The Circle Map

Lecture 35

Physics 311: Classical Mechanics
Fall 2011

2 December 2011

This week we have been studying discrete maps, both as interesting mathematical objects in their own right and as ways of studying physical systems. Remember a discrete map is any function that maps a point to a point:

$$\Psi : \qquad x \quad \rightarrow \quad \Phi(x) \tag{35.1}$$

and that allows us to create sequences of points in the form

$$\{x_0, \;\; x_1 = \Phi(x_0), \;\; x_2 = \Phi(\Phi(x_0)), \;\; \ldots\}$$

so that

$$x_{k+1} = \Phi(x_k) \tag{35.2}$$

In the case of the Poincaré map, each of these points was actually a point in phase space, so that for a one-dimensional system they would have two components (position and momentum.) The mapping Φ then corresponded to evolving the system forward by some fixed amount of time, so that the points formed a series of "snapshots" of the system in phase space. We also looked at the logistic map, which used points with only one component restricted to the unit interval $[0, 1]$. This led to interesting things such as period doubling bifurcation and (nearly) chaotic behavior.

35.1 Physical Motivation

35.1.1 The Damped Driven Pendulum

Consider a simple rigid pendulum, whose equation we well understand

$$m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0 \tag{35.3}$$

For example, we might imagine the motion of a child on a swing as being described by this equation. This system has a natural frequency $\omega_0 = \sqrt{\frac{g}{\ell}}$. It is already a non-linear system, by virtue of the $\sin\theta$ term. However, now let's make it more complicated and *not* ignore friction–we will add a damping term:

$$m\ell^2\ddot{\theta} + 2\beta\dot{\theta} + mg\ell\sin\theta = 0 \tag{35.4}$$

But then, when a child plays on the swings, they generally push themselves along by pumping their legs. Thus, we will add a driving torque $f(t)$. It makes sense to look at situations where this driving force is periodic (the child pumps their legs rhythmically with a regular period). And in fact, it turns out that the results we will be interested in are not particularly sensitive to the particulars of this driving torque, so we can assume a very simple form without losing any inherent complexity in the problem. Thus we will write

$$m\ell^2\ddot{\theta} + 2\beta\dot{\theta} + mg\ell\sin\theta = f_0 + f_1\sin\Omega t \tag{35.5}$$

When we non-dimensionalize this to make it simpler, we obtain

$$\ddot{\theta} + 2b\dot{\theta} + \sin\theta = r_0 + r_1\sin\omega\tau \tag{35.6}$$

where $\omega = \frac{\omega_0}{\Omega}$ is the ratio of the natural frequency of the system to the driving frequency. What we are going to find today is that the behavior of this system, and systems like it, depends in a dramatic way on whether ω is rational or irrational.

35.1.2 The Poincaré Map

Now, we might imagine that for small enough θ we could approximate this system using standard linearization techniques. This would leave us with a damped, driven simple harmonic oscillator. Or, if we carried our approximation out to one more term, we would have a damped, driven quartic oscillator. Both of these are systems we have studied, and the latter in particular had some lovely non-linear effects. But either of these approximations would completely miss the effects we want to study today. What's more, we can't access them numerically, either. The numbers in computers are *inherently rational*, so any effect that depends on the ratio of $\frac{\omega_0}{\Omega}$ being irrational will be invisible to numerical analysis.

Instead, let's use the tools we've been developing. We'll start by considering what the Poincaré map for this system will look like. We will use as the time lapse for the snapshots the period of the driving force, consistent with our earlier treatment. At every snapshot, we have the position in phase space of the pendulum bob, described by $(\theta_n, \dot{\theta}_n)$. And the time evolution function that takes us from one snapshot to the next we can write as

$$(\theta_{n+1}, \dot{\theta}_{n+1}) = \Phi(\theta_n, \dot{\theta}_n) \tag{35.7}$$

Since the driving force is periodic and exactly in synch with the snapshots, the function Φ that gives our time evolution will not depend explicitly on which snapshot we are at–it will only depend on the parameters $\{b, r_0, r_1, \omega\}$. By using the Poincaré map to analyze our system, we are eliminating a lot of extraneous complications, and turning differential equations into simple discrete maps.

