The Logistic Map

Lecture 34

Physics 311: Classical Mechanics
Fall 2011

30 November 2011

Last time we introduced the concept of the discrete map by discussing the Poincaré map. This can be thought of as a series of snapshots of the phase diagram of a physical system at even intervals in time, so that we get discrete points in the phase space. We generally look at Poincaré maps when a system either has periodic solutions, or where the equation of motion itself has periodic time dependence in it, and we use these periods as the time lapse between snapshots. For example, a simple harmonic oscillator always returns back to the original position and velocity after a full period, so the Poincaré map, using the period of the oscillator as the time lapse, will consist of a single point.

More interesting are cases where the equation of motion is periodic itself, such as with a damped driven harmonic oscillator. Then we use the period of the driving force as the time lapse. Then we know that the transient oscillating (at the natural frequency of the system) damps out over time, leaving the steady state oscillation at the driving frequency. In the Poincaré map, the steady state solution by itself gives a fixed point in the map, and the dying out of the transient solution appears as a convergence from any point in the phase space to the fixed point, after a large number of snapshots.

Now, we also discussed more generally the concept of the discrete map, which is a function that takes a discrete point x and replaces it with another discrete point $\Phi(x)$:

$$\Phi : \qquad x \quad \longrightarrow \quad \Phi(x) \tag{34.1}$$

In these terms, the action of the Poincaré map is to time-evolve a point in phase space forward by a fixed period. Furthermore, the point x is actually a point on the phase plane (so it has two components, position and momentum). We then spent a great deal of time discussing and classifying possible behaviors of the Poincaré map near fixed points, determining what would make a fixed point stable, or unstable.

34.1 Definition of the Logistic Map

Today we are going to study a particular example of a discrete map, where we are mapping a 1-dimensional point to another 1-dimensional point. Although this may seem like a rather stupid, simplistic idea, it can actually have some very interesting behaviors and will provide our first introduction to chaos. The particular mapping F_a will depend on a parameter a, and we will have

$$F_a : \qquad x \quad \longrightarrow \quad F_a(x) \tag{34.2}$$

with

$$F_a(x) = ax(1-x) \tag{34.3}$$

This is called the *logistic map*, and generally we discuss this with $x \in [0,1]$. First let's look at the shape of the function $F_a(x)$. We have

$$F_a(0) = F_a(1) = 0, \qquad F_a(x) > 0, \quad \text{for} \quad 0 < x < 1, \quad a > 0$$

It is quadratic in x, so it has one extremum $\tilde{x}$:

$$F_a'(\tilde{x}) = a(1-\tilde{x}) - a\tilde{x} = a - 2a\tilde{x} = 0, \qquad \Rightarrow \qquad \tilde{x} = \frac{1}{2}$$

and this must be a maximum. At this extremum we have

$$F_a(\tilde{x}) = F_a\left(\frac{1}{2}\right) = \frac{a}{4}$$

Now, if we want to think of F_a as producing an iterative mapping:

$$x_{k+1} = F_a(x_k) = ax_k(1-x_k) \tag{34.4}$$

where all of the x_k stay in the interval $[0,1]$, then we must have

$$0 < F_a(x) < 1, \qquad x \in [0,1]$$

which then means we must have

$$0 < a < 4 \tag{34.5}$$

The shape of the function $F_a(x)$ is shown in figure 34.1 for $a = 1,2,3,4$. What is fascinating about the discussion we are about to have about the mapping built from this function is something called *universality*. We have been very specific here in our definition of F and how it depends on the parameter a, but the qualitative behaviors we will find (and even

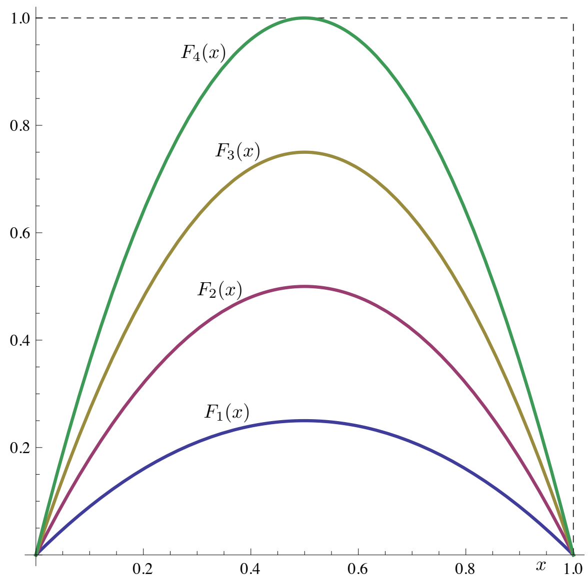

Figure 34.1: The function $F_a(x)$ on the unit interval $x \in [0, 1]$, for $a = 1, 2, 3, 4$.

