Phase Diagrams and Energy Loss

Lecture 32

Physics 311: Classical Mechanics
Fall 2011

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Last time we introduced the concept of the phase diagram, which is a good way to analyze the qualitative features of motion in a nonlinear system. A phase diagram plots the motion in the $\{x, p\}$ plane. We focused on phase diagrams in energy conserving systems, where using Hamiltonian dynamics is a natural fit for discussing the phase diagram. Today we will continue with some more phase diagrams, this time discussing systems which do not conserve energy. Here, although in some cases it is still possible to continue thinking in terms of Hamiltonians, we will primary just use "Newton's laws" language.

32.1 The Damped Harmonic Oscillator

A classic system which loses energy over time which we can analyze exactly is the damped harmonic oscillator, which satisfies the equation

$$\ddot{x} + 2\beta\dot{x} + \left(\frac{k}{m}\right)x = 0$$

Using the non-dimensionalization of time, momentum, and the damping parameter β given by

$$t = t_0\tau = \sqrt{\frac{m}{k}}\,\tau, \qquad p = m\frac{dx}{dt} = \frac{m}{t_0}\frac{dx}{d\tau} = \sqrt{mk}\,\frac{dx}{d\tau} = \sqrt{mk}\,\tilde{p}, \qquad \beta = b\sqrt{\frac{k}{m}}$$

and, as usual, converting the "dot" notation to mean derivatives with respect to τ, this is

$$\ddot{x} + 2b\dot{x} + x = 0 \tag{32.1}$$

which we could write in a form similar to Hamilton's equations as

$$\dot{\tilde{p}} = -2b\,\tilde{p} - x, \qquad \dot{x} = \tilde{p} \tag{32.2}$$

We can read right away that the only fixed point will have $(x, \tilde{p}) = (0, 0)$, which certainly matches our intuition into the system. However, we cannot create the same closed curves that we had before showing the motion of an object in phase space. Those closed curves showed the path in phase space that a particle traced out; since last time we always had energy conserving systems, everywhere on the path had to have the same energy, and this gave us an equation in $\tilde{p}$ and x satisfied at all times–the equation that defined the curves.

Now, we do not have energy conservation. In fact, we know that the term proportional to $\dot{x}$ in the equation of motion 32.1 represents energy loss, so we expect that for any solution we will have gradual energy loss with a path that ends at $(x, \tilde{p}) = (0, 0)$ (where the total energy is zero.) In fact, since we know how to solve this system exactly, we can simply determine $x(\tau)$ and $\tilde{p}(\tau)$ and plot them in phase space for a variety of initial conditions and values of b.

32.1.1 Underdamped Motion $b < 1$

As usual, the damped harmonic oscillator is different depending on the relative sizes of β and $\sqrt{\frac{k}{m}}$, that is, the size of b. We will first review the case where $b < 1$, the "underdamped" system. In this case, we know that the solution is

$$x(\tau) = Ae^{-b\tau} \cos\left(\tau\sqrt{1-b^2} + \phi\right) \tag{32.3}$$

and this has

$$\tilde{p}(\tau) = -Ae^{-b\tau}\left[b\cos\left(\tau\sqrt{1-b^2} + \phi\right) + \sqrt{1-b^2}\sin\left(\tau\sqrt{1-b^2} + \phi\right)\right]$$

with the definition

$$\cos\psi = b, \qquad \sin\psi = \sqrt{1-b^2} \tag{32.4}$$

this can be rewritten as

$$\tilde{p}(\tau) = -Ae^{-b\tau} \cos\left(\tau\sqrt{1-b^2} + \phi - \psi\right) \tag{32.5}$$

Both the position and momentum are set by the initial scale A, and both have an exponential decay factor $e^{-b\tau}$ and an oscillatory factor with frequency $\sqrt{1-b^2}$. However, the momentum

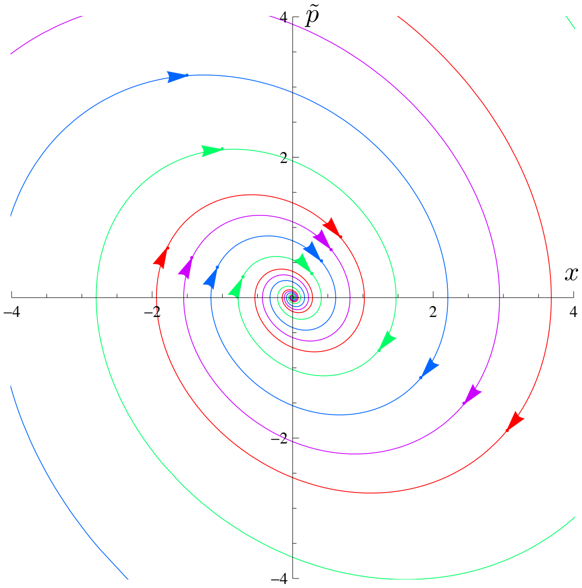

Figure 32.1: The phase diagram of the underdamped harmonic oscillator, with $b = 0.2$.

is out of phase with the position by $\psi = \cos^{-1} b$. In the limit where $b \to 0$ (the damping term disappears) the phase shift between x and $\tilde{p}$ becomes the $\pi/2$ we are familiar with from the undamped harmonic oscillator. (This creates the ellipse with major and minor axes aligned with the x and $\tilde{p}$ axes.) In the opposite limit, where $b \to 1$ (the critically damped oscillator) the phase shirt between x and $\tilde{p}$ disappears.

