Hysteresis

Lecture 30

Physics 311: Classical Mechanics
Fall 2011

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30.1 Review of Perturbation Theory

Last time we introduced the techniques of perturbation theory, which allow you to iteratively solve an equation (differential, algebraic, etc) which can be thought of as a simpler equation plus a small addition. We first applied this to the humble quadratic equation

$$\epsilon x^2 - x + 1 = 0. \tag{30.1}$$

The iterative procedure, shown as

$$\begin{array}{rclcrcl} 1 - x_0 &=& 0, & \rightarrow & x_0 &=& 1 \\ 1 - x_1 &=& \epsilon x_0^2, & \rightarrow & x_1 &=& 1 - \epsilon \\ 1 - x_2 &=& \epsilon x_1^2, & \rightarrow & x_2 &=& 1 - \epsilon + 2\epsilon^2 - \epsilon^3 \end{array} \tag{30.2}$$

consists of solving first the "unperturbed equation" $1 - x_0 = 0$, obtaining x_0 which is an approximation of x. Then we insert this answer into the original equation to obtain a better approximation x_1, and so on. At each step, we obtain x_k from x_{k-1} as

$$1 - x_k = \epsilon x_{k-1}^2 \tag{30.3}$$

and x_k is an approximation to x that is valid up to order ϵ^k, so that we have

$$x = x_k + \mathcal{O}(\epsilon^{k+1}) \tag{30.4}$$

We also noticed that this method had an important limitation: it can only find solutions close to the solution of the unperturbed equation $x_0 = 1$. Because of this it misses the second root of the quadratic equation entirely, which is very large for ϵ small, and moves out to infinity as $\epsilon \rightarrow 0$.

We also looked at the application of this technique to the differential equation

$$\ddot{z} + z + \epsilon z^3 = 0 \tag{30.5}$$

which arises naturally when we study motion in a generic potential expanded out to one more non-trivial term near a minimum. The basic iterative technique was the same. We began by solving the unperturbed equation

$$\ddot{z}_0 + z_0 = 0 \tag{30.6}$$

of which one solution is

$$z_0(\tau) = \cos\tau \tag{30.7}$$

We then inserted this solution into the original equation to find a slightly better approximation, as

$$\ddot{z}_1 + z_1 = -\epsilon z_0^3 = -\epsilon\cos^2\tau = -\frac{\epsilon}{4}\left[\cos 3\tau + 3\cos\tau\right] \tag{30.8}$$

Solving this equation required some care, as the most naive approach led to a divergence. What we have written above looks like a driven harmonic oscillator, where the driving force consists of a portion at the natural frequency and a portion at three times this frequency. The first causes us headaches, because when we drive an undamped harmonic oscillator at its natural frequency, we get a growing solution. In terms of the principle of the perturbative approach, this doesn't make sense because there's no sense in which (at large times) the solution $z_1(\tau)$ is close to the unperturbed solution $z_0(\tau)$. In terms of the physics, it makes no sense for this $z_1(\tau)$ to be a good approximation to the exact solution $z(\tau)$, since the original physical system has energy conservation, and should not support growing solutions.

The resolution to our troubles ended up being rethinking equation 30.7, and allowing it to take the form

$$z_0(\tau) = \cos(1 + \epsilon\eta)\tau \tag{30.9}$$

where it has a slight shift in the frequency for non-zero epsilon. Putting this into equation 30.8 then gave us an answer that was well controlled:

$$z_1(\tau) = \cos\left(\tau + \frac{3\epsilon\tau}{8}\right) + \frac{\epsilon}{32}\cos 3\tau + \mathcal{O}(\epsilon^2) \tag{30.10}$$

and then numerical calculations agreed well with this estimate.

30.2 The Damped Driven Quartic Oscillator

Today we are going to look at another system, somewhat related. Consider the damped, driven harmonic oscillator with a quartic interaction:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x + \delta x^3 = f_0 \cos \Omega t$$

