hat{z} + 2\hat{j} - 3\hat{k}\right)$$

$$\vec{r} = \left(\hat{z} + \hat{j} - 2\hat{k}\right) + \mu\left(\hat{z} + \hat{j} + \hat{k}\right)$$

Ans: $$d = \left|\frac{\left(\vec{a_2} - \vec{a_1}\right).\left(\vec{b_1} \times \vec{b_2}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|}\right|$$

On comparing

$$\vec{a_1} = 4\hat{z} - \hat{i} \qquad \vec{a_2} = \hat{z} + \hat{j} - 2\hat{k}$$

$$\vec{b_1} = \hat{z} + 2\hat{j} - 3\hat{k} \qquad \vec{b_2} = \hat{z} + \hat{j} + \hat{k}$$

$$\vec{a_1} - \vec{a_2} = \hat{z} + \hat{j} + 2\hat{k} - 4\hat{z} + \hat{j} = -3\hat{z} + 2\hat{j} + 2\hat{k}$$

$$\vec{b_1} - \vec{b_2} = \begin{vmatrix} \hat{z} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 1 & 1 & 1 \end{vmatrix} = \hat{z}(2+3) - \hat{j}(1+3) + \hat{k}(1-2)$$

$$= 5\hat{z} - 4\hat{j} - \hat{k}$$

$$b = \left|\frac{\left(-3\hat{z} + 2\hat{j} - 2\hat{k}\right).\left(5\hat{z} - 4\hat{j} - \hat{k}\right)}{\sqrt{25 + 16 + 1}}\right|$$

$$b = \left|\frac{-15 - 8 + 2}{\sqrt{42}}\right| = \frac{21}{\sqrt{2}\,\sqrt{21}} = \sqrt{\frac{21}{2}}$$

<u>Plane</u>:

(i) Eq. of plane passing through a given point (x_1, y_1, z_1) is:
$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c $\rightarrow$ constants.

Proof: General eq. of plane is $ax + by + cz + d = 0$

..... (i)
It is passes through (x_1, y_1, z_1)
$\Rightarrow ax_1 + by_1 + cz_1 + d = 0$ ..... (ii)
By (i) - (ii), we get
$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

<u>Intercept form of a plane</u>:

eq. of plane of intercepting lengths a, b & c with x, y & z-axis respectively is,

$$\boxed{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1}$$

<u>Illustration</u>: A variable plane moves in such a way that sum of reciprocals of its intercepts on 3 coordinate axes is constant. Show that plane passes through fixed point.

Ans: Let eq. of plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Then, intercepts of plane with axes are:
A(a, 0, 0), B(0, b, 0), c(0, 0, c)
$\therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ = constant (k) (given)
$\Rightarrow \frac{1}{ak} + \frac{1}{bk} + \frac{1}{ck} = 1$ & comparing with fixed point $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
$x = \frac{1}{k}, y = \frac{1}{k}, z = \frac{1}{k}$
This shows plane passes through fixed point $(\frac{1}{k}, \frac{1}{k}, \frac{1}{k})$

<u>Vector eq. of plane passing through a given point & normal to given vector</u>:

VEctor eq. of plane passing through u point of P.V. $\vec{a}$ & normal to vector $\vec{n}$ is $(\vec{r} - \vec{a}).\vec{n} = 0$

Dumb Question: What is normal to vector ?

Ans: Plane normal to vector means the every line in plane is $\perp$ to that given vector.

Proof: Let plane passes through A($\vec{a}$) & normal to vector $\vec{n}$ & $\vec{r}$ be P.V. of every point 'P' on palne.

Since $\overrightarrow{AP}$ lies in plane & $\vec{n}$ is normal to plane.

$\therefore\ \overrightarrow{AP} \perp \vec{n} \Rightarrow \overrightarrow{AP}.\hat{n} = 0$

$\Rightarrow (\vec{r} - \vec{a}).\vec{n} = 0 \quad (\because \overrightarrow{AP} = \vec{r} - \vec{a})$

eq. of plane $(\vec{r} - \vec{a}).\vec{n} = 0$

Eq. of plane in normal form:

Vector eq. of plane normal to unit vector $\hat{n}$ & at O distance d from origin is $\vec{r}.\hat{n} = d$.

Proof:

ON is $\perp$ to plane such that $\overrightarrow{ON} = d\hat{n}$ & $\overrightarrow{OP} = \vec{r}$

Since $\overrightarrow{NP} \perp . \overrightarrow{ON} = 0$

$\Rightarrow (\overrightarrow{OP} - \overrightarrow{ON}).\overrightarrow{ON} = 0$

$\Rightarrow (\vec{r} - d\hat{n}).d\hat{n} = 0 \quad \Rightarrow \vec{r}.d\hat{n} - d^2\hat{n}.\hat{n} = 0$

$\Rightarrow d(\vec{r}.\hat{n}) - d^2 = 0 \quad \Rightarrow \vec{r}.\hat{n} = d$

eq. of plane is $\vec{r}.\hat{n} = d$

<u>Cartesian Form</u>: Let l, m, n be d.r.'s of normal to given plane & P is length of $\perp$ from origin to plane, then eq. of plane is lx + my + nz = P.

<u>Dumb Question</u>: How this lx + my + nz = P is eq. of plane ?

Ans: Since $\vec{r}.\hat{n} = P = d$ is eq. of plane

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ & $\hat{n} = l\hat{i} + m\hat{j} + z\hat{k}$

$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(l\hat{i} + m\hat{j} + z\hat{k}\right) = P$$

$\Rightarrow$ lx + my + nz = P

<u>Illustration</u>: Find vector eq. of plane which is at a distance of 8 units from origin & which is normal to vector $2\hat{i} + \hat{j} + 2\hat{k}$.

Ans: d = 8 & $\vec{n} = 2\hat{i} + \hat{j} + 2\hat{k}$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

eq. of plane is, $\vec{r}.\hat{n} = d$

$$\vec{r}.\left(\frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}\right) = 8 \quad \Rightarrow \boxed{r.(2i + j + 2k) = 24}$$

<u>Eq. of plane passing through 3 given points</u>:

Eq. of plane passing through three points A, B, C having P.V.'s $\vec{a}$, $\vec{b}$, $\vec{c}$ respectivelt.

Let $\vec{r}$ be P>V> of any point P is the plane.

So, vectors,

$\overrightarrow{AP} = \vec{r} - \vec{a}$, $\overrightarrow{AB} = \vec{b} - \vec{a}$, $\overrightarrow{AC} = \vec{c} - \vec{a}$ are coplanar

Hence, $\overrightarrow{AP}.(\overrightarrow{AB} \times \overrightarrow{AC}) = 0$

$(\vec{r} - \vec{a}).\{(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\} = 0$

$\Rightarrow (\vec{r} - \vec{a}).(\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} - \vec{a} \times \vec{a}) = 0$

$\Rightarrow (\vec{r} - \vec{a}).(\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}) = 0$

$\Rightarrow \vec{r}.(\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}) = \vec{a}.(\vec{b} \times \vec{c}) + \vec{a}.(\vec{a} \times \vec{b}) + \vec{a}.(\vec{c} \times \vec{a})$

$\Rightarrow [\vec{r}\ \vec{b}\ \vec{c}] + [\vec{r}\ \vec{a}\ \vec{b}] + [\vec{r}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]$

<u>Note</u>: If P is length of $\perp$ from origin on this plane,

then $P = \dfrac{[\vec{a}\vec{b}\vec{c}]}{n}$ $n = |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$

Eq. of plane that passes through a point A with position vector $\vec{a}$ & is || to given vectors $\vec{b}$ & $\vec{c}$:

<u>Derivation</u>:

Let $\vec{r}$ be P.V. of any point P in plane

Then, $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{a}$

Sincen $\vec{b} \times \vec{c}$ are || to plane.

So, vector $\vec{r} - \vec{a}$, $\vec{b}$ & $\vec{c}$ are coplanar

$\therefore \ (\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$

$\Rightarrow \vec{r}.(\vec{b} \times \vec{c}) = \vec{a}.(\vec{b} \times \vec{c})$

$\Rightarrow [\vec{r}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]$

<u>Cartesian form</u>: Eq. of plane passing through a point (x_1, y_1, z_1) & || to two lines having direction ratios $(\alpha_1, \beta_1, \gamma_1)$ & $(\alpha_2, \beta_2, \gamma_2)$ is:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \lambda_2 \end{vmatrix} = 0$$

<u>Illustration</u>: Find eq. of plane passing through points P(1, 1, 1), Q(3, -1, 2), R(-3, 5, -4).

Ans: Let eq. of plane is $ax + by + cz + d = 0$

It passes through P(1, 1, 1). So,

eq. of plane

$a(x - 1) + b(y - 1) + c(z - 1) = 0$ ..... (i)

But it also passes through Q & R

$2a - 2b + c = 0] \times (-2)$

$-4a + 4b - 5c = 0$

$$\begin{array}{l} -4a + 4b - 2c = 0 \\ -4a + 4b - 5c = 0 \\ + \qquad\quad + \\ \hline \qquad\quad 3c = 0 \ \Rightarrow \ c = 0 \end{array}$$

Similarly $a = 6$, $b = 6$

Putting these in eq. (i)

We get, $6(x - 1) + 6(y - 1) = 0 \ \Rightarrow x + y = 2$

<u>Angle b/w two planes</u>:

$\vec{r}.\vec{n_1} = d_1 \quad \vec{r}.\vec{n_2} = d_2$ are two planers, then

$$\cos\theta = \frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}|\ |\vec{n_2}|}$$

Note: Angle b/w planes is defined as angle b/w their normals.

Cartesion Form: Let planes are $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$

$$\Rightarrow \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition for $\perp$:

$$\vec{n_1} \cdot \vec{n_2} = 0$$

or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Condition for parallelism:

$$\frac{\vec{n_1}}{\vec{n_2}} = \lambda \quad \text{or} \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$$

Angle b/w line & a plane:

If α, β, γ be direction ratios of line & $ax + by + cz + d = 0$ be eq. of plane in which normal has d.r's a, b, c. Q is angle b/w line & plane.

$$\Rightarrow \cos(90^0 - \theta) = \frac{a\alpha + b\beta + c\gamma}{\sqrt{a^2 + b^2 + c^2}\ \sqrt{\alpha^2 + \beta^2 + \gamma^2}}$$

Vector form: If θ is angle b/w line $\vec{r} = \vec{a} + \lambda\vec{b}$ & plane $\vec{r} \cdot \vec{n} = d$

$$\Rightarrow \sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}|\,|\vec{n}|}$$

<u>Illustration</u>: Find angle b/w line $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z+2}{2}$ & $3x + 2y - 2z + 3 = 0$.

Ans: D.r's of line are 2, 3, 2 & d.r's of normal to plane are 3, 2, - 2

$$\sin\theta = \frac{2 \times 3 + 3 \times 2 + 2 \times (-2)}{\sqrt{2^2 + 3^2 + 2^2}\,\sqrt{3^2 + 2^2 + (-2)^2}} = \frac{8}{\sqrt{17}}$$

$$\theta = \sin^{-1}\left(8/\sqrt{17}\right)$$

Eq. of plane passing through the Line of Intersection of planes
$a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is
$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

<u>Dumb Question</u>: How this eq. is that of required plane ?

Ans: Let $P(\alpha, \beta, \gamma)$ be point on line of intersection of two planes.
So, it lies on both the planes.
i.e. $a_1\alpha + b_1\beta + c_1\gamma + d_1 = 0$
& $a_2\alpha + b_2\beta + c_2\gamma + d_2 = 0$
So, point $P(\alpha, \beta, \gamma)$ should lie on required plane.
i.e. $(a_1\alpha + b_1\beta + c_1\gamma + d_1) + k(a_2\alpha + b_2\beta + c_2\gamma + d_2) = 0$
$\Rightarrow$ $P(\alpha, \beta, \gamma)$ lies on. plane $(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

<u>Vector Form</u>:

$\vec{r} \cdot \vec{n_1} = d_1$ & $\vec{r} \cdot \vec{n_2} = d_2$

So, eq. of required plane is $(\vec{r} \cdot \vec{n_1} - d_1) + k(\vec{r} \cdot \vec{n_2} - d_2) = 0$

<u>Illustration</u>: Find eq. of plane constaining line of intersection of plane $x - y + z + 7 = 0$ and $x + 3y + 2z + 5 = 0$ & passing through (1, 2, 2).

Ans: Eq. of plane through line of intersection of given planes is,

$(x - y + z + 7) + \lambda (x + 3y + 2z + 5) = 0$ ..... (i)

It passes through (1, 2, 3)

$(1 - 2 + 2 + 7) + \lambda (1 + 3 \times 2 + 2 \times 2 + 5) = 0$

$8 + \lambda (7 + 4 + 5) \Rightarrow \lambda = -\frac{8}{16} = -\frac{1}{2}$

Putting $\lambda = -\frac{1}{2}$ in eq. (i)

We get,

$(x - y + z + 7) + (-\frac{1}{2})(x + 3y + 2z + 5) = 0$

$2x - 2y + 2z + 14 - x - 3y - 2z - 5 = 0$

$x - 5y + 9 = 0$

<u>Two sides of plane</u>:

If $ax + by + cz + d = 0$ be a plane then points (x_1, y_1, z_1) & (x_2, y_2, z_2) are points lies on

Same side if $\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} > 0$ & opposite side if $\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} < 0$

<u>Distance of point from a plne</u>:

Length of $\perp$ from a point having P.V. $\vec{a}$ to plane $\vec{r}.\vec{n} = d$ is given by

$$P = \frac{\left|\vec{a}.\vec{n} - d\right|}{\left|\vec{n}\right|}$$

Proof: PM is length of $\perp$ from P to plane. Since line PM passes through P($\vec{a}$) & || to vector $\vec{n}$ which is normal to plane.

$\therefore$ eq. of line $\vec{r} = \vec{a} \times \lambda\,\vec{n}$ ..... (i)

Since point M is intersection of line & plane. So, it lies on line as well as plane

$$\left(\vec{a} + \lambda\vec{n}\right).\vec{n} = d \quad \Rightarrow \lambda = \frac{d - \left(\vec{a}.\vec{n}\right)}{\left|\vec{n}\right|^2}$$

Putting λ in eq. (i)

$$\vec{r} = \vec{a} + \left(\frac{d - \left(\vec{a}.\vec{n}\right)}{\left|\vec{n}\right|^2}\right)\vec{n}$$

$$\overrightarrow{PM} = \text{P.V. of M - P.V. of P} = \vec{a} + \left(\frac{d - \left(\vec{a}.\vec{n}\right)}{\left|\vec{n}\right|^2}\right)\vec{n} - \vec{a}$$

<u>Dumb Question</u>: How P.V. of M is $\vec{a} + \left(\frac{d - \vec{a}.\vec{n}}{\left|\vec{n}\right|^2}\right)\vec{n}$

Ans: M lies on line as well as plane. On solving value of λ for line eq. We get P.V. of M.

So, this is P.V. of M

$$PM = \left|\overrightarrow{PM}\right| = \left|\frac{\left(d - \vec{a}.\vec{n}\right)\vec{n}}{\left|\vec{n}\right|^2}\right| = \frac{\left|d - \left(\vec{a}.\vec{n}\right)\right|\left|\vec{n}\right|}{\left|\vec{n}\right|^2}$$

$$\boxed{PM = \frac{\left|d - \left(\vec{a}.\vec{n}\right)\right|}{\left|\vec{n}\right|}}$$

Cartesian Form: Length of ⊥ from point $P(x_1, y_1, z_1)$ to plane $ax + by + cz + d = 0$, Then eq. of PM is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$$

[Dumb Question: How this eq. of PM comes ?
Ans: It is passes through point (x_1, y_1, z_1) & || to normal of plane so, we get this eq.]

Coordinates of any point on PM are
$(x_1 + ar, y_1 + br, z_1 + cr)$
But this also coordinate of M & M also lies on plane
$\therefore \quad a(x_1 + ar) + b(y_1 + br) + c(z_1 + cr) + d = 0$

i.e. $r = -\dfrac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

$$PM = \sqrt{(x_1 + ar + x_1)^2 + (y_1 + br + y_1)^2 + (z_1 + cr + z_1)^2}$$

$$= \frac{|(ax_1 + by_1 + cz_1 + d)|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance b/w parallel planes:

Distance b/w || planes is difference of length of ⊥ from origin to two planes.
Let $ax + by + cz + d_1 = 0$ & $ax + by + cz + d_2 = 0$

$$D = \left| \frac{|d_1|}{\sqrt{a^2 + b^2 + c^2}} - \frac{|d_2|}{\sqrt{a^2 + b^2 + c^2}} \right| = \left| \frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Vector Form: $\vec{r} \cdot \vec{n} = d_1$ & $\vec{r} \cdot \vec{n} = {}_2$

$$\boxed{d = \frac{|d_1 - d_2|}{|\vec{n}|}}$$

Illustion: Find distance b/w parallel planes
$x + 2y + 2z + 2 = 0$ & $2x + 4y + 4z + 3 = 0$

Ans: Distance b/w $ax + by + cz + d_1 = 0$ & $ax + by + cz + d_2 = 0$ is

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

$$x + 2y + 2z + 2 = 0 \quad \& \quad x + 2y + 2z + \frac{3}{2} = 0$$

$$d = \frac{\left|2 - \frac{3}{2}\right|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{\frac{1}{2}}{3} = \frac{1}{6}$$

<u>Equation of planes Bisecting Angle b/w Two planes</u>:

Eq. of planes bisceting angle b/w planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is,

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Proof: Let P(x, y, z) be point on plane bisecting angle b/w two planes
7 PL & PM is length of ⊥ from P to planes.
By property of angle bisector.

PL = PM

Fig: 1

[<u>Dumb Question</u>: How PL = PM ?
Ans: In fig 1. △ OPL & OPM are congruent by AAA symmetry. So, all sides are equal.
So, PL = PM.]

$$\Rightarrow \left|\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right| = \left|\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

$$\Rightarrow \boxed{\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}}$$

<u>Note</u>: (i) Eq. of bisector of angle b/w two planes containing origin is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(ii) **<u>Bisector of acute & obtuse angles b/w</u>**:

Let $a_1x + b_1y + c_1z + d_1 = 0$
& $a_2x + b_2y + c_2z + d_2 = 0$ where $d_1, d_2 > 0$

(a) If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, origin lies in obtuse angle bisector & eq. of bisector of acute angle is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(b) If $a_1a_2 + b_1b_2 + c_1c_2 < 0$, origin lies in atute angle bisector & eq. of acute angle bisector is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

<u>Illustration</u>: Find eq. of bisector planes of angle b/w planes $2x + y - 2z + 3 = 0$ & $3x + 2y - 6z + 8 = 0$ specify obtuse & acute angle bisectors.

Ans: $2x + y - 2z + 3 = 0$ & $3x + 2y - 6z + 8 = 0$ where $d_1, d_2 > 0$
Now $a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + 1 \times 2 + 2 \times 6 > 0$

$$\therefore \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad \text{.....}$$

(i)
(i) is obtuse angle bisector plane

$$\& \quad \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

..... (ii)

(ii) is acute angle bisector plane.

$$\Rightarrow \frac{2x + y + 2z + 3}{\sqrt{4 + 1 + 4}} = \pm \frac{3x + 2y - 6z + 8}{\sqrt{9 + 4 + 36}}$$

$\Rightarrow 14x + 7y - 14z + 21 = \pm\ 9x + 6y - 18z + 24$

For obtuse angle bisector plane, Take -ve sign

$5x + y + 4z - 3 = 0$

<u>Intersection of line & plane</u>:

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \text{\& plane } ax + by + cz + d = 0$$

Let P be point of intersection coordinate of $P(x_1 + lr, y_1 + mr, z_1 + nr)$

But it satisfy eq. of plane.

$a(x_1 + lr) + b(y_1 + mr) + c(z_1 + nr)$

$$\Rightarrow r = -\frac{(ax_1 + by_1 + cz_1 + d)}{al + bm + cn}$$

<u>Condition for line to be || to a plane</u>:

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \text{ be || to plane } ax + by + cz + d = 0$$

if $al + bm + cm = 0$ or $\sin\theta = 0$

<u>Condition for a line to lie in the plane</u>:

line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ lie in plane $ax + by + cz + d = 0$

if $al + bm + cn = 0$ & $ax_1 + by_1 + cz_1 + d = 0$

<u>Dumb Question</u>: How line be || to plane if al + bm + cn = 0 ?

Ans: If θ is angle b/w line & plane, then

$$\sin\theta = \frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2}\,\sqrt{l^2 + m^2 + n^2}}$$ [Previously desired]

line is || to plane if $\sin\theta = 0$ or al + bm + cn = 0

<u>Sphere</u>:

<u>Equation of sphere whose centre is $\vec{c}$ & radius is a</u>:

In Δ OPC,

$\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$

$\vec{r} = \vec{c} + \overrightarrow{CP} \Rightarrow \overrightarrow{CP} = \vec{r} - \vec{c}$

$\Rightarrow |\vec{r} - \vec{c}| = CP = a$ (radius)

$\Rightarrow |\vec{r} - \vec{c}|^2 = a^2 \Rightarrow (\vec{r} - \vec{c}).(\vec{r} - \vec{c}) = A^2$

$\Rightarrow R^2 - 2\vec{r}.\vec{c} + c^2 = a^2 \Rightarrow r^2 - 2\vec{r}.\vec{c} + (c^2 - a^2) = 0$

<u>Cartesian Form</u>:

From Fig. $CP = R$

$CP^2 = R^2$

By distance formula.

$(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2$

<u>GEneral eq. of sphere</u>:

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is a sphere with centre $(- u, - v, - w)$ &

radius = $\sqrt{u^2 + v^2 + w^2 - d}$

<u>Illustration</u>: Find centre & radius of sphere $2x^2 + 2y^2 + 2z^2 - 2x - 4y + 2z + 3 = 0$

Ans: $2x^2 + 2y^2 + 2z^2 - 2x - 4y + 2z + 3 = 0$

or $x^2 + y^2 + z^2 - x - 2y + z + \frac{3}{2} = 0$

centre is $(- \frac{1}{2}$ coeff. of x, $- \frac{1}{2}$ coeff. of y, $- \frac{1}{2}$ coeff. of z)

$\therefore$ centre $(\frac{1}{2}, 1, - \frac{1}{2})$

radius = $\sqrt{\left(\frac{1}{2}\right)^2 + (1)^2 + \left(- \frac{1}{2}\right)^2 - 3/2} = 0$

Thus sphere is point circle.

<u>Diameter form of Eq. of sphere</u>:

<u>Vector Form</u>: Suppose $\vec{a}$ & $\vec{b}$ be P.V of extremities A & B of diameter AB of sphere. Let $\vec{r}$ be P.V. of any point P on sphere. Then,

$\overrightarrow{AP} = \vec{r} - \vec{a}$ & B $\overrightarrow{BP} = \vec{r} - \vec{b}$

Since, diameter of sphere subtends a right $\angle$ at any point of sphere.

$\therefore \angle APB = \pi/2 \Rightarrow \overrightarrow{AP}.\overrightarrow{BP} = 0$

$\Rightarrow (\vec{r} - \vec{a}).(\vec{r} - \vec{b}) = 0$

$\Rightarrow |\vec{r}|^2 - (\vec{a} + \vec{b}).\vec{r} + \vec{a}.\vec{b} = 0$

or $AP^2 + BP^2 = AB^2$

$$|\vec{r} - \vec{a}|^2 + |\vec{r} - \vec{b}| = |\vec{b} - \vec{a}|^2$$

<u>Cartesian Form</u>: If $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ are extremities of diameter then, eq. of sphere is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

<u>Section of sphere by plane</u>:

If sphere is intersection by a plane. We get circle by intersection.

PM is radius of circle

$PM = \sqrt{CP^2 + CM^2}$

Condition of Tangency of plane to a sphere:

Vector Form: Plane $\vec{r}.\vec{n} = d$ touches sphere $|\vec{r} - \vec{a}| = R$

if $\dfrac{|\vec{a}.\vec{n} - d|}{|\vec{n}|} = R$

Dumb Question: Why $\dfrac{|\vec{a}.\vec{n} - d|}{|\vec{n}|} = R$?

Ans: A plane touches to sphere if $\perp$ distaqnce from centre of sphere is equal to radius.

Cartesian Form: Plane ln + my + nz = p touches $(ul + vm + wn + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$

Illustration: Find eq. of sphere whose centre has P.V. $3\hat{i} + 2\hat{j} + 4\hat{k}$ & which touches plane $\vec{r}.\left(2\hat{i} + 2\hat{j} + \hat{k}\right) = 10$

Ans: Center = $3\hat{i} + 2\hat{j} + 4\hat{k}$

It touches plane $\vec{r}.\left(2\hat{i} + 2\hat{j} + \hat{k}\right) = 10$

$$\therefore \quad \frac{\left|\left(2\hat{i} + 2\hat{j} + \hat{k}\right).\left(2\hat{i} + 2\hat{j} + \hat{k}\right) - 10\right|}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{4}{3} = R$$

$\therefore$ eq. of sphere $|\vec{r} - 3\hat{i} + 2\hat{j} + 4\hat{k}| = \dfrac{4}{3}$

Easy Type

Q.1. Find ratio in which 2x + 3y + 5z = 1 divides line joining the points (1, 0, 2) & (1, 2, 5)

Ans: Let the ratio be k = 1 at point P

Then, P = $\left(\frac{K+1}{K+1}, \frac{2K}{K+1}, \frac{5K+2}{K+1}\right)$ must satisfy 2x + 3y + 5z = 1

$$2\left(\frac{K+1}{K+1}\right) + 3\left(\frac{2K}{K+1}\right) + 5\left(\frac{5K+2}{K+1}\right) = 1$$

$\Rightarrow$ 2(K + 1) + 6K + 25K + 10 = K + 1 $\Rightarrow$ K + 1 + 6K + 25K + 10 = 0

$\Rightarrow$ 32K + 11 = 0 $\Rightarrow$ K = $\frac{-11}{32}$

Thus line divides externally in rstio of 11:32

Q.2. What are direction cosines of a line which is equally inclined to axes ?

Ans: If α, β, γ are angles, if line is equally inclined

$\therefore$ $\alpha = \beta = \gamma$

$l^2 + m^2 + n^2$ $\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow 3\cos^2\alpha = 1$ $\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}} = \cos\beta = \cos\gamma$

$\therefore$ d.c's arw $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ or $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

Q.3. Find direction cosines of line which is $\perp$ to lines with d.r (1, -2, -2) & (0, 2, 1)

Ans: Let l, m, n be d.c's line 1 to given line, d.c's are proportional to d.r's

$\therefore$ d.c'r of lines are $(\lambda, -2\lambda, -2\lambda)$ & $(0, 2\mu, \mu)$

Since line is $\perp$ to given lines

$l\lambda + (-2m\lambda) + (-2n\lambda) = 0$ $\Rightarrow l - 2m - 2n = 0$

& $0 + 2m\mu + n\mu = 0$ $\Rightarrow 2m + n = 0$

By cross multiplication,

$$\frac{l}{2} = \frac{m}{-1} = \frac{n}{2} = R$$

l = 2R, m = - R & n = 2R

$l^2 + m^2 + n^2 = 1$

$4R^2 + R^2 + 4R^2 = 1$ $\Rightarrow 9R^2 = 1$ $\Rightarrow$ $\boxed{R = \pm \frac{1}{3}}$

$$\boxed{l = \pm \frac{2}{3}, m = \mp \frac{1}{3} \ \& \ n = \pm \frac{2}{3}}$$

Q.4. Prove that line x = ay + b, z = cy + d & x = a'y + b', z = c'y + d' are $\perp$ if aa' + cc' + 1 = 0

Ans: Ist line is x = ay + b, z = cy + d

$\frac{x-b}{a} = y, \frac{z-d}{c} = y \Rightarrow \frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$ ..... (i)

and IInd line $\frac{x-b'}{a'} = \frac{y-0}{1} = \frac{z-d'}{c'}$ ..... (ii)

These lines are $\perp$ if

aa' + |x| + cc' = 0 $\Rightarrow$ aa' + cc' + 1 = 0

<u>Dumb Question</u>: For what value of k, lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ & $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect?

Ans: $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = \lambda$

and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1} = \mu$

Since these lines have point of intersection in common. then

$(2\lambda + 1, 3\lambda - 1, 4\lambda + 1) = (\mu + 3, 2\mu + k, \mu)$

or $2\lambda + 1 = \mu + 3$ ..... (i), $3\lambda - 1 = 2\mu + k$ ..... (ii) & $4\lambda + 1 = \mu$ ..... (iii)

on solving (i) & (iii), we get $\lambda = -3/2$ & $\mu = -5$

Substituting in (ii)

$-\frac{a}{2} - 1 = -10 + k \Rightarrow k = \frac{a}{2}$

Q.6. Show that points $\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + 3\hat{j} + 3\hat{k}$ are equidistant from plane $\vec{r}.(5\hat{i} + 3\hat{j} - 7\hat{k}) + 9 = 0$

Ans: $\vec{r}.(5\hat{i}+3\hat{j}-7\hat{k}) = -9$

length of $\perp$ from $(3\hat{i}+3\hat{j}+3\hat{k})$

$$\Rightarrow \frac{-9-\left(3\hat{i}+3\hat{j}+3\hat{k}\right).\left(5\hat{i}+2\hat{j}-7\hat{k}\right)}{\sqrt{25+4+49}} = \frac{-9-15-6+21}{\sqrt{78}}$$

$$\Rightarrow \left|\frac{-9}{\sqrt{78}}\right| = \frac{9}{\sqrt{78}}$$

So, length of $\perp$ is equal.

Q.7. Find the point in which the plane; $\vec{r} = (\vec{a}-\vec{b}) + m(\vec{a}+\vec{b}+\vec{c}) + n(\vec{a}+\vec{c}-\vec{b})$ is cut by line through point $2\vec{a}+3\vec{b}$ & parallel to $\vec{c}$.

Ans: eq. of line through point $2\vec{a}+3\vec{b}$ & || to $\vec{c}$

$$\vec{r} = (2\vec{a}+3\vec{b}) + \lambda\vec{c}$$

Since line cuts plane

For one point,

$$(2\vec{a}+3\vec{b}) + \lambda\vec{c} = \vec{a}-\vec{b}+m(\vec{a}+\vec{b}+\vec{c}) + n(\vec{a}+\vec{c}-\vec{b})$$

Equating coeff. of $\vec{a}$, $\vec{b}$ & $\vec{c}$ on both side, we get

$$m+n=1, \quad m-n=4, \quad -m+n=\lambda$$

$\Rightarrow m = \frac{5}{2}$, $n = \frac{-3}{2}$ & $\lambda = -4$

$\therefore$ P.V. of required point is: $2\vec{a}+3\vec{b}-4\vec{c}$

Q.8. Show that line of intersection of planes $\vec{r}.(\hat{i}+2\hat{j}+3\hat{k}) = 0$ & $\vec{r}.(3\hat{i}+3\hat{j}+\hat{k}) = 0$ is equally inclined to $\hat{i}$ & $\hat{k}$.

Ans: **<u>Note</u>**: Line of intersection of two planes will be $\perp$ to normals to the planes. Hence it is || to vector

$$\vec{n_1} \times \vec{n_2} = \left(\hat{i}+2\hat{j}+3\hat{k}\right) \times \left(3\hat{i}+3\hat{j}+\hat{k}\right) = \left(-4\hat{i}+8\hat{j}+4\hat{k}\right)$$

Now,

$$\left(-4\hat{i}+8\hat{j}-4\hat{k}\right).\hat{i} = -4$$

& $\left(-4\hat{i}+8\hat{j}-4\hat{k}\right).\hat{i}=-4$

So, line is equally inclined to $\hat{i}$ & $\hat{k}$

Q.9. Projection of line segment on 3 axes 4, 5 & 13 respectively. Find length & direction cosines of line segment.

Ans:

$\overrightarrow{AB}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$

$\therefore$ Projection of $\overrightarrow{AB}$ on x-axis

$=\overrightarrow{AB}.\hat{i}=(x_2-x_1)=4$

Projection of $\overrightarrow{AB}$ on y-axis

$=\overrightarrow{AB}.\hat{j}=(y_2-y_1)=5$

Projection of $\overrightarrow{AB}$ on z-axis

$=\overrightarrow{AB}.\hat{k}=(z_2-z_1)=13$

$\therefore$ Length of AB $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}=\sqrt{4^2+5^2+13^2}$

$=\sqrt{210}$

$$\hat{AB}=\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}=\frac{4}{\sqrt{210}}\hat{i}+\frac{5}{\sqrt{210}}\hat{j}+\frac{13}{\sqrt{210}}\hat{k}$$

$\therefore$ d.r's $=\frac{4}{\sqrt{210}}, \frac{5}{\sqrt{210}}$ & $\frac{13}{\sqrt{210}}$

Q.10. Find locus of mid point of chords of sphere $r^2 - 2\vec{r}.\vec{c} + k = 0$ if chords being drawn || to vector $\vec{b}$.

Ans: $r^2 - 2\vec{r}.\vec{c} + k = 0$ is sphere of centre $\vec{c}$ chord AB || $\vec{b}$.

$\Rightarrow \overrightarrow{CM} \perp \overrightarrow{AB}$

$\Rightarrow (\vec{\alpha} - \vec{c}).\vec{b} = 0$

$\therefore$ locus of M is $(\vec{r} - \vec{c}).\vec{b} = 0$

$\Rightarrow \vec{r}.\vec{b} = \vec{c}.\vec{b} \rightarrow$ represents a plane.

<u>Dumb Question</u>: Why $\overrightarrow{CM} \perp \overrightarrow{AB}$ is same as $\overrightarrow{CM} \perp \vec{b}$?

Ans: Since $\overrightarrow{AB} \parallel \vec{b}$. So, if $\overrightarrow{CM}$ is $\perp \overrightarrow{AB}$. So, it will also $\perp$ to $\vec{b}$.

<u>Medium Type</u>

Q.1. If a variable plane forms a tetrahedron of constant volume $27k^3$ with coordinate planes, find locus of centroid of tetrahedron.

Ans: sLet variable plane cuts coordinate axes at A(a, 0, 0), B(0, b, 0), C(0, 0, c)

Then, eq. of plane will be

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = 1$$

Let Pα, β, γ) be centroid of terahedron OABC,

then, $\alpha = \frac{a}{4}$, $\beta = \frac{b}{4}$, $\gamma = \frac{c}{4}$

[<u>Dumb Question</u>: How this centroid tetrahedron OABC comes $\alpha = \frac{a}{4}$, $\beta = \frac{b}{4}$, $\gamma = \frac{c}{4}$?

Ans: Centroid of tetrahedrold

$$\alpha = \frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{a + 0 + 0 + 0}{4} = \frac{a}{4}$$

where (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) & (x_4, y_4, z_4) are coordinate of tetrahedron.]

Volume of tetrahedron = $\frac{1}{3}$ (Area of Δ AOB).OC

$$27k^3 = \frac{1}{3}\left(\frac{1}{2}ab\right)c = \frac{abc}{6}$$

$$\Rightarrow 27k^3 = \frac{(4\alpha)(4\beta)(4\gamma)}{6}$$

$$\Rightarrow \frac{27 \times 6k^3}{64} = \alpha\beta\gamma$$

Required locus of P(α, β, γ) is

$$\boxed{x\,y\,z = \frac{81}{32}\,k^3}$$

Q.2. Find vector eq. of straight line passing through intersection of plane

$$\vec{b} + \lambda_1(\vec{b} - \vec{a}) + \mu_1(\vec{a} + \vec{c}) = \vec{c} + \lambda_2(\vec{b} - \vec{c}) + \mu_2(\vec{a} + \vec{b})$$

$$\Rightarrow (-\lambda_1 + \mu_1 - \mu_2)\,\vec{a} + (1 + \lambda_1 - \lambda_2 - \mu_2)\,\vec{b} +$$

$\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors.

Ans: At points of intersection of two planes.

$$\vec{b} + \lambda_1(\vec{b} - \vec{a}) + \mu_1(\vec{a} + \vec{c}) = \vec{c} + \lambda_2(\vec{b} - \vec{c}) + \mu_2(\vec{a} + \vec{b})$$

$$\Rightarrow (-\lambda_1 + \mu_1 - \mu_2)\,\vec{a} + (1 + \lambda_1 - \lambda_2 - \mu_2)\,\vec{b} + (\mu_1 - 1 + \lambda_2)\,\vec{c} = 0$$

Since $\vec{a}, \vec{b}, \vec{c}$ are non coplanar, then

$$-\lambda_1 + \mu_1 - \mu_2 = 0, \quad 1 + \lambda_1 - \lambda_2 - \mu_2 = 0, \quad \mu_1 - 1 + \lambda_2 = 0$$

On solving, we get $\lambda_1 = 0$ & $\mu_1 = \mu_2$

$\therefore \quad \vec{r} = \vec{b} + \lambda_1(\vec{b} - \vec{a}) + \mu_1(\vec{a} + \vec{c})$

Since $\lambda_1 = 0$

$\Rightarrow \vec{r} = \vec{b} + \mu_1(\vec{a} + \vec{c})$ is required eq. of straight line.

Q.3. Prove that three lines from O with direction cosines l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are coplaner if

$$l_1(m_2n_3 - n_2m_3) + m_1(n_2l_3 - l_2n_3) + n_1(l_2m_3 - l_3m_2) = 0$$

Ans: **<u>Note</u>**: Three given lines are coplanar if they have common perpendicular. Let d.c's of common $\perp$ be l, m, n

$\Rightarrow ll_1 + mm_1 + nn_1 = 0$ ..... (i)

$ll_2 + mm_2 + nn_2 = 0$ ..... (ii)

$ll_3 + mm_3 + nn_3 = 0$ ..... (iii)

Soplving (ii) & (iii) by cross multiplication .....

$$\Rightarrow \frac{l}{m_2n_3 - n_2m_3} = \frac{m}{n_2l_3 - n_3l_2} = \frac{n}{l_2m_3 - l_3m_2} = k \Rightarrow$$

$l = k(m_2n_3 - n_2m_3)$, $m = k(n_2l_3 - n_3l_2)$, $n = k(l_2m_3 - l_3m_2)$

Substituting in (i), we get

$$k(m_2n_3 - n_2m_3)l_1 + k(n_2l_3 - n_3l_2)m_1 + k(l_2m_3 - l_3m_2)n_1 = 0$$

$\Rightarrow l_1(m_2n_3 - n_2m_3) + m_1(n_2l_3 - n_3l_2) + n_1(l_2m_3 - l_3m_2) = = 0$

Q.4. Solve the equation $x\vec{a} + y\vec{b} + z\vec{c} = \vec{d}$

Ans: $x\vec{a} + y\vec{b} + z\vec{c} = \vec{d}$ ..... (i)

Taking dot product by $\vec{b} \times \vec{c}$, we get

$$x\vec{a}.(\vec{b} \times \vec{c}) + y\vec{b}.(\vec{b} \times \vec{c}) + z\vec{c}.(\vec{b} \times \vec{c}) = \vec{d}.(\vec{b} \times \vec{c})$$

$$x[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{d}\ \vec{b}\ \vec{c}] \Rightarrow x = \frac{[\vec{d}\ \vec{c}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

& $z = \dfrac{[\vec{d}\ \vec{a}\ \vec{b}]}{[\vec{a}\ \vec{b}\ \vec{c}]}$

$\therefore\ x\vec{a} + y\vec{b} + z\vec{c} = \vec{d}$

$\Rightarrow [\vec{d}\ \vec{b}\ \vec{c}]\vec{a} + [\vec{d}\ \vec{c}\ \vec{a}]\vec{b} + [\vec{d}\ \vec{a}\ \vec{b}]\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$ is required solution.

Q.5. If planes $x - cy - bz = 0$, $cx - y + az = 0$ & $bx + ay - z = 0$ pass through a straight line, the find value of $a^2 + b^2 + c^2 + 2abc$.

Ans: Givem planes are:

$x - cy - bz = 0$ ..... (i)

$cx - y + az = 0$ ..... (ii)

& $bx + ay - z = 0$ ..... (iii)

Eq. of plane passing through line of intersection of plane (i) & (ii) is

$(x - cy - bz) + \lambda(cx - y + az) = 0$

$\Rightarrow x(1 + \lambda c) - \lambda(c + \lambda) + z(-b + a\lambda) = 0$ ..... (iv)

If plane (iv) & (iii) are same, then,

$$\frac{1 + c\lambda}{b} = \frac{-(c + \lambda)}{a} = \frac{-b + a\lambda}{-1}$$

$\Rightarrow \lambda = -\dfrac{(a + bc)}{(ac + b)}$ and $\lambda = -\dfrac{(ab + c)}{(1 - a^2)}$

$$\therefore\ -\frac{(a + bc)}{(ac + b)} = -\frac{(ab + c)}{(1 - a^2)}$$

$\Rightarrow a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$

$\Rightarrow a(2abc + c^2 + b^2 + a^2 - 1) = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

<u>Hard Type</u>

Q.1. Show that $\perp$ distance of a point A($\vec{a}$) to line $\vec{r} = \vec{b} + t\vec{c}$ is $\left|\vec{b} + \left(\frac{(\vec{a} - \vec{b}).\vec{c}}{c}\right)\vec{c} - \vec{a}\right|$ which is also equal to $\left|\frac{(\vec{a} - \vec{b}) \times \vec{c}}{|\vec{c}|}\right|$.

