The Vertically Driven Damped Planar Pendulum using RK4

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References and Acknowledgements

Introduction

- **The System:** A vertically driven damped planar pendulum
- The equation of motion:

$$\boxed{\ddot{\theta} = -(\alpha - \beta \cos \tau) \sin \tau \; - \gamma\dot{\theta}}$$

where, α is the square of the ratio of natural frequency to driving frequency, β is the square of the ratio of driving amplitude to the length of the pendulum and γ is the square of the ratio of damping constant to the driving frequency.

- This is a second order ODE which cannot be solved analytically hence we used Runge-Kutta 4 algorithm to solve it computationally.

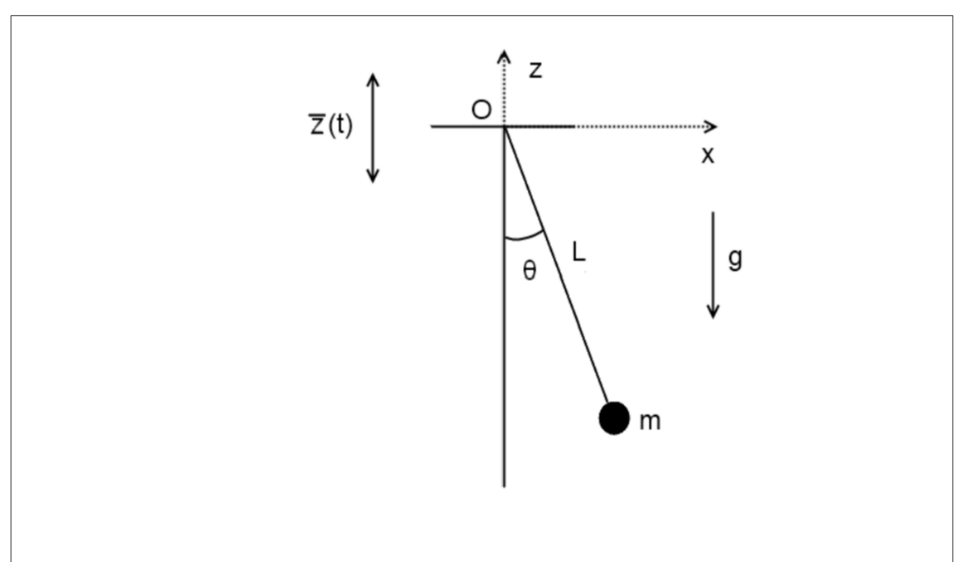


Figure 1: Rigid pendulum with motion $\bar{z}(t)$ of the Pivot

- The system tends to exhibit a very complex behavior caused by a very 'simple looking' equation with just a few parameters and initial conditions.
- Since it is a pendulum, the system has a downward equilibrium but surprisingly even with the driving and damping, it also has an upside-down equilibrium position.
- The system has been examined at parameter values:

  $\alpha = 0.02,\ \beta = 0.35,\ \gamma = 0.03.$
- At these parameters, the system has five different attractors such that after the transient has passed, the pendulum settles into a stable behavioral pattern or to a stable point depending on its initial condition.