Not surprisingly, when we plot this in phase space, we get inward spirals which end at the fixed point. This is shown in figure 32.1.

32.1.2 Critically Damped Motion $b = 1$

For the critically damped oscillator we have $b = 1$, and we know the solutions are

$$x(\tau) = (A + B\tau)e^{-\tau} \tag{32.6}$$

which has

$$p(\tau) = (B - A - B\tau)e^{-\tau} \tag{32.7}$$

Both the position and the momentum have the damping factor $e^{-\tau}$, and both multiply this by a factor with a constant term and a term linear in τ. Furthermore, we can see that at all

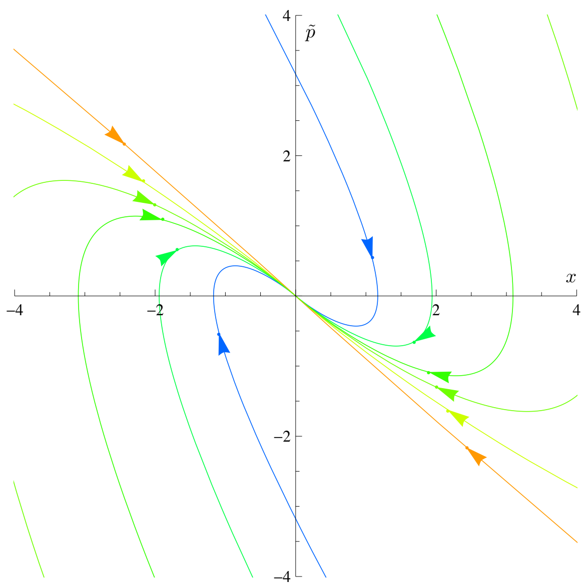

Figure 32.2: The phase diagram of the critically damped harmonic oscillator.

times we will have $x + p = Be^{-\tau}$. Notice that in the special case where $B = 0$ (a particular family of initial conditions), we always have $x + p = 0$, so the path is a straight line in phase space.

Because the oscillatory terms are gone, as the path of the particle moves inward it does not spiral. If it starts out moving away from the equilibrium (that is, with positive momentum and positive position, or negative momentum and negative position), it moves out to some maximal displacement from the equilibrium, before turning around and falling straight into the equilibrium point and staying there. This is shown in figure 32.2.

32.1.3 Overdamped Motion $b > 1$

When we have $b > 1$, the motion is overdamped, and the solutions are

$$x(\tau) = e^{-b\tau}\left[Ae^{-\tau\sqrt{b^2-1}} + Be^{\tau\sqrt{b^2-1}}\right] \tag{32.8}$$

which has

$$\tilde{p}(\tau) = -e^{-b\tau}\left[A\left(b + \sqrt{b^2-1}\right)e^{-\tau\sqrt{b^2-1}} + B\left(b - \sqrt{b^2-1}\right)e^{\tau\sqrt{b^2-1}}\right] \tag{32.9}$$

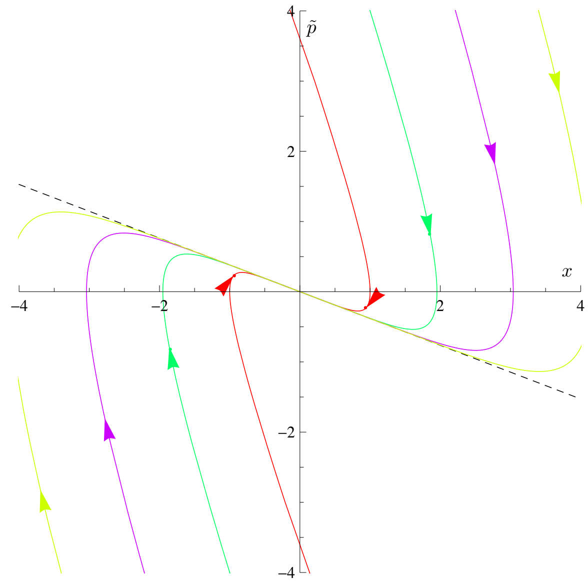

Figure 32.3: The phase diagram of the overdamped harmonic oscillator with $b = 1.5$.