Ans:

Let P is point on line such that $\overrightarrow{AP} \perp \overrightarrow{BP}$ & P.V. of P is $\vec{r} = \vec{b} + t\vec{c}$

$\therefore \quad \overrightarrow{AP}.\overrightarrow{BP} = 0$

$(\vec{r} - \vec{a}).(\vec{r} - \vec{b}) = 0$ ..... (i)

$(\vec{b} + t\vec{c} - \vec{a}).(\vec{b} + t\vec{c} - \vec{b}) = 0$

$(\vec{b} - \vec{a} + t\vec{c}).(t\vec{c}) = 0$

$\Rightarrow t\vec{b}.\vec{c} - t\vec{a}\,\vec{c} + t^2\vec{c}\,\vec{c} = 0$

$\Rightarrow t\vec{c}\,(\vec{b} - \vec{b}) + t^2c^2 = 0$

$\Rightarrow t = \frac{\vec{c}.(\vec{a} - \vec{b})}{c^2}$ ..... (ii)

From (i) & (ii)

$$\vec{r} = \vec{b} + \left(\frac{(\vec{a} - \vec{b}).\vec{c}}{c^2}\right)\vec{c}$$

$$\overrightarrow{AP} = \vec{r} - \vec{a} = \vec{b} + \left(\frac{(\vec{a} - \vec{b}).\vec{c}}{c^2}\right)\vec{c} - \vec{a}$$

$$|\overrightarrow{AP}| = |\vec{r} - \vec{a}| = \left|\vec{b} + \left(\frac{(\vec{a} - \vec{b}).\vec{c}}{c^2}\right)\vec{c} - \vec{a}\right| \quad \text{..... (iii)}$$

Also, $PA^2 = AB^2 - BP^2 = |\overrightarrow{AB}|^2 - |\overrightarrow{BP}|^2$

$$= |\vec{b} - \vec{a}|^2 = |t\vec{c}|^2 = |\vec{b} - \vec{a}|^2 - \left|\frac{(\vec{a} - \vec{b}).\vec{c}}{c^2}\vec{c}\right|^2$$

$$= |\vec{b} - \vec{a}|^2 - \frac{\{\vec{c}.(\vec{a} - \vec{b})\}^2}{c^4}c^2$$

$$= \frac{|\vec{b} - \vec{a}|^2 c^2 - (\vec{c}.(\vec{a} - \vec{b}))^2}{c^2} = \frac{|(\vec{b} - \vec{a}) \times \vec{c}|^2}{|\vec{c}|^2}$$

..... (iv)

<u>Dumb Question</u>: How this eq. (iv) const.

Ans: We know that $(AB)^2 - (\vec{A}.\vec{B})^2 = |\vec{A} \times \vec{B}|^2$

Q.2. If direction cosines of variable line in two adjacent points be l, m, n & $l + \delta l$, $m + \delta m$, $n + \delta n$ show that small anhgle 5θ b/w two position is

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$$

Ans: As we know $l^2 + m^2 + n^2 = 1$

and $(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$

$\Rightarrow l^2 + (\delta l^2)\ 2l\delta l + m^2 + (\delta m)^2 + 2m\delta m + n^2 + (\delta n)^2 + 2n\delta n = 1$

$\Rightarrow l^2 + m^2 + n^2 + (\delta l)^2 + (\delta m)^2 + (\delta n)^2 + 2(l\delta l + m\delta m + n\delta n) = 1$

$\Rightarrow (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = -2(l\delta l + m\delta m + n\delta n)$ ..... (i)

$\delta\theta$ is angle b/w two positions.

$$\cos\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$$

$$1 - 2\sin^2\frac{\delta\theta}{2} = 1 + l\delta l + m\delta m + n\delta n \text{ .....}$$

(ii)
From (i) & (ii)

$$(\delta l)^2 + (\delta m)^2 + (\delta n)^2 = 4 \sin^2 \frac{\delta\theta}{2}$$

$$\Rightarrow 4\left(\frac{\delta\theta}{2}\right)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

$$\boxed{(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2}$$

<u>Dumb Question</u>: How $\sin^2 \frac{\delta\theta}{2}$ is equal to $\left(\frac{\delta\theta}{2}\right)^2$?

Ans: Since $\delta\theta$ is very small so, $\frac{\delta\theta}{2}$ is very small
So, $\sin \frac{\delta\theta}{2} \rightarrow \frac{\delta\theta}{2}$ as $\delta\theta \rightarrow 0$
So, for approsmation we have taken.

Q.3. Find image of point P(3, 5, 6) in plane 2x + y + z = 0 & find eq. of sphere having point P(3, 5, 6) & its image point of extremities of diameter.

Ans:

2x + y + z = 0
P(3, 5, 6)
dr's iof normal to plane (i) are 2, 1, 1

Let θ be image of point P in plane (i)
Eq of P R is,

$$\frac{x-3}{2} = \frac{y-5}{1} = \frac{z-6}{1} = r$$

$\therefore$ R(2r + 3, r + 5, r + 6)

[<u>Dumb Question</u>: How eq. of line PR come ?
Ans: d.r's of normal to plane is same as d.r's of line PR i.e. 2, 1, 1
So, we get eq. of line PR]

But R lies on (i)

$\therefore$ 2(2r + 3) +(r + 5) + (r + 6) = 0

$6r + 17 = 0 \Rightarrow r = -\frac{17}{6}$

$$R \equiv \left(\frac{-34}{6}+3, \frac{-17}{6}+5, \frac{-17}{6}+6\right)$$

Since R lies mid point of PQ

$$\frac{-34+18}{6} = \frac{\alpha+3}{2}$$

$$\frac{-16}{3} = \alpha+3 \Rightarrow \alpha = -\frac{25}{3}$$

$$\frac{13}{6} = \frac{\beta+5}{2} \Rightarrow \frac{13}{2} - 5 = \beta \Rightarrow \beta = -\frac{2}{3}$$

$$\frac{19}{6} = \frac{\gamma+6}{2} \Rightarrow \gamma + 6 = \frac{19}{3} \Rightarrow \gamma = \frac{19}{3} - 6$$

$$\Rightarrow \gamma = 1/3$$

$$Q = \left(\frac{-25}{3}, -2/3, 1/3\right)$$

Eq. of sphere when extremities (x_1, y_1, z_1) & (x_2, y_2, z_2) are given:

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

$$(x-3)\left(x+\frac{25}{3}\right) + (y-5)\left(y+\frac{2}{3}\right) + (z-6)\left(y+\frac{1}{3}\right) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + \frac{16}{3}x - \frac{13}{3}y - \frac{19}{3}z - 25 - \frac{10}{3} + \frac{6}{3} = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + \frac{16}{3}x - \frac{13}{3}y - \frac{19}{3}z - 25 - \frac{4}{3} = 0$$

$$\Rightarrow 3x^2 + 3y^2 + 3z^2 + 16x - 13y - 19z - 79 = 0$$

<u>Key words</u>

- Direction COsines.
- Direction Ratios.
- Reflection of a point.
- Skew Lines.
- Shortest distance.
- Intersection of lines.
- Normal. FV
- Angle bisector.
- Tangency of plane to sphere.

Part-II

Functions

FUNCTION & GRAPH

Functions:

Let A $\triangle$ B be two non empty sets & F is a relation which associates each elemenet of set A with unique element of set B, then F is c/d a function from A to B.

Set A is called domain of F & B be the co domain of F.

Set of elements of B, which are images of elements of set A is c/d range of F.
F : A $\rightarrow$ B ("F is function of A into B")

If a $\in$ A then element in B which is assigned to 'a' is called image of 'a' & denoted by F(a).

let A = {a, b, c, d}, B = {1, 2, 3, 4, 5}
So, F(a) = 2, F(b) = 3, F(c) = 5, F(d) = 1

Dumb Question: What is non empty set ?

Ans: A set which contain at least one element
eg. A = {1, 2} is non empty set but B = { } is empty set.

No. of Function (or Mapping) From A to B:

Let A = $\{x_1, x_2, x_3, \ldots\ldots\ldots, x_m\}$ F : A $\rightarrow$ B
& B = $\{y_1, y_2, y_3, \ldots\ldots\ldots, y_n\}$

$F : A \longrightarrow B$

Then if each element in set A has n images in set B.
Thus, total no. of functions from A to B = n^m

<u>Dumb Question</u>: How total no. of function from A to B = n^m ?

Ans: x_1, element in set A take n images
x_2 element in set A take n images
.....
.....
x_m can take n images.

$\therefore$ Total no. of function from A to B
$\Rightarrow$ n x n x ..... m times = n^m

<u>Domain</u>: Domain of y = f(x) is set all real x for which f(x) is defined (real).

<u>How to find Domain</u>:

(i) Expression under even root (i.e. square root, 4th root) ≤ 0

(ii) Denominator $\neq 0$

(iii) If domain of y = (x) & y = y(x) are D_1 & D_2 respectively then domain of f(x) ± y(x) of f(x).g(x) is $D_1 \cap D_2$.

(iv) Domain of $\frac{f(x)}{g(x)}$ is $D_1 \cap D_2$ - {g(x) = 0}.

<u>Illustration</u>: Find domain of single valued function y = f(x) given by eq. $10^x + 10^y = 10$.

Ans: $10^x + 10^y = 10 \Rightarrow 10^y = 10 - 10^x$
$\Rightarrow y = \log_{10}(10 - 10^x)$ $\{a^m = b \Rightarrow m = \log_a b\}$
Now, $10 - 10^n > 0$
$\Rightarrow 10^1 > 10^x \Rightarrow 1 > x$
domain $x \in (-\infty, 1)$

<u>Dumb Question</u>: What is single value Function ?

Ans. For every point of domains, these is unique image only.

<u>Range</u>: Range of $y = f(x)$ is collection of all output & $\{F(x)\}$ corresponding to each real no. is domain.

<u>How to find range</u>:

First of all final domain of $y = f(x)$

(i) If domain $\in$ Finite no. of points $\Rightarrow$ range $\in$ set of corresponding f(x)values.

(ii) If domain $\in$ R or R - {some finite points}
Then express x in terms of y. From this find y for x to be defined.
(i.e. find values of y for which x exists)

(iii) If domain $\in$ a finite interval, find least and greatest value for range using mono tonicity.

<u>Illustration</u>: Find range of function $y = \log_e(3x^2 - 4x + 5)$.

Ans: y is defined if $3x^2 - 4x + 5 > 0$
[<u>Dumb Question</u>: Why $3x^2 - 4x + 5 > 0$?
Ans: ln x is defined only if $x > 0$]
if x is 0 ln x = $-\infty$ & for - ve x
ln is not defined,
$D = 16 - 4 \times 3 \times 5 = -44 < 0$
& coeff. of $x^2 = 3 > 0$
$\Rightarrow 3x^2 - 4x + 5 > 0 \ \forall\, x \in R$
$\therefore$ domain $\in$ R

$y = \log_e(3x^2 - 4x + 5) \Rightarrow 3x^2 - 4x + 5 = e^y$

$\Rightarrow 3x^2 - 4x + (5 - e^y) = 0$

Since x is real, So, $D \leq 0$

$(-4)^2 - 4(3)(5 - e^y) > 0 \Rightarrow e^y > 11/3$

$\Rightarrow y \leq \log 11/3$

$\therefore$ range $\in [\log 11/3, \infty)$

CLASSIFICATION OF FUNCTIONS

1. **Constant Function**: If range of function f consists of only one no. then f is c/d constant function.

Range = { a }

domain $x \in R$

2. **Polynomial function**: A function $y = f(x) = a_0x^n + a_1x^{n-} + ... + a_n$ where $a_0, a_1, ... a_n$ are real constants & n is non -ve integer, then f(x) is c/d polynomial function. If $a_0 = 0$, then n is degree of polynomial function.

Graph of $f(x) = x^2$

$f(x) = x^2$ is called square function. Domain $\in R$

Range $\in R^+ \cup \{0\}$ or $[0, \infty]$

<u>Graph of f(x) = x^3</u>:

$f(x) = x^2$ is cube function
domain $\in$ R
Range $\in$ R

(3) **<u>Rational Function</u>**: It is ratio of two polynomials
Let $P(x) = a_0x^n + a_1x^{n-1} + \dots\dots\dots + a_n$
$Q(x) = b_0x^m + b_1x^{m-1} + \dots\dots\dots + b_m$

Then $f(x) = \frac{f(x)}{g(x)}$ is a rational function if $Q(x) \neq 0$
Domain $\in$ R $\in \{x \mid Q(x) = 0\}$
i.e. Domain $\in$ R except those points for which denomiator = 0

<u>Graph of f(x) = $1/x$</u>

$f(x) = 1/x$ is called reciprocal function with coordinate axis as asymptotes.

Domain $\in$ R - {0}
Range $\in$ R - {0}

<u>Graph of f(x) = $1/x^2$</u>

f(x) = $1/x^2$
Domain $\in$ R - {0}
Range $\in$ (0, ∞)

<u>Dumb Question</u>: For y = $1/x$, how domain & range is R - {0} ?

Ans: For domains
$\lim_{x \to 0^+} f(x) = +\infty$ & $\lim_{x \to 0^-} f(x) = -\infty$
So, f(x) is not defined at x = 0
Similarly for Range
as $x \to \pm\infty \Rightarrow f(x) \to 0$
But we exclude $\pm\infty$
So, Range $\in$ R - {0}

(4) **<u>Irrational Function</u>**: Algebraic function containing terms having non-integral rational powers of x are c/d irrational functions.

<u>Graph of f(x) = $x^{1/2}$</u>

f(x) = $\sqrt{x}$ Domain $\in R^+ \cup \{0\}$ or $\{0, \infty)$
Range $\in R^+ \cup \{0\}$ of $[0, \infty)$

<u>Grafh of f(x) = $x^{1/3}$</u>

f(x) = $x^{1/3}$
domains $\in R$
Range $\in R$

(5) **<u>Identity Function</u>**: The function y = f(x) = x for all x $\in$ R c/d identity function on R

Domain $\in$ R
Range $\in$ R

<u>Absolute Value of Modulus Function</u>:

$$y = |x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Domain $\in$ R Range $\in (0, \infty)$

<u>Properties of modulus function</u>:

(i) $|x| \leq a \Rightarrow -a \leq x \leq a \quad (a \geq 0)$
(ii) $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a \quad (a \geq)$
(iii) $|x + y| \leq |x| + |y|$
(iv) $|x + y| \geq \big|\, |x| - |y| \,\big|$

<u>Illustration</u>: Find domain of $y = \dfrac{1}{\sqrt{|x| - x}}$

Ans: y is defined if $(|x| - x) > 0$
$\Rightarrow |x| > x$ which hold for -ve x only
Hence domain $(-\infty, 0)$

(7) **<u>Signum Function</u>**:

$$y = \text{Sgn}(x) = \begin{cases} \frac{|x|}{x} \text{ or } \frac{x}{|x|} & x \neq 0 \\ 0 \quad \text{if} \quad x = 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Domain $x \in R$
Range $\in \{-1, 0, 1\}$

(8) **<u>Greatest integer function</u>**:

[x] indicates integral part of x which is nearest & smaller integer to x.
It is also c/d floor of x or stepwise function.
[2.3] = 2 [5] = 5, [- 0.6] = - 1
In general

$n \leq n < n + 1$ ($n \in$ Integer)

$[n] = n$

x	[x]
$0 \leq x < 1$	0
$1 \leq x < 2$	1
$2 \leq x < 3$	2
$-1 \leq x < 0$	- 1
$-2 \leq x < -1$	- 2

<u>Properties of greatest integer function</u>:

(i) $[x] = x$ holds if $x \in I$ $\quad I \rightarrow$ Integer.

(ii) $[x + I] = [x] + I$ if I is integer.

(iii) $[x + y] \geq [x] + [y]$

(iv) If $[\phi(x)] \geq I$, then $\phi(x) \geq I$.

(v) If $[\phi(x)] \leq I$, then $\phi(x) < I + 1$

(vi) If $[x] > n \quad \Rightarrow x \geq n + 1$, $n \in I$.

(vii) $[-x] = -[x]$ if $x \in I$
& $[-x] = -[x] - 1$ if $x \notin I$.

(viii) $[x + y] = [x] + [x + y - [x]]$ for all $x, y \in R$.

(ix) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots\ldots\ldots\ldots + \left[x + \frac{n-1}{n}\right]$

$= [n\,x]$, $n \in N$ $\quad N \rightarrow$ natural no.

<u>Illustration</u>: $y = 2[x] + 3$ & $y = 3[x - 2] + 5$ then find value of $[x + y]$.

Ans: $2[x] + 3 = 3[x - 2] + 5$

$\Rightarrow 2[x] + 3 = 3[x] - 6 + 5 \quad \Rightarrow [x] = 4$

$\Rightarrow 4 \leq < 5$

$x = 4 + f$ $\quad (f \rightarrow$ fraction)

$\therefore \quad y = 2[x] + 3 = 11$

$\therefore \quad [x + y] = [4 + f + 11] = [15 + f] = 15$

(9) **<u>Fractional part function</u>**:

$y = \{x\}$

Let $x = I + f$, $\quad I = [x]$ & $f = \{x\}$

$\therefore y = \{x\} = x - [x]$

x	{x}
$0 \leq x < 1$	x
$1 \leq x < 2$	x - 1
$2 \leq x < 3$	x - 2
$-1 \leq x < 0$	x + 1
$-2 \leq x < -1$	x + 2

Domain $\quad x \in R$

Range $\quad (0, 1)$

<u>Properties of fraction part of x</u>

(i) $\{x\} = x$ if $0 \leq < 1$

(ii) $\{x\} = 0$ if $x \in I$

(iii) $\{-x\} = 1 - \{x\}$ if $x \notin I$

<u>Illustration</u>: Prove that $[x] + [y] \leq [x + y]$ where $x = [x] + \{x\}$

Ans: $x + y = [x] + \{x\} + [y] + \{y\}$

$[x + y] = [[x] + [y] + \{x\} + \{y\}]$

$= [x] + [y] + [\{x\} + \{y\}]$ [By $[x + I] = [x] + I$]

$[x + y] \geq [x] + [y]$

(10) **<u>Exponential Function</u>**:

$f(x) = a^x$, $a > 0$, $a \neq 1$

Domain $\in R$

Range $\in (0, \infty)$

<u>Case I</u>: $a > 1$

$f(x) = a^x$ increase with increase in x.

i.e. f(x) is increasing function ob R

<u>Case II</u>: $0 < a < 1$

$f(x) = a^x$ decrease with increase in x

<u>Dumb Question</u>: How range comes out $(0, \infty)$?

Ans: $y = a^x$

Let $a > 1$

As $x \to \infty$

Since $a > 1$

So, $a^x \to \infty$

and when $x \to -\infty$

$a^x \to 0$

$\therefore$ range $(0, \infty)$

<u>Logorithmic function</u>:

$f(x) = \log_a x \quad (x, a > 0) \quad \& \quad a \neq 1$

Domain $\in (0, \infty)$

Range $\in R$

<u>Properties</u>:

(i) $\text{Log}_a a = 1$ (ii) $\log_{b^m} a = \frac{1}{m}\log_b a$ $\{a, b > 0, b \neq 1 \quad m \in R\}$

(iii) $\log_a b = \frac{\log_m b}{\log_m a}$ $\{a, b > 0, a, b \neq 1 \quad \& \quad m > 1\}$

(iv) $a^{\log_a m} = m$ $\{a, m > 0 \quad \& \; a \neq 1\}$

(v) $a^{\log_c b} = b^{\log_c a}$ $\{a, b, c > 0 \quad \& \quad c \neq 1\}$

(vi) $\log_m a < b \Rightarrow \begin{cases} x > y & \text{if } m > 1 \\ x < y & \text{if } 0 < m < 1 \end{cases} \begin{cases} m, x, y > 0 \\ \& \quad m \neq 1 \end{cases}$

(vii) $\log_m a < b \Rightarrow \begin{cases} a < m^b & \text{if } m > 1 \\ a > m^b & \text{if } 0 < m < 1 \end{cases}$

<u>Dumb Question</u>: Why $b \neq 1$ in $\log_b a$?

Ans: $\log_b a = c \Rightarrow b^c = a$
if $b = 1$
$1^c \neq a$
Now whatever value of c, $1^c \neq a$, so, $b \neq 1$

<u>Illustration</u>: Find domain of $f(x) = \log_{10}(1 + x^3)$

Ans: $f(x) = \log_{10}(1 + x^3)$ exists if

$1 + x^3 > 0$

$\Rightarrow (1 + x)(1 - x + x^2) > 0$

But $1 - x^2 + x^2 > 0$ as $D < 0$ & $a > 0$

So, $1 + x > 0$

$x > -1$

$x \in (-1, \infty)$

<u>Trignometric Function</u>:

(1) **<u>Sine Function</u>**:

$f(x) = \sin x$

Domain $\in R$

Range [-1, 1]

(2) **<u>Cosione Function</u>**:

$f(x) = \cos x$

Domain $\in R$

Range [-1, 1]

(3) **<u>Tangent Function</u>**:

f(x) = tan x

Domain $\in$ R - {(2n + 1) $\pi/2$ }
Range $\in$ R

y = tan x increases strictly from - ∞ to ∞ as x increases from - $\pi/2$ to $\pi/2$, $\pi/2$ to $3\pi/2$

x = ± $\pi/2$, ± $3\pi/2$, ... are asymptotes to y = tan x.

<u>Dumb Question</u>: What is asymptotes ?

Ans: A curve which is tangent to given curve at infinity.

(d) **<u>Coecant function</u>**:

f(x) = cosec x
Domain $\in$ R - {nπ | n $\in$ I}
Range $\in$ R - (- 1, 1)
n = nπ n $\in$ I is asymptote to y = cosec x

(e) **<u>Secant Function</u>**:

f(x) = sec x

Domain $\in$ R - {(2n + 1) $\pi/2$ | n $\in$ I}
Range $\in$ R - (- 1, 1)
x = (2n + 1) $\pi/2$, n $\in$ I are asymptote to y = sec x.

(f) **<u>Cotangent Function</u>**:

f(x) = cot x
Domain $\in$ R - {nπ | n $\in$ I}
Range $\in$ R
It has x = nπ n $\in$ I as asymptotes.

<u>Inverse Function</u>:

(i) **Graph of y = $\sin^{-1}$ x;**

where

$x \in [-1, 1]$

and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

As the graph of f^{-1} is mirror image of f(x) about y = x.

(ii) **Graph of $y = \tan^{-1}x$;**

Here,

Domain $\in [-1, 1]$

Range $\in [0, \pi]$

(iii) **Graph of $y = \tan^{-1}x$;**

Here ,

Domain $\in$ R

Range $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(iv) **Graph of y = cot^{-1}x;**

We know that the function f:(0, π) $\rightarrow$ R, given by f(θ) = cot θ is invertible.

$\therefore$ Thus, domain of cot^{-1}x $\in$ R and Range $\in$ (0, π).

(v) **Graph for y = sec^{-1}x;**

The function f : [0, π] - $\left\{\frac{\pi}{2}\right\}$ $\rightarrow$ (- ∞, - 1) $\cup$ [1, ∞] given by f(θ) = sec θ is invertible.

$\therefore$ y = sec^{-1}x, has domain $\in$ R - (- 1, 1) and range $\in$ [0, π] - $\left\{\frac{\pi}{2}\right\}$: shown as

(vi) **Graph for y = cosec^{-1}x;**

As we know, f: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ - {0} $\rightarrow$ (- 1, 1) is invertible given by f(θ) = cosθ.

y = cosec^{-1}x; domain $\in$ R - (- 1, 1)

Range $\in$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ - {0}>

Sketch of y = sin(sin^{-1}x):

Domain x $\in$ [-1, 1]
and range y = x $\Rightarrow$ y $\in$ [-1, 1]
Sketch y = sin(sin^{-1}x) only
when x $\in$ [-1, 1] & y = x

Sketch of curve y = cos(cos^{-1}x):

Domain x $\in$ [-1, 1]
and range y = x $\Rightarrow$ y $\in$ [-1, 1]

Sketch of curve y = tan(tan^{-1}x):

Domain, x $\in$ R
Range y = x y $\in$ r
We should sketch
y = tan(tan^{-1}x) = x $\forall$ x $\in$ R

Sketch of curve y = cosec(cosec^{-1}x):

Domain x R - (-1, 1)
Range y = x $\Rightarrow$ y $\in$ R - (-1, 1)
y = cosec(cosec^{-1}x) = x only
when x $\in$ (-∞, -1) $\cup$ (1, ∞)

<u>Functions</u>:

Sketch of curve $y = \sec(\sec^{-1}x)$:

Domain $\in$ R - (-1, 1)
& range y = x $\Rightarrow$ y $\in$ R - (-1, 1)
$y = \sec(\sec^{-1}x) = x$, only when $x \in (-\infty, -1) \cup (1, \infty)$

Sketch of $y = \cot(\cot^{-1}x)$:

Domain $\in$ R - (-1, 1)
& range y = x $\Rightarrow$ y = R
We should sketch
$y = \cot(\cot^{-1}x) = x \forall x \in R$

Sketch of $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$:

Domain $[-1, 1]$ $\forall$ $x \in R$

& range $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

b/c $y = \sin^{-1}$

$\Rightarrow y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

<u>Dumb Question</u>: How domain is [-1, 1] $\forall$ x $\in$ R ?

Ans: $\left|\frac{2x}{1+x^2}\right| \le 1$ i.e. $-1 \le \left|\frac{2x}{1+x^2}\right| \le 1$

$\Rightarrow 2|x| \le 1 + x^2 \quad \Rightarrow |x|^2 - 2|x| + 1 \ge 0$

$\Rightarrow (|x| - 1)^2 \ge 0$

$x \in R$

Let $x = \tan\phi$

$$\frac{2x}{1+x^2} = \frac{2\tan\phi}{1+\tan^2\phi} = \sin 2\phi$$

$$y = \sin^{-1}(\sin 2\phi) = \begin{cases} \pi - 2\phi & ; \quad 2\theta > \pi \\ 2\theta & ; \quad -\pi/2 \le 2\theta \le \pi/2 \\ -\pi - 2\tan^{-1} x & ; \quad 2\theta < -\pi/2 \end{cases}$$

$$= \begin{cases} \pi - 2\tan^{-1} x & ; \quad \tan^{-1} x > \pi/4 \\ 2\tan^{-1} x & ; \quad -\pi/4 \le \tan^{-1} x \le \pi/4 \\ -\pi - 2\tan^{-1} x & ; \quad \tan^{-1} x < -\pi/4 \end{cases}$$

This curve has sharp edge at x = ± 1

So, not differentiable at x = ± 1

<u>Dumb Question</u>: How does $y = \pi - 2\theta$ for $2\theta > \pi/2$

Ans: Since our $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So, if $2\theta > \pi/2$ to make it we substract it from $\pi - 2\theta$ because $\sin(\pi - 2\theta) = \sin 2\theta$

Sketch of $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

For domain

$\left|\frac{1-x^2}{1+x^2}\right| \le 1 \quad \Rightarrow |1 - x^2| \le 1 + x^2$

Which is true for all n; as $1 + x^2 > 1 - x^2 \quad x \in R$
domain $\in [-1, 1]$
For range

$$y = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$$

$\Rightarrow y \in (0, \pi)$
Let $x = \tan\theta$

$$y = \cos^{-1}\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$$

$$= \begin{cases} 2\theta & 2\theta \geq 0 \\ -2\theta & 2\theta < 0 \end{cases} = \begin{cases} 2\tan^{-1}x & ; \quad \tan^{-1}x \geq 0 \\ -2\tan^{-1}x & ; \quad \tan^{-1}x < 0 \end{cases}$$

$$= \begin{cases} 2\tan^{-1}x & ; \quad x \geq 0 \\ -2\tan^{-1}x & ; \quad x < 0 \end{cases}$$

<u>Dumb Question</u>: How $\cos^{-1}(\cos 2\theta) = -2\tan^{-1}x$ when $x < 0$ or $\theta < 0$

Ans: Range of $y \in [0, \pi]$. So, when $\theta < 0$ or $x < 0$
Then we have to make it b/w $[0, \pi]$
So, $\cos^{-1}(\cos 2\theta) = -2\theta$ when $\theta < 0$
because

$$\boxed{\cos(-2\theta) = \cos 2\theta}$$

This curve has sharp edge at $x = 0$. So, not differentiable at $x = 0$.

Sketch of $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

For domain

$\left(\frac{2x}{1-x^2}\right) \in R$ except $1 - x^2 = 0$

i.e. $x \neq \pm 1$

or $x \in R - \{1, -1\}$

domain $\in R$

Range $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

as $y = \tan^{-1}\theta$ $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Let $x = \tan\theta$

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = \begin{cases} \pi + 2\theta, & 2\theta < -\pi/2 \\ 2\theta, & -\pi/2 < 2\theta < \pi/2 \\ -\pi + 2\theta, & 2\theta > \pi/2 \end{cases}$$

$$= \begin{cases} \pi + 2\tan^{-1}x; & \tan^{-1}x < -\pi/4 \\ 2\tan^{-1}x; & -\pi/4 < \tan^{-1}x < \pi/4 \\ -\pi + 2\tan^{-1}x; & \tan^{-1}x > \pi/4 \end{cases} = \begin{cases} \pi + 2\tan^{-1}x; & x < -1 \\ 2\tan^{-1}x; & -1 < x < 1 \\ -\pi + 2\tan^{-1}x; & x > 1 \end{cases}$$

<u>Dumb Question</u>: Why $y = \tan^{-1}(\tan 2\theta) = -\pi + 2\tan^{-1}x$, for $\theta > \pi/4$?

Ans: Range of $y \in \left(-\pi/2, \pi/2\right)$ & when $\theta > \pi/4$ So, $2\theta > \pi/2$.
To make it within range we substract π So, it comes under range.
Because of $\tan(-\pi + 2\theta) = \tan 2\theta$, this can be done.
This curve is neither continous nor differentiable at x = {-1, 1}

Sketch of curve $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

For domain,

$$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$\Rightarrow x \in r$ except $1 - 3x^2 = 0$ $\Rightarrow x \neq \pm \frac{1}{\sqrt{3}}$

$\therefore$ $x \in R - \{\pm \frac{1}{\sqrt{3}}\}$
domain $\in R$
For range

$$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$\Rightarrow y \in \left(-\pi/2, \pi/2\right)$
Let $x = \tan\theta$

$$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta) =$$

$$\begin{cases} \pi + 3\theta & ; \quad 3\theta < -\pi/2 \\ 3\theta & ; \quad -\pi/2 < 3\theta < \pi/2 \\ -\pi + 3\theta & ; \quad 3\theta > -\pi/2 \end{cases}$$

$$= \begin{cases} \pi + 3\tan^{-1}x & ; \quad \tan^{-1}x < -\pi/6 \\ 3\tan^{-1}x & ; \quad -\pi/6 < \tan^{-1}x < \pi/6 \\ -\pi + 3\tan^{-1}x & ; \quad \tan^{-1}x > \pi/6 \end{cases} = \begin{cases} \pi + 3\tan^{-1}x & ; \quad x < -\frac{1}{\sqrt{3}} \\ 3\tan^{-1}x & ; \quad -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & ; \quad x > \frac{1}{\sqrt{3}} \end{cases}$$

<u>Dumb Question</u>: Why $\tan^{-1}(\tan 3\theta) = \pi + 3\theta$ if $\theta < -\pi/6$?

Ans: Range of $\tan^{-1}(\tan 3\theta)$ is $(-\pi/2, \pi/2)$ but if $\theta < -\pi/6$ So, $3\theta < -\pi/2$ So, to make it within given range. We add π to it.
and in IIIrd quadrant.
$\tan(\pi + 3\theta) = \tan 3\theta$
So, this is done.

This curve is neither cont. nor differentiable at $x = \pm \frac{1}{\sqrt{3}}$.

Sketch of $y = \sin^{-1}(3x - 4x^3)$:

For domain
$y = \sin^{-1}(3x - 4x^3)$
$\Rightarrow x \in [-1, 1]$
For range

$y = \sin^{-1}(3x - 4x^3) \quad \Rightarrow y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Let $x = \sin\theta$

$$\Rightarrow y = \sin^{-1}(\sin 3\theta) = \begin{cases} \pi - 3\theta & ; \ \pi/2 \leq 3\theta \leq 3\pi/2 \\ 3\theta & ; \ -\pi/2 \leq 3\theta \leq \pi/2 \\ -\pi - 3\theta & ; \ -3\pi/2 \leq 3\theta \leq -\pi/2 \end{cases}$$

$$= \begin{cases} \pi - 3\sin^{-1}x & ; \ \pi/6 \leq \sin^{-1}x \leq \pi/2 \\ 3\sin^{-1}x & ; \ -\pi/6 \leq \sin^{-1}x \leq \pi/6 \\ -\pi - 3\sin^{-1}x & ; \ -\pi/2 \leq \sin^{-1}x \leq -\pi/6 \end{cases} = \begin{cases} \pi - 3\sin^{-1}x & ; \ \frac{1}{2} \leq x \leq 1 \\ 3\sin^{-1}x & ; \ -\frac{1}{2} \leq x \leq \frac{1}{2} \\ -\pi - 3\sin^{-1}x & ; \ -1 \leq x \leq -\frac{1}{2} \end{cases}$$

This curve has sharp edge at $x = \pm 1/2$ so, not differentiable at $x = \pm 1/2$.

ketch of curve $y = \cos^{-1}(4x^3 - 3x)$

Domain $\in [-1, 1]$ range $\in [0, \pi]$

Let $x = \cos\theta$

$$\Rightarrow y = \cos^{-1}(\cos 3\theta) = \begin{cases} 2\pi - 3\theta & ; \ \pi \leq 3\theta \leq 2\pi \\ 3\theta & ; \ 0 \leq 3\theta \leq \pi \\ -2\pi + 3\theta & ; \ -\pi \leq 3\theta \leq 0 \end{cases}$$

$$= \begin{cases} 2\pi - 3\cos^{-1}x & ; \ \pi/3 \le \cos^{-1}x \le 2\pi/3 \\ 3\cos^{-1}x & ; \ 0 \le \cos^{-1}x \le \pi/3 \\ -2\pi + 3\cos^{-1}x & ; \ -\pi/3 \le \cos^{-1}x \le 0 \end{cases} = \begin{cases} 2\pi - 3\cos^{-1}x & ; \ -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & ; \ \frac{1}{2} \le x \le 1 \\ -2\pi + 3\cos^{-1}x & ; \ -1 \le x \le \frac{1}{2} \end{cases}$$

This curve has sharp edge at x = ± 1/2. So, not differentiable at x = ± 1/2.

<u>Dumb Question</u>: Why $y = \cos^{-1}(\cos 3\theta) = 2\pi - 3\theta$ if $\pi/3 \le \theta \le 2\pi/3$.

Ans: Range of $y \in [0, \pi)$ if $\pi \le 3\theta \le 2\pi$. So, it s out of range so, it make it within range, we subtract it from 2π
and $\cos(2\pi - 3\theta) = \cos 3\theta$

<u>Odd and Even functions</u>: A function is said to be an odd function if, $f(-x) = -f(x)$ for all x.
Graph of odd function is symmetrical in opposite quadrants. eg.

<u>Even function</u>: A function f(x) is said to be an even of f(-x) = f(x) for all x.
Graph is symmetrical about y-axis.

<u>Properties</u>:

(i) Product of two odd functions or two even functions is an even function.

(ii) Product of odd and even function is an odd function.

(iii) Every function y = f(x) can be expressed as sum of an even and odd function.

<u>Illustration</u>: IF f is an even function, find real values of x satisfying eq. $f(x) = f\left(\frac{x+1}{x+2}\right)$

Ans: Since f(x) is even function
$\therefore$ f(x) = f(-x)

Thus, $x = \left(\frac{x+1}{x+2}\right)$ or $-x = \left(\frac{x+1}{x+2}\right)$

$\Rightarrow x^2 + x - 1 = 0$ or $-x^2 - 3x - 1 = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$ or $x = \frac{-3 \pm \sqrt{5}}{2}$

$\therefore \quad x = \left\{\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}\right\}$

<u>Periodic Function</u>: A function f(x) is said to be periodic function if, there exists a +ve real no. such that,

$f(x + T) = f(x), \forall x \in R$

Then, f(x) is periodic with period T, where T is least +ve value.

eg. sin x is priodic with period 2π.

<u>Illustration</u>: Prove that f(x) = x - [x] is periodic function. Also find its period.

Ans: Let T > 0,

Then, $f(x + T) = f(x), \forall x \in R$

$\Rightarrow (x + T) - [x + T] = x - [x] \forall x \in R$

$\Rightarrow [x + T] - [x] = T \; v \; x \in R$

$\Rightarrow T = 1, 2, 3, 4, ...$ {Since subtraction of two integer}

Smallest value of T satisfying

f(x + T) = f(x) is 1

So, period with period 1.

<u>Properties of periodic function</u>:

(i) If f(x) is periodic with period T, then,

c.f(x), f(x + c) & f(x) ± c is periodic with period T.

eg. if sin x has period 2π. Then f(x) = 5 sinx + 4 is also periodic with period 2π.

(ii) If f(x) is periodic with period T, then, kf(cx + d) has period $\frac{T}{|c|}$

f(x) = 7 sin2x - 12 has period $\frac{2\pi}{|2|} = \pi$ as sinx is periodic with period

2^{π}.

(iii) If $f_1(x)$, $f_2(x)$ are periodic functions with periods T_1, T_2 respectively then; $h(x) = af_1(x) \pm bf_2(x)$ has period as

$$= \begin{cases} \text{LCm of } \{T_1, T_2\}; \text{ If } h(x) \text{ is not an even function} \\ \text{or} \\ \frac{1}{2} \text{ LCM of } \{T_1, T_2\}, f_1(x) \ \& f_2(x) \text{ are complementar pair wise comparable functions} \end{cases}$$

LCM of $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) = \frac{\text{LCM of } (a, b, c)}{\text{HCF of } (b, d, f)}$

<u>Note</u>: LCM of rational and irrational is not possible. eg. 2^{π} is irrational & 1 is rational.

<u>Composite Function</u>:

Let two function,
$f : x \rightarrow Y_1$ and $g : Y_1 \rightarrow Y$

Let another function $h : X \rightarrow Y$
Such that
$h(x) = g(f(x)) = (gof)(x)$
domain (gof) = $\{x : x \in \text{domain } (f), f(x) \in \text{domain } (a)\}$
domain fog(x) = $\{x : x \in \text{domain } (g), g(x) \in \text{domain } f\}$
gof exist iff range of f $\subseteq$ domain of g
fog exist iff range of g $\subseteq$ domain of f.

Properties of composite function:

(i) f is even, g is even $\Rightarrow$ fog is even function.

(ii) f is odd, g is odd $\Rightarrow$ fog is odd function.

(iii) f is even, g is odd $\Rightarrow$ fog is even function.

(iv) f is odd, g is even $\Rightarrow$ fog is even function.

Illustration: Let g(x) = 1 + x - [x] & f(x) = $\begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$. Find fog(x).

Ans: g(x) = 1 + x - [x] = 1 {x}
$\Rightarrow$ g(x) is greater than J.
So, f(g(x)) = 1. since f(x) = 1 for all x > 0
$\therefore$ f(g(x)) = 1 for all x $\in$ R.

Mapping of function: A function exists only if," to every element in domain there exists unique image in co domain.
f : A $\rightarrow$ B

One - One Mapping (Injective): A function f: A $\rightarrow$ B is said to bwe one-one mapping if different elements of A have different images in B. $\Rightarrow$ no two elements of set A can have same 'f' image
eg.

f(x) is one-one

Methods to find one-one mapping:

(1) **Graphically**: A fun. is one-one if no line 11 to x-axis meets graph of function at more than one point.

By Calculus: If f(x) is function and if $f'(x) \geq 0 \ \forall \ x \in$ domain i.e. increasing or $f'(x) \leq 0 \ \forall \ x \in$ domain i.e. decreasing, then one-one.

No. of one-one Mapping: If A & B are finite sets having m & h elements, then, no. of one-one function from A to B

$$= \begin{cases} {}^{n}P_m & \text{if} \quad n \geq m \\ 0 & \text{if} \quad n < m \end{cases}$$

Dumb Question: How No. of one-one mapping is ${}^{n}P_m$ if $n \geq m$.

Ans: x_1 can take n images
x_2 can take (n - 1) images
.....
.....
.....
.....
x_m can take (n - m + 1) images
$\therefore$ No. of mapping = n(n - 1)(n - 2) ..... (n - m + 1) = ${}^{n}P_m$

Many one Mapping: A mapping $f : A \rightarrow B$ is many one if two or more elements of set A have the same image in B.

i.e. $f(x^3) = f(x_4) = y_3$

<u>Onto Function (Surjective)</u>: If function $f : A \rightarrow B$ is such that each element of B is the 'f' image of at least one element in A.
i.e. range of f = co domain $\Rightarrow f(A) = B$

<u>Into Function</u>: A function of $f : A \rightarrow B$ is into if there exists an element in B having no pre image in A.

<u>No. of one-one onto mapping (bijection)</u>: If A & B are finite sets & $f : A \rightarrow B$ is a bijection.
Then A & B have same no. of elements.

Total no. of bijection $A \rightarrow B = n\,!$

<u>Dumb Question</u>: How total no. of bijection $A \rightarrow B = n\,!$?

Ans: x_1 can take n images
x_2 can take (n - 1) images
x_3 can take (n - 2) images

.....
.....
.....
x_m can take 1 images

$\therefore$ No. of mapping = n(n - 1)(n - 2) ..... 1 = n !

<u>Functions</u>:

<u>Illustration</u>: Find no. of surjection from A $\rightarrow$ B
A = {1, 2, 3, 4} & B = {a, b}

Ans:

1 $\xrightarrow{\text{can have}}$ 2 images
2 $\xrightarrow{\text{can have}}$ 2 images
3 $\xrightarrow{\text{can have}}$ 2 images
4 $\xrightarrow{\text{can have}}$ 2 images

$\therefore$ Total no. of function = 2 x 2 x 2 x 2 = 2^4
But there are two function f(x) = a for all x $\in$ A & g(x) = b for all x $\in$ A are not surjective.
$\therefore$ Total no. of surjection from A to B = 2^4 - 2 = 14

<u>Inverse function</u>: Let f : A $\rightarrow$ B be one-one & onto function then there exists a unique function.

