Phase Diagrams

Lecture 31

Physics 311: Classical Mechanics
Fall 2011

21 November 2011

For the past few days we have been considering nonlinear systems, and using perturbation theory to study the behavior of these systems. However, perturbation theory doesn't tell us everything. It's really only useful in cases where we can consider the nonlinearity of the system to be a small perturbation on a linear system. There are plenty of circumstances in which this just isn't true. One tool when a system is just fundamentally nonlinear is the *phase diagram*, which is a good way to represent important qualitative and quantitative aspects of a system. Although this was introduced in the fall of physics 200, we will reintroduce it here as if it is brand new.

Although they are also informative in dissipative systems and there are analogs in higher dimensional systems, for today let's restrict ourselves to systems with one degree of freedom where energy is conserved. It can be very useful to think of phase diagrams in terms of Hamiltonian dynamics, so we will begin by reviewing this.

31.1 Review of Hamiltonians

Recall that, for a system in which energy is conserved described by a single (scleronomic) coordinate q the Lagrangian is some function

$$L = L(q, \dot{q}) = T(q, \dot{q}) - V(q) \tag{31.1}$$

Note that since we assume the coordinate q is scleronomic, the energy is equal to the Hamiltonian, and since we want to have energy conserved, we must have no explicit time dependence in the Lagrangian L. Furthermore, we assume that the potential does not depend on $\dot{q}$. Then, the conjugate momentum associated to q is

$$p = \frac{\partial L}{\partial \dot{q}} \tag{31.2}$$

Since the coordinates are scleronomic, we can write transformation rules between the coordinate q and cartesian coordinates $\{x, y, z\}$ as

$$x = x(q), \qquad y = y(q), \qquad z = z(q)$$

(think of the rigid pendulum, where $q = \theta$, the angle off the vertical, and the pendulum bob swings in the xy plane with a fixed relationship between x and y.) Then we must have

$$\dot{x} = \frac{\partial x}{\partial q}\dot{q}, \qquad \dot{y} = \frac{\partial y}{\partial q}\dot{q}, \qquad \dot{z} = \frac{\partial z}{\partial q}\dot{q}$$

so that

$$T(q, \dot{q}) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left[\left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 + \left(\frac{\partial z}{\partial q}\right)^2\right]\dot{q}^2$$

which means that we can always write it in the form

$$T(q, \dot{q}) = \frac{1}{2}mf(q)\,\dot{q}^2 \tag{31.3}$$

Now, this means that the conjugate momentum p will be

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}} = mf(q)\,\dot{q}$$

Recall that we define the Hamiltonian as

$$H = \sum_{j=1}^{J} \dot{q}_j p_j - L \tag{31.4}$$

when we have J generalized coordinates $\{q_j\}$. In our case, we only have one coordinate q, so this is

$$H = \dot{q}p - L = mf(q)\,\dot{q}^2 - T + V = \frac{1}{2}mf(q)\,\dot{q}^2 + V(q)$$

and when we transform this to write the Hamiltonian as a function of p and q, we get

$$H(p, q) = \frac{p^2}{2mf(q)} + V(q) \tag{31.5}$$

Hamilton's equations then give us back the dynamics of the system. They are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{2mf(q)}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -\frac{p^2 f'(q)}{2mf(q)^2} - \frac{\partial V}{\partial q} \tag{31.6}$$

Note that since energy is conserved, we always have

$$\frac{p^2}{2mf(q)} + V(q) = E = \text{constant}. \tag{31.7}$$

That is, the quantities $\{p, q\}$ always satisfy this relationship. Phase diagrams are essentially graphical representations of the paths of particles in the $\{p, q\}$ plane. Since in an energy conserving system the above equation is always satisfied, the particle path must trace out a curve that satisfies this equation. This will become clearer when we look at some examples.