We notice immediately, of course, that both x and $\tilde{p}$ are exponentially suppressed. (Keep in mind that $\sqrt{b^2-1} < b$.) There is no oscillatory factor, so the mass will never move through the equilibrium point. As with the critically damped motion, if the mass begins moving away from the equilibrium, it will move out to some maximal displacement before turning around and falling into the fixed point.

We also notice that at large times τ, the terms proportional to B will always win, so that we will have

$$\text{large } \tau : \qquad x \approx B e^{-b\tau} e^{\tau\sqrt{b^2-1}}, \qquad \tilde{p} \approx -B\left(b - \sqrt{b^2-1}\right) e^{-b\tau} e^{\tau\sqrt{b^2-1}}$$

which gives

$$\text{large } \tau : \qquad \tilde{p} = -\left(b - \sqrt{b^2-1}\right) x$$

so that at large times any solution *without* $B = 0$ will asymptote to a straight line in phase space, which moves to the origin and has slope $-\left(b - \sqrt{b^2-1}\right)$. For very large b, this comes closer and closer to zero, while in the opposite limit, where $b \to 1$, it becomes equal to negative one–the same as the special case line in phase space for $b = 1$. Figure 32.3 shows the phase diagram for the overdamped oscillator with $b = 1.5$.

In all three of these cases, the fixed point in phase space looks rather different than it did for the energy conserving cases of last time. For energy conserving systems, we have two kinds of fixed points: unstable fixed points, near which the phase diagram has hyperbolas, and stable fixed points, near which the phase diagram has ellipses. The ellipses show stability because a small displacement from the fixed point stays near that fixed point always–orbiting round and round on the ellipses.

For the damped harmonic oscillator we have another kind of stable fixed point, which we could call an *attractor*. The attractor is stable because a small displacement from it will stay near the fixed point always–in fact, it will eventually fall into the fixed point!

32.2 Limit Cycles

Now let's consider the damped, driven harmonic oscillator. This will allow us to introduce another common object one encounters in phase diagrams, called a *limit cycle*. Recall that the equation of motion for the damped, driven harmonic oscillator is

$$\ddot{x} + 2\beta\dot{x} + \left(\frac{k}{m}\right)x = f_0 \cos \Omega t$$

We can non-dimensionalize this (the way we did before) using $\sqrt{\frac{m}{k}}$ as the natural time scale of the system and $\frac{mf_0}{k}$ as the natural length scale. This gives us

$$t = t_0\tau = \sqrt{\frac{m}{k}}\,\tau, \qquad x = \frac{mf_0}{k}\,y, \qquad \beta = b\sqrt{\frac{k}{m}}, \qquad \Omega = \sqrt{\frac{k}{m}}\,\omega$$

and with these definitions our equation of motion becomes

$$\ddot{y} + 2b\dot{y} + y = \cos\omega\tau \tag{32.10}$$

and our non-dimensionalized momentum will be

$$\tilde{p} = \dot{y}. \tag{32.11}$$

We know that the solutions to this equation have two parts: a transitory part and a steady state piece.

$$y(\tau) \;=\; (\text{transitory}) \;+\; (\text{steady state}) \;=\; y_t(\tau) + y_s(\tau)$$

The transitory part is the solution to the undriven damped harmonic oscillator which we have just been talking about. The steady state part oscillates at the driving frequency ω

and has a particular amplitude that is a function of b and ω, and exhibits a resonance. (it is largest where the driving frequency is the same as the natural frequency of the system, that is, where $\omega = 1$.) Specifically, we have

$$y_s(\tau) = f\cos\omega\tau, \qquad f = \frac{1}{\sqrt{(1-\omega^2)^2 + 4b^2\omega^2}} \tag{32.12}$$

Let's ignore the transitory part for now, since we know that $y_s(\tau)$ is, itself a solution of equation 32.10 (with a particular set of initial conditions.) The momentum associated with this solution is

$$p_s(\tau) = -f\omega\sin\omega\tau \tag{32.13}$$

and we know that this particular solution traces out an ellipse in phase space:

$$\omega^2 y_s^2 + p_s^2 = f^2\omega^2. \tag{32.14}$$

In some ways this looks exactly like what we had for the simplest of systems, the plain simple harmonic oscillator. However, remember that $y_s(\tau)$ is *only* a solution for the particular f given in equation 32.12. And $y_s(\tau)$ is the only solution to our equation of motion that does form an ellipse. Notice that the eccentricity of the ellipse is determined by ω, which is the ratio of the driving frequency to the natural frequency of the system. For $\omega = 1$, the ellipse is actually a circle. On the other hand, the size of the ellipse is determined by f, the amplitude of the steady-state oscillation. This is a function of both ω and b, which is determined by the damping coefficient. For a particular b, the largest f can be is $\frac{1}{2b}$.

