The Hénon Map

Lecture 37

Physics 311: Classical Mechanics
Fall 2011

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We have been discussing discrete maps as a way to understand the dynamics of non-linear systems, and have been gradually increasing the complexity of the maps themselves, focusing on common features, such as fixed points, linearization, bifurcations, and the transition into chaos. Today I want to present one last famous example of a discrete map. You will notice that it shares many of these features, as well as some new, interesting behaviors. Today's map is called the Hénon map, and it is based around a kicked simple harmonic oscillator system (just as last time we used the kicked rotator to the develop a discrete map.)

37.1 The Kicked Simple Harmonic Oscillator

37.1.1 The Poincaré Map

Suppose we consider a simple harmonic oscillator in one dimension, which we know has the non-dimensionalized Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2,$$

equations of motion

$$\dot{x} = \frac{\partial H}{\partial p} = p, \qquad \dot{p} = -\frac{\partial H}{\partial x} = -x,$$

and solutions

$$x(\tau) = A\sin(\tau + \phi), \qquad p(\tau) = A\cos(\tau + \phi)$$

Now, we imagine that at discrete time intervals T, this system receives a "kick," which we will idealize as a delta function force (that is, an impulse force.) Suppose we say that the magnitude of the kick is proportional to the position squared, x^2, and is negative. This can be derived from the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{37.1}$$

which gives equations of motion

$$\begin{aligned} \dot{x} &= \tfrac{\partial H}{\partial p} &&= p \\ \dot{p} &= -\tfrac{\partial H}{\partial x} &&= -x - x^2 \textstyle\sum_{n=-\infty}^{\infty} \delta(t - nT) \end{aligned} \tag{37.2}$$

Because of the kicks, energy is *not* conserved in this system. Notice that, unlike the kicked rotator from yesterday, we are not assigning the magnitude of the kick a variable that we can play with later. However, the value T which gives the time intervals between kicks we will allow to vary. One way to justify this is to recall that when studying the driven simple harmonic oscillator (a similar system) we non-dimensionalize away the magnitude of the driving force (so changing it can't have any physical effect other than rescaling the oscillations) but we do retain information about the comparison of the driving frequency to the natural frequency.)

Now, suppose we say that the position and momentum just before the nth kick is (x_n, p_n), so we have:

$$x(nT - \epsilon) = x_n, \qquad p(nT - \epsilon) = p_n \tag{37.3}$$

over the time of the kick $[nT - \epsilon, nT + \epsilon]$, $x(\tau)$ will not change significantly. But $p(\tau)$ will, because of the kick. We will have

$$p(nT + \epsilon) = p_n - \int_{nT-\epsilon}^{nT+\epsilon} x^2 \delta(t - nT)\, dt = p_n - x_n^2$$

Now let's think about the time between then and the next kick. In this time, we just have a normal simple harmonic oscillator, so we must have

$$x(\tau) = A\sin(\tau + \phi), \qquad p(\tau) = A\cos(\tau + \phi)$$

where A and ϕ are fixed by "initial conditions" (by which what we really mean are the $x(nT + \epsilon)$ and $p(nT + \epsilon)$ values.) This means we define A and ϕ through the equations

$$x_n = A\sin(nT + \phi), \qquad p_n - x_n^2 = A\cos(nT + \phi)$$

Then, just before the next kick we should have

$$\begin{aligned} x_{n+1} &= x(nT + T - \epsilon) \\ &= A\sin(nT + \phi + T) \\ &= A\sin(nT + \phi)\cos T + A\cos(nT + \phi)\sin T \\ &= x_n \cos T + (p_n - x_n^2)\sin T \end{aligned}$$

and

$$\begin{aligned} p_{n+1} &= p(nT + T - \epsilon) \\ &= A\cos(nT + \phi + T) \\ &= A\cos(nT + \phi)\cos T - A\sin(nT + \phi)\sin T \\ &= (p_n - x_n^2)\cos T - x_n \sin T \end{aligned}$$

Now we have the Poincaré mapping of this system (using the period T as the time between snapshots):

$$\begin{aligned} x_{n+1} &= x_n \cos T + (p_n - x_n^2)\sin T \\ p_{n+1} &= (p_n - x_n^2)\cos T - x_n \sin T \end{aligned} \tag{37.4}$$

37.1.2 The Standard Hénon Map

The above is equivalent to a mapping called the "Hénon map," but it is usually written somewhat differently. First, notice that it is possible to eliminate the p_n dependence in equation 37.4. In order to do this, we multiply the top equation by $\cos T$ and the bottom by $\sin T$, and then we take the difference:

$$x_{n+1}\cos T = x_n \cos^2 T + (p_n - x_n^2)\sin T \cos T$$

$$p_{n+1}\sin T = -x_n \sin^2 T + (p_n - x_n^2)\sin T \cos T$$

so we get

$$x_{n+1}\cos T - p_{n+1}\sin T = x_n$$

which can be rewritten as

$$p_n \sin T = x_n \cos T - x_{n-1}$$

and this can be inserted back into the first part of equation 37.4, giving

$$x_{n+1} = x_n \cos T + x_n \cos T - x_{n-1} - x_n^2 \sin T = 2x_n \cos T - x_{n-1} - x_n^2 \sin T$$

