Indian Institute of Technology Roorkee

Department of Electronics and Communication Engineering

ECN-360: Introduction to Information and Communication Theory	Mid-term Exam
Duration: 1.5 Hrs	Spring 2022-2023
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Max. Marks: 50	TAs: Ankita, Ragini, and Akhilesh

Instructions:

1. All questions are compulsory.
2. Assume suitable data if necessary.

Q1. Consider a quantum key distribution system, in which the transmitter (Alice) sends binary quantum bits (qubits) '0' and '1' with the probability γ and 1- γ, respectively, to the receiver (Bob) in the presence of an Eavesdropper (Eve) through zero mean AWGN quantum channel as shown in Fig. 1.

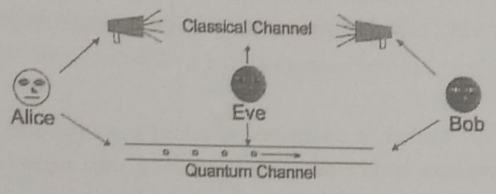

Figure 1

To ensure the security of a system, Bob uses a dual threshold detection to receive qubit '0' and qubit '1'. The probability density function (PDF) of Bob's received sample can be illustrated in Fig. 2.

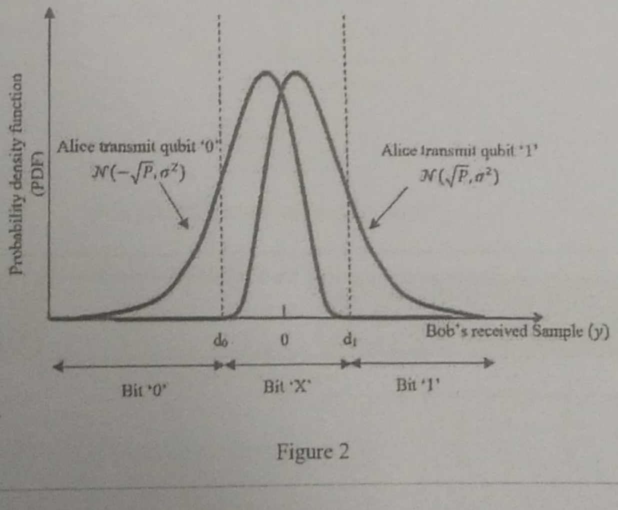

Figure 2

The decision region for the received sample (y) used by Bob can be given by

$$\text{Decision} = \begin{cases} 0 & y \le d_0 \\ 1 & y \ge d_1 \\ X & \text{otherwise} \end{cases}$$

where $d_0 = -d_1$. Conditional probabilities that Bob received qubit '0' when Alice transmitted qubit '0' and qubit '1' are $P_B(0|0) = Pr_B(y \le d_0|0) = \alpha$ and $P_B(0|1) = Pr(y \le d_0|1) = \beta$, respectively. Similarly, the conditional probability that Eve received qubit '0' when Alice transmitted qubit '0' is $P_E(0|0) = \delta$. The PDF of Eve's received sample is similar to Fig. 2 with the modified decision region as $d_E = d_0 = d_1 = 0$.

a) Draw the discrete memoryless channel (DMC) representation for Alice-Bob. (6 Marks)

b) Calculate the mutual information between Alice and Bob using the channel deduced in part (a) in terms of α and β, for $\gamma = \frac{1}{2}$. (2 Marks)

c) Draw the DMC representation for Alice-Eve. (6 Marks)

d) Find out at what value of γ, the mutual information between Alice and Eve becomes maximum. Also, calculate the capacity of the channel deduced in part (c) in terms of δ. (2 Marks)

Q2. The downlink FDMA system is composed of a single source and K users as shown in Fig. 3, where total transmission bandwidth (B) is divided into many non-overlapping, but equal, K frequency bands, as shown in Fig. 4.

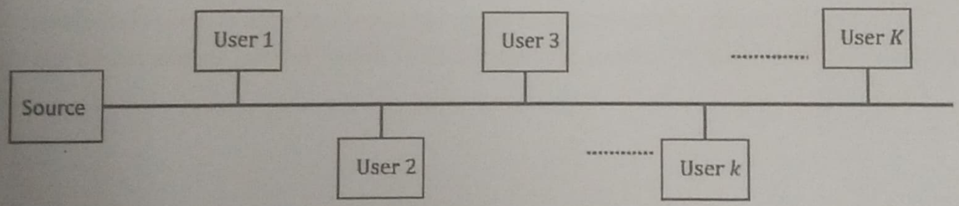

Figure 3

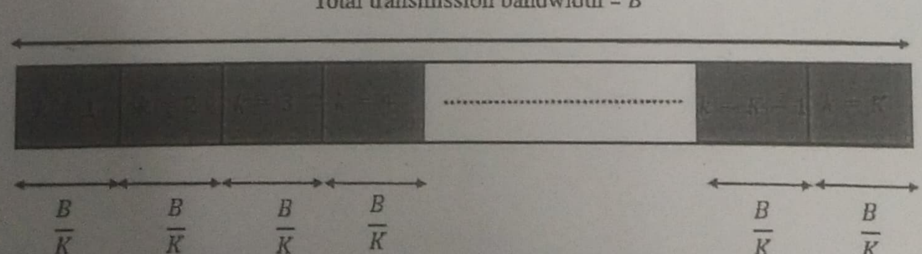

Figure 4

Each band is dynamically assigned to a specific user to transmit data. In an FDMA system, signals, while occupying their assigned frequency bands, can be transmitted simultaneously and continuously without interfering with each other.