some of the quantitative behaviors) turn out to be strikingly independent of the details of $F_a(x)$. For example, here are two other mappings we could use, that would have quite similar implications:

$$\begin{aligned} G_a(x) &= xe^{a(x-1)} \\ H_a(x) &= \begin{cases} ax & \text{if} \quad x < \frac{1}{2} \\ a(1-x) & \text{if} \quad x > \frac{1}{2} \end{cases} \end{aligned}$$

Now, remember that we are using this function to define a mapping, with equation 34.4. Thus, starting from a point x_0, we generate a series of points:

$$\{x_0, x_1, x_2, \dots\} = \{x_0, F_a(x_0), F_a[F_a(x_0)], \dots\} \tag{34.6}$$

We can picture the action of this graphically in a straightforward way. We take a plot of $F_a(x)$, and start at some point x_0 on the x-axis. then we go up vertically until we intersect the function $F_a(x)$. Then we move across horizontally back to the y-axis. This is $F_a(x_0) = x_1$ on the y-axis. We then repeat the process start at x_1 on the x-axis, and so on. This is shown in figure 34.2 on the left.

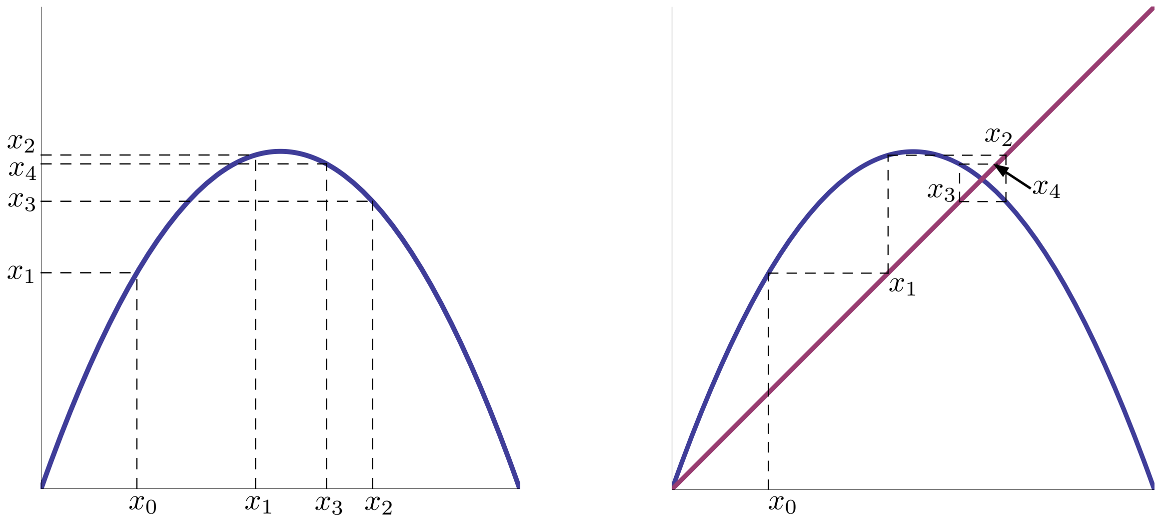

Figure 34.2: The first few points in the series generated by $x_0 = 0.2$ with $a = 2.8$.

Or, even better, we can take a plot of $F_a(x)$ and show it against a plot of just the straight line diagonal $f(x) = x$. Then, we start on the x-axis at some point x_0, and move vertically until we hit $F_a(x)$. Then we move horizontally (staying at height $F_a(x_0) = x_1$) until we hit the diagonal $f(x) = x$. At this point we are at the point (x_1, x_1). We then move vertically until we hit $F_a(x_1)$ and then again horizontally (staying at height $F_a(x_1) = x_2$ until we hit the diagonal, at the point (x_2, x_2). And so on. This is shown in figure 34.2 on the right.