First, we want to simplify things by nondimensionalizing. We will use the natural frequency ω_0 to define a time scale, and we will use a combination of this with the driving amplitude f_0 to define a length scale:

$$x = \frac{f_0 z}{\omega_0^2}, \qquad \tau = \omega_0 t \tag{30.11}$$

so that the equation becomes

$$f_0 \frac{d^2 z}{d\tau^2} + \frac{2\beta f_0}{\omega_0}\frac{dz}{d\tau} + f_0 z + \frac{f_0^3 \delta}{\omega_0^6} z^3 = f_0 \cos \frac{\Omega\tau}{\omega_0}$$

and with the further definitions

$$\epsilon = \frac{f_0^2 \delta}{\omega_0^6}, \qquad b = \frac{\beta}{\omega_0}, \qquad \omega = \frac{\Omega}{\omega_0} \tag{30.12}$$

this becomes

$$\ddot{z} + 2b\dot{z} + z + \epsilon z^3 = \cos \omega\tau \tag{30.13}$$

This system is interesting to us for two reasons. First, it is another equation that we can approach using perturbation theory, so we can get more practice with those techniques. But also, this system exhibits some very strange nonlinear phenomena that are accessible even perturbatively. Specifically, it contains an example of *hysteresis*, where the response of a system to a driving term depends in a nontrivial way on the history of what the driving term has been. The most common examples of hysteresis occur in circuits, and you may remember encountering one in physics 200 (the Schmidt trigger).

30.3 The Unperturbed Equation

We know that the first step of analyzing equation 30.13 perturbatively is to find the solution to the unperturbed equation,

$$\ddot{z}_0 + 2b\dot{z}_0 + z_0 = \cos \omega\tau \tag{30.14}$$

It may have been a while since you looked at this type of expression, and anyways it's useful to review salient features so that we can compare them to what we get in the perturbed case, so let's go through this carefully. We recall that the right technique to apply to this equation is the method of general and specific solutions, so we write $z_0 = x + y$ where $x(\tau)$ is a *general* solution to the equation

$$\ddot{x} + 2b\dot{x} + x = 0 \tag{30.15}$$

and y is *any one* solution to the equation

$$\ddot{y} + 2b\dot{y} + y = \cos\omega\tau \tag{30.16}$$

30.3.1 Undriven Damped Harmonic Oscillator

First we'll tackle the general equation 30.15, which is the equation for an undriven damped harmonic oscillator. We write the basic ansatz

$$x(\tau) = Ae^{\lambda\tau} \tag{30.17}$$

and then when we insert this into equation 30.15, we obtain a quadratic equation in λ:

$$\lambda^2 Ae^{\lambda\tau} + 2b\lambda Ae^{\lambda\tau} + Ae^{\lambda\tau} = 0$$

so that we just have

$$\lambda^2 + 2b\lambda + 1 = 0$$

This has solutions

$$\lambda_+ = \frac{-2b + \sqrt{4b^2 - 4}}{2} = -b + \sqrt{b^2 - 1} \tag{30.18}$$

and

$$\lambda_- = \frac{-2b + \sqrt{4b^2 - 4}}{2} = -b + \sqrt{b^2 - 1} \tag{30.19}$$

We build our general solution then as

$$x(\tau) = Ae^{\lambda_+\tau} + Be^{-\lambda_-\tau} \tag{30.20}$$

where A and B are fixed by initial conditions.

Overdamped Behavior: $\omega_0 < \beta$

We recall that the physical implications of this are different depending on the relative sizes of β and ω_0. If we have $1 < b$, which means $\omega_0 < \beta$, then both of these are real, and we simply have two real solutions, both of which give exponential decay. This is referred to as the *overdamped* situation. We can write

$$\eta = \sqrt{b^2 - 1} \in \mathbb{R}$$

and we get

$$x(\tau) = Ae^{-(b-\eta)\tau} + Be^{-(b+\eta)\tau}$$

with

$$b \pm \eta > 0.$$

Underdamped Behavior: $\omega_0 > \beta$

On the other hand, if $1 > b$, which means $\omega_0 > \beta$, then both solutions are complex, and we can write

$$\chi = \sqrt{1 - b^2} \in \mathbb{R}$$

so we get

$$x(\tau) = e^{-b\tau}\left[Ae^{i\chi\tau} + Be^{-i\chi\tau}\right] = e^{-b\tau}\left[\tilde{A}\cos\chi\tau + \tilde{B}\sin\chi\tau\right]$$

which is damped oscillatory behavior. This is the *underdamped* situation.

