Orbit Stabilizer Theorem

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1 Motivation

The orbit stabilizer theorem is a somewhat combinatorial result in group theory. By analyzing the stabilizer and orbit of a given element in a permutation group, you can compute its order, this has some direct application for finding certain groups(rotation groups). It also comes in useful when proving important theorems in group theory, such as Burnside's lemma, and the Sylow theorems, but these are beyond the scope of this handout.

2 Prerequisites

You should have some idea about what a permutation group, and by extension what group is. You should also have a general knowledge of algebra up to Lagrange's theorem. The last part of the handout uses free groups and such. Everything else will be covered in this handout.

3 One last note before beginning!

In making this I essentially wrote down the definition of an orbit and a stabilizer, and I tried to prove the theorem myself. I then applied it to some problem I found in a textbook(which is cited at the bottom of this handout). I did the needed work, and now I'll show you what I did, in hopes that by highlighting my thought process in proving the result you may gain some newfound insight. Also, if there are any errors you should contact me, this is my first handout after all... Also(again), I included proof ideas here, I first encountered that while reading Introduction to the Theory of Computation by Sipser.

4 Lagrange's Theorem,Orbits, and Stabilizers

I know I said you should already know Lagrange's theorem, but I'm just going to state it because it's important.

Theorem 1. *For every group G and H, which is a subgroup of G, $|H|||G|$, and $\frac{|G|}{|H|}$ is the number of left/right cosets of H in G*

This is a well known result. We now define orbits and stabilizers, for reference, in these definitions we're working with a group of permutations G over some set S.

Definition 4.1. The stabilizer of i in G, where $i \in S$, which we call $Stab_G(i)$, is the set of all permutations f in G such that $f(i) = i$.

It's clear that the stabilizer is always a subgroup of G, since clearly it contains the identity and it is closed under permutation composition(if f fixes i, g fixes i then fg would also fix it).

Definition 4.2. The orbit of i in G is the set of all possible elements that a member of G can take i to. We say $Orb_G(i) = \{n | f(i) = n, f \in G\}$.

5 Orbit Stabilizer Theorem

We now state the main focus of this handout, and obviously then prove it.

Theorem 2. *Let G be a permutation group on a set S, and $i \in S$, then $|G| = |Orb_G(i)| \cdot |Stab_G(i)|$.*

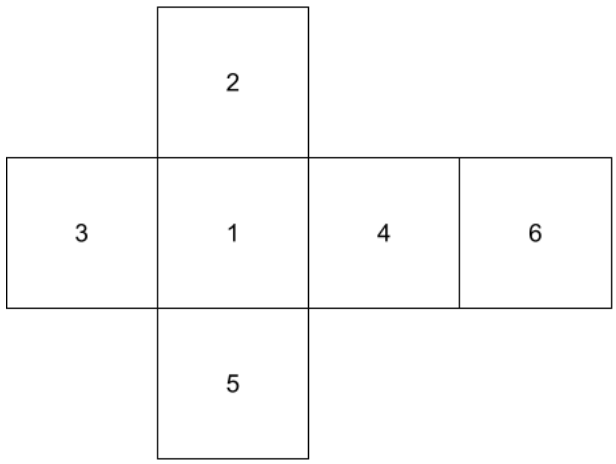
Proof Idea 5.1. *Since $Stab_G(i)$ is a subgroup, we seek to apply Lagrange's theorem. The way in which we seek to do this is by showing that $|Orb_G(i)|$ is the number of left cosets of $Stab_G(i)$ in G. Once this has been verified, we then apply Lagrange's theorem directly and prove the orbit stabilizer theorem.*

Proof. Clearly every permutation in G takes i to some element within $Orb_G(i)$, we show that if two permutations of G take i to the same element then they have the same left coset with regards to $Stab_G(i)$. Assume that $f(i) = j, g(i) = j, f, g \in G$, then clearly $fStab_G(i) = gStab_G(i)$, since $f^{-1}g$ fixes i, thus $f^{-1}gStab_G(i) = Stab_G(i)$. Similar logic yields that if two permutations take i to different elements within $Orb_G(i)$ then they have different left cosets, thus by Lagrange's theorem, we get $\frac{|G|}{|Stab_G(i)|} = |Orb_G(i)| \rightarrow |G| = |Stab_G(i)| \cdot |Orb_G(i)|$ □

6 Rotation Group of Cube

Now in this section we compute the rotation group of a cube, which we denote as G, although we do get a little carried away in how we do it. Now, right off the bat, we find its order. The rotation group of a cube must be the set of all rotations which leave the cube looking idential to what it once was. We number each face, now the rotation group is just a set of permutations of all 6 faces. Pick face 1, it's orbit has size 6, since you can rotate it and get it to any face you need, and its stabilizer has size 4, since you have 4 rotations about the perpendicular line though face 1.
Thus, by the orbit stabilizer theorem, $|G| = 4 \cdot 6 = 24$. Now things get interesting. Keeping with the notion that $|G|$ is just a permutation group on the 6 faces, we find that G is isomorphic to the subgroup of S_6 generated by

$(1265), (1463), (3245)$, we get these numbers from the following figure:



In essence, each permutation corresponds to a $90°$ rotation with respect to one of the three axes. We now note the following: $(1463)(3245) = (3245)(1265)$, thus we can drop (1463) from our generating set. We now do some algebra to get elements of good orders. We know $(1265)(1463) = (326)(145)$, and $(1265)(326)(145) = (14)(52)(63)$, finally, we get that $(1265)(14)(52)(63) = (635)(142)$. In all of this, we have shown that G can be generated by (1265) and $(14)(52)(63)$, and that their product has order 3. If we denote (1265) as a and $(14)(52)(63)$ as b, we get that G is the homomorphic image of $G' = \langle a, b | a^4 = b^2 = (ab)^3 = e) \rangle$, but looking at this we see that this is S_4, which has the same order as G, so clearly $G = S_4$. Now this is my first time dealing with generators and relations so it may be off, so don't hesitate to contact me and tell me if I'm wrong. Regardless, I believe my presentation of the orbit stabilizer theorem was fairly accurate, and I hope that you were able to take away something from this handout.

7 Works Cited

Contemporary Abstract Algebra by Gallian
http://www.weddslist.com/groups/misc/serpres.html