Now, remember that a more general solution also has the transitory piece, which is a solution to the undriven damped harmonic oscillator. Suppose (for example) that the system is underdamped, so that $b < 1$. In this case we have a solution in the form

$$y(\tau) = Ae^{-b\tau}\cos\left(\tau\sqrt{1-b^2} + \phi\right) \; + \; f\cos\omega\tau \tag{32.15}$$

which has momentum

$$\tilde{p}(\tau) = -Ae^{-b\tau}\cos\left(\tau\sqrt{1-b^2} + \phi - \psi\right) \; - \; f\omega\sin\omega\tau, \qquad \cos\psi = b \tag{32.16}$$

We know, whatever the initial conditions (set by ϕ and A) are, eventually the transitory part dies off and leaves us with the steady state solution. We also know that, if we start out initially with a very large y and $\tilde{p}$, then at least initially the solution must look very much like the transitory part, alone. Thus, what we expect to see for those cases in phase space is inward spirals (much like we had without the driving term) but that end up converging with

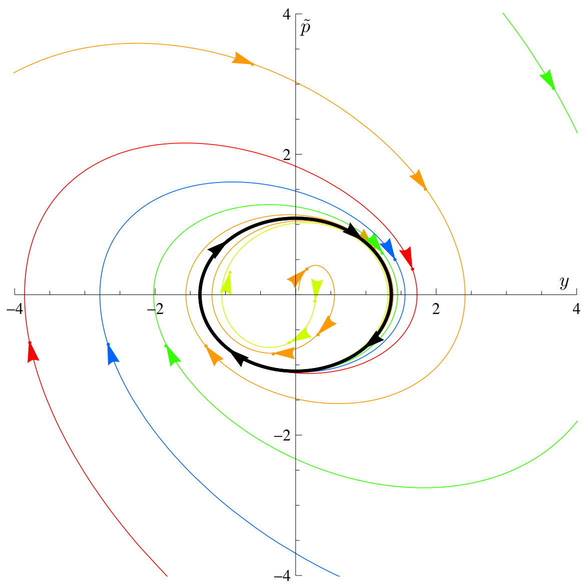

Figure 32.4: The phase diagram of the underdamped driven harmonic oscillator with $b = 0.4$ and $\omega = 0.8$. The limit cycle is shown in solid black.

the stead state ellipse. On the other hand, we might have initial conditions where y and $\tilde{p}$ start out small. But they, too, after a large amount of time, must end up on the steady state ellipse. The phase diagram is shown in figure 32.4.

The steady state ellipse is what we call a *limit cycle*. Although there are no fixed points in this system, it exhibits another type of stability. For large enough times, we know that any solution will be arbitrary close to some point on the limit cycle. We could repeat this argument for the critically damped or overdamped cases. Both times, the limit cycle would still be there. And since every solution has an oscillator part, they all have spirals, but the greater the damping term is, the faster they converge onto the limit cycle.

32.3 The Van Der Pol Oscillator

32.3.1 Energy Loss and Gain

The van der Pol oscillator is a famous nonlinear system which has a limit cycle. The equation of motion for this system is

$$\ddot{x} + \beta(x^2 - a^2)\dot{x} + \left(\frac{k}{m}\right)x = 0$$

where a is some constant with the dimensions of length, and β is another constant (positive) with dimensions of $\frac{1}{\text{time}\cdot\text{length}^2}$, and k and m are, as usual, a spring constant and a mass. We can use a as the natural length scale of the system, and $\sqrt{\frac{m}{k}}$ as the natural time scale, and we define

$$x = ay, \qquad t = \sqrt{\frac{m}{k}}\,\tau, \qquad \beta = \frac{\epsilon}{a}\sqrt{\frac{k}{m}} \tag{32.17}$$

so that our equation of motion and momentum become simply

$$\ddot{y} + \epsilon(y^2 - 1)\dot{y} + y = 0, \qquad \tilde{p} = \dot{y} \tag{32.18}$$

This looks a lot like a damped harmonic oscillator, except that the "damping coefficient" is actually position dependent. However, things are a little more complicated because of this dependence. Perhaps most significantly is that the "damping term" doesn't actually always damp! Notice that the term is positive for $y^2 > 1$ and *negative* for $y^2 < 1$. We can see the implications of this most clearly if we consider the energy of the system, which should be

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{ka^2}{2}\left[\dot{y}^2 + y^2\right]$$

which we can write

$$e = \dot{y}^2 + y^2 = \tilde{p}^2 + y^2, \qquad E = \frac{ka^2}{2}\,e \tag{32.19}$$

Now consider the time derivative of the non-dimensionalized energy e:

$$\frac{de}{d\tau} = 2\dot{y}\ddot{y} + 2y\dot{y} = 2\dot{y}\left[\ddot{y} + y\right]$$

which, using the equation of motion 32.18, is

$$\frac{de}{d\tau} = -2\epsilon\dot{y}^2(y^2 - 1)$$

or finally

$$\frac{de}{d\tau} = -2\epsilon\tilde{p}^2(y^2 - 1) \tag{32.20}$$

Since β is positive, so is ϵ, and we will always have $\tilde{p}^2 \geq 0$. Thus, the sign of the change in energy is determined by the factor $y^2 - 1$. For $y^2 > 1$, the energy in the system is decreasing with time, but for $y^2 < 1$, it is actually increasing!