Critically Damped Behavior: $\omega_0 = \beta$

The third case, where $b = 1$, ($\omega_0 = \beta$) is the *critically damped* system, where the ansatz process leads only to one value of λ, so we actually only find one family of solutions.

$$x(\tau) = Ae^{-\tau}$$

Since we know that there can be generically two initial conditions we need to fit, we are clearly not done. It turns out that a second family of solutions can be found in the form

$$x(\tau) = B\tau e^{-\tau}.$$

This has

$$\dot{x} = Be^{-\tau} - B\tau e^{-\tau}$$

and

$$\ddot{x} = -2Be^{-\tau} + B\tau e^{-\tau}$$

so if we insert this into equation 30.15 (using $b = 1$) we get

$$\ddot{x} + 2\dot{x} + x$$

$$= -2Be^{-\tau} + B\tau e^{-\tau} + 2\left(Be^{-\tau} - B\tau e^{-\tau}\right) + B\tau e^{-\tau} = 0$$

Finally, in this special case we write the general solution.

$$x(\tau) = Ae^{-\tau} + B\tau e^{-\tau}$$

Notice that all three cases, even this last one, share the property that they die out for large times. (The linear growth with τ in the last case is easily killed by the exponential suppresion.)

30.3.2 Specific Solution

Now we need to find *at least one* solution to equation 30.16. Since we only need one solution, if we can make an intelligent guess as to what one solution will look like, we can work from this. In this case, it makes sense to suppose that a solution might exist which oscillates at the same frequency as the driving force. Therefore we make the ansatz

$$y(\tau) = f\cos(\omega\tau + \phi) \tag{30.21}$$

where f and ϕ are unknown constants that I will need to fix in order to make this a valid solution of equation 30.16. We note that we will have

$$\dot{y} = -f\omega\sin(\omega\tau + \phi), \qquad \ddot{y} = -f\omega^2\cos(\omega\tau + \phi)$$

so that if we plug our ansatz into equation 30.16, we obtain

$$\ddot{y} + 2b\dot{y} + y = -f\omega^2\cos(\omega\tau + \phi) - 2bf\omega\sin(\omega\tau + \phi) + f\cos(\omega\tau + \phi) = \cos\omega\tau$$

In order to see how this might be satisfied, it is necessary to expand out the trigonometric functions, using

$$\cos(\omega\tau + \phi) = \cos\omega\tau\cos\phi - \sin\omega\tau\sin\phi$$

$$\sin(\omega\tau + \phi) = \sin\omega\tau\cos\phi + \cos\omega\tau\sin\phi$$

Then, we obtain

$$f(1-\omega^2)\cos\phi\cos\omega\tau - f(1-\Omega^2)\sin\phi\sin\omega\tau - 2b\omega f\cos\phi\sin\omega\tau - 2b\omega f\sin\phi\cos\omega\tau = \cos\omega\tau$$

and this is

$$f\left[(1-\omega^2)\cos\phi - 2b\omega\sin\phi\right]\cos\omega\tau - f\left[(1-\omega^2)\sin\phi + 2b\omega\cos\phi\right]\sin\omega\tau = \cos\omega\tau$$

In order for this equation to be satisfied at all times, we require

$$(1-\omega^2)\cos\phi - 2b\omega\sin\phi = \frac{1}{f}$$

and

$$(1-\omega^2)\sin\phi + 2b\omega\cos\phi = 0$$

We can use the second equation to tell us what ϕ is, since

$$\tan\phi = \frac{2b\omega}{\omega^2-1} \tag{30.22}$$

So the response of the system is out of phase with the driving force, with a phase shift ϕ that depends on b and ω. We also want to determine what f is, which we can do by squaring both equations and adding them:

$$\frac{1}{f^2} + 0 = (1-\omega^2)^2\cos^2\phi + 4b^2\omega^2\sin^2\phi - 4b\omega(1-\omega^2)\sin\phi\cos\phi$$

$$+(1-\omega^2)^2\sin^2\phi + 4b^2\omega^2\cos^2\phi + 4b\omega(1-\omega^2)\sin\phi\cos\phi$$

which gives

$$(1-\omega^2)^2 + 4b^2\omega^2 = \frac{1}{f^2}$$

or

$$f = \frac{1}{\sqrt{(1-\omega^2)^2 + 4b^2\omega^2}} \tag{30.23}$$

Notice that this, the amplitude of the oscillation of $y(\tau)$, has some very interesting features. If we think of it as a function of ω, the ratio of the driving frequency Ω to the natural frequency ω_0, then it is maximized at $\omega = \sqrt{1-2b^2}$, which when the damping coefficient is small is very close to $\omega = 1$ (that is, the driving frequency is close to the natural frequency of the system.) This is a *resonance*.