35.1.3 The Sine Circle Map

Now, the damping in the system will mean that after a large amount of time, when the system has reached a steady state solution, the phase diagram will end up converging to a single curve in phase space, such as a limit cycle. At this point, we have a natural relationship between $\dot{\theta}$ and θ (given by the curve), and we can replace the Poincaré map (which maps points in phase space to points in phase space) with one that is even simpler, and just maps an angle θ_n to an angle θ_{n+1}. The general structure of this map should be

$$\theta_{n+1} = \Big(\theta_n + \Theta + g(\theta_n)\Big) \bmod 2\pi$$

Note that the "mod 2π" is there to remind us that the angle itself is only defined up to modulo 2π, so that every time we acquire $\theta > 2\pi$ we should reinterpret it as lying somewhere in the region $[0, 2\pi]$. For the same reason, we can assume that $\Theta \in [0, 2\pi]$, and that g is a periodic function, with $g(\theta + 2\pi) = g(\theta)$. Again, what is amazing is that the important qualitative features of what we are discussing do not care what the exact form of g is, so we will make the very simple assumption

$$g(\theta) = K \sin\theta$$

At this point, we have boiled down our complicated damped driven pendulum to a discrete map of one-dimensional points θ defined on the circle:

$$\theta_{n+1} \equiv C_{\Theta,K}(\theta_n) = \Big(\theta_n + \Theta + K\sin\theta_n\Big) \bmod 2\pi \tag{35.8}$$

Note that by using the "mod 2π" rule, we can assume $0 \leq \Theta \leq 2\pi$. Furthermore, we will restrict our discussion to $0 \leq K \leq 1$.

This mapping is known as the *standard sine circle map*. At this point, we have made so many simplifying assumptions and approximations that you may not believe it has anything at all to do with the original physical system, as typified by equation 35.5. Our argument has been heuristic at best, and it is not possible to prove a rigorous connection between the two. Nonetheless, experimentally we find excellent qualitative agreement between the features of the sine circle map and the damped driven pendulum.

35.2 The Circle Map with $\mathbf{K} = \mathbf{0}$

Consider first the case where $K = 0$, so that the mapping is simply

$$\theta_{n+1} = \Big(\theta_n + \Theta\Big) \bmod 2\pi$$

Furthermore, we will write

$$\rho = \frac{\Theta}{2\pi}, \qquad 0 \leq \rho \leq 1 \tag{35.9}$$

so we get

$$\theta_{n+1} = \Big(\theta_n + 2\pi\rho\Big) \bmod 2\pi$$

If $\rho = 0$ the mapping is trivial, and the same is true if $\rho = 1$:

$$\theta_{n+1} = \Big(\theta_n + 2\pi\Big) \bmod 2\pi = \theta_n$$

it just rotates us once around the circle, bringing us back to where we started. Every angle θ is a fixed point of $C_{2\pi,0}$ or $C_{0,0}$.

Suppose $\rho = \frac{1}{2}$. Then the mapping takes us halfway around the circle. Applying the mapping twice brings us back to our original position. (That is, the circle map $C_{\pi,0}$ has no fixed points, but $C^2_{\pi,0}$ is trivial, and every point is a fixed point.) If $\rho = \frac{2}{3}$, then the mapping takes any point 2/3 of the way around the circle. If we perform it again, we are taken $\frac{4}{3}$ of the way around, which is the same as $\frac{1}{3}$ of the way around. Performing the mapping a third time then brings any point back to its original position, so now $C^3_{\frac{4\pi}{3},0}$ is shown to be trivial. This is shown in figure 35.1.

It is not hard to see how this generalizes. Whenever ρ is a rational number, we can write it as

$$\rho = \frac{j}{k}, \qquad j, k \in \mathbb{Z}, \qquad \mathrm{GCD}(j, k) = 1$$

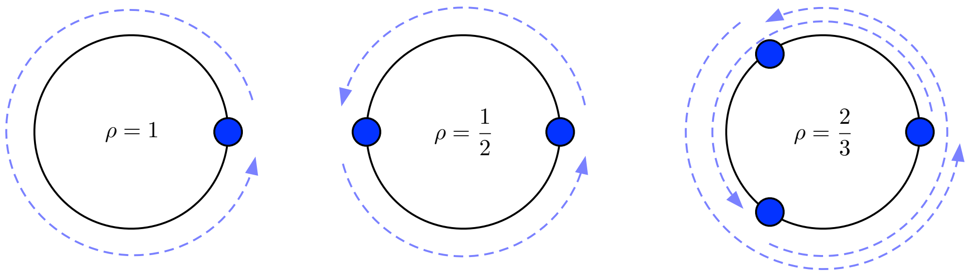

Figure 35.1: The mapping cases with $\rho = 1$, $\rho = \frac{1}{2}$, and $\rho = \frac{2}{3}$.