$g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x, \forall x \in A$ & $y \in B$
Then g is c/d inverse of F.
So, $g = f^{-1} : B \to A = \{(f(x), x) \mid (x, f(x) \in f\}$
Domain of $f = \{1, 2, 3, 4\}$ = range of f^{-1}
Range of $f = \{2, 4, 6, 8\}$ = domain of f^{-1}.

<u>Illustration</u>: If $f(x) = 3x - 5$, then find $f^{-1}(x)$.

Ans: $f(x) = 3x - 5$ which linear function so bijective.
Let $f(x) = y \Rightarrow Y = 3x - 5$
$\Rightarrow x = \frac{y+5}{3} \Rightarrow f^{-1}(y) = \frac{y+5}{3}$
$\Rightarrow f^{-1}(x) = \frac{x+5}{3}$

<u>Note</u>: (1) Graph of $y = f(x)$ and $y = f^{-1}(x)$ are mirror images to each other.

(2) If $f : A \to B$ & $g : B \to C$ are two bijections, then $gof : A \to C$ is bijection & $(gof)^{-1} = (f^{-1}og^{-1})$.

(3) If fog = gof then either $f^{-1} = g$ or $g^{-1} = f$ and $(fog)(x) = (gof)(x) = (x)$

<u>Easy Type</u>

Q.1. Find domain of $f(x) = \log_{10}\log_{10}(1 + x^3)$

Ans: Let $\log_{10}(1 + x^3) = y$
$f(x) = \log_{10}y \Rightarrow y > 0$
$\Rightarrow \log_{10}(1 + x^3) > 0 \Rightarrow (1 + x^3) > 10^0$
$\Rightarrow x^3 > 0 \Rightarrow x \in (0, \infty)$

Q.2. Find domain of $y = \sqrt{\sin^{-1}(\log_2 x)}$

Ans: Let $\log_2 x = z$

z is defined if $x > 0$ ..... (i)
and $-1 \le z \le 1$ or $-1 \le \log_2 x \le 1$
$\Rightarrow 1/2 \le x \le 2$ ..... (ii)
But y is defined if
$\sin^{-1} z \ge 0$ or $\sin^{-1}\log_2(x) \ge 0$ ..... (iii)
$\Rightarrow \Rightarrow \log_2 x \ge 0 \quad \Rightarrow x \ge 2^0 \quad \Rightarrow x \ge 1$ ..... (iv)
From (i), (ii) & (iv)
$1 \le x \le 2$
$\therefore$ domain $x \in [1, 2]$

Q.3. Find values of 'a' for which $f(x) = \tan^{-1}(x^2 - 18x + 9) > 0$ for $x \in R$.

Ans: $f(x) > 0 \ \forall\, x \in R$
$\Rightarrow x^2 - 18x + a > 0 \ \forall\, x \in R$
$ax^2 + bx + c > 0$ if $D < 0$ & $a > 0$
$D = (18)^2 - 49 < 0 \quad \Rightarrow a > 81$
$\Rightarrow a \in (81, \infty)$

Q.4. $f(x) = \dfrac{1}{\sqrt{[x] - x}}$ find domain of f9x).

Ans: $f(x) = \dfrac{1}{\sqrt{[x] - x}}$ exists if
$[x] - x > 0 \quad \Rightarrow [x] > x$
But by definition of greatest integer function
$[x] \le x$ as $x = [x] + \{x\}$
So, it is not possible that $[x] > x$
$\therefore$ domain $f(x) = \phi$

Q.5. Solve $4\{x\} = x + [x]$

Ans: $x = [x] + \{x\}$

$4\{x\} = [x] + \{x\} + [x] \quad \Rightarrow \{x\} = \dfrac{2[x]}{3}$
But $0 \le \{x\} < 1$

$\therefore 0 \le \frac{2[x]}{3} < 1 \quad \Rightarrow 0 \le [x] < 3/2$

$\therefore [x] = 0$ or 1

If $[x] = 1$, then $\{x\} = 2/3$

Thus, $x = [x] + \{x\} = 1 + 2/3 = 5/3$

If $[x] = 0$, $\{x\} = 0$

$\therefore x = 0$

$\therefore$ Solutions $x \in \{0, 5/3\}$

Q.6. Find range for $y = \frac{x - [x]}{1 - [x] + x}$.

Ans: $y = \frac{x - [x]}{1 - [x] + x} = \frac{\{x\}}{1 + \{x\}}$

Domain $\in R$

$y = \frac{\{x\}}{1 + \{x\}} \quad \Rightarrow \{x\} = \frac{y}{1 - y}$

Since $0 \le \{x\} < 1 \quad \Rightarrow 0 \le \frac{y}{1 - y} < 1 \quad \Rightarrow 0 \le y < 1/2$

range = (0, 1/2)

Q.7. Let f(x) be periodic & k bca +ve real no. such that $f(x + k) + f(x) = 0 \ \forall \ x \in R$. Prove that f(x) f(x) is periodic with period 2k.

Ans: $f(x + k) + f(x) = 0 \ \forall \ x \in R$

$f(x + k) = - f(x) \ \forall \ x \in R$ ..... (i)

put $x = x + k$

$\Rightarrow f(x + 2k) = -f(x + k) \ \forall \ x \in R$

from (i)

$f(x + 2k) = f(x), \ \forall \ x \in R$

$\therefore$ f(x) is peruiodic with period 2k.

Q.8. Find period of $f(x) = \tan 3x + \sin\left(\frac{x}{3}\right)$

Ans: Period for tan 3x is $\frac{\pi}{|3|} = \pi/3$

Period for sin $\pi/3$ is $|2\pi \times \frac{3}{1} = 6\pi$

$\therefore$ LCM of $\pi/6 \le \sin^{-1}\left(\frac{x^2 - a}{x^2 + 1}\right) < \pi/2$

$\Rightarrow \frac{1}{2} \le \frac{x^2 - a}{x^2 + 1} < 1 \quad \Rightarrow \frac{1}{2} \le 1 - \frac{(a + 1)}{x^2 + 1} < 1, \forall x \in R$

$\Rightarrow -\frac{1}{2} \le -\frac{(a + 1)}{x^2 + 1} < 0$

$\Rightarrow a + 1 > 0$

$\Rightarrow a \in (-1, \infty)$

Q.10. If f : R $\rightarrow$ R f(x) = $x^2 + 1$, then find value of $f^{-1}(17)$.

Ans: $f(x) = x^2 + 1$

$f^{-1}(17) \quad \Rightarrow f(x) = 17$

$\Rightarrow \quad x^2 + 1 = 17 \quad \Rightarrow x = \pm 4$

$x = \{4, -4\}$

<u>Medium Type</u>

Q.1. Find domain of function. $f(x) = \frac{\log\sqrt{x^2 - x + 1}}{\log(x^2 - x + 1)}$

Ans: $\log(x^2 - x + 1)$ is defined when

$x^2 - x + 1 > 0 \quad \& \quad x^2 - x + 1 \ne 1$

$\Rightarrow x \in R$ and $x \ne 0, 1$

$\Rightarrow x \in R - \{0, 1\}$ ..... (i)

[<u>Dumb Question</u>: How $x^2 - x + 1 > 0 \ \forall x \in R$ & why $x^2 - x + 1 \ne 1$?

Ans: $ax^2 + bx + c > 0 \ \forall x \in R$ if $a > 0$ & $D < 0$

For $x^2 - x + 1$, $a = 1 > 0$ & $D = 1 - 4 = -3 < 0$

So, $x^2 - x + 1 > 0$ for all $x \in R$

& If $x^2 - x + 1 = 1$ then $\log x^2 - x + 1 = 0$

& $\frac{1}{\log x^2 - x + 1} = \infty$ which is not desired.]

Again $\log(\sin^{-1}\sqrt{x^2 - x + 1})$ exists when $x^2 + x + 1 > 0$ & $x^2 + x + 1 \leq 1$
i.e. $0 < x^2 + x + 1 \leq 1$
$\Rightarrow x^2 + x \leq 0 \quad \Rightarrow x \in [-1, 0]$ ..... (ii)
From (i) & (ii)
$x \in [-1, 0)$

Q.2. Find domain of $f(x) = \frac{1}{[|x - 1|] + [|7 - x|] - 6}$ where {} is greatest integer function.

Ans: f(x) is defined when
$[|x - 1|] + [|7 - x|] - 6 \neq 0$

$$\Rightarrow \begin{cases} [1 - x] + [7 - x] \neq 6 & \text{when } x \leq 1 \text{ ..... (i)} \\ [x - 1] + [7 - x] \neq 6 & \text{when } 1 \leq x \leq 7 \text{ ..... (ii)} \\ [x - 1] + [x - 7] \neq 6 & \text{when } x \geq 7 \text{ ..... (iii)} \end{cases}$$

Taking (i)
$[1 - x] + [7 - x] \neq 6 \quad \Rightarrow 1 + [-x] + 7 + [-x] \neq 6$
$\Rightarrow 2[-x] \neq -2 \quad [-x] \neq -1$
$\Rightarrow x \notin (0, 1]$ ..... (1)
From (ii)
$[x - 1] + [7 - x] \neq 6 \quad \Rightarrow [x] - 1 + 7 + [-x] \neq 6$
$\Rightarrow [x] + [-x] \neq 0 \quad x \notin$ Integer
$\Rightarrow x \notin \{1, 1, 3, 4, 5, 6, 7\}$ ..... (2)
From (iii)
$x \in [7, 8)$ ..... (3)
From (1), (2) & (3)
Domain $f(x) \in R - \{(0, 1] \cup \{1, 2, 3, 4, 5, 6, 7\} \cup [7, 8)\}$

Q.3. f(x, y) is periodic function satisfying condition $f(x, y) = f((2x + 2y),(2y - 2x))$ $\forall$ $x,y \in R$. Now $g(x) = f(2^x, 0)$ then prove that g(x) is periodic function, find its period.

Ans: $f(x, y) = f(2x + 2y, 2y - 2x)$ ..... (i)
Put $x = 2x + 2y$ & $y = 2y - 2x$
$\Rightarrow f(2x + 2y, 2y - 2x) = f(2(2x + 2y) + 2(2y - 2x), 2(2y - 2x) - 2(2x +$

2y)) [By (i)]

$\Rightarrow f(x, y) = f(2x + 2y, 2y - 2x) = f(8y, - 8x)$

$\Rightarrow f(x, y) = f(8y, - 8x)$ ..... (ii)

$\Rightarrow f(8y, - 8x) = f\{8(- 8x), - 8(8y)\}$ [By (ii)]

$\Rightarrow f(x, y) = f(2x + 2y, 2y - 2x) = f(8y, - 8x) = f(- 64x, - 64y)$

$\Rightarrow f(x, y) = f(- 64x, - 64y)$ ..... (iii)

$\Rightarrow f(- 64x, - 64y) = f(64 \text{ x } 64 \text{ x}, 64 \text{ x } 64y) = f(2^{12}x, 2^{12}y)$ [By (iii)]

$\Rightarrow f(x, y) = f(2^{12}x, 2^{12}y)$

$\Rightarrow f(2^x, 0) = f(2^{12}.2^x, 0) = f(2^{12+x}, 0)$

$\Rightarrow g(x, 0) = f(2^x, 0) = f(2^{12+x}, 0)$

$\Rightarrow g(x, 0) = g(x + 12, 0)$

$\Rightarrow g(x)$ is periodic with period $\underline{\underline{12}}$

Q.4. Let $f(x) = \frac{g^x}{g^x + 3}$. Show that $f(x) + f(1 - x) = 1$ & hence, evaluate

$$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + f\left(\frac{3}{1996}\right) + \ldots\ldots\ldots\ldots\ldots\ldots + f\left(\frac{1995}{1996}\right)$$

Ans: $f(x) = \frac{g^x}{g^x + 3}$ ..... (i)

and $f(1 - x) = \frac{g^{1-x}}{g^{1-x} + 3} = \frac{g}{3(3 + g^x)}$ ..... (ii)

Adding (i) & (ii), we get

$$f(x) + f(1 - x) = \frac{g^x}{g^x + 3} + \frac{g}{3(3 + g^x)} = \frac{3.g^x + g}{3(g^x + 3)} = \frac{3(g^x + 3)}{3(g^x + 3)}$$

$\therefore f(x) + f(1 - x) = 1$ ..... (iii)

Now, putting $x = \frac{1}{1996}, \frac{2}{1996}, \frac{3}{1996}, \ldots\ldots\ldots\ldots, \frac{996}{1996}$ in (iii).

$$f\left(\frac{1}{1996}\right) + f\left(\frac{1995}{1996}\right) = 1$$

$$f\left(\frac{1}{1996}\right) + f\left(\frac{1994}{1996}\right) = 1$$

.....

.....

.....

.....

.....

.....

$$f\left(\frac{997}{1996}\right) + f\left(\frac{999}{1996}\right) = 1$$

$$f\left(\frac{998}{1996}\right) + f\left(\frac{998}{1996}\right) = 1 \quad \text{or} \quad f\left(\frac{998}{1996}\right) = \frac{1}{2}$$

Adding all.

$$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + \ldots\ldots\ldots\ldots\ldots\ldots + f\left(\frac{1995}{1996}\right) = (1 + 1 + \ldots\ldots\ldots\ldots \text{997 times}) + \frac{1}{2}$$

[<u>Dumb Question</u>: For what $\frac{1}{2}$ is added ?

Ans: For $f\left(\frac{998}{1996}\right) = \frac{1}{2}$]

$= 997 + \frac{1}{2} = 997.5$

Q.5. Let g(x) be inverse of f(x) and $f'(x) = \frac{1}{1 + x^3}$. Then find g'(x) in terms of g(x).

Ans: Since g(x) is inverse of f(x)

$\therefore$ (gof)(x) = x

$\Rightarrow g\{f(x)\} = x$

$\Rightarrow g'\{f(x)\}\ f'(x) = 1$

$\Rightarrow g'\{f(x)\} = \frac{1}{f'(x)} = 1 + x^3$

$\Rightarrow g'\{f(g(x))\} = 1 + (g(x))^3$

$\Rightarrow g'(x) = 1 + (g(x))^3$

<u>Dumb Question</u>: How f(g(x)) = x ?

Ans: Since g(x) is inverse of f(x)

$\therefore$ f(x) is also inverse of g(x)

By inverse property

f(g(x)) = x

Hard Type

Q1. Prove that $a^2 + b^2 + c^2 + 2abc < 2$ where a, b, c, are sides of Δ ABC such that $a + b + c = 2$

Ans: $a + b + c = 2$
$\Rightarrow 1 - a + 1 - b + 1 - c = 1$
$\Rightarrow x + y + z = 1$ where $x = 1 - a, y = 1 - b, z = 1 - c$
Sine in Δ
Sum of two sides > third side
$a + b > c \Rightarrow 0 < c < 1$
Similarly $0 < a, b < 1$
hence $0 < x, y, z < 1$
Now,
$a^2 + b^2 + c^2 + 2abc = (1 - x)^2 + (1 - y)^2 + (1 - z)^2 + 2(1 - x)(1 - y)(1 - z)$

$= 3 - 2(x + y + z) + x^2 + y^2 + z^2 + 2\{1 - (x + y + z) + (xy + yz + zx) - xyz\}$

$= 1 + x^2 + y^2 + z^2 - 2xyz + 2(xy + yz + zx)$

$= 1 + (x + y + z)^2 - 2xyz$

$= 2 - 2xyz < 2$ as $0 < x, y, z < 1$
$\therefore a^2 + b^2 + c^2 + 2abc < 2$

Dumb Question: How $0 < c < 1$?

Ans: Since g_n Δ,
Sum of two sides > third side
$a + b > c$ ..... (i)
or $a + b + c > 2c$
$2 > 2c \Rightarrow c < 1$ ..... (ii)
Since c is side of Δ
$\therefore c > 0$ ..... (iii)
$\therefore$ From (ii) & (iii)
$0 < c < 1$

Q.2. A function R $\rightarrow$ R is f(x) = $\frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Find integral value of α for which f is onto.

Ans: Since f : R $\rightarrow$ R is onto mapping

$\therefore$ Range = codomain = R

$\Rightarrow \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ assumes all real values of x

Let y = $\frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

$\Rightarrow x^2(\alpha + 8y) + 6x(1 - y) - (8 - \alpha y) = 0 \ \forall \ y \in R$

$\Rightarrow D \geq 0$

$36(1 - y)^2 + 4(\alpha + 8y)(8 + \alpha y) \geq 0$

$\Rightarrow [y^2(8\alpha + 9) + y(\alpha^2 + 46) + (8\alpha + 9) \geq 0$

$ax^2 + bx + c \geq 0 \ \forall \ x \in R$ if $a > 0$ & $D \leq 0$

$(\alpha^2 + 46)^2 - 4(8\alpha + 9)(8\alpha + 9) \leq 0$ & $(8\alpha + 9) > 0$

$\Rightarrow (\alpha^2 + 46)^2) - [2(8\alpha + 9)]^2 \leq 0$ & $\alpha > -9/8$

$\Rightarrow (\alpha^2 + 46 - 16\alpha - 18)(\alpha^2 + 46 + 16\alpha + 18) \leq 0$ & $\alpha > -9/8$

$\Rightarrow (\alpha - 14)(\alpha - 2)(\alpha + 8)^2 \leq 0$ and $\alpha > -9/8$

$\Rightarrow \alpha [2, 14] \cup \{-8\}$ and $\alpha > -9/8$

$\therefore \alpha \in [2, 14]$

Q.3. Solve the eq. $[x]\{x\} = x$

Ans: $x = [x] + \{x\}$ ..... (i)

$\Rightarrow [x]\{x\} = [x] + \{x\}$ $\Rightarrow \{x\} = \frac{[x]}{[x] - 1}$ ..... (ii)

$[x] \neq 1$ in eq. (ii)

But if [x] = 1, then

$\{x\} = x$ which is donje only when,

$x \in [0, 1)$ ..... (iii)

& $[x] = 1 \Rightarrow \in [1, 2)$ ..... (iv)

$\therefore$ From (iii) & (iv) no value of x,

when [x] = 1

As we know

$\{x\} \in [0, 1)$

$\therefore \quad 0 \le \frac{[x]}{[x]-1} < 1$

$\frac{[x]}{[x]-1} < 1 \quad \& \quad \frac{[x]}{[x]-1} \ge 0$

$\Rightarrow \frac{1}{[x]-1} < 0 \quad \& \quad \{[x] \le 0 \quad \text{or} \quad [x] > 1\}$

$\Rightarrow [x] < 1 \quad \& \quad \{[x] \le 0 \quad \text{or} \quad [x] > 1\}$

$\Rightarrow [x] \le 0$

$\therefore \quad x = [x] + \{x\} = [x] + \frac{[x]}{[x]-1}$

$x = \frac{[x]^2}{[x]-1}$, where x takes values less than 1.

$\therefore \quad x = \frac{[x]^2}{[x]-1}$, where $x < 1$

$x = \frac{[x]^2}{[x]-1}$, where [x] is any non positive integer.

Q4. <u>Sketch</u> $y = (x - 1)(x - 2)$

Ans: $y = (x - 1)(x - 2)$

(i) put $y = 0 \Rightarrow x = 1, 2$

(ii) $y = x^2 - 3x + 2$

$\Rightarrow \frac{dy}{dx} = 2x - 3 \quad \& \quad \frac{d^2y}{dx^2}$

as $\frac{d^2y}{dx^2} > 0$

$\therefore$ minimaq at $x = 3/2$.

(iii) Increases when $x > 3/2$ & decreases when $x < 3/2$.

$y = x^2 - 3x + 2$

when $x = 3/2$

$y = \left(\frac{3}{2}\right)^2 - 3 \times \frac{3}{2} + 2 = -\frac{1}{4}$

Q5. Sketch curve $y = (x - 1)(x - 2)(x - 3)$

Ans: $y = (x - 1)(x - 2)(x - 3)$
(i) Put $y = 0 \Rightarrow x = 1, 2, 3$
(ii) $y = x^3 - 6x^2 + 11x - 6$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 12x + 11$ & $\frac{d^2y}{dx^2} = 6x - 12$

when $\frac{dy}{dx} = 0 \Rightarrow x = \frac{6 \pm \sqrt{3}}{3}$

Maxima when $x = \frac{6 - \sqrt{3}}{3}$ as $\frac{d^2y}{dx^2} = -2\sqrt{3}$

Minima when $x = \frac{6-\sqrt{3}}{2}$ as $\frac{d^2y}{dx^2} = 2\sqrt{3}$

(iii) $\frac{dy}{dx} = 3x^2 - 12x + 11$

$= 3(x - \frac{6 - \sqrt{3}}{3})(x - \frac{6-\sqrt{3}}{2})$

$\frac{dy}{dx} > 0$ or Increases when

$x < \frac{6 - \sqrt{3}}{3}$ or $x > \frac{6-\sqrt{3}}{2}$

Decreases when $\frac{dy}{dx} < 0$

or $\frac{6 - \sqrt{3}}{3} < x < \frac{6-\sqrt{3}}{2}$

(iv) Concave upwards when x > 2 & concave down when x < 2.

x = 2 is point where concavity of curve changes.

<u>Key Words</u>:

* Function.
* Domain.
* Co domain.
* Range.
* Identity Function.
* Constant Function.
* Logarithmic Function.
* Modulus Function.
* Signum Function.
* Greatest Integer Function.
* Fractional Part Function.
* Odd & Even Function.
* Periodic Function.
* Composite Function.
* Mapping.
* Bijective.
* Surjective.

* Injective.
* Inverse of Function.

Limits and Continuity

INTRODUCTION

To find limit of a function is an interesting concept where it may be possible that value of the function does not exist at a point but we try to find the value in the neighborhood of the point. We will talk about this in more detail in the chapter. In the other part of the chapter we will discuss continuity of a function which is closely related to the concept of limits. There are some functions for which graph is continuous while there are others for which this is not the case…

Definition of Limit:

We sometime come across situations when the values of the function 'f' for values of 'x' near a point 'c' lie near a number 'l' which is not equal to f (c) or that value lie near no number at all.

Dumb Question:

1) How is it possible that function has different value near point c and at c?

Ans: Let us explain it with help of an example, Consider

Fig (1)

In this function clearly the value of function near 1/2 lies near 1/2 but the value of

function at 1/2 is 1 which is not equal to 1/2.

So, there is need to introduce the notion of limit. A function f is said to tends to a limit l as x tends to 'a', if on approaching the point x=a from the values just greater than or just smaller than x=a, f(x) has tendency to move more closer to value l.

Mathematically we write this as

$\lim_{x \to a} f(x) = l$ Which is equivalent to,

$|f(x)-l| < \varepsilon \ \forall$ x whenever $0 < |x-a| < \delta$, and ε and δ are sufficiently small positive numbers.

Right and Left hand limits:

<u>Right hand limit:</u>

$\lim_{x \to a^+} f(x) = l$ If for every numbers $\varepsilon>0$ there exists corresponding number $\delta>0$ such that for all x satisfying

$a < x < a+\delta$

$\Rightarrow |f(x)-l)| < \varepsilon$

Example: $\lim_{x \to 1^+} [x] = 1$

<u>Left hand limit:</u>

$\lim_{x \to a^-} f(x) = l$ If for every number $\varepsilon>0$ there exists a corresponding $\delta>0$ such that for all x satisfying $a-\delta< x < a \Rightarrow |f(x)-l|<\varepsilon$

Example: $\lim_{x \to 1^-} [x] = 0$

Fig (2)

Note that limit of a function exists at any point if and only if left hand limit is equal to right hand limit at that point.

Illustration 1:

Find the value of $\lim_{x \to 1} f(x)$ Where

$$f(x) = \begin{cases} 8x^3 - 7 & 0 < x \leq 1 \\ 2x - x^2 & 1 < x < 2 \end{cases}$$

Solution:

$$\lim_{x \to 1^-} f(x) = \lim_{n \to 0^+} f(1-h)$$
$$= \lim_{n \to 0^+} 8(1-h)^3 - 7$$
$$= 8 \times 1^3 - 7$$
$$= 1$$
$$\lim_{x \to 1^+} f(x) = \lim_{n \to 0^+} f(1+h)$$
$$= \lim_{n \to 0^+} 2(1+h) - (1+h)^2$$
$$= 2 - 1^2$$
$$= 1$$

So, right hand limit and left hand limit are equal. Hence $\lim_{x \to 1} f(x) = 1$

Discovering infinity:

Infinity (∞) is a symbol and not a number it is symbol for something which keeps increasing and passes all limits. Similarly $-\infty$ is symbol of a variable that continuously decrease and passes all limits.

Remember the following points:

1) We cannot plot ∞ on paper.

2) $\infty + \infty = \infty$.

3) $\infty - \infty$ is indeterminate.

4) $\infty \times \infty = \infty$.

5) $0 \times \infty$ is indeterminate.

6) $\frac{a}{\infty} = 0$ If a is finite.

7) $\frac{\infty}{\infty}$, ∞^0, 1^∞ are all indeterminate.

Dumb Question:

1) It is often said '$\infty + a$' where a is finite is ∞. How is this possible?

Ans: Consider ∞ like an ocean. Now 'a' is like a bucket of water. If you add a bucket of water to ocean will the volume of ocean increase? It will remain ocean. So similarly ∞ remains ∞ on adding a finite number to it.

Illustration 2:

Find the value of $\lim_{x\to\infty} x + x^2$?

Solution:

$$\lim_{x\to\infty} x + x^2 = \infty + \infty$$

$$= \infty$$

$\therefore$ The value of $\lim_{x\to\infty} x + x^2 = \infty$.

Theorem on Limits:

Let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. If l and m exist then,

1) $\lim_{x\to a} [f(x) \pm g(x)] = l \pm m$.

2) $\lim_{x \to a} [f(x) \times g(x)] = l \times m$.

3) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}$ Provided m ¹ 0.

4) $\lim_{x \to a} k\, f(x) = k \lim_{x \to a} f(x)$ Where k is constant.

5) If f(x) £ g(x) then l £ m.

6) $\lim_{x \to a} [f(x)]^{g(x)} = l^{m}$.

7) If f(x) £ g(x) £ h(x) for all x.

and $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$ then $\lim_{x \to a} g(x) = l$ (Squeeze play/ Sandwitch Theorem).

Illustration 3:

If [x] denotes the integral part of x, then find

$$\lim_{n \to \infty} \frac{[1^2 x] + [2^2 x] + - - - - + [n^2 x]}{n^3}.$$

Solution:

Let $S_n = [1^2 x] + [2^2 x] + - - - - - - - + [n^2 x]$

Now we know

$x - 1 < [x] \le x.$

$1^2 x - 1 < [1^2 x] \le 1^2 x.$

$2^2 x - 1 < [2^2 x] \le 2^2 x$

$- - -$

$- - -$

$n^2 x - 1 < [n^2 x] \le n^2 x$

\ Adding them all gives us

$$(1^2+2^2+-----+n^2)x-n<S_n\le(1^2+2^2+---+n^2)x$$

$$\text{Or } \frac{n(n+1)(2n+1)}{6n^3}-\frac{n}{n^3}<\frac{S_n}{n^3}\le\frac{n(n+1)(2n+1)x}{6n^3}$$

$$\Rightarrow\lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)}{6n^3}-\frac{1}{n^2}\right)<\lim_{n\to\infty}\frac{S_n}{n^3}\le\lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)x}{6n^3}\right)$$

$$\Rightarrow\frac{x}{3}<\lim_{n\to\infty}\frac{S_n}{n^3}\le\frac{x}{3}$$

By using squeeze play theorem we get,

$$\lim_{n\to\infty}\frac{S_n}{n^3}=\frac{x}{3}.$$

Some important expansions (Power Series):

1) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+----+\frac{x^r}{r!}+---$.

2) $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+---+\frac{(-1)^{r-1}x^r}{r}+---$ Here -1 < x £ 1.

3) $a^x=1+x(\ln a)+\frac{x^2}{2!}(\ln a)^2+----+\frac{x^r(\ln a)^r}{r!}+---$.

4) $\text{Sin}x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+-----\frac{(-1)^{r-1}.x^{2r+1}}{(2r+1)!}+----$.

5) $\text{Cos}x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+-----\frac{(-1)^{r-1}x^{2r}}{2r!}+-----$.

6) $\text{Tan}x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + ------$.

7) $\text{Sin}^{-1}x = x + \frac{1^2x^3}{3!} + \frac{1^2 3^2 x^5}{5!} + \frac{1^2 3^2 5^2 x^7}{7!} + ----$.

8) $\text{Tan}^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ----$

9) $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + ------$ (For rational or integral n).

Illustration 4:

Find the series expansion of Sin^2x?

Solution:

Now we know that

$$\text{Sin}^2x = \frac{1 - \text{Cos}2x}{2}$$

$$\therefore \text{Sin}^2x = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} - ---\right)$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2}\left(\frac{2^2x^2}{2!} - \frac{2^4x^4}{4!} - ---\right)$$

$$= \frac{2.x^2}{2!} - \frac{2^3.x^4}{4!} + \frac{2^5.x^6}{6!} + ---$$

Note that many other series could be found in that way as we found the series for Sin^2x.

Some Standard results on limits:

1) $\lim_{x\to 0} \text{Sin}x = \lim_{x\to 0} \tan x = 0$

2) $\lim_{x\to 0} \frac{\text{Sin}x}{x} = \lim_{x\to 0} \frac{x}{\text{Sin}x} = \lim_{x\to 0} \frac{\text{Sin}^{-1}x}{x} = \lim_{x\to 0} \frac{x}{\text{Sin}^{-1}x} = 1$

3) $\lim_{x\to 0} \text{Cos}x = \lim_{x\to 0} \frac{1}{\text{Cos}x} = 1$

4) $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{\tan^{-1}x}{x} = \lim_{x\to 0} \frac{x}{\tan^{-1}x} = 1$

5) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

6) $\lim_{x\to a} \frac{x^n - a^n}{x-n} = na^{n-1}, n \in Q$

7) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1; \lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$

8) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1; \lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e$

9) $\lim_{x\to\infty} \frac{\text{Sin}x}{x} = \lim_{x\to\infty} \frac{\text{Cos}x}{x} = 0$

10)

$\lim_{x\to a} (f(x))^{g(x)} = e^{\lim_{x\to a} (f(x)-1)g(x)}$ Where $g(x) \to \infty$ and $f(x) \to 1$ for $x \to a$

Note: These limits could be derived using the series expansion or by L^1 Hospital's rule which will discussed in a later section.

Illustration 5:

Find the value of $\lim_{x \to a} \frac{x\text{Sin}a - a\text{Sin}x}{x - a}$?

Solution:

$$\lim_{x \to a} \frac{x\text{Sin}a - a\text{Sin}x}{x - a} = \lim_{x \to a} \frac{x\text{Sin}a - a\text{Sin}a - a\text{Sin}x + a\text{Sin}a}{x - a}$$

$$= \lim_{x \to a} \frac{(x - a)\text{Sin}a - a(\text{Sin}x - \text{Sin}a)}{x - a}$$

$$= \lim_{x \to a} \left(\text{Sin}a - \frac{a\ 2\text{Sin}\frac{(x - a)}{2}\text{Cos}\frac{(x + a)}{2}}{2\left(\frac{x - a}{2}\right)} \right)$$

$$= \text{Sin}a - a.1.\text{Cos}a$$

$$= \text{Sin}a - a\text{Cos}a$$

Evaluation of Limits:

1) <u>Direct Substitution:</u>

If we get a finite number by direct substitution of point we are done.

Illustration 6: Find $\lim_{x \to 5} \frac{x^2 - 1}{5x + 3}$?

Solution:

$$\lim_{x \to 5} \frac{x^2 - 1}{5x + 3} = \frac{5^2 - 1}{5.5 + 3}$$

$$= \frac{24}{28} = \frac{6}{7}$$

2) <u>Algebraic Limits:</u>

A) Factorization process of finding limit.

If direct substitution of x=a in a rational function takes useless form like $\frac{0}{0}, \frac{\infty}{\infty}$ then a factor of (x-a) can be cancelled from both numerator and denominator. Now again use direct substitution to see if limit can be evaluated.

Illustration 7:

Find $\lim_{x\to 2} \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$?

Solution:

$$\lim_{x\to 2} \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8} = \lim_{x\to 2} \frac{(x-1)(x-2)(x-3)}{(x-2)(x-4)}$$

$$= \lim_{x\to 2} \frac{(x-1)(x-3)}{x-4}$$

$$= \frac{-1}{-2} = \frac{1}{2}$$

B) Using the result $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Illustration 8: Evaluate $\lim_{x\to 1} \frac{x^{\frac{1}{3}} - 1}{x^{\frac{1}{2}} - 1}$?

Solution:

$$\lim_{x\to 1} \frac{x^{\frac{1}{3}} - 1}{x^{\frac{1}{2}} - 1} = \lim_{x\to 1} \left(\frac{x^{\frac{1}{3}} - 1}{x - 1} \right) \left(\frac{x - 1}{x^{\frac{1}{2}} - 1} \right)$$

$$= \frac{\frac{1}{3} \times 1^{\frac{-2}{3}}}{\frac{1}{2} \times 1^{\frac{-1}{2}}}$$

$$= \frac{2}{3}$$

C) Using rationalization:

Some factor which creates $\frac{0}{0}$ form is cancelled by rationalization and limit is evaluated.

Illustration 9: Evaluate $\lim_{a \to 0} \frac{\sqrt{y+a} - \sqrt{y}}{6a}$?

Solution:

$$\lim_{a \to 0} \frac{\sqrt{y+a} - \sqrt{y}}{6a} = \lim_{a \to 0} \frac{\left(\sqrt{y+a} - \sqrt{y}\right)\left(\sqrt{y+a} + \sqrt{y}\right)}{6a\left(\sqrt{y+a} + \sqrt{y}\right)}$$

$$= \lim_{a \to 0} \frac{(y+a) - y}{6a\left(\sqrt{y+a} + \sqrt{y}\right)}$$

$$= \lim_{a \to 0} \frac{1}{6\left(\sqrt{y+a} + \sqrt{y}\right)}$$

$$= \frac{1}{6\left(2\sqrt{a}\right)} = \frac{1}{12\sqrt{a}}$$

D) Limits where x® ∞

We write down expression as a rational function and then divide each term by highest power of x obtained from numerator and denominator.

Illustration 10: Find the value of $\lim_{x \to \infty} \frac{x^2 + 4}{4x^2 - 1}$?

Solution:

$$\lim_{x \to \infty} \frac{x^2 + 4}{4x^2 - 1} = \lim_{x \to \infty} \frac{1 + \frac{4}{x^2}}{4 - \frac{1}{x^2}}$$

$$= \frac{1+0}{4-0}$$

$$= \frac{1}{4}$$

3) <u>Trigonometric Limits:</u>

The trigonometric limits are those which involve trigonometric ratios.

The use of the result $\lim_{x \to 0} \frac{\text{Sin}x}{x}$ is a integral part of evaluation of such limits.

Illustration 11: Evaluate $\lim_{x \to 0} \frac{\text{Sin}\left(\pi \text{Cos}^2 x\right)}{x^2}$?

$$\lim_{x\to 0}\frac{\text{Sin}\left(\pi\,\text{Cos}^2 x\right)}{x^2}=\lim_{x\to 0}\frac{\text{Sin}(\pi(1-\text{Sin}^2 x))}{x^2}$$

$$=\lim_{x\to 0}\frac{\text{Sin}(\pi-\pi\,\text{Sin}^2 x)}{x^2}$$

$$=\lim_{x\to 0}\frac{\text{Sin}\ \pi\,\text{Sin}^2 x}{x^2}$$

$$=\lim_{x\to 0}\frac{\text{Sin}(\pi\,\text{Sin}^2 x)}{(\pi\,\text{Sin}^2 x)}\times\pi\times\lim_{x\to 0}\frac{\text{Sin}^2 x}{x^2}$$

= 1´ p ´ 1

= p.

4) <u>Exponential and Logarithmic Limits:</u>

These are the limits which involve use of logarithmic and exponential functions. Some of the standard limits help us in solving these limits.

Illustration 12:

Find the following limits $\lim_{x\to 0}\frac{e^{2x}x\text{Cos}x-x\log(1+x)-x}{x^2}$

Solution:

$$\lim_{x\to 0}\frac{e^{2x}x\text{Cos}x-x\log(1+x)-x}{x^2}$$

$$=\lim_{x\to 0}\frac{e^{2x}x\text{Cos}x-xe^{2x}+xe^{2x}-x-x\log(1+x)}{x^2}$$

$$=\lim_{x\to 0}\left(\frac{-e^{2x}.x\left(2\text{Sin}^2\frac{x}{2}\right)}{x^2}+\frac{e^{2x}-1}{x}-\frac{\log(1+x)}{x}\right)$$

$$=\lim_{x\to 0}\frac{-e^{2x}2\text{Sin}^2\frac{x}{2}}{x}+\lim_{x\to 0}\frac{e^{2x}-1}{x}-\lim_{x\to 0}\frac{\log(1+x)}{x}$$

$$=\lim_{x\to 0}-e^{2x}\text{Sin}\frac{x}{2}\left(\frac{\text{Sin}(x/2)}{(x/2)}\right)+\lim_{x\to 0}2\left(\frac{e^{2x}-1}{2x}\right)-\lim_{x\to 0}\frac{\log(1+x)}{x}$$

$$=1.0+2.1-1=2.1$$

L' Hospital's Rule:

L' Hospital's Rule is applied only to the two forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.The rule

states that if $\lim_{x\to a}\frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f^{1}(x)}{g^{1}(x)}$ provided later limit exists. But if still it takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then we differentiate numerator and denominator again.

i.e. $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f^{11}(x)}{g^{11}(x)}$ and this process continues till we find that $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form is removed.

Illustration 13:

Evaluate $\lim_{x\to 0}(\text{Cosec}x)^{x}$

Solution:

Let $A = \lim_{x\to 0}(\text{Cosec}x)^{x}$

$\log A = \lim_{x\to 0} x\log \text{Cosec}x$

$$= \lim_{x\to 0}\frac{\log \text{Cosec}x}{\frac{1}{x}} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{\frac{1}{\text{Cosec}x}(-\text{Cosec}x.\text{Cot}x)}{\frac{-1}{x^{2}}} \qquad \text{(By L' Hospital' s Rule)}$$

$$= \lim_{x\to 0}\frac{x^{2}}{\tan x}$$

$= 0$

$\therefore \log A = 0 \text{ or } A = 1$

So, $\lim_{x\to 0}(\text{Cosec}x)^{x} = 1$

Continuity at point:

A function 'f' is said to be continuous at a point 'a' in domain of f, if following conditions are met.

1) f (a) exists.

2) $\lim_{x \to a} f(x)$ exists finitely which essentially means that $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist finitely and are equal.

3) $\lim_{x \to a} f(x) = f(a)$. If anyone or more conditions are not met than function f is said to be discontinuous at point 'a'. Geometrically graph of function exhibit a break at point x=a.

Illustration 14: Discuss the continuity of function y = f(x) as shown by graph below at the point x=1.

Fig (3)

Solution:

Now clearly the function exhibits a break at point x=1 and is thus discontinuous at x=1. But if we go by the rules mentioned in above article we observe that.

$\lim_{x \to 1^-} f(x) = 1$ And $\lim_{x \to 1^+} f(x) = 2$

So $\lim_{x \to 1} f(x)$ does not exists and thus the function is discontinuous at x=1.

<u>Continuity of function in open interval:</u>

A function f is said to be continuous in (a, b) if f is continuous at each and every point belonging to interval (a, b).

<u>Continuity of function in closed interval:</u>

A function f is said to be continuous in closed interval [a, b] if

1) f is continuous in open interval (a, b) and

2) $\lim_{x \to a^+} f(x)$ exists finitely and $\lim_{x \to a^+} f(x) = f(a)$ and

3) $\lim_{x \to b^-} f(x)$ exists finitely and $\lim_{x \to b^-} f(x) = f(b)$.

Illustration 15:

Consider a function f(x) which is continuous in [1, 2] and given that f (1) = -1 and f (1) = 2, prove that there exists a point x such that f(x) =1.

Solution:

Let us arbitrarily choose some graph for f(x)

Fig (4)

For that matter function f(x) can take any shape. But since the function f is continuous between 1 and 2 it must be defined on all values lying between 1 and 2 and the function's curve will go from y = -1 to y = 2 through some path. Whatever path be chosen y =1 will lie in between (Remember there cannot be a sudden jump because function is continuous)

Hence there exists a point x-where f(x) = 1.

<u>Properties of continuous functions:</u>

Let f(x) and g(x) are continuous functions at x=a then

1) K f(x) is continuous where K is constant
2) f(x) ± g(x) is continuous at x=a.
3) f(x) ´g(x) is continuous at x=a.
4) f(x)/g(x) is continuous at x=0 provided g (a) ¹ 0.
5) If m = f(x) is continuous at $x=x_0$ and f (m) is continuous at the point $m_0 = f(x_0)$, then composite function f (f(x)) is continuous at point x_0.
6) If f(x) is continuous on [a, b] such that f (a) and f (b) are of opposite signs. Then there exists at least one solution of equation f(x) = 0 in the open interval (a, b).
7) Functions like sinx, cosx, tanx, cotx, secx, cosecx, logx, e^x etc are continuous in their domain.

Dumb Question:

1) How can function like tanx be continuous?

Ans: Please read the point mentioned above carefully. It is written that tanx is continuous in its domain. The points (2n+1)p/2 when tanx®∞ do not lie in the domain of function f.

Illustration 16:

Let function $f = e^x + \cos x + 3\sin x - \frac{5}{2}$ Prove that there is a solution for equation f(x) =0 in the interval [0, 1].
Solution:

We observe that e^x, cosx, sinx, $\frac{5}{2}$ all are continuous functions in interval [0, 1]. So the function f is also continuous in [0, 1].