34.2 Fixed Points

We now want to determine what the fixed points of this mapping are, and whether or not they are stable. A fixed point $\bar{x}$ will satisfy the equation

$$\bar{x} = F_a(\bar{x}) = a\bar{x}(1 - \bar{x}) \tag{34.7}$$

This gives us the quadratic equation

$$a\bar{x}^2 + (1 - a)\bar{x} = \bar{x}[a\bar{x} + 1 - a] = 0$$

which has two solutions:

$$\bar{x}_1 = 0, \qquad \bar{x}_2 = \frac{a-1}{a}$$

Notice that this second solution only gives a point on the unit interval if we have $a \geq 1$. What can we say about the stability of these fixed points? We analyze this exactly the same way we analyzed it for the stability of fixed points in the Poincaré map, only it is easier here because our points are one-dimensional. We look at

$$x = \bar{x} + \epsilon$$

for a small distance ϵ away from a fixed point $\bar{x}$. In this case

$$F_a(x) = F_a(\bar{x} + \epsilon) = F_a(\bar{x}) + F_a'(\bar{x})\epsilon = \bar{x} + F_a'(\bar{x})\epsilon$$

where we use the Taylor series to make our usual approximation. Then, out of a series x_k we can make a series $\epsilon_k = x_k - \bar{x}$, with

$$\epsilon_{k+1} = x_{k+1} - \bar{x} = F_a(x_k) - \bar{x} = F_a(\bar{x} + \epsilon_k) - \bar{x} = \bar{x} + F_a'(\bar{x})\epsilon_k - \bar{x} = F_a'(\bar{x})\epsilon_k$$

We want to know if this series goes to zero for k large, or not. If we start with a displacement ϵ_0 from the fixed point, we have

$$\epsilon_k = [F_a'(\bar{x})]^k \epsilon_0$$

so the entire issue is what $F_a'(\bar{x})$ is. Remember we have

$$F_a'(x) = a - 2ax \tag{34.8}$$

$\bar{\mathbf{x}}_1 = 0$ Fixed Point

First we will look at the fixed point at the origin. Plugging this in gives

$$F_a'(\bar{x}_1) = a \tag{34.9}$$

so that near this fixed point we have the series

$$\epsilon_k = a^k \epsilon_0 \tag{34.10}$$

It's clear from this that if $a < 1$ this will be a stable fixed point, but for $a > 1$ (the same region where we have the second fixed point) it will be unstable. This is shown in figure 34.3.

What happens with $a = 1$ is not clear from this analysis, but it does turn out to be stable: In this case we have

$$F_1(x) = x - x^2$$

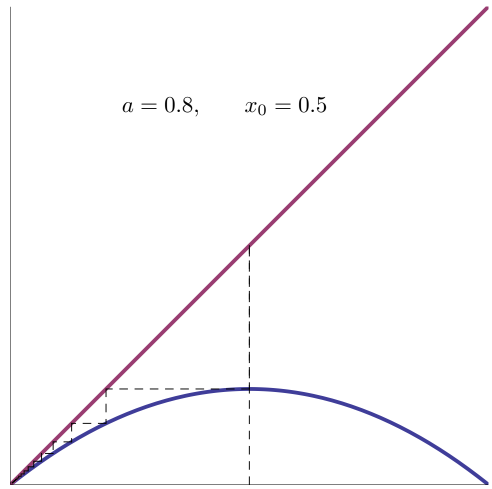

Figure 34.3: Convergence to the first fixed point for $a = 0.8$.

and for a number less than 1, we always have

$$F_1(x) < x$$

(so the series, starting at any point on the unit interval, must decrease until we hit the origin.)

$\mathbf{\bar{x}_2 = \frac{a-1}{a}}$ Fixed Point

Now let's consider the second fixed point (assuming that we have $a > 1$, so that it exists and is distinct from the first fixed point.) In this case we have

$$F_a'(\bar{x}_2) = a - 2a\bar{x}_2 = a - 2(a-1) = 2 - a \tag{34.11}$$

so that near this fixed point we have the series

$$\epsilon_k = (2-a)^k \epsilon_0 \tag{34.12}$$

I will break down the possibilities for this into three sections.

