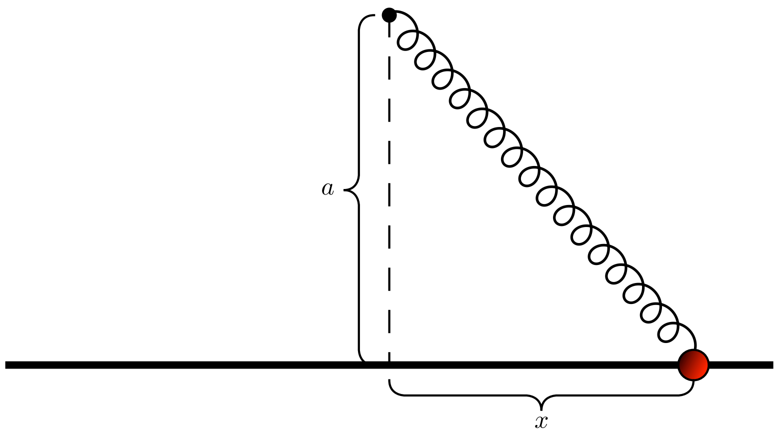

Figure 29.1: The bead slides on a rod and is attached to a spring whose other end is fixed.

Perturbation Theory

Lecture 29

Physics 311: Classical Mechanics
Fall 2011

16 November 2011

Last time we began our discussion of nonlinear dynamics by studying the motion of a bead on a rod, which is attached to a spring whose other end is fixed to a point some distance away from the rod. (figure 29.1.) This allowed us to discuss how we can study motion that stays near the minimum of some potential, and how for a generic potential this motion will be that of a simple harmonic oscillator. The frequency of oscillation then depends on the curvature of the potential at its minimum–that is, the second derivative. Today we want to continue our discussion of this simple system, and consider what happens beyond this linear approximation.

29.1 Reminders of Last Time, and Beyond

Recall that we started with a the distance from the rod to the fixed point and ℓ the equilibrium extent of the spring, m the mass of the bead and k the spring constant, and we immediately nondimensionalized the problem, using

$$y = \frac{x}{\ell}, \qquad b = \frac{a}{\ell}, \qquad \tilde{V} = \frac{V}{k\ell^2}, \qquad t = \tilde{\tau}\sqrt{\frac{m}{k}} \tag{29.1}$$

so that the non-dimensionalized potential is

$$\tilde{V}(y) = \frac{1}{2}\left[\sqrt{y^2+b^2} - 1\right]^2 \tag{29.2}$$

and the equation of motion is

$$\ddot{y} = \frac{d^2y}{d\tilde{\tau}^2} = -\frac{\partial\tilde{V}}{\partial y} \tag{29.3}$$

where dots refer to derivatives with respect to $\tilde{\tau}$, the non-dimensionalized time. The dependence everywhere on b, the ratio of a to ℓ, reflects the fact that qualitatively different results occur depending on this ratio. In the case where $b < 1$, the potential has two minima, at $y_0 = \pm\sqrt{1-b^2}$, and a maximum at $y_0 = 0$. In the case where $b > 1$, the potential has a single minimum at $y_0 = 0$. Last time we also considered briefly the special case $b = 1$, where the motion is fundamentally nonlinear, even for very small oscillations. For today we will focus on the case $b > 1$, but we will not stop by approximating the potential near $y_0 = 0$ up to quadratic terms. Recall that we calculated derivatives

$$\left.\frac{\partial\tilde{V}}{\partial y}\right|_{y_0=0} = \frac{y_0\left(\sqrt{b^2+y_0^2}-1\right)}{\sqrt{b^2+y_0^2}} = 0, \tag{29.4}$$

$$\left.\frac{\partial^2\tilde{V}}{\partial y^2}\right|_{y_0=0} = 1 - \frac{b^2}{(y_0^2+b^2)^{3/2}} = \frac{b-1}{b}, \tag{29.5}$$

$$\left.\frac{\partial^3\tilde{V}}{\partial y^3}\right|_{y_0=0} = \frac{3b^2y_0}{(y_0^2+b^2)^{5/2}} = 0, \tag{29.6}$$

and

$$\left.\frac{\partial^4\tilde{V}}{\partial y^4}\right|_{y_0=0} = \frac{3b^2(b^2-4y_0^2)}{(b^2+y_0^2)^{7/2}} = \frac{3}{b^3} \tag{29.7}$$