Notice the fact that the $(n+1)$th point now depends on both the nth point and the $(n-1)$th point. This is equivalent to having replaced two coupled first order differential equations in $x(\tau)$ and $p(\tau)$ with one second order differential equation in $x(\tau)$. This is still not quite the usual form of the Hénon map. Suppose we make the definitions

$$X_n = \alpha x_n + \beta, \qquad \alpha = -\frac{\tan T}{2-\cos T}, \qquad \beta = \frac{1}{2-\cos T} \tag{37.5}$$

then we can write

$$\frac{1}{\alpha}(X_{n+1}-\beta) = \frac{2}{\alpha}(X_n-\beta)\cos T - \frac{1}{\alpha}(X_{n-1}-\beta) - \frac{1}{\alpha^2}(X_n^2 - 2\beta X_n + \beta^2)\sin T$$

or

$$X_{n+1} + X_{n-1} = \left[2\beta - 2\beta\cos T - \frac{\beta^2}{\alpha}\sin T\right] + \left[2\cos T + \frac{2\beta}{\alpha}\sin T\right] X_n - \frac{\sin T}{\alpha}X_n^2$$

which, using that $\frac{\beta}{\alpha} = -\cot T$, is

$$X_{n+1} + X_{n-1} = \beta\left[2 - 2\cos T + \cos T\right] + \left[2\cos T - 2\cos T\right]X_n - \frac{\sin T}{\alpha}X_n^2$$

or

$$X_{n+1} + X_{n-1} = \beta\left[2 - \cos T\right] - \frac{\sin T}{\alpha}X_n^2$$

and then finally

$$X_{n+1} + X_{n-1} = 1 - aX_n^2, \qquad a = \frac{\sin T}{\alpha} = \cos T(\cos T - 2)$$

Notice that with these redefinitions we have eliminated the X_n term from the mapping–this is the main point. Notice also that a is a number with a limited range:

$$a \in [-1, 3]$$

that is a "re-packaging" of the constant T. At this point we can rewrite this back in terms of a mapping of two-dimensional points (X_n, P_n) (so that it is again a straightforward discrete map.) To do this we just write

$$\begin{aligned} X_{n+1} &= 1 - aX_n^2 - P_n \\ P_{n+1} &= X_n \end{aligned} \tag{37.6}$$

And this is the usual form that the Hénon map is presented in. Note that we have

$$P_n = X_{n-1} = \alpha x_{n-1} + \beta = \alpha(x_n\cos T - p_n \sin T) + \beta$$

Our new space (X, P) is thus a warped version of our old phase space (x, p). The "position" coordinate has been shifted and rescaled, and the "momentum" coordinate has been rotated and shifted and rescaled.

37.1.3 Fixed Points, Linearization, and Area Preservation

We now have the map

$$\Phi(X, P) = \Big(\Phi^X(X, P),\, \Phi^P(X, P)\Big)$$

with

$$\Phi^X(X, P) = 1 - aX^2 - P, \qquad \Phi^P(X, P) = X$$

First let's work out what the linearized map $\mathbb{A}$ will be near an arbitrary point. We will have

$$\mathbb{A} = \begin{pmatrix} \frac{\partial \Phi^X}{\partial X} & \frac{\partial \Phi^X}{\partial P} \\ \\ \frac{\partial \Phi^P}{\partial X} & \frac{\partial \Phi^P}{\partial P} \end{pmatrix} = \begin{pmatrix} -2aX & -1 \\ \\ 1 & 0 \end{pmatrix} \tag{37.7}$$

where the exact matrix depends on the point (X, P) that we are linearizing *around*. We notice immediately that

$$\det \mathbb{A} = 1 \tag{37.8}$$

which means that this map will be area preserving. Just like the kicked rotator system of last time, we *do not* have a system with energy conservation (because of the periodic kicks), but we *do* have a form of conservation law here. And we expect this to affect the way fixed points of this mapping work, just as we found last time. Now, the fixed points must satisfy

$$\Phi^X(\bar{X}, \bar{P}) = \bar{X}, \qquad \Phi^P(\bar{X}, \bar{P}) = \bar{P}$$

The second condition gives me $\bar{X} = \bar{P}$. Then, the first gives me

$$\bar{X} = 1 - a\bar{X}^2 - \bar{X}, \qquad \Rightarrow \qquad a\bar{X}^2 + 2\bar{X} - 1 = 0$$

which has solutions

$$\bar{X} = \bar{P} = \frac{-2 \pm \sqrt{4 + 4a}}{2a} = \frac{-1 \pm \sqrt{1 + a}}{a}$$

Everywhere in the range of $a \in [-1, 3]$ we have two real solutions, except at $a = 0$ where $\frac{-1-\sqrt{1+a}}{a}$ blows up, so we only have one.