This basic pattern *suggests* that a limit cycle might exist. For y very small, we will have an oscillatory system where the oscillations want to grow, while for y very large, we will have an oscillator system where the oscillations want to shrink (in fact, we will have an overdamped system.) It is not inconceivable that there might be some sort of steady state path in phase space. However, what is not obvious based on this is exactly what this limit cycle will look like.

32.3.2 Phase Diagrams and Solutions

Now, we know that for $\epsilon = 0$, this is just the simple harmonic oscillator, which (non-dimensionalized) just has a bunch of circles in phase space. Qualitatively, we should expect the phase space to not look *too* different from this, if ϵ is very small. (However, the system is rather hard to study using the traditional tools of perturbation theory which we worked with last week.) For very small ϵ, it then makes sense that the limit cycle should be approximately a circle. In fact, one can use perturbation theory to show that a solution exists for very small ϵ that is approximately a circle of radius 2 in phase space.

In figure 32.5, we show the results of numerical work with $\epsilon = 0.1$. On the right, $y(\tau)$ is plotted for a couple of different sets of initial conditions. Notice that over time, both settle down to the same steady state solution, which (to the naked eye) looks essentially like a sine wave. The presence of that steady state solution produces the limit cycle in phase space. On the left we show the phase diagram for $\epsilon = 0.1$. Here the limit cycle is obvious, but it is also clear that it is *not quite* a perfect circle.

Notice also that the van der Pol oscillator has a fixed point–at $(y, \tilde{p}) = (0, 0)$. However, this is an unstable fixed point. In fact, for small y we should have

$$\ddot{y} - \epsilon\dot{y} + y \approx 0$$

which is essentially the equation for a damped harmonic oscillator–except that the damping term has the wrong sign. We can read off what the solutions will look like near $(0, 0)$ from our previous work with the damped harmonic oscillator, replacing b with $-\epsilon/2$. We clearly get

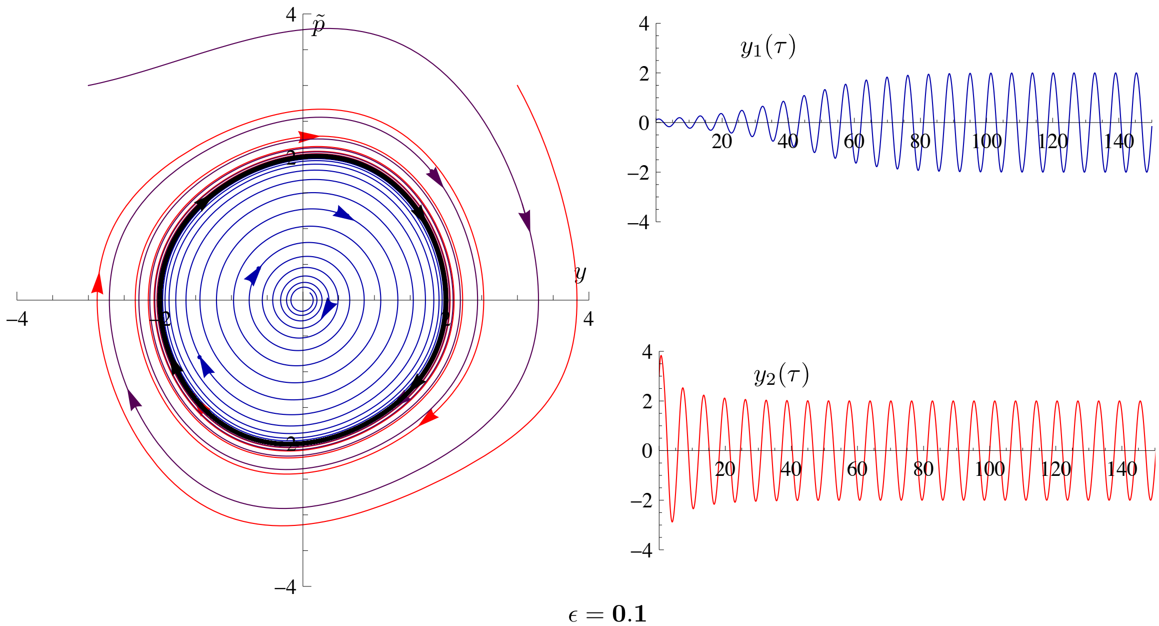

Figure 32.5: On the left, the phase diagram of the van der Pol oscillator with $\epsilon = 0.1$. the limit cycle is shown in solid black. On the right, two solutions $y(\tau)$ for the same value of ϵ, with different initial conditions. The first begins with initial conditions $(y, \tilde{p}) = (0.1, 0.1)$, the second with $(3, 3)$.