And last time, we were content then to approximate the potential as

$$\tilde{V}(z+y_0) = \tilde{V}(y_0) + \frac{1}{2}\frac{\partial^2\tilde{V}}{\partial y^2}\Big|_{y_0=0} z^2 + \cdots \approx \frac{(b-1)^2}{2} + \frac{(b-1)}{2b}z^2 \tag{29.8}$$

which resulted in the linearized equation of motion for $z(t)$:

$$\ddot{z} \approx -\left(\frac{b-1}{b}\right) z \approx -\tilde{\omega}^2 z, \qquad \tilde{\omega} = \sqrt{\frac{b-1}{b}} \tag{29.9}$$

Suppose that we now want to do a somewhat better job of approximating the motion than we can do just using the linearized equation. We might consider expanding out the potential to include terms of higher order in z. Since $V'''(y_0) = 0$, the next correction is the fourth derivative, which gives

$$\tilde{V}(z+y_0) \approx \frac{(b-1)^2}{2} + \frac{(b-1)}{2b}z^2 + \frac{1}{8b^3}z^4 \tag{29.10}$$

which gives equation of motion

$$\ddot{z} \approx -\tilde{\omega}^2 z - \frac{1}{2b^3}z^3 \tag{29.11}$$

For small z, the second term on the right is a small correction to the equation of motion. But as we increase the size of the oscillations we study, we enter a regime where this term can no longer be safely ignored. We are therefore interested in solving differential equations in the general form of 29.11. In general there are no obvious paths to a solution–the equation is determinedly nonlinear. However, suppose we *also* have $b \gg 1$. (That is, the distance between the rod and the fixed point is very large compared to the equilibrium extent of the spring). In this case

$$\delta = \frac{1}{2b^3} \tag{29.12}$$

can be assumed to be very small. The equation we want to solve becomes

$$\ddot{z} + \tilde{\omega}^2 z + \delta z^3 = 0. \tag{29.13}$$

This type of equation, which can be thought of as a linear equation *plus* a small nonlinear correction, can be approached using a technique coming from what is known as *perturbation theory*.

29.2 A Quadratic Equation, Perturbatively

Perturbation theory is a general idea which can be applied to all sorts of problems, including algebraic equations (as opposed to differential equations.) We will introduce it as a way to approach a very simple equation:

$$\epsilon x^2 + x - 1 = 0 \tag{29.14}$$

Now, this is an equation that (hopefully) we can all solve. It has two solutions which we can find exactly, namely

$$x = \frac{-1 + \sqrt{1 + 4\epsilon}}{2\epsilon}, \qquad x = \frac{-1 - \sqrt{1 + 4\epsilon}}{2\epsilon} \tag{29.15}$$

However, suppose for a moment that you did not know the quadratic equation, and were truly stumped on how to solve equation 29.14. Let us further suppose that ϵ is very small. Then you might reason that this equation is very close to an even simpler one:

$$x_0 - 1 = 0$$

which has solution $x_0 = 1$. You might then argue that, whatever the solution of the "hard" equation 29.14 is, it must be very close to x_0, and must get even closer as $\epsilon \to 0$. In fact, we should have

$$x = x_0 + \mathcal{O}(\epsilon)$$

Now, suppose we rewrite equation 29.14 to take advantage of this fact:

$$1 - x = \epsilon x^2 = \epsilon(x_0 + \mathcal{O}(\epsilon))^2 = \epsilon(x_0^2 + \mathcal{O}(\epsilon)) = \epsilon x_0^2 + \mathcal{O}(\epsilon^2) \tag{29.16}$$