Post-Transient Behaviors

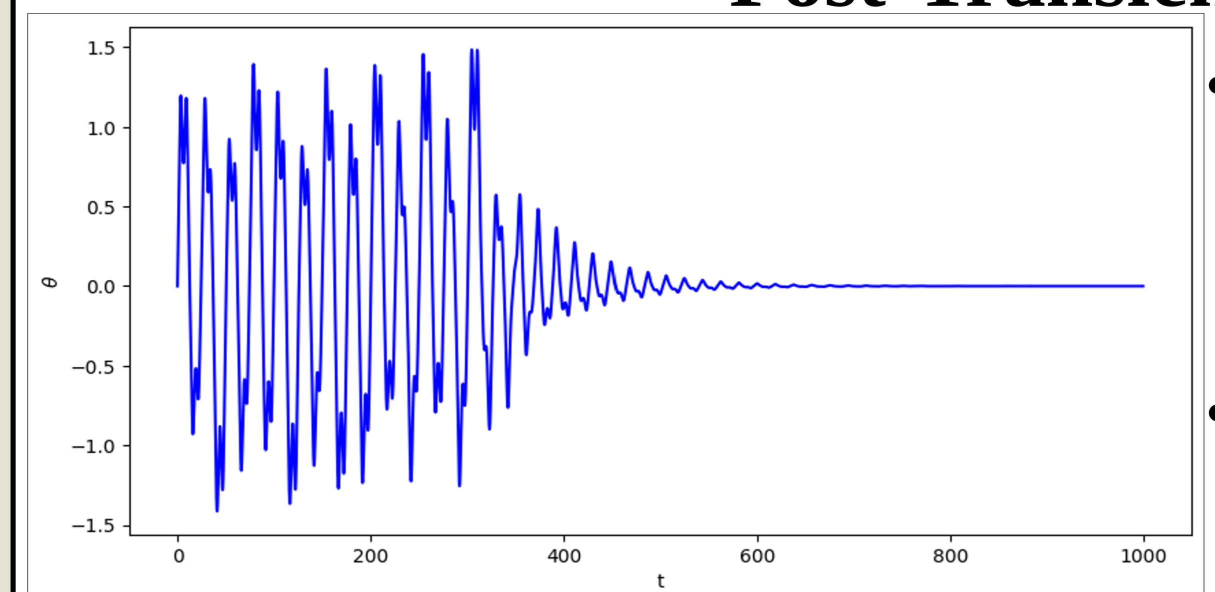


Figure 2: The pendulum settles to a stationary state at (0, 0) with Initial conditions $(\theta, \dot{\theta})$=(0,0.4)

- Transient behavior refers to behavior that dies out over time and settles into a steady state behavior that continues indefinitely, here dies down to a point.
- Figure 2 corresponds to the downward attractor which we know settles at $\theta = 0$.

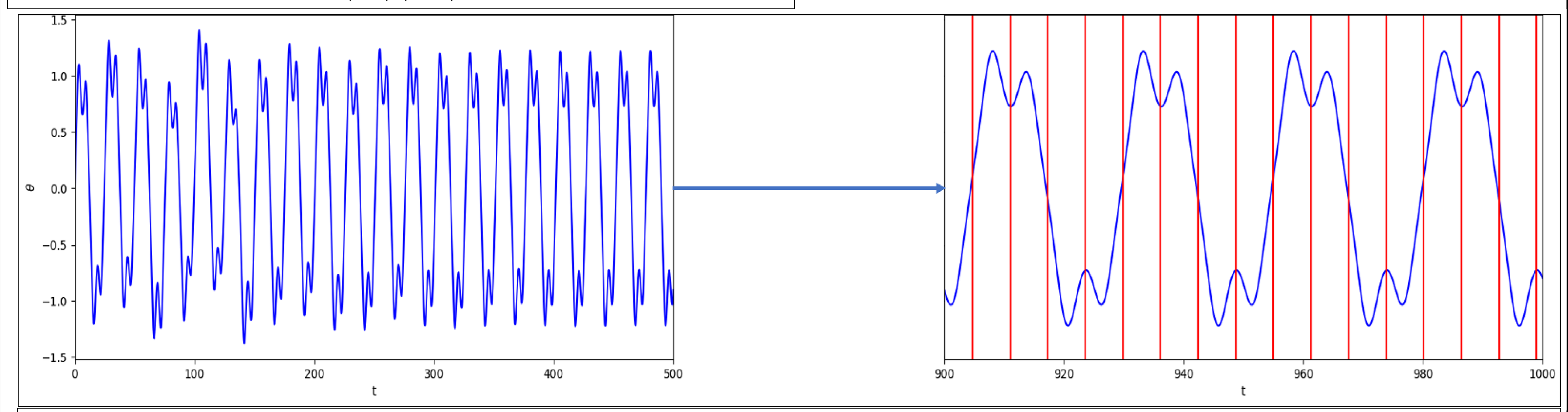
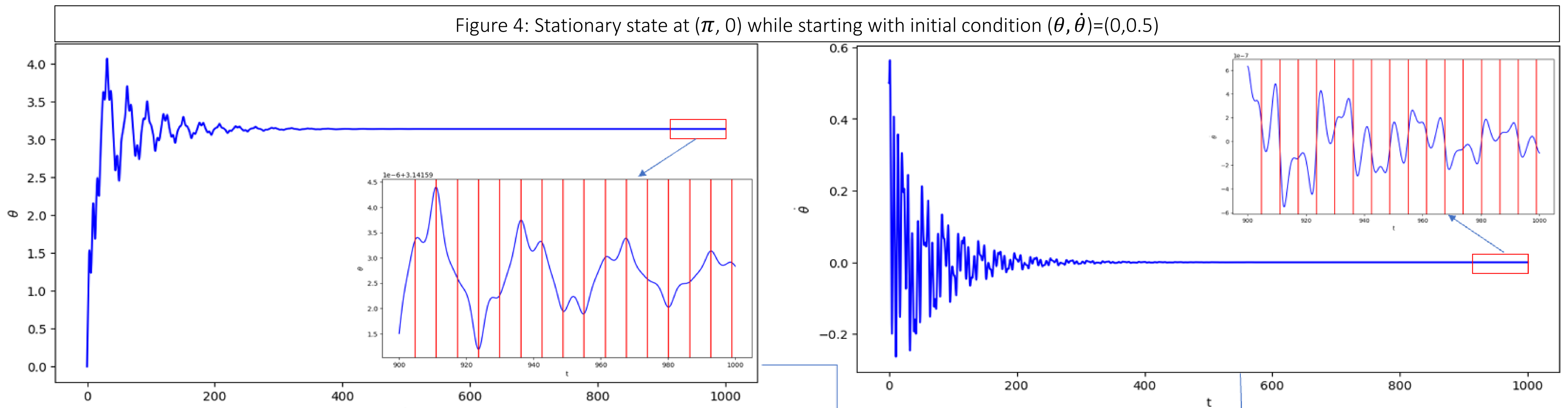


