The Kicked Rotator System

Lecture 36

Physics 311: Classical Mechanics
Fall 2011

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We spent last week discussing *discrete maps* and their relationships to physical systems. The Poincaré map is perhaps the most important discrete map to us, and describes the way a system evolves in phase space in discrete time steps. Our first example of a discrete map beyond this was the logistic map, which was a simple mapping of one dimensional points to one dimensional points. This gave us a chance to explore fixed points and bifurcation, as well as a type of chaotic behavior. Next we considered the circle map, another mapping of one dimensional points to one dimensional points, which can be thought of as a simplified version of the damped, driven pendulum. There we explored more fixed points, and the idea of frequency locking.

Today we want to study another system by using discrete maps. This is called the *kicked rotator* system. When we construct the Poincar e map for this system, we will create a discrete map of 2-dimensional points to 2-dimensional points, that is simple enough to study in the same way we've been analyzing our other mappings, but leads to very interesting behaviors, including chaos.

36.1 Physical Motivation

Imagine we have a two-dimensional rigid body which rotates around a fixed point, but *without* gravity. It does not really matter what shape the rigid body is, just that it has some moment of inertia I as it rotates around the fixed point. If it is left to rotate, in absence of gravity or any other force acting other than at the fixed point, the angular momentum will be conserved:

$$J = I\dot{\phi} = \text{ conserved} \tag{36.1}$$

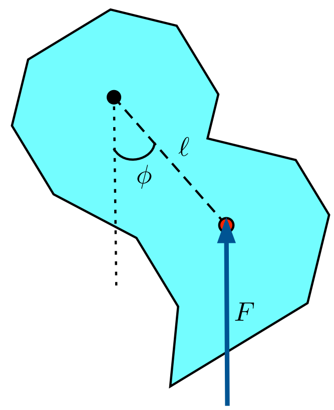

Figure 36.1: The rigid body with fixed point and impulse force applied at a point distance ℓ away from the fixed point, always in the same direction.

as will the Hamiltonian:

$$E = H = \frac{1}{2} I \dot{\phi}^2 = \frac{J^2}{2I} = \text{ conserved} \tag{36.2}$$

Now suppose that we perturb the system by providing a force at regular periodic intervals, a "kick" of fixed magnitude and intervals. We will idealize this as a delta function, an impulse that acts at the instants in time $\{T, 2T, 3T, \dots\}$, although of course this is a rather non-physical simplification. Now, if the force of magnitude F always acts on the same point, which is a distance ℓ away from the fixed point, then it provides a torque which will change depending on the angle the body is at:

$$\tau = F\ell \sin\phi \tag{36.3}$$

(see figure 36.1.) In this case, the perturbed Hamiltonian (which will give the correct equations of motion) will be

$$H = \frac{J^2}{2I} + F\ell\cos\phi \sum_{n=0}^{\infty} \delta(t - nT) \tag{36.4}$$

Then we obtain

$$\begin{aligned} \dot{\phi} &= \frac{\partial H}{\partial J} &&= \frac{J}{I} \\ \dot{J} &= -\frac{\partial H}{\partial \phi} &&= F\ell\sin\phi \textstyle\sum_{n=0}^{\infty} \delta(t - nT) \end{aligned} \tag{36.5}$$

We see that most of the time we have $\dot{J} = 0$ (angular momentum *between* kicks is conserved), but at the exact moments nT, $\dot{J}$ is infinite. However, it is infinite in such a way that the angular momentum changes by a discrete amount time we move from $t = nT - \varepsilon$ to $t = nT + \varepsilon$:

$$\Delta J = \int_{nT-\varepsilon}^{nT+\varepsilon} \dot{J}\, dt = F\ell \sin\phi \int_{nT-\varepsilon}^{nT+\varepsilon} \delta(t - nT)\, dt = F\ell \sin\phi \tag{36.6}$$

Now let's make a Poincaré map with the phase space (ϕ, J). The above change means that the nth kick changes the angular momentum from J_n to J_{n+1} with

$$J_{n+1} = J_n + F\ell \sin\phi_n$$

where ϕ_n is the angle the rigid body is at at time nT. Then, considering the first of Hamilton's equations, the angle ϕ increases at a regular rate between times nT and $(n+1)T$ so that at time $(n+1)T$ we have

$$\phi_{n+1} = \left(\phi_n + \frac{J_{n+1}T}{I}\right) \bmod 2\pi$$

With appropriate non-dimensionalization, we can simplify this somewhat to form the mapping

$$\begin{aligned} \phi_{n+1} &= (\phi_n + J_{n+1}) \bmod 2\pi \\ J_{n+1} &= J_n + \epsilon \sin\phi_n \end{aligned}$$

which we can also write as

$$Z_\epsilon : \quad (\phi_n, J_n) \;\rightarrow\; Z_\epsilon(\phi_n, J_n) \equiv (\phi_{n+1}, J_{n+1}) \tag{36.7}$$