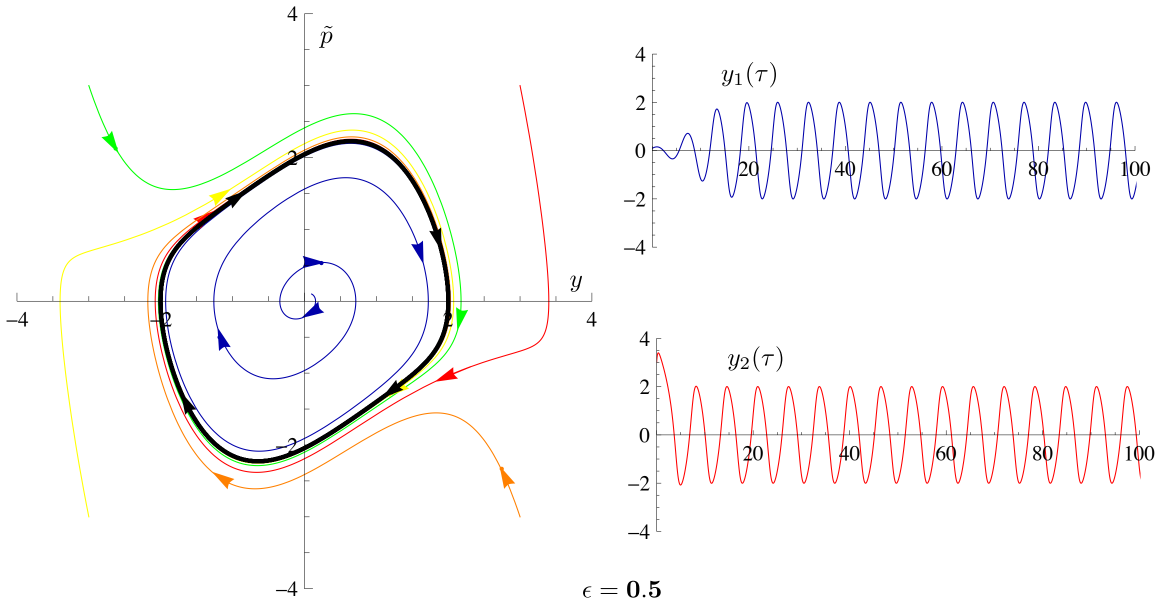

Figure 32.6: On the left, the phase diagram of the van der Pol oscillator with $\epsilon = 0.5$. the limit cycle is shown in solid black. On the right, two solutions $y(\tau)$ for the same value of ϵ, with different initial conditions. The first begins with initial conditions $(y, \tilde{p}) = (0.1, 0.1)$, the second with $(3, 3)$.

exponentially growing $y(\tau)$ and $\tilde{p}(\tau)$–in fact, very near this fixed point the phase diagram should look essentially like figures 32.1, 32.2, and 32.3, except the mirror image, and the arrows reversed. This is a new kind of unstable fixed point (like the attractor was new), which we will call a *repulsor*.

In figure 32.6, we show the same graphs for $\epsilon = 0.5$. From the plots of $y(\tau)$, we can start to see some (barely) visible deviations from a sine wave in the steady state, oscillatory behavior. We also see that with the larger value of ϵ, the steady state is approached much more quickly, with the transitory behavior almost entirely gone by $\tau = 20$ here, while before it persisted to around $\tau = 80$ for equivalent initial conditions. In the phase diagram, we can see very clearly the distortion of the limit cycle–it's beginning to look more like a diamond than a circle.

In figure 32.7, we increase the value of ϵ to $\epsilon = 1.5$. Now the steady state oscillatory solutions $y(\tau)$ are noticeably distorted, looking a little bit more like a sawtooth. In phase space, the limit cycle no longer looks remotely circular. We also notice that whatever initial conditions

