Poincaré Maps and Discrete Maps

Lecture 33

Physics 311: Classical Mechanics
Fall 2011

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We have gotten a fair amount of interesting intuition from studying the phase space diagrams for various systems. They give us a different way to visualize what is occurring for the possible motions of a physical system with interesting features like limit cycles, separatrixes, or equilibrium points. However, there are circumstances where the phase diagram may simply lead to more confusion, rather than making the qualities of the motion more apparent. Today I want to talk about an alternative way of representing what occurs in a physical system that can be useful in cases where the phase diagram becomes obscure.

33.1 Time Dependence and Phase Diagrams

Phase diagrams are particularly useful in cases where the equation of motion has no explicit time dependence in it (for example, through a driving force term). The standard damped driven oscillator (non-dimensionalized)

$$\ddot{z} + 2b\dot{z} + z = 0 \tag{33.1}$$

is an example of this type of system, as is the somewhat more complicated van der Pol oscillator

$$\ddot{z} + b(z^2 - a^2)\dot{z} + z = 0 \tag{33.2}$$

What makes the phase diagrams of these systems nice is that no two lines in the phase diagram can ever intersect (except at a fixed point.) A point in a phase diagram represents particular values for z and $\dot{z}$, and an intersection would indicate that it was possible for a mass, at a given location and with a given velocity, to evolve in two different ways. But in both of the equations above, we could solve for $\ddot{z}$ as a function of just z and $\dot{z}$–so a particular set $\{z, \dot{z}\}$ should always give the same $\ddot{z}$, and the position of the mass $z(t)$ should only be able to evolve in one way.

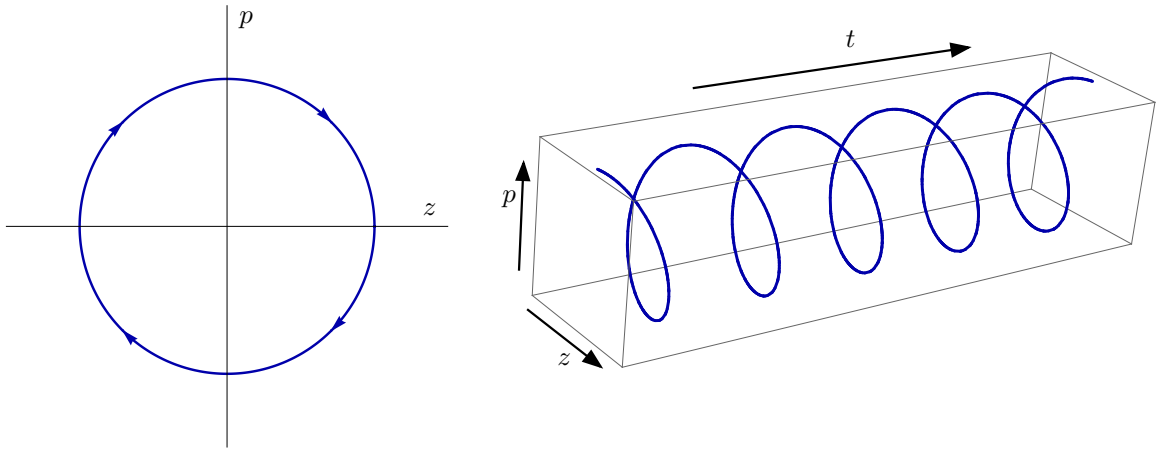

Figure 33.1: The time evolution of the phase diagram for a simple harmonic oscillator.

Now consider the equation

$$\ddot{z} + 2b\dot{z} + z = \cos\omega\tau \tag{33.3}$$

for the damped driven oscillator, or

$$\ddot{z} + 2b\dot{z} + z + \epsilon z^3 = \cos\omega\tau \tag{33.4}$$

which we considered for the damped, driven quartic oscillator. What changes when we have equations with explicit time dependence is that we would then solve for $\ddot{z}$ in terms of z, $\dot{z}$ *and* t...so depending on the time we are evaluating $\ddot{z}$ at, we can get two different answers for the same set $\{z, \dot{z}\}$. In a phase diagram, this leads to an intersection between two lines, and when we plot many trajectories in phase space for such a system we inevitably end up with a tangled mess that it very difficult to read. This is because the phase diagram *doesn't* show time dependence.

One way to deal with this problem is to imagine drawing a phase diagram with a third axis, representing time, so that the evolution was represented as a three-dimensional graph. For example, our old favorite the plain non-dimensionalized simple harmonic oscillator

$$\ddot{z} + z = 0 \tag{33.5}$$

has a phase space consisting of circles:

$$e = \frac{1}{2}\dot{z}^2 + \frac{1}{2}z^2 \tag{33.6}$$

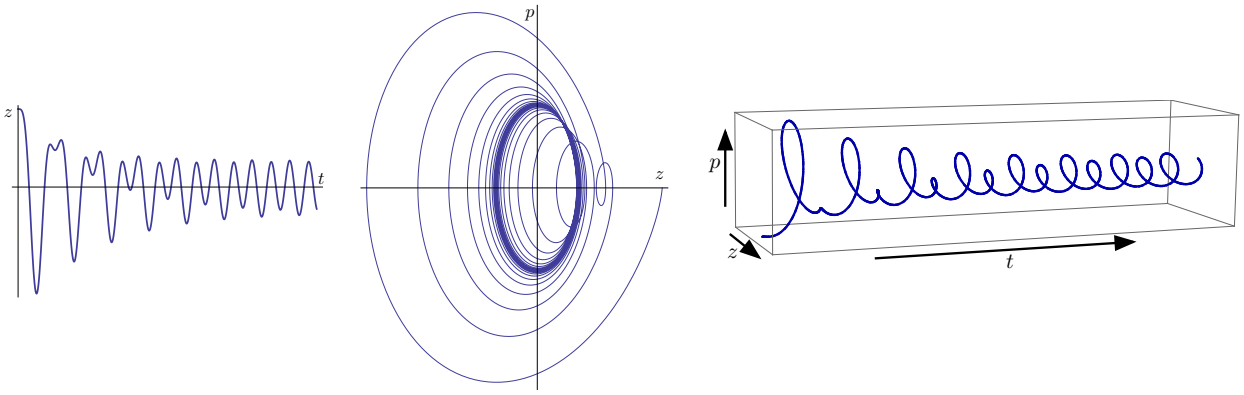

Figure 33.2: On the right, the time evolution of the phase diagram for a damped driven harmonic oscillator, where the driving frequency is twice the natural frequency. In the middle we see the phase diagram itself, which is somewhat confusing. On the left is $z(\tau)$.

If you consider one specific trajectory (at a specific energy), then you can draw the time evolution of the phase diagram, which becomes a spiral. (Figure 33.1.)

Let's consider a slightly more interesting example–the damped driven harmonic oscillator:

$$\ddot{z} + 2b\dot{z} + z = \cos\omega\tau \tag{33.7}$$

We are by now quite familiar with the structure of the solution, which will be a sum of two terms: a decaying oscillation at frequency ω_0 (providing β isn't too large), and a steady state oscillation at frequency Ω with some particular amplitude f_0 (determined by the resonance equation.) The time evolved phase diagram here makes these features particularly clear. In figure 33.2 we can see the the result when the driving frequency is twice the natural frequency. Note that we start out with a large oscillation with a larger period, that appears to have a tiny blip in it at about half the period. Over time, the large oscillation decreases, and the blip grows, until it ends up with a steady state oscillation at twice the period. By contrast, the phase diagram looks a bit muddled, though the limit cycle is still quite clear.

Finally, we might look at the damped driven quartic oscillator, which when we non-dimensionalized it, was

$$\ddot{z} + 2b\dot{z} + z + \epsilon z^3 = \cos\omega\tau \tag{33.8}$$

Recall that we approached this problem using perturbative methods (assuming ϵ to be small), and found that including the first set of corrections due to the quartic bit gave us a steady-

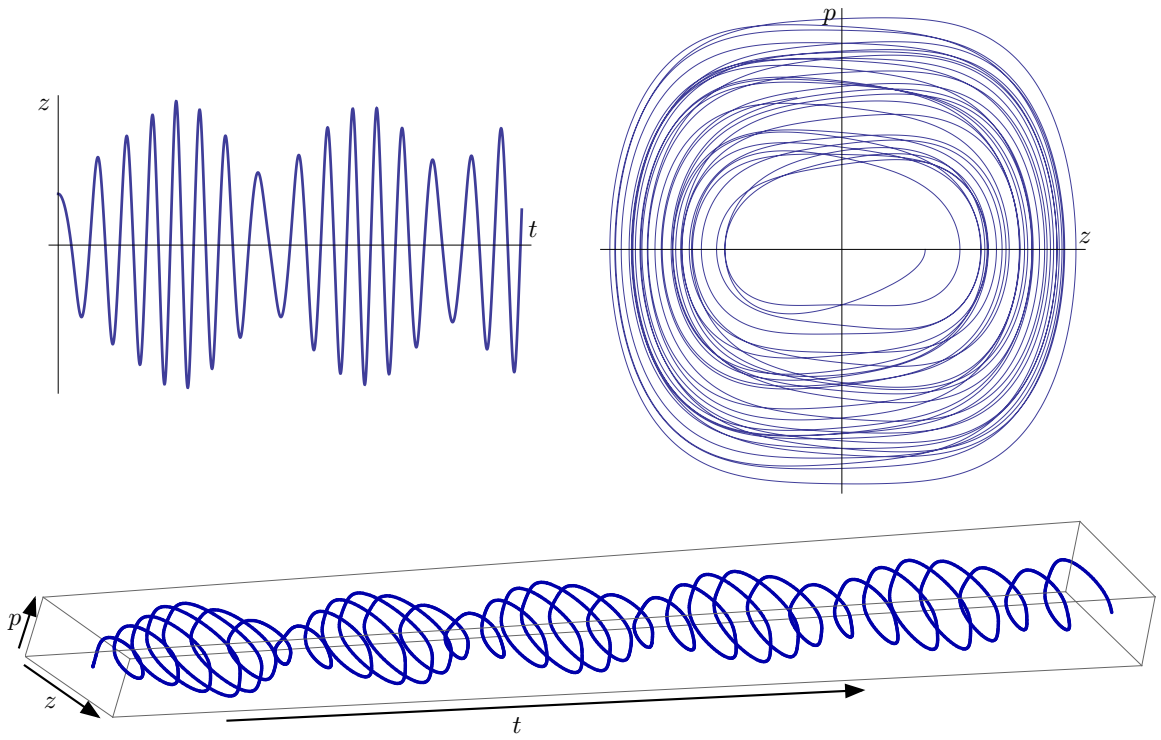

Figure 33.3: On the right, the time evolution of the phase diagram for a damped driven quartic harmonic oscillator, where the driving frequency is twice the natural frequency. In the middle we see the phase diagram itself, which is somewhat confusing. On the left is $z(\tau)$.

state solution in the form

$$z(\tau) \approx f\cos(\omega\tau + \phi) + g\cos(3\omega\tau + \psi) \tag{33.9}$$

where f was determined by the rather complicated resonance equation, which gave the possibility of hysteresis, and g, ϕ, and ψ would be determined by similar methods. For us now, the important thing to notice is the presence of the two separate terms, even in the steady-state solution, which have two different frequencies. Then there's a decaying part, with still a *third* frequency. When we plot the phase diagram for this system, we get a big hairy mess, that doesn't really give us much. But consider the time-evolved phase diagram. Here we clearly see the cycles (somewhat distorted) growing and shrinking, so that there are clearly two competing frequencies at work.