Now $f(0) = 1+1+0-\frac{5}{2} = -\frac{1}{2} < 0$

$f(1) = e + \cos 1 + 3\sin 1 - \frac{5}{2}$

Since 0<1< (p/2) so cos1 and sin1 are positive quantities and e> $\frac{5}{2}$.
So, f (1)>0.
Now f (0) and f (1) are of opposite signs. So f(x) =0 has a solution in interval [0, 1].
<u>Classification of discontinuities:</u>
1) Removable discontinuity:

Here at a point x=a of discontinuity, $\lim_{x\to a} f(x)$ exists but f (a) is either undefined or if defined it does not tally with $\lim_{x\to a} f(x)$.
In this case the removal of discontinuity is achieved by modifying definition of function suitably so that f (a) tallies with $\lim_{x\to a} f(x)$ which is found existing.
2) Irremovable discontinuity:

Here $\lim_{x\to a} f(x)$ does not exist even though f (a) exists or additionally f (a) also does not exist.

Here, since $\lim_{x\to a} f(x)$ does not exist, no modification of definition at x=a can succeed to make $\lim_{x\to a} f(x)$ exist. Hence discontinuity remains irremovable.
Illustration 17:
Consider the function

$$f(x) = \begin{cases} x-1, & x<0 \\ \frac{1}{5} & x=0 \\ x^3-1, & x>0 \end{cases}$$

Discuss the continuity of function at x=0. If discontinuous suggest how to remove discontinuity.

Solution:

Let us plot the function f(x)

Fig (5)

Now $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x-1 = -1$

And $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^3-1 = -1$

So, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = -1$

But f (0) = $\frac{1}{5}$.

\f (x) has removable discontinuity and f(x) can be made continuous by taking f (0) = -1.

$$\therefore f(x) = \begin{cases} x-1, & x<0 \\ -1 & x=0 \\ x^3-1, & x>0 \end{cases}$$

PROBLEMS (EASY TYPE)

1) Prove that $\lim_{x\to 0}\frac{\text{Sin}x}{x}=1$

Solution:

$\lim_{x\to 0}\text{Sin}x=0$ And $\lim_{x\to 0}x=0$ so $\frac{\text{Sin}x}{x}$ assumes the indeterminate form 0/0 when x tends to 0.

We know for $x\in\left[0,\frac{\pi}{2}\right]$

Sinx < x < Tanx

$$\Rightarrow 1<\frac{x}{\text{Sin}x}<\frac{1}{\text{Cos}x}$$

$$\Rightarrow 1>\frac{\text{Sin}x}{x}>\text{Cos}x$$

This is true for x>0, the same holds when x<0 in that Sin (-x)/ (-x) = $\frac{\text{Sin}x}{x}$ and

Cos (-x) = Cosx.

Now since $\lim_{x\to 0}\text{Cos}x=1$, it follows from sandwitch theorem,

$\lim_{x\to 0}\frac{\text{Sin}x}{x}=1$.

2) Find the value of $\lim_{x\to\frac{\pi}{2}}\frac{\left[\frac{x}{2}\right]}{\log(\text{Sin}x)}$.

Solution: As $\lim_{x\to\frac{\pi}{2}}\frac{\left[\frac{x}{2}\right]}{(\text{Sin}x)}$

\ $x\to\frac{\pi}{2}\Rightarrow\frac{x}{2}\to\frac{\pi}{4}$

Where $\frac{x}{2}\to\frac{\pi}{4}<1$ for $x\to\frac{\pi^+}{2}$ or $x\to\frac{\pi^-}{2}$

$\therefore\left[\frac{x}{2}\right]=0$, Since $0<\left(\frac{x}{2}\to\frac{\pi}{4}\right)<1$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{\left[\frac{x}{2}\right]}{(\text{Sin}x)} = \lim_{x\to\frac{\pi}{2}} 0 = 0$$

3) Evaluate $\lim_{h\to 0} \frac{\log_{10}(1+h)}{h}$.

Solution:

$$\lim_{h\to 0} \frac{\log_{10}(1+h)}{h} = \lim_{h\to 0} \frac{\log_e(1+h)\times\log_{10}e}{h}$$

$$= \lim_{h\to 0} \frac{\log_e(1+h)}{h}\times\log_{10}e$$

$$= \log_{10}e\times 1 = \log_{10}e$$

4) Evaluate $\lim_{x\to 0} \frac{x-\int_0^x \text{Cos}t^2 dt}{x^3-6x}$.

Solution:

$$\lim_{x\to 0} \frac{x-\int_0^x \text{Cos}t^2 dt}{x^3-6x} \quad \left\{\text{form } \frac{0}{0}\right\}$$

Applying L-Hospital's Rule we get,

$$= \lim_{x\to 0} \frac{1-\frac{d}{dx}\int_0^x \text{Cos}t^2 dt}{3x^2-6}$$

$$= \lim_{x\to 0} \frac{1-\text{Cos}(x^2)}{3(x^2-2)} = \frac{1-\text{Cos}0}{3(0-2)}$$

$$= \frac{0}{-6} = 0$$

5) Find $\lim_{x\to\sqrt{2}} \frac{x^9-3x^8+x^6-9x^4-4x^2-16x+84}{x^5-3x^4-4x+12}$.

Solution:

For $x=\sqrt{2}$ both the numerator and denominator are 0.

Thus, $\lim_{x\to\sqrt{2}} \frac{x^9-3x^8+x^6-9x^4-4x^2-16x+84}{x^5-3x^4-4x+12}$

$$= \lim_{x\to\sqrt{2}} \frac{(x^2-2)[x^7-3x^6+2x^5-5x^4+4x^3-19x^2+8x-42]}{(x^2-2)[x^3-3x^2+2x-6]}$$

Given [x^2-2 ¹ 0].

$$= \lim_{x\to\sqrt{2}} \frac{x^7 - 3x^6 + 2x^5 - 5x^4 + 4x^3 - 19x^2 + 8x - 42}{x^3 - 3x^2 + 2x - 6}$$

$$= \frac{(\sqrt{2})^7 - 3(\sqrt{2})^6 + 2(\sqrt{2})^5 - 5(\sqrt{2})^4 + 4(\sqrt{2})^3 - 19(\sqrt{2})^2 + 8(\sqrt{2}) - 42}{(\sqrt{2})^3 - 3(\sqrt{2})^2 + 2(\sqrt{2}) - 6}$$

$$= \frac{8\sqrt{2} - 24 + 8\sqrt{2} - 20 + 8\sqrt{2} - 38 + 8\sqrt{2} - 42}{2\sqrt{2} - 6 + 2\sqrt{2} - 6}$$

$$= \frac{8\sqrt{2} - 31}{\sqrt{2} - 3}$$

6) If $\lim_{x\to\infty}\left(\frac{x^2-1}{x+1} - ax - b\right) = 2$ then find the values for a and b.

Solution:

We have

$$\lim_{x\to\infty}\left(\frac{x^2-1}{x+1} - ax - b\right) = \lim_{x\to\infty} \frac{x^2(1-a) - x(a+b) + (1+b)}{(x+1)} = 2$$

Since, limit of above expression is a finite non-zero number.

\ degree of numerator = degree of denominator

Þ 1-a = 0 Þ a = 1

\ Putting a = 1 in above limit we get

$$\lim_{x\to\infty} \frac{-x(1+b) + (1+b)}{1+x} = 2$$

Þ - (1+b) = 2 Þ b = -3

Hence a = 1 and b = -3.

7) Let

$$f(x) = \begin{cases} \{1 + |\sin x|\}^{\frac{a}{|\sin x|}} & -\frac{\pi}{6} < x < 0 \\ b; & x = 0 \\ e^{\frac{\tan 2x}{\tan 3x}} & 0 < x < \frac{\pi}{6} \end{cases}$$

Determine a and b such that f(x) is continuous at x=0.

Solution:

Since f(x) is continuous at x=0.

Therefore R.H.L = L.H.L = f (0)

R.H.L at x = 0.

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} e^{\frac{\tan 2h}{\tan 3h}}$$

$$= \lim_{h \to 0} e^{\frac{\tan 2h}{2h} \times \frac{3h}{\tan 3h} \times \frac{2}{3}}$$

$$= e^{\frac{2}{3}} \text{ ------(1)}$$

Again L.H.L at x = 0

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \{1+|\operatorname{Sin}(0-h)|\}^{\frac{a}{|\operatorname{Sin}(0-h)|}}$$

$$= \lim_{h \to 0} \{1+|\operatorname{Sinh}|\}^{\frac{a}{|\operatorname{Sinh}|}}$$

$$= e^{\lim_{h \to 0} |\operatorname{Sinh}| \cdot \frac{a}{|\operatorname{Sinh}|}}$$

$$= e^a$$

And f (0) = b ------ (2)

$$\text{Thus, } e^{\frac{2}{3}} = e^a = b$$

$$\Rightarrow a = \frac{2}{3} \quad \text{and } b = e^{\frac{2}{3}}$$

8) A function is defined as follows,

$$f(x) \begin{cases} 1; & \text{When } -\infty < x < 0 \\ 1+\operatorname{Sin}x & \text{When } 0 \le x < \frac{\pi}{2} \\ 2+\left(x-\frac{\pi}{2}\right)^2 & \text{When } \frac{\pi}{2} \le x < \infty \end{cases}$$

Discuss continuity of f.

Solution:

Continuity at x = 0

L.H.L at x = 0 $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1) = 1$

R.H.L at x = 0 $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1 + \text{Sin}x) = 1$

f (0) = 1+Sin0 = 1

= L.H.L = R.H.L = f (0) so f (x) is continuous at x = 0.

Continuity at $x = \frac{\pi}{2}$

L.H.L at $x = \frac{\pi}{2} = \lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} (1 + \text{Sin}x) = 1 + 1 = 2$

R.H.L at $x = \frac{\pi}{2} = \lim_{x \to \frac{\pi}{2}^+} f(x) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 2$

$f\left(\frac{\pi}{2}\right) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 2$

\ R.H.L = L.H.L = $f\left(\frac{\pi}{2}\right)$

So, f(x) is continuous at $x = \frac{\pi}{2}$.

Hence f(x) is continuous over the whole real number.

9) Discuss the continuity of the function:

g (x) = [x] + [-x] at integral values of x.

Solution:

Here x can assume two values a) Integers b) Non-Integers

a) If x is an integer

[x] = x and [-x] = -x Þ g(x) = x-x = 0.

b) If x is not an integer.

Let x = n+f Where n is an integer and f Î (0, 1).

Þ [x] = [n+f] = n.

Þ [-x] = [-n-f] = [(-n-1) + (1-f)] = (-n-1). (Because 0<f<1 Þ (1-f) <1)

Hence g(x) = [x] + [-x] = n + (-n-1) = -1.

So we get;

g(x) = 0; if x is an integer

= -1; if x is not an integer

Let us discuss the continuity of g(x) at a point x=a (Where a Î integer).

L.H.L = $\lim_{x \to a^-} g(x) = -1$ (as x®a$^-$, x is not an integer)

R.H.L = $\lim_{x \to a^+} g(x) = -1$ (as x®a$^+$, x is not an integer)

But g (a) = 0 because a is an integer.

Hence g(x) has a removable discontinuity at integral values of x

\ g (x) = [x] + [-x], xÏ integer.

= -1, x Î integer.

10) Let f(x) = x- |x-x^2|, -1£ x £ 1. Discuss the continuity of f(x) in the closed interval

[-1, 1]. Draw the graph of f(x) in that interval.

Solution:

Here f(x) = x-|x (1-x)|

= x- |x| |x-1|, -1£ x £ 1.

Now using definition of modulus function we have

$$\therefore f(x) = x - (-x)(1-x)$$

$$= 2x - x^2, \qquad -1 \le x < 0$$

$$f(x) = x - x(1-x)$$

$$= x^2 \qquad 0 \le x \le 1$$

We know that polynomials are continuous every where, so only doubtful point is the turning point x=0 of definition.

$$\lim_{h \to 0} f(0+h) = \lim_{h \to 0} (0+h)^2 = 0$$

$$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} 2(0-h) - (0-h)^2 = 0$$

$$f(0) = 0^2 = 0$$

So f(x) is continuous at x=0.

Hence f(x) is continuous in [-1, 1].

Hence the graph of f(x) is continuous in [-1, 1] and

f (x) = 2x-x^2 in -1£ x < 0

x^2 in 0£ x £ 1.

\ The graph of f(x) is as follows;

11) A function f(x) is defined by

$$f(x) = \begin{cases} \dfrac{[x^2]-1}{x^2-1} & \text{for } x^2 \neq 1 \\ 0 & \text{for } x^2 = 1 \end{cases}$$

Discuss the continuity of f(x) at x=1.

Solution:

We have

$$f(x) = \begin{cases} \dfrac{[x^2]-1}{x^2-1} & \text{for } x^2 \neq 1 \\ 0 & \text{for } x^2 = 1 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \dfrac{-1}{x^2-1} & \text{for } 0 < x^2 < 1 \\ 0 & \text{for } x^2 = 1 \\ \dfrac{1-1}{x^2-1} & \text{for } 1 < x^2 < 2 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \dfrac{-1}{x^2-1} & \text{for } 0 < x^2 < 1 \\ 0 & \text{for } x^2 = 1 \\ 0 & \text{for } 1 < x^2 < 2 \end{cases}$$

\ R.H.L at $x^2 = 1$

Þ $\lim_{x \to 1^+} f(x) = 0$

Also L.H.L at $x^2=1$

Þ $\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \dfrac{-1}{(1-h)^2 - 1} = -\infty$

$\therefore \lim_{x \to 1} f(x)$ does not exist {as R.H.L ¹ L.H.L}

Hence f(x) is not continuous at x=1.

PROBLEMS (MEDIUM TYPE)

1) Without expansion or using L-Hospital's Rule prove that

$$\lim_{\theta\to 0}\frac{\theta-\operatorname{Sin}\theta}{\theta^3}=\frac{1}{6}.$$

Solution:

Let the limit be P then

$$P=\lim_{\theta\to 0}\frac{\theta-\operatorname{Sin}\theta}{\theta^3}$$

$$=\lim_{\theta\to 0}\frac{3\theta-\operatorname{Sin}3\theta}{(3\theta)^3}\qquad \text{(replacing } \theta \text{ by } 3\theta\text{)}$$

$$=\lim_{\theta\to 0}\frac{3\theta-3\operatorname{Sin}\theta+4\operatorname{Sin}^3\theta}{(3\theta)^3}$$

$$=\lim_{\theta\to 0}\left\{\frac{\theta-\operatorname{Sin}\theta}{9\theta^3}+\frac{4}{27}\left(\frac{\operatorname{Sin}\theta}{\theta}\right)^3\right\}$$

$$=\frac{1}{9}\lim_{\theta\to 0}\frac{\theta-\operatorname{Sin}\theta}{\theta^3}+\frac{4}{27}\lim_{\theta\to 0}\left(\frac{\operatorname{Sin}\theta}{\theta}\right)^3$$

$$=\frac{1}{9}P+\frac{4}{27}1^3$$

$$\therefore\ P-\frac{P}{9}=\frac{4}{27}$$

$$\therefore\ \frac{8P}{9}=\frac{4}{27}$$

$$\therefore\ P=\frac{1}{6}$$

Dumb Question:

1) How can we replace q by 3q in the limit without changing it?

Ans: Since q ® 0, 3q will also tend to zero and thus there is no effect on the limit and thus it remains same.

2) Prove that

$$\lim_{m\to\infty}\left\{1+\operatorname{Cos}^{2m}(n!\,\pi x)\right\}=2 \quad \text{If x is rational.}$$

$$\lim_{m\to\infty}\left\{1+\operatorname{Cos}^{2m}(n!\,\pi x)\right\}=\mathbf{1} \quad \text{If x is irrational.}$$

Solution:

We know that |Cosq| £ 1 for all q.

If |Cos (n!p x) | = P <1,

limit = $\lim_{m\to\infty}(1+P^{2m}) = 1 ----(1)$

As P^{2m} ® 0 When m® ∞.

If |Cos (n!p x) | =1.

limit = $\lim_{m\to\infty}(1+1^{2m}) = \lim_{m\to\infty} 2 = 2 ---(2)$

Now |Cos (n!p x) | =1

Þ n!px = kp, Where k is an integer.

\x = $\frac{k}{n!}$ which is a rational number.

\ From (2) limit = 2 if x is rational.

Again |Cos (n!p x) |<1

Þ |Cos (n!p x) | ¹ 1 for all n Î N.

Þ n!px ¹ kp for all nÎN, k Î Z

Þ x ¹ $\frac{k}{n!}$ for all n Î N, k Î Z.

Þ x is not rational i.e. x is irrational.

\From (1) limit = 1 if x is irrational.

3) Given $f(x) = \frac{\sqrt{2}\text{Cos}x - 1}{\text{Cot}x - 1}$ for all $x \in \left(0, \frac{\pi}{2}\right)$ except at $x = \frac{\pi}{4}$. Define $f\left(\frac{\pi}{4}\right)$ so that f(x) may be continuous at $x = \frac{\pi}{4}$.

Solution:

f (x) will be continuous at $x = \frac{\pi}{4}$,

If $\lim_{x\to\frac{\pi}{4}} f(x) = f\left(\frac{\pi}{4}\right)$

$$\therefore f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}\text{Cos}x - 1}{\text{Cot}x - 1}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\left(\sqrt{2}\text{Cos}x - 1\right)\text{Sin}x}{\text{Cos}x - \text{Sin}x}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\left(\sqrt{2}\text{Cos}x - 1\right)\left(\sqrt{2}\text{Cos}x + 1\right)(\text{Cos}x + \text{Sin}x)\text{Sin}x}{\left(\sqrt{2}\text{Cos}x + 1\right)(\text{Cos}x - \text{Sin}x)(\text{Cos}x + \text{Sin}x)}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\left(2\text{Cos}^2x - 1\right)}{\left(\text{Cos}^2x - \text{Sin}^2x\right)} \times \frac{(\text{Cos}x + \text{Sin}x)\text{Sin}x}{\left(\sqrt{2}\text{Cos}x + 1\right)}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\text{Sin}x\,(\text{Cos}x + \text{Sin}x)}{\left(\sqrt{2}\text{Cos}x + 1\right)}$$

$$= \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)}{\sqrt{2} \times \frac{1}{\sqrt{2}} + 1}$$

$$= \frac{1}{2}$$

Dumb Question:

1) Why did we multiply by $\left(\sqrt{2}\text{Cos}x + 1\right)(\text{Cos}x + \text{Sin}x)$ in both numerator and denominator?

Ans: The basic aim to do the multiplication was to cancel out Cos2x which was making the limit an indeterminate $\frac{0}{0}$ form.

4) Prove that $f(x) = [\tan x] + \sqrt{\tan x - [\tan x]}$ (where [] denotes greatest integer function) is continuous in $\left(0, \frac{\pi}{2}\right)$.

Solution:

Here $f(x) = [\tan x] + \sqrt{\tan x - [\tan x]}$ or $g(x) = [x] + \sqrt{x - [x]}$, Where x = tanx ³ 0.

Then for a ÎN we discuss continuity of f(x) as L.H.L at x=a,

$$\lim_{x\to a^-} g(x) = \lim_{h\to 0} [a - h] + \sqrt{a - h - [a - h]}$$

$$= \lim_{h \to 0} [a-1] + \sqrt{a-h-a+1}$$

$$= a-1+1$$

$$= a \qquad \begin{cases} \text{Where } a-1 < a-h < a \\ \therefore [a-h] = a-1 \end{cases}$$

Now R.H.L at x=a.

$$\lim_{x \to a^+} g(x) = \lim_{h \to 0} [a+h] + \sqrt{a+h-[a+h]}$$

$$= \lim_{h \to 0} a + \sqrt{a+h-a}$$

$$= a \qquad \begin{cases} \therefore a < a+h < a+1 \\ \Rightarrow [a+h] = a \end{cases}$$

$$\text{and } g(a) = [a] + \sqrt{a-[a]}$$

$$= a \quad \text{as } a \in N$$

So, g(x) is continuous for all a ÎN, Now g(x) is clearly continuous in (a-1, a) for all
a ÎN.
Hence g(x) is continuous in [0,∞].

Now let f(x) = tanx which is continuous in $\left(0, \frac{\pi}{2}\right)$.

So g {f(x)} is continuous in $\left(0, \frac{\pi}{2}\right)$.

Hence $f(x) = [\tan x] + \sqrt{\tan x - [\tan x]}$ is continuous in $\left(0, \frac{\pi}{2}\right)$.

Find the point of discontinuity of $y = \frac{1}{u^2 + u - 2}$ where $u = \frac{1}{x-1}$.

Solution:

The function $u = f(x) = \frac{1}{x-1}$ is discontinuous at the point x=1 -----

(1)

The function $y = g(x) = \frac{1}{u^2 + u - 2} = \frac{1}{(u+2)(u-1)}$ is discontinuous at u=-2 and u=1.

When u = -2, $\frac{1}{x-1} = -2 \Rightarrow x = \frac{1}{2}$

$u = 1 \Rightarrow \frac{1}{x-1} = 1 \Rightarrow x = 2$

Hence composite function y = g(x) is discontinuous at three points $x = \frac{1}{2}, 1, 2$.

6) If $f(x) = \lim_{n \to \infty} \frac{\log(x+2) - x^{2n}\sin x}{x^{2n} + 1}$ examine the continuity of f(x) at x=1.

Solution:

To examine the continuity at x=1, we are required to derive the definition of f(x) in the intervals x<1, x>1 and at x=1, i.e. on and around x=1.

Now if 0<x<1

$$f(x) = \lim_{n\to\infty} \frac{\log(x+2) - x^{2n}\text{Sin}x}{x^{2n}+1}$$

$$= \frac{\log(x+2) - 0\times\text{Sin}x}{0+1} = \log(x+2)$$

$$\text{if } x = 1\, f(x) = \lim_{n\to\infty} \frac{\log(x+2) - 1\times\text{Sin}x}{1+1}$$

$$= \frac{\log(x+2) - \text{Sin}x}{2}$$

$$\text{if } x > 1\, f(x) = \lim_{n\to\infty} \frac{\log(x+2) - x^{2n}\text{Sin}x}{x^{2n}+1}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{x^{2n}}\log(x+2) - \text{Sin}x}{1+\frac{1}{x^{2n}}} = -\text{Sin}x$$

$$\text{Thus we have } f(x) = \log(x+2), \quad 0 < x < 1$$

$$= \frac{1}{2}\{\log(x+2) - \text{Sin}x\},\ x = 1$$

$$-\text{Sin}x, \quad x > 1$$

$$\therefore f(1+0) = \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0} \{-\text{Sin}(1+h)\} = -\text{Sin}1$$

$$f(1-0) = \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0} \log(1-h+2) = \log 3$$

$$\therefore f(1+0) \neq f(1-0)$$

So f(x) is not continuous at x=1.

Dumb Question:

1) How does $\lim_{n\to\infty} \frac{x^{2n}}{x^{2n}+1}$ varies with the value of x?

Ans: Now suppose x<1

So, $\lim_{n\to\infty} x^{2n} = 0$

$$\therefore \lim_{n\to\infty} \frac{x^{2n}}{x^{2n}+1} = 0$$

If x =1 then $\lim_{n\to\infty} \frac{x^{2n}}{x^{2n}+1} = \frac{1}{1+1} = \frac{1}{2}$

If x>1 then $\lim_{n\to\infty} x^{2n} \to \infty$

$$\therefore \lim_{n\to\infty} \frac{x^{2n}}{x^{2n}+1} = \lim_{n\to\infty} \frac{1}{1+x^{-2n}}$$
$$= \frac{1}{1+0}$$
$$= 1$$

7) Show $\left(1+\sum_{k=1}^{n} \frac{2}{{}^{n}C_{k}}\right)^{n} \to e^{2}$ as $n\to\infty$(for $n \geq 6$)

Solution:

Let $a_n = 1+2\sum_{k=1}^{n} \frac{1}{{}^{n}C_{k}}$ while for $n \geq 6$.

$$a_n = 1 + 2\frac{1}{{}^nC_1} + 2\frac{1}{{}^nC_2} + \left(\frac{1}{{}^nC_3} + --- + \frac{1}{{}^nC_n}\right)$$

$$a_n \le 1 + \frac{2}{n} + \frac{4}{n(n-1)} + \frac{n-2}{{}^nC_3}$$

$$a_n \le 1 + \frac{2}{n} + \frac{4}{n(n-1)} + \frac{6(n-2)}{n(n-1)(n-2)}$$

$$a_n \le 1 + \frac{2}{n} + \frac{10}{n(n-1)} ----(1)$$

$$\text{also } a_n = 1 + 2\sum_{k=1}^{n}\frac{1}{{}^nC_k} \ge 1 + \frac{2}{{}^nC_1} ----(2)$$

$$\Rightarrow 1 + \frac{2}{n} \le a_n \le 1 + \frac{2}{n} + \frac{10}{n(n-1)}$$

$$\Rightarrow \left(1 + \frac{2}{n}\right)^n \le \mathbf{a}_n^n \le \left(1 + \frac{2}{n} + \frac{10}{n(n-1)}\right)^n$$

$$\Rightarrow \lim_{n\to\infty}\left(1 + \frac{2}{n}\right)^n \le \lim_{n\to\infty} \mathbf{a}_n^n \le \lim_{n\to\infty}\left[1 + \frac{2}{n}\left(1 + \frac{5}{n-1}\right)\right]^n$$

$$\Rightarrow e^2 \le \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{{}^nC_k}\right)^n \le e^2$$

$\therefore$ By squeeze principle for limits,

$$\lim_{n\to\infty}\left(1 + \sum_{k=1}^{n}\frac{2}{{}^nC_k}\right)^n = e^2 .$$

KEY WORDS

- Limit
- Extending / Approaching
- Right Hand Limit

- Left Hand Limit
- Infinity ∞
- Sandwitch Theorem / Squeeze play Theorem.
- Power Series
- L-Hospital's Rule
- Continuity
- Removable discontinuity
- Irremovable discontinuity

DIFFERENTIABILITY

INTRODUCTION

For many years the path in which planets were revolving round the sun was not known. After many years of observation Kepler conducted that planets move around the sun in elliptical orbits. But he could not give logical reasoning for his claim. But once Newton and Leibniz gave the fundamental theorem of calculus, the reasoning for this and many more things became quiet clear. Since then differential calculus has proved itself to be indispensable in development of mathematics and physics sciences.

In this section we will talk about some techniques of differentiation and their application to various question

So let us start with DIFFERENTIATION.

Definition of Derivative:

The derivative $f^1(x)$ of a function y = f(x) at a given point x is defined as

$$f^1(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Where $f^1(x)$ is called derivative of f(x) with respect to x.

Derivation of f(x) from first principle (i.e. Definition on ab-init):

Let y = f(x) for an increment dx in x, let dy be the corresponding increment in y.

y + dy = f (x+dx).

dy = f (x+dx)-f(x)

$$\text{So, } \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\lim_{\partial x \to 0} \frac{\delta y}{\delta x} = \lim_{\partial x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= f^1(x)$$

$f^1(x)$ if exists is called derivative of f(x) w.r.t x.

Note: $f^1(x), \frac{dy}{dx}, y^1, y_1$ all represents the same thing.

Illustration 1:

Find the derivative of $x^2 + x + 3$ w.r.t x using the first principle?

Solution:

Let $y = x^2 + x + 3$

$$\text{So, } \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\{(x+\Delta x)^2 + (x+\Delta x) + 3 - (x^2 + x + 3)\}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x.\ \Delta x + \Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + 1 + \Delta x$$

$$= 2x + 1$$

$$\therefore \frac{d}{dx}(x^2 + x + 3) = 2x + 1$$

Right and Left Derivative of a function:

The Right hand derivative of a function at point 'a' denoted by $f^1(a^+)$ is defined as

$$f^1(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Similarly left hand derivative of function at 'a' denoted by $f^1(a^-)$ is defined as

$$f^1(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

Differentiability of a function at a point:

A function f(x) is said to be differentiable at a point 'a' if

1) Both $f^1(a^+)$ and $f^1(a^-)$ are exist and are finite.

2) $f^1(a^+) = f^1(a^-)$.

Illustration 2:

Prove that function |x-1| is not differentiable at x=1.

Solution:

Let us find the right hand derivative and left hand derivative of this function.

$$f^1(1^+) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0^+} \frac{|h|}{h}$$
$$= \lim_{h\to 0^+} \frac{h}{h}$$
$$= 1$$

Similarly

$$f^1(1^-) = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0^-} \frac{|h|}{h}$$
$$= \lim_{h\to 0^-} \frac{-h}{h}$$
$$= -1$$

Now since $f^1(1^+) \neq f^1(1^-)$, so the function is not differentiable at x = 1.

Geometrical Interpretation of a Derivative:

Let us take two point P[c, f(c)] and Q[c+h, f(c+h)] on curve y = f(x).

Fig (1)

We join PQ so that it is a secant to curve.

Now $\tan\angle XRQ = \tan\angle NPQ$

$$= \frac{NQ}{PN}$$
$$= \frac{f(c+h)-f(c)}{h}$$

Now as h approaches O the point Q moving along curve approaches point P, the chord PQ approaches tangent line TP and ÐXRQ approaches ÐXTP denoted by Q.

So, $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \tan Q$

Hence $f^{1}(c)$ is slope of tangent to the curve y = f(x) at point (c, f(c)).

Illustration 3:

Show that tangent to hyperbola $y = \frac{1}{x}$ at (1, 1) makes an angle $\frac{3\pi}{4}$ with x-axis.

Solution:

Let the angle the tangent makes beq.

So, we know that

$$\tan\theta = f^{1}(1) \quad \text{where } f(x) = y = \frac{1}{x}$$

$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{1+h} - \frac{1}{1}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(1+h)}$$

$$= \lim_{h \to 0} \frac{-1}{1+h}$$

$$= -1$$

$$\therefore \tan\theta = -1$$

$$\text{Or } \theta = \frac{3\pi}{4}$$

Derivability of f(x) on an interval

f(x) is said to be derivable on the closed interval [a, b] if

1) For the points a and b, $f^{1}(a^{+})$, and $f^{1}(b^{-})$ finitely exist.

2) For any point c such that a<c<b, $f^1(c^+)$ and $f^1(c^-)$ finitely exist and equal.
3) f(x) is said to be derivable on open interval (a, b) for any point c such that a<c<b,

$f^1(c^+)$ And $f^1(c^-)$ finitely exists and are equal.

Illustration 4:
Consider the function;

$$g(x) = \begin{cases} x^4 - 4x^3 + 4x^2 - 9 & -1 \le x \le 0 \\ -9 & 0 < x \le 1 \\ x - 10 & x > 1 \end{cases}$$

Discuss differentiability of g(x) in [-1,¥].
Solution:

$$g^1(x) = \begin{cases} 4x^3 - 12x^2 + 8x & -1 \le x < 0 \\ 0 & 0 < x < 1 \\ +1 & 1 < x \end{cases}$$

Now clearly at $x = 1$

$g^1(1^+) = 1$ and $g^1(1^-) = 0$

So, g(x) is not differentiable at x=1.

Relationship between continuity and differentiability:
If a function f is differentiable at a point 'a' then it is continuous at that point but if a function f is continuous at a point 'a' then it need not necessarily be differentiable at that point.
For Example:
y = |x| + 1 is continuous at x=0 but not differentiable.
Dumb Question:
1) How do we know that |x|+1 is not differentiable at x=0?
Ans:

Fig (2)

The graph of function y = |x| + 1 looks like the figure mode above.

Now remember that derivative of a function represents the slope of graph at that point. So if there is no tangent line at certain point, function is not differentiable at that point or in other words function is not differentiable at corner point of a curve. And hence not differentiable at x=0.

Tips to check differentiability and continuity:

1) It is advantageous to check differentiability first because every differentiable function is continuous.
2) On the other hand every discontinuous function is non-differentiable.
3) If right hand derivative is not equal to left hand derivative but both exist finitely at x=a then function is not differentiable but continuous at x=a.
4) Continuity of a function does not imply its differentiability.
5) Also, non-differentiability of a function does not mean that it is discontinuous at point a.

Illustration 5:

Function f is defined by;

$$f(x) = \begin{cases} \dfrac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, x \neq 0 \\ 0, x = 0 \end{cases}$$

Show that f is continuous but not derivable at x=0.

Solution:

$$f^1(0^+) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{\frac{he^{\frac{1}{h}}}{1+e^{\frac{1}{h}}} - 0}{h}$$

$$= \lim_{h\to 0^+} \frac{e^{\frac{1}{h}}}{1+e^{\frac{1}{h}}}$$

$$= \lim_{h\to 0^+} \frac{1}{e^{\frac{-1}{h}}+1}$$

$$= 1$$

$$f^1(0^-) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{\frac{he^{\frac{1}{h}}}{1+e^{\frac{1}{h}}} - 0}{h}$$

$$= \lim_{h\to 0^-} \frac{1}{e^{\frac{-1}{h}}+1}$$

$$= 0$$

$f^1(0^-) \neq f^1(0^+)$ So, f(x) is not derivable at x=0.

But since both $f^1(0^-)$ and $f^1(0^+)$ exist finitely the function is continuous at x=0.

Dumb Question:

1) Why $\lim_{h\to 0^-} \frac{1}{e^{\frac{-1}{h}}+1}$ is 0 and $\lim_{h\to 0^+} \frac{1}{e^{\frac{-1}{h}}+1}$ is 1?

Ans:

Since $h \to 0^+$ So, $\frac{1}{h} \to \infty$

and Thus $e^{\frac{-1}{h}} \to 0$

So, $\lim_{h \to 0^+} \frac{1}{e^{\frac{-1}{h}} + 1} = 1$

But when $h \to 0^-$ So, $\frac{1}{h} \to -\infty$ and thus

$e^{\frac{-1}{h}} \to \infty$

So, $\lim_{h \to 0^-} \frac{1}{e^{\frac{-1}{h}} + 1} = 0$

General Theorem on Differentiation:

1) $\frac{d}{dx}$ (Constant) = 0 why?

Proof: Let f(x) = k.

Then $f^1(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{k - k}{\Delta x} = 0$

Therefore $\frac{d}{dx}$ (Constant) = 0.

2) $\frac{d}{dx}$ [c f(x)] = c $\frac{d}{dx}$ (f(x)); where c is a constant why?

Let g(x) = c.f(x)

Where c is constant and f(x), xÎR be a differential function of x. We then have,

$g^1(x) = \lim_{\Delta x \to 0} \frac{c(f(x) + \Delta f(x)) - c.f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} c.\frac{\Delta f(x)}{\Delta x}$

$= c \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$

Thus $\frac{d}{dx}$ [c f(x)] = c $\frac{d}{dx}$ (f(x)).

3) $\frac{d}{dx}[f_1(x) \pm f_2(x)] = \frac{d}{dx}(f_1(x)) \pm \frac{d}{dx}(f_2(x))$ Why?

Proof:

Let $y = f_1(x) + f_2(x)$, Where $f_1(x)$ and $f_2(x)$ are differentiable functions of x.

Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{[(f_1(x) + \Delta f_1(x)) \pm (f_2(x) + \Delta f_2(x))] - [f_1(x) \pm f_2(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left[\frac{\Delta f_1(x)}{\Delta x} \pm \frac{\Delta f_2(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \frac{\Delta f_1(x)}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{\Delta f_2(x)}{\Delta x}$$

$$= \frac{df_1(x)}{dx} \pm \frac{df_2(x)}{dx}$$

Thus $\frac{d}{dx}[f_1(x) \pm f_2(x)] = \frac{d}{dx}(f_1(x)) \pm \frac{d}{dx}(f_2(x))$.

4) $\frac{d}{dx}[f_1(x).f_2(x)] = f_1(x)\frac{d}{dx}(f_2(x)) + f_2(x)\frac{d}{dx}(f_1(x))$ Why?

Proof:

Let $y = f_1(x).f_2(x)$ where both $f_1(x)$ and $f_2(x)$ are differentiable function of x.

Then, $y + \Delta y = [f_1(x) + \Delta f_1(x)][f_2(x) + \Delta f_2(x)]$

Therefore $\Delta y = [f_1(x) + \Delta f_1(x)][f_2(x) + \Delta f_2(x)] - [f_1(x).f_2(x)]$

$$= f_1(x).\Delta f_2(x) + f_2(x).\Delta f_1(x) + \Delta f_1(x).\Delta f_2(x)$$

Thus, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left[\frac{f_1(x).\Delta f_2(x) + f_2(x).\Delta f_1(x) + \Delta f_1(x).\Delta f_2(x)}{\Delta x}\right]$

$$= \lim_{\Delta x \to 0}\left[f_1(x)\frac{\Delta f_2(x)}{\Delta x} + f_2(x)\frac{\Delta f_1(x)}{\Delta x} + \frac{\Delta f_1(x)}{\Delta x}.\Delta f_2(x)\right]$$

$$= \lim_{\Delta x \to 0} f_1(x) \frac{\Delta f_2(x)}{\Delta x} + \lim_{\Delta x \to 0} f_2(x) \frac{\Delta f_1(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta f_1(x)}{\Delta x} . \Delta f_2(x)$$

$$= f_1(x) \lim_{\Delta x \to 0} \frac{\Delta f_2(x)}{\Delta x} + f_2(x) \lim_{\Delta x \to 0} \frac{\Delta f_1(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta f_1(x)}{\Delta x} . \lim_{\Delta x \to 0} \Delta f_2(x)$$

$$= f_1(x) \frac{df_2(x)}{dx} + f_2(x) \frac{df_1(x)}{dx} + \frac{df_1(x)}{dx} . 0$$

$$\frac{d}{dx}\left[f_1(x).f_2(x)\right] = f_1(x) \frac{d}{dx}(f_2(x)) + f_2(x) \frac{d}{dx}(f_1(x))$$

5) $$\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right] = \frac{f_2(x) \frac{d}{dx}(f_1(x)) - f_1(x) \frac{d}{dx}(f_2(x))}{[f_2(x)]^2}, f_2(x) \neq 0$$ Why?

Proof:

Let $y = \frac{f_1(x)}{f_2(x)}$, where $f_1(x)$ and $f_2(x)$ are differentiable functions at all points in its domain and $f_2(x)$ ¹ 0.

$$\text{Therefore, } y + \Delta y = \frac{f_1(x) + \Delta f_1(x)}{f_2(x) + \Delta f_2(x)}$$

$$\text{Or } \Delta y = \frac{f_1(x) + \Delta f_1(x)}{f_2(x) + \Delta f_2(x)} - \frac{f_1(x)}{f_2(x)}$$

$$= \frac{f_2(x)\, \Delta f_1(x) - f_1(x)\, \Delta f_2(x)}{f_2(x)(f_2(x) + \Delta f_2(x))}$$

$$\text{Therefore, } \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{f_2(x) \lim_{\Delta x \to 0} \frac{\Delta f_1(x)}{\Delta x} - f_1(x) \lim_{\Delta x \to 0} \frac{\Delta f_2(x)}{\Delta x}}{f_2(x) \lim_{\Delta x \to 0} (f_2(x) + \Delta f_2(x))}$$

$$\text{i.e. } \frac{dy}{dx} = \frac{f_2(x) \frac{df_1(x)}{dx} - f_1(x) \frac{df_2(x)}{dx}}{f_2{}^2(x)}$$

$$\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right] = \frac{f_2(x) \frac{d}{dx}(f_1(x)) - f_1(x) \frac{d}{dx}(f_2(x))}{[f_2(x)]^2}$$

6) $$\frac{d}{dx}(f(g(x))) = \frac{d(f(g(x)))}{d(g(x))} \times \frac{d(g(x))}{dx} \text{ (chain)}$$ Why?

Proof:
Let y = f (t), t = g(x)
Then y = f (g(x)) is a function of x.
Dy = f (t+Dt) - f (t)
And Dt = g (x+Dx) - g(x)

Therefore, $\frac{\Delta y}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$

and $\frac{\Delta t}{\Delta x} = \frac{g(x+\Delta x) - g(x)}{\Delta x}$

Assuming that for sufficient small values of Dx, Dt ¹ 0 this will necessarily be the case if $\frac{dt}{dx} \neq 0$ then since g is differentiable it is continuous and hence when Dx®0, x+Dx®x any g(x+Dx) ® g(x). Therefore t+Dt®t and Dt®0, since Dt ¹ 0 and Dx®0 we may write,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t} \times \frac{\Delta t}{\Delta x}.$$

Clearly f and g are both continuous functions because they are differentiable thus Dt®0 when Dx®0 and Dy®0 when Dt®0.