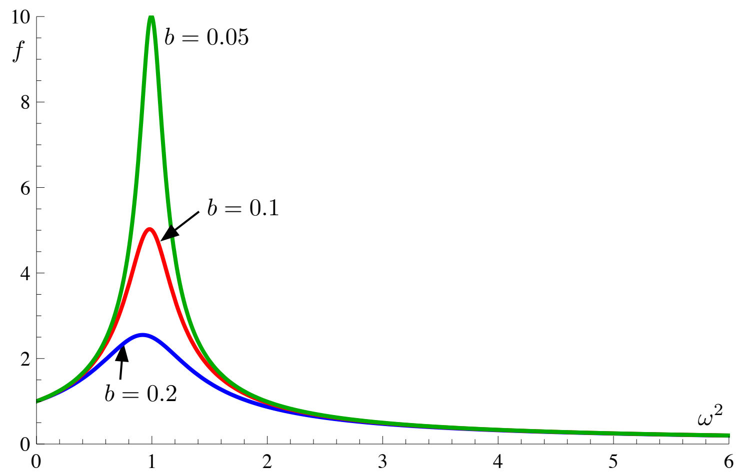

Figure 30.1: The amplitude of the steady state solution for the damped, driven harmonic oscillator exhibits resonance.

Let us now put all the pieces together. We started out trying to find $z_0(\tau)$ the general solution to equation 30.14, and we said it was the sum of two parts: $x(\tau)$ which was the general solution to equation 30.15 and $y(\tau)$ which was any one specific solution to equation 30.16. Then we found both $x(\tau)$ and $y(\tau)$. Therefore, all told we have

$$z_0(\tau) = Ae^{\lambda_+\tau} + Be^{\lambda_-\tau} + \frac{\cos(\omega\tau + \phi)}{\sqrt{(1-\omega^2)^2 + 4b^2\omega^2}} \tag{30.24}$$

with ϕ given by equation 30.22 and $\lambda_\pm$ given by equations 30.18 and 30.19. However, we showed that, whatever $\lambda_\pm$ are, the real parts of both are negative, so the first two terms are *transient*. After sufficient time has passed, we can ignore them. (Or, if we happen to pick just the right initial conditions, we will have both vanish exactly.) The important part of the solution is the last term, which is the *steady state* solution of $z_0(\tau)$. And the most striking aspect of it is its resonance behavior.

The amplitude f is plotted in figure 30.1 as a function of ω, for various values of b. Notice that the smaller b is (which corresponds to less damping in the system) the higher and narrower the resonance peak is. The most familiar physical example of this occurs in circuit analysis for a driven RCL circuit. If you think back to physics 200 labs, fall semester, you may remember building an RCL circuit and driving it with a signal from the function generator. As you gradually increase the frequency that the function generator is set to, the response signal in the circuit increases until you're at resonance, and then decreases again.

30.4 The First Perturbation

Now, of course, we want to use our solution $z_0(\tau)$ in equation 30.13, to find a better approximation $z_1(\tau)$. What we want is to solve the system

$$\ddot{z}_1 + 2b\dot{z}_1 + z_1 + \epsilon z_0^3 = \cos\omega\tau \tag{30.25}$$

To somewhat simplify things, we will focus on the steady state part of z_0, so that we will simply say

$$z_0(\tau) = f\cos(\omega\tau + \phi) \tag{30.26}$$

so that equation 30.25 becomes

$$\ddot{z}_1 + 2b\dot{z}_1 + z_1 = \cos\omega\tau - \epsilon f^3\cos^3(\omega\tau + \phi)$$

(with f and ϕ given by equations 30.22 and 30.23) and we learned last time that

$$\cos^3\theta = \frac{1}{4}\left[\cos 3\theta + 3\cos\theta\right]$$

so that we have

$$\ddot{z}_1 + 2b\dot{z}_1 + z_1 = \cos\omega\tau - \frac{3\epsilon f^3}{4}\cos(\omega\tau + \phi) - \frac{\epsilon f^3}{4}\cos(3\omega\tau + 3\phi) \tag{30.27}$$

