Step-by-Step Guide to Reproducing Figure 2 from Counsell et al. (2025)

Based on the paper "Interface modes in inspiralling neutron stars" [arXiv:2504.06181v1]

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Overview

This guide outlines the computational steps required to generate the data points for a plot similar to Figure 2 in Counsell et al. (2025). Each data point $(f, |\Delta\Phi|)$ on the plot represents the signature of an interfacial mode (i-mode) from a neutron star with a first-order phase transition. The color of the point is determined by the strength of this transition, $\Delta\epsilon/\epsilon_i$. The process involves stellar structure calculations, solving perturbation equations, and calculating the resulting gravitational-wave signature.

1 Step-by-Step Guide

1.1 Step 1: Identify the Phase Transition in Your EoS

First, you must analyze your Equation of State (EoS) data to locate and quantify the first-order phase transition.

1. **Locate the transition:** Plot pressure p versus energy density ϵ. A first-order phase transition manifests as a region where p remains constant ($p = p_{\mathrm{trans}}$) while ϵ increases from a value ϵ_1 to ϵ_2.

2. **Calculate the energy density jump ($\Delta\epsilon$):** This is the magnitude of the discontinuity in energy density.
$$\Delta\epsilon = \epsilon_2 - \epsilon_1$$

3. **Determine the relative jump:** The color axis in Figure 2 is the relative jump in energy density. This is calculated as:
$$\frac{\Delta\epsilon}{\epsilon_i}$$
where ϵ_i is the energy density at the start of the interface, i.e., $\epsilon_i = \epsilon_1$.

1.2 Step 2: Calculate Stellar Structure

For a chosen stellar mass M (e.g., $1.4M_\odot$), solve the general relativistic Tolman-Oppenheimer-Volkoff (TOV) equations to determine the star's equilibrium structure.

- **Inputs:** Your EoS in the form $p(\epsilon)$.
- **Outputs:** The star's total radius R and the internal profiles for pressure $p(r)$, energy density $\epsilon(r)$, enclosed mass $m(r)$, and the metric potentials $\nu(r)$ and $\lambda(r)$.

- **TOV Equations** (in units where $c = G = 1$):

$$\frac{dp}{dr} = -\frac{(\epsilon(r) + p(r))\big(m(r) + 4\pi r^3 p(r)\big)}{r\,(r - 2m(r))}$$
$$\frac{dm}{dr} = 4\pi r^2 \epsilon(r)$$

1.3 Step 3: Calculate the i-Mode Oscillation

This is the most complex computational step. You must solve the general relativistic fluid perturbation equations for the stellar model obtained in Step 2 to find the i-mode. The paper uses the relativistic Cowling approximation.

- **Inputs:** The stellar profiles ($p(r)$, $\epsilon(r)$, $m(r)$, etc.) and the speed of sound profile $c_s^2(r) = dp/d\epsilon$.
- **Outputs:** The eigenfrequency ω of the interfacial mode and its corresponding radial and angular eigenfunctions, $W_l(r)$ and $V_l(r)$.
- **Identification:** The i-mode is identified by its characteristic eigenfunctions, which exhibit a sharp "kink" or discontinuity at the radial location of the phase transition.

1.4 Step 4: Calculate Tidal Overlap (Q_l) and Normalization ($\mathcal{A}^2$)

Using the i-mode eigenfunctions and the background stellar structure, compute the tidal overlap integral Q_l and the mode normalization constant $\mathcal{A}^2$. The paper uses the ‘l=2‘ mode.

- **Tidal Overlap Integral** Q_l (from Eq. 4):

$$Q_l = \frac{l}{c^2}\int_0^R e^{(\nu+\lambda)/2}(\epsilon + p) r^l [W_l + (l+1)V_l] dr$$

- **Mode Normalization** $\mathcal{A}^2$ (from Eq. 5):

$$\mathcal{A}^2 = \frac{1}{c^2}\int_0^R e^{(\lambda-\nu)/2}(\epsilon + p)[e^\lambda W_l^2 + l(l+1)V_l^2] dr$$

1.5 Step 5: Calculate Plot Coordinates (f, $|\Delta\Phi|$)

With all components calculated, you can now find the final coordinates for your plot.

1. **Gravitational-Wave Frequency (f):** The GW frequency of the resonance is half the mode frequency.
$$f = \frac{\omega}{2\pi}$$
2. **Orbital Phase Shift ($|\Delta\Phi|$):** This is the primary observable, estimated using Equation (6) in the paper. For an equal-mass binary ($q = M'/M = 1$) and the $l = 2$ mode, the formula is:

$$|\Delta\Phi| \approx 2\pi \times \frac{5\pi}{4096}\left(\frac{c^2 R}{GM}\right)^5 \frac{1}{(1+1)}\frac{GM/R^3}{\omega^2}\left(\frac{Q_2}{MR^2}\right)^2 \frac{MR^2}{\mathcal{A}^2}$$

1.6 Step 6: Plot the Results

1. Create a scatter plot, typically with logarithmic axes, with the GW frequency f on the x-axis and the phase shift $|\Delta\Phi|$ on the y-axis.

2. For each data point corresponding to a specific EoS, set its color based on the value of $\Delta\epsilon/\epsilon_i$ calculated in Step 1.

3. (Optional) To complete the plot, you can overlay the detector sensitivity curves. These are calculated using Equation (7):

$$|\Delta\Phi(f)| = \frac{\sqrt{S_n(f)}}{2A(f)\sqrt{f}}$$

where $S_n(f)$ is the detector's public noise power spectral density and $A(f)$ is the gravitational-wave amplitude for a binary at a given distance (the paper uses 40 Mpc).