31.2 The Simple Harmonic Oscillator

As our first example of a phase diagram, let's look at the time-honored simple harmonic oscillator in one dimension. In this case, as we know, our Lagrangian is simply

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

our momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

and our Hamiltonian is

$$H = p\dot{x} - L = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

In terms of p and x, our Hamiltonian is then

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{31.8}$$

and Hamilton's equations give

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial H}{\partial x} = -kx \tag{31.9}$$

Now, we know that we can combine these equations into the very familiar equation of motion for the system:

$$m\ddot{x} = -kx$$

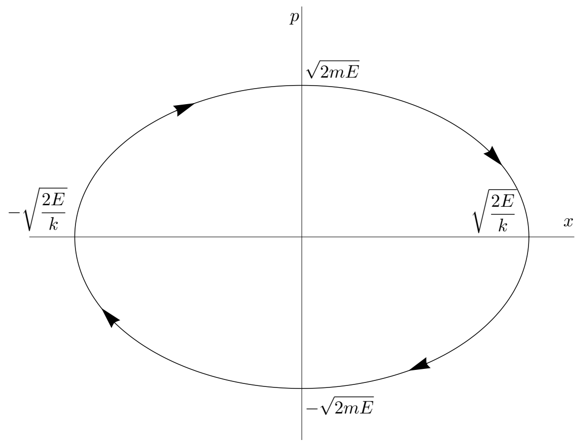

Figure 31.1: The path of the simple harmonic oscillator in phase space

which can be solved to give

$$x(t) = A\cos\left(t\sqrt{\frac{k}{m}} + \phi_0\right)$$

and then this implies

$$p(t) = m\dot{x} = -A\sqrt{km}\ \sin\left(t\sqrt{\frac{k}{m}} + \phi_0\right)$$

as well. However, suppose that (for whatever reason) we could not solve the equation of motion. We still have that energy is conserved in the system, which means that if the initial energy in the system is some constant E, then at all times we must have the relationship between p and x

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2. \tag{31.10}$$

This is the equation of an ellipse in $\{p, x\}$ space (also called *phase space*). Notice that the ellipse has its major and minor axes on the p and x axes, with lengths $2\sqrt{2mE}$ and $2\sqrt{\frac{2E}{k}}$. (Which is major and which is minor depend on the sizes of m and k.) The ellipse is shown in figure 31.1. Notice that it neatly summarizes what we know qualitatively about the motion of the SHO–momentum is greatest (either positive or negative) when the position is zero (the mass is passing through the equilibrium point), and momentum is zero when the displacement is greatest (either positive or negative). As the mass moves, it goes from

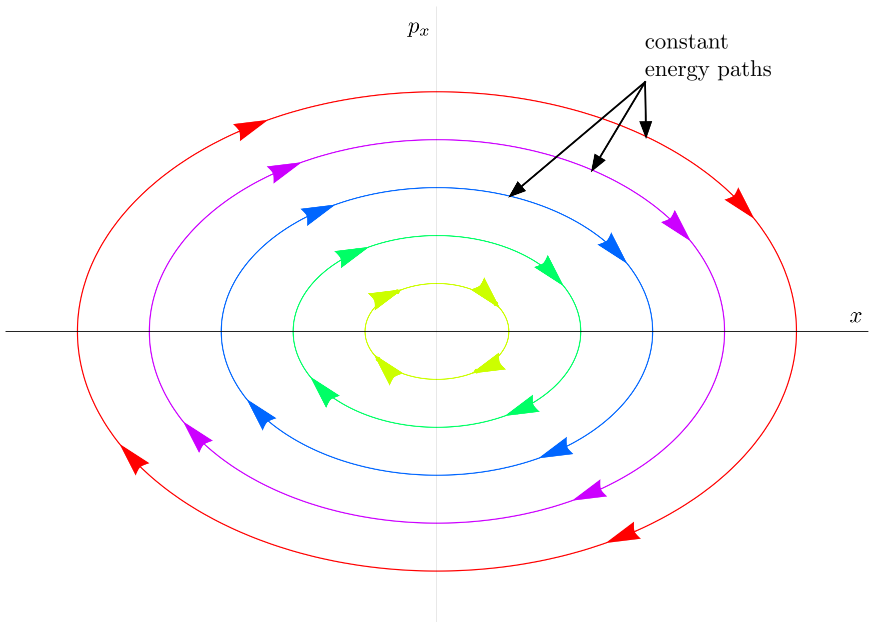

Figure 31.2: The phase diagram of the simple harmonic oscillator

having lots of momentum near the equilibrium point to having very little near the position extrema, and then back.