1. $\mathbf{1 < a \leq 2}$:

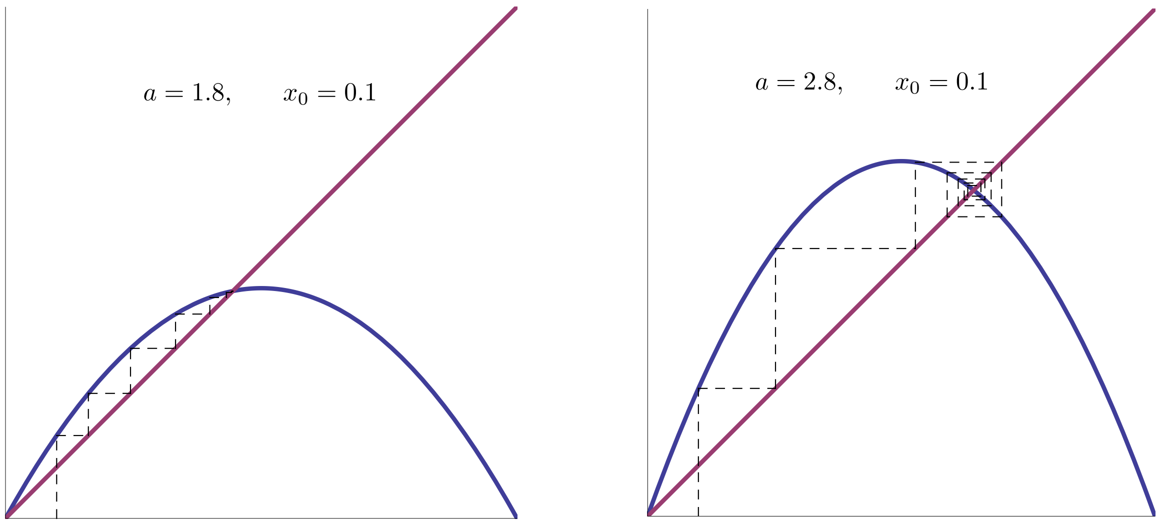

Figure 34.4: Convergence to the second fixed point for $a = 1.8$ on the left, and for $a = 2.8$ on the right.

If we have $1 < a < 2$, then $0 < 2 - a < 1$. This means two things: first, we always have

$$|\epsilon_{k+1}| < |\epsilon_k|$$

so that the series of points converges to the fixed point. Second, if we start with $\epsilon_0 > 0$, then the entire series will have $\epsilon_k > 0$, and conversely if $\epsilon_0 < 0$, then the entire series will have $\epsilon_k < 0$. This means that if we start on one side close to the fixed point, we will stay on that side as we converge to it (although the first few points may not satisfy this, if we do not start out particularly close to the fixed point.) This is shown on the left in figure 34.4.

In the special case where $a = 2$, the linearization near $\bar{x}$ isn't quite good enough to understand what happens, but here we find

$$\epsilon_{k+1} = F_2(\bar{x}_2 + \epsilon_k) - \bar{x}_2 = F_2\left(\frac{1}{2} + \epsilon_k\right) - \frac{1}{2} = 2\left(\frac{1}{2} + \epsilon_k\right) - 2\left(\frac{1}{2} + \epsilon_k\right)^2 - \frac{1}{2} = -2\epsilon_k^2$$

so that after ϵ_0, which can be either positive or negative, each successive ϵ_k after that must be negative, which means the convergence stays on one side of the fixed point.

2. $\mathbf{2 < a \leq 3}$:

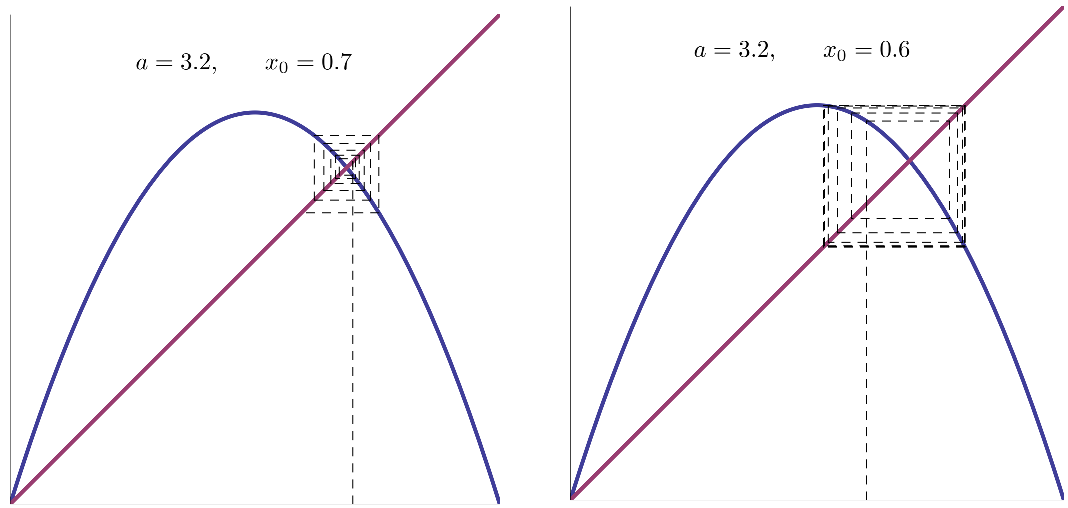

Figure 34.5: Divergence from the second fixed point for $a = 3.2$. Here, the fixed point should be at 0.6875. On the left, we start at 0.7, and we see that the mapping moves us away from this point. On the right, we start at 0.6, and see that after many iterations the mapping starts flipping us back and forth between two points.