Therefore, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \times \lim_{\Delta x \to 0} \frac{\Delta t}{\Delta x}$

Hence $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Standard Formulae of Differentiation:

1) $\frac{d}{dx}(x^n) = nx^{n-1}$

Proof:

Let $y = x^n$

Therefore, $y + \Delta y = (x + \Delta x)^n$

Or $\Delta y = (x + \Delta x)^n - x^n$

$$\text{Or } \frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x}$$

Therefore, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x} \right]$

$$\text{Or } \frac{dy}{dx} = nx^{n-1} \quad [\text{because } \lim_{x \to a} \frac{(x^n - a^n)}{(x-a)} = na^{n-1}$$

Thus $\frac{d}{dx}(x^n) = nx^{n-1}$

2) $\frac{d}{dx}(e^x) = e^x$

Proof:

If $f(x) = e^x$ then,

$$f^1(x) = \lim_{x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x . 1 \quad \left[\text{Since, } \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \right]$$

Thus $\frac{d}{dx}(e^x) = e^x$

3) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

Proof:

Let $f(x) = \log_e x \quad (x > 0)$ then,

$$f^1(x) = \lim_{\Delta x \to 0} \frac{\log_e (x+\Delta x) - \log_e x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\log_e \left(1 + \frac{\Delta x}{x}\right)}{\Delta x}$$

$$= \frac{1}{x}.1 \qquad \left[\text{Since } \lim_{h\to 0}\frac{\log_e(1+h)}{h} = 1\right]$$

So, $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

4) $\frac{d}{dx}(a^x) = a^x \ln a.$

Proof:

Let $f(x) = a^x$, then

$$f^1(x) = \lim_{\Delta x\to 0}\frac{a^{x+\Delta x} - a^x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} a^x\left(\frac{a^{\Delta x} - 1}{\Delta x}\right)$$

$$= \lim_{\Delta x\to 0} a^x\left(\frac{e^{\log_e a^{\Delta x}} - 1}{\log_e a^{\Delta x}}\right)\log_e a$$

$$= a^x.1.\log_e a \qquad (\because \text{ as } \Delta x \to 0, \log_e a^{\Delta x} \to 0)$$

$$= a^x \log_e a$$

5) $\frac{d}{dx}(\sin x) = \cos x$

Proof:

Let f(x) = sinx then,

$$f^1(x) = \lim_{\Delta x\to 0}\frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} 2\cos\left(x + \frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)$$

$$= \lim_{\Delta x\to 0}\cos\left(x + \frac{\Delta x}{2}\right)\lim_{\Delta x\to 0}\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

$$= \cos x.1 \qquad \left[\text{Since } \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

Hence, $\frac{d}{dx}(\sin x) = \cos x$

6) $\frac{d}{dx}(\cos x) = -\sin x$

7) $\frac{d}{dx}(\tan x) = \sec^2 x$

8) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x.\cot x$

9) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

10) $\frac{d}{dx}(\sec x) = \sec x.\tan x$ Why?

Proof:

Let f(x) = secx

$$f^1(x) = \lim_{\Delta x \to 0} \frac{\sec(x+\Delta x) - \sec x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{1}{\cos(x+\Delta x)} - \frac{1}{\cos x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\cos x - \cos(x+\Delta x)}{\Delta x.\cos(x+\Delta x)\cos x}\right]$$

$$= \lim_{\Delta x \to 0} \frac{2\sin\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x.\cos(x+\Delta x)\cos x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin\left(x + \frac{\Delta x}{2}\right)}{\cos(x+\Delta x)\cos x} \times \lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$= \frac{\sin x}{\cos^2 x}.1 \qquad \left[\text{Since } \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1\right]$$

$$= \tan x.\sec x$$

Hence, $\frac{d}{dx}(\sec x) = \sec x.\tan x$

11) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

Proof:

Let $y = \sin^{-1}x$ then x = siny.

Differentiating w.r.t x we get,

$$\cos y.\frac{dy}{dx} = 1$$

Therefore, $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

Hence, $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

12) $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

13) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

14) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

15) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

16) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$ Why?

Proof:

Let $y = \sec^{-1}x$ then secy = x.

Differentiating w.r.t x we have,

$$\sec y.\tan y.\frac{dy}{dx} = 1$$

$$\text{Or } \frac{dy}{dx} = \frac{1}{\sec y.\tan y}$$

$$= \frac{1}{\sec y\sqrt{\sec^2 y - 1}}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$

Hence, $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$

Dumb Question:

Why $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$ and not $\frac{1}{x\sqrt{x^2-1}}$?

Ans:

Fig (4)

Note that $y = \sec^{-1}x$ is an increasing function in its domain.

So, $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$

Illustration 6:

Differentiate $y = 2^{\ln \sin e^{\tan^{-1}\sqrt{x^2-1}}}$

Solution:

$$y = 2^{\ln \sin e^{\tan^{-1}\sqrt{x^2-1}}}$$

$$\frac{dy}{dx} = 2^{\ln \sin e^{\tan^{-1}\sqrt{x^2-1}}} \ln 2 \frac{1}{\sin(e^{\tan^{-1}\sqrt{x^2-1}})} \times \cos\left(e^{\tan^{-1}\sqrt{x^2-1}}\right) \times$$

$$e^{\tan^{-1}\sqrt{x^2-1}} \frac{1}{1+(x^2-1)} \cdot \frac{1}{2\sqrt{x^2-1}} \cdot 2x$$

$$= 2^{\ln \sin e^{\tan^{-1}\sqrt{x^2-1}}} \ln 2 \frac{1}{\sin(e^{\tan^{-1}\sqrt{x^2-1}})} \times \cos\left(e^{\tan^{-1}\sqrt{x^2-1}}\right) \times e^{\tan^{-1}\sqrt{x^2-1}} \frac{1}{x\sqrt{x^2-1}}$$

Different Methods of Differentiation:

1) Differentiation of a function defined parametrically:

Let x, y be function of parameter t, i.e.

x = f (t), y = f (t) then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\varphi^1(t)}{f^1(t)}$$

Illustration:

If $x^{2/3} + y^{2/3} = a^{2/3}$ then find $\frac{dy}{dx}$?

Solution:

$$\text{Put } x = a\cos^3\theta,\ y = a\sin^3\theta$$

$$\text{Then } \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$\text{And } \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\text{Therefore } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta din\theta}$$

$$= -\tan\theta = -\sqrt[3]{y/x}$$

2) Logarithmic Differentiation:

The process of taking logarithm on both sides and then differentiating is called logarithmic differentiation.

Illustration 8:

Differentiate $x^{\sin x}$ w.r.t x.

Solution:

Let $y = x^{\sin x}$.

Taking logarithm on both sides, then we have

logy = sinx.logx

$$\text{Therefore, } \frac{1}{y}\frac{dy}{dx} = \sin x\ \frac{d}{dx}(\log x) + \log x\ \frac{d}{dx}(\sin x)$$

$$\text{or } \frac{1}{y}\frac{dy}{dx} = \sin x\ \frac{1}{x} + \log x.\cos x$$

$$\text{Or } \frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x.\log x\right]$$

$$= x^{\sin x - 1}.\sin x + x^{\sin x}.\cos x.\log x\ .$$

3) Differentiation of Implicit function:
If the relation between x and y is given by equation containing both and this equation is not immediately solvable for y, then y is called implicit function of x.
Illustration 9:

If $e^x + e^y = e^{x+y}$ Then prove that $\frac{dy}{dx} = -\frac{e^x(e^y - 1)}{e^y(e^x - 1)}$
Solution:

Given that $e^x + e^y = e^{x+y}$ differentiating on both sides w.r.t x we have,

$$e^x + e^y \frac{dy}{dx} = e^{(x+y)}\left(1 + \frac{dy}{dx}\right)$$

or $\left(e^y - e^{(x+y)}\right)\frac{dy}{dx} = e^{(x+y)} - e^x$

Hence, $\frac{dy}{dx} = -\frac{e^x(e^y - 1)}{e^y(e^x - 1)}$

4) Differentiation of a function w.r.t other functions:

$$\frac{d\,f(x)}{d\varphi(x)} = \frac{\frac{d}{dx}f(x)}{\frac{d}{dx}\varphi(x)} = \frac{f^1(x)}{\varphi^1(x)}$$

Illustration:

1) Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\tan^{-1}x$.
Solution:

$$\frac{\frac{d}{dx}\sin^{-1}\left(\frac{2x}{1+x^2}\right)}{\frac{d}{dx}\tan^{-1}(x)} = \frac{\frac{d}{dx}(2\tan^{-1}x)}{\frac{d}{dx}\tan^{-1}(x)}$$

$$= \frac{2\frac{d}{dx}\tan^{-1}(x)}{\frac{d}{dx}\tan^{-1}(x)}$$

$$= 2$$

2) Differentiate ln tanx with respect to $\sin^{-1}(e^x)$
Solution:

$$\frac{d(\ln \tan x)}{d(\sin^{-1}(e^x))} = \frac{\frac{d}{dx}(\ln \tan x)}{\frac{d}{dx}\sin^{-1}(e^x)}$$

$$= \frac{\cot x.\sec^2 x}{e^x \frac{1}{\sqrt{1-e^{2x}}}}$$

$$= \frac{e^{-x}\sqrt{1-e^{2x}}}{\sin x.\cos x}$$

5) Higher Order derivatives:
Let y = f(x) be differentiable function such that z = $f^1(x)$ is also differentiable. Then second derivative of y = f(x) is denoted by

$\frac{d^2y}{dx^2}$ or $f^{11}(x)$ or y^{11} and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$.

In general $\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right), n \in N$

Illustration 11:

For the curve $y = (2x+1)^3$ find the rate of change of slope at (4, 27).
Solution:

We know that slope of a curve is given by $\frac{dy}{dx}$. Now let g(x) denote slope of curve.

So, $g(x) = \frac{dy}{dx} = 3 \times 2(2x+1)^2$

Now rate of change of slope = $g^1(x)$

$$g^1(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(6(2x+1)^2)$$

$$= 24(2x+1)$$

So rate of change of slope at (4, 27)
= 24(2´4+1)
= 216.

DIFFERENTIATION (EASY TYPE)

1) Differentiate $\frac{\sin x + \cos x}{\sin x - \cos x}$ with respect to x.

Solution:

Let y = $\frac{\sin x + \cos x}{\sin x - \cos x}$ then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)$$

$$= \frac{(\sin x - \cos x)\frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x)\frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-(\sin^2 x + \cos^2 x - 2\sin x.\cos x) - (\sin^2 x + \cos^2 x + 2\sin x.\cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

2) Let f be twice differentiable such that $f^{11}(x) = -f(x)$ and $f^{1}(x) = g(x)$. If $h(x) = \{f(x)\}^2 + \{g(x)\}^2$ Where h (5) = 11, find h (10).

Solution:

Given $h(x) = \{f(x)\}^2 + \{g(x)\}^2$ on differentiating both sides w.r.t x we get

$h^1(x) = 2f(x).f^1(x) + 2.g(x).g^1(x) -----(1)$

Now $f^1(x) = g(x)$ then,

$f^{11}(x) = g^1(x) \Rightarrow -f(x) = g^1(x) -----(2)$ $\{\because f^{11}(x) = -f(x), \text{ given}\}$

From (1) and (2) we get,

$h^1(x) = 2f(x).g(x) + 2.g(x).\{-f(x)\}$ $\{\text{using } f^1(x) = g(x) \text{ and } g^1(x) = -f(x)\}$

Therefore $h^1(x) = 0$.

So, h(x) must be constant [as d/dx constant = 0]

But h (5) = 11 so h (x) = 11

Hence, h (10) = 11.

3) Find the sum of series $\sum_{r=1}^{n} r.x^{r-1}$ using calculus.

Solution:

Let $S = 1 + x + x^2 + x^3 + ------ + x^n$ which is a geometric progression.

Therefore $S = 1 + x + x^2 + x^3 + ------ + x^n = \dfrac{1(1-x^{n+1})}{(1-x)}$

On differentiating both sides, we get,

$$0 + 1 + 2x + 3x^2 + 4x^3 + ---- + nx^{n-1}$$

$$= \frac{(1-x)[-(n+1)x^n] - (1-x^{n+1})(-1)}{(1-x)^2}$$

$$\sum_{r=1}^{n} r.x^{r-1} = \frac{1}{(1-x)^2} \quad \left\{1 - (n+1)x^n + nx^{n+1}\right\}$$

4) If $\mu = f(x^2),\ v = g(x^3), f^1(x) = \sin x$ and $g^1(x) = \cos x$ then find $\dfrac{du}{dv}$.

Solution:

Differentiating $\mu = f(x^2)$ and $v = g(x^3)$ w.r.t x, we get,

$$\frac{du}{dx} = f^1(x^2).2x = \sin(x^2).2x$$

$$\left\{\because f^1(x) = \sin x \Rightarrow f^1(x^2) = \sin(x^2)\right\}$$

$$\frac{dv}{dx} = g^1(x^3).3x^2 = \cos(x^3).3x^2$$

$$\left\{\because g^1(x) = \cos x \Rightarrow g^1(x^3) = \cos(x^3)\right\}$$

$$\therefore \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\sin(x^2).2x}{\cos(x^3).3x^2} = \frac{2}{3x} \times \frac{\sin x^2}{\cos x^3}.$$

5) If $e^x = \dfrac{\sqrt{1+t} - \sqrt{1-t}}{\sqrt{1+t} + \sqrt{1-t}}$ and $\tan\dfrac{y}{z} = \sqrt{\dfrac{1-t}{1+t}}$ then find $\dfrac{dy}{dx}$ at $t = \dfrac{1}{2}$.

Solution:

Putting t = cosq.

$$\text{At } t = \frac{1}{2}, \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$e^x = \frac{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{1 - \tan\left(\frac{\theta}{2}\right)}{1 + \tan\left(\frac{\theta}{2}\right)} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$\tan\left(\frac{y}{z}\right) = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \tan\left(\frac{\theta}{2}\right)$$

$$\therefore\ y = \theta$$

$$\therefore\ x = \log\ \tan\left(\frac{\pi}{4} - \frac{y}{z}\right)$$

$$\frac{dx}{dy} = \frac{1}{\tan\left(\frac{\pi}{4} - \frac{y}{z}\right)}.\sec^2\left(\frac{\pi}{4} - \frac{y}{z}\right) \times \left(-\frac{1}{2}\right)$$

$$= -\frac{1}{2\sin\left(\frac{\pi}{4} - \frac{y}{z}\right).\cos\left(\frac{\pi}{4} - \frac{y}{z}\right)}$$

$$= -\frac{1}{\sin\left(\frac{\pi}{2} - y\right)}$$

$$= -\frac{1}{\cos y}$$

$$\frac{dx}{dy} = -\frac{1}{\cos\theta} = -\frac{1}{\cos\left(\frac{\pi}{3}\right)} = -2$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

6) If $y^2 = P(x)$ a polynomial of degree 3 then find $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ in terms of P(x) and its derivative.

Solution:

We have $P^1(x) = 2yy^1;\ P^{11}(x) = 2yy^{11} + 2y^{12}$

And $P^{111}(x) = 2yy^{111} + 6y^1y^{11} ----(1)$

Also

$$2\frac{d}{dx}(y^3y^{11}) = 2\left[y^3y^{111} + 3y^2y^1y^{11}\right]$$

$$= y^2\left[2yy^{111} + 6y^1y^{11}\right]$$

$$= P(x)P^{111}(x), \quad by(1)$$

7) If $x^y y^x = 1$ find $\frac{dy}{dx}$

Solution:

Taking log on both sides

ylogx + xlogy = log1.

Differentiating on both sides we get,

$$y\frac{d}{dx}(\log x) + \left\{\frac{d}{dx}y\right\}\log x + \left\{\frac{d}{dx}x\right\}\log y + x\left\{\frac{d}{dx}\log y\right\} = 0$$

$$\text{Or } y.\frac{1}{x} + \log x.\frac{dy}{dx} + 1.\log y + x.\frac{1}{y}\frac{dy}{dx} = 0$$

$$\text{Or } \left[\log x + \frac{x}{y}\right]\frac{dy}{dx} = -\left[\frac{y}{x} + \log y\right]$$

$$= \frac{dy}{dx} = -\frac{(y + x\log y)}{(x + y\log x)}.\frac{y}{x}$$

8) If x = a (t+sint) and y = a (1-cost), Then find $\frac{d^2y}{dx^2}$

Solution:

Here x = a (t+sint) and y = a (1-cost).

Differentiating both sides w.r.t t we get,

$$\frac{dx}{dt} = a(1+\cos t) \text{ and } \frac{dy}{dt} = a(\sin t)$$

$$\therefore \frac{dy}{dx} = \frac{a\sin t}{a(1+\cos t)}$$

$$= \frac{2\sin\frac{t}{2}.\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2}$$

Again differentiating both sides we get,

$$\frac{d^2y}{dx^2} = \sec^2\frac{t}{2} \times \frac{1}{2}\frac{dt}{dx}$$

$$= \frac{1}{2}\sec^2\left(\frac{t}{2}\right).\frac{1}{a(1+\cos t)}$$

$$= \frac{1}{2a} \times \frac{\sec^2\left(\frac{t}{2}\right)}{2\left(\cos^2\frac{t}{2}\right)}$$

Hence $\frac{d^2y}{dx^2} = \frac{1}{4a}.\sec^4\left(\frac{t}{2}\right)$

9) Find $\frac{dy}{dx}$ if $y = (\sin x)^{\sin x^{\sin x \cdots\cdots \infty}}$

Solution:

We have $y = (\sin x)^y$

Therefore by taking log on both sides we get,

logy = y log sinx.

Differentiating both sides w.r.t x we get,

$$\frac{1}{y}\frac{dy}{dx} = y.\frac{d}{dx}(\log \sin x) + (\log \sin x)\frac{dy}{dx}$$

$$= y\frac{\cos x}{\sin x} + \log \sin x \frac{dy}{dx}$$

$$\text{Or}\left(\frac{1}{y} - \log \sin x\right)\frac{dy}{dx} = y\cot x$$

$$\text{Or } \frac{dy}{dx} = \frac{y^2\cot x}{1 - y\log \sin x} = \frac{y^2\cot x}{1 - \log y}$$

10) Is f(x) differentiable at x=0 if f(x) is defined as follow

$$f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0? \end{cases}$$

Solution:

$$f^1(0+0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{\dfrac{0+h}{1+e^{\frac{1}{0+h}}} - 0}{h}$$

$$= \lim_{h\to 0}\frac{1}{e^{1/h}+1} = \frac{1}{1+e^{\infty}} = 0$$

$$f^1(0-0) = \lim_{h\to 0}\frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0}\frac{\dfrac{0-h}{1+e^{\frac{1}{0-h}}} - 0}{-h}$$

$$= \lim_{h\to 0}\frac{1}{1+e^{-1/h}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$$\therefore f^1(0+0) \neq f^1(0-0)$$

Here f(x) is not differentiable at x=0.

11) Determine the values of x for which the following function fails to be continuous or differentiable.

$$f(x) = \begin{cases} (1-x), & x < 1 \\ (1-x)(2-x), & 1 \leq x \leq 2 \\ (3-x), & x > 2 \end{cases}$$

Justify your Answer.

Solution:

By given definition it is clear that the function f is continuous and differentiable at all points except possibility at x=1 and x=2.

Continuity at x=1,

L.H.L = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(1-x) = \lim_{h\to 0}(1-(1-h)) = 0$

R.H.L = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(1-x)(2-x) = \lim_{h\to 0}\{1-(1+h)\}\{2-(1+h)\} = 0$

Also f (1) = 0, Hence f(x) is continuous at x=1.

Now differentiability at x=1.

$$Lf^{1}(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{h} = \lim_{h\to 0}\frac{1-(1-h)-0}{-h} = -1$$

$$Rf^{1}(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\{1-(1+h)\}\{2-(1+h)\}-0}{h} = -1$$

Since $L\{f^{1}(1)\} = R\{f^{1}(1)\}$ so we get f(x) is differentiable at x=1.

Continuity at x=2.

$$L.H.L = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(1-x)(2-x) = \lim_{h\to 0}(1-(2-h))(2-(2-h)) = 0$$

$$R.H.L = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3-x) = \lim_{h\to 0} 3-(2+h) = 1$$

Since R.H.L ¹ L.H.L

Therefore f(x) is not continuous at x=2. As such f(x) can not be differentiable at x=2. Hence f(x) is continuous and differentiable at all points except at x=2.

12) If

$$f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$$ Then find a) Constant term b) Coefficient of x.

Solution:

Here

$$f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix} = A + Bx + Cx^2 + ---(1)$$

Putting x =0 we get

$$f(0) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = A + B(0) + C(0)^2 + ----$$

Þ A=0

Again Differentiating (1) w.r.t x we get,

$$f^{1}(x) = \begin{vmatrix} a(1+x)^{a-1} & 2b(1+2x)^{b-1} & 0 \\ 1 & (1+x)^{a} & (1+2x)^{b} \\ (1+2x)^{b} & 1 & (1+x)^{a} \end{vmatrix} + \begin{vmatrix} (1+x)^{a} & (1+2x)^{b} & 1 \\ 0 & a(1+x)^{a-1} & 2b(1+2x)^{b-1} \\ (1+2x)^{b} & 1 & (1+x)^{a} \end{vmatrix}$$

$$+ \begin{vmatrix} (1+x)^{a} & (1+2x)^{b} & 1 \\ 1 & (1+x)^{a} & (1+2x)^{b} \\ 2b(1+2x)^{b-1} & 0 & a(1+x)^{a-1} \end{vmatrix} = B + 2C_x + ----$$

$$f^{1}(0) = \begin{vmatrix} a & 2b & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 2b \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2b & 0 & a \end{vmatrix} = B$$

Therefore Coefficient of constant term = Coefficient of x = 0.

(MEDIUM TYPE)

1) Let f(x) be a real function not identically zero such that $f(x + y^{2n+1}) = f(x) + \{f(y)\}^{2n+1}$, nÎN and x, y are any real numbers and $f^{l}(0)$ ³ 0. Find the values of f (5) and $f^{l}(10)$.

Solution:

Here $f(x + y^{2n+1}) = f(x) + \{f(y)\}^{2n+1}$ ------ (1)

Putting x=0, y=0 we get f (0) = f (0) + $\{f(0)\}^{2n+1}$.

\ f (0) = 0.

$$f^{1}(0) \geq 0 \Rightarrow \lim_{h \to 0} \frac{f(x) - f(0)}{x} \geq 0$$

$$\Rightarrow \lim_{x \to 0} \frac{f(x)}{x} \geq 0$$

\ If x ³ 0, f(x) ³ 0 -------- (2)

Putting x = 0, y = 1 in (1)

$f(1) = f(0) + \{f(1)\}^{2n+1}$

Or $f(1)\left[1 - \{f(1)\}^{2n}\right] = 0$

$\therefore f(1) = 0 \text{ or } 1 \quad \{\text{from } (2)\}$

Putting y=1 in (1) for all real x,

f (x+1) = f(x) + $\{f(1)\}^{2n+1}$------ (3)

\f (1) = 0 Þ f(x+1) = f(x)

Þ f (1) = f (2) = f (3) = ------ = 0

i.e. f(x) is identically zero.

\ f (1) ¹ 0 Hence f (1) =1.

So from (3) f(x+1) = f (x) +1 ---- (4)
\ f (5) = f (4) +1 = {f (3) + 1}+ 1
= {f (2) +1} +2
= {f (1) +1} +3
=f (1) +4
=1+4
=5
Also (4)Þ f(x) is a function whose value increases by 1.When variable x is increased by 1.
\f (x) = x \f^1(x) = 1
\ f^1(10) =1.

Dumb Question:

1) Why does the equation $f(1)\left[1-(f(1))^{2n}\right]=0$ has the solution as f (1) = 0, 1? Some complex solution could also be possible.
Ans: It is mentioned in the question that function f is real valued function, so the value of f (1) has to be real and cannot be complex.
2) If f be a function such that f (xy) = f (x). f (y), " yÎR and f (1+x) = 1+x (1+ g(x)). Where $\lim_{x\to 0} g(x) = 0$, find the value of $\int_1^2 \frac{f(x)}{f^1(x)} \times \frac{1}{1+x^2}\,dx.$
Solution:
We know,

$$f^1(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x)}{h}$$

$$f^1(x) = \lim_{h\to 0}\frac{f(x).f\left(1+\frac{h}{x}\right)-f(x)}{h} \qquad \{\text{given } f(xy) = f(x).f(y)\}$$

$$f^1(x) = \lim_{h\to 0}\frac{f(x)\left\{1+\frac{h}{x}\left(1+g\frac{h}{x}\right)\right\}-f(x)}{h} \qquad \left[\text{given } f(1+x) = 1+x(1+g(x))\right]$$

$$f^1(x) = \lim_{h\to 0}\frac{f(x)\left\{1+\frac{h}{x}\left(1+g\frac{h}{x}\right)-1\right\}}{h}$$

$$f^1(x) = \frac{f(x)}{x} ----(1) \qquad \left[\text{as } \lim_{x\to 0} g(x) = 0\right]$$

To find $\int_1^2 \frac{f(x)}{f^1(x)} \times \frac{1}{1+x^2} dx.$

$$\Rightarrow \int_1^2 \frac{x}{1+x^2} dx$$

$$= \frac{1}{2}\left[\log |1+x^2|\right]_1^2$$

$$= \frac{1}{2}\log[5/2]$$

) Let f be differentiable function such that,

$f^1(x) = f(x) + \int_0^2 f(x)dx$, $f(0) = \frac{4 - e^2}{3}$ find f(x)?

Solution:

Given that

$$f^1(x) = f(x) + \int_0^2 f(x)dx$$

$$\Rightarrow f^1(x) = f(x) + c \quad \left[c = \int_0^2 f(x)dx \quad (\text{say})\right]$$

$$\Rightarrow \frac{f^1(x)}{f(x) + c} = 1$$

Integrating above expression both sides,

$$\Rightarrow \log_e\{f(x) + c\} = x$$

$$\Rightarrow f(x) + c = ke^x ----(1)$$

Where k is a constant of integration

Since $f(0) = \frac{4 - e^2}{3} \Rightarrow f(0) + c = k$

$$\Rightarrow k = \frac{4 - e^2}{3} + c \Rightarrow c = k - \left(\frac{4 - e^2}{3}\right) --- (2)$$

From (1) and (2) we get,

$$f(x) + k - \left(\frac{4 - e^2}{3}\right) = k(e^x)$$

$$\Rightarrow f(x) = k(e^x - 1) + \left(\frac{4 - e^2}{3}\right) --- (3)$$

Now to find constant of integration k.
Integrating both sides from 0 to 2 we get,

$$\int_0^2 f(x)dx = k\int_0^2 (e^x - 1)dx + \frac{4-e^2}{3}\int_0^2 dx$$

$$\Rightarrow c = k(e^2 - 2 - 1) + \frac{4-e^2}{3}(2)$$

$$\Rightarrow c = \frac{2}{3}(4-e^2) + k(e^2 - 3) --- (4)$$

From (2) and (4) we get,

$$k - \left(\frac{4-e^2}{3}\right) = \frac{2}{3}(4-e^2) + k(e^2-3) \Rightarrow k = 1$$

Putting k=1 in (3) we get,

$$f(x) = (e^x - 1) + \left(\frac{4-e^2}{3}\right)$$

$$\text{Hence } f(x) = e^x - \frac{(e^2-1)}{3}$$

4) Let f(x+y) = f(x) –f (y) + 2xy -1 for all x, y ÎR, if f(x) is differentiable and $f^1(0) = \sqrt{-3+a-a^2}$ prove that f(x)>0 for all xÎR.
Solution:
Putting x = 0 = y in given functional equation, we get f (0) = -1.

$$f^1(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x) - f(h) + 2xh - 1 - f(x)}{h}$$

$$= 2x - \lim_{h\to 0}\frac{f(h) - f(0)}{h} = 2x - f^1(0) = 2x - \sqrt{-3+a-a^2}$$

$$\Rightarrow f(x) = x^2 - \sqrt{-3+a-a^2}.x + c$$

Putting $x = 0$ we have $c = -1$ Hence,

$$f(x) = x^2 - \sqrt{-3+a-a^2}.x - 1$$

The discriminant

$$\Delta = -3 + a - a^2 + 4$$

$$= -a^2 + a + 1$$

$$= -\left\{\left(a + \frac{1}{2}\right)^2 + \frac{3}{4}\right\} < 0$$

Hence, f(x) > 0 for all xÎR.

Dumb Question:

1) How does $\lim_{h\to 0}\frac{f(x)-f(h)+2xh-1-f(x)}{h}$ transform to $2x-\lim_{h\to 0}\frac{f(h)-f(0)}{h}$?

Ans:

$$\lim_{h\to 0}\frac{f(x)-f(h)+2xh-1-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{2xh-(f(h)-(-1))}{h}$$

$$=\lim_{h\to 0}2x-\frac{(f(h)-f(0))}{h}\quad(\because f(0)=-1)$$

$$=2x-\lim_{h\to 0}\frac{(f(h)-f(0))}{h}$$

5) If $2x=y^{1/5}+y^{-1/5}$ then express y as an explicit function x and prove that $(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=25y$.

Solution:

$$\left(y^{1/5}-y^{-1/5}\right)^2=\left(y^{1/5}+y^{-1/5}\right)^2-4=4x^2-4\,\{\because\ y^{1/5}-y^{-1/5}=2x\}$$

$$\therefore\ 2y^{1/5}=\left(y^{1/5}+y^{-1/5}\right)+\left(y^{1/5}-y^{-1/5}\right)=2x+2\sqrt{x^2-1}$$

$$\therefore\ y=(x+\sqrt{x^2-1})^5$$

Differentiating w.r.t x,

$$\frac{dy}{dx}=5\left(x+\sqrt{x^2-1}\right)^4\left\{1+\frac{1}{2}.\frac{2x}{\sqrt{x^2-1}}\right\}$$

$$=\frac{5\left(x+\sqrt{x^2-1}\right)^5}{\sqrt{x^2-1}}=\frac{5y}{\sqrt{x^2-1}}$$

$$\therefore\ (x^2-1)\left(\frac{dy}{dx}\right)^2=25y^2$$

Differentiating w.r.t x,

$$2x\left(\frac{dy}{dx}\right)^2+(x^2-1).2\frac{dy}{dx}.\frac{d^2y}{dx^2}=50y\frac{dy}{dx}$$

$$\text{Or } x\frac{dy}{dx}+(x^2-1)\frac{d^2y}{dx^2}=25y$$

Therefore $\frac{dy}{dx} \neq 0$ for then y is constant.
Dumb Question:

1) Why is $2y^{1/5} = 2x + 2\sqrt{x^2 - 1}$?
Ans:

$$\left(y^{1/5} - y^{-1/5}\right)^2 = 4x^2 - 4$$

$$\text{So, } y^{1/5} - y^{-1/5} = 2\sqrt{x^2 - 1}$$

$$\text{And also } y^{1/5} + y^{-1/5} = 2x \text{ (given)}$$

$$\text{Hence } 2y^{1/5} = 2x + 2\sqrt{x^2 - 1}$$

6) The function y = f(x) is defined as follows, x = 2t-|t|, y = t^2 + t |t|, tÎR. Then discuss the continuity and differentiability of the function at x=0 also draw the graph of the function in the interval [-1, 1].
Solution:
When t < 0, x = 2t -(-t) = 3t < 0.
y = t^2 + t (-t) = 0.
\y = 0 when x<0.
When t ³ 0, x = 2t-t = t ³ 0.
y = t^2+t (t) = $2t^2$ = $2x^2$
\ y = $2x^2$ when x ³ 0.
Thus the function is as follows.
f (x) = 0, x<0
=$2x^2$, x ³ 0
Now,

$$f^1(0+0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 - 0}{h}$$

$$= \lim_{h \to 0} 2h = 0$$

$$f^1(0-0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \left(\frac{0-0}{-h}\right) = 0$$

$$\therefore f^1(0+0) \neq f^1(0-0)$$

\f (x) is differentiable (finitely) at x=0.
\ f (x) is also continuous at x = 0.

Being a polynomial function f(x) is continuous at other points of the interval [-1, 1].
Now in the interval [-1, 1] we have y = 0 in [-1, 0], y = $2x^2$ in [0, 1]
So the graph is as below.

Fig (5)

(HARD TYPE)

1) $f(x) = \begin{cases} 1+x, & 0 \le x \le 2 \\ 3-x, & 2 \le x \le 3 \end{cases}$

Determine f {f(x)} and hence find the points of discontinuity and non-differentiability. Also draw the graph of f {f(x)} in [0, 3].
Solution:
Clearly f {f(x)} = 1+ f(x), 0 £ f(x) £ 2
3 - f(x), 2 < f(x) £ 3
When 0 £ f(x) £ 2
0 £ 1+x £ 2, if 0 £ x £ 2 {using first piece of definition}
Þ -1 £ x £ 1 if 0 £ x £ 2
Þ 0 £ x £ 1 (taking the interval of common points)
\ 0 £ f(x) £ 2 when 0 £ x £ 1 and f(x) = 1+x --------- (1)
When 0 £ f(x) £ 2
0 £ 3-x £ 2 If 2 < x £ 3 (using the second piece of definition)
Þ -3 £ -x £ -1 If 2 < x < £ 3
Þ 1 £ x £ 3 If 2 < x £ 3
Þ 2 < x £ 3 (taking interval of common points)
\0 £ f(x) £ 2 when 2 < x £ 3 and f(x) = 3-x ----- (2)
When 2 < f(x) £ 3

2<1+x £ 3 If 0 £ x £ 2 (using the first piece of definition)
Þ 1< x £ 2 If 0 £ x £ 2
Þ 1< x £ 2
\2< f(x) £ 3 when 1< x £ 2 and f(x) = 1+x ------- (3)
When 2< f(x) £ 3
2<3-x £ 3 If 2 < x £ 3 (using second piece of definition)
Þ -1 < -x £ 0 If 2 < x £ 3
Þ 0 £ x <1 If 2 < x £ 3
Þ xÎf.

Thus we get

f {f(x)} = 1+1+x = 2+x,	0 £ x £ 1	{from (1)}
= 1+3-x = 4-x,	2 < x £ 3	{from (2)}
= 3-(1+x) = 2-x,	1 < x £ 2	{from (3)}

Hence the function is

f {f(x)} = 2+x,	0 £ x£ 1
2-x,	1< x £ 2
4-x,	2 < x £ 3

As the polynomial functions are continuous and differentiable every where f {f(x)} is continuous and differentiable in [0, 3] except perhaps at the turning points of definition, namely, x = 1, 2 denote f {f(x)} by g(x).
Now g (1+0) = 2-1=1
g (1-0) = 2+1 = 3
So g(x) is not continuous at x=1.
g (2+0) = 4-2 =2
g (2-0) = 2-2 = 0
So, g(x) is not continuous at x=2.
Hence g(x) is also not differentiable at x = 1, 2.
\The points of discontinuity and non-differentiability are x=1, 2.
Now we have to draw the graph for y = f {f(x)} where
y = 2+x in [0, 1],
y = 2-x in [1, 2]
y = 4-x in [2, 3]
The graph is discontinuous at x = 1, 2.

Fig (6)

Dumb Question:

1) In the above solution, there is a condition 0 £1+x £2 if 0 £ x £ 2 from that it is inferred that 0 £ x£ 1 How?

Ans:

0 £1+x £2 if 0 £ x £ 2

Þ -1 £ x £ 1 if 0 £ x £ 2

Now clearly both the conditions need to be satisfied simultaneously. So the intersection of these two conditions will give the desired results.

i.e. 0 £ x £ 1

2) If $f(x) = \log_{\sec 2x} |\cos 4x| + |\sin x|$ then find $\frac{dy}{dx}$ at $x = -\frac{\pi}{6}$ from the first principle.

Solution:

Clearly the definition of the function f(x) is $\left[-\frac{\pi}{6} - h, -\frac{\pi}{6} + h\right]$ where h is a very small positive quantity, remains the same, so $x = -\frac{\pi}{6}$ is not a turning point of definition.

Now,

$$\left(\frac{dy}{dx}\right)_{x=-\frac{\pi}{6}} = \lim_{h\to 0}\frac{f\left(-\frac{\pi}{6}+h\right)-f\left(-\frac{\pi}{6}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\left\{\log_{\sec\left(-\frac{\pi}{3}+2h\right)}\left|\cos 4\left(-\frac{\pi}{6}+h\right)\right|+\left|\sin\left(-\frac{\pi}{6}+h\right)\right|-\log_{\sec\left(-\frac{\pi}{3}\right)}\left|\cos-\frac{2\pi}{3}\right|-\left|\sin\ -\frac{\pi}{6}\right|\right\}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left\{\log_{\sec\left(-\frac{\pi}{3}+2h\right)}\left|\cos\left(-\frac{2\pi}{3}+4h\right)\right|-\log_{\sec-\frac{\pi}{3}}\left|\cos-\frac{2\pi}{3}\right|\right\}+$$

$$\lim_{h\to 0}\frac{1}{h}\left\{\left|\sin\left(-\frac{\pi}{6}+h\right)\right|-\left|\sin\ -\frac{\pi}{6}\right|\right\}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\log\left\{-\cos\left(-\frac{2\pi}{3}+4h\right)\right\}}{\log\sec\left(-\frac{\pi}{3}+2h\right)}-\log_2\left(\frac{1}{2}\right)\right]+\lim_{h\to 0}\frac{1}{h}\left\{-\sin\left(-\frac{\pi}{6}+h\right)-\frac{1}{2}\right\}$$

(Using definition of mod)

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\log\cos\left(\frac{\pi}{3}+4h\right)}{-\log\cos\left(-\frac{\pi}{3}+2h\right)}+1\right]+\lim_{h\to 0}\frac{1}{h}\left\{\frac{1}{2}\cosh-\frac{\sqrt{3}}{2}\sinh\ -\frac{1}{2}\right\}$$

Now $\lim_{h\to 0}\frac{1}{h}\left\{\frac{1}{2}\cosh-\frac{\sqrt{3}}{2}\sinh\ -\frac{1}{2}\right\} = \lim_{h\to 0}\frac{1}{h}\left\{-\frac{\sqrt{3}}{2}\sinh\ -\frac{1}{2}.2\sin^2\frac{h}{2}\right\}$

$$= \lim_{h\to 0}\left[-\frac{\sqrt{3}}{2}\frac{\sinh}{h}-\frac{1}{2}\left(\frac{\sin(h/2)}{h/2}\right).\sin\left(\frac{h}{2}\right)\right] = -\frac{\sqrt{3}}{2}\lim_{h\to 0}\frac{1}{h}\left[\frac{\log\cos\left(\frac{\pi}{3}+4h\right)}{-\log\cos\left(-\frac{\pi}{3}+2h\right)}+1\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\frac{\left\{-\log\left(\frac{1}{2}\cos 4h-\frac{\sqrt{3}}{2}\sin 4h\right)+\log\left(\frac{1}{2}\cos 2h+\frac{\sqrt{3}}{2}\sin 2h\right)\right\}}{\log\left(\frac{1}{2}\cos 2h+\frac{\sqrt{3}}{2}\sin 2h\right)}$$

$$= \lim_{h\to 0} \frac{\left\{ -\frac{-2\sin 4h - 2\sqrt{3}\cos 4h}{\frac{1}{2}\cos 4h - \frac{\sqrt{3}}{2}\sin 4h} + \frac{-\sin 2h + \sqrt{3}\cos 2h}{\frac{1}{2}\cos 2h + \frac{\sqrt{3}}{2}\sin 2h} \right\}}{\left\{ \log\left(\frac{1}{2}\cos 2h + \frac{\sqrt{3}}{2}\sin 2h\right) + h \times \frac{-\sin 2h + \sqrt{3}\cos 2h}{\frac{1}{2}\cos 2h + \frac{\sqrt{3}}{2}\sin 2h} \right\}}$$

(Using L-Hospital Rule)

$$= \frac{4\sqrt{3} + 2\sqrt{3}}{\log\frac{1}{2}} = \frac{6\sqrt{3}}{-\log 2}$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=-\frac{\pi}{6}} = \frac{-6\sqrt{3}}{\log 2} - \frac{\sqrt{3}}{2}$$

3) Let a+b =1, $2a^2+2b^2$ =1 and f(x) be a continuous function such that f (2+x) +f(x) = 2 for all xÎ(0, 2) and p = $\int_0^4 f(x)dx - 4$, $q = \frac{\alpha}{\beta}$ then find the least positive integral values of 'a' for which the equation ax^2-bx+c = 0 has both roots lies between p and q;
Where a, b, c ÎN.

Solution:

Given a+b =1 --------- (1)

$2a^2+2b^2$ =1 -------- (2)

Solving equation (1) and (2) we get,

a = b = $\frac{1}{2}$.

Þ $q = \frac{\alpha}{\beta} = 1$ ---------- (3)

Given f(x+2) +f(x) = 2 for all x Î[0, 2] -------- (4)

Now p = $\int_0^4 f(x)dx - 4$

$= \int_0^2 f(x)dx + \int_2^4 f(x)dx - 4$

$= \int_0^2 f(x)dx + \int_0^2 f(t+2)dt - 4$

(let x = t+2 for second integration}

$= \int_0^2 f(x)dx + \int_0^2 \{2 - f(x)\}\, dx - 4$

$= \int_0^2 f(x)dx + 2\int_0^2 dx - \int_0^2 f(x)dx - 4 = 0$

Then p =0, q=1

Let the roots of equation ax^2-bx+c = 0 be a and b

\f (x) = ax^2-bx+c= a(x-a) (x-b) -------- (5)

Since equation f(x) = 0 has both roots between 0 and 1.

\f (0).f (1) > 0 ------- (6)

But f (0).f (1) = c (a-b+c) = an integer ------- (7)

Therefore least value of f (0). f (1) = 1 -------- (8)

Now from equation (5)

f (0).f (1) = a a b a (1- a) (1-b)

= a^2ab (1- a) (1- b) --------- (9)

As we know

a (1- a) has greatest values $\frac{1}{4}$ at $\alpha = \frac{1}{2}$ and b (1- b) has greatest value $\frac{1}{4}$ at $\beta = \frac{1}{2}$

But b ¹ a

Thus from equation (8) greatest value of

f (0). f (1) < $\frac{a^2}{16}$ ------ (10)

From (8) and (10)

$1 < \frac{a^2}{16}$

$\Rightarrow a^2 - 16 > 0$

$\Rightarrow a < -4 \text{ or } a > 4 \quad (\because a \in N)$

$\Rightarrow \text{least value of } a = 5$

{As a Î Natural number}

Dumb Question:

1) Why c (a-b+c) is an integer?

Ans: It is given in the question that a, b, c ÎN

So, cÎN and a-b+cÎI

So, c (a-b+c) ÎI
Hence c (a-b+c) is an integer.