This is just *another* damped, driven harmonic oscillator. Only this time, we are driving at two different frequencies: ω and 3ω. The method for solving it is the same. First we look for a general solution to

$$\ddot{z}_1 + 2b\dot{z}_1 + z_1 = 0$$

and then we add this to any specific solution to equation 30.27. However, the "general solution" part is the same as it was in the previous problem, and we know that it is transient. We will ignore it, imagining we are studying our systems at late times. (This is consistent with having thrown out the transient parts to z_0 as well.) Our steady state specific solution we guess should consist of two parts: one oscillating at ω and one oscillating at 3ω. Thus our ansatz is

$$z_1(\tau) = \tilde{f}\cos(\omega\tau + \tilde{\phi}) + g\cos(3\omega\tau + \psi) \tag{30.28}$$

where we expect this to only work for specific choices of $\tilde{f}$, $\tilde{\phi}$, g, and ψ. We also then have

$$\dot{z}_1 = -\omega\tilde{f}\sin(\omega\tau + \tilde{\phi}) - 3\omega g\sin(3\omega\tau + \psi)$$

$$\ddot{z}_1 = -\omega^2\tilde{f}\cos(\omega\tau + \tilde{\phi}) - 9\omega^2 g\cos(3\omega\tau + \psi)$$

and then when we insert these into the left-hand-side of equation 30.27, we get

$$\ddot{z}_1 + 2b\dot{z}_1 + z_1 =$$
$$\tilde{f}(1-\omega^2)\cos(\omega\tau+\tilde{\phi}) + g(1-9\omega^2)\cos(3\omega\tau+\psi) - 2b\tilde{f}\omega\sin(\omega\tau+\tilde{\phi}) - 6bg\omega\sin(3\omega\tau+\psi)$$

Then we want to rewrite this, expanding out the trig functions and getting four terms, proportional to sines and cosines of $\omega\tau$ and $3\omega\tau$. Using our earlier work as a guide, we obtain

$$= \tilde{f}\left[(1-\omega^2)\cos\tilde{\phi} - 2b\omega\sin\tilde{\phi}\right]\cos\omega\tau - \tilde{f}\left[(1-\omega^2)\sin\tilde{\phi} + 2b\omega\cos\tilde{\phi}\right]\sin\omega\tau$$
$$+g\left[(1-9\omega^2)\cos\psi - 6bg\omega\sin\psi\right]\cos 3\omega\tau - g\left[(1-9\omega^2)\sin\psi + 6bg\omega\cos\psi\right]\sin 3\omega\tau$$

Now consider the right-hand-side of equation 30.27, which we want in the same form so that we can determine how to set them equal to each other. To do this we need to expand out $\cos(\omega\tau+\phi)$ and $\cos(3\omega\tau+3\phi)$. This gives

$$\cos\omega\tau - \frac{3\epsilon f^3}{4}\cos(\omega\tau+\phi) - \frac{\epsilon f^3}{4}\cos(3\omega\tau+3\phi)$$
$$= \left[1 - \frac{3\epsilon f^3}{4}\cos\phi\right]\cos\omega\tau + \frac{3\epsilon f^3}{4}\sin\phi\sin\omega\tau - \frac{\epsilon f^3}{4}\cos 3\phi\cos 3\omega\tau + \frac{\epsilon f^3}{4}\sin 3\phi\sin 3\omega\tau$$

Finally setting the right-hand-side and left-hand-side of equation 30.27 equal gives

$$\tilde{f}\left[(1-\omega^2)\cos\tilde{\phi} - 2b\omega\sin\tilde{\phi}\right]\cos\omega\tau - \tilde{f}\left[(1-\omega^2)\sin\tilde{\phi} + 2b\omega\cos\tilde{\phi}\right]\sin\omega\tau$$
$$+g\left[(1-9\omega^2)\cos\psi - 6bg\omega\sin\psi\right]\cos 3\omega\tau - g\left[(1-9\omega^2)\sin\psi + 6bg\omega\cos\psi\right]\sin 3\omega\tau$$
$$= \left[1 - \frac{3\epsilon f^3}{4}\cos\phi\right]\cos\omega\tau + \frac{3\epsilon f^3}{4}\sin\phi\sin\omega\tau - \frac{\epsilon f^3}{4}\cos 3\phi\cos 3\omega\tau + \frac{\epsilon f^3}{4}\sin 3\phi\sin 3\omega\tau$$