This suggests we consider the quantity x_1 that satisfies equation

$$1 - x_1 = \epsilon x_0^2$$

With the knowledge that $x_0 = 1$, this has solution $x_1 = 1 - \epsilon$. x_1 is now also close to the desired (exact) quantity x. In fact, it must be even closer than our first approximation x_0. Based on equation 29.16 we must have

$$x = x_1 + \mathcal{O}(\epsilon^2)$$

But now, we can rewrite equation 29.14 *again*, using our new knowledge:

$$1 - x = \epsilon x^2 = \epsilon(x_1 + \mathcal{O}(\epsilon^2))^2 = \epsilon(x_1^2 + \mathcal{O}(\epsilon^2)) = \epsilon x_1^2 + \mathcal{O}(\epsilon^3) \tag{29.17}$$

We consider the quantity x_2 that satisfies the equation

$$1 - x_2 = \epsilon x_1^2$$

and, knowing that $x_1 = 1 - \epsilon$, this gives $x_2 = 1 - \epsilon + 2\epsilon^2 - \epsilon^3$. We expect this is an *even better* approximation of our exact quantity x:

$$x = x_2 + \mathcal{O}(\epsilon^3) = 1 - \epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{29.18}$$

Notice that our procedure gave us a term in x_2 that was cubic in ϵ. However, when we interpret this as an approximation of x, since we expect it only to be good up to quadratic terms, we ignore the cubic term (lumping it into the $\mathcal{O}(\epsilon^3)$ expression.)

Clearly, we could keep going, at every stage producing a better approximation of the true x. The procedure is *iterative*, and each stage uses the results of the previous stage; yet, no stage requires that we solve any equation harder than linear. To summarize, we write

$$\begin{array}{rclcrcl} 1 - x_0 &=& 0, & \to & x_0 &=& 1 \\ 1 - x_1 &=& \epsilon x_0^2, & \to & x_1 &=& 1 - \epsilon \\ 1 - x_2 &=& \epsilon x_1^2, & \to & x_2 &=& 1 - \epsilon + 2\epsilon^2 - \epsilon^3 \end{array} \tag{29.19}$$

If we know $\{x_0, \ldots, x_{k_1}\}$ then to find x_k we write

$$1 - x_k = \epsilon x_{k-1}^2 \tag{29.20}$$

and solve this linear equation. This gives an approximation of x that is valid up to order ϵ^k, so we can write

$$x = x_k + \mathcal{O}(\epsilon^{k+1}) \tag{29.21}$$

Now, let's compare this result to what we obtain by taking the exact solutions (equations 29.15) and Taylor expanding them in ϵ. We'll start by looking at

$$x = \frac{-1 + \sqrt{1 + 4\epsilon}}{2\epsilon}$$

To expand this, we need to know the expansion for $f(x) = \sqrt{1+x}$. We note that

$$\begin{array}{rclcrcl} f(x) &=& \sqrt{1+x} & \quad & f(0) &=& 1 \\ f'(x) &=& \frac{1}{2}(1+x)^{-1/2} & & f'(0) &=& \frac{1}{2} \\ f''(x) &=& -\frac{1}{4}(1+x)^{-3/2} & & f''(0) &=& -\frac{1}{4} \\ f'''(x) &=& \frac{3}{8}(1+x)^{-3/2} & & f'''(0) &=& \frac{3}{8} \end{array}$$

which allows me to write the Taylor series as

$$f(x) = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$

then we have

$$\sqrt{1+4\epsilon} = 1 + 2\epsilon - 2\epsilon^2 + 4\epsilon^3 + \mathcal{O}(\epsilon^4)$$

so that we obtain

$$x = \frac{-1+\sqrt{1+4\epsilon}}{2\epsilon} = \frac{-1+1+2\epsilon-2\epsilon^2+4\epsilon^3+\mathcal{O}(\epsilon^4)}{2\epsilon} = 1 - \epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{29.22}$$

which clearly agrees with equation 29.18.