KEYWORDS

- Derivative
- First Principle
- $\frac{dy}{dx}$
- Right Hand Derivative
- Left Hand Derivative
- Parametric
- Logarithmic Differentiation
- Implicit function
- Chain Rule

Applications of Derivatives

Derivative as rate of change:

If variable quantity y is function of t i.e. y = f(t), then small change in time Δy in y

$\therefore$ Average rate of change = $\frac{\Delta y}{\Delta t}$

when $\Delta t \to 0$, rate of change becomes instantaneous

i.e. $\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$

Illustration: If radius of circle increasing at uniform rate of 2 cm/s, find rate of increasing of area of circle, at instant when radius is 20 cm.

Ans: $\frac{dr}{dt} = 2$ cm/s

Area of circle = πr^2

Differentiating w.r.t. to t,

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\frac{dA}{dt} >= 2\pi$ x 20 x 2 = 80 π cm^2/s

Slopes of tangent & Normal:

Slope of tangent:

Let y = f(x) be cont. & $P(x_1, y_1)$ be point on it.

Then $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ is slope of tangent to curve y = f(x) at point P.

<u>Note</u>: (1) If tangent is 11 to x-axis, then

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

(2) If tangent is $\perp$ to x-axis, then

$$\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

<u>Dumb Question</u>: How $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$ when tangent is 11 to x-axis.

Ans: We know that slope of line = $\tan\theta$ where $\theta \to 0$ with x-axis in anticlockwise direction.
If tangent is 11 to x-axis
$\therefore\ \theta = 0$ or $\tan\theta = 0$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

<u>Slope of Normal</u>: Normal to curve at P is line $\perp$ to tangent at P & passing through $P(x_1, y_1)$

$$\therefore \text{Slope of normal at P} = -\frac{1}{\text{Slope of tangent at P}}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$

<u>Note</u>: (i) If normal is 11 to x-axis

$$\Rightarrow -\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0 \Rightarrow \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

(ii) If normal is $\perp$ to x-axis

$\Rightarrow - \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

<u>Illustration</u>: Find point on curve $y = x^3 - 3x$ at which tangent is 11 to x-axis.

Ans: Let the point be $P(x_1, y_1)$ on curve $y = x^3 - 3x$ ..... (i)

$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 3x) = 3x^2 - 3$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2 - 3$

But tangent is 11 to x-axis

$\therefore \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

$\Rightarrow 3x_1^2 - 3 = 0 \quad \Rightarrow x_1 = \pm 1$ ..... (ii)

Since $P(x_1, y_1)$ lies on curve.

$\therefore \quad y_1 = x_1^3 - 3x_1$

$x_1 = 1$ where $x_1 = -1$

$y_1 = 1 - 3 = -2$ $\quad y_1 = -1 + 3 = 2$

$\therefore$ Points are $(1, -2)$ & $(-1, 2)$

<u>Equations of tangent & Normals</u>:

<u>Eq. of tangent</u>:

Slope of tangent at $P(x_1, y_1) = \tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

Since it passes through $P(x_1, y_1)$

$\therefore$ eq. of tangent is

$(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$

Eq. of Normal:

Slope of Normal = - $\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$
Eq. of normal is

$(y - y_1) = - \left(\frac{dx}{dy}\right)_{(x_1, y_1)} (x - x_1)$

Illustration: Find eq. of tangent & normal to curve $2y = 3 - x^2$ at (1,1)

Ans: Eq. of given curve is $2y = 3 - x^2$ ..... (i)

differentiating (i) w.r.t. x $2\left(\frac{dy}{dx}\right) = - 2x \Rightarrow \left(\frac{dy}{dx}\right) = - x$

$\left(\frac{dy}{dx}\right)_{(1, 1)} = - 1$

eq. of tangent at (1, 1) is

$y - 1 = - 1(x - 1) \Rightarrow x + y = 2$

& eq. of normal at (1,1) is

$y - 1 = 1(x - 1) = y - x = 0$

Angle of intersection of two curves:

It is angle b/w tangent s to the two curves at this point of intersection.

Let C_1 & C_2 be two curves of eq. is = f(x) & y = g(x) respectively.
Let θ is angle b/w two tangents of two curves & tangents PT_1 & PT_2 makes angle θ_1 & θ_2 respectively with x-axis.

$$\therefore \quad \tan\theta = \left| \frac{\left(\frac{dy}{dx}\right)_{C_2} - \left(\frac{dy}{dx}\right)_{C_1}}{1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}} \right|$$

troduction

<u>Dumb Question</u>: How this derived ?

Ans: Let $m_1 = \tan\theta_1 = \left(\frac{dy}{dx}\right)_{C_1}$

$m_2 = \tan\theta_2 = \left(\frac{dy}{dx}\right)_{C_2}$

From fiq, $\theta = \theta + \theta_1$

$\Rightarrow \theta = \theta_2 - \theta_1$

$$\tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2}$$

$$\tan\theta = \left| \frac{\left(\frac{dy}{dx}\right)_{C_2} - \left(\frac{dy}{dx}\right)_{C_1}}{1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}} \right|$$

<u>Orthogonal curves</u>: If angle of intersection of two curves is right angle, two curves are c/d orthogonal curves.

If curves are orthogonal, $\theta = \frac{\pi}{2}$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = -1$$

<u>Illustration</u>: Find angle of intersection of curves $y = x^2$ & $y = 4 - x^2$

For intersection points of given curves,

$$\left(\frac{dy}{dx}\right)_{C_1} = \frac{d}{dx}(x^2) = 2x$$

$$\left(\frac{dy}{dx}\right)_{C_2} = \frac{d}{dx}(4 - x^2) = -2x$$

At $x = -\sqrt{2}$,

$$\left(\frac{dy}{dx}\right)_{C_1} = 2 \times \sqrt{2} = 2\sqrt{2} \quad \& \quad \left(\frac{dy}{dx}\right)_{C_2} = -2\sqrt{2}$$

$$\therefore \tan\theta_1 = \left|\frac{2\sqrt{2} - \left(-2\sqrt{2}\right)}{1 + 2\sqrt{2}\left(-2\sqrt{2}\right)}\right| = \left|\frac{4\sqrt{2}}{-2}\right|$$

$$\Rightarrow \theta_1 = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

$$\therefore \tan\theta_2 = \left|\frac{-2\sqrt{2} - 2\sqrt{2}}{1 + \left(-2\sqrt{2}\right)\left(2\sqrt{2}\right)}\right|$$

$$\theta_2 = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

Both angles are equal.

<u>Length of Tangent, Sub-Tangent, Normal & sub-Normal</u>:

Length of Tangent: Length of segment PT of tangent b/w point of tangent & x-axis is c/d length of tangent.

PT = $\sqrt{y_1^2 + y_1^2\left(\frac{dx}{dy}\right)^2_{(x_1, y_1)}}$

Subtangent: Projection of segment PT along x-axis i.e. St c/d subtangent.

Length of Normal: Length of segment PN intercapted b/w point on curve r x-axis.

PN = $\sqrt{y_1^2 + y_1^2\left(\frac{dx}{dy}\right)^2_{(x_1, y_1)}}$

Subnormal: Projection of segment PN along x-axis i.e. c/d subnormal.

SN = |y, $\left(\frac{dx}{dy}\right)_{(x, y_1)}$

Dumb Question: How these relation derived ?

Ans: Since PT makes angle θ with x-axis, then

$\tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

$\therefore \quad \frac{ST}{PS} = \cot\theta$

Subtangent = ST = PS $\cot\theta$

But PS = y_1 & $\cot\theta = \left(\frac{dx}{dy}\right)_{(x_1, y_1)}$ [see in fig.]

$\therefore \quad ST = \left| y_1 \left(\frac{dx}{dy}\right)_{(x_1, y_1)} \right|$

$\frac{SN}{PS} = \cot(90 - \theta) \Rightarrow PS \tan\theta$

$\Rightarrow$Subnormal = SN = $\left| y_1 \left(\frac{dx}{dy}\right)_{(x_1, y_1)} \right|$

Length of tangent = PT = $\sqrt{PS^2 + TS^2} = \sqrt{y_1^2 + y_1^2 \left(\frac{dx}{dy}\right)^2_{(x_1, y_1)}}$

Length normal = PN = $\sqrt{PS^2 + SN^2} = \sqrt{y_1^2 + y_1 \left(\frac{dx}{dy}\right)^2_{(x_1, y_1)}}$

<u>Illustration</u>: Show that curve $y = be^{x/a}$. subnormal varies as square of ordinate ?

Ans:

[**<u>Dumb Question</u>**: What is ordinate ?

Ans: y-axis component is c/d ordi i.e. in $P(x_1, y_1)$ y_1 is ordinate of point P.

$y = be^{x/a}$ ..... (i)

differentiating curve $y = be^{x/a}$ w.r.t. x.

$\left(\frac{dy}{dx}\right) = be^{x/a} . \frac{1}{a}$

$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{b}{a} e^{x_1/a}$ ..... (ii)

& Let $P(x_1, y_1)$ lie on curve

$\therefore \quad y_1 = be^{x_1/a}$

Length of subnormal = $\left| y_1 \left(\frac{dy}{dx} \right)_{(x, y_1)} \right|$

$$= be^{x_1/a} \cdot \frac{be^{x_1/a}}{a} = \frac{1}{a}\left(be^{x_1/a}\right)^2$$

$$= \frac{1}{a} y_1^2$$

So, subnormal varies as square of ordinate.

<u>Rolle's Theorem</u>:

<u>Statement</u>: Let f be real valued function defined on closed interval [a, b] such that,

(i) f(x) is continuous in closed interval [a, b].

(ii) f(x) is differentiable in open interval (a, b).

(iii) f(a) = f(b)

Then there is at least one value of C of x in (a, b) for which f'(c) = 0

Proof: <u>Case I</u>: f(x) is constant function in interval [a, b] then f'(x) = 0 for all x $\in$ [a, b]

Hence, follows Rolle's theorem, we can say that f'(c) = 0 where a < c < b.

<u>Case II</u>: f(x) is not constant in interval [a, b] & sionce f(a) = f(b)

Let f(x) increases for x > a.
Since f(a) = f(b), so, function must increase to some value x = c & decreasing upto x = b.
∴ At x = c function has maximum value.
Let h be small quantity, then,

$$f(c + h) - f(c) < 0 \quad \& \quad f(c - h) - f(c) < 0$$

$$\therefore \frac{f(c+h) - f(c)}{h} < 0 \quad \& \quad \frac{f(c-h) - f(c)}{-h} > 0$$

$$\therefore \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \le 0 \quad \& \quad \frac{f(c-h) - f(c)}{-h} \ge 0$$

But if $\therefore \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \ne \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$

Then Rolle's theorem cannot applicable b/c. f(x) is not differentiable at x = c
∴ Only possible way for Rolle's theorem, when

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = 0$$

$\Rightarrow f'(c) = 0$ where $a < c < b$

<u>Note</u>: (i) Polynomial function is everywhere cont. & differentiable.

(ii) Exponential function, sine & cosine function are everywhere cont. & differentiable.

(iii) Logarithmic function is cont. & differentiable in its domain.

(iv) tanx is not cont. & differentiable at $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$, ...

(v) |x| is not differentiable at x = 0.

Illustration: Verify Rolle's theorem for function $f(x) = x^3 - 3x^2 + 2x$ in interval [0, 2].

Ans: (a) f(x) is polynomial so, it is cont. & differentiable everywhere.

(b) $f'(x) = 3x^2 - 6x + 2$ clearly exista for all x $\frac{3\pi}{2}$ (0,2)

(c) $f(0) = 0$, $f(2) = 2^3 - 3(2)^2 + 2(2) = 0$
$\therefore f(0) = f(2)$
So, All conditions of Rolle's theorem are satisfied.
So, there exist some c $\in$(0,2) such that $f'(c) = 0$

$\Rightarrow f'(c) = 3c^2 - 6c + 2 = 0 \quad \Rightarrow c = 1 \pm \frac{1}{\sqrt{3}}$

Lagrange's Mean Value Theorem:

Statement:

(i) f(x) is cont. in closed interval [a, b].

(ii) is differentiable in open interval (a, b)

Then there is at least one value c $\in$(a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical interpretation:

Let A, B be points on curve y = f(x) corresponding to x = a & x = b, so that
A = [a, f(a)] & B = [b, f(b)]

$\therefore$ Slope of chord AB = $\dfrac{f(b) - f(a)}{b - a}$

But slope of chord AB = f'(c), the slope of tangent to curve at x = c.
Accordin to LMVT, if a curve has tangent at each of its points then there is a point 'c' on this curve in between A & B, the tangent at which is paralled to chord AB.

<u>Illustration</u>: Find c of LMVT for which f(x) = $\sqrt{25 - x^2}$ in [1, 5].

Ans: (1) f(x) has definite & unique value of each x $\in$ [1, 5]
So, every point in interval [1, 5] the value of f(x) is equal to limit of f(x).
$\therefore$ f(x) is cont. in [1, 5]

(2) f'(x) = - $\dfrac{x}{\sqrt{25 - x^2}}$ exists for all x $\in$ [1, 5]
$\therefore$ f(x) is differentiable in (1, b)
So, there must br some c such that

f'(c) = $\dfrac{f(5) - f(1)}{5 - 1} = \dfrac{0 - \sqrt{24}}{4} = \dfrac{-\sqrt{6}}{2}$

But f'(c) = $\dfrac{-c}{\sqrt{25 - c^2}} = \dfrac{-\sqrt{6}}{2} \Rightarrow c = \pm\sqrt{15}$

<u>Application of derivative in determining the nature of roots of cubic polynomial</u>:

Let f(x) = $x^3 + ax^2 + bx + c$ be given cubic polynomial.
Δ f(x) = 0
f'(x) = $3x^2 + 2ax + b$ ..... (i)
Let D = $4a^2 - 12b = 4(a^2 - 3b)$ be discriminate of eq. f'(x) = 0

<u>Case I</u>: If D < 0 $\Rightarrow$ f'(x) > 0 $\forall$ x $\in$ R
$\Rightarrow$ f(x) = - ∞ and $\lim_{x\to\infty} f(x) = \infty$

from fig. graph of y = f(x) cut x-axis only once. So, we have only real root (say x_0)

$\therefore x_0 > 0$ if $c < 0$ & $x_0 < 0$ if $c > 0$

<u>Case II</u>: If $D > 0$, $f'(x) = 0$
have two real roots.

$\Rightarrow f'(x) = 3(x - x_1)(x - x_2)$

$f'(x) < 0, x \in (x_1, x_2)$

$f'(x) > 0, x \in (-\infty, x_1) \cup (x_2, \infty)$

$\Rightarrow f(x)$ would increase in $(-\infty, x_1)$ & (x_2, ∞) and would decrese in (x_1, x_2)

$\therefore x = x_1$ would be point of local maxima & $x = x_2$ would be point of local minima.

(I)

(II)

3 distict roots $x = \alpha, \beta, \gamma$ One real root $x = \alpha$ & two imaginary root

(III)

One real root $x = 9$ & two imaginary roots.

(IV)

Three roots $x = \alpha, n_2, x_2$ (x_2 is repeated root)

(V)

Threen real roots $x = n_1, x_2, \alpha$
(x_1 is repeated roots)

<u>Results</u>:

(a) From I graph, $f(x_1) > 0, f(x_2) < 0$
$\therefore f(x_1)\, f(x_2) < 0$, $f(x) = 0$ would have 3 real & distinct roots.

(b) Fromn II & III graph $f(x_1)\, f(x_2) > 0$, $f(x) = 0$ have one real & two imaginary roots.

(c) $f(x_1\, f(x_2) = 0$, $f(x) = 0$ have 3 real roots but of root would be repeated.

<u>Case III</u>: $D = 0$, $f'(x) = 3(x - x_1)^0$ where x_1 is root of $f'(x)$.
$\Rightarrow f(x) = (x - x_1)^3 + K$
Then, $f(x) = 0$, has three real roots if $K = 0$

$f(x) = 0$ has one real root if $K \neq 0$

<u>Illustration</u>: Find values of a so that $x^3 - 3x + a = 0$ has three real & distinct roots.

Ans: Let $f(x) = 3x^3 - 3x + a$
$\Rightarrow f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$
$x_1 = 1, x_2 = -1$
$f(x_1) = f(1) = a - 2$
$f(x_2) = f(-1) = a + 2$
Since roots would be real & distinct if
$f(1)\, f(-1) < 0 \quad \Rightarrow (a - 2)(a + 2) < 0$
$\Rightarrow -2 < a < 2$

<u>Increasing function</u>:

(a) **<u>Strictly increasing function</u>**:

A function f(x) is c/d strictly increasing function in its domain if $x_1 < x_2$

For strictly increasing function $f'(x) > 0 \ \forall\, x \in$ domain

<u>Dumb Question</u>: How $f'(x) > 0 n \ \forall\, x \in$ domain for strictly increasin function.

Ans: As $x_1 < x_2$

$\Rightarrow f(x_1) < f(x_2)$

So, $f(x) < f(n + h)$

$$f'(x) = \lim_{h \to 0} \frac{f(n+h) - f(x)}{h} = \lim_{h \to 0} \frac{+ve}{+ve}$$

$\Rightarrow f'(x) > 0$

<u>Types of strictly increasing function</u>:

(1) <u>Concave up</u>: When $f'(x) > 0$ & $f''(x) > 0 \ \forall \ x \in$ domain

(2) <u>Concave down</u>: $f'(x) > 0$ & $f''(x) < 0 \ \forall \ x \in$ domain

(3) <u>When $f'(x) > 0$ & $f''(x) = 0 \ \forall \ x \in$ domain</u>

$f'(x) > 0$ & $f''(x) = 0$

Increasing function:

A function f(x) is said to be non decreasing if for $x_1 < x_2$
$\Rightarrow f(x_1) \leq f(x_2)$
Laet us see in fig.

For portion ABCD, $x_1 < x_2$

$\Rightarrow f(x_1) < f(x_2)$

Δ for BC, $x_1 < x_2$

$\Rightarrow f(x_1) = f(x_2)$

Dumb Question: What is diff. b/w strictly increasing & increasing function ?

Ans: Strictly function for $x_1 < x_2$, $f(x_2)$ is always greater than $f(x_1)$ but in increasing function for $x_1 < x_2$, $f(x_2$ may be greater or equal to $f(x_1)$.

Decreasing functions:

(a) **Strictly decreasing function**:

A function f(x) ic c/d strictly decreasimg in its domain if $x_1 < x_2$
$\Rightarrow f(x_1) > f(x_2)$
For strictly decreasing function.
$f'(x) < 0$

<u>Dumb Question</u>: How $f'(x) < 0$ for strictly decreasing ?

Ans: As $x_1 < x_2 \quad \Rightarrow f(x_1) > f(x_2)$
So, $f(x + h) < f(x)$

$$\Rightarrow f'(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-ve}{+ve}$$

$\Rightarrow f'(x) < 0$

<u>Types of strictly decreasing function</u>:

(i) <u>Concave up</u>:

When $f'(x) < 0$
$\Delta f''(x) > 0 \; \forall \; x \in$ domain

(ii) <u>Concave down</u>:

When $f'(x) < 0$ & $f''(x) < 0 \forall x \in$ domain

(iii)

When $f'(x) < 0$ & $f'(x) = 0 \forall x \in$ domain

(5) **<u>Non-increasing function</u>**:

A function f(x) is c/d non-increasing if for $x_1 < x_2$
for $x_1 < x_2$

$\Rightarrow f(x_1) \leq f(x_2)$

For AB & CD, $x_1 < x_2$

$\Rightarrow f(x_1 > f(x_2)$

Bc, $x_1 < x_2$

$\Rightarrow f(x_1 = f(x_2)$

<u>Illustration</u>: Find interval in which $f(x) = x^3 - 3x^2 - 9x + 20$ is strictly increasing or decreasing.

Ans: $f(x) = x^3 - 3x^2 - 9x + 20$

$f'(x) = 3x^2 - 6x - 9$

$\Rightarrow f'(x) = 3(x - 3)(x + 1)$

For strictly increasing

$f'(x) > 0$

$\therefore\ 3(x - 3)(x + 1) > 0 \quad \Rightarrow (x - 3)(x + 1) > 0$

$\Rightarrow (x + 1) < 0$ or $(x - 3) > 0$

$\Rightarrow x < -1$ or $x > 3$

$x \in (-\infty, -1) \cup (3, \infty)$

For strictly decreasing

$f'(x) < 0$

$(x + 1)(x - 3) < 0$

$\Rightarrow x \in (-1, 3)$

<u>Properties of Monotonic function</u>:

(1) If f(x) is strictly increasing function on [a, b]

$\Rightarrow$ { f'(x) exists & f'(x) is also strictly increasing on [a, b].

(2) If f(x) & g(x) are also two continous & differentiable functions & fog(x) & gof(x) exists . then,

$$\Rightarrow \begin{cases} f'(x) > 0,\ g'(x) > 0 & \Rightarrow (fog)'(x) > 0 \ \&\ (gof)'(x) > 0 \\ f'(x) > 0,\ g'(x) < 0 & \Rightarrow (fog)'(x) < 0 \ \&\ (gof)'(x) < 0 \\ f'(x) < 0,\ g'(x) > 0 & \Rightarrow (fog)'(x) < 0 \ \&\ (gof)'(x) < 0 \\ f'(x) < 0,\ g'(x) < 0 & \Rightarrow (fog)'(x) > 0 \ \&\ (gof)'(x) > 0 \end{cases}$$

In other words

(i) If f(x) & g(x) are both strictly increasing or strictly decreasing.

$\Rightarrow$(fog)(x) & (gof)(x) both are strictly inctreasing.

(ii) If amongst two functions f(x) & g(x), one is strictly increasing other is strictly decreasing.

$\Rightarrow$(fog)(x) & (gof)(x) both are strictly decreasing.

<u>Illustration</u>: Let Ø(x) = sin(cosx) then check it whethe5r increasing ordecreasing in [0, $\frac{\pi}{2}$].

Ans: Ø(x) will be increasing if Ø'(x) > 0
Ø(x) = sin(cosx)
Ø'(x) = cos(cosx).(- sinx)

So, it is clearly decreasing for x $\in$[0, $\frac{\pi}{2}$] as Ø'(x) $\leq$0

<u>Other method</u>: f(x) = sinx & I(x) = cosx are increasing & decreasing in [0, $\frac{\pi}{2}$]
$\Rightarrow$fog(x) = Ø(x) = sin(cosx) is decreasing.

<u>Critical Points</u>:

It is collection of points for which,
(i) f(x) does not exists.

(ii) f'(x) does not exists.

(iii) f'(x) = 0
All values of x obtained from above conditions are c/d critical points.

<u>Illustration</u>: Find critical points for f(x) = $(x - 2)^{2/3}(2x + 1)$

Ans: f(x) = $(x - 2)^{2/3}(2x + 1)$

f'(x) = $\frac{2}{3}(x - 2)^{-1/3}(2x + 1) + (x - 2)^{2/3} - 2$

f'(x) = $2\left[\frac{(2x + 1)}{3(x - 2)^{1/3}} + \frac{(x - 2)^{2/3}}{1}\right]$

Clearly f'(x) doesnot defined at x = 2 so, x = 2 is critical point.

Dumb Question: Why f'(x) is not defined at x = 2 ?

Ans: Putting x = 2, f'(x) = ∞ which is not defined.
Another critical point
f'(x) = 0

$$2\left[\frac{2x + 1 + 3(x - 2)}{(x - 2)^{1/3}}\right] = 0 \quad \Rightarrow x = 1$$

So, x = 1 & x = 2 are two critical points of f(x)

Maxima & Minima:

Maxima: A function f(x) is said have maxima at x = a if f(a) $\geq$ f(a + h) where h $\geq$ 0 (very small & a lies in its domain.

Minima: A function f(x) is said to have minima at x = b if f(b) $\leq$ f(b + b) & f(b) $\leq$ f(b - b) where h $\geq$ 0 (very small)

Methods for finding extremum of continuone functions:

(a) **First derivative test**:

(i) When f(x) attains maximum at x = a:

From graph,
For x < a, $\theta_1 < 90^0$ $\Rightarrow \tan\theta_1 > 0$ or increasing for x < a
For x = a, $\tan\theta = 0$ or neither increasing nor decreasing for x = a.
For x > a, $\theta_2 > 90^0$ $\Rightarrow \tan\theta_2 < 0$ or decreasing for x > a
So,
f(x) is maximum at x =a

$$\Rightarrow \begin{cases} f(x) \text{ is increasing } x < a \\ f(x) \text{ is decreasing for } x > a \end{cases}$$

(ii) <u>When f(x) attains minimum at x = b</u>:

From graph,
For x < b, $\theta_1 > 90^0$ $\tan\theta_1 < 0$ or decreasing when x - < b
For x = b, $\tan\theta = 0$ or neither increasing nor decreasing for x = b
For x > b, $\theta_2 < 90^0$ $\Rightarrow \tan\theta_2 > 0$ or increasing when x > b
So,
f(x) is minimum at x = b

$$\Rightarrow \begin{cases} f(x) \text{ is decreasing } x < a \\ f(x) \text{ is increasing for } x > a \end{cases}$$

<u>Illustration</u>: Let $f(x) = x^3 - 3x^2 + 6$, find point at which f(x) have

local maximum & local minimum.

Ans: $f(x) = x^3 - 3x^2 + 6$
$f'(x) = 3x^2 - 6x$
Critical points are if $f'(x) = 0$
$3x^2 - 6x = 0 \Rightarrow x^2 - 2x = 0$
$\Rightarrow x = 0, 2$
At $x = 0$, $f'(x)$ change sign from + ve to - ve & $f'(x)$ changes sign from - ve to + ve at $x = 2$.
So, at $x = 0$, function has local maxima 7 at $x = 2$ it has local minima.

<u>Second Derivative Test</u>:

(1) First we find root of $f'(x) = 0$.
Let $x = a$ is one roots of $f'(x) = 0$

(2) Now find $f''(x)$ at $x = a$

(i) If $f''(a) = -ve$, then $f(x)$ is max. at $x = a$.

(ii) If $f''(a) = +ve$, then $f(x)$ is min. at $x = a$

(iii) If $f''(a) = 0$
Then find $F'''(x)$ at $x = 0$
If $f'''(a) \neq = 0$ then $f(x)$ has neither maximum nor minimum at $x = a$.
But if $f'''(a) = +$ ve, then find $f^{v}(a)$,
(a) If $f^{v} = +$ ve, then $f(x)$ is min. at $x = a$.
(b) If $f^{v}(a) = -$ ve, then $f(x)$ is max. at $x = a$.

<u>Dumb Question</u>: How $y = |x^2 - 3| = -(x^2 - 3)$ for neighbour hood of $x = \sqrt{2}$.

Ans: $y = |x^2 - 3|$ since $|x^2 - 3|$ is + ve quantity.
If we put $x = \sqrt{2}$, $x^2 - 3 = -1$ but is + ve quantity so, we put - ve to make it + ve.
So, $y = |x^2 - 3| = -(x^2 - 3)$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{c_1} = 2x = 2\sqrt{2}$$

$$\& \quad \left(\frac{dy}{dx}\right)_{c_2} = -2x = -2\sqrt{2}$$

$$\tan\theta = \left|\frac{2\sqrt{2} - \left(-2\sqrt{2}\right)}{1 + 2\sqrt{2}\left(-2\sqrt{2}\right)}\right| = \left|\frac{4\sqrt{2}}{-7}\right| = \left(\frac{4\sqrt{2}}{7}\right)$$

$$\theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

Q.6. If $ax^2 + bx + c = 0$, $a,b,c \in R$. Find condition that this eq. would have at least one root in (0, 1).

Ans: Let $f'(x) = ax^2 + bx + c$
Integrating both sides,

$$\Rightarrow f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d \text{ ..... (i)}$$

$$\Rightarrow f(0) = d \quad \& \quad f(1) = \frac{a}{3} + \frac{b}{2} + c + d$$

Since, Rolle's theorem is applicable.
[<u>Dumb Question</u>: Why Rolle's theorem is applicable ?
Ans: Rolle's theorem is applicable b/c f(x) is polynomial function which is continuous & differentiable everywhare.]

$$\Rightarrow f(0) = f(1)$$

$$\Rightarrow \cancel{d} = \frac{a}{3} + \frac{b}{2} + c + \cancel{d}$$

$$\boxed{\Rightarrow 2a + 3b + 6c = 0}$$

$\rightarrow$ required condition.

Q.7. Let f(x) & g(x) be differentiable for $0 \le x \le 2$ such that f(0) = 2, g(0) = 1 & f(2) = 8. Let there exists a real no. c in [0, 2] such that f'(c) = 3g'(c) find value of g(2).

Ans: As f(x) & g(x) are cobt. & differentiable in ... then there exists

atleast one value 'c' such that

$$f'(c) = \frac{f(2) - f(c)}{2 - 0}$$

$$\Delta g'(c) = \frac{g(2) - g(0)}{2 - 0}$$

But $\frac{f'(c)}{g'(c)} = 3$

$$\frac{\frac{f(2) - f(0)}{2 - 0}}{\frac{g(2) - g(0)}{2 - 0}} = \frac{8 - 2}{g(2) - 1} \Rightarrow \frac{8 - 2}{g(2) - 1} = 3$$

$\Rightarrow g(2) = 3$

Q.8. f(x) is a polynomial of degree 4 with real coeff. such that f(x) = 0 is satisfied by x = 1, 2, 3 only, them and f'(1) f'(2) f'(3) ?

Ans: f(x) = 0 has only roots 1, 2, 3 only
But it is degree 4 eq.
$\Rightarrow$Any one of 1, 2, or 3 is 4 repeated root of f(x) = 0
$\Rightarrow$f'(1) or f'(2) or f'(3) any one of them must be zero.
$\Rightarrow$f'(1) f'(2) f'(3) = 0

Q.9. Find interval for which f(x) = x - cosx is increasing or dectreasing.

Ans: f(x) = x - cosx
Differentiating w.r.t. x
f'(x) = 1 - sinx
we know
$-1 \leq \sin x \leq 1$ for all $x \in R$
.... $\Rightarrow \sin\theta \geq 0$ for $x \forall R$.

Q.10. If a < 0, & $f(x) = e^{ax} + e^{-ax}$ is monotonically decreasing. Find interval to which x belongs.

Ans: Given a < o &

$f(x) = e^{ax} + e^{-ax}$ is decreasing

$\Rightarrow f'(x) < 0 \quad \Rightarrow ae^{ax} - ae^{-xx} < 0$

$$\Rightarrow a\left(\frac{e^{2ax} - 1}{e^{ax}}\right) < 0$$

But $a < 0$

$\Rightarrow (e^{2ax} - 1) > 0 \quad \Rightarrow e^{2ax} > 1$

$\Rightarrow 2ax > 0 \quad \Rightarrow ax > 1$

$\Rightarrow x < 0$ as $a < 0$

<u>Medium Type</u>:

Q.1. Find values of 'k' for which point min. of function $f(x) = 1 + k^2x - x^3$ satisfy inequality $\frac{x^2 + x + 2}{(x^2 + 5x + 6)} < 0$.

Ans: $\frac{x^2 + x + 2}{(x^2 + 5x + 6)} < 0 \quad \Rightarrow \frac{\left(x + \frac{1}{2}\right)^2 + \frac{7}{4}}{(x + 2)(x + 3)} < 0$

Since $\left(x + \frac{1}{2}\right)^2 + \frac{7}{4}$ it always +ve

$\Rightarrow (x + 2)(x + 3) < 0$

$\Rightarrow -3 < x < -2$ ..... (i)

$f(x) = 1 + k^2x - x^3$

$f'(x) = k^2 - 3x^2$

$f''(x) = -6x$

For max/min, $f(x) = 0$

$\Rightarrow x = \pm\frac{|k|}{\sqrt{3}}$

Let $x_1 = \frac{|k|}{\sqrt{3}}$ & $x_2 = -\frac{|k|}{\sqrt{3}}$

$\therefore f''(x_1) < 0$ & $f''(x_2) > 0$

$\therefore f(x)$ is max. at $x = x_1$ & min. at $x = x_2$

But x_2 is min. which lies b/w

$-3 < x_2 < -2$ (from relation (i))

$\Rightarrow -3 < -\frac{|k|}{\sqrt{3}} < -2$

$\Rightarrow 3\sqrt{3} > |k| > 2\sqrt{3}$

$\Rightarrow k \in (-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$

Q.2. If $f(x) = x^{1/x}$, $x > 0$ determine bigger of two no. e^{π} & π^{e}.

Ans. $y = f(x) = x^{1/x}$, $x > 0$

Taking log on both sides

$\ln y = \frac{1}{x} \ln x$

Differentiating both side

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2} + \ln x\left(-\frac{1}{x^2}\right)$$

$$\frac{dy}{dx} = y\left[\frac{1 - \ln x}{x^2}\right]$$

$\therefore f'(x) = \frac{x^{1/x}}{x^2}[1 - \ln x]$

Let $f'(x) = 0$

$\Rightarrow \log x = 0$ or $x = e$

$$f''(x) = \frac{x^{1/x}}{x^2}\left[0 - \frac{1}{x}\right] + (1 - \log x)\frac{d}{dx}\left(\frac{x^{1/x}}{x^2}\right)$$

$$\therefore f''(e) = \frac{e^{1/e}}{e^2}\left[-\frac{1}{e}\right] + 0$$

$\Rightarrow f''(e) < 0$

$\Rightarrow f(x)$ has maxima at $x = e$

$\Rightarrow f(e) > f(\pi)$ for all $x > 0$

$\Rightarrow e^{1/e} > \pi^{1/\pi}$

$\Rightarrow e^{\pi} > \pi^{e}$

Q.3. If $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are real no.s & $3b^2 <$

c^2, is an increasing cubic function & $g(x) = af'(x) + bf''(x) + c^2$, then prove that $\int_a^x g(t)\,dt$ is an increasing function.

Ans: $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

Since f(x) is increasing

$f'(x) > 0 \quad \Rightarrow 3ax^2 + 2bx + c > 0$

$\Rightarrow 3a > 0 \quad \& \quad D < 0$

$\Rightarrow 3a > 0 \quad \& \quad b^2 - 3ac < 0$

$\Rightarrow a > 0 \quad \& \quad b^2 < 3ac$ ..... (i)

$g(x) = af'(x) + bf''(x) + c^2$

$g(x) = 3a^2x^2 + 2abx + ac + 6abx + 2b^2 + c^2$

$g(x) = 3a^2x^2 + 8abx + (2b^2 + c^2 + ac)$

$D = 64a^2b^2 - 4.3a^2(2b^2 + c^2 + ac)$

$= 4a^2(10b^2 - 3c^2 - 3ac)$

Since $3ac > b^2$ from (i)

$\Rightarrow -3ac < -b^2$

$\Rightarrow 4a^2(10b^2 - 3c^2 - 3ac) < 4a^2(10b^2 - 3c^2 - b^2)$

$= 4a^2(9b^2 - 3c^2)$

Since $3b^2 < c^2$

$= 12a^2(3b^2 - c^2) = -ve$

$\therefore D < 0$

$\Rightarrow g(x) > 0 \ \forall\, x \in R$

$\therefore \int_a^x g(t)\,dt$ is increasing function.

Q.4. If at each point of curve $y = x^3 - ax^2 + x + 1$ tangent is inclined at acute angle with +ve direction of x-axis. Find interval in which a lies.

Ans: $y = x^3 - ax^2 + x + 1$

Sine tangent is inclined at an acute angle with the direction of x-axis.

$\Rightarrow \frac{dy}{dx} \geq 0$

$\frac{dy}{dx} = 3x^2 - 2ax + 1 \geq 0$ for all $x \in R$

But if $ax^2 + bx + c \geq 0$ for $x \in R$

$\Rightarrow a > 0 \quad \& \quad D \leq 0$

$D = (2a)^2 - 4(3)(11)$

$= 4(a^2 - 3)$

$D \leq 0$

$\Rightarrow (a^2 - 3) \leq 0 \quad \Rightarrow (a - \sqrt{3})(a + \sqrt{3}) \leq 0$
$\Rightarrow -\sqrt{3} \leq a \sqrt{3}$

<u>Dumb Question</u>: How $\frac{dy}{dx} \geq 0$ if angle is acute.

Ans: Since $\frac{dy}{dx} = \tan\theta$ & if θ is lessthan $\frac{\pi}{2}$. So, $\tan\theta \geq 0$. So, $\frac{dy}{dx} \geq 0$

Q.5. If $f(x) = \log_e x$ & $g(x) = x^2$ & $c \in (4, 5)$, then find value of $c \log\left(\frac{4^{25}}{5^{16}}\right)$

Ans: $\log(4^{25}) - \log 5^{16}$
$25 \log 4 - 16 \log 5$
Since domain is [4, 5]
So, let
$Ø(x) = x^2 (\log 4) - 16 \log x$ i9s cont. [45, 5] & differentiable on (4, 5)
By LMUT,
$\frac{Ø(5) - Ø(4)}{5 - 4} = Ø'(c)$, $c \in (4, 5)$
$Ø(5) - Ø(4) = \log\left(\frac{4^{25}}{5^{16}}\right)$
But, $Ø'(c) = \frac{2}{c}(c^2 \log 4 - 8)$ ..... (i)
also $Ø'(c) = \frac{Ø(5) - Ø(4)}{5 - 4}$ ..... (ii)
$\Rightarrow$By (i) & (ii)
$\frac{2}{c}(c^2 \log_e 4 - 8) = \log_e\left(\frac{4^{25}}{5^{16}}\right)$
$\Rightarrow c \log\left(\frac{4^{25}}{5^{16}}\right) = 2(c^2 \log 4 - 8)$

Hard Type:

Q.1. Let S be square of unit area. Consider aany quadrilateral which has one vertex on each side of s. If a, b, c, & d denote lengths of side of quadrilateral, prove that
$2 \leq a^2 + b^2 + c^2 + d^2 \leq 4$

Ans:

Let S be square of unit area & ABCD be quadrilateral of sides a, b, c & d

$a^2 = (1 - x)^2 + z^2$

$b^2 = w^2 + (1 - z)^2$

$c^2 = (1 - w)^2 + (1 - y)^2$

$d^2 = x^2 + y^2$

$a^2 + b^2 + c^2 + d^2 = \{x^2 + (1 - x)^2\} + \{y^2 + (1 - y)^2\} + \{z^2 + (1 - z)^2\}$
$+ \{w^2 + (1 - w)^2\}$

where $0 \leq x, y, z, w \leq 1$

Let $f(x) = x^2 + (1 - x)^2$, $0 \leq x \leq 1$

Then $f'(x) = 2x - 2(1 - x)$

$f'(x) = 0$ for min/max.

$\Rightarrow 4x - 2 = 0 \quad \Rightarrow x = \frac{1}{2}$

Again $f''(x) = 4 > 0$ when $x = \frac{1}{2}$

$\therefore$ f(x) is min. at $x = \frac{1}{2}$ & m,ax. at x = 1

$a^2 + b^2 + c^2 + d^2 = 4\{x^2 + (1 - x)^2\}$

Max. value of $x^2 + (1 - x)^2 = 1^2 + (1 - 1)^2 = 1$

Min. value of $x^2 + (1 - x)^2 = \left(\frac{1}{2}\right)^2 + \left(1 - \frac{1}{2}\right)^2 = \frac{1}{2}$

$$\boxed{2 \leq a^2 + b^2 + c^2 + d^2 \leq 4}$$

Q.2. Let $a + b = 4$ & $a < 2$ & let g(x) be differentiable function. If $\frac{dg}{dx} > 0 \ \forall\ x$ prove that $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as (b - a) increases.

Ans: Let $(b - a) = t$ & since $a + b = 4$

$\Rightarrow a = \frac{4 - t}{2}$ & $b = \frac{4 + t}{2}$ & $t > 0$

[<u>Dumb Question</u>: Why $t = b - a > 0$?

Ans: Since $a < 2$ &n & $a + b = 4$. So, $b > 2$

$\therefore$ $b - a > 0$ or $t > 0$]

$$\int_0^4 g(x)\,dx + \int_0^b g(x)\,dx = \varnothing(t)$$

$$\therefore\ \varnothing(t) = \int_0^{4 - \frac{t}{2}} g(x)\,dx + \int_0^{\frac{4+t}{2}} g(x)\,dx$$

$$\varnothing(t) = g\left(\frac{4 - t}{2}\right)\left(-\frac{1}{2}\right) + g\left(\frac{4 + t}{2}\right)\left(\frac{1}{2}\right)$$ [By Leibniz rule]

[<u>Dumb Question</u>: What is Leibniz rule ?

Ans: $\varnothing(t) = \int_0^{f(x)} g(x)\,dx$]

$\varnothing'(t) = g(f(x))\,f'(x)$

Since g(x) is increasing

$x_2 > x_1 \Rightarrow g(x_2) > g(x_1)$

Now, $\frac{4 + t}{2} > \frac{4 - t}{2}$ & g(x) is increasing

$$\therefore\ g\left(\frac{4 + t}{2}\right) > \left(\frac{4 - t}{2}\right)$$

$\therefore$ Ø'(t) = $\frac{1}{2}\left[g\left(\frac{4+t}{2}\right) \cdot g\left(\frac{4-t}{2}\right)\right] > 0$

$\Rightarrow$Ø'(t) > 0

$\Rightarrow$Ø'(t) is increases as t increases.

Q.3. Tangent represented by graph of function y = ... at the point with absciss a x = 1 from an angle $\pi/6$ & at point x = 2 an angle of $\pi/3$ & at the point x an angle of $\pi/4$. Find value of

$\int_1^3 f'(x)\, f''(x)\, dx + \int_2^3 f''(x)\, dx$.