The arrows on the diagram indicate the direction in phase space which the particle moves in. Although we can get them from the functions $\{x(t), p(t)\}$, it is easier to read them off of Hamilton's equations: equation 31.9. When p is positive (the upper half of the graph), x is increasing, so the arrow must point to the right. When p is negative (the lower half of the graph), x is decreasing, so the arrow must point to the left. Similarly, when x is positive (the right hand side of the graph), p is decreasing so the arrow must point down. When x is negative (the left hand side of the graph), p is increasing so the arrow must point up. Putting these facts together makes it obvious that, for example, in the upper right quadrant, the arrow points to the right and downward.

A phase diagram typically shows multiple paths, which in this case means multiple values of E, the total energy. Notice that when we change the energy E, the ratio of the major axis to the minor axis does not change (assume that, as in figure 31.1, the major axis is along the x-axis):

$$\frac{\text{major axis}}{\text{minor axis}} = \frac{\sqrt{2mE}}{\sqrt{\frac{2E}{k}}} = \sqrt{km} \tag{31.11}$$

All that changes when we increase or decrease E is the overall scale of the ellipse. Multiple ellipses for multiple possible Es are shown in figure 31.2.

Non-dimensionalization

I have not gone to the trouble of non-dimensionalizing this problem, mainly because it is so simple. Furthermore, the classic SHO has no natural length scale (think of the oscillatory motion–it is qualitatively identical, no matter what the amplitude of the oscillation is), so there is no obvious choice for how to replace x with a non-dimensionalized displacement. We *could* use the natural time scale $t_0 = \sqrt{\frac{m}{k}}$ to nondimensionalize the time. If we defined an arbitrary length scale a, then we would say

$$t = t_0\tau = \sqrt{\frac{m}{k}}\,\tau, \qquad x = ay,$$

where τ is our non-dimensionalized time, and y is our non-dimensionalized length. Then, we can naturally construct a non-dimensionalized momentum $\tilde{p}$ and energy e as

$$p = m\frac{dx}{dt} = \frac{ma}{t_0}\frac{dy}{d\tau} = a\sqrt{km}\,\frac{dy}{d\tau} \equiv a\sqrt{km}\,\tilde{p}, \qquad E = \frac{1}{2}ka^2 \times e$$

With these definitions, the equation that defines the ellipses in phase space becomes

$$e = \tilde{p}^2 + y^2$$

and Hamilton's equations become

$$\frac{dy}{d\tau} = \tilde{p}, \qquad \frac{d\tilde{p}}{d\tau} = -y$$

Our non-dimensionalization has turned the ellipses in phase space into circles, absorbing the information about the relative sizes of m and k into the definition of $\tilde{p}$.

Fixed Points

Finally, notice that there is a fixed point, where x and p are constant, only at $\{x, p\} = \{0, 0\}$. A fixed point in phase space is the same as an equilibrium point. The stability of the equilibrium point can be seen in the ellipses–a small displacement off of the equilibrium point orbits around and around it in phase space, never getting too far away.

For any system described by $\{q, p\}$ (where q is scleronomic, as in the previous section), fixed points require $\dot{p} = \dot{q} = 0$ (the right-hand-sides of both of Hamilton's equations vanish.) Since p is proportional to $\dot{q}$, the only possible fixed points must be on the x-axis, with $p = 0$. But of course this by itself is not enough. By looking at the second of Hamilton's equations, we see that we must also have $\frac{\partial V}{\partial q} = 0$.

31.3 The Rigid Pendulum

Consider a slightly more interesting case: the rigid pendulum. If we describe the motion of the pendulum in terms of the angle θ off of the vertical, we know that the Lagrangian is

$$L = \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell \cos\theta$$

Before we go further, we can non-dimensionalize right from the start. The natural time scale for this problem will be the familiar $t_0 = \sqrt{\frac{\ell}{g}}$. In addition, notice that the Lagrangian has the dimensions of energy. Our natural energy scale for this problem is $E_0 = mg\ell$. Then we write

$$t = t_0 \tau = \sqrt{\frac{\ell}{g}}\,\tau, \qquad L = E_0 \tilde{L} = mg\ell \tilde{L}$$

and we get

$$\tilde{L} = \frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2 + \cos\theta = \frac{1}{2}\dot{\theta}^2 + \cos\theta$$

as a non-dimensionalized Lagrangian. (From here on out, dots will indicate time derivatives with respect to τ.) The conjugate momentum (now also non-dimensional)

$$\tilde{p}_\theta = \frac{\partial \tilde{L}}{\partial \dot{\theta}} = \dot{\theta}$$

so our Hamiltonian is

$$\tilde{H} = \tilde{p}_\theta \dot{\theta} - \tilde{L} = \dot{\theta}^2 - \frac{1}{2}\dot{\theta}^2 - \cos\theta = \frac{1}{2}\dot{\theta}^2 - \cos\theta$$

