Linear Approximations

Lecture 28

Physics 311: Classical Mechanics
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We are now moving on from central force systems and beginning the fourth section of the course: nonlinear dynamics. Most of the physical systems in the world around us (and many of the systems we have encountered in the course so far) result in nonlinear differential equations: if we want to fully understand the motions of the system, we should understand what the solutions to these equations look like. But most of the differential equations we know how to solve are *linear*. A second order linear differential equation is always in the form

$$A(t)\ddot{x} + B(t)\dot{x} + C(t)x = 0.$$

What is nice about a differential equation like this one is that if we have two solutions $x_1(t)$ and $x_2(t)$, then in general any combination in the form

$$x(t) = ax_1(t) + bx_2(t)$$

will also be a solution. This allows us to build an entire family of solutions from just knowing one or two. For example, with the simple harmonic oscillator equation

$$\ddot{x} + \omega^2 x = 0$$

we typically solve the problem by finding two different solutions $x_1(t) = \cos\omega t$ and $x_2(t) = \sin\omega t$ and then we know that

$$x(t) = a\cos\omega t + b\sin\omega t$$

will also be a solution. Since a second order differential equation's solution is determined by two boundary conditions, we can then typically find a and b to fit any set of boundary conditions we desire–and we are done. One of the most interesting (or frustrating, depending on your point of view) things about nonlinear systems is that knowing one, two, or even a

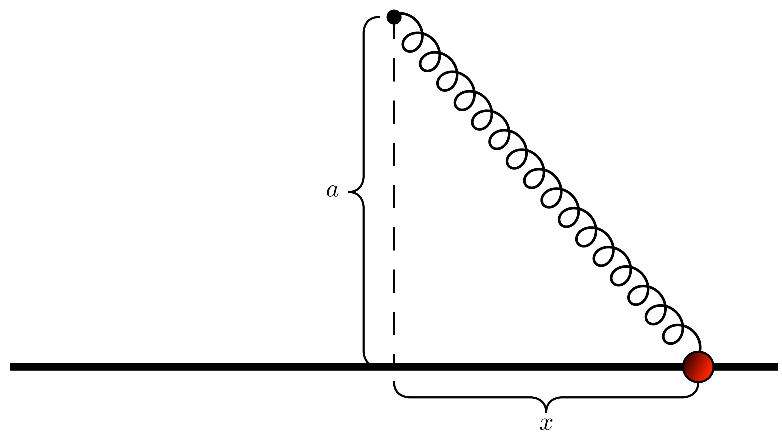

Figure 28.1: The bead slides on a rod and is attached to a spring whose other end is fixed.

large set of solutions doesn't necessarily tell you anything about other solutions. Sometimes multiple solutions can be found with wildly different behaviors. Sometimes, we find a system where even very similar initial conditions will lead (in the far future) to wildly disparate motions. In such cases, we refer to the system as *chaotic*.

For the next few weeks we are going to discuss what tools we *can* use to study fundamentally nonlinear systems, when our traditional analytical skills fail us. These include qualitative analysis, perturbation theory, and phase diagrams.

28.1 The Potential For a Nonlinear System

We will often use variations on oscillator systems to discuss and illustrate examples. This is for two reasons: first, we are all very familiar with the basic simple harmonic oscillator and having it as a touchstone will be helpful, and second, nature abounds with systems we can model, at least in some regime, as being some type of oscillator.

For our first example, consider a bead of mass m that slides back and forth along a straight rod (figure 28.1). This bead is also attached to a spring with spring constant k whose other end is fixed. There are two distance scales in the problem: a, the perpendicular distance from the rod to the fixed point of the spring, and ℓ the equilibrium extent of the spring. If we use x to describe the position of the bead along the rod, the extension of the spring is

$\sqrt{x^2+a^2}$, and this makes the potential

$$V(x) = \frac{1}{2}k\left[\sqrt{x^2+a^2}-\ell\right]^2. \tag{28.1}$$

The equation of motion for this system is

$$m\ddot{x} = -\frac{\partial V}{\partial x}$$

which gives

$$m\ddot{x} = -\frac{kx\left(\sqrt{x^2+a^2}-\ell\right)}{\sqrt{x^2+a^2}} \tag{28.2}$$

a clearly nonlinear equation.