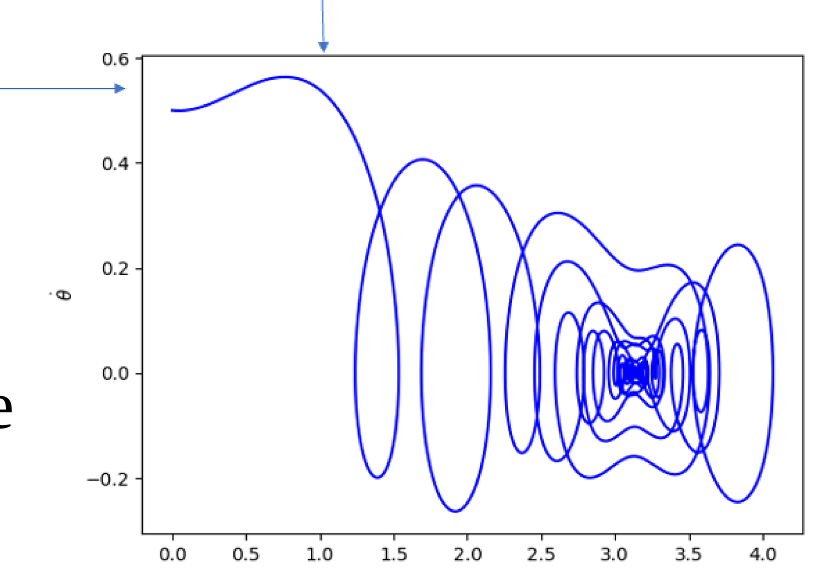
Figure 3: The pendulum settles into a stable point with Initial conditions $(\theta, \dot{\theta})$=(0,0.370)

- Figure 3 is an example in which the transient dies down to a steady state behavior that is not a point, but in-turn shows oscillatory behavior.
- The red lines here are at a 2π interval which means that it take exactly 4 cycles for the pendulum to come back to its initial position and start all over again.

Phase Maps

Figure 4: Stationary state at $(\pi, 0)$ while starting with initial condition $(\theta, \dot{\theta})$=(0,0.5)





- Phase diagrams are essentially graphical representations of the path of particles (here the mass of the pendulum) in the $(\theta, \dot{\theta})$ plane.
- While phase maps help us to visualize our system better, we do tend to lose information about the time dependence.