On the other hand, x has another solution,

$$x = \frac{-1-\sqrt{1+4\epsilon}}{2\epsilon}.$$

What happens to this solution in our approximation? Applying the Taylor series expansion to it, we obtain

$$x = \frac{-1-\sqrt{1+4\epsilon}}{2\epsilon} = \frac{-1-1-2\epsilon+2\epsilon^2-4\epsilon^3+\mathcal{O}(\epsilon^4)}{2\epsilon} = -\frac{1}{\epsilon} - 1 + \epsilon - 2\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{29.23}$$

and we can see from this what the trouble is. The first term in this expansion is $\frac{1}{\epsilon}$, which is large when ϵ is small. In fact, in the limit $\epsilon \to 0$ we have $x \to \infty$. As ϵ gets smaller and smaller, this solution moves off to infinity, until it disappears completely: the linear equation $1 - x_0 = 0$ has only one solution.

Our perturbative approach looked for solutions that were close to the solution of the "unperturbed equation" $1 - x_0 = 0$. Therefore, it only found solutions close to $x_0 = 1$. It missed an *entire solution*, because this solution disappeared in the limit $\epsilon \to 0$. This is an important limitation of the perturbative approach that we will run up against again.

29.3 Perturbation Theory for a Differential Equation

Let us return to the differential equation we are interested in solving, which I will write as

$$\ddot{z} + \tilde{\omega}^2 z + \delta z^3 = 0$$

Now, although we already non-dimensionalized the complicated system we started with, now that we have reduced the problem to the above equation, it is somewhat useful to redo the non-dimensionalization. In particular, we can redefine both the time scale and the small parameter δ to make the problem simpler. Setting

$$\tau = \tilde{\tau}\tilde{\omega}, \qquad \epsilon = \frac{\delta}{\tilde{\omega}^2} \tag{29.24}$$

we obtain the somewhat prettier equation

$$\ddot{z} + z + \epsilon z^3 = 0 \tag{29.25}$$

We have essentially rescaled the natural frequency of the system to be equal to one.

This is qualitatively similar to equation 29.14, in that it can be thought of as a much simpler equation, plus a correction that is small. The perturbative approach dictates that we start by solving the much simpler equation:

$$\ddot{z}_0 + z_0 = 0 \tag{29.26}$$

whose solutions we know well:

$$z_0(\tau) = A\cos\tau + B\sin\tau$$

We then expect that we have

$$z(\tau) = z_0(\tau) + \mathcal{O}(\epsilon) = A\cos\tau + B\sin\tau + \mathcal{O}(\epsilon).$$

That is, $z_0(\tau)$ is a reasonable approximation to $z(\tau)$. The constants A and B would generally be fixed using initial conditions. In this circumstance, they take on particular importance because of what we have learned of the perturbative approach. We are going to use the solution $z_0(\tau)$ to find a better approximation to $z(\tau)$, and in doing so we will find a solution close to $z_0(\tau)$. Whatever values of A and B we choose, they will affect (in a fundamentally nonlinear way) the next stage of the approximation, because they determine *which* $z_0(\tau)$ we will be close to. For now, let's make the simple choice

$$z_0(\tau) = \cos\tau \tag{29.27}$$

which is $A = 1, B = 0$. Now, following the same logic we used on the quadratic equation, we will have

$$\ddot{z} + z = -\epsilon z_0^3 + \mathcal{O}(\epsilon^2)$$

and because of this we consider the function $z_1(\tau)$ which satisfies the equation

$$\ddot{z}_1 + z_1 = -\epsilon z_0^3 = -\epsilon \cos^3 \tau \tag{29.28}$$

The function $z_1(\tau)$ will be a better approximation for $z(\tau)$, with

$$z(\tau) = z_1(\tau) + \mathcal{O}(\epsilon^2).$$

Now we want to solve equation 29.28, which at first glance might seem like a rather daunting task. However, it turns out to simply be an undamped, driven simple harmonic oscillator. This is a type of equation you encountered in the fall of physics 200. Finding the solution is a little tedious, but essentially straightforward. First we note that we can rewrite the "driving force" term $-\epsilon \cos^3 \tau$ as a sum of two linear trigonometric functions. To do so we note that we have