If we have $2 < a < 3$, then $-1 < 2 - a < 0$. Again we have

$$|\epsilon_{k+1}| < |\epsilon_k|$$

so again the series of points converges to the fixed point. But this time if we start with $\epsilon_0 > 0$, then we will have $\epsilon_1 < 0$, and $\epsilon_2 > 0$, and so on, so we will move back and forth across the fixed point as we converge to it. This is shown on the right in figure 34.4.

In the special case where $a = 3$, our linearized treatment of the fixed point again fails, but in fact we will see that the fixed point stays stable.

3. $\mathbf{3 < a \leq 4}$:

   If we have $3 < a \leq 4$, then $2 - a < -1$. This means that we will have

   $$|\epsilon_{k+1}| > |\epsilon_k|$$

   so that the fixed point is unstable, and a small separation from it grows after repeated mappings. This is shown in figure 34.5 on the left. The transition that occurs at $a = 3$ turns out to be a very interesting feature for the logistic map.

34.3 Bifurcation

Let's consider what happens past the point $a = 3$. If we take the mapping farther on, beyond our analysis near the fixed point, we can see that something interesting occurs. We don't converge to a single point, but rather we start to alternate between two. We can see this in figure 34.5 on the right. Can we show analytically that this behavior exists? Suppose we consider what happens after two consecutive mappings, so that we define

$$F_a^2(x) \equiv F_a \circ F_a(x) = aF_a(x)[1 - F_a(x)] = a^2x(1-x)[1-ax(1-x)] \tag{34.13}$$

If our assessment of what is occurring in figure 34.5 is correct, we should look for fixed points of $F_a^2(x)$–because if the mapping is creating an alternation between two points, then doing the mapping twice should keep us fixed at one or the other of these points. A fixed point $\bar{x}$ of $F_a^2(x)$ should satisfy

$$\bar{x} = a^2\bar{x}(1-\bar{x})[1-a\bar{x}(1-\bar{x})]$$

which we can simplify as

$$\begin{aligned} 0 &= a^2\bar{x}(1-\bar{x})[1-a\bar{x}(1-\bar{x})] - \bar{x} \\ &= a^2\bar{x}(1-\bar{x})[1-a\bar{x}+a\bar{x}^2] - \bar{x} \\ &= a^2\bar{x}[1-\bar{x}-a\bar{x}+a\bar{x}^2+a\bar{x}^2-a\bar{x}^3] - \bar{x} \\ &= \bar{x}[a^2-1-a^2(1+a)\bar{x}+2a^3\bar{x}^2-a^3\bar{x}^3] \\ &= -\bar{x}[a\bar{x}-a+1][a^2\bar{x}^2-a^2\bar{x}-a\bar{x}+a+1] \end{aligned}$$

where we are helped out in working out the factorization by the fact that a fixed point of $F_a(x)$ must also be a fixed point of $F_a^2(x)$–so we must be able to factor out a $\bar{x}(a\bar{x}-a+1)$. In addition to the fixed points of $F_a(x)$, we now have two new fixed points, which are solutions to the quadratic equation

$$a^2\bar{x}^2 - a(a+1)\bar{x} + a + 1 = 0$$

so they are

$$\bar{x}_{\pm} = \frac{a(a+1) \pm \sqrt{a^2(a+1)^2 - 4a^2(a+1)}}{2a^2} = \frac{a+1 \pm \sqrt{(a+1)(a-3)}}{2a}$$