In terms of $\tilde{p}_\theta$ and θ, this is

$$\tilde{H} = \frac{1}{2}\tilde{p}_\theta^2 - \cos\theta \tag{31.12}$$

and Hamilton's equations give us

$$\dot{\theta} = \frac{\partial \tilde{H}}{\partial \tilde{p}_\theta} = \tilde{p}_\theta, \qquad \dot{\tilde{p}}_\theta = -\frac{\partial \tilde{H}}{\partial \theta} = -\sin\theta. \tag{31.13}$$

Just as in the simple harmonic oscillator, the Hamiltonian is equal to the energy, and is conserved. This gives us a relationship between $\tilde{p}_\theta$ and θ that must be satisfied on any path $\{\theta(t), \tilde{p}_\theta(t)\}$:

$$e = \frac{1}{2}\tilde{p}_\theta^2 - \cos\theta \tag{31.14}$$

(so the total energy is $E = E_0 e$.) This is a somewhat less familiar equation than that of an ellipse, so let's think about the shapes that will be traced out in phase space carefully. First, note that the angle θ varies between $-\pi$ and π, and actually that the point $\theta = \pi$ is the same as the point $\theta = -\pi$. Thus, our phase space diagram only covers $[-\pi, \pi]$, and we should then think of the θ coordinate as looping back on itself.

31.3.1 Fixed Points

Next, let's consider the fixed points in phase space. These require

$$\tilde{p}_\theta = 0, \qquad \sin\theta = 0, \qquad \rightarrow \qquad \theta = 0, \pm\pi$$

Thus, there are two fixed points in phase space: $\{\theta, \tilde{p}_\theta\} = \{0, 0\}$ and $\{\theta, \tilde{p}_\theta\} = \{\pm\pi, 0\}$ (notice this second one is one point because $\theta = \pi$ is the same as $\theta = -\pi$.) If we examine the motion near the first fixed point (small θ and $\tilde{p}_\theta$), we find the familiar small angle approximation $\theta = \epsilon$ of the rigid pendulum, which gives

$$e \approx \frac{1}{2}\tilde{p}_\theta^2 - 1 + \frac{1}{2}\epsilon^2.$$

Near this fixed point, we have circles in phase space with radii $\sqrt{2(e+1)}$–which matches our long-standing understanding that for small oscillations a rigid pendulum (just like nearly any system near the minimum of a the potential) is the same as a simple harmonic oscillator.

On the other hand, near the other fixed point $\theta = \pi - \epsilon$, we have

$$e = \frac{1}{2}\tilde{p}_\theta^2 - \cos(\pi - \epsilon) = \frac{1}{2}\tilde{p}_\theta^2 + \cos\epsilon \approx \frac{1}{2}\tilde{p}_\theta^2 + 1 - \frac{1}{2}\epsilon^2$$

The relative sign between the $\tilde{p}_\theta^2$ term and the ϵ^2 term means that this is the equation for a hyperbola with asymptotes

$$\tilde{p}_\theta = \pm\epsilon$$

The hyperbolas reflect the fact that this is an unstable equilibrium point–even small displacements from $\{\theta, p_\theta\} = \{\pm\pi, 0\}$ will follow paths that take the pendulum bob far away from the fixed point.