This, then, is the discrete map we are interested in studying today. We know that the points ϕ are periodically identified: we want to say $\phi = \phi + 2\pi$. What is interesting is that we can *also* periodically identify J without changing the dynamics in any real way. Note that if we consider two starting points (J_0, ϕ_0) and (J_0', ϕ_0'), with $J_0' = J_0 + 2\pi$ and $\phi_0' = \phi_0$, then we will have

$$\phi_1 = (\phi_0 + J_0) \bmod 2\pi$$

$$J_1 = \epsilon \sin\phi_0 + J_0$$

and

$$\phi_1' = (\phi_0' + J_0') \bmod 2\pi = (\phi_0 + J_0 + 2\pi) \bmod 2\pi = (\phi_0 + J_0) \bmod 2\pi = \phi_1$$

$$J_1' = J_0' + \epsilon \sin\phi_0' = J_0 + 2\pi + \epsilon \sin\phi_0 = J_1 + 2\pi$$

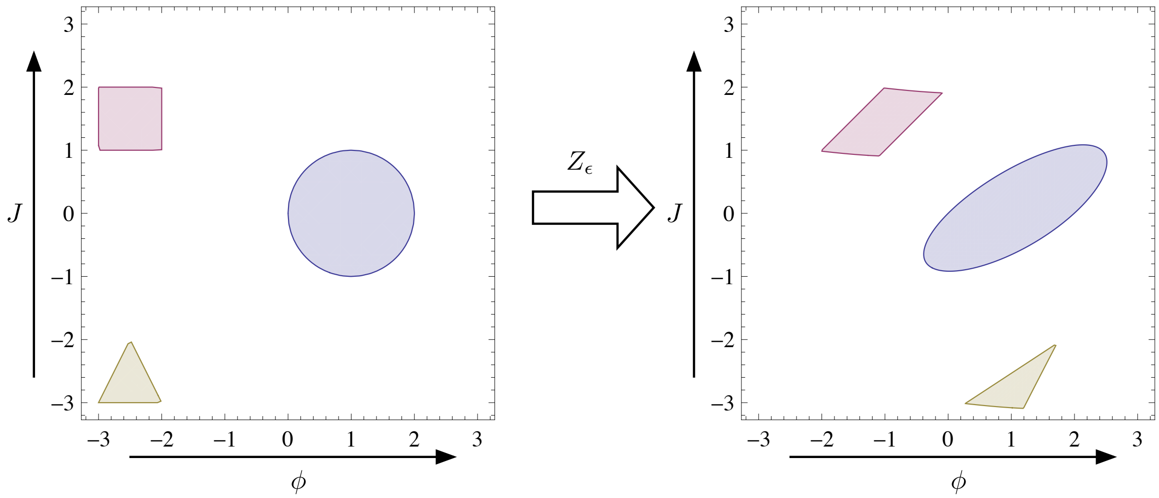

Figure 36.2: The mapping of Z_ϵ from one region of (ϕ, J) space to another preserves area. (This particular mapping is $Z_{0.1}$.)

if we continued this way, we would always have $\phi'_n = \phi_n$ and $J'_n = J_n + 2\pi$, so nothing really changes if we periodically identify *both* ϕ and J, and restrict both to the region $[-\pi, \pi]$:

$$\begin{aligned} \phi_{n+1} &= (\phi_n + J_{n+1}) \bmod 2\pi \\ J_{n+1} &= (J_n + \epsilon \sin \phi_n) \bmod 2\pi \end{aligned} \tag{36.8}$$

and

$$-\pi \le \phi \le \pi, \qquad -\pi \le J \le \pi \tag{36.9}$$

36.2 An Area Preserving Map

Now, when $\epsilon \neq 0$, we *don't* have conserved energy in this system–every time the rigid body is *kicked*, we impart energy to it. However, this mapping is sometimes called conservative anyways. It conserves something else: *area*. Imagine a region V of (ϕ, J) space, and then the mapping of this region under the mapping Z_ϵ: $\tilde{V}$ (that is, we take the set of all points inside the original region, and we act on these points with Z_ϵ, and then the set of all points that are the results of these mappings is the region $\tilde{V}$. I claim that the areas of region V and $\tilde{V}$ are the same.