And we can always assume $0 < j < k$. Each mapping will take a point $\frac{j}{k}$ of the way around the circle. After k rotations, we will then be back where we started. In the process, a point that begins, say at $\theta = 0$, will reach every point in the form $\frac{2\pi}{k}$, though not in order (unless $j = 1$.) In this case we call ρ the *winding number* of the mapping.

Now, if ρ isn't rational, then no number of mappings will ever bring us back to where we started. Instead, if we start at one point, then after an infinite number of mappings we will be able to get arbitrarily close to every point on the circle. Thus, for $K = 0$, there are *no* fixed points, unless ρ is an integer, in which case all points are fixed points.

35.3 Fixed Points of the Circle Map

Let's now assume that K is not zero, and consider what fixed points the circle map will have.

Non-zero K, $\Theta = 0$

We now want to think about what happens when we allow K to be non-zero. For example, suppose we consider $\Theta = 0$. In this case we have

$$\theta_{n+1} = \Big(\theta_n + K \sin\theta_n\Big) \bmod 2\pi$$

Note that there are two fixed points of this mapping: $\bar{\theta} = 0$ and $\bar{\theta} = \pi$. Let's do a quick analysis of the stability of these fixed points. The function that defines our mapping is

$$F(\theta) = \theta + K \sin\theta$$

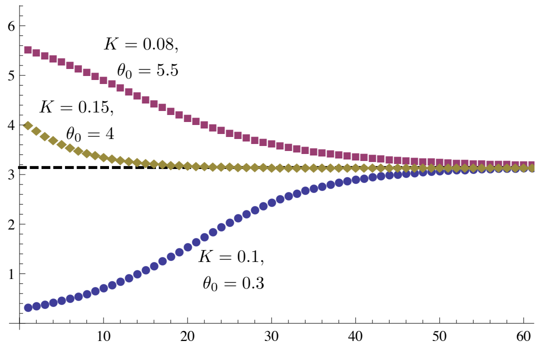

Figure 35.2: Three different values of K and initial points θ_0, all converging after a large number of mappings to the stable fixed point $\theta = \pi$.

(if we are near either fixed point, we can assume that we don't stray too far from it in a single mapping, so we will ignore the "mod 2π" part.) The derivative of this is

$$F'(\theta) = 1 + K\cos\theta$$

which has

$$F'(0) = 1 + K, \qquad F'(\pi) = 1 - K$$

so we see that (since $0 < K < 1$), we have $F'(0) > 1$, and $F'(\pi) < 1$. This makes the fixed point at π stable–so that, starting from any point on the circle, we will eventually converge to π. This is shown in figure 35.2.

Non-zero K, non-zero Θ

Now let's allow Θ to be non-zero. Using the "mod 2π" rule, we note that we can always assume $0 < \Theta < 2\pi$. If we again look for fixed points $\bar{\theta}$, we are looking for points where

$$\bar{\theta} + 2\pi N = F(\bar{\theta}) = \bar{\theta} + \Theta + K\sin\bar{\theta}, \qquad N \in \mathbb{Z} \tag{35.10}$$

Typically $N = 0$ or $N = 1$ for $0 < K < 1$ and $0 < \Theta < 2\pi$. There can be two angles $\bar{\theta}$ on the circle that satisfy this. First we can have

$$\sin(\bar{\theta} - \pi) = -\sin\bar{\theta} = \frac{\Theta}{K}, \qquad \Rightarrow \qquad \bar{\theta} = \pi + \sin^{-1}\frac{\Theta}{K}$$