Now, the eigenvalues of $\mathbb{A}$ evaluated at one of the fixed points will be determined by

$$\det[\mathbb{A} - \lambda \mathbb{I}] = \begin{vmatrix} (-2a\bar{X} - \lambda) & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda(\lambda + 2a\bar{X}) + 1 = \lambda^2 + 2a\bar{X}\lambda + 1 = 0$$

which has solutions

$$\lambda_\pm = \frac{-2a\bar{X} \pm \sqrt{4a^2\bar{X}^2 - 4}}{2} = -a\bar{X} \pm \sqrt{a^2\bar{X}^2 - 1}$$

Behavior Near Lower Fixed Point

We'll start with the lower fixed point $\bar{X} = \frac{-1-\sqrt{1+a}}{a}$. Here, we have

$$\lambda_{\pm} = 1 + \sqrt{1+a} \pm \sqrt{1 + 1 + a + 2\sqrt{1+a} - 1} = 1 + \sqrt{1+a} \pm \sqrt{\sqrt{1+a}\left(\sqrt{1+a} + 2\right)}$$

These are clearly both real and positive for $a \in [-1, 3]$. We also know that we will have $\lambda_+ \lambda_- = 1$, so $\lambda_+ > 1$ and $0 < \lambda_- < 1$. We know that near this fixed point we should have a hyperbolic structure arise.

Behavior Near Upper Fixed Point

Now with the upper fixed point $\bar{X} = \frac{-1+\sqrt{1+a}}{a}$. Here we have

$$\lambda_{\pm} = 1 + \sqrt{1+a} \pm \sqrt{1 + 1 + a - 2\sqrt{1+a} - 1} = 1 + \sqrt{1+a} \pm \sqrt{\sqrt{1+a}\left(\sqrt{1+a} - 2\right)}$$

These are clearly both imaginary for $a \in [-1, 3]$. Since they are then complex conjugates, and we know $\lambda_+ \lambda_- = 1$, we must have $|\lambda_+| = |\lambda_-| = 1$. We know that near this fixed point we should have an elliptical structure.

37.1.4 Pretty Pictures

Now let's see what we can do with this mapping. We want to graphically represent what happens to series after a large number of mappings (so we see convergent behavior, but ignore transient stuff that depends on the initial point. So what we'll do is start with a randomly chosen point, then perform the mapping a large number of times, and then throw out most of the points we end up with, keeping only the last section of the list. We will then repeat this process a large number of times, and then combine all of the final points, plotting them together.

We'll start with a value of a near the upper limit: $a = 2.9$. This is shown on the left in figure 37.1. What we see is mostly empty space–most randomly chosen initial points are taken far away after many mappings, out effectively to infinity. Since we are truncating the initial points in the series, we never see the points in these series at all. However, near the predicted stable fixed point (marked with a red dot) we see a small region where points stay, even after a large number of mappings. And, just as predicted, we see that in its immediate

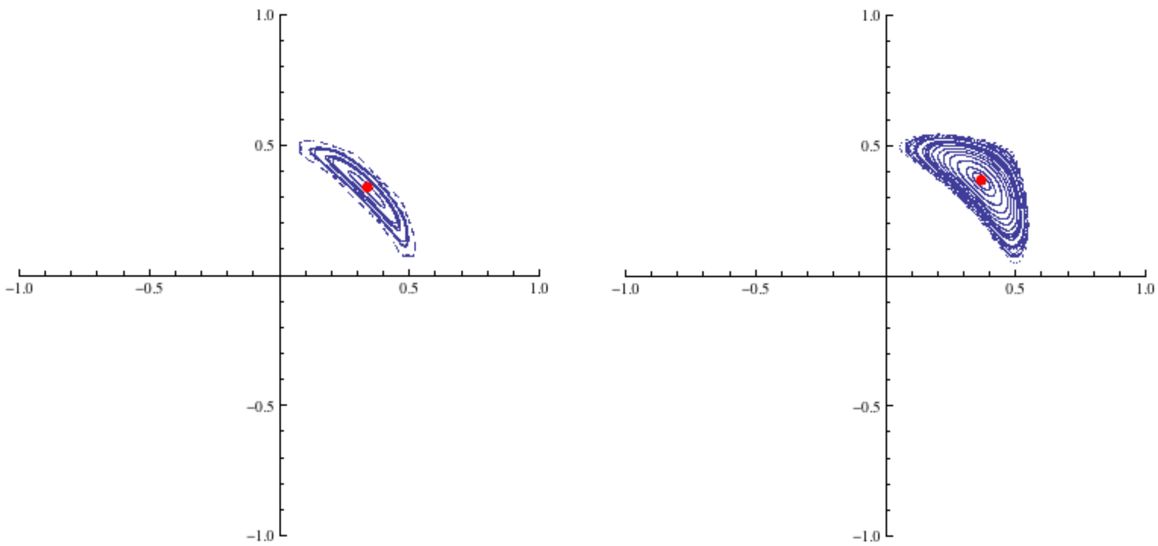

Figure 37.1: The late-stage behavior of series for $a = 2.9$ on the left, and for $a = 2.0$ on the right.

neighborhood there are little ellipses that have formed. On the right in figure 37.1, we have the value $a = 2$. We can now see the region of stability has increased, and is now taking on a peculiar, triangular shape.