Notice that for $a < 3$ this doesn't have real solutions, so these two fixed points don't exist. Exactly at $a = 3$, these two fixed points are actually the same:

$$\frac{a+1}{2a} = \frac{2}{3}$$

which is the same as the second fixed point of $F_a(x)$:

$$\frac{a-1}{a} = \frac{2}{3}$$

So exactly at $a = 3$, these are all the same point. Then slightly above $a = 3$ we get three separate fixed points for $F_a^2(x)$. Let's also analyze their stability, by calculating $(F_a^2)'(\bar{x}_\pm)$. We note that

$$F_a^2(x) = a^2 x - a^2(1+a)x^2 + 2a^3x^3 - a^3x^4$$

so

$$(F_a^2)'(x) = a^2 - 2a^2(1+a)x + 6a^3x^2 - 4a^3x^3$$

and we have

$$\begin{aligned}
(F_a^2)'(\bar{x}_\pm) &= a^2 - 2a^2(1+a)\bar{x}_\pm + 6a^3\bar{x}_\pm^2 - 4a^3\bar{x}_\pm^3 \\
&= a^2 - 2a^2(1+a)\bar{x}_\pm + 6a^3\bar{x}_\pm^2 - 4a\bar{x}_\pm[a^2\bar{x}_\pm^2] \\
&= a^2 - 2a^2(1+a)\bar{x}_\pm + 6a^3\bar{x}_\pm^2 - 4a\bar{x}_\pm[a(a+1)\bar{x}_\pm - a - 1] \\
&= a^2 - 2a^2(1+a)\bar{x}_\pm + 6a^3\bar{x}_\pm^2 + 4a(a+1)\bar{x}_\pm - 4a^2(a+1)\bar{x}_\pm^2 \\
&= a^2 + 2a(a+1)(2-a)\bar{x}_\pm + 2a^2(a-2)\bar{x}_\pm^2 \\
&= a^2 - 2a(a+1)(a-2)\bar{x}_\pm + 2(a-2)[a^2\bar{x}_\pm^2] \\
&= a^2 - 2a(a+1)(a-2)\bar{x}_\pm + 2(a-2)[a(a+1)\bar{x}_\pm - a - 1] \\
&= a^2 - 2(a-2)(a+1) \\
&= 4 + 2a - a^2
\end{aligned}$$

Based on our earlier analysis, we know that the fixed point will be stable if this quantity has magnitude less than 1:

$$|4 + 2a - a^2| < 1$$

If we plot it in the region $3 < a < 4$ that we are interested, we see that it start out equal to 1, and then decreases over the range, ending up at -4. (Figure 34.6.) For a close to 3 but slightly larger, the fixed points $\bar{x}_\pm$ of $F_a^2(x)$ will be stable. For some value of a we have $4 + 2a - a^2 = -1$, and for a larger than this they will be unstable. We see that

$$4 + 2a - a^2 = -1 \qquad \rightarrow \qquad a = 1 + \sqrt{6} = 3.44949$$

(there is another root, but it gives a negative value of a.)

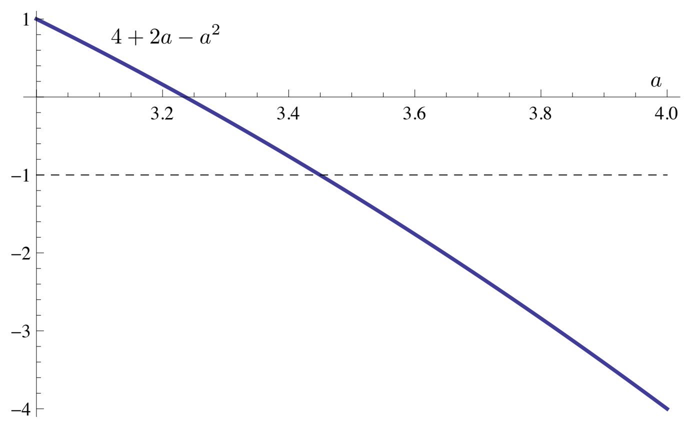

Figure 34.6: A plot of $4 + 2a - a^2$ in the region $a \in [3, 4]$

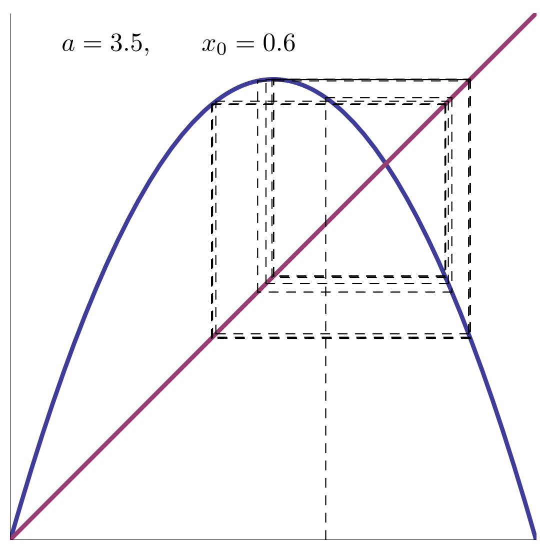

Figure 34.7: The mapping with $a = 3.5$ and $x_0 = 0.6$.