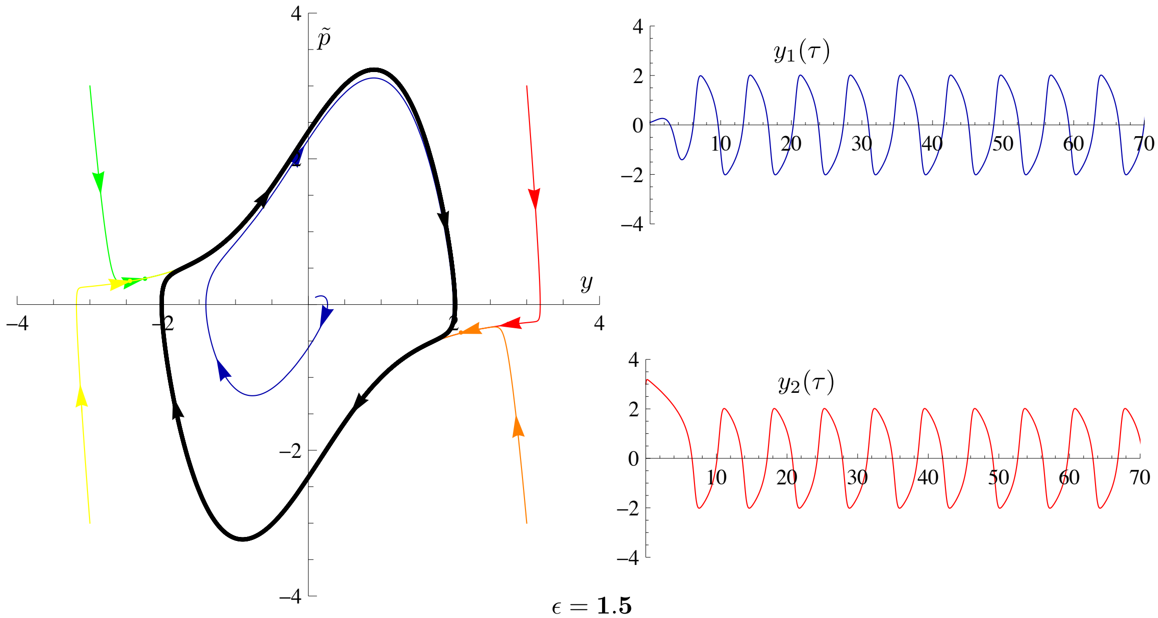

Figure 32.7: On the left, the phase diagram of the van der Pol oscillator with $\epsilon = 1.5$. the limit cycle is shown in solid black. On the right, two solutions $y(\tau)$ for the same value of ϵ, with different initial conditions. The first begins with initial conditions $(y, \tilde{p}) = (0.1, 0.1)$, the second with $(3, 3)$.

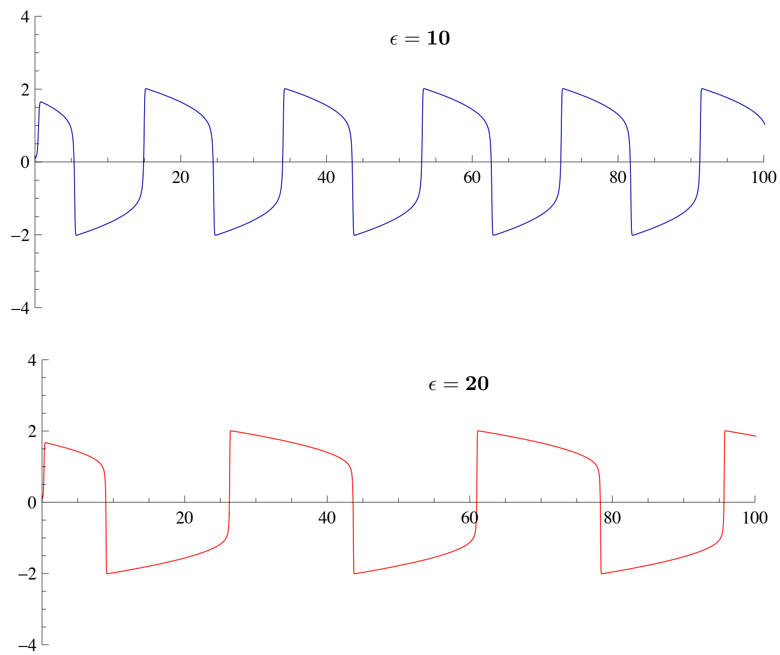

Figure 32.8: Two solutions $y(\tau)$ for very large values $\epsilon = 10$ and $\epsilon = 20$. Both use the initial conditions $(y, \tilde{p}) = (0.1, 0.1)$. The solutions are clearly becoming more like square waves for very large ϵ.

we give the system, it approaches the limit cycle along one of two “approach curves” on either side of the diamond shape. Finally, figure 32.8 shows a solution $y(\tau)$ for two very large values of ϵ: $\epsilon = 10$ and $\epsilon = 20$. From this we can see that the motion of the mass is approaching a square wave with amplitude 2 for very large values of epsilon. Notice that the larger ϵ is, the larger the period of the square wave. In phase space this would make a very elongated diamond shape for a limit cycle.