31.3.2 The Separatrix

Consider the particular value of energy $E = mg\ell$. Near the unstable equilibrium fixed point, this value of energy actually gives the straight line asymptotes for the hyperbolas. That is, if the bob has exactly this amount of energy, it might start arbitrarily close to the unstable fixed point, swing down and away from it, and then back up to it from the other direction (though it would take an infinite amount of time for the bob to actually come back up the the fixed point.) This value for the energy has a special physical significance because it divides two qualitatively different types of motion. Notice that we can use the energy equation to solve for p_θ:

$$\tilde{p}_\theta = \dot{\theta} = \pm\sqrt{2\,(e + \cos\theta)}$$

(also using Hamilton's equations.) For energies such that $e > 1$ (or $E > mg\ell$), we will never have $\dot{\theta} = 0$, and thus whatever sign appears before the square root will never change. This describes the motion of the bob swinging round and round the pivot point. For energies such that $e < 1$ ($E < mg\ell$), we have $\dot{\theta} = 0$ for some angles $\pm\theta_0$ that satisfy

$$\tilde{p}_\theta = 0 = \sqrt{(e + \cos\theta_0)}, \qquad \cos\theta_0 = -e$$

This describes motion where the pendulum bob swings back and forth.

31.3.3 The Phase Diagram

In figure 31.3 we show all of this in the phase diagram for the rigid pendulum. The lines $\theta = \pm\pi$ are shown with dotted lines–we should really imagine rolling up the diagram in a

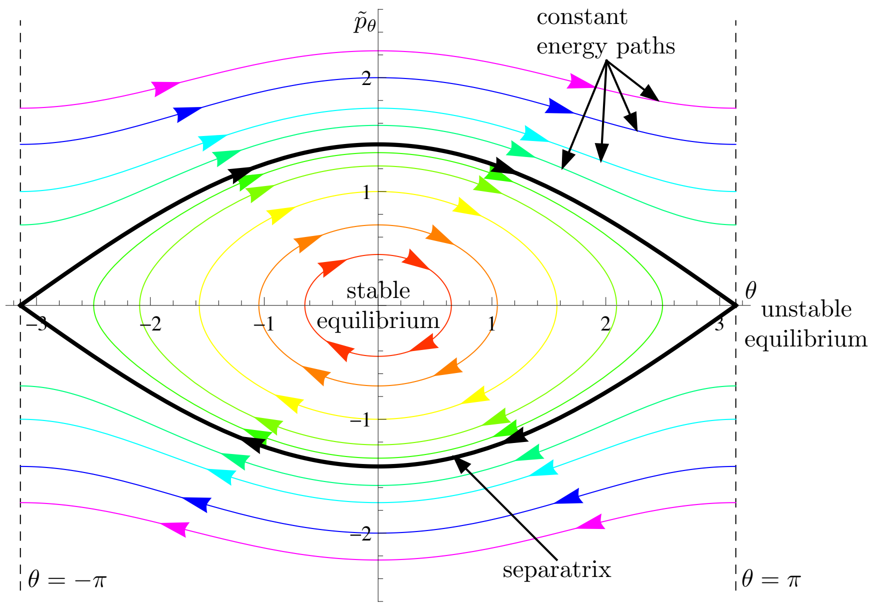

Figure 31.3: The phase diagram of the rigid pendulum

tube until these lines meet. Near the stable fixed point, at the center, we can see ellipses, showing the traditional SHO motion. Near the unstable fixed point, at the edges, we can see the hyperbolas. Note that the arrows again indicate the flow along the lines, and again for $\tilde{p}_\theta > 0$ we have arrows going right while for $\tilde{p}_\theta < 0$ we have arrows going left.

The separatrix is shown in solid black–it goes from the unstable fixed point around and *back to the unstable fixed point.* The separatrix is actually two separate lines–one in the top half and one in the bottom half. Since it takes the bob an infinite amount of time to get up to the fixed point along one of these lines, it never actually goes from one to the other. Finally, note that inside the separatrix are closed curves, which start out as ellipses and become increasingly distorted as they get closer to the separatrix. Outside the separatrix are actually *also* closed curves, but these go around and around the tube–indicating the motion of the bob going around and around the pivot point and never changing direction. The ones in the top half rotate one way, the ones in the bottom half rotate the other way.

The motion of the pendulum bob, described by $\{\theta(\tau), \tilde{p}(\tau)\}$ can only be written in terms of elliptic integrals. However, we can get almost anything qualitative we want to know about it from the phase diagram.