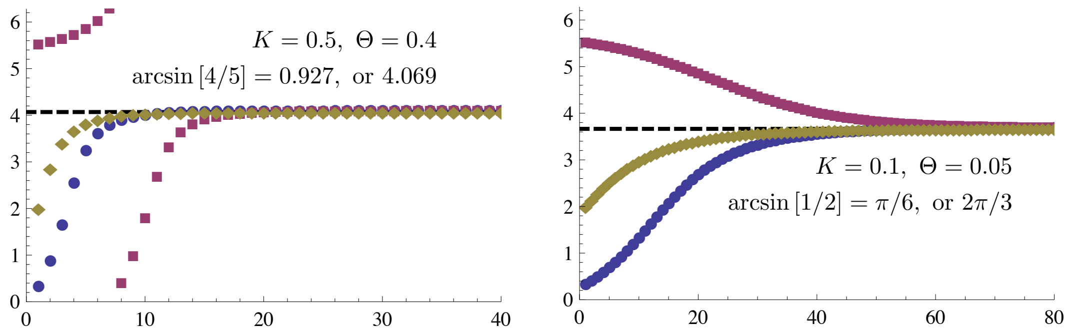

Figure 35.3: Two different cases of convergence to stable fixed points with different choices for K and Θ.

or, we can have

$$2\pi = \Theta + K\sin\bar{\theta}, \qquad \Rightarrow \qquad \bar{\theta} = \sin^{-1}\frac{2\pi - \Theta}{K}$$

In summary, we have

$$\bar{\theta} = \pi + \sin^{-1}\frac{\Theta}{K}, \qquad \text{or} \qquad \bar{\theta} = \sin^{-1}\frac{2\pi - \Theta}{K} \tag{35.11}$$

As it happens, each of these two fixed points are stable. Each equation they come from has a second solution which is *unstable*, just as in the simpler situation where $\Theta = 0$, where we had one stable and one unstable fixed point. However, it is immediately obvious that there will not always *be* fixed points: in order to have them, we require

$$\frac{\Theta}{K} \leq 1, \quad \Rightarrow \quad \Theta \leq K, \qquad \text{or} \qquad \frac{2\pi - \Theta}{K} \leq 1, \quad \Rightarrow \quad 2\pi - \Theta \leq K$$

Two different cases of K and Θ are shown in figure 35.3. In both cases, there is a clear convergence, regardless of starting point, to a fixed point that is one of the solutions of equation 35.11. We also show the regions of (K, Θ) space that allow for fixed points, in figure 35.4.

35.4 Fixed Points of $\mathbf{C}^{\mathbf{k}}_{\mathbf{K},\boldsymbol{\Theta}}$

We now want to consider when any iteration of the circle map $C^k_{K,\Theta}$ will have fixed points. First we will consider just $C^2_{K,\Theta}$.

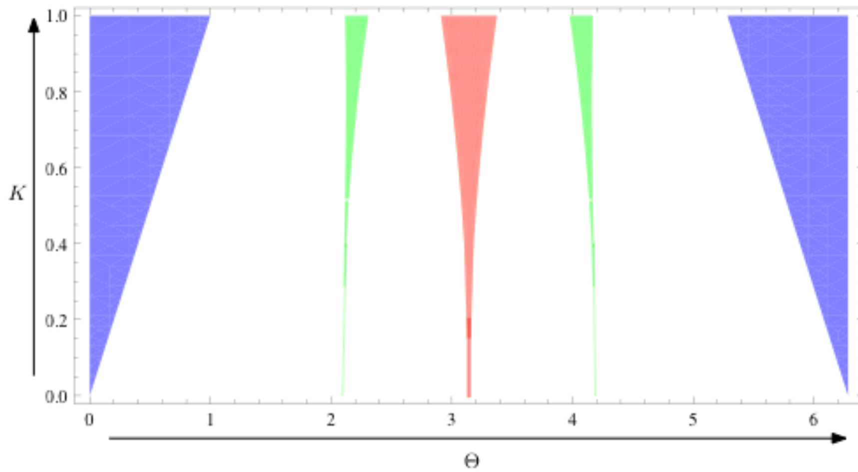

Figure 35.4: The regions of K and Θ space that allow for fixed points of the circle map $C_{K,\Theta}$ are shown in blue. The additional region that allows for fixed points of $C^2_{K,\Theta}$ is shown in red. Two more regions that allow for fixed points of $C^3_{K,\Theta}$ are shown in green.