The motion of this system depends qualitatively on the relative sizes of a and ℓ. For example, we can see that there will always be an equilibrium point at $x=0$, where the spring exerts a force perpendicular to the rod. However, if $a<\ell$ this force pulls the bead towards the fixed point, so if the bead is slightly displaced it will tend to return to $x=0$, while if $a>\ell$, this force pushes the bead away from the fixed point, so if the bead is slightly displaced it will tend to move further off $x=0$. We want to begin by examining this in more detail.

28.1.1 Non-dimensionalization

First we non-dimensionalize the potential, using ℓ as the fundamental length scale. This will make the number of symbols we have to keep track of slightly smaller, as well as highlighting the essentials of the system. We can write

$$V(x) = \frac{1}{2}k\ell^2\left[\sqrt{\left(\frac{x}{\ell}\right)^2+\left(\frac{a}{\ell}\right)^2}-1\right]^2$$

and then we define

$$y=\frac{x}{\ell}, \qquad b=\frac{a}{\ell}, \qquad \tilde{V}=\frac{V}{k\ell^2} \tag{28.3}$$

so that the non-dimensionalized potential is

$$\tilde{V}(y) = \frac{1}{2}\left[\sqrt{y^2+b^2}-1\right]^2 \tag{28.4}$$

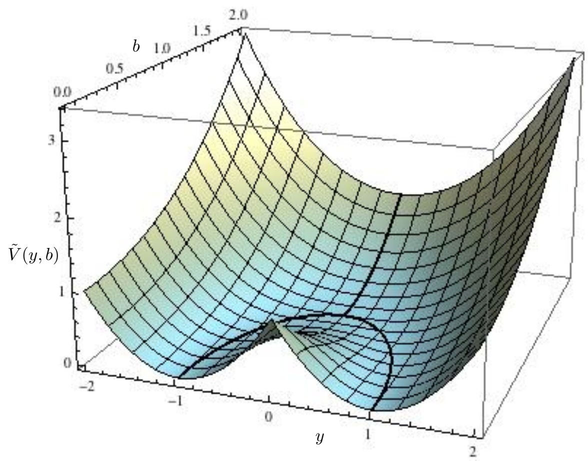

Figure 28.2: The potential plotted as a function of $\frac{x}{\ell}$ and $\frac{a}{\ell}$. The minima if the potential (as a function of $\frac{a}{\ell}$) is shown as a thick black line.

Using these definitions, we can rewrite the equation of motion as

$$m\ell\ddot{y} = -\frac{k\ell y\left(\sqrt{y^2+b^2}-1\right)}{\sqrt{y^2+b^2}}$$

or

$$\ddot{y} = -\frac{k}{m}\frac{\partial \tilde{V}}{\partial y}$$

If we further non-dimensionalize time as

$$t = t_0\tilde{\tau}, \qquad t_0 = \sqrt{\frac{m}{k}} \tag{28.5}$$

then we get

$$\ddot{y} = \frac{d^2y}{d\tilde{\tau}^2} = -\frac{\partial \tilde{V}}{\partial y} \tag{28.6}$$

(in what follows, we will always use dots to represent derivatives with respect to $\tilde{\tau}$.)