If we decrease a further, to $a = 1.0$, this triangular shape warps unit it has three long, skinny "legs." In addition, we can "zoom out" a bit, and plot a great deal more series. The addition of all the new points obscures the structure inside the triangular region, but now we see that it has developed wispy "tails"–curves that branch out away from the stable region that points tend to converge to. These curves now form a shape reminiscent of the Starfleet symbol from Star Trek.

Figure 37.3 shows $a = 0.1$ and $a = 0.01$. Note that the region of stability has grown even more. In addition, new structures are forming. We can see four new regions where ellipses are forming, inside the larger stable region. These ellipses are actually forming around stable points of Φ^4. Recall that Φ has two fixed points, one of which is stable (and is at the center of the stable region.) We could show mathematically that while Φ^2 has four fixed points, in the region $a \in [-1, 3]$ that we are interested in two of these are complex (and thus non-physical) and the other two are just the fixed points of Φ. Now, Φ^4 turns out to have 16 fixed points, of which 4 are fixed points of Φ^2. The remaining 12 fall into three sets of four, and a single

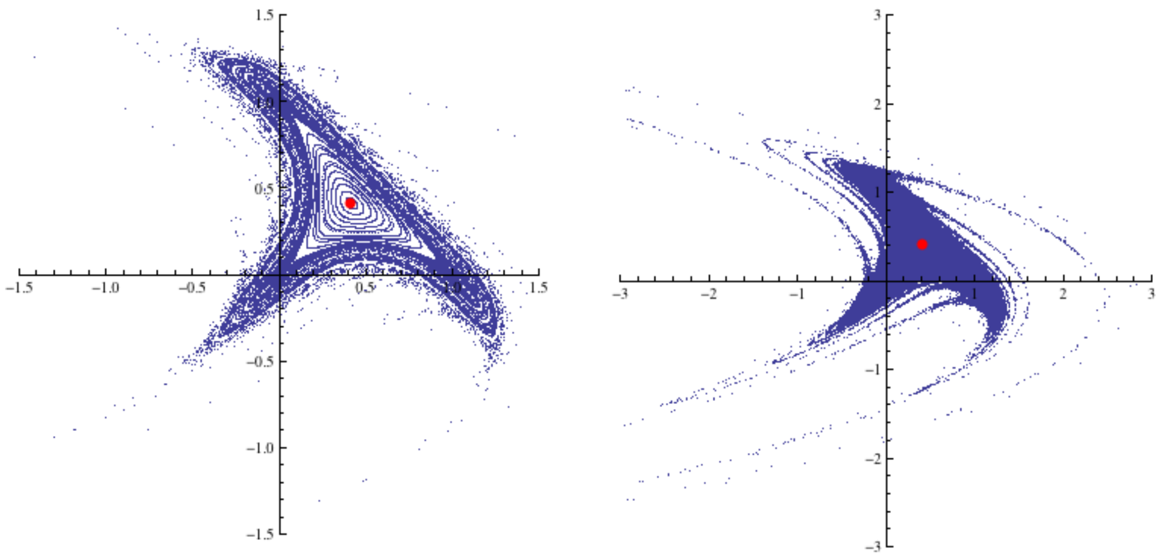

Figure 37.2: The late-stage behavior of series for $a = 1.0$. On the left is a close-up of the structure of the stable region, on the right is a zoomed out image with many more points shown.

mapping Φ will take any point within one of these sets to another point within the same set. Of these three sets, one gives us the four points around which ellipses are forming in figure 37.3. Mathematically we can show that this set is stable in the region $a \in [0, 0.217404]$(so in fact these ellipses should first appear at $a = 0.217404$ as we decrease a. A second of these three sets gives complex points for $a < 2$ (so they aren't physical), and for $a > 2$ gives unstable points. The third gives unstable points for any $a > 0$, and complex points for $a < 0$. This third set is also shown in figure 37.3–and we can see the hyperbolic structure near them, just as we should.