These expressions must equal term-by-term, in order for them to be true at all times τ. Thus we must have

$$\begin{aligned}
1 - \tfrac{3\epsilon f^3}{4}\cos\phi &= \tilde{f}\left[(1-\omega^2)\cos\tilde{\phi} - 2b\omega\sin\tilde{\phi}\right] \\
-\tfrac{3\epsilon f^3}{4}\sin\phi &= \tilde{f}\left[(1-\omega^2)\sin\tilde{\phi} + 2b\omega\cos\tilde{\phi}\right] \\
-\tfrac{\epsilon f^3}{4}\cos 3\phi &= g\left[(1-9\omega^2)\cos\psi - 6bg\omega\sin\psi\right] \\
-\tfrac{\epsilon f^3}{4}\sin 3\phi &= g\left[(1-9\omega^2)\sin\psi + 6bg\omega\cos\psi\right]
\end{aligned} \tag{30.29}$$

The last two equations can be solved to find g and ψ exactly the same way we proceeded for the original driven damped harmonic oscillator. There is nothing particularly unexpected about the results. We find that at a particular value of ω, we can maximize g–it exhibits a resonance. Not surprisingly, the frequency that works is where $1 = 9\omega^2$. Remembering that $\omega = \frac{\Omega}{\omega_0}$ (the ratio of the driving frequency to the natural frequency), our perturbed system has a big resonance near the natural frequency ω_0 and a smaller one near $\frac{1}{3}\omega_0$.

Let's focus on the first two equations–it is here that we are going to find truly interesting things happening. If we proceed following our noses, we can solve these equations for $\tilde{f}$ and $\tilde{\phi}$. Considering our ansatz for $z_1(\tau)$ (equation 30.28) and comparing it to our solution for $z_0(\tau)$ (equation 30.26), we expect that $\tilde{f}$ and $\tilde{\phi}$ will be close to f and ϕ, with small corrections in ϵ. (By similar logic, we expect g to be proportional to ϵ, which it is.)

This is a perfectly rational course to follow. However, it turns out that we can do slightly better than this. Recall the drawback of the perturbative method as exhibited by the quadratic equation–sometimes the straightforward application of it misses solutions. Furthermore, recall how when we applied the perturbative method to the undamped perturbed oscillator, we used a slight variation–we allowed the solution to the unperturbed equation itself to have dependence on ϵ (which vanishes in the lowest approximation.) Using that as our guide, suppose we replace f and ϕ from the solution for $z_0(\tau)$ with

$$\tilde{f} = f + \mathcal{O}(\epsilon), \qquad \tilde{\phi} = \phi + \mathcal{O}(\epsilon)$$

and we try to solve for $\tilde{f}$ and $\tilde{\phi}$. Our equations are now

$$1 = \tilde{f}\left[1 - \omega^2 + \frac{3\epsilon\tilde{f}^2}{4}\right]\cos\tilde{\phi} - 2b\omega\tilde{f}\sin\tilde{\phi}$$

$$0 = \tilde{f}\left[1 - \omega^2 + \frac{3\epsilon\tilde{f}^2}{4}\right]\sin\tilde{\phi} + 2b\omega\tilde{f}\cos\tilde{\phi}$$

We will focus on $\tilde{f}$–the amplitude of our response $z_1(\tau)$ is more interesting than its phase shift. In order to eliminate $\tilde{\phi}$ from these equations, we square them and add. This gives

$$1 = \tilde{f}^2\left[1 - \omega^2 + \frac{3\epsilon\tilde{f}^2}{4}\right]^2\cos^2\tilde{\phi} + 4b^2\omega^2\tilde{f}^2\sin^2\tilde{\phi} - 4\tilde{f}^2b\omega\left[1 - \omega^2 + \frac{3\epsilon\tilde{f}^2}{4}\right]\sin\tilde{\phi}\cos\tilde{\phi}$$

$$+\tilde{f}^2\left[1 - \omega^2 + \frac{3\epsilon\tilde{f}^2}{4}\right]^2\sin^2\tilde{\phi} + 4b^2\omega^2\tilde{f}^2\cos^2\tilde{\phi} + 4\tilde{f}^2b\omega\left[1 - \omega^2 + \frac{3\epsilon\tilde{f}^2}{4}\right]\sin\tilde{\phi}\cos\tilde{\phi}$$

and this gives

$$1 = \tilde{f}^2 \left[1 - \omega^2 + \frac{3\epsilon \tilde{f}^2}{4}\right]^2 + 4b^2\omega^2\tilde{f}^2$$

This equation can be thought of as giving a relationship between $\tilde{f}$ (which is a nondimensionalized version of the amplitude of the response, $z_1(\tau)$), and ω^2, which is a non-dimensionalized version of the Ω^2, the square of the driving frequency. It is cubic in $\tilde{f}^2$, and quadratic in ω^2. This means for any particular ω^2, we expect there to be as many as three values of $\tilde{f}^2$ that satisfy this equation (six values of $\tilde{f}$, but half of them are negative and we ignore those). If we proceeded by straightforward perturbation instead (solving for $\tilde{f}$ as a function of f), we would only find one of the three–the other two would be hidden, just as the second solution to the quadratic equation is hidden by the perturbative method.