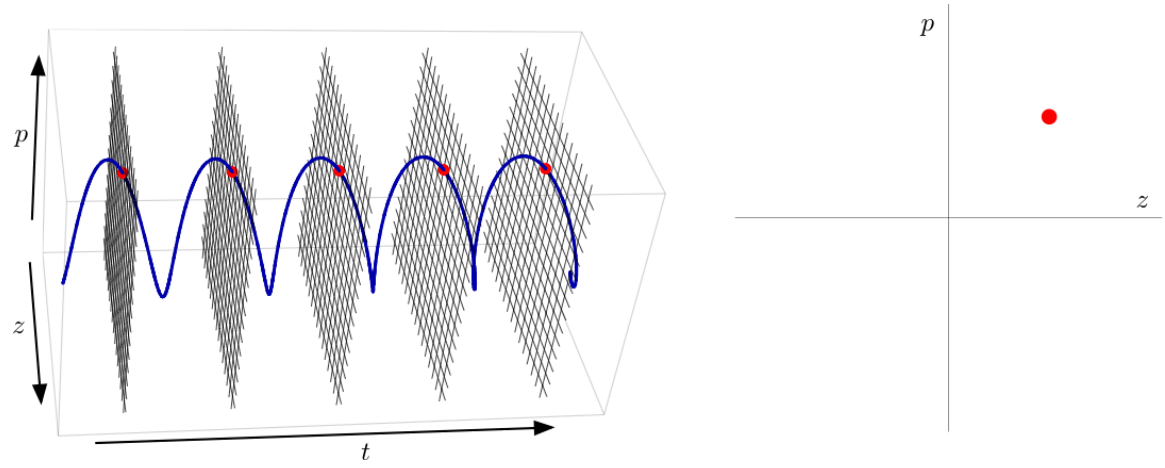

Figure 33.4: The Poincaré map for the simple harmonic oscillator, with the period of oscillation chosen as the time lapse between "snapshots".

33.2 Poincaré Maps

On the other hand, the 3-dimensional "time-evolution" plots are a bit cumbersome. Another option is what is called a "Poincaré" map, which is essentially a set of overlaid snapshots of the location of the mass in phase space at evenly spaced times. One way to imagine building a Poincaré map is by taking the time-evolved phase space diagram and putting in a series of planes at evenly spaced times, and looking at the points where the curve intersects the planes. Then stack these planes on top of each other. This results in a series of points on the plane.

The Poincaré map is of particular use in systems where either the solution oscillates harmonically (or nearly harmonically), or there is some periodic driving force. The point is that in order to create a Poincaré map we need to know how much time to elapse between "snapshots." Either a periodicity in the solution or a periodicity in the equation of motion itself will give us a natural choice. To return to our favorite ultra-simple example, the SHO, we know that the phase diagram is a simple circle, and the time evolved phase diagram is a spiral. With the non-dimensionalized equation

$$\ddot{z} + z = 0 \tag{33.10}$$

we have a non-dimensionalized period $\tilde{T} = 2\pi$. If we take our snapshots exactly $\tilde{T}$ apart, then every time we take our snapshot, the mass will be at the same point in phase space–so

the poincaré map will be a single dot on the plane. This is shown in figure 33.4. Now, if we don't choose $\tilde{T}$ as the time between snapshots, we will not necessarily end up with the same Poincaré map. For example, if we picked $\tilde{T}/2$ between snapshots, then two adjacent snapshots will show two different points–on opposite sides of the phase diagram circle. Stacked together, the Poincaré map becomes *two* dots on the plane, rather than one. And if we chose some random value between snapshots that wasn't a rational multiple of $\tilde{T}$, then we would end up with an infinite number of dots. Because each is somewhere on the original phase diagram circle, as we stack them all up on top of each other we end up with the same circle in the Poincaré map as we had in the phase diagram.

However, the whole point of the Poincaré map is to visually capture the periodicity of the solution–so we would not usually choose a random number for the time between snapshots, not when the solution gives us a natural choice for this time.

Now let's look at another example–the damped driven harmonic oscillator. Non-dimensionalized, this is

$$\ddot{z} + 2b\dot{z} + z = \cos\omega\tau \tag{33.11}$$

We know what the solutions look like here–a general solution is a sum of a transitory part oscillating at frequency 1 and a steady state part oscillating at frequency ω. If we assume the system is underdamped, this is

$$z(\tau) = Ae^{-b\tau}\cos[\tau\sqrt{1-b^2} + \phi] + f\cos(\omega\tau + \psi) \tag{33.12}$$

with

$$f = \frac{1}{\sqrt{(1-\omega^2)^2 + 4b^2\omega^2}}, \qquad \psi = \tan^{-1}\left[\frac{2b\omega}{\omega^2 - 1}\right] \tag{33.13}$$