Conclusions and Outlooks

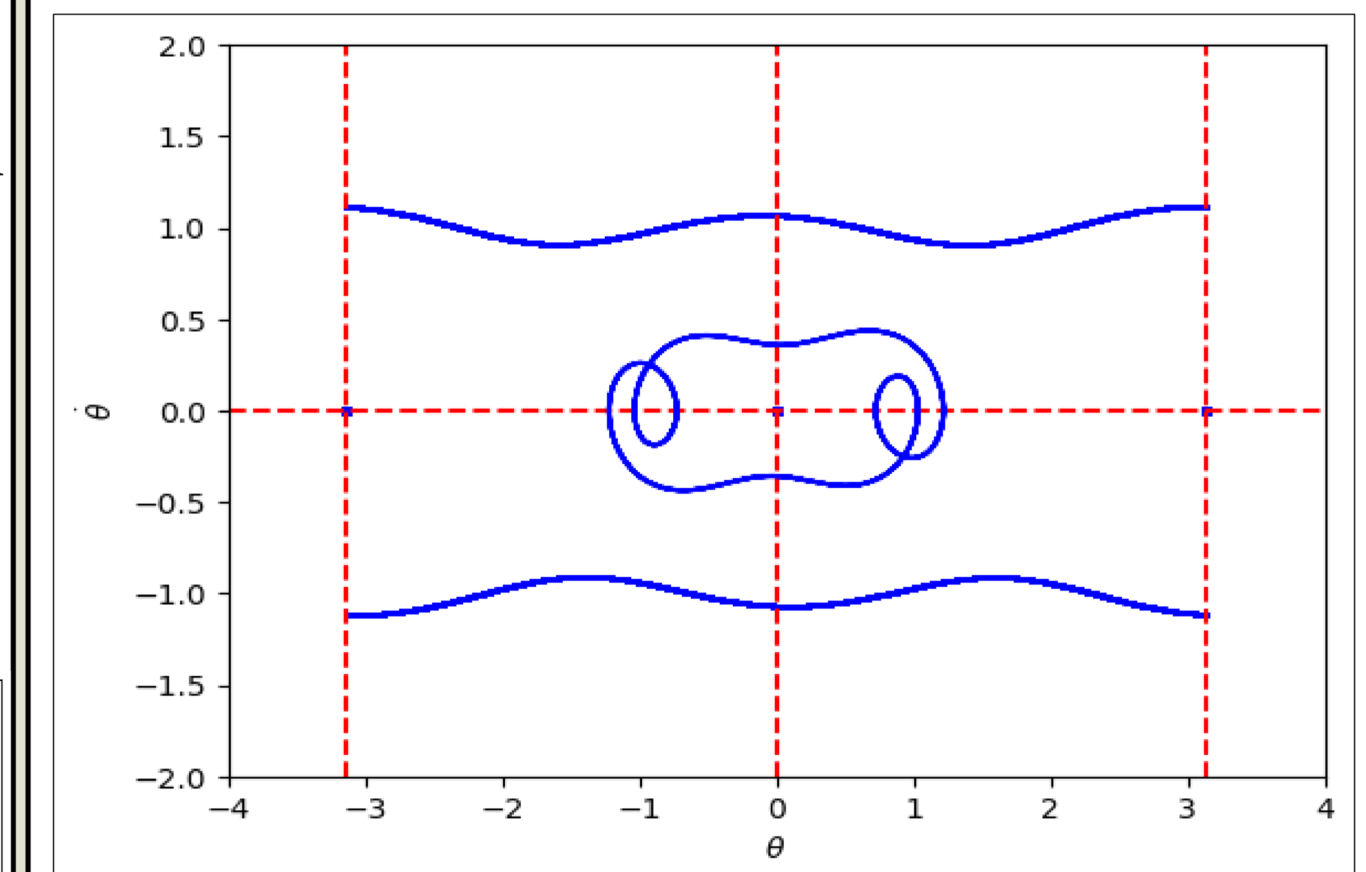


Figure 5: All attractors at parameter $\alpha = 0.02,\ \beta = 0.35,\ \gamma = 0.03$

- After removing transient behavior at various initial conditions, I was able to plot all the attractors as shown in figure 5.
- I was able to find each attractors basin of attraction by randomly selecting a combination of initial conditions in the range $(\theta, \dot{\theta}) = ([-\pi, \pi], [-\frac{\pi}{2}, \frac{\pi}{2}])$ and see what they settle into as shown in figure 6.