For the considered FDMA system, signal received at i^{th} user receiver is given by

$$y_i = \sqrt{P_i}x_i + n_i, \qquad i \in \{1, 2, .., K\}$$

where n_i is the sampled value of zero mean Gaussian random process with double sided power spectral density $\frac{N_o}{2}$, x_i and P_i denote the transmitted symbol and power allocated to user i, respectively, $E[|x_i|^2] = 1$, $\sum_{i=1}^{K} P_i = P$, and P is the total transmit power.

a) Determine mean and variance of n_i. (2 Marks)

b) Determine the capacity of i^{th} user, C_i. (3.5 Marks)

c) It is intuitive that $\sum_{i=1}^{K} C_i \leq B\log_2\left(1+\frac{P}{BN_o}\right)$. For $K = 3$, find the value of P_1, P_2, and P_3 at which this equality hold? (3.5 Marks)

Q3. Consider a zero mean AWGN channel with power constraint P, where the signal takes n different paths and the received signals are added together at the receiver as shown in Fig. 5.

a) What is the capacity in bits/sample (bits/transmission) of this channel if all the noise are jointly normal with an $n \times n$ identity covariance matrix? (4.5 Marks)

b) What is the capacity of the channel for a Nyquist rate of 54×10^3 samples/second? (1.5 Marks)

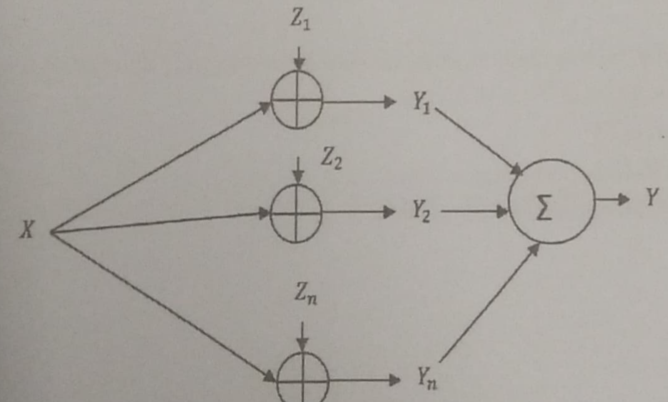

Figure 5

Q4. A memoryless Cauchy source X with probability density function

$$f_X(x) = \frac{1}{\pi\gamma\left[16+\frac{(x-p_o)^2}{\gamma}\right]}, \qquad -\infty < x < \infty$$

is quantized using a five-level quantizer defined by

$$\hat{X} = q(X) = \begin{cases} -1.5, & -4 \leq X < -2 \\ -0.5, & -2 \leq X < 0 \\ 0.5, & 0 \leq X < 1 \\ 1.5, & 1 \leq X < 2 \\ 6, & \text{otherwise} \end{cases}$$

a) Determine the entropy (in bits) of quantized source $\hat{X}$ given that $p_o = -1$ and $\gamma = 1$. (7 Marks)

b) Now let $\check{X} = i + 0.5, i \leq X < i + 1$, for $i = 0,1,2, \ldots$ Which random variable has higher entropy, $\hat{X}$ or $\check{X}$? (There is no need to compute entropy of $\check{X}$, just give your intuitive reasoning.) (2 Marks)

Q5. The Kullback-Leiber (KL) divergence for distributions $p(x)$ and $q(x)$ of a continuous random variable, is defined to be the integral as shown below:

$$D_{KL}(p(x)||q(x)) = \int_{-\infty}^{\infty} p(x) \ln\left(\frac{p(x)}{q(x)}\right) dx$$

Find the KL-divergence, if $p(x)$ and $q(x)$ are $\mathcal{N}(0,1)$ and $\frac{1}{4}e^{-\frac{|x|}{2}}$, respectively. (4 Marks)

Q6. Let X and Y be two random variables. Let X be uniformly distributed over $\{1, 2, 3, \ldots, 16\}$, and let $\Pr(Y = k) = 2^{-k}$; $k = 1, 2, 3, \ldots$

a) Find $H_4(X)$ (1.5 Marks)

b) Find $H_e(Y)$ (1.5 Marks)

c) Determine $I_2(X; Y)$, when $H_{10}(X, Y) = 1.2$ (3 Marks)