This has

$$p(\tau) = \dot{z} = -Abe^{-b\tau}\cos[\tau\sqrt{1-b^2} + \phi] - A\sqrt{1-b^2}e^{-b\tau}\sin[\tau\sqrt{1-b^2} + \phi] - f\omega\sin(\omega\tau + \psi)$$

so that

$$p(\tau) = -Ae^{-b\tau}\cos[\tau\sqrt{1-b^2} + \tilde{\phi}] - f\omega\sin(\omega\tau + \psi) \tag{33.14}$$

with

$$\tilde{\phi} = \phi - \cos^{-1} b \tag{33.15}$$

And this gives us our first dilemma–what amount of time should we allow to lapse between snapshots? There seem to be two sensible choices: the "natural period" $\tilde{T}_n = 2\pi$ which is the period of the transitory oscillation, and the "driving period" $\tilde{T}_d = \frac{2\pi}{\omega}$, which is the period of the steady state solution. In general, it makes sense to favor the driving period over the

natural period–because the steady state solution is more significant than the transitory part. Suppose we take the first snapshot at $\tau_0 = 0$, the second at $\tau_1 = \frac{2\pi}{\omega}$, the third at $\tau_2 = \frac{4\pi}{\omega}$, and so on. The first snapshot will have the point in phase space

$$\begin{aligned} z_0 &= A\cos\phi + f\cos\psi \\ p_0 &= -A\cos\tilde{\phi} - f\omega\sin\psi \end{aligned} \tag{33.16}$$

The second snapshot will have the point in phase space

$$\begin{aligned} z_1 &= Ae^{-2\pi b/\omega}\cos\left[\phi + \frac{2\pi\sqrt{1-b^2}}{\omega}\right] + f\cos\psi \\ p_1 &= -Ae^{-2\pi b/\omega}\cos\left[\tilde{\phi} + \frac{2\pi\sqrt{1-b^2}}{\omega}\right] - f\omega\sin\psi \end{aligned} \tag{33.17}$$

and so on, with the nth snapshot having the point

$$\begin{aligned} z_n &= Ae^{-2n\pi b/\omega}\cos\left[\phi + \frac{2n\pi\sqrt{1-b^2}}{\omega}\right] + f\cos\psi \\ p_n &= -Ae^{-2n\pi b/\omega}\cos\left[\tilde{\phi} + \frac{2n\pi\sqrt{1-b^2}}{\omega}\right] - f\omega\sin\psi \end{aligned} \tag{33.18}$$

We notice something immediately about this. These points drift in phase space, so that the Poincaré map will show an infinite series of dots. However, they also *converge*. For large values of n, we have

$$\begin{aligned} z_\infty &= f\cos\psi \\ p_\infty &= -f\omega\sin\psi \end{aligned} \tag{33.19}$$

Now, the point $\{z_0, p_0\}$ depends on the initial conditions we chose for our system. (In point of fact, we could say that it *is* the initial conditions.) However, notice that whatever we chose for A and ϕ, we would end up with the same $\{z_\infty, p_\infty\}$. This is shown in figure 33.5.

Recall that part of our motivation for developing this was that when our equation of motion depended explicitly on time then our phase diagram could have intersections corresponding to different times when the particle had the same position and momentum. Because the driving force wasn't equal the mass evolved differently from that location in phase space. By choosing to use the driving force period as our time-lapse between snapshots, we have eliminated this issue from the Poincaré map–all of our snapshots are taken at moments when the driving force is the same. If we think of the series of dots as a discrete path, then this

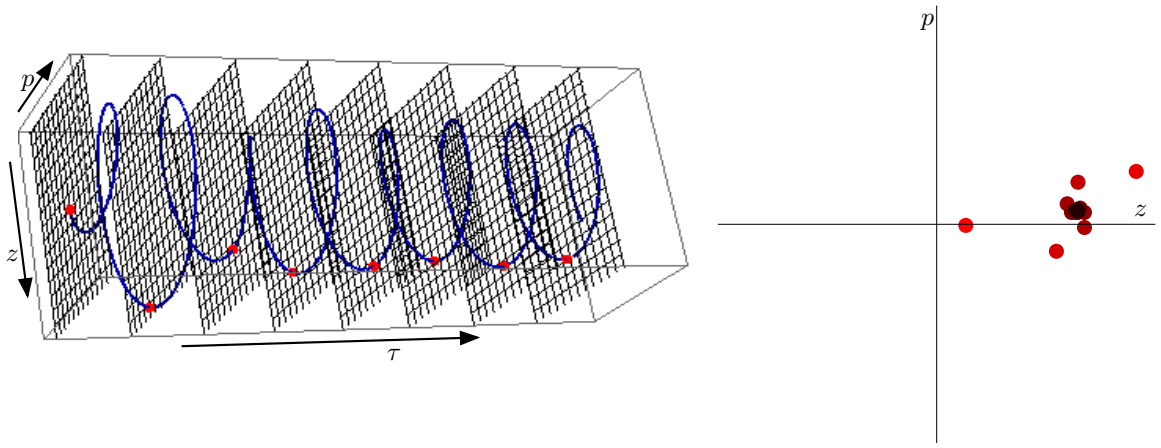

Figure 33.5: The Poincaré map for the damped, driven harmonic oscillator, with the period of the driving term chosen as the time lapse between "snapshots". The earlier dots are represented by lighter color, so that the convergence also corresponds to darker dots.

path cannot have intersections. If two snapshots are identical, then the two snapshots that immediately follow them must also be identical, and the two after that, and so on. This is another reason why we choose to use the driving force period as our time-lapse and *not* the natural period–if we chose the natural period we would not have this feature.

In our current example, the damped driven harmonic oscillator, there is only one way for this to happen–the snapshot catches the mass at the point $\{z_\infty, p_\infty\}$. Then, the next snapshot will also have $\{z_\infty, p_\infty\}$, and the one after that, and so on. This is a *fixed point* of the Poincaré map, and it corresponds to one location on the limit cycle of the phase diagram. (Note that if we shifted the times we took the snapshots slightly, but kept the amount of time between snapshots $2\pi/\omega$, then we would end up with a very similar story, but a different fixed point. This would correspond to a different location on the limit cycle of the phase diagram.)