$$\cos^3 \tau = \left(\frac{e^{i\tau} + e^{-i\tau}}{2}\right)^3 = \frac{e^{3i\tau} + 3e^{i\tau} + 3e^{-i\tau} + e^{-3i\tau}}{8}$$

$$= \frac{1}{4}\left[\left(\frac{e^{3i\tau} + e^{-3i\tau}}{2}\right) + 3\left(\frac{e^{i\tau} + e^{-i\tau}}{2}\right)\right] = \frac{1}{4}\cos 3\tau + \frac{3}{4}\cos\tau$$

so that the equation we are trying to solve becomes

$$\ddot{z}_1 + z_1 = -\frac{\epsilon}{4}\cos 3\tau - \frac{3\epsilon}{4}\cos\tau \tag{29.29}$$

29.3.1 Reminder: Undamped Driven Oscillators

Now, let's review the general approach to solving an equation in the form

$$\ddot{f} + \omega^2 f = f_0 \cos \Omega\tau. \tag{29.30}$$

We know we can solve this using the method of "general and particular solutions." First we write the solution to the general equation

$$\ddot{g} + \omega^2 g = 0, \quad \rightarrow \quad g(\tau) = A\cos\omega\tau + B\sin\omega\tau$$

and then we look for a particular solution (any solution) to the equation

$$\ddot{h} + \omega^2 h = f_0 \cos \Omega\tau. \tag{29.31}$$

Finally, we write

$$f(\tau) = g(\tau) + h(\tau) = A\cos\omega\tau + B\sin\omega\tau + h(\tau)$$

where we can use A and B to agree with initial conditions. Still, we need to find at least one solution to equation 29.31. We guess that we might find a solution with the same oscillating frequency as the driving force, so we write the "ansatz" (guess)

$$h(\tau) = h_0\cos(\Omega\tau + \phi)$$

and we plug this into the equation, getting

$$-\Omega^2 h_0\cos(\Omega\tau+\phi) + \omega^2 h_0\cos(\Omega\tau+\phi) = (\omega^2 - \Omega^2)h_0\cos(\Omega\tau+\phi) = f_0\cos\Omega\tau$$

which will work if we have

$$\phi = 0, \qquad h_0 = \frac{f_0}{\omega^2 - \Omega^2}$$

Finally, we can write the general solution to equation 29.30 as

$$f(\tau) = A\cos\omega\tau + B\sin\omega\tau + \frac{f_0\cos\Omega\tau}{\omega^2 - \Omega^2} \tag{29.32}$$

Note that technically equation 29.30 is nonlinear–it is not possible to add two solution in the form of 29.32 and obtain a third solution. However, this nonlinearity is no real obstacle in this case because the "general solution" part is linear, so we can obtain the whole family of solutions, dependent on the constants A and B.

Special Case: $\Omega = \omega$

Unfortunately for us, one of the driving terms in equation 29.29 has the same frequency as the natural oscillation frequency ω. Following the basic procedure above will fail in this particular case, and we need to treat it separately. We have equation

$$\ddot{f} + \omega^2 f = f_0\cos\omega\tau. \tag{29.33}$$

and we again can use the method of finding general and particular solutions $g(\tau)$ and $h(\tau)$, with the general solution

$$\ddot{g} + \omega^2 g = 0, \qquad \to \qquad g(\tau) = A\cos\omega\tau + B\sin\omega\tau.$$

Then we need to find at least one solution to the equation

$$\ddot{h} + \omega^2 h = f_0\cos\omega\tau$$

and a guess that $h(\tau) = h_0 \cos(\omega\tau + \phi)$ will not work, because when we plug it in, the left side of the above equation will simply vanish, no matter what h_0 is. It turns out that in this case what we need is the guess

$$h(\tau) = h_0 \tau \sin \omega\tau.$$

Notice that this solution actually grows with time–which matches our physical intuition into what happens when we drive an oscillator at its resonance frequency (see for, for example, the Tacoma Narrows Bridge disaster of 1940.) Plugging this ansatz in, we obtain

$$\frac{d^2}{d\tau^2}\left[h_0 \tau \sin \omega\tau\right] + \omega^2 h_0 \tau \sin \omega\tau = f_0 \cos \omega\tau$$