28.1.2 Extrema

A plot of the non-dimensionalized potential is shown in figure 28.2. In order to analyze it, let's consider where the minima and maxima of this potential are. We calculate the first derivative as

$$\frac{\partial \tilde{V}}{\partial y} = \frac{y\left(\sqrt{b^2+y^2}-1\right)}{\sqrt{b^2+y^2}} \tag{28.7}$$

Extrema y_0 of the potential occur where this vanishes, so we have

$$0 = \frac{y_0\left(\sqrt{b^2+y_0^2}-1\right)}{\sqrt{b^2+y_0^2}}$$

There are multiple ways to have this equation satisfied. $y_0 = 0$ is always a solution. However, under some circumstances, $1 = \sqrt{b^2+y_0^2}$ can also yield a solution: specifically, when $b < 1$. Remember that b is the ratio of a (the distance from the fixed point of the spring to the rod) to ℓ (the equilibrium extent of the rod). When $b < 1$, the equilibrium extent of the spring is longer than its extent when the bead is at the center, which we earlier argued would make $y_0 = 0$ an unstable equilibrium point. This has to do with whether an extremum of the potential is a maximum or a minimum, so to discuss this we need to calculate the second derivative of the potential. This is

$$\frac{\partial^2 \tilde{V}}{\partial y^2} = 1 - \frac{b^2}{(y^2+b^2)^{3/2}}. \tag{28.8}$$

Now we will look at each extremum in each case separately.

Case 1: $b > 1$

If we have $b > 1$, then there is only one extremum for the minimum: $y_0 = 0$. In this situation we have

$$\left.\frac{\partial^2 \tilde{V}}{\partial y^2}\right|_{y_0=0} = 1 - \frac{1}{b} > 0$$

so this is a minimum, making this equilibrium point stable.

Case 2: $b < 1$

If $b < 1$, we have three extrema for the potential:

$$y_0 = 0, \qquad y_0 = \sqrt{1-b^2}, \qquad y_0 = -\sqrt{1-b^2} \tag{28.9}$$

at $y_0 = 0$, the second derivative is

$$\left.\frac{\partial^2 \tilde{V}}{\partial y^2}\right|_{y_0=0} = 1 - \frac{1}{b} < 0$$

so this is now a maximum. At each of the other two we have

$$\left.\frac{\partial^2 \tilde{V}}{\partial y^2}\right|_{y_0=\pm\sqrt{1-b^2}} = 1 - b^2 > 0$$

so these are minima. In figure 28.2, the locations of the minima of $\tilde{V}(x)$ is shown as a function of b as a solid black line. We can see that for $b > 1$ there is only one minimum, at $y = 0$, but when b reaches $b = 1$, this line bifurcates into two, which for $b < 1$ branch out on either side of a maximum that forms at $y = 0$.

28.2 Linearized Motion

A common approach to analyzing the motion of the bead in this system is to assume that the bead's motion consists of a small oscillation near one of the minima of the potential. In this situation we can approximate the potential using a Taylor series expansion

$$\tilde{V}(z + y_0) = \tilde{V}(y_0) + \left.\frac{\partial \tilde{V}}{\partial y}\right|_{y_0} z + \frac{1}{2}\left.\frac{\partial^2 \tilde{V}}{\partial y^2}\right|_{y_0} z^2 + \cdots \tag{28.10}$$

Since we are at a minimum, the second term in this expansion vanishes, and we get

$$\tilde{V}(z + y_0) = \tilde{V}_0 + \frac{1}{2}\left(\left.\frac{\partial \tilde{V}}{\partial y^2}\right|_{y_0}\right) z^2 + \cdots \tag{28.11}$$

if z is small enough, we can ignore any further terms. Furthermore, the dynamics of the system will not care about the constant term $\tilde{V}_0$, since the force is proportional to the gradient of the potential. Physically, we know that the potential energy is always only defined up to a constant–what point we choose to identify with $V = 0$ is always arbitrary. And the second derivative $\tilde{V}''(y_0)$ is just a constant. Thus, we are saying we can approximate the potential near y_0 as just a quadratic potential–which we know leads to simple harmonic motion.