31.4 The Rod and the Spring

For a third example of an energy conserving system where we can study the phase diagram, let's return to the system of the bead of mass m sliding along a rod, attached by a spring of equilibrium extent ℓ and spring constant k to some fixed point a distance a away from the rod. We will use the same non-dimensionalization we used when we first introduced the system:

$$y = \frac{x}{\ell}, \qquad b = \frac{a}{\ell}, \qquad \tilde{L} = \frac{L}{k\ell^2}, \qquad \tau = t\sqrt{\frac{k}{m}}$$

The only new thing here is that I am defining a non-dimensionalized lagrangian $\tilde{L}$ the same way we defined the non-dimensionalized potential $\tilde{V}$ before. Then we have

$$\tilde{L} = \frac{1}{2}\dot{y}^2 - \frac{1}{2}\left[\sqrt{y^2+b^2} - 1\right]^2$$

(and again, I am using dots to indicate time derivatives with respect to the non-dimensional τ.) The conjugate momentum is then

$$\tilde{p} = \frac{\partial \tilde{L}}{\partial \dot{y}} = \dot{y}$$

so the Hamiltonian is

$$\tilde{H} = \dot{y}\tilde{p} - \tilde{L} = \dot{y}^2 - \frac{1}{2}\dot{y}^2 + \frac{1}{2}\left[\sqrt{y^2+b^2} - 1\right]^2 = \frac{1}{2}\dot{y}^2 + \frac{1}{2}\left[\sqrt{y^2+b^2} - 1\right]^2$$

which we write as

$$\tilde{H} = \frac{1}{2}\tilde{p}^2 + \frac{1}{2}\left[\sqrt{y^2+b^2} - 1\right]^2 \tag{31.15}$$

so that Hamilton's equations are

$$\dot{y} = \frac{\partial \tilde{H}}{\partial \tilde{p}} = \tilde{p}, \qquad \dot{\tilde{p}} = -\frac{\partial \tilde{H}}{\partial y} = -\frac{y}{\sqrt{y^2+b^2}}\left[\sqrt{y^2+b^2} - 1\right] \tag{31.16}$$

and the statement of energy conservation becomes

$$e = \frac{1}{2}\tilde{p}^2 + \frac{1}{2}\left[\sqrt{y^2+b^2} - 1\right]^2 \tag{31.17}$$

Based on our earlier analysis of this system, we expect to get qualitatively different types of motions depending on the value of b.

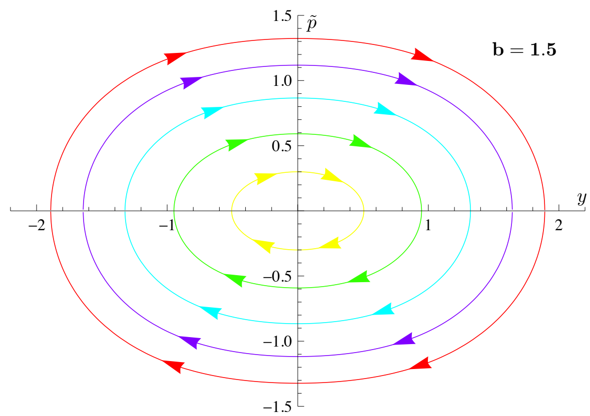

Figure 31.4: A phase diagram of the rod and spring system with $b = 1.5$. Note that there is one equilibrium point at $(y, \tilde{p}) = (0, 0)$, and near this point the phase diagram has ellipses with the major axis on the y axis.

31.4.1 b > 1

First consider the case where $b > 1$. We previously showed that in this case the potential has exactly one minimum at $y = 0$. Correspondingly, in "phase space" language we say that there is only one fixed point, such that $\dot{\tilde{p}} = \dot{y} = 0$, and this is where

$$(y, \tilde{p}) = (0, 0)$$

For small perturbations about this fixed point, we should get small ellipses, having previously shown that this is a stable equilibrium point. And indeed, expanding equation 31.17 for small $\tilde{p}$ and y gives us the relationship

$$\text{small y :} \qquad e \approx \frac{1}{2}\tilde{p}^2 + \frac{(b-1)^2}{2} + \frac{b-1}{2b}y^2$$

which are ellipses for whom the ratio of the two axes is $\sqrt{\frac{b-1}{b}}$. The axis given by $\tilde{p} = 0$ is always the major axis, while that given by $y = 0$ is always the minor axis. Notice that for very large b, this ratio is almost 1, giving approximately a perfect circle, while for $b \gtrsim 1$ this is significantly smaller than 1, making an ellipse with large eccentricity. In figure 31.4, we show a phase diagram for $b = 1.5$.