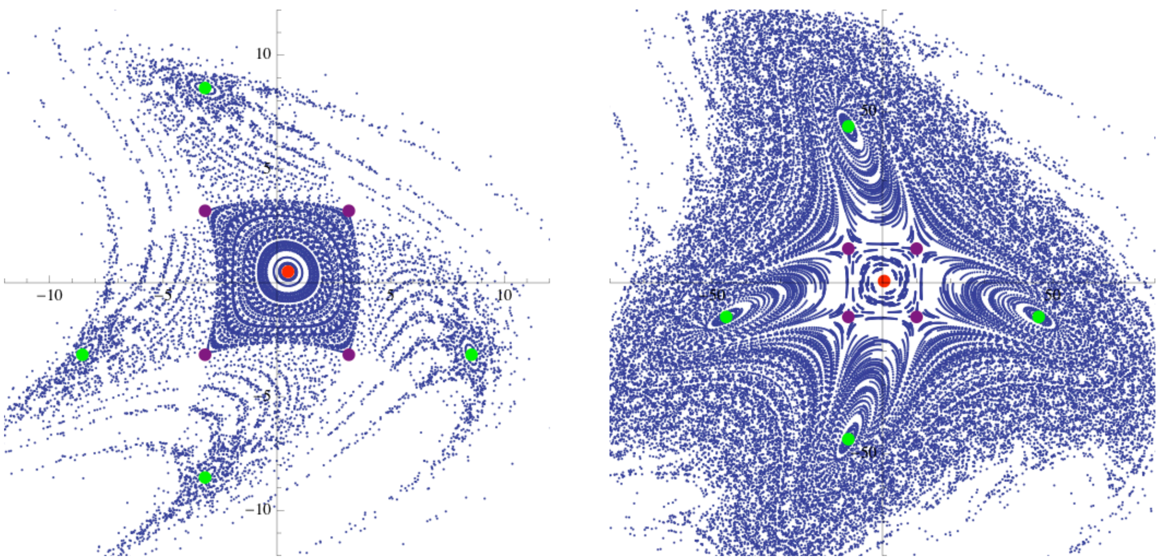

Figure 37.3: The late-stage behavior of series for $a = 0.1$ on the left, and $a = 0.01$ on the right. The fixed point of Φ is shown in red, while the stable fixed points of Φ^4 are in green, and the unstable fixed points of Φ^4 are in purple.

37.2 The Kicked, Damped Simple Harmonic Oscillator

The Hénon map is often generalized by adding a damping term to the original system. Let's see how this will work. Before we allow the system to be kicked, we have a standard damped harmonic oscillator, which has equations of motion

$$\dot{x} = p, \qquad \dot{p} = -x - 2bp$$

(non-dimensionalized.) The has solutions (assuming the system is underdamped)

$$x(\tau) = Ae^{-b\tau} \sin\left(\sqrt{1-b^2}\tau + \phi\right),$$

$$p(\tau) = -bAe^{-b\tau} \sin\left(\sqrt{1-b^2}\tau + \phi\right) + \sqrt{1-b^2}Ae^{-b\tau} \cos\left(\sqrt{1-b^2}\tau + \phi\right)$$

Now, if we put in the periodic kicks, just as before, we change the equations of motion to become

$$\begin{aligned} \dot{x} &= p \\ \dot{p} &= -x - 2bp - x^2 \textstyle\sum_{n=-\infty}^{\infty} \delta(t - nT) \end{aligned} \tag{37.9}$$

37.2.1 The Poincaré Map

We can analyze this system, and determine its Poincaré map, just as we did before. We'll make the time-lapse between snapshots T, as usual, and define (x_n, p_n), the nth point in the Poincaré map, as the point in phase space just before the kick:

$$x_n = x(nT - \epsilon), \qquad p_n = p(nT - \epsilon) \tag{37.10}$$

An instant after the kick we will have

$$x(nT + \epsilon) = x_n, \qquad p_n(nT + \epsilon) = p_n - x_n^2$$

(the position does not change discontinuously, but the momentum does, because of the kick.) For the remainder of the time before the next kick, x and p must satisfy the equations without the kick. We can use the values at $t = nT + \epsilon$ to set A and ϕ, then determine $x(nT + T - \epsilon)$ and $p(nT + T - \epsilon)$. We have

$$x_n = Ae^{-bnT} \sin\left(nT\sqrt{1-b^2} + \phi\right)$$

$$p_n - x_n^2 = -bAe^{-bnT}\sin\left(nT\sqrt{1-b^2}+\phi\right) + A\sqrt{1-b^2}e^{-bnT}\cos\left(nT\sqrt{1-b^2}+\phi\right)$$

notice that we can rewrite this as

$$Ae^{-bnT}\sin\left(nT\sqrt{1-b^2}+\phi\right) = x_n$$

$$Ae^{-bnT}\cos\left(nT\sqrt{1-b^2}+\phi\right) = \frac{p_n - x_n^2 + bx_n}{\sqrt{1-b^2}}$$

In this case, we get

$$\begin{aligned}
x_{n+1} &= x(nT+T-\epsilon) \\
&= Ae^{-b(n+1)T}\sin\left((n+1)T\sqrt{1-b^2}+\phi\right) \\
&= e^{-bT}Ae^{-bnT}\left[\sin\left(nT\sqrt{1-b^2}+\phi\right)\cos\left(T\sqrt{1-b^2}\right)+\cos\left(nT\sqrt{1-b^2}+\phi\right)\sin\left(T\sqrt{1-b^2}\right)\right] \\
&= e^{-bT}\left[x_n\cos\left(T\sqrt{1-b^2}\right)+\frac{p_n - x_n^2 + bx_n}{\sqrt{1-b^2}}\sin\left(T\sqrt{1-b^2}\right)\right]
\end{aligned}$$