This procedure would work for any potential with a minimum, except those special cases for which $\tilde{V}''(y_0) = 0$ (the second derivative of the potential *also* vanishes where the first derivative vanishes.) This is at the root of why so many physical systems are modeled using simple harmonic motion–basically *no* real-world system is exactly a simple harmonic oscillator, but *almost any* real-world system can be approximated as one near a (stable) equilibrium point.

Case 1: $b > 1$ near $y_0 = 0$

First let's look at the case $b > 1$ near the minimum $y_0 = 0$. Here we have

$$\tilde{V}(z + y_0) = \tilde{V}(0) + \frac{1}{2}\left(1 - \frac{1}{b}\right) z^2 + \cdots$$

or

$$\tilde{V}(z + y_0) \approx \frac{(b-1)^2}{2} + \frac{(b-1)}{2b} z^2 \tag{28.12}$$

And since $\ddot{y} = \ddot{z}$ and $\frac{\partial \tilde{V}}{\partial y} = \frac{\partial \tilde{V}}{\partial z}$, our (approximate) equation of motion becomes

$$\ddot{z} = -\frac{b-1}{b} z = -\tilde{\omega}^2 z \tag{28.13}$$

which is a simple harmonic oscillator whose frequency is determined in terms of b. Putting back in the dimensionful parameters, the frequency will be

$$\omega = \frac{\tilde{\omega}}{t_0} = \sqrt{\frac{k}{m}\left(\frac{a-\ell}{a}\right)} \tag{28.14}$$

and essentially all of the information about the shape of the potential influences these equations in only two ways: the oscillations occur near the location of the minimum, and the frequency is determined by the second derivative of the potential at the minimum. This second derivative is often referred to as the *curvature* of the potential. The equation of motion for z, unlike our original equation of motion for y (equation 28.6), is linear. Because of this, we refer to this approximation technique as *linearization*.

Case 2: $b < 1$ near $y_0 = \pm\sqrt{1-b^2}$

Now let's look at the case $b < 1$ near either of the minima $y_0 = \pm\sqrt{1-b^2}$. Here we have

$$\tilde{V}(z + y_0) = \tilde{V}(0) + \frac{1}{2}\left(1 - b^2\right) z^2 + \cdots$$

or

$$\tilde{V}(z+y_0) = \frac{1}{2}(1-b^2)z^2 \tag{28.15}$$

which gives the equation of motion

$$\ddot{z} = -(1-b^2)z = -\tilde{\omega}^2 z \tag{28.16}$$

so that the frequency of the small oscillations about these minima are

$$\omega = \frac{\tilde{\omega}}{t_0} = \sqrt{\frac{k}{m}\left(1-\frac{a^2}{\ell^2}\right)} \tag{28.17}$$

and we see that the small oscillations in this case are different in that they are oscillating about different points, and with a different frequency.

Case 3: $b < 1$ **near** $y_0 = 0$

Let us briefly talk about what happens if you attempt this procedure near a maximum of the potential. If we have $b < 1$ then $y_0 = 0$ is a maximum, as we have shown. In this case, the potential can be approximated as

$$\tilde{V}(z+y_0) = \tilde{V}(0) + \frac{1}{2}\left(1-\frac{1}{b}\right)z^2 + \cdots$$

which is

$$\tilde{V}(z+y_0) = \frac{(b-1)^2}{2} - \frac{(1-b)}{2b}z^2 \tag{28.18}$$

and this gives equation of motion

$$\ddot{z} = \left(\frac{1-b}{b}\right)z = \lambda^2 z, \qquad \lambda = \sqrt{\frac{1-b}{b}} \tag{28.19}$$

This has exponential solutions, so a generic solution looks like

$$z(\tilde{\tau}) = Ae^{\lambda\tilde{\tau}} + Be^{-\lambda\tilde{\tau}}$$

For generic initial conditions A will be non-zero, which will ensure that over time exponential growth with dominate. Our entire approach relies on being able to assume that z is small– the motion stays close to the equilibrium point. But this shows us that even if z starts out small, it very shortly will not be–so we shouldn't really trust our solution past the first very short amount of time. In contrast, we had oscillating solutions before, and for them would can always limit z by the amplitude of the oscillation–if the amplitude is small, then z will be small for all time. We refer to this type of equilibrium point as an unstable equilibrium point, and we generally don't use linearization to analyze it beyond the basic observation that it *is* unstable.