The left hand side of this gives

$$\frac{d^2}{d\tau^2}\left[h_0 \tau \sin \omega\tau\right] + \omega^2 h_0 \tau \sin \omega\tau = \frac{d}{d\tau}\left[h_0 \sin \omega\tau + \omega h_0 \tau \cos \omega\tau\right] + \omega^2 h_0 \tau \sin \omega\tau$$

$$= h_0 \omega \cos \omega\tau + h_0 \omega \cos \omega\tau - \omega^2 h_0 \tau \sin \omega\tau + \omega^2 h_0 \tau \sin \omega\tau = 2h_0 \omega \cos \omega\tau$$

so that we get

$$2h_0 \omega \cos \omega\tau = f_0 \cos \omega\tau$$

or

$$h_0 = \frac{f_0}{2\omega}$$

and we finally write the general solution

$$f(\tau) = A \cos \omega\tau + B \sin \omega\tau + \frac{f_0 \tau}{2\omega} \sin \omega\tau$$

29.3.2 The Naive Solution for $z_1(\tau)$

Armed with this refresher, we now consider equation 29.32. We guess that the general solution will be in the form

$$z_1(\tau) = A \cos(\tau + \phi) + C\tau \sin \tau + D \cos 3\tau$$

and we need to determine what restrictions to put on the constants $\{A, C, D, \phi\}$. The first term should give us our "general solution" with two constants A and ϕ. The second will give us the particular solution with driving frequency equal to the natural frequency, and the third term will give us the 3ω harmonic. Notice that this ansatz has

$$\dot{z}_1 = -A \sin(\tau + \phi) + C \sin \tau + C\tau \cos \tau - 3D \sin 3\tau$$

and also

$$\ddot{z}_1 = -A\cos(\tau+\phi) + 2C\cos\tau - C\tau\sin\tau - 9D\cos 3\tau$$

so that we get

$$\ddot{z}_1 + z_1 = -A\cos(\tau+\phi) + 2C\cos\tau - C\tau\sin\tau - 9D\cos 3\tau$$

$$+A\cos(\tau+\phi) + C\tau\sin\tau + D\cos 3\tau$$

which gives

$$\ddot{z}_1 + z_1 = 2C\cos\tau - 8D\cos 3\tau.$$

Plugging this into equation 29.32 gives

$$2C\cos\tau - 8D\cos 3\tau = -\frac{\epsilon}{4}\cos 3\tau - \frac{3\epsilon}{4}\cos\tau$$

which clearly requires that

$$D = \frac{\epsilon}{32}, \qquad C = -\frac{3\epsilon}{8}$$

so that we write

$$z_1(\tau) = A\cos(\tau+\phi) - \frac{3\epsilon\tau}{8}\sin\tau + \frac{\epsilon}{32}\cos 3\tau$$

However, we are not quite done. Consider $z_1(\tau)$. We should have that this is close to $z_0(\tau)$, which was

$$z_0(\tau) = \cos\tau$$

At first glance, all this seems to require is that $A = 1$ and $\phi = 0$, since both the later terms are proportional to ϵ. However, the second term presents a problem: in grows linearly with time. Even if for small τ we can say it is a small correction to $z_0(\tau)$, eventually it will be a *large* correction. This violates everything we have been assuming. And the equation we set out to solve, equation 29.25 *shouldn't* ever have a growing solution–the system is fundamentally energy conserving. What are we to do about it?