If, as usual, we define

$$\cos\phi_0 = b, \qquad \sin\phi_0 = \sqrt{1-b^2} \tag{37.11}$$

this becomes

$$x_n = \frac{e^{-bT}}{\sin\phi_0}\left[x_n\sin\left(\phi_0+T\sqrt{1-b^2}\right)+(p_n-x_n^2)\sin\left(T\sqrt{1-b^2}\right)\right] \tag{37.12}$$

and similarly we have

$$\begin{aligned}
p_{n+1} &= p(nT+T-\epsilon) \\
&= -bAe^{-b(n+1)T}\sin\left((n+1)T\sqrt{1-b^2}+\phi\right)+\sqrt{1-b^2}Ae^{-b(n+1)T}\cos\left((n+1)T\sqrt{1-b^2}+\phi\right) \\
&= -be^{-b(n+1)T}A\Big[\sin\left(nT\sqrt{1-b^2}+\phi\right)\cos\left(T\sqrt{1-b^2}\right)+\cos\left(nT\sqrt{1-b^2}+\phi\right)\sin\left(T\sqrt{1-b^2}\right)\Big] \\
+\sqrt{1-b^2}e^{-b(n+1)T}A&\Big[\cos\left(nT\sqrt{1-b^2}+\phi\right)\cos\left(T\sqrt{1-b^2}\right)-\sin\left(nT\sqrt{1-b^2}+\phi\right)\sin\left(T\sqrt{1-b^2}\right)\Big] \\
&= -be^{-bT}\left[x_n\cos\left(T\sqrt{1-b^2}\right)+\tfrac{p_n-x_n^2+bx_n}{\sqrt{1-b^2}}\sin\left(T\sqrt{1-b^2}\right)\right] \\
&\quad +\sqrt{1-b^2}e^{-bT}\left[\tfrac{p_n-x_n^2+bx_n}{\sqrt{1-b^2}}\cos\left(T\sqrt{1-b^2}\right)-x_n\sin\left(T\sqrt{1-b^2}\right)\right]
\end{aligned}$$

which is

$$\begin{aligned}
p_{n+1} &= e^{-bT}\Bigg[\frac{p_n-x_n^2+bx_n}{\sqrt{1-b^2}}\left\{\sin\phi_0\cos\left(T\sqrt{1-b^2}\right)-\cos\phi_0\sin\left(T\sqrt{1-b^2}\right)\right\} \\
&\quad -x_n\left\{\cos\phi_0\cos\left(T\sqrt{1-b^2}\right)+\sin\phi_0\sin\left(T\sqrt{1-b^2}\right)\right\}\Bigg] \\
&= e^{-bT}\left[\frac{p_n-x_n^2+bx_n}{\sqrt{1-b^2}}\sin\left(\phi_0-T\sqrt{1-b^2}\right)-x_n\cos\left(\phi_0-T\sqrt{1-b^2}\right)\right] \\
&= \frac{e^{-bT}}{\sin\phi_0}\Bigg[(p_n-x_n^2)\sin\left(\phi_0-T\sqrt{1-b^2}\right) \\
&\quad +x_n\left\{\cos\phi_0\sin\left(\phi_0-T\sqrt{1-b^2}\right)-\sin\phi_0\cos\left(\phi_0-T\sqrt{1-b^2}\right)\right\}\Bigg] \\
&= \frac{e^{-bT}}{\sin\phi_0}\left[(p_n-x_n^2)\sin\left(\phi_0-T\sqrt{1-b^2}\right)-x_n\sin\left(T\sqrt{1-b^2}\right)\right]
\end{aligned}$$

so that we have

$$p_{n+1}=\frac{e^{-bT}}{\sin\phi_0}\left[(p_n-x_n^2)\sin\left(\phi_0-T\sqrt{1-b^2}\right)-x_n\sin\left(T\sqrt{1-b^2}\right)\right] \tag{37.13}$$

Now, equations 37.12 and 37.13 define our Poincaré map. Note that if we make the notational definition

$$\tilde{T} = T\sqrt{1-b^2} \tag{37.14}$$

we can write them somewhat more simply as

$$\begin{aligned} \Phi^x(x,p) &= \tfrac{e^{-bT}}{\sin\phi_0}\left[x\sin\left(\phi_0+\tilde{T}\right)+(p-x^2)\sin\tilde{T}\right] \\ \Phi^p(x,p) &= \tfrac{e^{-bT}}{\sin\phi_0}\left[(p-x^2)\sin\left(\phi_0-\tilde{T}\right)-x\sin\tilde{T}\right] \end{aligned} \tag{37.15}$$

37.2.2 The Standard Map

Now we will follow the same steps as before, to convert the Poincaré map into standard form. First, we want to eliminate p_n from the Poincaré map to write a single relationship between x_{n+1}, x_n, and x_{n-1}. We write

$$\begin{aligned} \sin\tilde{T}\,p_{n+1}-\sin(\phi_0-\tilde{T})\,x_{n+1} &= -\frac{x_n\,e^{-bT}}{\sin\phi_0}\left[\sin^2\tilde{T}+\sin^2\phi_0\cos^2\tilde{T}-\cos^2\phi_0\sin^2\tilde{T}\right] \\ &= -\frac{x_n\,e^{-bT}}{\sin\phi_0}\left[\sin^2\phi_0\sin^2\tilde{T}+\sin^2\phi_0\cos^2\tilde{T}\right] \\ &= -x_n\,\sin\phi_0 e^{-bT} \end{aligned}$$

and then we have

$$p_n\sin\tilde{T} = x_n\sin(\phi_0-\tilde{T})-x_{n-1}e^{-bT}\sin\phi_0 \tag{37.16}$$