More generically we could imagine that if we had a differential equation with explicit periodic time dependence of period T, we could build a Poincaré map using snapshots with lapse-time T, and the discrete path marked out by the dots would not have intersections. However, it can have closed cycles, where the zeroeth dot is the same as the nth dot, which is the same as the $2n$th dot, and so on. In this case, the 1st dot would be the same as the $(n+1)$th dot, and the $(2n+1)$th dot, etc.

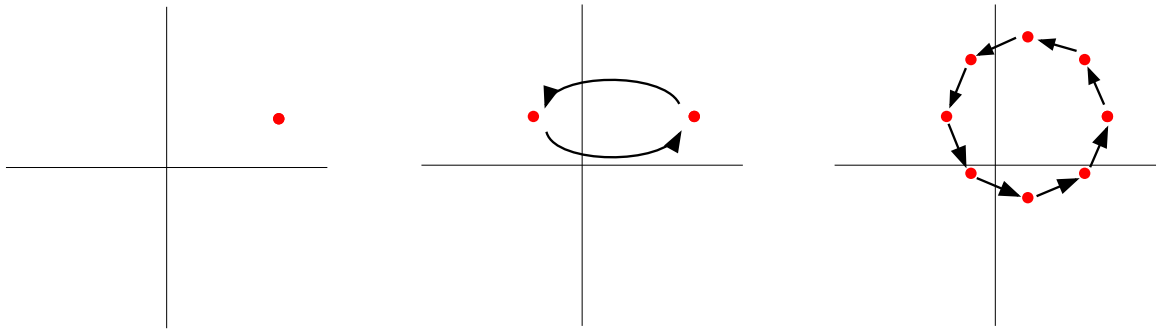

Figure 33.6: Possible cycles in a Poincaré map. The simplest such is a fixed point, shown on the far left. The middle consists of a cycle of two points, so that $\{z_{n+2}, p_{n+2}\} = \{z_n, p_n\}$. The far right consists of a cycle of 8 points.

33.3 The Discrete Map and its Linearization

We can also think of the Poincaré map as an example of a type of *discrete map*. A discrete map is a function that sends one point to another:

$$\Phi : \quad x_k \quad \longrightarrow \quad x_{k+1} \tag{33.20}$$

In the case of the Poincaré map for a one-dimensional system the point has two components, $x_k = (z_k, p_k)$, but in general we might consider x_k to have any number of components. In addition, for the Poincaré map Φ is defined as a function that *time-evolves* the position of an object in phase space forward by some fixed amount of time–it is determined, at the end of the day, by the equations of motion. Note that if we have

$$x_{k+1} = \Phi(x_k) \tag{33.21}$$

then we should also be able to write

$$x_k = \Phi^k(x_0) \tag{33.22}$$

In this case, a fixed point $\bar{x}$ satisfies

$$\text{fixed point:} \qquad \bar{x} = \Phi(\bar{x}) \tag{33.23}$$

and a point $\tilde{x}$ in a cycle with n points in it satisfies

$$\text{cycle of } n \text{ points:} \qquad \tilde{x} = \Phi^n(\tilde{x}) \tag{33.24}$$

It is particularly useful to consider the action of the Poincaré map in the vicinity of a fixed point–particularly keeping in mind (for comparison) the fixed points of phase space diagrams and how we analyze motion near them using linearized equations of motion. There, we were able to classify them as stable or unstable, based on whether a small displacement resulted in motion that stayed close to the fixed point, or took us far away. For concreteness, we will assume that our map is a Poincaré map for a one-dimensional system, so that each point has two components $x_k = (z_k, p_k)$, though very similar analysis is possible for more general maps.

We start with a fixed point

$$\bar{x} = (\bar{z}, \bar{p}) \tag{33.25}$$

Now, remember that $\Phi(z,p)$ is actually a two-component object:

$$\Phi(z,p) = \Big(\Phi^z(z,p), \Phi^p(z,p)\Big) \tag{33.26}$$

where Φ^z and Φ^p are just functions of two variables. Then if we suppose that

$$z = \bar{z} + \epsilon^z, \qquad p = \bar{p} + \epsilon^p \tag{33.27}$$

then we should be able to Taylor expand Φ^z and Φ^p near the point $(\bar{z}, \bar{p})$

$$\Phi^z(\bar{z}+\epsilon_z, \bar{p}+\epsilon_p) = \Phi^z(\bar{z},\bar{p}) + \frac{\partial \Phi^z}{\partial z}\epsilon_z + \frac{\partial \Phi^z}{\partial p}\epsilon_p + \cdots = \bar{z} + \frac{\partial \Phi^z}{\partial z}\epsilon_z + \frac{\partial \Phi^z}{\partial p}\epsilon_p + \cdots$$

and

$$\Phi^p(\bar{z}+\epsilon^z, \bar{p}+\epsilon^p) = \Phi^p(\bar{z},\bar{p}) + \frac{\partial \Phi^p}{\partial z}\epsilon^z + \frac{\partial \Phi^p}{\partial p}\epsilon^p + \cdots = \bar{p} + \frac{\partial \Phi^p}{\partial z}\epsilon^z + \frac{\partial \Phi^p}{\partial p}\epsilon^p + \cdots$$

We can rewrite this using matrices, to say that near a fixed point $\bar{x}$, the Poincaré map takes a point $\bar{x} + \epsilon$ and maps it to another point using a linear transformation:

$$\Phi: \qquad \bar{x} + \epsilon \qquad \longrightarrow \qquad \bar{x} + \mathbb{A}\epsilon \tag{33.28}$$

where $\mathbb{A}$ and ϵ are

$$\mathbb{A} = \begin{bmatrix} \frac{\partial \Phi^z}{\partial z} & \frac{\partial \Phi^z}{\partial p} \\ \\ \frac{\partial \Phi^p}{\partial z} & \frac{\partial \Phi^p}{\partial p} \end{bmatrix}, \qquad \epsilon = \begin{bmatrix} \epsilon^z \\ \epsilon^p \end{bmatrix} \tag{33.29}$$