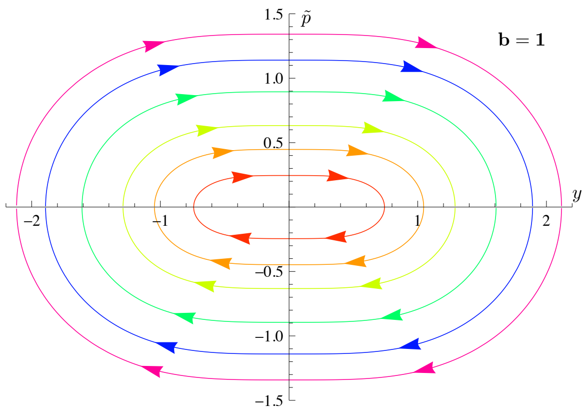

Figure 31.5: A phase diagram of the rod and spring system with $b = 1$. Note the very elongated curves of constant energy–even near the equilibrium at $(\tilde{p}, y) = (0, 0)$, they don't really form ellipses, but something narrower and flatter.

31.4.2 b = 1

Now consider the special case $b = 1$, which we spent time with before because in this case even small motions near the equilibrium point $y = 0$ are fundamentally nonlinear. This was because the second derivative of the potential vanishes at the equilibrium point $y = 0$, so we can't approximate it as quadratic, even for very small oscillations. Correspondingly, if we approximate equation 31.17 for small y in this case, we obtain

$$\text{small y :} \qquad e \approx \frac{1}{2}\tilde{p}^2 + \frac{1}{2}\left[1 + \frac{1}{2}y^2 - 1\right]^2 \approx \frac{1}{2}\tilde{p}^2 + \frac{1}{8}y^4$$

which does not give us ellipses–it gives us something far flatter and more eccentric. In figure 31.5, we show a phase diagram for $b = 1$.

31.4.3 b < 1

The interesting case, from the point of view of phase diagrams, is when $b < 1$. In this case, there are three different fixed points, corresponding to our three equilibrium points from our previous treatment.

$$\text{fixed points } (y, \tilde{p}) : \qquad (0,0), \quad \left(\sqrt{1-b^2}, 0\right), \quad \left(-\sqrt{1-b^2}, 0\right)$$

Near $\mathbf{y = \pm\sqrt{1 - b^2}}$

Consider first what the phase diagram will look like near the fixed points with $y = \pm\sqrt{1-b^2}$. Remember that we showed before these are stable equilibria, so we expect ellipses. Defining $y = \epsilon \pm \sqrt{1-b^2}$, we can make the approximations

$$\begin{aligned} & y^2 + b^2 = 1 \pm 2\epsilon\sqrt{1-b^2} + \mathcal{O}(\epsilon^2) \\ \rightarrow \qquad & \sqrt{y^2+b^2} - 1 = \pm\epsilon\sqrt{1-b^2} + \mathcal{O}(\epsilon^2) \\ \rightarrow \qquad & \left[\sqrt{y^2+b^2} - 1\right]^2 = \epsilon^2(1-b^2) + \mathcal{O}(\epsilon^3) \end{aligned}$$

so that we get

$$e \approx \frac{1}{2}\tilde{p}^2 + \frac{(1-b^2)}{2}\epsilon^2$$

which does, indeed, give us ellipses. Notice that since $1 - b^2 < 1$, the major axis of the ellipse will always be along the y axis (the ϵ axis). The smaller b is, the closer to perfect circles they will be.

Near $\mathbf{y = 0}$

On the other hand, suppose we look near the fixed point $(0,0)$. Remember that this was an unstable equilibrium point. Recalling the results from looking at the rigid pendulum, we expect the curves of constant energy to be hyperbolas near this point. Expanding equation 31.17 about this point, gives us the same equation we got when we were considering $b > 1$, except this time we write

$$\text{small } y: \qquad e \approx \frac{1}{2}\tilde{p}^2 + \frac{(1-b)^2}{2} - \left(\frac{1-b}{2b}\right) y^2$$

to emphasize the fact that the coefficient in front of the y^2 term is negative. This is the equation for a hyperbola with asymptotes

$$\tilde{p} = \pm\sqrt{\frac{1-b}{b}}\, y$$

Thus for very small b, these asymptotes have very sharp slopes, while for $b \lesssim 1$, the slopes are very shallow.