Using this, we can write

$$x_{n+1} = e^{-bT}\left[x_n\sin(\phi_0+\tilde{T})+x_n\sin(\phi_0-\tilde{T})-x_{n-1}e^{-bT}\sin\phi_0-x_n^2\sin\tilde{T}\right]$$

and then

$$x_{n+1}+e^{-2bT}x_{n-1} = \left(2e^{-bT}\cos\tilde{T}\right)x_n-\left(\frac{e^{-bT}\sin\tilde{T}}{\sin\phi_0}\right)x_n^2 \tag{37.17}$$

Now we make the definitions

$$X_n = \alpha x_n+\beta, \qquad \alpha = -\frac{\tan\tilde{T}}{\sin\phi_0\left[1+e^{-2bT}-e^{-bT}\cos\tilde{T}\right]}, \qquad \beta = \frac{1}{1+e^{-2bT}-e^{-bT}\cos\tilde{T}} \tag{37.18}$$

Notice that these are the same as those in equation 37.5 if we set $b = 0$. If we plug this in, we get

$$\frac{X_{n+1} - \beta}{\alpha} + e^{-2bT}\left(\frac{X_{n-1} - \beta}{\alpha}\right) = \left(2e^{-bT}\cos\tilde{T}\right)\left(\frac{X_n - \beta}{\alpha}\right) - \left(\frac{e^{-bT}\sin\tilde{T}}{\sin\phi_0}\right)\left(\frac{X_n - \beta}{\alpha}\right)^2$$

which is

$$\begin{aligned} X_{n+1} + e^{-2bT}X_{n-1} = \beta\left[1 + e^{-2bT} - 2e^{-bT}\cos\tilde{T} - \frac{\beta e^{-bT}\sin\tilde{T}}{\alpha\sin\phi_0}\right] \\ +2e^{-bT}\left[\cos\tilde{T} + \frac{\beta\sin\tilde{T}}{\alpha\sin\phi_0}\right]X_n - \frac{e^{-bT}\sin\tilde{T}}{\alpha\sin\phi_0}X_n^2 \end{aligned}$$

Now, since we have

$$\frac{\beta}{\alpha} = -\frac{\sin\phi_0}{\tan\tilde{T}}$$

this is

$$X_{n+1} + e^{-2bT}X_{n-1} = \beta\left[1 + e^{-2bT} - e^{-bT}\cos\tilde{T}\right] - \frac{e^{-bT}\sin\tilde{T}}{\alpha\sin\phi_0}X_n^2$$

which is

$$X_{n+1} + e^{-2bT}X_{n-1} = 1 + e^{-bT}\cos\tilde{T}\left(1 + e^{-2bT} - e^{-bT}\cos\tilde{T}\right)X_n^2$$

Then if we define

$$c = -e^{-2bT}, \qquad a = e^{-bT}\cos\tilde{T}\left(e^{-bT}\cos\tilde{T} - 1 - e^{-2bT}\right) \tag{37.19}$$

we get

$$X_{n+1} = 1 - aX_n^2 + cX_{n-1}$$

This we can finally convert back into a mapping of two-dimensional points to two-dimensional points, with

$$\begin{aligned} X_{n+1} &= 1 - aX_n^2 + cP_n \\ P_{n+1} &= X_n \end{aligned} \tag{37.20}$$

This is the most common form in which the Hénon map is presented. And although our physical system would require $-1 \le c \le 0$ and $-1 \le a \le 3$, we can also generalize this and discuss *other* values of c and a. Now, our mapping is

$$\Phi^X(X, P) = 1 - aX^2 + cP, \qquad \Phi^P(X, P) = X$$

and we can notice immediately that the linearized mapping is

$$\mathbb{A} = \begin{bmatrix} -2aX & c \\ 1 & 0 \end{bmatrix}$$

which has

$$\det \mathbb{A} = -c$$

Evidently, the mapping is only area preserving for $c = -1$ (which corresponds to $b = 0$, the undamped system.) This is interesting–we are used to the idea that a dissipative system has fewer conserved quantities than a non-dissipative system. The plain simple harmonic oscillator conserves energy, but the damped simpler harmonic oscillator doesn't. Now, our kicked simple harmonic oscillator doesn't conserve energy. And yet it's Poincaré map *is* area preserving. And this feature is destroyed by adding friction to the original system.

37.2.3 More Pretty Pictures

Now, the generalized Hénon map is a little more complicated than the mapping we were working with earlier. Depending on the values of a and c, it's behavior can change dramatically. Even for values of c that are pretty close to -1, the images we create are quite different from those we had before. There will be no more ellipses, no matter how close to a fixed point we are (because we don't have an area preserving map.) In figure 37.4 is a mapping with $c = -0.8$ and $a = 0.1$. Notice how, though it shares *some* features in common with the graph in figure 37.3, it is markedly different. The ellipses close to the main fixed point have disappeared, to be replaced with spirals. The outer fixed points of Φ^4 seem to have vanished, and the entire shape has shrunk noticeably. Furthermore, far more initial points are creating series that move out to infinity quickly, making a much sparser graph.