28.3 Fundamentally Nonlinear Behavior

Consider for a moment the case $a = \ell$, where the distance from the rod to the fixed point of the spring is exactly the equilibrium extent of the spring. In our nondimensionalized equations this is $b = 1$. In this case, we get only one minimum $y_0 = 0$ (this is the point of bifurcation, where the one minimum becomes two). However, we also have

$$\frac{\partial^2 \tilde{V}}{\partial y^2}\Big|_{y_0=0} = 1 - \frac{1}{(y_0^2+1)^{3/2}} = 1 - \frac{1}{1} = 0$$

which means that if we are approximating our potential using a Taylor series, we must go out farther than quadratic order to get non-trivial dependence on $\tilde{y}$. We can calculate

$$\frac{\partial^3 \tilde{V}}{\partial y^3}\Big|_{y_0=0} = \frac{3b^2 y_0}{(y_0^2+b^2)^{5/2}} = 0 \tag{28.20}$$

which also vanishes, so we need

$$\frac{\partial^4 \tilde{V}}{\partial y^4}\Big|_{y_0=0} = \frac{3b^2(b^2-4y_0^2)}{(b^2+y_0^2)^{7/2}} = \frac{3b^4}{b^7} = \frac{3}{b^3}. \tag{28.21}$$

Now, putting all the pieces together for $b = 1$, we write the Taylor series approximation of $\tilde{V}(y)$ near $y_0 = 0$ as

$$\tilde{V}(z+y_0) = \frac{1}{24}\frac{\partial^4 \tilde{V}}{\partial y^2}\Big|_{y_0} z^4 + \cdots \approx \frac{1}{8}z^4 \tag{28.22}$$

(Note that *all* of the lower terms in the Taylor series, even $\tilde{V}(y_0)$, vanish in this case.) This makes the equation of motion

$$\ddot{z} = -\frac{1}{2}z^3 \tag{28.23}$$

This is fundamentally nonlinear–there is no way to approximate it using linear techniques. On the other hand, in this case there are still some tricks we can use. First we notice that the force, though not linear, is a restoring force–for positive z the force makes z want to decrease and for negative z the force makes z want to increase–towards zero. Therefore the motion will be oscillatory–though not in a simple way. Thus we have reason to trust that if we start out z small, it will never get too large, and we can trust the solutions to equation 28.23 even at late times.

Second we notice that this system has energy conservation. (In fact, the original system, before we made any approximations, did as well.) This means we should be able to replace this second order differential equation with a first order differential equation. In order to do this, we (as usual) multiply both sides by $\dot{\tilde{y}}$ and discover we have total derivatives of time:

$$\ddot{z}\dot{z} = -\frac{1}{2}z^3\dot{z}$$

is also

$$\frac{d}{d\tilde{\tau}}\left[\frac{1}{2}\dot{z}^2\right] = -\frac{d}{d\tilde{\tau}}\left[\frac{1}{8}z^4\right]$$

so that we can write

$$\frac{d}{d\tilde{\tau}}\left[\frac{1}{2}\dot{z}^2 + \frac{1}{8}z^4\right] = 0$$

which can also be written

$$\frac{1}{2}\dot{z}^2 + \frac{1}{8}z^4 = \tilde{E} \tag{28.24}$$

(and in fact the constant $\tilde{E}$ is actually a nondimensionalized energy.) This can be solved for $\frac{dz}{d\tilde{\tau}}$ to give