Conversely, for any particular value of $\tilde{f}$ we expect there to be as many as two values of ω^2 that satisfy the equation. And if we *plot* the solution, we get a rather interesting shape (which varies, depending on the value of b.) This is shown in figure 30.2, for $\frac{3\epsilon}{4} = 0.1$ and $\frac{3\epsilon}{4} = 0.01$. Notice how the original (unperturbed) resonance shape has *warped*.

Now consider what happens if we have a system like this and we slowly vary the driving frequency. (Again, picture yourself back in physics 200 lab adjusting the signal coming from the function generator.) But now, look at figure 30.3. If we start with the frequency at zero, and start increasing it, we will gradually move from point $\mathcal{A}$ to point $\mathcal{B}$ to point $\mathcal{C}$–as long as the change is gradual enough, we must follow the curve continuously. But then, if we increase the frequency further, the only thing the system can do is drop suddenly down to point $\mathcal{E}$, and then continue on to point $\mathcal{F}$. So we have the path $\mathcal{ABCEF}$.

But now imagine that we start with the frequency high, and gradually decrease it. We will move continuously from point $\mathcal{F}$ to point $\mathcal{E}$ to point $\mathcal{D}$. And then, if we decrease the driving frequency further, the system must suddenly jump up to point $\mathcal{B}$ before continuing on to point $\mathcal{A}$. So we have the path $\mathcal{FEDBA}$.

This gives two important features to the system: first, that a small change in driving frequency will occasionally result in a large, discontinuous change in the response amplitude. Second, that what this change is, and where it occurs, depends on whether you were gradually increasing the driving frequency from a small value, or gradually decreasing the driving frequency from a large value. This is *hysteresis*. Notice that the red part of the curve (between points $\mathcal{C}$ and $\mathcal{D}$) are never reached is never accessed in the process. These values for Γ are physically possible–if you set up your initial conditions just right you may be able to find a response corresponding to them–but the arrangement is unstable. A small change will then force you onto the lower leg, between $\mathcal{D}$ and $\mathcal{E}$.