Now, consider what we have. For $a \leq 1$ there is a single stable fixed point at $x = 0$. For $1 < a \leq 3$, there are two fixed points, one at $x = 0$ that is unstable, and another at $x = \frac{a-1}{a}$ that is stable. For $3 < a < 1 + \sqrt{6}$, neither fixed point is stable, but after many iterations the mapping begins to alternate between two points, so that $F_a^2(x)$ has two fixed points, $\bar{x}_\pm$, both of which are stable, but for $1 + \sqrt{6} < a$, they too become unstable. If we plot the mapping for a just above this, say for $a = 3.5$ we notice that now after many iterations, instead of alternating between two points, we are alternating between four!

It is not hard to guess how to proceed from there: we consider

$$F_a^4(x) \equiv F_a \circ F_a \circ F_a \circ F_a(x) \tag{34.14}$$

and look for fixed points of it that aren't fixed points of $F_a^2(x)$. We should find four (which we would have to solve a quartic equation in order to do.) They will be stable between $a = 1 + \sqrt{6}$ and some new value of a, at which point they will also become unstable, and we will end up alternating between eight points. And so on, and so on.

34.4 Descent into Chaos

We're starting to get a good sense for how this works. When we have $a \in [1, 3]$ there is a single non-zero fixed point of $F_a(x)$, and it is stable. At a critical value $a_0 = 3$ suddenly the fixed point becomes unstable, but we obtain two stable fixed points for $F_a^2(x)$. This is called *period doubling bifurcation*, because after many iterations we must apply F_a twice to come back to our original position. This is true for $a \in [3, 1 + \sqrt{6}]$, but at a second critical value $a_1 = 1 + \sqrt{6}$ they become unstable and we obtain four new stable fixed points for $F_a^4(x)$ (again an example of period doubling bifurcation. This gives us a series of critical values

$$\begin{array}{rcl} a_0 & = & 3 \\ a_1 & = & 1 + \sqrt{6} = 3.4495 \\ a_2 & = & 3.5441 \\ \vdots & \vdots & \vdots \end{array} \tag{34.15}$$

But what happens as we continue increasing a? Does the pattern of bifurcation continue until we hit $a = 4$? Famously, it does not. Instead, the critical values of a approach a limit

$$a_\infty = \lim_{n \to \infty} a_n \approx 3.5699456 \tag{34.16}$$

Beyond this point, we generally don't have stable fixed points of $F_a^k(x)$ for any k. This means that, starting with an arbitrary initial point x_0, even after a large number of mappings N

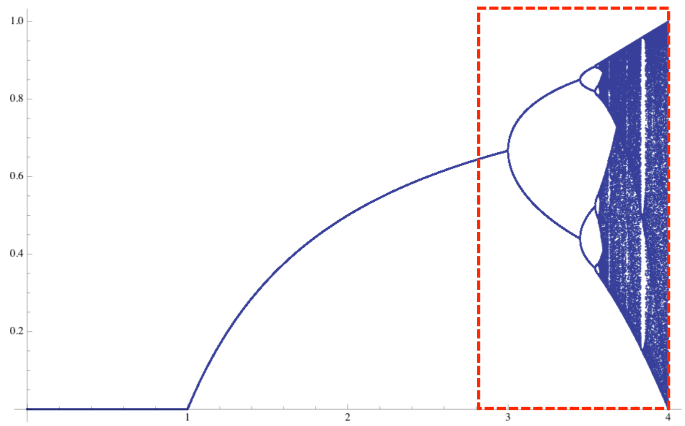

Figure 34.8: A plot of the "final endpoints" for a range of x_0, plotted against a in the region $a \in [0, 4]$. the red outlined section is "zoomed in on" in figure 34.9.

we will end up at some random point x_N that doesn't converge. This behavior is sometimes called *quasi-chaotic*, though it is not strictly speaking chaotic.