The Separatrix

Just like in the case of the rigid pendulum, for $b < 1$ this system has a separatrix–a curve in phase space which divides qualitatively different motions. Looking at the hyperbola near the fixed point $(0, 0)$, we see that for the energy $e = \frac{(1-b)^2}{2}$ the paths are the asymptotes of the hyperbolas. Elsewhere, such a path satisfies the equation

$$\frac{(1-b)^2}{2} = \frac{1}{2}\tilde{p}^2 + \frac{1}{2}\left[\sqrt{b^2 + y^2} - 1\right]^2$$

If we plot this in phase space, we get a "figure 8" lying on its side that passes through the origin at the "X". It crosses the y-axis in three places: at $y = 0$, and at $y = \pm 2\sqrt{1-b}$. Similar to what we saw with the separatrix curve for the rigid pendulum, approaching the fixed point $(0, 0)$ along this curve corresponds to the motion of the bead climbing up to the unstable equilibrium point at the center of the rod, slowing down as it goes so that it takes infinitely long to get to the center, by which time it isn't moving at all. Thus, the two loops of the "8" are actually two separate motions.

For $e < \frac{(1-b)^2}{2}$, the bead does not have enough energy to climb up past the local maximum of the potential–it stays bound near one of the local minimum at $\pm\sqrt{1-b^2}$, going in closed curves in phase space around it. For $e > \frac{(1-b)^2}{2}$, the bead has enough energy to climb all the way up the local maximum of the potential and down into the other well, so it oscillates around both local minima of the potential. In phase space, it traces out a large closed curve which has all three fixed points inside it. All of this information is shown in figure 31.6, which shows the phase diagram for $b = 0.5$.

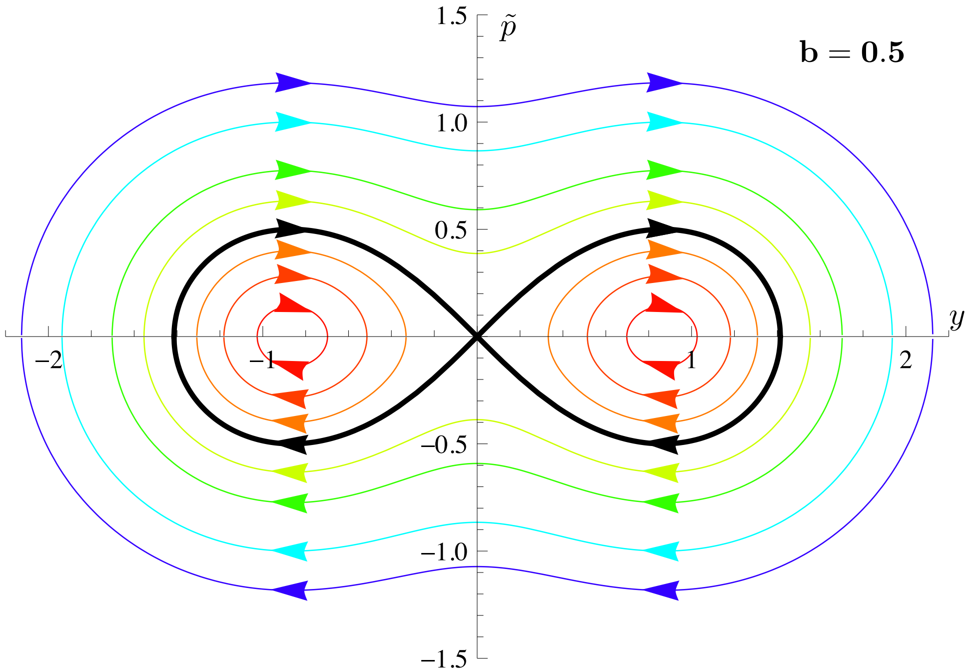

Figure 31.6: A phase diagram of the rod and spring system with $b = 0.5$. Note the three fixed points, two with ellipses near them and the third with hyperbola near it. The separatrix is shown in black.