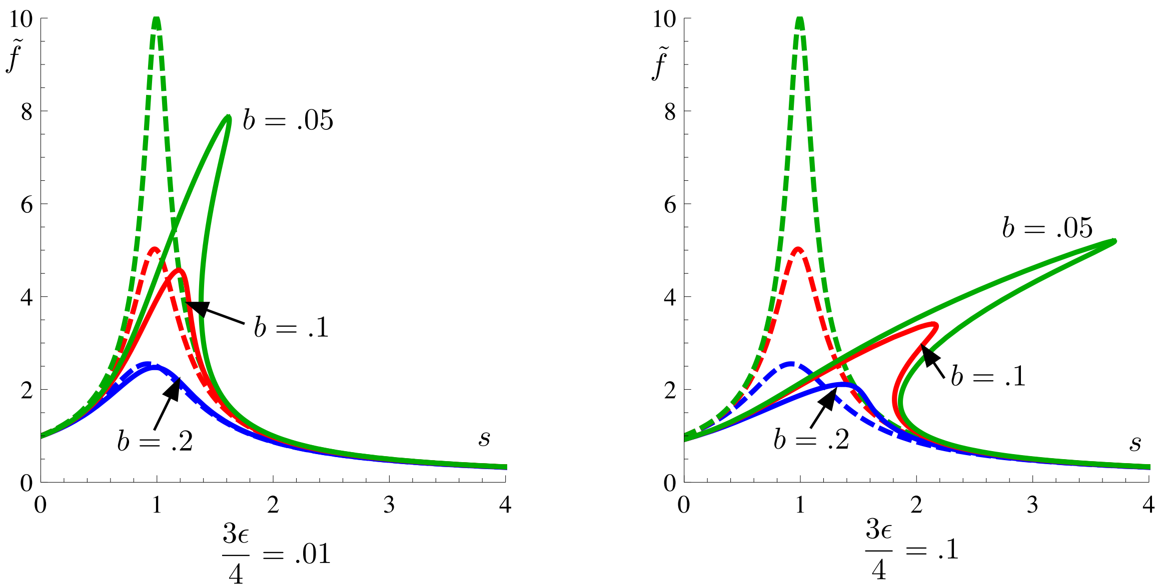

Figure 30.2: The warped resonance curve relating $\tilde{f}$ to ω^2. Note how for increasing perturbation, the resonance peak becomes increasingly bent over. Note also how a particular value of ω^2 can have as many as three values of $\tilde{f}$, and a particular value of $\tilde{f}$ can have as many as two values of ω^2.

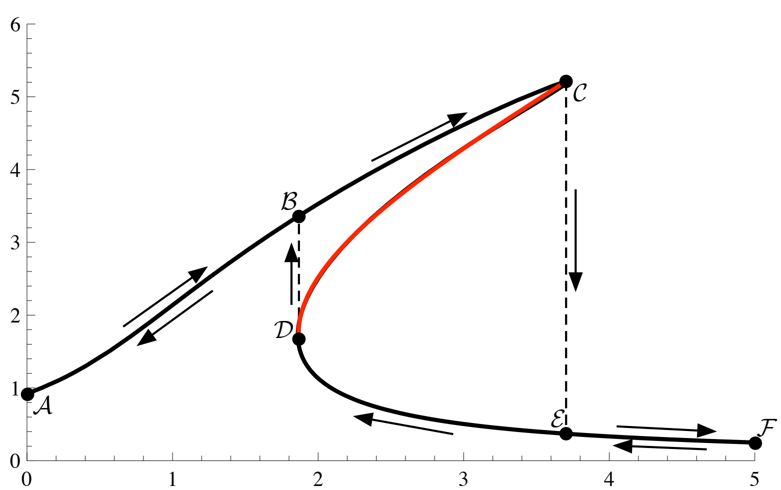

Figure 30.3: Slowly changing the driving frequency will give different results, depending on the history of the system.

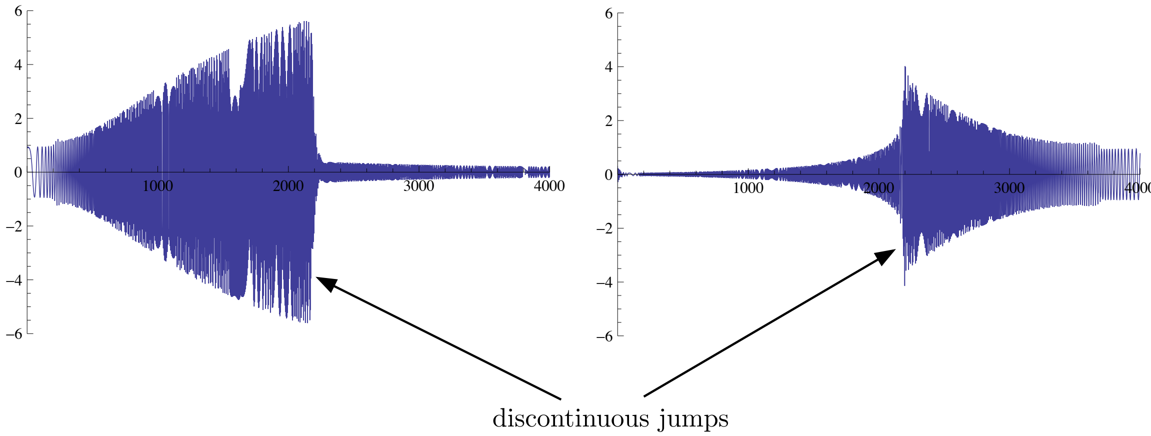

Figure 30.4: Numerically generated solutions $z(\tau)$, with slowly varying driving frequency ω (increasing for the left graph, decreasing for the right graph.).

In figure 30.4, I plot the solution, numerically generated, to equation 30.13, for a very slowly varying ω (increasing in the left graph, decreasing in the right graph.) Note how at some point the amplitude of the solution suddenly shifts, and note that the shift is different, whether we are increasing or decreasing. (You can't tell from the figures exactly what frequency the shifts happen at, but you can tell that the amplitude jump is different, and that the jump is small when we are turning down the driving frequency than it is when we are turning up the driving frequency, which is consistent with figure 30.3.