Ans: Given

at x = 1, $\frac{dy}{dx} = \tan \pi/6 = \frac{1}{\sqrt{3}}$

at x = 1, $\Rightarrow f'(1) = \tan \pi/6 = \frac{1}{\sqrt{3}}$

at x = 2 $\Rightarrow f'(2) = \tan \pi/3 = \sqrt{3}$

at x = 3 $\Rightarrow f'(3) = \tan \pi/4 = 1$

Then, $\int_1^3 f'(x)\, f''(x)\, dx + \int_2^3 f''(x)\, dx$

Let f'(x) = t

f''(x) dx = dt

$\int_{f(1)}^{f(3)} t\, dt + f'(x)\,|_2^3$

$\Rightarrow \frac{1}{2} t^2 \,|_{f(1)}^{f(3)} + f'(x)\,|_2^3$

$$\Rightarrow \frac{1}{2}\left[\left(f'(3)\right)^2 - \left(f'(1)\right)^2\right] + f'(3) - f'(2)$$

$$\Rightarrow \frac{1}{2}\left[1^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right] + \left(1 - \sqrt{3}\right)$$

$$\Rightarrow \frac{1}{2}\left[\frac{2}{3}\right] + 1 - \sqrt{3} \quad \Rightarrow 1 + \frac{1}{3} - \sqrt{3}$$

$$\Rightarrow \frac{4}{3} - \sqrt{3}$$

<u>Key Words</u>:

* Derivative.
* TAngent.
* Normal.
* Orthogonal Curves.
* Sub Tangent.
* Sub Normal.
* Rolle's Theorem.
* Layrange's Mean Value Theorem.
* Maxima.
* Minima.
* Monotonicity.

Indefinite Integration

Indefinite Integration

Basic Concept

Let $F(x)$ be a differentiable function of x such that $\frac{d}{dx}[F(x)] = f(x)$.
Then $F(x)$ is called the integral of $f(x)$. Symbiotically, it is written as
$\int f(x)dx = F(x)$.
$f(x)$, the function to be integrated is called the integrand.
$F(x)$ is also called the anti-derivate (or primitive function) of $f(x)$.

Constant of Integration:
As the differential coefficient of a constant is zero, we have

$$\frac{d}{dx}(F(x)) = f(x)$$

$$\Rightarrow \frac{d}{dx}[F(x) + c] = f(x).$$

Therefore, $\int f(x)\,dx = F(x) + c$.

This constant c is called the constant of integration and can take any real value.

Properties of Indefinite Integration

(*i*) $\int af(x)\,dx = a\int f(x)\,dx$ (Here 'a' is a constant)

(*ii*) $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

(*iii*) If $\int f(u)\,du = F(u) + c$, then $\int f(ax + b)\,dx = \frac{1}{a}F(ax + b) + c$, $a \neq 0$.

Basic Formulae

1. If $K \in R$, $\int K dx = Kx + C$

2. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

3. $\int \frac{1}{x} \, dx = \log |x| + C$

4. $\int e^x \, dx = e^x + C$

5. $\int a^x \, dx = \frac{a^x}{\log a} + C$ (for $a > 0$, $a \neq 1$)

6. $\int \text{Sinx} \, dx = -\text{Cosx} + C$

7. $\int \text{Cossdx} = \text{Sinx} + C$

8. $\int \text{tanxdx} = \log |\text{Secx}| + C$ or $= -\log|\text{Cosx}| + C$

9. $\int \text{Cotxdx} = \log|\text{Sinx}| + C$

10. $\int \text{Secxdx} = \log|\text{Secx} + \text{tanx}| + C$

$$= \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + C$$

11. $\int \text{Cosxdx} = \log|\text{Cosecx} - \text{Cotx}| + C$

$$= \log \left| \tan \frac{x}{2} \right| + C$$

$= -\log|\text{Cosecx} + \text{Cotx}| + C$

12. $\int \text{Sec}x \tan x \, dx = \text{Sec}x + C$

13. $\int \text{Cosec } x \cot x \, dx = -\text{Cosec}x + C$

14. $\int \text{Sec}^2 x \, dx = \tan x + C$

15. $\int \text{Cosec}^2 x \, dx = -\text{Cot}x + C$

16. $\int \sinh x \, dx = \text{Cosh } x + C$

17. $\int \text{Cosh } x \, dx = \text{Sinh } x + C$

18. $\int \tanh x \, dx = \log|\text{Cosh } x| + C$

19. $\int \coth x \, dx = \log|\text{Sinh } x| + C$

20. $\int \text{Sech } x \, dx = 2\tan^{-1}(e^x) + C$

21. $\int \text{Co sech } x \, dx = \log\left|\tanh \frac{x}{2}\right| + C$

22. $\int \text{Sech}^2 x \, dx = \tanh x + C$

23. $\int \text{Co sech}^2 x \, dx = -\text{Coth } x + C$

24. $\int \text{Sech } x \tanh x \, dx = -\text{Sech } x + C$

25. $\int \text{Cosech } x \text{ Coth } x \, dx = -\text{Cosech } x + C$

26. $\int \frac{dx}{\sqrt{1 - x^2}} = \text{Sin}^{-1} x + C$ or $-\cos^{-1} x + C$

27. $\int \frac{dx}{\sqrt{1 + x^2}} \tan^{-1} x + C$ or $-\cot^{-1} x + C$

28. $\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + C$ or $-\text{cosec}^{-1} x + C$

29. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ or $-\cos^{-1}\left(\frac{x}{a}\right) + C$

30. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \text{Cosh}^{-1}\left(\frac{x}{a}\right) + c$ (or) $\log\left|x + \sqrt{x^2 - a^2}\right| + c$

31. $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \text{Sinh}^{-1}\left(\frac{x}{a}\right) + c$ (or) $\log\left|x + \sqrt{x^2 + a^2}\right| + c$

32. $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

33. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a + x}{a - x}\right| + c$

34. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right| + c$

35. $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$

36. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + c$

$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log|x + \sqrt{x^2 - a^2}| + c$

37. $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + c$

$= \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2 + a^2}| + c$

Method of Integration:
If the integrand is not a derivative of a known function, then the corresponding integrals cannot be found directly. In order to find the integral of complex problems, generally three rules of integration are used.

- Integration by substitution or by change of the independent variable.
- Integration by parts.
- Integration by partial fractions.

Integration by substitution

There are following types of substitutions

- **Direct Substitution**

— If integral is of the form , then put

$g(x) = t$

, provided exists

e.g. Evaluate $\int \frac{\sin(\ln x)}{x} dx$

Sol. Let $\ln x = t$

Then $dt = \frac{1}{x}$

Hence $I = \int \sin t \, dt = -\cos t + c$

- **Standard Substitutions**

- For terms of the form $x2 + a2$ or $\sqrt{x^2 + a^2}$, put $x = a \tan \Theta$ or $a \cot \Theta$
- For terms of the form $x2 - a2$ or $\sqrt{x^2 - a^2}$, put $x = a \sec \Theta$ or a cosec Θ
- For terms of the form $a2 - x2$ or $\sqrt{a^2 - x^2}$, put $x = a \sin \Theta$ or $a \cos \Theta$
- If both $\sqrt{a + x}$, $\sqrt{a - x}$ are present then put $x = a \cos \Theta$
- For the type $\sqrt{(x - a)(b - x)}$, put $x = a \cos^2 \Theta + b \sin^2 \Theta$
- For the type $\left(\sqrt{x^2 + a^2} + x\right)$' or $\left(x + \sqrt{x^2 - a^2}\right)$' 'put the expression within the bracket = t
- For the type $(x + a)^{-1 - 1/n} (x + b)^{-1 + 1/n}$ or $\left(\frac{x + b}{x + a}\right)^{1/n - 1} \frac{1}{(x + a)^2}$ (n □ N, $n > 1$), put
- For $\frac{1}{(x + a)^{n_1} (x + b)^{n_2}}$, $n_1, n_2 \in N$ (and > 1), again put $(x + a) = t(x + b)$

- **Indirect Substitution**

— If the integrand is of the form

$F(x)g(x)$, where

$g(x)$ is a function of the integral of $f(x)$, then put integral of $f(x) = t$

.

e.g. Evaluate

$$\int \frac{\sqrt{x}}{\sqrt{x^3 + a^3}}\, dx$$

Sol. Integral of the numerator = $\frac{x^{3/2}}{3/2}$

Put

$x^{3/2}$

$= t$

We get

$$l = \frac{2}{3}\int \frac{dt}{\sqrt{t^2 + a^3}} = \frac{2}{3}\ln\left| x^{3/2} + \sqrt{x^3 + a^3} \right| + c$$

- **Derived Substitution:**

— Some time it is useful to write the integral as a sum of two related integrals which can be evaluated by making suitable substitutions.

Examples of such integrals are:

A. Algebraic Twins

$$\int \frac{2x^2}{x^4+1}\,dx = \int \frac{x^2+1}{x^4+1}\,dx + \int \frac{x^2-1}{x^4+1}\,dx$$

$$\int \frac{2}{x^4+1}\,dx = \int \frac{x^2+1}{x^4+1}\,dx - \int \frac{x^2-1}{x^4+1}\,dx$$

$$\int \frac{2x^2}{\left(x^4+1+kx^2\right)}\,dx, \quad \int \frac{2}{\left(x^4+1+kx^2\right)}\,dx$$

Method:

* Make the integration in the form

$$\int \left(1+\frac{1}{x^2}\right) f\left(x-\frac{1}{x}\right) dx$$

or

$$\int \left(1-\frac{1}{x^2}\right) f\left(x+\frac{1}{x}\right) dx$$

*If

$$\left(1+\frac{1}{x^2}\right)$$

is present then put

$$x-\frac{1}{x}=t \;\Rightarrow\; \left(1+\frac{1}{x^2}\right)dx = dt$$

If

$$1-\frac{1}{x^2}$$

is present then put

$$x+\frac{1}{x}=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx=dt$$

B. Trigonometric twins

$$\int\sqrt{\tan x}\,dx, \int\sqrt{\cot x}\,dx$$

$$\int\frac{1}{\left(\sin^4 x+\cos^4 x\right)}dx, \int\frac{1}{\left(\sin^6 x+\cos^6 x\right)}dx$$

$$\int\frac{\pm\sin x\pm\cos x}{a+b\sin x\cos x}dx$$

Integration by Parts

1. If

u and v be two function of x

, then integral of product of these two functions is given by

$$\int u.v dx = u\int v\,dx-\int\left[\frac{du}{dx}\int v dx\right]dx$$

Note:

In applying the above rule care has to be takne in the selection of the first function (u) and the second function (v). Normally we use the following methods:

(i) If in the product of two functions, one of the functions in not directly integrable (

e.g.

$\sin^{-1}x$

, $\cos^{-1}x$

, $\tan^{-1}x$

etc.) then we take it as the first function and the remaining function is taken as the second function

e.g.

In the integration

$\int x \tan^{-1}x dx$, $\tan^{-1}x$ is taken as the first function and x

as the second function.

(ii) If there is no other function, then unity is taken as the second function

e.g.

In the integration of

$\int \tan^{-1}x dx$, $\tan^{-1}$ is taken as the first function and 1 as the second function.

(*iii*) If both of the function are directly integrable then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable. Usually we use the following preference order for the first function (inverse, Logarithmic, Algebraic, Trigonometric, Exponential)

In the aove stated order, the function on the left is always chosen as the first function. This rule is called as ILATE

e.g.

In the integration of

$\int$ x sin x dx, x is taken as the first function and sin*x* is taken as the second function.

Important Result:
*

In the integral , if

$g(x)$ ex dx, if g(x) cna be expressed as $g(x) = f(x) + f'(x)$

then

$$\int e^x \left[f(x) + g(x) \right] dx = e^x f(x) + c$$

Some times to solve integral of the form

$$\int \frac{f(x)}{\{p(x)\}^2} dx$$

we write it as

$$\int \frac{f(x)}{p'(x)} \cdot \frac{p'(x)}{\{p(x)\}^2} dx$$

and solve the integral with the help of integration by parts, taking

$$\frac{f(x)}{p'(x)}$$

as the first function.

e.g.

$$\int \frac{x^2}{(x\sin x + \cos x)^2} dx$$

is solved by writting it as

$$\int \underset{(i)}{(x\sec x)} \underset{(ii)}{\frac{(x\cos x)}{(x\sin x + \cos x)^2}} dx$$

and this integral can be solved by parts.

Algebraic Integrals

I. Integral of the form

$$\int \frac{px+q}{(ax^2+bx+c} dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx, \int (px+q)\sqrt{ax^2+bx+c} dx$$

In these types of integrals we write

$px + q = l$

(diff. coefficient of $ax^2 + bx + c$) + m

Find l and m

by comparing the coefficient of x

and constant term on both sides of the identify. In this way the question will reduce the sum of two integrals which can be integrated easily.

Integral of the type

$$\frac{ax^2 + bx + c}{px^2 + qx + r} \text{ or } \frac{ax^2 + bx + c}{\sqrt{px^2 + qx + r}}$$

In this case substitute

$ax^2 + bx + c = M(px^2 + qx + r) + N$

$(2px + q) + R$

Find M, N and R

. The integration reduces to integration of three independent functions.

II. Integration of Irrational Algebraic Fractions

1. Rational fucntion of $(x + b)^{1/n}$

and x

can be easily evaluated by the substitution

$t^n = ax + b$

. Thus

$$\int f(x), (ax+b)^{1/n})dx] \int f\left(\frac{t^n - b}{a}, t\right)\frac{nt^{n-1}}{a}dt.$$

2. In the integration of

$$\frac{1}{(x-k)}\frac{1}{\sqrt{ax^2+bx+c}},$$

the substitution

$x - k = 1/t$

reduces the integration

$$\frac{1}{(x-k)\sqrt{ax^2+bx+c}}$$

to the problem of integrating an expression of the form

$$\frac{1}{\sqrt{At^2+Bt+C}}.$$

3. .

$$\int \frac{dx}{(x-k)^r\sqrt{ax^2+bx+c}}$$

Here we substitute,

$x - k = 1/t$

.

This substitution will reduce the given integral to

$$\int \frac{t^{r-1}dt}{\sqrt{At^2 + Bt + C}}.$$

4. To integrate

$$\frac{1}{(Ax^2 + B)\sqrt{Cx^2 + D}},$$

we first put

$x = 1/t$, so that

$$\int \frac{dx}{(Ax^2 + B)\sqrt{Cx^2 + D}} = \int \frac{-1/t^2 dt}{(A/t^2 + B)\sqrt{C/t^2 + D}} = \int \frac{tdt}{(A + Bt^2)\sqrt{C + Dt^2}}.$$

Now the substitution

$C + Dt^2 = u^2$

reduces it to the form

$$\int \frac{du}{u^2 \pm a^2}.$$

III. Integration of the function of the type

,

$$\int x^m (a + bx^n)^p \, dx$$

Where m, n, p are rational numbers

This integral is expressed through elementary functions only if one of the following conditions is fulfilled:

(1) If p is an integer,

(2) If

$$\frac{m+1}{n}$$

is an intger,

(3) If

$$\frac{m+1}{n}$$

+ p is an integer.

1st case :

(a) If p is a positive integer, remove the brackets (a + bxn)p according to the Newton binomial and calculate the integrals of powers.

(b) If p is a negative integer, then the substitution x = tk, where k is the common denominator of the fractions m and n, leads to the integral of a rational fraction;

2nd case :

If

$$\frac{m+1}{n}$$

is an integer, then the substitution a + bx

n

= t

k

is applied, where k is the denominator of the fraction p;

3rd case :

If

$$\frac{m+1}{n}$$

+ p is an integer, then the substitution a + bx

n

= xnt

k

is applied, where k is the denominator of the fraction p.

Example :

(i)

$$I = \int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\, dx$$

(ii)

$$I = \int x^{-11}(1+x^4)^{-\frac{1}{2}}\, dx$$

(iii)

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx$$

(iv)

$$\frac{dx}{\sqrt[4]{1+x^4}}$$

Trignometric Integrals

I. Integral of the form $\int R(\sin x, \cos x)\,dx$

Universal substitution tan .

$$\frac{x}{2} = t$$

In this case sin

x

$=$

$$\frac{2t}{1+t^2}$$

; cos

x

$=$

$\frac{1-t^2}{1+t^2}$

,

x

= 2 tan- 1

t

;

dx

$=$

$\frac{2dt}{1+t^2}$

If R (- sin x, cos x) = - R (sin x, cos x), then the substitution cos x = t is applied.

If R (sin x, - cos x) = - R (sin x, cos x), then the substitution sin x = t is applied.

If R (- sin x, - cos x) = R (sin x, cos x), then the substitution tan x = t is applied.

II. Integral of the form

(i)

$$\int \frac{A\sin x}{B\sin x + C\cos x}\,dx$$

(ii)

$$\int \frac{A\cos x}{B\sin x + C\cos x}$$

dx

(iii)

$$\int \frac{A\sin x + B\cos x}{C\sin x + D\cos x}$$

dx

(iv)

$$\int \frac{A\sin x + B\cos x + C}{D\sin x + E\cos x + F}\,dx$$

This type of integration can be solved by converting the Nr in the form Nr = P(Dr) + Q + R the value of P, Q, R can be findout by comparing the coeffcient of both sides.

III. Integral of the form

(i)

$$\int \sin ax \sin bx\,dx$$

(ii)

$$\int \sin ax \cos bx dx$$

(iii)

$$\int \cos ax \cos bx dx$$

Transform the product of trigonometric function into a sum or difference, using one of the following formulas:

sin ax sin bx = [cos(a-b) x - cos (a+b)x]

cos ax cos bx = [cos (a - b) x + cos (a + b) x]

sin ax cos bx = [sin (a - b) x + sin (a + b) x]

IV. Integral of the form

$$\int \sin^m x \cos^n x dx$$

, where m and n are integers.

(i) If m is an odd positive number, then apply the substitution cos x = t.

(ii) If n is an odd positive number, apply the substitution sin x = t.

(iii) If m and n are even non-negative numbers, use the formulas

$$\sin^2 x = \frac{1-\cos 2x}{2}; \cos^2 x = \frac{1+\cos 2x}{2}$$

(iv)

$$\int \sin^p x \cos^q x \, dx$$

where (o <

x

< ?/2) and

p

and

q

are rational numbers.

Substitute sin

x

= t

$$\int \sin^p x \cos^q x\,dx = \int t^p (1-t^2)^{q-1}\,dt$$

V. Integral of the form

–

$$\int \frac{dx}{a\sin x + b\cos x}$$

$$= \frac{1}{\sqrt{a^2+b^2}} \int \operatorname{cosec}(x \pm \theta)\,dx$$ **when** $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

VI. Integral of the form

(i)

$$\int \frac{dx}{a + b\sin^2 x}$$

(ii)

$$\int \frac{dx}{a + b\cos^2 x}$$

(iii)

$$\int \frac{dx}{a\sin^2 x + b\cos^2 x + c\sin x\cos x}$$

(iv)

$$\int \frac{dx}{a\sin^2 x + b\cos^2 x}$$

This type of integration can be solved by multiplying $\sec^2 x$, in N_r and D_r and substituting $\tan x = t$, or $\cot x = t$.

VII. Integral of the form

(i)

$$\int \frac{dx}{a + b\sin x}$$

(ii)

$$\int \frac{dx}{a + b\cos x}$$

(iii)

$$\int \frac{dx}{a\sin x + b\cos x + c}$$

To solve this type of integration

1) convert sinx and cosx in terms of tan x/2 by putting.

$$\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}, \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$$

2) Write N

r

in the form sec

2

x/2 and Dr in the form tan x/2.

3) Substitute tan x/2 = t, so that sec

2

x/2 dx = 2dt

VIII.

If D

r

is in the form K + L sin x cos x, then Nr must be in the form of sin x + cos x, or sin x - cos x.

(1) If N

r

has sin*x*+ cos*x*

then substitute sin

$x - \cos x = t$

$\Rightarrow$

$(\cos x + \sin x)dx = dt$

(2) Nr

has $\sin x - \cos x$

then substitute sin

$x + \cos x = t$

$\Rightarrow$

$(\cos x - \sin x)dx = dt$

Note: If $\sin x - \cos x =$

$t \Rightarrow \quad 1 - \sin 2x = t^2$

$\Rightarrow \quad \sin 2x = 1 - t^2$ If $\sin x + \cos x = t$

$\Rightarrow \quad 1 + \sin 2x = t^2$

$\Rightarrow \quad \sin 2x = t^2 - 1$

INTEGRATION BY Partial fraction

Let

$\frac{P(x)}{Q(x)}, \; Q(x) \neq 0$

is a proper algebric function.

The partial fractions depend on the nature of the factors of $Q(x)$. We have deal with the following different type when the factors of

$Q(x)$ are

(i) Linear and non-repeated

(ii) Linear and repeated

(iii) Quadratic and non-repeated

(iv) Quadratic and repeated

Case I :

When denominator is expressible as the product of non-repeated linear factors :

Let $Q(x) = (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n)$.

Then we assume that ;

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \frac{A_3}{(x-a_3)} + \ldots + \frac{A_n}{(x-a_n)}$$

where A_1, A_2, ..., A_n

are constants and can be determined by equating numerator on R.H.S to numerator on L.H.S. and then substituting $x = a_1, a_2, \ldots a_n$,

Case II :

When the denominator

$Q(x)$ is expressible as the product of the linear factors such that some of them are repeating. (Linear and Repeated)

Let, $Q(x) = (x-a)^k (x-a_1)(x-a_2) \ldots (x-a_r)$. Then we assume that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \ldots \frac{A_k}{(x-a)^k} + \frac{B_1}{(x-a_1)} + \frac{B_2}{(x-a_2)} + \ldots + \frac{B_r}{(x-a_r)}$$

Case III :

When some of the factors in denominator are quadratic but non-repeating.

Corresponding to each quadratic factor $ax^2 + bx + c$, we assume the partial fraction of the type

$$\frac{Ax+B}{ax^2+bx+c}$$

, where A and B are constants to be determined by comparing coefficients of similar powers of x in numerator of both sides.

Case IV :

When some of the factors of the denominator are quadratic and repeating. For every quadratic repeating factor of the type

$(ax^2 + bx + c)^k$

, we assume :

$$\frac{A_1x + A_2}{ax^2 + bx + c} + \frac{A_3x + A_4}{(ax^2 + bx + c)^2} + \ldots + \frac{A_{2k-1}x + A_{2k}}{(ax^2 + bx + c)^k}$$

Short cut Method of Finding the Constant of a Non-repeated Linear Factor in Denominator

Let

$$\frac{f(x)}{(x-a_1)(x-a_2)\ldots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots + \frac{A_n}{x-a_n}$$

$$\therefore \quad A_1 = \lim_{x \to a_1} (x-a_1)\left[\frac{f(x)}{(x-a_1)(x-a_2)\ldots(x-a_n)}\right]$$

$$= \frac{f(a)}{\prod_{i=2}^{n}(a_1 - a_i)}$$

$$A_2 = \lim_{x \to a2} (x - a_2)\left[\frac{f(x)}{(x-a_1)(x-a_2)....(x-a_n)}\right]$$

$$\frac{f(x)}{\prod_{i=1}^{n}(a_2 - a_i) i \neq 2}$$

$$A_n = \lim_{x \to a_n} (x - a_n)\left[\frac{f(x)}{(x-a_1)(x-a_2)....(x-a_n)}\right]$$

$$\frac{f(a_n)}{\prod_{i=1}^{n-1}(a_n - a_i)}$$

Definite Integrals

1. If f and F are two continuous functions defind on [a, b] such that

$$\frac{d}{dx}[F(x)] = f(x).$$

then the number

F(b) - F(a) is called definite integration of f between a and b and is denoted by

$$\int_a^b f(x)dx$$

i.e.,

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

This is called fundamental theorem of integral calculus. Here 'a' is called lower limit (LL) and 'b' is upper limit (U.L) and

$a \leq b$

always.

Also

$$\int_a^b f(x)\, dx$$

denotes algebric sum of the area bounded by curve

$y = f(x)$, ordinates

$x = a, x = b$

and

x - axis.

2.

$$\int_a^b f(x)dx$$

is always unique.

- $\int_a^b f(x)dx$ is also defined as an infinite limit sum..

4.

$$\int_a^b f(x)dx = \int_a^b f(\theta)d\theta = \int_a^b f(t)dt = ...$$

Properties of Definite Integration:

1.

$$\int_a^a f(x)dx = 0$$

2.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

3.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

where a < c < b

4.

$$f(x) = g(x) \Rightarrow \int_a^b f(x)dx = \int_a^b g(x)dx$$

but converse need not be true

5.

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

6.

$$\int_a^b k\, f(x)\, dx = k\int_a^b f(x)dx$$

7.

$$\int_a^b 0\, dx = k$$

8.

$$\int_a^b k\, dx = k(b-a)$$

9.

$$\left|\int_a^b f(x)\, dx\right| \le \int_a^b |f(x)|dx$$

10.

$$f(x) \ge 0 \Rightarrow \int_a^b f(x)dx \ge 0$$

If but converse need not be true.

11.

$$f(x) \geq g(x) \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

If but converse need not be true.

12.

$$\int_a^b f[g(x)]g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx$$

(change limit Theorem)

13.

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

14.

$$\int_{-a}^a f(x)dx = \begin{cases} 0, & \text{if } f \text{ is odd} i.e. f(-x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{if } f \text{ is even } f(-x) = f(x) \end{cases}$$

15.

$$\int_0^a f(x)dx = \begin{cases} 0, & \text{if } f(a-x) = -f(x) \\ 2\int_0^{a/2} f(x)dx, & \text{if } f(a-x) = f(x) \end{cases}$$

16.

$$\int_0^a f(x)dx = \int_0^{a/2} f(x)dx + \int_0^{a/2} f(a-x)dx$$

17.

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

18.

$$\frac{d}{dx}\int_{\phi(x)}^{\Psi(x)} f(t)dt = f[\Psi(x)]\Psi'(x) - f[\phi(x)].\phi'(x)$$

This is called

Leibnitz rule

.19.

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

where m and M are respectively the minimum and maximum values of

$f(x)$ in $[a, b]$

20. If f is a periodic function with period

T

, i.e., if $f(x + T) = f(x)$ then

a)

$$\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$$

b)

$$\int_{mT}^{nT} f(x)dx = (n-m)\int_{0}^{T} f(x)dx$$

c)

$$\int_{a+mT}^{b+nT} f(x)dx = \int_{0}^{T} f(x)dx$$

where m, n are integers

d)

$$\int_{a}^{a+T} f(x)dx = \int_{0}^{T} f(x)dx$$

i.e., it is not dependent on 'a'

21.

$$\left|\int_{a}^{b} f(x)g(x)dx\right| \le \sqrt{\left(\int_{a}^{b} f^2(x)dx\right)\left(\int_{a}^{b} g^2(x)dx\right)}$$

This is called cauchy - schwartz inequality

WALLI'S FORMULAE :

22.

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \begin{cases} \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \ldots\ldots \times \frac{1}{2} \times \frac{\pi}{2}\ ;\ if\ \ n\ is\ even \\ \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \ldots\ldots \times \frac{2}{3} \times 1\ ;\ if\ \ n\ is\ odd \end{cases}$$

23. $$I_{m,n} = \int_0^{\pi/2} \sin^m x.\cos^n x\,dx$$

Case (1)

: If m is odd and n is either even or odd

$$I_{m,n} = \frac{m-1}{m+n} \times \frac{m-3}{m+n-2} \times \frac{m-5}{m+n-4} \times \ldots\ldots \times \frac{2}{n+3} \times \frac{1}{n+1}$$

Case (2)

: If m is even and n is even

$$I_{m,n} = \frac{m-1}{m+n} \times \frac{m-3}{m+n-2} \times \frac{m-5}{m+n-4} \times \ldots\ldots \times \frac{1}{n+2} \times \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \ldots\ldots \times \frac{1}{2} \times \frac{\pi}{2}$$

Case (3)

: If m is even and n is odd.

$$I_{m,n} = \frac{m-1}{m+n} \times \frac{m-3}{m+n-2} \times \frac{m-5}{m+n-4} \times \ldots\ldots \times \frac{1}{n+2} \times \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \ldots\ldots \times \frac{2}{3} \times 1$$

Reduction Formulae on Definite Integration

:

24. If

$$I_n = \int_0^{\pi/4} \tan^n x\,dx$$

then

$$I_n + I_{n-1} = \frac{1}{n-1}$$

(where n > 2)

25. If

$$I_n = \int_0^{\pi/4} \sec^n x\,dx$$

then

$$I_n = \frac{\left(\sqrt{2}\right)^{n-2}}{n-1} + \left(\frac{n-2}{n+1}\right) I_{n-2}$$

26. If

$$I_n = \int_{\pi/2}^{\pi/4} \cot^n x\,dx$$

then

$$I_n + I_{n-2} = \frac{1}{1-n}$$

27. If

$$I_n = \int_{\pi/6}^{\pi/4} \operatorname{cosec}^n x\,dx$$

then

$$I_n = \frac{2^{n-2}\sqrt{3}}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

28.

By parts Formulae on Definite Integration :

$$\int_a^b f(x)\,g(x)\,dx = \left[f(x)\int g(x)dx \right]_a^b - \int_a^b \left[f^{|}(x)\int g(x)dx \right] dx$$

29. If a function has finite number of points of discontinuties in [a, b] then the function is definite integrable in that interval.

30.

Definite integration as infinite limit sum :

1)

$$\underset{n\to\infty}{Lt}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)\frac{1}{n}=\int_{0}^{1} f(x)dx$$

2)

Working rule :

Step I

: First reduce the given infinite limt into the form

$$\underset{n\to\infty}{Lt}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)\frac{1}{n}$$

Step II

: Replace r/n with x and 1/n with dx

Step III

: Replace

$$\underset{n\to\infty}{Lt}\sum_{r=1}^{n} by\int_{0}^{1}$$

$$\underset{n\to\infty}{Lt}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)\frac{1}{n}=\int_{0}^{1} f(x)dx$$

Note :

1)

$$\underset{n\to\infty}{Lt}\sum_{r=0}^{n} f\left(\frac{r}{n}\right)\frac{1}{n}=\int_{0}^{1} f(x)dx$$

2)

$$\underset{n\to\infty}{Lt}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\frac{1}{n}=\int_{0}^{1} f(x)dx$$

3)

$$\underset{n\to\infty}{Lt}\sum_{r=1}^{kn} f\left(\frac{r}{n}\right)\frac{1}{n}=\int_{0}^{k} f(x)dx$$

Matrices

Matrices and determinants are very nice ways to vepresent bunch of number. matrices as well as determinants dlong with some very nice properties that each of then have, prove to be very useful in solving equations, finding area of polygons etc. Matrices a also find use in Linear Algebra which is beyond the scope of JEE syllabus.
So, Let us study these in details.

Matrices And Determinants

MATRICES
A rectangular away of elements or symbols.along sows and wlumns in colled a matricx .

Equal MAtrices :- Two matrices are eqnal if they have the same order and each element of ove is equal to the corresponding of the other.

Classification (Secondaty Information):-

Row Matrix

A mstrix lhaving a single row .

Column Matrix

A matrix with a single column .

Square Matrix

An m X n matrix is said to be a square matrix if m = n i.e. no. of rows = no. o of columns.

eg: A - $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}_{3 \times 3}$

Trace of a Matrix

The sum of the elements of O square matrix A lying along the principal diagonal .

Properties of trace of a matrix(Secondary Information) :-

(i) tr (λ A) = λ tr (A)

(ii) tr (A + B) = tr (A) + tr (B)

(iii) tr (A B) = tr (B A)

Diagonal Matrix (Secondary Information) :-

A square matrix with all the elemente O except the diagonal elements.

Scalar Matrix:- A diagonal matrix whose all the leading elements are equal.

Unit or Identify Matrix:-

A diagonal matrix of order n which has only unity for all its diagonal elements. is called unit or Identify matrix of order n and is denoted by In.

Triangnlar Matrix (Secondary Information):

A square matrix in which all the elements below the diagonal are zero is called upper triangnlar and square matrix in which all the elements above diagonal are zero is called lower triangnlar matrix
Null Matrix :-

MAtrix with all elements O.

Transpose of a Matrix:- The matrix obtaired from a given matrix A, by inter changing rows and columns is called transpose of A and is denoted by a^T . or A^1.

Properties of Transpose (Secondary In formation):-

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(\alpha A)^T = \alpha . A^T$

(iv) $(A\,B)^T = B^T\,A^T$.

<u>Conjugate of a Matrix</u>:- A matrix obtained from any given matrix A coutaining complex number as its elements, or replaing its elements by the corresponding conjugate complex no is called conjugate of A and is denoted by $\bar{A}$.

<u>Properties of conjugate (Secondary Information</u>:-

(i) $\overline{(\bar{A})} = A$

(ii) $\overline{(A+B)} = \bar{A} + \bar{B}$

(iii) $\overline{(\alpha\,A)} = \bar{\alpha}\;\bar{A}$

(iv) $\overline{(A+B)} = \bar{A}\;\bar{B}$

<u>Transpose Conjugate of a Matrix</u>:-

The transpose of conjugate of a matrix denoted by A^θ.

<u>Properties of Transpose of conjugate</u>:-

(Secondary Information):-

(i) $(A^\theta)^\theta = A$

(ii) $(A+B)^\theta = A^\theta + B^\theta$

(iii) $(KA)^\theta = \bar{K}\,A^\theta$

(iv) $(A\,B)^\theta = B^\theta\,A^\theta$

<u>ALGEBRA OF MATRICES</u>:-

<u>Addition and Subtraction</u>:- Any two matrices can be added lif they are of same order and the (or Subtrected)

resulting matrix is of same order, corresponding elements are added or subtracted .

Scalar Multiplication:-

The matrix obtained by multiplleying every element of by a scalar α is called the scalar multiple of A by α and is denoted by α A .

Multipliecation of Matrices:-

Two matrices can be multiplied lonly when the no of columns in the first is equal to the no . of rows inm the second. Such matrices are called conformable for multiplication.

If A B = c
A - $[a_{ij}]_{mXn}$ B - $[b_{ke}]_{nXp}$

$$C_{ij}^{2} \sum_{k=1}^{n} a_{ik} b_{kj}$$

Special Matrices (Secondary Information:-

Symmetric and Skew Symmetric Matrices:-

A square matriex is A is said to be symmetric if
$A = A^T$.

and skew symmetric if $A = - A^T$.

Unitary Matrix

A matrix is unitary if $\overline{A^T}$ A = 1
Hermition and skew - Harmition Matrix :-

A square matrix is said to mermition if A = A^{θ} .
and skew Hermition if A^{θ} = - A .

Singular Matrix :- Ahy square matrix A is singular if | A | = O .

Orthogonal Matrix :- Any squrae matrix A of order n is said to be orthogonal if $A A^T = A^T A = I_n$.

Idempotent Matrix :- A square matrix is called idmpotent provided it satisfies the selation $A^2 = A$.

In volutary Matrix :-

A matrix such $A^2 = I$.

Nilpotent Matrix

A square matrix such that $A^m = O$ where m is Q posetive integer .

Adjoint of a Square Matrix
Let A be a square matrix of order n and let C_{ij} ;be cofactor of a_{ij} in A. Then the transpose of the matrix of cofacturs of elements of A is called the adjoint of A and is denoted by adj A .

We have
$A (\text{adj } A) = | A | I_n = (\text{adj } A) A$

Onverse of a Matrix
A no n - singular square matrix of ordere n is invertible if there exists a square matrix B of the same order such that $AB = I_n = BA$.

$A^{-1} = B$

$A^{-1} = \frac{1}{|A|} \text{adj } A$.

properties of inverse of a matrix :-

(Secondary Information).

(i) $(A B)^{-1} = B^{-1} A^{-1}$

(ii)$(ABC \ldots\ldots\ldots)^{-1} = \ldots\ldots\ldots C^{-1} B^{-1} A^{-1}$

(iii) $(A^T)^{-1} = (A^{-1})^T$

(iv) A is a no n - singular matrix of order n . There $|\text{adj } A| = |A|^{1-n}$

(v) (A B) = adj B . adj A .

(vi) A is are invetible square matrix . There
adj $(A^T) = (\text{adj}^- A)^T$

(vii) If A is a no n - singular square matrix, there
$\text{adj}(\text{adj}^- A) = |A|^{n-2} A$.

<u>System of Simultaulons Linear Equations</u> :-

$a_{11} x_1 + a_{12} x_2 + ... + a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + ... + a_{2n} x_n = b_2$

:

:

$a_{n1} x_1 + a_{n2} x_2 + ... + a_{nn} x_n = b_n$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{14} \\ a_{21} & a_{22} & \cdots\cdots & a_{24} \\ \vdots & & & \vdots \\ 1 & & & . \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix}_{nxn} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{nx1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}_{nx1}$$

AX = B
$X = A^{-1} B$

(i) If A is non singular, the system of equation AX = B has a uonique so1[4] given by $X = A^{-1} B$.

(ii) If A is singular and (adj A) B = O, there system of equation given by AX = B is cousistent with. Infinefely many solutions.

(iii) If A is a singular matrix,and (adj A) . B $\neq$ O, then the system of equation given by AX = B is inconsistent

<u>DUMB QUESTION</u>

Q1. If co-ordinates (x, y)one rotated by are augle θ about the arigin the new coordinetes are represnted by matrix . How ?

Sol^n :- $\begin{aligned} x &= x\cos\theta - y\sin\theta \\ y &= x\sin\theta + y\cos\theta \end{aligned}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta - & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

<u>Illustration</u> :-

Q1. A = $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$ and B 2 $\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$ show that A B ≠ B A .

<u>Sol^n</u> :- A . B = $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 10 \\ 9 & 11 \end{bmatrix}$

B . A = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 5 \\ 5 & 9 & 4 \\ 5 & 12 & 7 \end{bmatrix}$

Q 2. Show taht A = $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is ...of index 3.

<u>Solution</u> :- $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -2 & -1 & -3 \end{bmatrix}$

$A^3 = A^2 - A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 = 0$ i.e. $A^k = 0$

<u>k = 3</u>

<u>DETERMINAATS</u> :-

The experssion $\begin{bmatrix} a_1 & b_2 \\ a_2 & b_2 \end{bmatrix}$ is colled a determinant.

<u>Calculation of determinants</u>

$$\begin{bmatrix} a_1 & b_2 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1 (b_2 c_3 - b_2 b_3) - b_1 (a_2 c_3 - c_2 a_3) + c_1 (a_2 b_3 - b_2 c_3)$

<u>Properties of determinants</u> (Secondary Information) :-

1. If rows be changed into columns and columns into the rows, the determinant remain unaltered .

2. If any two rows or columns of a determinant are interchanged the resulting determinant is the negative of the original determinant .

3. If two rows (or ltwo columns) in a determinant have corresponding entries that are equal, the value of the determinant K, then determinant is multiplied by K .

4. If each of the entries in a row (or column) of a determinant is multiplied by K, then determinant is multiplied by K .

5. If each entry in a row (or column) of a determinant is written as the sum of two or more trems than the determinant can be written as the

sum of two more determinants .

6. If to each element of a line (row or column) of a determinant be adeled the equimultiples of the corresponding elements of one or more parallel lines, the determinant remains unaltered .

$$\begin{vmatrix} a_1 + la_2 + ma_3 & a_2 & a_3 \\ b_1 + lb_2 + mb_3 & b_2 & b_3 \\ c_1 + lc_2 + mc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & b_2 & b_3 \\ a_1 & c_2 & c_3 \end{vmatrix}$$

7. If each entry in any cow (or column) of determinant is zero, then the volue of detrminant equal to zero .

8. If determinant varishes for X = a, then (X - a) is a factor of D .

<u>Product of two Determinants</u> :-

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1 \alpha_1 + b_1 \beta_1 + c_1 \gamma_1 & a_1 \alpha_2 + b_1 \beta_2 + c_1 \gamma \\ a_2 \alpha_1 + b_2 \beta_1 + c_2 \gamma_1 & a_2 \alpha_2 + b_2 \beta_2 + c_2 \\ a_3 \alpha_1 + b_3 \beta_1 + c_3 \gamma_1 & a_3 \alpha_2 + b_3 \beta_2 + c_3 \end{vmatrix}$$

<u>Differentiation of a determinant</u> (Sexondary Information) :-

$$\Delta(X) = \begin{vmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{vmatrix} \text{ then } \Delta(X) = \begin{vmatrix} a_1^1(x) & b_1^1(x) \\ a_2(x) & b_2(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & b_1(x) \\ a_2^1(x) & b_2^1(x) \end{vmatrix}$$

Similarly if

$$\Delta(X) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}, \text{ then } \Delta^1(X) = \begin{vmatrix} R_1^1 \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2^1 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R_3^1 \end{vmatrix}$$

<u>Summation of Determinants</u> :-
(Secondary Information)

Let $\Delta r = \begin{vmatrix} f(r) & a & l \\ g(r) & b & m \\ h(r) & c & n \end{vmatrix}$

Then $$\sum_{r=1}^{n} \Delta r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & a & l \\ \sum_{r=1}^{n} g(r) & b & m \\ \sum_{r=1}^{n} h(r) & c & n \end{vmatrix}$$

<u>Integration of Determinants</u> (Secondary Information) :-

$$\Delta(X) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ l & m & n \end{vmatrix}$$

$$\int_{a^1}^{b^1} \Delta(x)\, dx = \begin{vmatrix} \int_{a^1}^{b^1} f(x)\, dx & \int_{a^1}^{b^1} g(x)\, dx & \int_{a^1}^{b^1} h(x)\, dx \\ a & b & c \\ l & m & n \end{vmatrix}$$

<u>Special Determinants</u> (Secondary Information) :-

1. Symmetric Determinant :-
The elements situted at equal distance from the diagonal are equal both in magnitude and sign.

eg. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

<u>Skew symmetric determinant</u> :- Determinant of skew symmetric matrix .

3. <u>Circulant</u> :- The elements of the rows (or columns) are in cyclic arangements .

eg . $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

<u>Solution of system of Linear Equations</u> :-

$a_1X + b_1 y + c_1 z = d_1$

$a_2X + b_2 y + c_2 z = d_2$

$a_3X + b_3 y + c_3 z = d_3.$

D =

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

X = D1/D Y = D2/D Z = D3/D

The following cases arise

(1) $D \neq O$, the system has one unique solution .
$D = O$, $D_1, D_2, D_3 \neq O$, then system is incories tant .