29.3.3 The Real Answer

The problem came in because our perturbation technique gave us a driven simple harmonic oscillator with a driving frequency that was the same as the natural frequency. Suppose we re-examine our first approximation

$$z_0(\tau) = \cos\tau$$

Since this is only a valid solution for $z(\tau)$ up to order ϵ, we can always change it in ways that depend on ϵ, as long as it agrees at lowest order. Keeping that it mind, let's write

$$z_0(\tau) = \cos\left[\tau\Big(1 + \epsilon\eta + \mathcal{O}(\epsilon^2)\Big)\right],$$

(that is, the frequency becomes $1 + \epsilon\eta + \mathcal{O}(\epsilon^2)$–we allow it to be shifted by the perturbation. Then the equation for $z_1(\tau)$ becomes

$$\ddot{z}_1 + z_1 = -\frac{\epsilon}{4}\cos 3(1+\epsilon\eta)\tau - \frac{3\epsilon}{4}\cos(1+\epsilon\eta)\tau$$

Now we have a two driving frequencies $1 + \epsilon\eta$ and $3(1 + \epsilon\eta)$, neither of which is exactly the same as the natural frequency 1. We therefore make the ansatz that

$$z_1(\tau) = A\cos(\tau + \phi) + B\cos(1+\epsilon\eta)\tau + C\cos 3(1+\epsilon\eta)\tau$$

which has

$$\ddot{z}_1(\tau) = -A\cos(\tau+\phi) - B(1+2\epsilon\eta)\cos(1+\epsilon\eta)\tau - 9C(1+2\epsilon\eta)\cos 3(1+\epsilon\eta)\tau.$$

so that we get

$$\ddot{z}_1 + z_1 = -A\cos(\tau+\phi) - B(1+2\epsilon\eta)\cos(1+\epsilon\eta)\tau - 9C(1+2\epsilon\eta)\cos 3(1+\epsilon\eta)\tau$$
$$+A\cos(\tau+\phi) + B\cos(1+\epsilon\eta)\tau + C\cos 3(1+\epsilon\eta)\tau$$

$$= -2B\epsilon\eta\cos(1+\epsilon\eta)\tau - C(8+18\epsilon\eta)\cos 3(1+\epsilon\eta)\tau$$

and when we plug it into our equation, we obtain

$$-2B\epsilon\eta\cos(1+\epsilon\eta)\tau - C(8+18\epsilon\eta)\cos 3(1+\epsilon\eta)\tau = -\frac{\epsilon}{4}\cos 3(1+\epsilon\eta)\tau - \frac{3\epsilon}{4}\cos(1+\epsilon\eta)\tau$$

We see that we need

$$B = \frac{3}{8\eta}, \qquad C = \frac{\epsilon}{8(4+9\epsilon\eta)} = \frac{\epsilon}{32}$$

(where I am dropping terms higher order in ϵ.) Now, we write our full solution $z_1(\tau)$ as

$$z_1(\tau) = A\cos(\tau+\phi) + \frac{3}{8\eta}\cos(1+\epsilon\eta)\tau + \frac{\epsilon}{32}\cos 3\tau$$

Finally we require that at $\epsilon = 0$, $z_1(\tau)$ agrees with $z_0(\tau) = \cos\tau$. We notice immediately that our $z_1(\tau)$ has two terms that are now both proportional to $\cos\tau$. We can therefore make $z_1(\tau)$ close to $z_0(\tau)$ by requiring

$$A = 0, \qquad \eta = \frac{3}{8}$$

so that we write

$$z_1(\tau) = \cos\left(\tau + \frac{3\epsilon\tau}{8}\right) + \frac{\epsilon}{32}\cos 3\tau + \mathcal{O}(\epsilon^2) \tag{29.34}$$

This satisfies all of our requirements. It gives an approximation for $z(\tau)$ that is valid up to linear order in ϵ, and it is close to our initial approximation $z_0(\tau)$. Note that it consists of a large oscillating term that is slightly offset from the natural frequency of the system, and a small oscillating term that is at three times the natural frequency (this is a "harmonic.")