A particular set of values, $a = 1.4$ and $c = 0.3$, exhibit a famous property: this mapping has what is called a *strange attractor*. The late-stage points in a set of series with random initial points are shown for this mapping in figure 37.5. Notice, on the left, what appears to be a set of several, perhaps 4 or 5, curves. These are the *attractor*, where a generic series will converge to. But if we zoom in on what appears to be three or four curves of the attractor (on the right) we see it is actually, seven or eight. Zooming in again then shows that it breaks into more curves. In fact, this attractor is a fractal–we could zoom in over and over again, and see it break up into more and more lines.

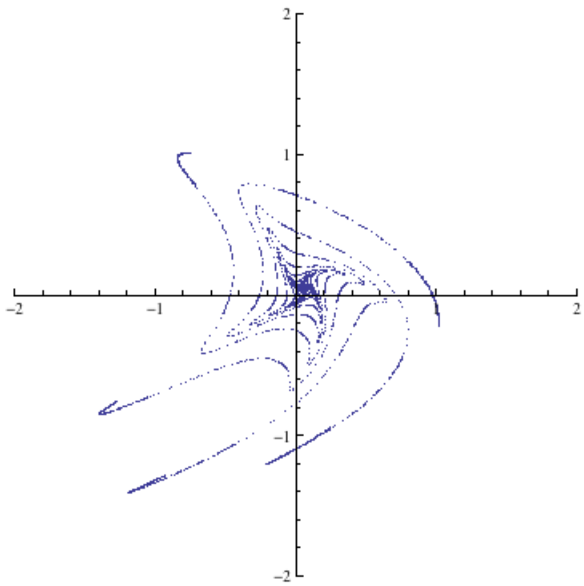

Figure 37.4: The Hénon map with $a = 0.1$ and $c = -0.8$.

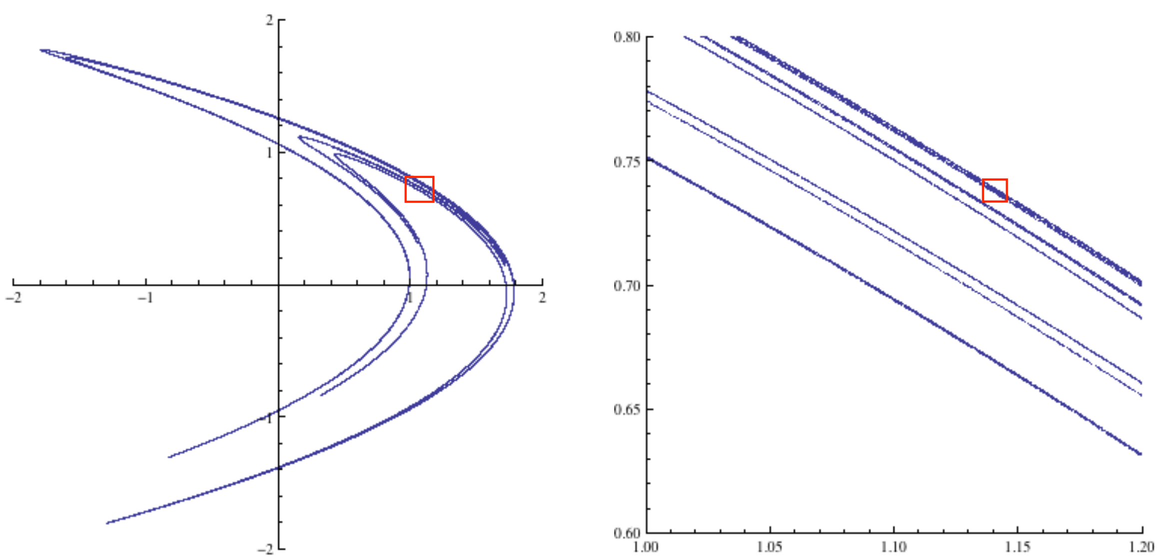

Figure 37.5: The strange attractor of the Hénon map, with $a = 1.4$ and $c = 0.3$.

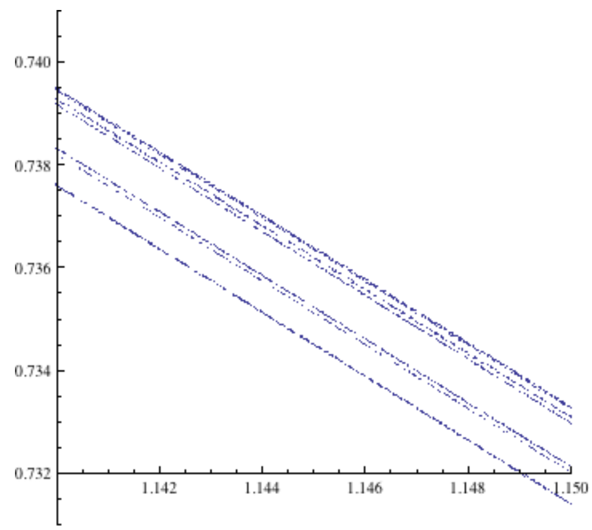

Figure 37.6: The strange attractor of the Hénon map, with $a = 1.4$ and $c = 0.3$, zoomed in further.