$\mathbf{\Theta = 0}$

We have already discussed that if $K = 0$ and $\Theta = \pi$, every point is a fixed point of the "doubled mapping," so that $C^2_{0,\pi}$ is the trivial mapping. Now let's allow $0 < K < 1$ but keep $\Theta = 0$ for now. The function $F^2(\theta)$ is

$$F^2(\theta) = \theta + K\sin\theta + K\sin[\theta + K\sin\theta]$$

so a fixed point now must satisfy

$$\bar{\theta} + 2\pi N = \bar{\theta} + K\sin\bar{\theta} + K\sin[\bar{\theta} + K\sin\bar{\theta}], \qquad N \in \mathbb{Z}$$

or

$$2\pi N = K\sin\bar{\theta} + K\sin[\bar{\theta} + K\sin\bar{\theta}], \qquad N \in \mathbb{Z}$$

Looking at the function

$$g_K(\theta) = K\sin\theta + K\sin[\theta + K\sin\theta]$$

It is immediately apparent that

$$-2K < g_K(\theta) < 2K$$

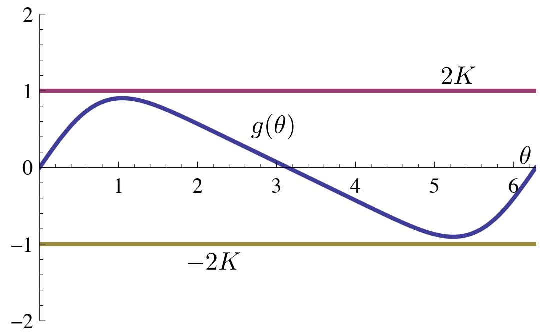

Figure 35.5: The function $g_K(\theta)$ has zeros for the fixed points of $C^2_{K,0}$.

so that if $0 < K < 1$, we have $-2 < g_K(\theta) < 2$. This means that a fixed point will always have $N = 0$. (It is the only value for which we can have $g_K(\theta) = 2\pi N$.) Now, we know that

$$g_K(0) = g_K(\pi) = g_K(2\pi) = 0$$

But these are the fixed points of $C_{K,0}$ that we already knew about. Are there any new fixed points? Actually, no. Plotting $g_K(\theta)$ reveals quickly that in increases above $\theta = 0$ to some maximum, decreases smoothly, passing through 0 at $\theta = \pi$, then hits a negative minimum, and increases again to 0 at $\theta = 2\pi$ (figure 35.5.)

But what if $\Theta \neq 0$? Then we have the mapping

$$F^2(\theta) = \theta + \Theta + K\sin\theta + \Theta + K\sin[\theta + \Theta + K\sin\theta]$$

so the fixed points $\bar{\theta}$ now require

$$\bar{\theta} + 2\pi N = \bar{\theta} + 2\Theta + K\sin\bar{\theta} + K\sin[\bar{\theta} + \Theta + K\sin\bar{\theta}]$$

or

$$2\pi N = g_{\Theta,K}(\bar{\theta}) = 2\Theta + K\sin\bar{\theta} + K\sin[\bar{\theta} + \Theta + K\sin\bar{\theta}] \tag{35.12}$$

We know for $\Theta = 0$, there are no $\bar{\theta}$ that satisfy this equation that don't satisfy equation 35.10. And certainly for a small enough Θ we don't expect this to change suddenly. But what about for larger Θ? And what happens as we vary K? We would like to be able to determine what region of (Θ, K) space allows for fixed points of $C^2_{K,\Theta}$ that aren't fixed points of $C_{K,\Theta}$.

Take, for example, $K = 1$ and $\Theta = \pi$. In this case we get

$$2\pi N = g_{\pi,1} = 2\pi + \sin\bar{\theta} + \sin[\bar{\theta} + \pi + \sin\bar{\theta}] = 2\pi + \sin\bar{\theta} - \sin[\bar{\theta} + \sin\bar{\theta}]$$

which we immediately see has solutions

$$\sin\bar{\theta} = \sin[\bar{\theta} + \sin\bar{\theta}] \qquad \Rightarrow \qquad \bar{\theta} = 0, \pi$$

These are familiar values...but remember that for these values of K and Θ there weren't any fixed points of $C_{K,\Theta}$. Just as you would expect, one of these values proves to be stable, and the other unstable. Now, suppose we keep $\Theta = \pi$, but allow K to vary. This gives us the condition

$$2\pi N = g_{\pi,K} = 2\pi + K\sin\bar{\theta} + K\sin[\bar{\theta} + \pi + K\sin\bar{\theta}] = 2\pi + K\sin\bar{\theta} - K\sin[\bar{\theta} + K\sin\bar{\theta}]$$

which is still satisfied by $\bar{\theta} = 0, \pi$. Now, suppose we allow Θ to be *near* π, but not at π. For some region around $\Theta = \pi$, we should still have these fixed points. In fact, if we numerically plot the region in Θ and K space where $C^2_{K,\Theta}$ has stable fixed points, in addition to the regions we had for $C_{K,\Theta}$ we have a new, roughly triangular "tongue" that extends from some finite range of Θ at $K = 1$ to a point $\Theta = \pi$ at $K = 0$. This is shown in figure 35.4 as well.