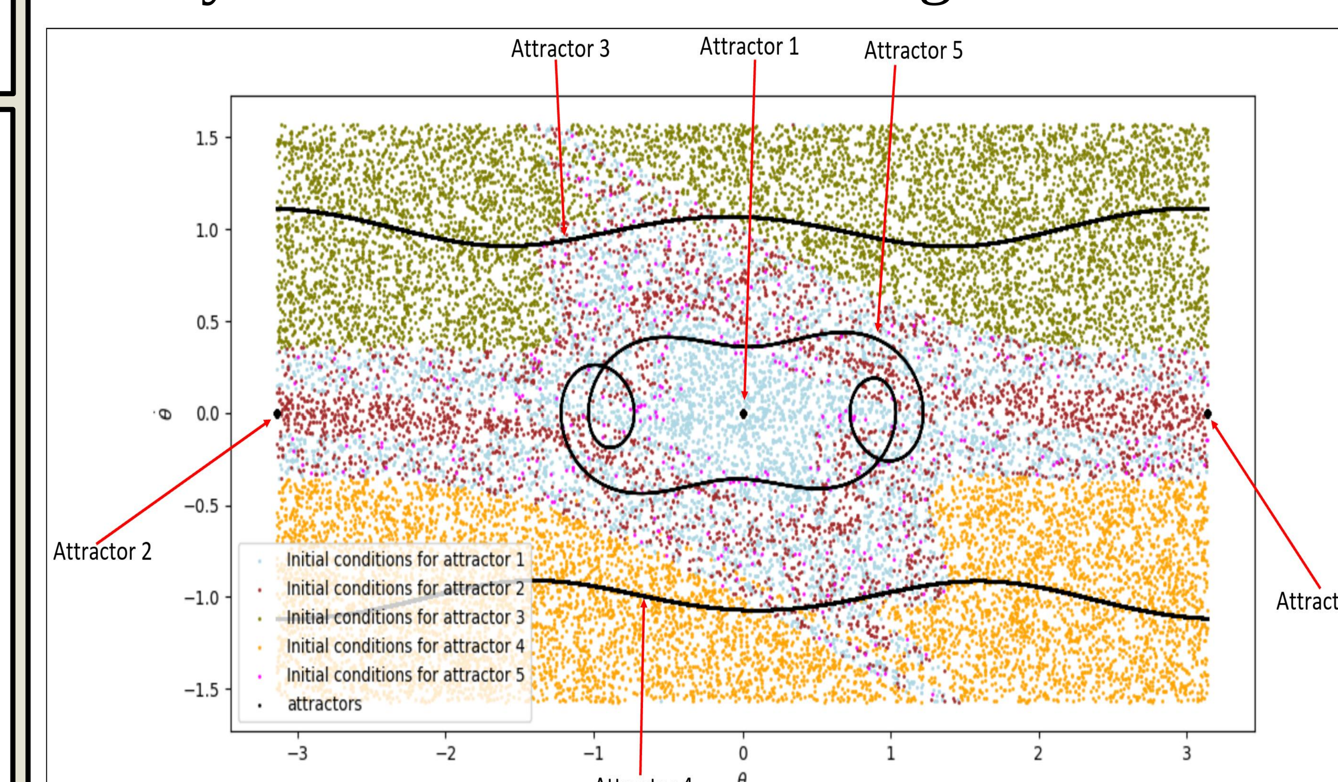


Figure 6: Initial conditions that settle into a given attractor.

Future Work

- The same study can be performed for another interesting parameter sets such as
  - $\alpha = 0.5,\ \beta = 0.1,\ \gamma = 0.03.$
  - $\alpha = 0.1,\ \beta = 0.545,\ \gamma = 0.08$

  as the configuration of attractor changes by changing the initial parameters.