Notice finally that for small τ, it is possible to expand out the first term, giving

$$\cos\left(\tau + \frac{3\epsilon\tau}{8}\right) = \cos\tau\cos\frac{3\epsilon\tau}{8} - \sin\tau\sin\frac{3\epsilon\tau}{8} \approx \cos\tau - \frac{3\epsilon\tau}{8}\sin\tau$$

so that our solution becomes

$$z_1(\tau) = \cos\tau - \frac{3\epsilon\tau}{8}\sin\tau + \frac{\epsilon}{32}\cos 3\tau + \mathcal{O}(\epsilon^2)$$

which is what we were getting with the naive approach to perturbation theory. It is fine, so long as we don't allow the clock to run too long. (That is, it's ok for small τ.) Figure 29.2 shows a plot (generated numerically) of the solution to the equation 29.14, plotted against the solution to the unperturbed equation (with $\epsilon = 0$.) The frequency shift of the primary oscillatory part is quite clear. However, less obvious is the harmonic parts of the perturbed solution. To see them we take a Fourier transform of the numeric solution, shown in figure 29.3.

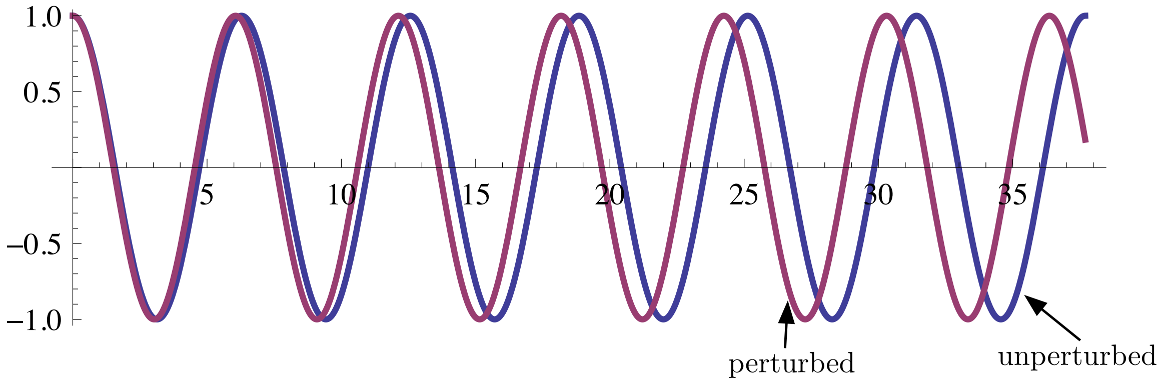

Figure 29.2: The solution to equation 29.14 with $\epsilon = 0.1$, plotted against the solution with $\epsilon = 0$ (the unperturbed equation.)

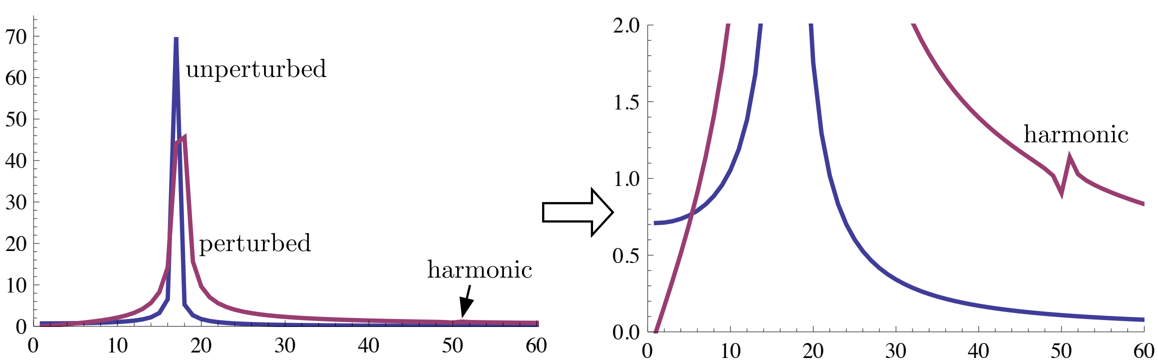

Figure 29.3: The Fourier transforms of the $\epsilon = 0$ and $\epsilon = 0.1$ solutions to equation 29.14. The harmonic at roughly 3 times the natural frequency is present as a tiny (but recognizable) bump in the plot.