We could keep going, looking at fixed points of $C^k_{K,\Theta}$ for higher and higher values of k. We would find more "tongue" regions in (Θ, K) space that are of finite extension at $K = 1$, and converge to points at $K = 0$. On the $K = 0$ line, each of these points corresponds to a rational fraction of 2π. For example, for $k = 3$, we find fixed points for $K = 0$ at

$$\bar{\theta} = \frac{2\pi}{3}, \qquad \bar{\theta} = \frac{4\pi}{3}$$

which corresponds exactly to our earlier analysis of the circle map for $K = 0$.

35.5 Rotation Number and the Devil's Staircase

Let's return for a moment to the $K = 0$ case, where we defined the quantity $\rho(\Theta) = \frac{\Theta}{2\pi}$, as the fraction of the circle that we went around in a single tracing. With $K = 0$, the mapping is

$$F(\theta) = \Big(\theta + 2\pi\rho\Big) \bmod 2\pi$$

and this is the same regardless of what θ we start with. Now, we see immediately that with $K \neq 0$ we have

$$F(\theta) = \Big(\theta + \Theta + K\sin\theta\Big) \bmod 2\pi$$

and the amount of rotation will be different, depending on the initial point θ. However, it is still possible to define a generalization of ρ. Let's start by calling $\Delta\theta$ the amount that a point at θ is rotated.

$$\Delta\theta = F(\theta) - \theta = \Theta + K \sin\theta \tag{35.13}$$

Now, if we begin at a point θ_0, then act with the mapping n times, $\Delta\theta_n$ is the amount it changes by on the nth mapping:

$$\Delta\theta_n = \Theta + K \sin\theta_n \tag{35.14}$$

Suppose we perform the mapping m times. We can define an *average* change $\Delta\theta_{ave}$ as

$$\Delta\theta_{ave} = \frac{1}{m}\sum_{n=0}^{m} \Delta\theta_n \tag{35.15}$$

Notice that when $K = 0$, $\Delta\theta_n = 2\pi\rho$ for every n, so $\Delta\theta_{ave} = 2\pi\rho$ as well. Finally, we will imagine that we do this averaging over an *infinite* number of mappings. Then we will define

$$\rho(\Theta, K) = \frac{1}{2\pi}\lim_{m\to\infty}\left[\frac{1}{m}\sum_{n=0}^{m}\Delta\theta_n\right] \tag{35.16}$$

This is called the *rotation number* for the circle map. Obviously, we still have

$$\rho(\Theta, 0) = \frac{\Theta}{2\pi}$$

as usual. This is a very simple function of Θ. But for any other value of $K \in [0, 1]$, the function $\rho(\Theta, K)$ is very odd, indeed.

First, notice that for any values of (Θ, K) that give the circle map $C_{\Theta,K}$ a stable fixed point, we should have either $\rho(\Theta, K) = 0$ or $\rho(\Theta, K) = 1$. This is because after a large number of mappings, any series of points $\{\theta_n\}$ must be converging to the stable fixed point–either by moving an increasingly small amount in each mapping, or by rotating by increasingly close to once around the circle each mapping. That is, for large enough n, we have $\Delta\theta_n \approx 0$ or $\Delta\theta_n \approx 2\pi$. Now, if we take equation 35.15 for some finite m, this doesn't do much. But as we make m larger and larger, more and more terms give increasingly close to the same value of $\Delta\theta_n$, until the average is completely dominated by them. It is not hard to see, based on continuity, that we should have

$$\rho(\Theta, K) = 0, \qquad \text{for} \qquad \Theta < K$$

and

$$\rho(\Theta, K) = 1, \qquad \text{for} \quad 2\pi - \Theta < K$$

This implies that if we fix $K \neq 0$, the rotation number $\rho(\Theta, K)$ as a function of Θ goes through regions where it is simply a constant.