32.3.3 Lienard Variables

Let’s see if we can make some qualitative sense out of what is happening in this limit. We can do this through a change of variables. First we introduce the function

$$X(y) = \frac{1}{3}y^3 - y \tag{32.21}$$

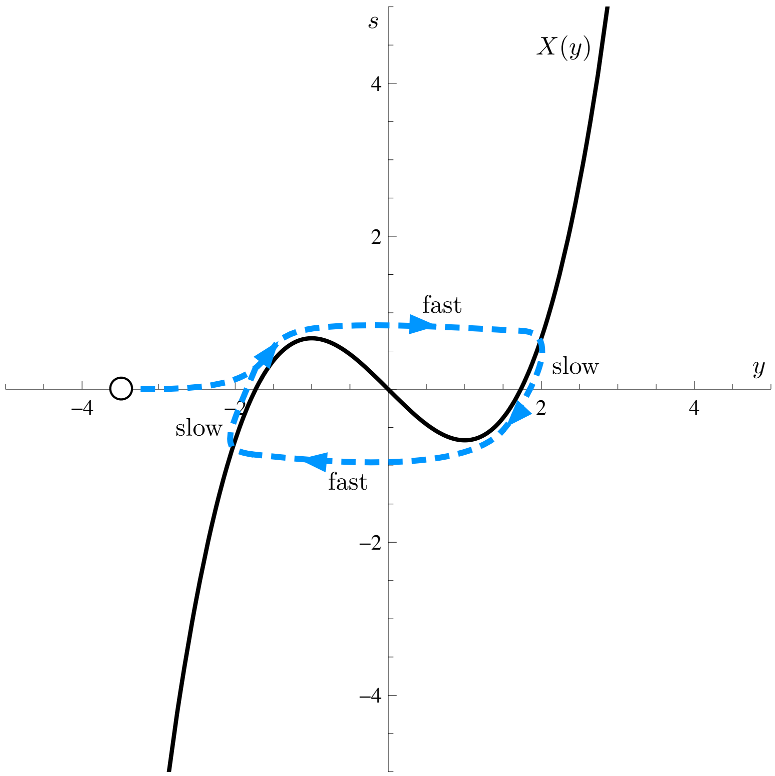

Figure 32.9: The path of a mass in the (y, s) plane.

and the new variable

$$s = \frac{1}{\epsilon}(\tilde{p} + \epsilon X) \tag{32.22}$$

Because s depends on $\tilde{p}$, we can think of (y, s) as being (deformed) coordinates in phase space. Using equation 32.18, they satisfy the relations

$$\dot{s} = -\frac{y}{\epsilon}, \qquad \dot{y} = \epsilon[s - X(y)] \tag{32.23}$$

Now, consider the phase space (y, s) with the curve $s = X(y)$ plotted on it (figure **??**). Remember that ϵ is a large number (where our numerics are telling us the steady state solutions are square waves.)

Suppose we start out with initial conditions such that we are somewhere above the curve $X(y)$. In this case, $s - X(y) > 0$, which means that $\dot{y}$ is positive. If, on the other hand, we start out below the curve, we have $\dot{y}$ negative. Remembering that we are really interested in the steady state behavior, for which the initial conditions should be fairly irrelevant, let us suppose we begin with $s = 0$ and y to the left of the curve (negative), not too far away. Then $\dot{s}$ is small (because ϵ is large) and positive, and $\dot{y}$ is positive. The path of the particle must therefore move to the right and slightly upward, towards the curve.

As we approach the curve, we will have $s - X(y)$ get smaller, and thus $\dot{y}$ gets smaller–then the path in phase space angles up more. We are now moving *very slowly* along a trajectory that is very close to the curve and points upward and to the right. Because both $\dot{y}$ and $\dot{s}$ are small, this portion of the path in phase space takes a long time.

Eventually, the path moves up far enough that the curve $X(y)$ bends down and away from it. Then $s - X(y)$ gets large again, so $\dot{y}$ becomes large, and the path moves quickly horizontally across the phase space plane. During this part, y goes from negative to positive, which means that $\dot{s}$ changes sign as well–it goes from positive to negative. As the path approaches the curve again, $\dot{y}$ approaches zero, and eventually the $X(y)$ curve is crossed, with $\dot{y} = 0$ and $\dot{s} < 0$. Now below the curve, we have $\dot{y} < 0$ and $\dot{s} < 0$, and we are in the symmetric position to where we started. The bottom half of the path is then the same (except opposite) from the top half, and then we go back to the top half, and so on.

This is the limit cycle, presented in (y, s) space. In terms of interpreting this as a square wave, note that the legs where $\dot{s}$ and $\dot{y}$ are both very small take a long time to traverse. Thus for a very long time y is approximately constant, near ± 2. The legs where y is changing significantly have $\dot{y}$ large (either positive or negative) so they are traversed very quickly–thus y changes very quickly from $+2$ to -2, or the other way around. This is the “square wave” profile!