Conditional Probability

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1 Motivation

Since conditional probability has appeared on the COMC for two consecutive years(2020,2021), I think that making a handout on it would be appropriate. I present the main ideas, then analyze the two COMC problems which make use of it. In doing this, I present an incorrect solution and explain what's wrong with it before giving the correct solutions, because conditional probability is somewhat counter intuitive, and many people make the same mistakes. Don't hesitate to contact me if this handout has any issues.

2 Preliminaries

All that is needed to understand this is some familiarity with basic probability(high school level). Obviously no conditional probability is needed.

3 Conditional Probability Basics

So let's start with defining what a conditional probability is. Here's an example, say we analyze two events let's say A and B, and we then want to find the probability of A given that B has occurred, this would be a conditional probability, and we would use the following notation to denote this:
$P(A|B)$
How would we calculate these? Well there are two main ways, the first way is by using the following result:

Theorem 1. *Let A and B be two events in the same sample space, then* $P(A|B) = \frac{P(A \cap B)}{P(B)}$

We won't prove this, but to understand the general idea behind it, just think of it as almost reducing the sample space to just a space where B has occurred, in this space, A only happens if $A \cap B$ has happened(trivially), and all probabilities are scaled down by a factor of $P(B)$. To get an intuitive idea of what I mean by this scaling, just think of B, in the entire sample space it occurs with probability $P(B)$, but in the sample space where B is guaranteed to occur, the probability of B is $1 = \frac{P(B)}{P(B)}$, it's scaled down.

The other way to compute conditional probability is to use Bayes's theorem. Which is:

Theorem 2. *Let, A and B be two events in the same sample space, then* $P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}$

Proof. We note that(by theorem 1) $P(B|A) = \frac{P(B \cap A)}{P(A)}$, plugging this in to the expression specified by the theorem gives us:
$\frac{P(A) \cdot (\frac{P(B \cap A)}{P(A)})}{P(B)} = \frac{P(B \cap A)}{P(B)} = \frac{P(A \cap B)}{P(B)}$ Again using theorem 1, we see that this is $P(A|B)$, thus Bayes's theorem has been verified. $\square$

4 COMC Problems

Now that we have the basics, let's analyze the two COMC problems. As outlined earlier, I will present an incorrect solution first.

Example 4.1. *2020 COMC B2*
Alice places a coin, heads up, on a table then turns off the light and leaves the room. Bill enters the room with 2 coins and flips them onto the table and leaves. Carl enters the room, in the dark, and removes a coin at random. Alice reenters the room, turns on the light and notices that both coins are heads. What is the probability that the coin Carl removed was also heads?

Fake Solution 4.1. We notice that there are three possible arrangements of coins when Carl enters, HHH, HTH, and HHT, where the letter corresponds to the face up. Each of these is equally likely, and the only once where Carl takes a heads is the first, thus the probability that the coin Carl removed was a heads is $\frac{1}{3}$

Now, why is this wrong? Well, we didn't adequately consider the two conditions that the problem provided, that being that both coins left are heads, and that Carl took a heads. With these conditions in mind it's clear that, although having HHH, HTH, and HHT are equally likely, the fact that there are two heads remaining means that, with our conditions, they're not. For example, in the HHH case, Carl can take anything and end up with two heads left, but for the others, he must take the tails. Considering this leads to the real solution presented below:

Real Solution 4.1. We denote the event of Carl picking up a heads as A and the event of there being two heads left as B. We know $P(A|B) = \frac{P(A \cap B)}{P(B)}$. $P(A \cap B)$ is easy to compute, it's just the probability of Carl picking up a heads and leaving two heads, this only happens when we have three heads in a row, which happens with probability $\frac{1}{4}$. Then we find the probability of B, clearly B can only happen with HHH, HTH, and HHT, each of these has probability $\frac{1}{4}$(the other possibility is HTT). Furthermore, for the HTH and HHT case the probability is multiplied by $\frac{1}{3}$, since Carl needs to remove the tails, and he choses at random. Adding everything up gets us $P(B) = \frac{1}{4} + \frac{1}{12} + \frac{1}{12} = \frac{5}{12}$. Then, computing $\frac{P(A \cap B)}{P(B)}$ gets us $\frac{\frac{1}{4}}{\frac{5}{12}} = \frac{3}{5}$, which is indeed the correct answer.

We consider the 2021 problem now, for the sake of brevity we omit a false solution.

Example 4.2. *2021 COMC B1*
A bag contains two regularly shaped (cubic) dice which are identical in size. One die has the number 2 on every side. The other die has the numbers 2 on three sides and number 4 on each side opposite to one that has number 2. You pick up a die and look at one side of it, observing the number 2. What is the probability the opposite side of the die has the number 2 as well?

Real Solution 4.2. Let A be the event that we picked the die with only 2's, and let B be the event that we pick a die at random and see the number 2. Clearly the question is asking for $P(A|B)$. We use Bayes's theorem, which states $P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}$. $P(B|A)$ is easy to find, it's asking for the probability that we see a 2 given that we picked the dice with only 2's on it. Thus it's clear $P(B|A) = 1$. $P(A) = \frac{1}{2}$, you have two dice and you pick one at random. Lastly, $P(B) = (1 \cdot \frac{1}{2}) + (\frac{1}{2} \cdot \frac{1}{2}) = \frac{3}{4}$, we have a 50% chance of picking a die with only 2's, and a 50% chance of picking a die with half 2's, half 4's, hence we obtain that expression. Putting everything together gives us $P(A|B) = \frac{1 \cdot \frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$

5 Conclusion

I hope that this handout has proven helpful, and that walking you through each solution has given you an idea of how to apply condition probability in math problems. Make sure to avoid the errors shown in the false solution, errors which I used to make quite a bit. Again, contact me if there are any issues with this handout, and hopefully if there's a conditional probability question on the 2022 COMC, you answer it correctly!

6 Works Cited

https://www2.cms.math.ca/Competitions/COMC/2021/
https://www2.cms.math.ca/Competitions/COMC/2020/
Probability Theory: A Concise Course by Y.A. Rozanov