But now, remember that we also have stable fixed points of $C^2_{\Theta,K}$ to consider. These (that is, the ones that aren't fixed points of $C_{\Theta,K}$ anyways) imply that after a large number of mappings, the series $\{\theta_n\}$ is alternating between two points, so that $\Delta\theta_n + \Delta\theta_{n+1} \approx 2\pi$. This will mean that the as we take an average of $\Delta\theta_n$ over more and more points in the series, we will eventually get $\Delta\theta_{ave} = \pi$, so that $\rho(\Theta, K) = \frac{1}{2}$. And again, for any fixed value of $K \neq 0$, there is a range of values of Θ (near $\Theta = \pi$) that have this behavior. Thus we get

$$\rho(\Theta, K) = \frac{1}{2}, \qquad \text{for} \quad \pi - \varepsilon(K) < \Theta < \pi + \varepsilon(K)$$

where $\varepsilon(K)$ is essentially the shape of the red "tongue" in figure 35.4. In fact, we can see that for any k, we will have regions of (Θ, K) space where $C^k_{K,\Theta}$ has stable fixed points that $C^\ell_{K,\Theta}$ doesn't, for $\ell < k$. These fixed points will naturally give

$$\rho(K, \Theta) = \frac{j}{k}, \qquad j \in \{1, 2, \ldots, k-1\}$$

and for any value of $K \neq 0$, there will be finite regions of Θ that give this constant value of $\rho(K, \Theta)$. Looking at figure 35.4, and imagining a great many more tongues displayed on it, we notice that this is going to give the function $\rho(\Theta, K)$ some very strange qualities. For $K = 0$, it is a simple linear function, from $\Theta = 0$ to $\Theta = 2\pi$. If we make K small but non-zero, then suddenly tiny plateaus appear in the function $\rho(K, \Theta)$, centered around every rational fraction of 2Π. As we increase K, the size of each of these plateaus grows. This is shown in figure 35.6.

Now, remember that the set of rational fractions of 2π form a set of measure zero, as compared to the dense irrational fractions of 2π. For any specific value of $0 < K < 1$, the function $\rho(K, \Theta)$ must be constant in the immediate neighborhood of a rational fraction of Θ, but elsewhere it will be continuously increasing. Still, as K grows, the sizes of the "neighborhoods" where ρ is constant grows. By the time we reach $\rho(\Theta, 1)$ the points where ρ is *not* constant has become a set of measure zero! The function $\rho(\Theta, 1)$ is rather famous, and is known as the "devil's staircase." Among other interesting features, it is self-similar. If you "zoom in" on the graph, you will find structures identical to those at larger length scales. This is shown in figure 35.7.

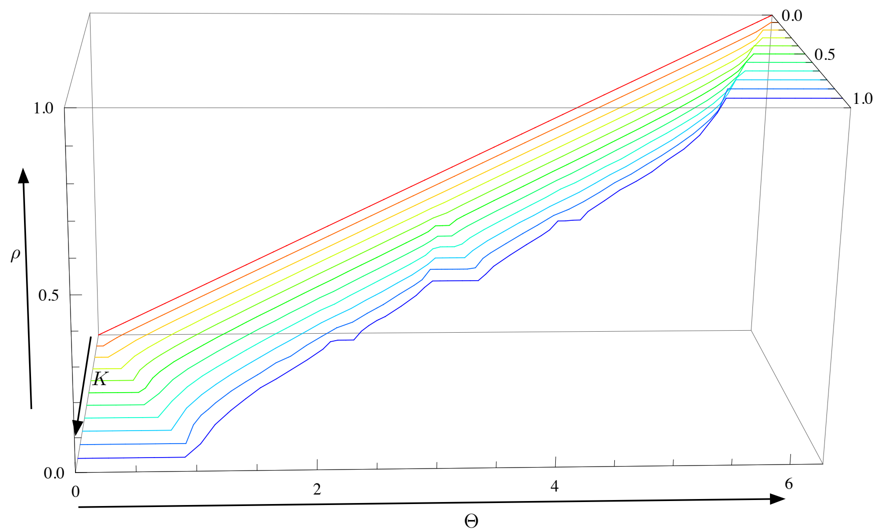

Figure 35.6: The function $\rho(K, \Theta)$.