Now let's return to imagining a series of points x_k that are produced by successive action of the Poincaré map. If we just look at the series of points ϵ which describe how far from the fixed point we are, then we have

$$\epsilon_{k+1} = \mathbb{A}\epsilon_k, \qquad \epsilon_k = \mathbb{A}^k \epsilon_0 \tag{33.30}$$

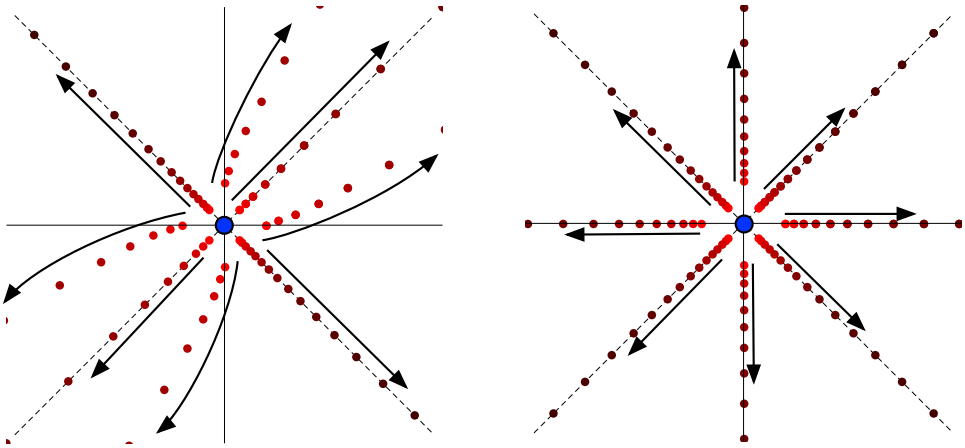

Figure 33.7: On the left. series of points near a fixed point (shown in blue) where the matrix $\mathbb{A}$ has two distinct real eigenvalues with $\lambda_a, \lambda_b > 1$. Note that series of points along the dotted lines stay on the dotted lines–they correspond to eigenvectors of $\mathbb{A}$. Series of points elsewhere will eventually converge onto one dotted line–the one corresponding to the larger eigenvalue. On the right, series of points where $\lambda_a = \lambda_b > 1$. Note that here all series move away from the fixed point along straight lines.

It turns out that in order to analyze the stability of a fixed point $\bar{x}$, what we want to do is find the eigenvalues of the matrix $\mathbb{A}$ that describes how points near the fixed point transform. $\mathbb{A}$ should have two eigenvalues λ_a and λ_b, with two associated eigenvectors $a = (z_a, p_a)$ and $b = (z_b, p_b)$. Note that for each of these, we have

$$\mathbb{A}a = \lambda_a a, \qquad \mathbb{A}b = \lambda_b b \tag{33.31}$$

so that if we start with a point at an eigenvector $x_0 = a_0$ or $x_0 = b_0$ (remember that an eigenvector can have any length, so a_0 or b_0 corresponds to a particular length) then we have

$$a_k = \mathbb{A}^k a = \lambda_a^k a, \qquad b_k = \mathbb{A}^k b = \lambda_b^k b \tag{33.32}$$

Note that a_k and b_k are essentially rescalings of a_0 and b_0. Now, depending on what the eigenvalues of $\mathbb{A}$ are, we get different behaviors.

Two Distinct Real Eigenvalues, $\lambda_a, \lambda_b > 1$

Suppose that λ_a and λ_b are distinct real numbers with $\lambda_a, \lambda_b > 1$. We can show the Poincaré map by putting the fixed point in the center. Then we draw dotted lines along the directions

the eigenvectors a and b point in, away from the center. If we start out with a point somewhere on the line corresponding to eigenvector a, then we know we will have a series

$$\left\{a_0, \quad a_1 = \lambda_a a_0, \quad a_2 = \lambda_a^2 a_0, \quad \ldots\right\}$$

Each successive point is farther away from the fixed point than the one before it, still on the dotted line, because $\lambda_a^{k+1} > \lambda_a^k$, and similarly for λ_b. We get a similar series of dots for points on the other diagonal. A generic point x_0 we should be able to write as a linear combination of a and b:

$$\epsilon_0 = c_1 a + c_2 b$$

and we will have

$$\epsilon_1 = \mathbb{A}[c_1 a + c_2 b] = c_1 \mathbb{A} a + c_2 \mathbb{A} b = c_1 \lambda_a a + c_2 \lambda_b b$$

and similarly

$$\epsilon_k = c_1 \lambda_a^k a + c_2 \lambda_b^k b$$

Notice that whichever eigenvalue is larger will start to dominate after a number of mappings, so that the points will end up moving closer and closer to that dotted line. However, we cannot act with the mapping to many times, because we will leave the region close to the fixed point where this analysis is valid. This is shown in figure 33.7 on the left. This fixed point is clearly unstable.

Two Identical Real Eigenvalues, $\lambda_a = \lambda_b = \lambda > 1$

What changes if our eigenvalues are equal to each other? Not much, qualitatively. If we have a 2×2 matrix with identical eigenvalues, then it must be proportional to the identity–and this means that every vector is an eigenvector of $\mathbb{A}$. This means that for any point x_0 we will have

$$\epsilon_k = \lambda^k \epsilon_0$$

so that every series of points moves outward along straight lines. This is shown in figure 33.7 on the right.