Suppose we consider a series of iterative mappings, starting from initial points

$$x_0 \in \{0.01, 0.02, 0.03, \ldots, 0.99\}$$

and look at where they end up after a large number of mappings (say 1000). Furthermore, we do this for a range of values of $a \in [0, 4]$. For each value of a we have a collection of 99 final ending points. For $a < 1$ we know all of these final ending points will be very close to zero, since this is the stable fixed point. For $1 < a < a_0$, we know that all of these final ending points will be very close to the stable fixed point $\frac{a-1}{a}$. For $a_0 < a < a_1$, about half of them should end up at one of the stable fixed points of F_a^2, and the other half at the other stable fixed point. And so on. Then we plot these "final ending points" as a function of a. This will give us a graphical representation of the bifurcation process, and its breakdown at a_∞.

A full map of the region $a \in [0, 4]$ is shown in figure 34.8. We can clearly see the behavior with respect to a up through the first couple of bifurcations. Beyond this, we need to "zoom in" to get a better look. This is done in figure 34.9. Notice that even above a_∞, we occasionally have regions where $F_a^k(x)$ again has stable fixed points, so that the graph has vertical stripes of mostly empty space. In fact, if we zoom on the largest such stripe, what is revealed is another set of bifurcations, leading to the return of "chaos." This is shown in figure 34.10.

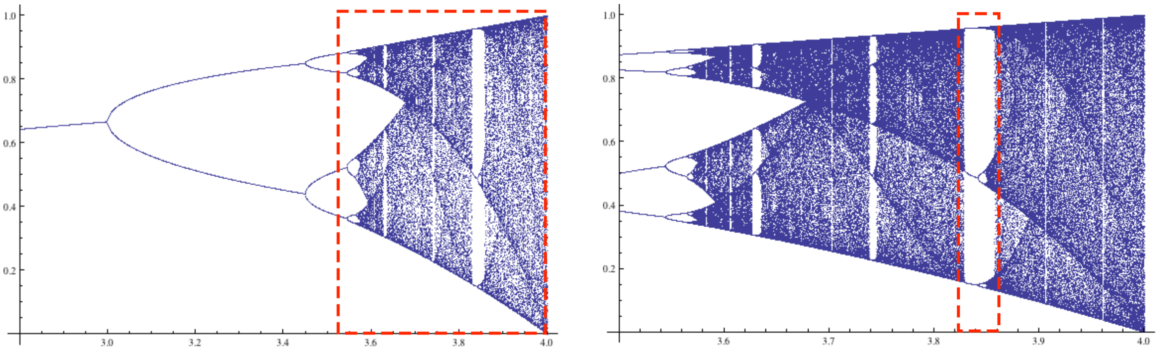

Figure 34.9: A plot of the "final endpoints" for a range of x_0, plotted against a. On the left, we plot in the region $a \in [2.8, 4]$, where we get a good view of the first few bifurcations. On the right, we plot in the region $a \in [3.5, 4]$ which gives us a better view of the breakdown into "chaos" that occurs above a_∞. The red dotted region on the left is shown in closer detail on the right; the red dotted region on the right is shown in closer detail in figure 34.10.

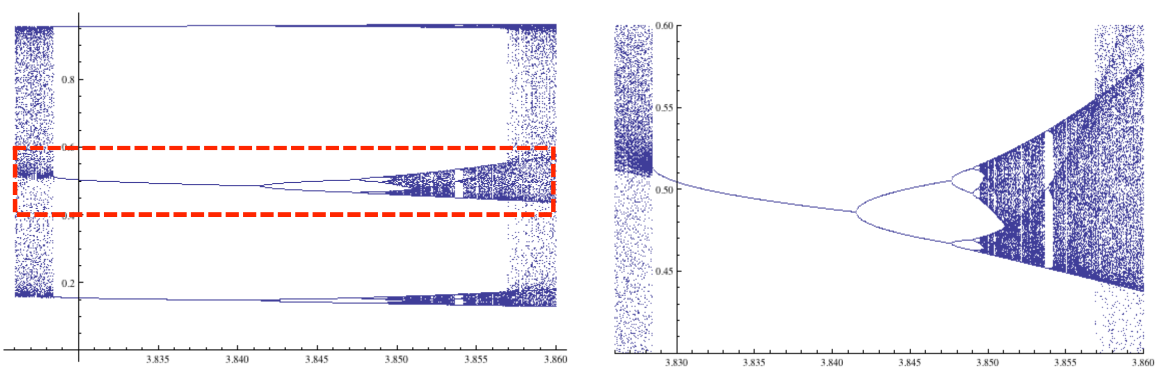

Figure 34.10: A plot of the "final endpoints" for a range of x_0, plotted against a. On the left, we plot in the region $a \in [3.82, 3.86]$, where we see another region of stable fixed points and bifurcations. On the right, we zoom in vertically to get a better view of these bifurcations.