So what does this all mean in terms of the physical system we began by considering (the damped driven rigid pendulum)? Remember that Θ and K relate, originally to the parameters of the periodic driving force used on the damped, driven pendulum. And the value of ρ is going to indicate some periodicity of the resulting motion. Since our circle map started as a Poincaré map, using the period of the driving force as the time lapse, a motion with $\rho = 1$ indicates motion that is periodic with the same periodicity as the driving force. A motion with $\rho = \frac{1}{2}$ indicates motion that has twice the period as the driving force, a motion with $\rho = \frac{1}{3}$ indicates motion that has three times the period as the driving force, and so on.

Clearly, the most important qualitative feature of ρ, the rotation number, are these plateaus. They indicate regions of (Θ, K) space where, averaged over a large number of cycles, we get the same basic behavior. This means that, if we have some periodic motion with a particular periodicity that is related by a rational number to the driving force period, then we can actually change the details of the driving force (the quantities Θ and K) by some small amount, and expect that the periodicity *will not* change. This is known as frequency locking, and it is observed in many physical systems, including the damped driven pendulum.

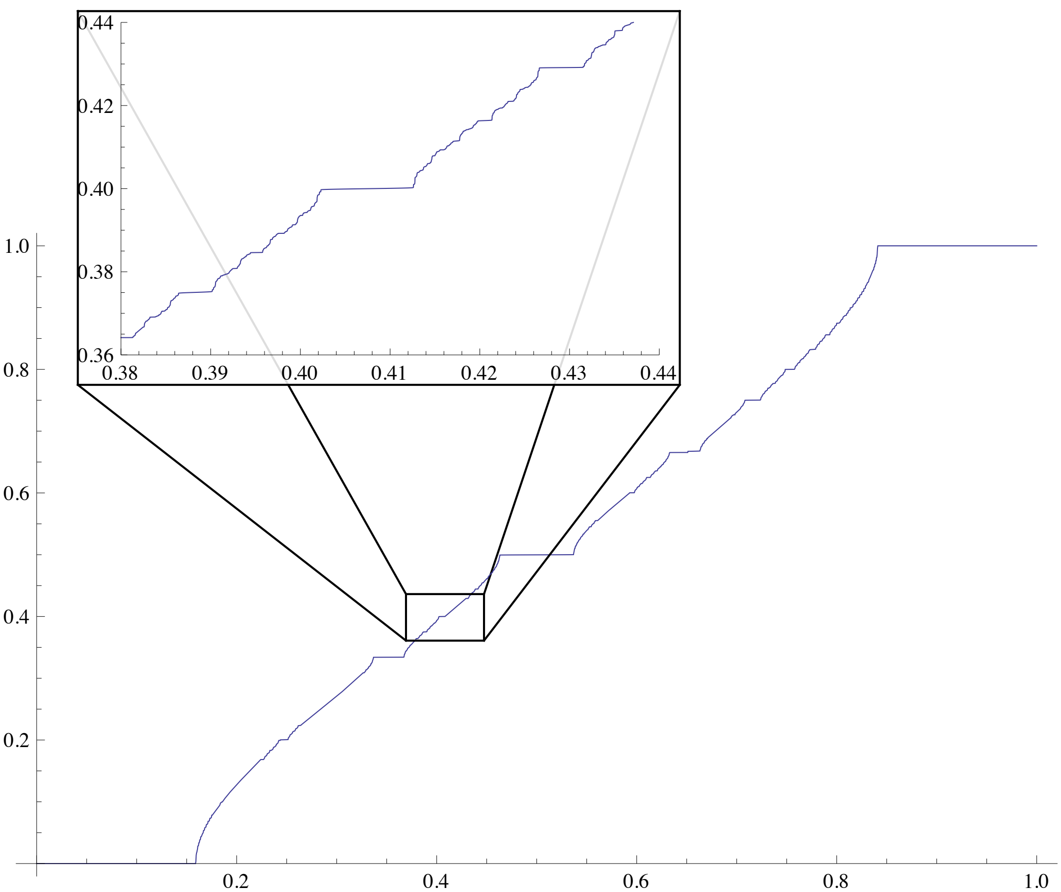

Figure 35.7: The self-similarity of the devil's staircase $\rho(1, \Theta)$.