Two Real Eigenvalues, $\lambda_a, \lambda_b < -1$

Suppose instead of two real eigenvalues greater than $+1$, we have two real eigenvalues less than -1. Then if we start with a point a_0, an eigenvector with eigenvalue λ_a, we will create

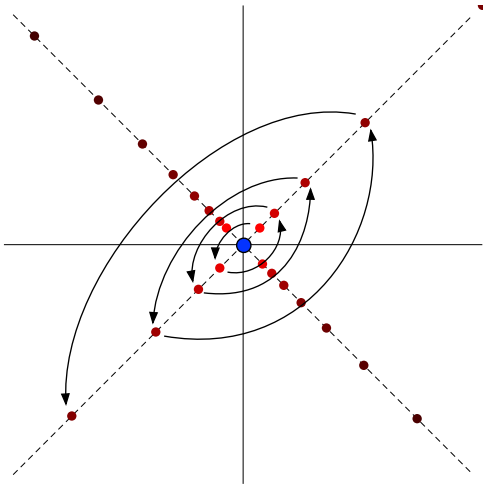

Figure 33.8: The mapping near a fixed point where the eigenvalues are real and $\lambda_a, \lambda_b < -1$. One set of arrows is shown for the eigenvalues moving across the fixed point in the center; the other set of dots works identically.

the series of points

$$\Big\{a_0, \qquad a_1 = -|\lambda_a| a_0, \qquad a_0 = |\lambda_a|^2 a_0, \qquad \dots \Big\}$$

Notice that each successive term flips sign. Since the a_k are points on the plane, this means the series of points is flipping back and forth across the fixed point. However, the distance between a_k and the origin increases each time (since $|\lambda_a| > 1$), so the points are also moving farther away from the fixed point. We have a similar story for points corresponding to eigenvectors with eigenvalue λ_b. And just as before, points in between can be thought of as a linear combination of eigenvectors, and they will flip back and forth but not along a straight line–with many mappings they begin to converge to the line corresponding to the larger eigenvalue. Unless, of course, the eigenvalues are equal to each other, in which case all series of points lie along straight lines. This is shown in figure 33.8.

Summary with Real Eigenvalues $|\lambda_a|, |\lambda_b| > 1$

From here it is clear that whenever the eigenvalues are real and have $|\lambda_a|, |\lambda_b| > 1$, we have an unstable fixed point, where any starting point not actually on top of the fixed point will move successively farther away from the fixed point with each mapping. (We are neglecting the

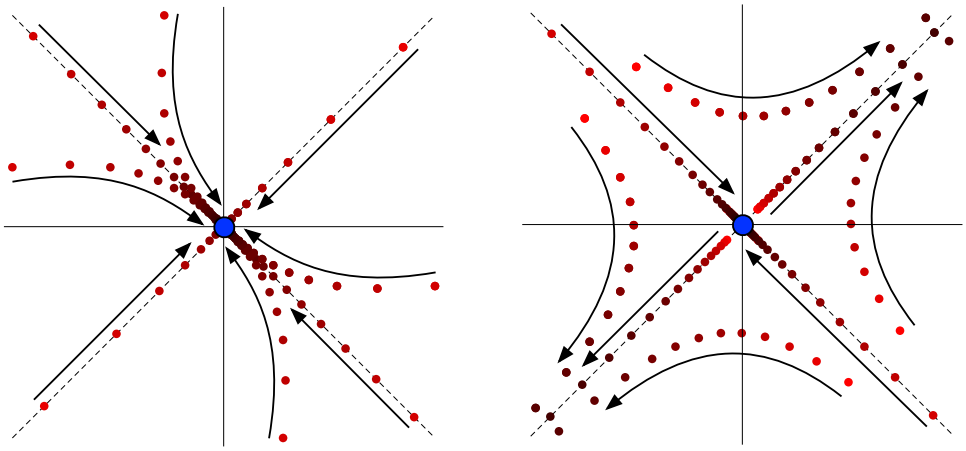

Figure 33.9: On the left a series of points near a fixed point (shown in blue) where the matrix $\mathbb{A}$ has two distinct real eigenvalues with $0 < \lambda_a, \lambda_b < 1$. Note that series of points converge to the fixed point. On the right, series of points where $\lambda_a > 1$, and $\lambda_b < 1$. Here, only points along the dotted line corresponding to the smaller eigenvalue actually reach the center–all others eventually move away and converge to the dotted line associated to the larger eigenvalue.

case where one eigenvalue is positive and the other negative for brevity; nothing unexpected happens there.)

Real Eigenvalues with $|\lambda_a|, |\lambda_b| < 1$

Now suppose we have real eigenvalues, but we have $|\lambda_a|, |\lambda_b| < 1$. For simplicity let's assume both eigenvalues are positive (having one or both negative just adds some flipping over the fixed point, as above.) In this case, if we pick an initial point that corresponds to an eigenvector, a_0, then our series

$$\left\{a_0, \qquad a_1 = \lambda_a a_0, \qquad a_2 = \lambda_a^2 a_0, \qquad \ldots\right\}$$

consists of points where each successive point is closer to the fixed point than the previous one:

$$|a_{k+1}| = \lambda_a^{k+1}|a_0| = \lambda_a|a_k| < |a_k|$$

along the line of eigenvectors. Again, initial points that aren't eigenvectors can be thought of as linear combinations of those that are:

$$\epsilon_0 = c_1 a + c_2 b$$

so that

$$\epsilon_k = c_1 \lambda_a^k a + c_2 \lambda_b^k b$$

If $\lambda_a > \lambda_b$ then after many mappings the second term will be small compared to the first, so again the points tend to converge along the line corresponding to the larger eigenvalue. This is shown in figure 33.9 on the left. But all of the points will eventually end up at the fixed point at the center–so in this case the fixed point is stable.