$$\frac{dz}{d\tilde{\tau}} = \pm\frac{1}{2}\sqrt{8\tilde{E} - z^4} \tag{28.25}$$

and this we can (at least in principle) solve using direct integration. We write

$$d\tilde{\tau} = \pm\frac{2dz}{\sqrt{8\tilde{E} - z^4}}$$

and then we integrate to obtain

$$\tilde{\tau}(z) = \pm\int\frac{2dz}{\sqrt{8\tilde{E} - z^4}} \tag{28.26}$$

which in principle can be inverted to give $z(\tilde{\tau})$. Now, this actual integral is an elliptic integral, so at this point we are forced to rely on numerics to give an actual path–however, we have in some sense solved the problem. One common question we might ask about an oscillating system is: what is its period? For this system we can find the answer by performing the integral over definite endpoints. We will suppose we start with initial conditions $z(0) = \tilde{A}$, $\dot{z}(0) = 0$ (that is, with the mass at its farthest point from the center. Then the period $\tilde{T}$ is four times the amount of time it takes the mass to reach the center:

$$\int_0^{\tilde{T}/4} d\tilde{\tau} = -\int_{\tilde{A}}^0 \frac{2dz}{\sqrt{8\tilde{E} - z^4}}$$

(just as in radial infall, we now take the negative root because we know the mass is falling towards the center–decreasing z with time.) Using initial conditions, we also have

$$\tilde{E} = \frac{1}{8}\tilde{A}^4$$

so this is

$$\frac{\tilde{T}}{4} = 2\int_0^{\tilde{A}} \frac{dz}{\sqrt{\tilde{A}^4 - z^4}} = \frac{2}{\tilde{A}^2}\int_0^{\tilde{A}} \frac{dz}{\sqrt{1 - \frac{z^4}{\tilde{A}^4}}}$$

with the substitution

$$s = \frac{z}{\tilde{A}}, \qquad dz = \tilde{A}\ ds$$

we get

$$\tilde{T} = \frac{8}{\tilde{A}}\int_0^1 \frac{ds}{\sqrt{1 - s^4}} = \frac{8 \times 1.31103}{\tilde{A}} = \frac{10.4882}{\tilde{A}}$$

where in the last step, having extracted all the dependence on $\tilde{A}$ from the integral (so that it is simply a number) we perform the integral numerically. With the dimensionful parameters put back in, assuming the initial value of $x = \ell y$ is $A = \ell\tilde{A}$, we finally obtain the period

$$T = t_0\tilde{T} = \frac{10.4882\,\ell}{A}\sqrt{\frac{m}{k}}$$

The most important feature of this period is its dependence on A: the simple harmonic oscillator famously has a period that is independent of amplitude–but this oscillator has a period that varies inversely with amplitude: the larger the initial displacement, the shorter the period. This is shown in figure 28.3.

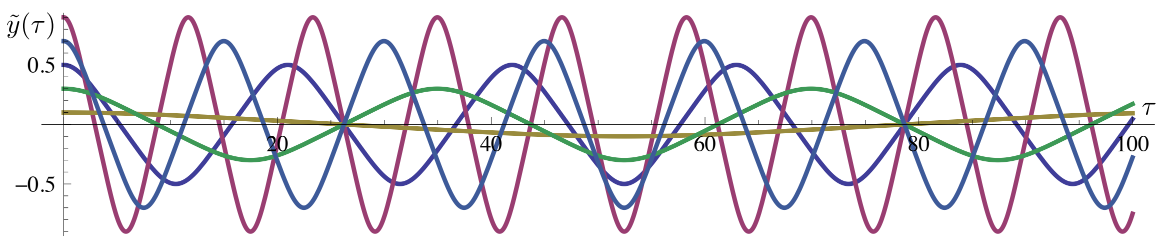

Figure 28.3: The functions $z(\tilde{\tau})$ for a variety of initial displacements $\tilde{A}$. Note that as the amplitude grows, the period decreases.