(2) $D_1 = O$, and $D_1 = D_2 = D_3 = O$, then the system has infinite solutions .

<u>Illustrations</u>

<u>Q 1</u>. Without expanding, show taht $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = o$

Solution :- L H S = $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & b_c \\ 1 & b & c_a \\ 1 & c & a_b \end{vmatrix} =$

$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$ [$R_1 \rightarrow a R_1$, $R_2 \rightarrow b R_2$, $R_3 \rightarrow c R_3$ and abc taken common]

$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

Q 2. Prove that

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$
=

Solution : L H S :-

$$\begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix} \begin{vmatrix} 1 & \gamma+\delta & \gamma\delta \\ 1 & \alpha+\beta & \alpha\beta \\ 0 & 0 & o \end{vmatrix} = 0$$

ASSIGNMENTS

EASY

E 1. If $a_1 b_1$ c and d are real constants and A

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,prove that $A^2 - (a + b)A + (ab - bc) I = 0$

Sol :- $A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$

$A^2 = A^2 - (a + b) A + (ab + bc)I$

$$= \begin{bmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab - bd + o \\ ac + cd - ac - cd + 0 & bc + a^2 - ad - a^2 + ad - bc \end{bmatrix}$$

<u>E 1</u> $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ show taht A^2 4 A - 5 I = o

<u>SOl</u> :- $A^2 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 - 4 - 5 & 8 - 8 - 0 & 8 - 8 - 0 \\ 8 - 8 - 0 & 9 - 4 - 5 & 8 - 8 - 0 \\ 8 - 8 - 0 & 8 - 8 - 0 & 9 - 4 - 5 \end{bmatrix}$$

E 3. Solve syste the equation .
X + 3y - 2z = 0., 2x - x - 4z = 0, x - 11y + 14z = 0

Sol :- $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & -4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ => A X = B .

|A| = 0

Solving the terme of z .

x = $\frac{-10z}{7}$, $y = \frac{8z}{7}$.

=> x = - 10 k , y = 8 k , z = 7 K
E 4 . For what value of x, the matrix

$$A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$$ is singular .

<u>Solution</u> :- | A | = 0

=> x [(3 - x) (1 + x - 4) - 0 + 2 (2 - 2)] = 0

x (3 - x) (x - 3) = 0 img x = 0, 3

E 5 :- Evaluate $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

<u>Sol</u> :- Applying $c^1 \rightarrow c_1 + c_2\ c_3$

$$= \begin{vmatrix} 1+w & w^2 & w\ w^2 \\ 1+w & w^2 & w^2\ 1 \\ 1+w & w^2 & 1\ w \end{vmatrix} = \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix} = 0$$

<u>E 6</u> :- For what value of K do the following homogenous system of equations passes a row trivial solution .

x + ky + 3z = 0

3x + ky - 2z = 0

2x + 3y - 4z = 0

Sol :- D = 0

$$D = \begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

$R_1 - 3R_1$ and $R_3 - 2R_1$

=> $\begin{vmatrix} 1 & k & 3 \\ 0 & -2k & -11 \\ 0 & 3-2k & -10 \end{vmatrix} = 0$

k = 33/2

E 7. Evoluate $\begin{vmatrix} 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \\ 4^2 & 5^2 & 6^2 \end{vmatrix}$

<u>Sol</u> :- $R_2 - R_1$, $R_3 - R_2$ then $C_2 - C_1$ and $C_3 - C_2$

D = $\begin{vmatrix} 4 & 5 & 7 \\ 5 & 2 & 2 \\ 7 & 2 & 2 \end{vmatrix}$

$c_3 - c_2$

$\begin{vmatrix} 4 & 5 & 7 \\ 5 & 2 & 0 \\ 7 & 2 & 0 \end{vmatrix}$ = 2 [10 - 14] = - 8

E 8. If $\begin{vmatrix} y+2 & x & x \\ y & z+x & y \\ 2 & z & x+y \end{vmatrix}$ = K X Y 2 , Find K .

Solution :- Putting x = 1, y = 1, z = 1 [As if in true for any volues of x_1 y_1 z_1

k = 4

<u>MEDIUM</u>

M 1. Determine $\alpha_1\beta$ and γ if $\begin{bmatrix} O & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is or thogonal

Sol^n:- $AA^1 = 1$

$$\Rightarrow \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \gamma^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \gamma^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

=> 4 $\beta^2 + \gamma^2 = 1, \quad 2\beta^2 - \gamma^2 = 0, \quad \alpha^2 + \beta^2 + \gamma^2 = 1$

$$\alpha = \pm\frac{1}{\sqrt{2}}\beta = \pm\frac{1}{\sqrt{6}}, \quad \gamma = \pm\frac{1}{\sqrt{3}}$$

M 2. Using elementary row trans formations find the inverse of the matrix A = $\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$

Sol^n :- A = I A

$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

. $R_1 \rightarrow R_1 - R_2$

. $R_2 \rightarrow -2R_1$ and $R_3 \rightarrow R_3 - 3R_1$

. $R_2 \rightarrow 1/2\ R_2$

. $R_1 \rightarrow R_1 + R_1$ and $R_3 \rightarrow R_3 + 2R_2$

. $R_3 \rightarrow 1/2\ R_3$ and .$R_1 \rightarrow R_1 + 1/2\ R_3$ and $R_2 \rightarrow R_2 - 1/2\ R_3$

We get :-

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} 2 \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & \cdot 1/4 \end{bmatrix}_A$$

$$A^{-1} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/\cdot 8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix}$$

<u>M 3</u> If a, b and c are $P^{th}1$ q^{th} and r^{th} terms of are

H.P., prove taht $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$

Solution :- A : First term, D : Common difference
1/a = A +(p -1) D, 1/b = A +(q - 1) D, 1/c = A + (r - 1)D

$$\Delta abc \begin{vmatrix} 1/a & 1/b & 1/c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$R_1 \rightarrow R_1 - D\ R_2 - (A - D)r$,

$$\Delta = abc \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

<u>M 4 .</u> for what valume of m, does the system of equation 3x + my - mand 2x - 5y = 20 lhas a solution satisfying the condstion. x > 0, y > 0.

Solution $\Delta = \begin{vmatrix} 3 & m \\ 2 & -5 \end{vmatrix} = -(15 + 2_m)$

$\Delta_1 = -25m$ $\Delta 2 = 60 - 2m$

$$X = \frac{25m(15+2m)}{(15+2m)^2} > 0, \quad y = \frac{2(m-30)(15+2m)}{(15+2m)} > 0$$

=> m > 30 or m < -15/2

<u>HARD</u>

<u>H 1</u>. Show that the produst of two triangular matrices is itself triangular.

Sol^n:- A = $[Aij]_{nxn}$ and B = $[bjk]_{nxn}$

aij = 0 , i > j and bjk = 0 j > k

AB = $[cik]_{nxn}$ then cik = $\sum_{j=1}^{1}$ aij bik

Suppose i > k
if j < i, then aij = 0 => aik = 0

if i < j, then j > k , because i > k , => bjk = 0
=> cik = 0
Hence c i k = 0 for i > k

Hence AB is also triangular .

H 2 . Without expanding at any stage show taht

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = AX + B$$

Where A and B are determinants of order 3 not involving X:

Solution:- L.H.S. = $\begin{vmatrix} x^2+x & x+x & x-2 \\ 2x^2+3x-1 & 3x & 2x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

"$R_1 + R_3 - R_2$" and then "$C_1 - C_3$" and $C_2 - C_3$"

L.H.S. = $\begin{vmatrix} 4 & 0 & 0 \\ 2x^2 + 2 & 3 & 3x - 3 \\ x^2 + 4 & 0 & 2x - 1 \end{vmatrix}$

"$R_2 - \frac{x^2}{2} R_1$ and $R_3 - \frac{x^2}{4} R_1$"

L.H.S. = $\begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3x - 3 \\ 4 & 0 & 2x - 1 \end{vmatrix}$

$= x \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3 \\ 4 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix}$

= Ax + B = R.H.S.

<u>KEY TERMS</u>

1. Matrix
2. Diagonal Matrix
3. Row Matrix
4. Column Matrix
5. Sclar Matrix
6. Unit or Identity Matrix
7. Triangular Matrix
8. Null Matrix
9. Tranpose Matrix
10. Conjnagte of a Matrix
11. Symmetri and Skew- Symmetric Matrices
12. Hermitian and Skew - Hermitian Matrices
13. Unit any Matrix
14 Singular Matrix
15. Orthogovel Matrix
16. Idem potent Matrix
17. Involutary Matrix

18. Nilpotent Matrix
19. Square Matrix
20. Adioint of a Matrix
21. Cofacto
22. Determinant
23. Symmetric and skew - Symmetric determinant
24. Circulant

Probability

Dispute in 1654 led to the creation of a mathematical theory of prabability by two famous french mathematicians Blaise Pascal and Pierrede Fermet . First fundamental principles of prabobility theory were formulated for Ist time In popular dice game. Consisted in throwing a pair of dice 24 time, the problem was to decide whether or not to bet even money on occurrence of at least one "double six" during 24 throws. By gambling rule chevalier de mere believed it would be profitable but c caluclation shows just opposite .
Dutch scientist christion Huygens in 1657 published the first book of probability entitled De Ratiociniis in Ludo Aleac.
In 1812 pierse de laplace introduced a host of new ideas and mathematical techniques in his p book .

<u>Equally likely</u> - If two events are called equally likely if none of the events have preference of occurence of other .

<u>Nutually exclusive</u> :- If occurence of one event rules out lthe ocurrence of other .

<u>Exhaustive</u> :- Set of event in experiment is said to exhaustive it nothing happen than those listed possible out comes can occur as a consequence of the experiment .

Total out comes of experiment is called sample space .

A; 'm' out comes favour the occurence of event A .

$$P(A) = \frac{n(A)}{n(S)} = \frac{m}{n(\text{sample space})}$$

$P(\bar{A})$ = complement of A $P(A^c)$

$$P(\bar{A}) = \frac{n - m}{n} = 1 - P(A)$$

$P(A) + P(\bar{A}) = 1$

$\swarrow 0 \leq P(A) \leq 1 \rightarrow$ sure ivent

Impossible event img sure event

Example of kequally likely events :-
When a unbiased coin is tossed then occrenice of head or tail are equally likely .

Example of mutually exclusive event -

Let sample space of unbised die is die is S = {1, 2, 3, 4, 5, 6} in which E_1 = {1, 2, 3,} = ecent of occurrence lof no. less than 4 & E_2 = {5, 6} = event of occurrence of no. greater than 4.

cleaarly . $E_1 \cap E_2 = \phi$

so, E_1 and E_2 are mutually exclusive .

Examples of exhaustiv event :-
Let sample space of unbiased die is S = {1, 2, 3, 4,5, 6} in which E_1 = { 1, 2, 3, 4} = Event of occurence of no . less tahn 5.

& E_2 = {3, 4, 5, 6} = Event of occurence of no. Greater than 2.
Then $E_1 U E_2$ = {1, 2, 3, 4, 5, 6} = 5
Hence E_1 & E_2 are exhaustive events .

Odds in favour :- If 'a' is no of cases favourable to odds in favour of E are a : b & odds against of E are b : a.

$$P(E) = \frac{a}{a+b}$$

$$\& \ P(E^1) = \frac{a}{a+b}$$

$\therefore P(E) + P(E^1) = 1$
Illustration :- A person while dialing 7 digit phone no. forget last two digit and he randonly dials 2 numbers .
Find the chance of currect no.

Ans - There are 10 digit on phone.
1st digit can be dialled in 10 ways.
2^{n} digit can be dialled in 10 way .
sample space (total no. of ways two digit can be dialled)

= 10 X 10 = 100
But there is only 1 currect no.
So, P (A) = 1/100

<u>Independent and dependedt event</u> :- Two event are said to be independent if occurence or non - occurence of are does affects the probabitity of occurence or non-occurence of the other .
eg. Occurence of head or tail is independent on each other.

<u>Venn diagram</u> :-
I $\rightarrow A\,\overline{B}$ (A but not B)

II $A \cap B$

III $\rightarrow B \cap \overline{B}$ (B but not A

IV $\rightarrow \overline{A} \cap \overline{B}$
P(A or B) = P(at least one of A or B)
P(A img B) = P(A) + P(B) - P(A img B)
= I + II +III
= (I + II) + (III + II) - II
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A \cup B) = P(A \cap \overline{B}) + (\overline{A} \cap B) + P(A \cap B)$

$= P(A) + P(\overline{A} \cap B)$

=| - P ($\overline{A} \cap \overline{B}$)

$P(A \cup B) + P(\overline{A} \cap \overline{B})$

P (exactly one A or B) = I + III = P ($P(A \cup \overline{B}) + P(\overline{A} \cap B)$)

$=P(A) + P(B) - 2P(A \cap B)$

$= P(A \cup B) - 2P(a \cap B)$

<u>Note</u> :- For mutually exclusive event

$P(A \cap B) = 0$

For exhaustive event

(A img B) = Sample space

<u>Conditional probability</u> :-

Probability of ocurence of an event B when it is known that some event A has already occured .It is P(B/A)

$$P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{n(A \cap B)/n(S)}{n(A)/n(S)} = \frac{P(A \cap B)}{P(A)}$$

$$P(B/A) = \frac{n(A \cap B)}{n(B)}, \quad 0 \leq P(B) \leq 1$$

$$P(B \cap A) = \begin{cases} P(A)\,P(B/A), & P(A) > 0 \\ P(B)\quad (B/A), & P(B) > 0 \end{cases}$$

$$P(B/A) = P(A \cap \bar{B})$$

<u>Note</u> - For independent events -

$P(A \cap B) = P(A).\ P(B)$

Dumb Question :- How $P(A \cap B) = P(A)\ P(A/B)$

<u>Ans</u> - P(A/B) = probabilituy of ocurence of A in B

ie, simultanous occurence

Probability of simultaneous occurence of A and B

P(A/B) = Probability of B

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

<u>Illustration</u> - If $P(A) = \frac{1}{2}$ $P(B) = \frac{1}{3}$ $P(A \cap B) = \frac{3}{4}$

Find $P(A \cap B)$, $P(A/B)$, $P(\overline{A}/B)$, $P(A \cap \overline{B})$, $P(\overline{A}/\overline{B})$

<u>Ans</u> - $P(A \cap B) = P(A) + P(B) - P(A \cap B)$

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{3} - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = \frac{1}{12}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/12}{1/3} = \frac{1}{4}$$

$$P(A \cap B) = P(A) - P(A \cap B)$$

$$= \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$$

$$P(\overline{A}/\overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{1 - P(A \cap B)}{1 - P(B)}$$

$$= \frac{1 - 3/4}{1 - 1/3} = \frac{1/4}{2/3} = \frac{3}{8}$$

$P(A/B) + = 1$ $P(\overline{A}/B) = 1$

$P(\overline{A}/B) = 1 - P(A/B)$

$$= 1 - \frac{1}{4}$$

$= \frac{3}{4}$

<u>Bernoulli strial</u> :- An experiment $\frac{\text{n independent}}{\text{trial}}$ out come

$p + q = 1$

P(getting r' s success) = $n\,C_r\, p^r\, q^{n-r}$

$P\ \underbrace{SSS.....S}_{r}\ \underbrace{FF.....F}_{n-r}$

$= p^r\, q^{n-r}\, nC_r$ img arrangement in a row

P (getting r + 1 success) = $nC_{r+1}\, q^{n-r-1} p^{r+1}$

<u>Bagrang's relation ship</u>

$$\frac{P(r+1)}{P(r)} = \frac{n!}{(r+1)!(n-r-1)} pr+1_{qn-r-1} \quad \frac{r!\,(n-r)!}{n!\,p^r\ q^{n-r}}$$

$$\frac{P(r+1)}{P(r)} = \frac{n-r}{n+r} \cdot \frac{P}{q}$$

<u>Dumb Question</u> :- How relation P (getting r' s success)

$= P^r\, q^{n-r}\, nC_r$

<u>Ans</u> - In this P is probability of getting success and q is probability of getting failure and $p + q = 1$

Sclection of r success out of n out comes = nC_r

img..... P(getting r' s success)

= P x P x P.....r times x q x q x.....(n - r) times x nC_r

<u>Illustration</u> :- A die is rolled n times. Find the value of n if probability of getting at least one ace is 91/216

<u>Ans</u> P(getting at least one ace) = 1 - P(no ace)

P(1) = 1/6 P(not 1) = 5/6

$$\frac{91}{216} = 1 - \left(\frac{5}{6}\right)^n$$

=> n = 3

<u>Probability theorem</u> - (Baye's theorem)

If an event A can occur with n mutually exclusive, exhaustive event B_1, B_2, ...B_n and probability of $P(A/B_1)$, $P(A/B_2)$, ...$P(A/B_n)$ is known. Then probability .

$$P(Bi/A) = \frac{P(Bi).\ P(A/Bi)}{\sum_{r=1}^{n} P(Br).\ P(A/B_r)}$$

P(A img B_i) = P(Bi). P(A/Bi)
= P (A). P(Bi/A)

P(A) = P(A img B_1) + P(A img B_2) + ...+ $P(B_n)$ $P(A/B_n)$

$= P(A \cap B_i)$ $P(B_i)\ P(A / B_i)$

$=P(A).\ P(B_i / A)$

$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots\ldots\ldots + P(A \cap B_n)$

$=P(B_1)\ P(A / B_1)\ P\ (A / B_2) + \ldots\ldots\ldots\ldots + P(B_n)\ P(A / B_n)$

$$\sum_{r=1}^{n} P(B_r)\ P\ (A / B_r)$$

<u>Illustration</u> :- A bag cintains 5 balls of unknown color. Two balls are drqwn and both are found to be white. Find the chance that there are 4 white ball in box . Assume any no. of white ball equally likely .

<u>Ans</u> - A : Two balls drawn from a bag contuins 5 balls.
both are white
B_0 : 0 white ball
B_1 : 1 white ball
B_2 : 2 white ball
B_3 : 3 white ball
B_4 : 4 white ball
B_5 : 5 white ball
$P(B_0) = P(B_0) = P(B_1) = P(B_3) = P(B_4) = P(B_5) = P(B_6) = 1/6$

$P(A/B_0) = 0$ $P(A/B_1) = 0$

$$P(A/B_2) = \frac{2\ C_2}{5\ C_2} = \frac{1}{10}$$

$$P(A/B_3) = \frac{3\ C_2}{5\ C_2} = \frac{3}{10}$$

$$P(A/B_4) = \frac{4\ C_2}{5\ C_2} = \frac{6}{10}$$

$$P(A/B_5) = \frac{5\ C_2}{5\ C_2} = 1$$

$P(B_4/A) =$

$$\frac{P(B_4).P(A/B_4)}{P(B_2).P(A/B_2) + P(B_3).P(A/B_3) + P(B_4).\ P(A/B_4) + P(B_5)P(A/B_5)}$$

$$= \frac{6/60}{20/60} = \frac{3}{10}$$

<u>Dumb question</u> - Why probability of $P(A/B_0)$ and $P(A/B_1)$ is zero ?

<u>Ans</u> - Because in event we are getting 2 white ball.

and when in sample we have none or one white we can not get 2 white ball.

<u>Stockist approach</u> :-

<u>Q</u>- A bag contains 5 fair coin, 2 doulaly headed coin, 2 doubly tailed and are biased coin. Probability of head with biased = 2/3 . A coin is selected and tossed. If head appeared same coin is tossed and if tail appears another coin selected from remaining and tosed. Find the chasce -

(1) Same .coin is tessed twice .

(2) Chance .head appears in both toss.

(3)If head appears in both wins. find the chance that it is doubly headed coin .

<u>Ans</u> - Total coin = 5 + 2 + 2 + 1 = 10

(1) P(getting H in I st toss) $= \frac{2}{10} \times 1 + \frac{5}{10} \times \frac{1}{2} + \frac{1}{10} \times \frac{2}{3} = \frac{31}{60}$

(2) P(getting both (H H) in 2 toss) =

$$\frac{2}{10} \times 1 \times 1 + \frac{5}{10} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{10} \times \frac{2}{3} \times \frac{2}{3} = \frac{25}{72}$$

(3) P(Doubly head / H H) $\frac{2/10}{P(HH)}$

$$= \frac{2/10}{25/72} = \frac{2 \times 72}{25 \times 10} = \frac{72}{125}$$

<u>In finite samble space (Geometrical probaility)</u> :-

(1) If a point is randomly sclected from area 'S' which indudes an area 6 then chance that it is sclected from area 6

$$\eta = \frac{6}{5}$$

(2) If a point is randomly taken from line AB of length L then chance that it is selected from the segment PQ(l) contained by AB is = l/2

<u>Illustration</u> :- In a circle of radius 'a'. Find the probalility that point is close to the circumference than to its centre.

<u>Ans</u> - n(S) = $\frac{\pi}{a_2}$

E : Point is close to circumference than oits centre.

P(E) = $\frac{\pi a^2 \pi \left(\frac{a}{2}\right)^2}{\pi a_2} = 1 - \frac{1}{4} = \frac{3}{4}$

<u>Dumb question</u> - How probability of " point is close to circumference than its centre " is 3/4 .

<u>Ans</u> - A point is close to circumference than its centre when it lies in region of outside the circle of r = a/2 and inside the circle of r = a .

$$P(E) = \frac{\pi a^2 - \pi\left(\frac{a}{2}\right)^2}{\pi a^2} = \frac{3}{4}$$

<u>Easy level</u>

<u>Q. 1</u> - A natural no. is randomly selected. Find the chance that digit in unit place of its square is 4.

<u>Ans</u> :-

0	1	4	5	6	9
↑	↑	↑	↑	↑	↑
0	(1,9)	(2,8)	(5)	(4,6)	(3,7)

There are only two digit (2 and 8). Whose square's unit place is 4.
So, P(E) = 2/10 = 1/5

<u>Q. 2</u> 4 red and 3 white ball are arranged | n a row. Find chance that two ball at extreme are whit .

<u>Ans</u> :- <u>Case I</u>:- alike balls

$$n(S) = \frac{.y!}{3!\ 4!}$$

$$n(A) = \frac{5!}{4!}$$

P(A)= $\frac{n(A)}{n(S)} = \frac{5}{7 \times 5} = \frac{1}{7}$

<u>Case</u>2:- Different balls-

n(S) = 7!

n(A) = $3C_2$. 2!, 5!

P(A) = $\frac{n(A)}{n(S)} = \frac{3C_2 \times 2!\ 5!}{7!} = \frac{1}{7}$

<u>Q. 3</u>- P(A) = 1/2 ,P(B) = 1/2 . Find the least and greatest valume of intersection P(A img B).

<u>Ans</u> - P(A $\cap$ B) = P(A). P(B/A)
P(A $\cap$ B) greatest = 1/2 if P(B/A) = 1

$P(A \cap B) = P(A) + P(B) - P(A \cap B) \leq 1$

$= \frac{1}{2} + \frac{2}{3} - P(A) + P(B) - P(A \cap B) \leq 1$

$P(A \cap B) \geq \frac{1}{2} + \frac{2}{3} - 1$

$\frac{1}{6}\ P(A \cap B) \geq \frac{1}{2}$

<u>Q. 4</u>- A and B are independent then show that $\bar{A}$ and $\bar{B}$ are also independent .

<u>Ans</u> - <u>Note</u> - If and are independent. then $P(A \cap B)$ P(A) . P(B)
<u>T.PT</u> :- $P(\bar{A})\,P(\bar{B}) = P(\bar{A} \cap \bar{B})$
<u>L. H. S.</u> :- $P(\bar{A})\,P(\bar{B})$ { 1 - P(A) } { 1 - P(B)}

= 1 - P(A) - P(B) + P(A) . P(B)
= 1 - [P(A) + P(B) - P(A $\cap$ B)]
= 1 - P(A $\cup$ B)

$= P(\overline{A} \cap \overline{B})$

<u>Q. 5</u> - Two cords are drawn from 52 cards pack. Find the chance that both the cards are aces.

<u>Ans</u>- n(S) = no. of ways drawing 2 cards from 52 cards.

$= 52\ C_2$

n(A) = no. of ways drawing 2 ace cards from 4 ace card

$= 4\ C_2$

$$P(E) = \frac{n(A)}{n(S)} = \frac{4C_2}{52C_2} = \frac{4 \times 3}{52 \times 51} = \frac{1}{13} \times \frac{1}{17}$$

$$P(E) = \frac{1}{221}$$

<u>Q. 6</u>- In a bag 4 cards morked with 'I' and. 4 marked with 'T' are present . 3 time a card is drawn without .

replacement. Find the chance of forming wird 'I I T'.

<u>Ans</u> - Total no. of way
$= n(S) = 8C_3$
drawing 3 cards

$n(A) = 4C_2 \text{ X } 4C_1$

$$P(E) = \frac{n(A)}{n(S)} = \frac{4C_2 \times 4C_1}{8C_3} = \frac{3}{7}$$

<u>Dumb question</u> :-

How does $n(A) = 4C_2 \text{ X } 4C_1$?

<u>Ans</u>:- For wird 'I I T' we reguire 2I's and I 'T' So, out of 4 I' s we require 2 I' s and out of 4 T's ,we require 1 T. So, $n(A) = 4C_1$

Q. 7 - Probability of a no. appearing if a clice is rolled is proportipnal lto the ,no. Find the chance of getting aprime no ?

Ans- Probability . P(1) = λ P(2) = 2 λ

P(3) = 3 λ P(4) = 4 λ P(5) = 5 λ

P(6) = 6 λ

P(1) $\rightarrow$ probability of getting 1 if clice is ralled

P(1) + P(2) + P(3) + P(4) + P(5) + P(6) =
$\lambda + 2\lambda + 3\lambda + 4\lambda + 5\lambda + 6\lambda = 21\lambda$

= 21 λ

But 21 λ = 1 $\leftarrow$ total prabability

$$\lambda = \frac{1}{21}$$

P(prime) = P(2) + P(3) + P(5) = 10 $\lambda = \frac{10}{21}$

Q. 8 - A Coin is tossed l4 time if head appeared 1 point awarded and tail appeared 2 point given. Find expectation of player ?

Ans- P(4) = P(HHHH) = $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

P(5) = P (HHHT/HHTH/HTHH/THHH)

= 4 X $\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$

= $\frac{6}{16}$

P(6) = P (HHTT/HTHT/HTTH/TTHH/THTH/THHT)

= 6 X (1/2 X 1/2 X 1/2 X 1/2 X)

= 6/16

P(7) = P(HTTT/THTT/TTHT/TTTH)

= 4 x $\frac{6}{16} = \frac{4}{16}$

P(8) = P(TTTT) = $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

P(4) $\rightarrow$ probability of getting 4 points.

Expectation =

$$4 \times \frac{1}{16} + 5 \times \frac{4}{16} + 6 \times \frac{1}{16} + 7 \times \frac{4}{16} + 8 \times \frac{1}{16}$$

$$= \frac{1}{16} [4 + 20 + 36 + 28 + 8]$$

$$= \frac{96}{16} = 6$$

<u>Dumb question</u> :- What is expectati0n ?

<u>Ans</u> - If P be probability of success of a person in any event and M be prize money which be will receive in case of sussess, themn,

Expectation = PM = probability of success X prize money

<u>Q. 9</u>- There are 3 compartment in a box. A ball is randomly drawn from anyone compartment. Find chance it is ded

<u>Ans</u> - $E_1 \rightarrow$ frawn from I_{st} compartment

$E_2 \rightarrow$ frawn from II_{nd} compartment

$E_3 \rightarrow$ frawn from III_{rd} compartment

E . $\rightarrow$ bll is red

$P(E_1) = P(E_3) = 1/3$ (equally likely event)

$P(E) = P(E \text{ img } E_1) + P(E \text{ img } E_2) + P(E \text{ img } E_3)$

$= P(E_1) P(E / E_1) + P(E / E_2) + P(E / E_2) + P(E/E_3). P(E / E_3)$
$= 3 / 7$

$$P(E) = \left[\frac{3}{5} + \frac{4}{7} + \frac{3}{7}\right] = \frac{8}{15}$$

<u>Q. 10</u> A coin is tossed till head and tail appears, it can be tossed 5 time max . First 2 throws are H H . Find the chance that coin tossed 5 time.?

<u>Ans</u>- I^{st} two throws are H H and if 3rd and 4th throw are also H H . So we have to go for 5th throw for getting tail.

$\underbrace{HH} \mid \underbrace{H}, \underbrace{H}$

P(A) = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

<u>Medium level</u>

<u>Q. 1</u>- A, B and C are three event such that -

P(exatly one A or B) = P(exactly one B C)

=> P(exactly one C or A) = P

and $P(A \cap B \cap C) = p^2$ and A, B and C are exhaustive. Find = value of P .

<u>Ans</u> - <u>Not</u> :- In exhaustive events

$P(A \cup B \cup C) = 1$

P = P (a) + P (B) - 2P (A $\cap$ B)

P = P (B) + P (C) - 2P (B $\cap$ C)

P = P (C) + P (A) - 2P (C $\cap$ A)

3 P = 2 (P(A) + P(B) + P(C)) - 2P(A $\cap$ B) + P(A $\cap$ B) + P(C $\cap$ A))

$\frac{3P}{2}$ P(A) + P(B) + P(C)[(A $\cap$ B) + P(B $\cap$ C) +P(C $\cap$ A)]

$\frac{3P}{2}$ + [P(A $\cap$ B) + P(B $\cap$ C) +P(C $\cap$ A)] P(A) + P(B) + P(C) .....(1)

$P(A \cap B \cup C) = P(A) + P(B) + P(C) - [\, P(A \cap B) + P(B \cap C) + P(C \cap A) + P(A \cap B \cap C)$.....(2)

$$\frac{3P}{2} + P\,(A \cap B \cap C) = P(A \cup B \cup C)$$

If A, B and C are exhaustive .

then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$ holds -

From eqn ..... (2)

$P(A \cap B \cap C) = P(A \cap B) + P(B \cap C) + P(C \cap A)$

P utting in eqn ..... (1)

$$\frac{3P}{2} + P(A \cap B \cap C) = P(A \cup B \cup C)$$

$$\frac{3P}{2} + P^2 = 1 \text{ .....(3)}$$

On solving

$$P = \frac{1}{2}$$

<u>Q. 2</u>. 12 face cards are removed from 52 cards and remaining 40 cards are well shuffled and 4 cards are drawn from pack of 40 cards. Find the chance that all four are of different suit and different denomination .

<u>Ans</u>- Heart Diamond club spade

$n(s) = 40\ C_4$

No. of way drawing I^{st} card = 40 ways

No. of way drawing 2nd card = 27 ways

No. of way drawing 3rd card = 16 ways

No. of way drawing 4th card = 7 ways

Total no. of way of drawing all 4 cards = 40 X 27 X 16 X 7

<u>Dumb question</u> - Why NO. of ways of drawing 2nd card is 27 ?

<u>Ans</u>- Let I^{st} card drawn is ace of Heart. So, no, Heart and no ace card is further selected. So remaining cards are (40 - 10 - 3) = 27
So, 27 ways

$$P(E)= i \frac{40 \times 27 \times 16 \times 7}{40C_2} = \frac{40 \times 27 \times 16 \times 7}{40 \times 39 \times 38 \times 37}$$

<u>Q. 3</u> - There are two luts of article one lot contains 3 defective and 5 good article and other lot contains 4 defective and 8 good articles. A lot is randomly seleste a and 3 articles are drawn . this lot will be rejected if 2 or mor than 2 article are found to be defective. Find the chance that this be rejected.

<u>And</u> - $E_1 \rightarrow I^{st}$ lot is chosen

$E_2 \rightarrow 2^{nd}$ lot is chosen

$E \rightarrow$ lot is rejected

$P(E) = P(E^1)\, P(E/E_1) + P(E_2).\, P(E/E_2)$

$P(E_1) = P(E_2) = 1/2$ (equally likely events)

$$P(E) \quad \frac{1}{2} \times \left[\frac{3C_2 \times 5C_1}{\underbrace{8C_3}_{\substack{2\text{ defective}\\+1\text{ good}}}} + \frac{3C_3}{\underbrace{8C_3}_{\substack{\text{all 3 are}\\\text{defective}}}} \right] + \frac{1}{2} \left[\frac{4C_2 \times 8C_1}{\underbrace{12C_3}_{\substack{2\text{ defective}\\+1\text{ good}}}} + \frac{4C_3}{\underbrace{12C_3}_{\text{all 3 defective}}} \right]$$

Q. 4 A boy contains | coin of worth M rupees and n coins of total . Worth of M rupees. Coins are drawn without replace -ment till coin whose valime is M, is drawn, I Find the expectaion of draw .

Ans - P(drawing M in Ist draw) = $\frac{1}{n+1}$

P(drawing M in 2nd) drawn) = $\frac{1}{n+1} \times \frac{1}{n} = \frac{1}{n+1}$ ($\frac{1}{n+1}$ $\rightarrow$ other coin draw, $\frac{1}{n+1} \times M \rightarrow$ drawing of M)

P(drawing M in (r + 1)th draw) =

Expectation = $\frac{1}{n+1} \times M \rightarrow$ M coin draw In Istdraw, + $\frac{1}{n+1}\left[M + \frac{m}{n}\right] \rightarrow$ Mcoin draw in IInd draw , + $\frac{1}{n+1}\left[M + \frac{2m}{n}\right] \rightarrow$ In 3rd draw

$$= \frac{1}{n+1} \times (n+1)M + \frac{1}{n+1} \times \frac{m}{n}\left[1 + 2 + \ldots\ldots\ldots + n\right]$$

$$= \frac{n+1}{n+1} m + \frac{1}{(n+1)} \times \frac{m}{n} \times \frac{n(n+1)}{2}$$

Expectation = $m + \frac{m}{2}$

Q. 5 - A letter is known to khave either from Agra or from Mathura or. Satara. on the stamp two consecultive word 'RA'are legible. Find the chance that letter from Agra .

Ans- A : .Two consecutive word 'RA'are legible .

B_1: . letter is from $\underline{\underline{A}\overline{G}\overline{R}\underline{A}}$

B_2: . letter is from $\overline{M}\underline{A}\overline{T}\overline{H}\overline{U}\overline{R}\underline{A}$

B_3: . letter is from $\underline{\underline{S}}\overline{A}\overline{T}\underline{A}\overline{R}\underline{A}$

$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$

$P(B/B_1) = \frac{1}{3(3 \text{ Consecutive } 3 \text{ letter word})} = \frac{1}{3}$

$P(A/B_2) = 1/6$. $P(A/B_2) = 1/5$.

$P(A) = P(B_1).\ P(A/B_1) + P(B_2) + P(A/B_2) + P(B_3)\ P(A/B_3)$

$= \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{5}$

$P(B_1 / A) = \frac{P(B_1)\ .\ P(A / B_1)}{P(A)}$

$$\frac{\frac{1}{3} \times \frac{1}{3}}{\frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{5}}$$

$P(B_1 / A) = \frac{10}{21}$

Dumb question - Q. How . $P(A/B_1) = \frac{1}{3}$?

<u>And</u> - B_1 Letter from AGRA
So, in AGRA there are 3 two letter consecutive word AG, Ga, RA

and A $\rightarrow$ RA is legible

So, there is only one 'RA'

$\therefore P(A / B_1) = \frac{1}{3}$

<u>Hard level</u>

<u>Q. 1</u> Aline of length L is divided (broken) in to there parts randomly . Find the chance that these parts are the possible sides of Δ

<u>Ans</u>- For to be sides of Δ,

x > o, y > o
L > X + y

Sum of two sides .> third side
x + y > L -(x + y)

$(x + y) > \frac{L}{2}$ .....(1)

L - x > X

$\Rightarrow \frac{L}{2}$ .....(2)

L - y > y

$\frac{L}{2}$ .....(3)

n(s) $\frac{1}{2}$ x L x L = $\frac{L^2}{2}$

(from eqn. L > x + y)

n(A) = Area of ΔABC

$\frac{1}{2}$ x $\frac{L}{2}$ x $\frac{L}{2}$

= $\frac{1}{2}$ x $\frac{L^2}{4}$

P(E)= $\frac{n(A)}{n(S)}$

$$= \frac{\frac{1}{2}\frac{L^2}{4}}{\frac{1}{2}L^2} = \frac{1}{4}$$

<u>Q. 2</u> A box contains 6 balls. It is equally likcly that ball is white or black . Two balls are drawn from this bag one is foond black and one is whit.

find the chance that there is 3 black ball in the box .

<u>Ans</u> . 1st ball can be white of black = 2 way

2nd ball can be white of black = 2 way

img .....Total no. of ways in which 6 balls can be white of black

= 2 X 2 X 2 X 2 X 2 X 2 = 2^6 ways

n(s) = 2^6

$P(B_o) = \frac{1}{2^6}$, $P(B_1) = \frac{6C_1}{2^6} = \frac{6}{2^6}$

$P(B_2) = \frac{6C_1}{2^6} = \frac{15}{2^6}$, $P(B_3)$ $\frac{6C_1}{2^6} = \frac{20}{2^6}$

$P(B_4)$ $\frac{6C_4}{2^6} = \frac{15}{2^6}$, $P(B_5)$ $\frac{6C_5}{2^6} = \frac{6}{2^6}$

$P(B_6) = \frac{6C_6}{2^6} = \frac{1}{2^6}$

A $\rightarrow$ Two balls drawn from bag & of diff. color

$P(A / B_0)$ = o [b/c in draw of 2 balls one black ball is there]

$P(A / B_1) = \frac{5C_1 \times 1C_1}{6C_2} = \frac{5}{15}$

$$P(A / B_2) = \frac{4C_1 \times 2C_1}{6C_2} = \frac{8}{15}$$

$$P(A / B_3) = \frac{3C_1 \times 3C_1}{6C_2} = 9/15$$

$$P(A / B_4) = \frac{4C_1 \times 2C_1}{6C_2} = \frac{8}{15}$$

$$P(A / B_5) = \frac{5C_1 \times 1C_1}{6C_2} = \frac{5}{15}$$

$$P(A / B_6) = 0$$

$P(A) = P(B_o)\, P(A / B_o) + P(B_1)\, P(A / B_1) + P(B_2)\, P(A / B_2) + P(B_3)\, P(A / B_3) + P(B_4)\, P(A / B_4) + P(B_5)\, P(A / B_5) + P(B_6)\, P(A / B_6)$

$$= \frac{1}{2^6} \times 0 + \frac{6}{2^6} \times \frac{5}{15} + \frac{15}{2^6} \times \frac{8}{15} + \frac{20}{2^6} \times \frac{9}{15}$$

$$+ \frac{15}{2^6} \times \frac{8}{15} + \frac{6}{2^6} \times \frac{5}{15} + \frac{1}{2^6} \times 0$$

$$P(B_3 / A) = \frac{P(B_3)\ P(A / B_3)}{P(A)}$$

$$= \frac{\frac{15}{2^6} \times \frac{8}{15}}{\frac{6}{2^6} \times \frac{5}{15} + \frac{15}{2^6} \times \frac{8}{15} + \frac{20}{2^6} \times \frac{9}{15} + \frac{15}{2^6} \times \frac{8}{15} + \frac{6}{2^6} \times \frac{5}{15}}$$

$$= \frac{1/8}{\frac{1}{16} + \frac{4}{16} + \frac{3}{16}}$$

$P(B_3 / A) = \frac{1/8}{8/16} = \frac{1}{4}$

<u>Q. 3</u>- Three whole no. is selected at random and multipliea find the chance that-

(1) Their product is divisible by 5 but not 10.

(2)Divisible by 10 .

(3) is an even no.

<u>Ans</u>- $E_1 \rightarrow$ Digits at unit plce of 3 whole no. 1, 3, 7, 9

$E_2 \rightarrow$ digits at unit place of whole no. is 1, 3, 5, 7, 9

$E_3 \rightarrow$ digits at unit place of 3 whole no;s is 1, 2, 3, 4, 5, 6, 7, 8, 9 there are 10 digits -

$P(E_1) = \frac{4}{10} \times \rightarrow$ Ist Whole no. , $\frac{4}{10} \times \rightarrow$ 2nd Whole no., $\frac{4}{10} \times \rightarrow$ 3rd Whole no., $= \frac{8}{125}$

$P(E_2) = \frac{5}{10} \times \frac{5}{10} \times = \frac{1}{8}$

$P(E_3) = \frac{8}{10} \times \frac{8}{10} \times \frac{8}{10} = \frac{64}{125}$

divisible by 5 not by 10 $\rightarrow$ at unit place only 5 is there .

(1) P(divisible by 5 not by 10) = $P(E_2) - P(E_1)$

$= \frac{1}{8} - \frac{8}{125} = \frac{125 - 64}{1000}$

$= \frac{61}{1000}$

(2) (divisible by 10 means last digit is zers)

$= 1 - \frac{61}{1000} - \frac{64}{125} = \frac{427}{1000}$

Dumb question -

Q. How P(disivible by 5 not by 10) = $P(E_2) - P(E_1)$

<u>Ans</u>- Divisible by 5 not 10 only when at uniy place only 5 is there.

$P(E_1)$ = Prob. of having 1, 3, 7, 9 at unit place

$P(E_2)$ = Prob. of having '5' at unit place

$= P(E_2) - P(E_1)$

=P (divisible by 5 not by 10)

(3) P(even) = 1 - $P(E_2)$ [have digit 2, 4, 6, 8 at unit place]

$= 1 - \frac{1}{8}$

$= \frac{7}{8}$

<u>Key words</u>

- Sample space
- Equally likely events
- Mutually exclusive events
- Exhaustive events
- Independent events
- Expectation value
- Bernoulli's trial
- Boye's theorem (Total frobability theorem)
- Infinite sample space