Real Eigenvalues with $|\lambda_a| > 1$, $|\lambda_b| < 1$

And of course we should consider the case where we have two real eigenvalues, one with magnitude greater than 1 and one with magnitude less than 1. Again working for simplicity with both positive, suppose $\lambda_a > 1$ and $0 < \lambda_b < 1$. Points initially along the line corresponding to eigenvector a move increasingly farther away from the fixed point along this line. Points initially along the line corresponding to eigenvector b move increasingly closer to the fixed point along this line. But for any point that is not along one of these lines, we write, as usual, it as a linear combination of the two eigenvectors. But then after many mappings, the term corresponding to the larger eigenvalue dominates, and this takes the points in the series farther and farther away. So even if only one of the eigenvalues has magnitude greater than 1, the fixed point is still unstable.

Complex Eigenvalues

What if the eigenvalues are complex? Well, first think about how we find the eigenvalues–they will be the solutions to the quadratic equation

$$\det[\mathbb{A} - \lambda \mathbb{I}] = 0$$

We know the form of a solution to the quadratic equation is

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and this is complex if

$$b^2 - 4ac < 0$$

But in this case, both solutions are complex, and they are a pair of complex conjugates, that we can write as

$$\lambda_a = \lambda e^{i\phi}, \qquad \lambda_b = \lambda e^{-i\phi}$$

where λ is a positive real number. I'm sure you won't be surprised to learn that the important distinction to make is whether λ is greater than 1 or less than 1. The eigenvectors that correspond to these eigenvalues are complex–so they don't lie anywhere on the phase plane. This will mean there aren't any lines in the phase plane where a series of points always lies along the line. But we can still think of any point in the phase plane as a linear combination of the eigenvectors a and b:

$$\epsilon_0 = c_1 a + c_2 b$$

(except that this time we must choose c_1 and c_2 complex as well, so that ϵ_0 comes out real.) But then we have

$$\epsilon_k = c_1 \lambda^k e^{ik\phi} a + c_2 \lambda^k e^{-ik\phi} b = \lambda^k [c_1 e^{ik\phi} a + c_2 e^{-ik\phi} b]$$

If $\lambda < 1$, then the magnitudes of the ϵ_k get smaller and approach the fixed point–so the fixed point is stable. If $\lambda > 1$, then the magnitudes of the ϵ_k get larger, so the fixed point is unstable. At the same time, they rotate around (as you might expect given that there are no lines that they stay on), so they form inward, or outward spirals. This is shown in figure 33.10.

Eigenvalues of magnitude 1

Finally there is the case where one or both eigenvalue has magnitude 1. There are several subcases here, but no real surprises:

1. $\lambda_a = \lambda_b = 1$: The matrix $\mathbb{A}$ is equal to the identity, and all points are fixed points.
2. $\lambda_a = \lambda_b = -1$: We have $\mathbb{A} = -\mathbb{I}$ and after every mapping each point simply reflects across the center, so that after two mappings each point is back where it started.
3. $\lambda_a = -\lambda_b = 1$: After every mapping every point reflects across the line corresponding to a, so that a point on this line doesn't move. Every other point comes back to it's original position after two mappings.
4. Eigenvalues real, $|\lambda_a| = 1$, $|\lambda_b| > 1$: points on the line corresponding to a either stay where they are or reflect back and forth, points on the line corresponding to b move farther and farther away from the fixed point (flipping back and forth if $\lambda_b < 0$), generic points move farther and farther away, and get closer to the b line.

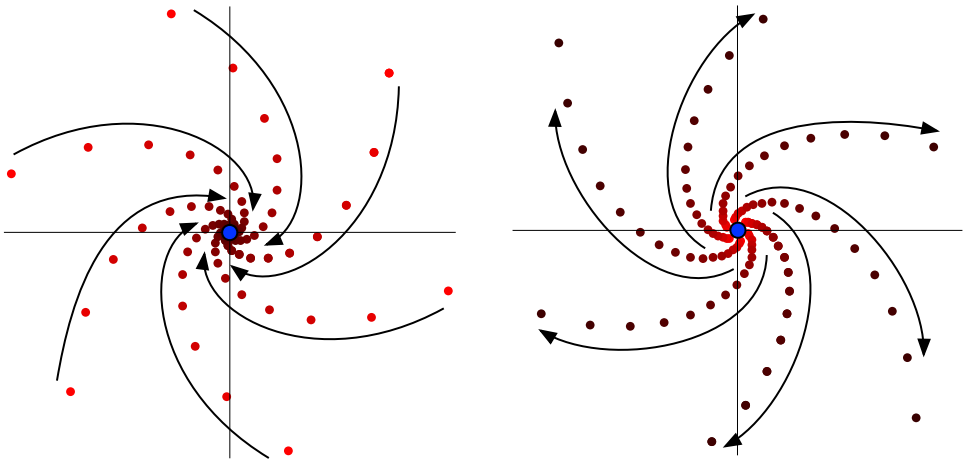

Figure 33.10: The mapping when the eigenvalues are complex. Both form spirals; when the magnitude of the eigenvalues are greater than one they spiral outward (right) so that the fixed point is unstable, when less than one they spiral inward (left) so that the fixed point is stable.

5. Eigenvalues real, $|\lambda_a| = 1$, $|\lambda_b| < 1$: points on the line corresponding to a either stay where they are or reflect back and forth, points on the line corresponding to b move closer and closer to the fixed point (maybe flipping back and forth), generic points move closer and closer, and get closer to the a line.

6. Eigenvalues complex, $|\lambda_a| = |\lambda_b| = 1$: points rotate on ellipses around the fixed point.