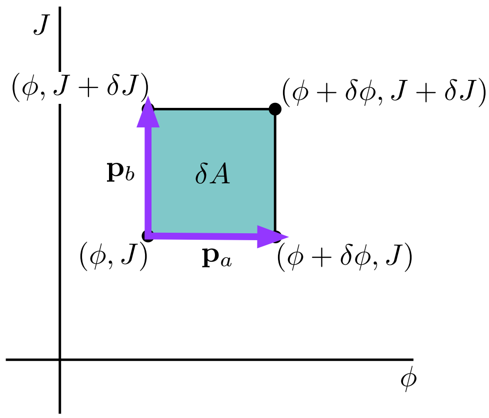

Figure 36.3: The infinitesimal area before we perform the mapping

How should we go about proving this? The easiest way to do it is infinitesimally, and since we haven't considered this before, I will set it up as generally as possible, only using the particulars of this mapping at the last minute. Imagine a tiny region–a square defined by the points

$$(\phi, J), \qquad (\phi + \delta\phi, J), \qquad (\phi, J + \delta J), \qquad (\phi + \delta\phi, J + \delta J)$$

We know the area of this region is $\delta A = \delta\phi\delta J$. We can think of this in terms of two infinitesimal vectors

$$\mathbf{p}_a = (\delta\phi, 0) = (p_a^\phi, p_a^J), \qquad \mathbf{p}_b = (0, \delta J) = (p_b^\phi, p_b^J)$$

that point from the corner (ϕ, J) to each adjacent corner. Then the area between these two vectors will be

$$\delta A = |\mathbf{p}_a \times \mathbf{p}_b| = |p_a^\phi p_b^J - p_a^J p_b^\phi| = \left| \det \begin{bmatrix} p_a^\phi & p_b^\phi \\ p_a^J & p_b^J \end{bmatrix} \right| \tag{36.10}$$

This is all shown in figure 36.3. Now consider the mapping, which acts on the lower left corner as

$$(\phi, J) \to \Big(Z_\epsilon^\phi(\phi, J), Z_\epsilon^J(\phi, J) \Big)$$

and the two adjacent corners as

$$(\phi + \delta\phi, J) \to \Big(Z_\epsilon^\phi(\phi + \delta\phi, J), Z_\epsilon^J(\phi + \delta\phi, J) \Big) = \Big(Z_\epsilon^\phi(\phi, J), Z_\epsilon^J(\phi, J) \Big) + \Big(\frac{\partial Z_\epsilon^\phi}{\partial \phi}, \frac{\partial Z_\epsilon^J}{\partial \phi} \Big) \delta\phi$$

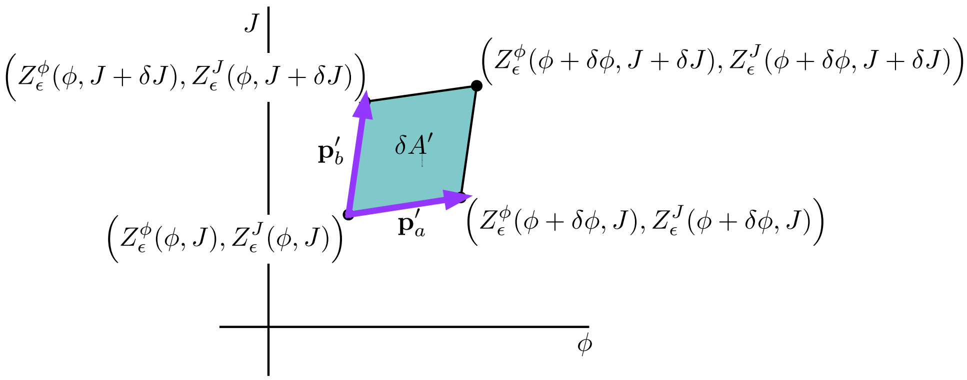

Figure 36.4: The infinitesimal area after we perform the mapping

and

$$(\phi, J+\delta J) \to \left(Z_\epsilon^\phi(\phi, J+\delta J), Z_\epsilon^J(\phi, J+\delta J) = \right) = \left(Z_\epsilon^\phi(\phi, J), Z_\epsilon^J(\phi, J)\right) + \left(\frac{\partial Z_\epsilon^\phi}{\partial J}, \frac{\partial Z_\epsilon^J}{\partial J}\right)\delta J$$

Now, we have a new infinitesimal area $\delta A'$, which we can think of in terms of the new vectors $\mathbf{p}'_a$ and $\mathbf{p}'_b$ that point from the mapping of the "lower left corner" $\left(Z_\epsilon^\phi(\phi, J), Z_\epsilon^J(\phi, J)\right)$ to the mappings of each adjacent corner. These are

$$\mathbf{p}'_a = \left(\frac{\partial Z_\epsilon^\phi}{\partial \phi}, \frac{\partial Z_\epsilon^J}{\partial \phi}\right)\delta\phi, \qquad \text{and} \qquad \mathbf{p}'_b = \left(\frac{\partial Z_\epsilon^\phi}{\partial J}, \frac{\partial Z_\epsilon^J}{\partial J}\right)\delta J$$

so we will have

$$\delta A' = |\mathbf{p}'_a \times \mathbf{p}'_b| = |p_a'^\phi p_b'^J - p_a'^J p_b'^\phi| = \left|\det \begin{bmatrix} p_a'^\phi & p_b'^\phi \\ p_a'^J & p_b'^J \end{bmatrix}\right| = \left|\det \begin{bmatrix} \frac{\partial Z_\epsilon^\phi}{\partial \phi}\delta\phi & \frac{\partial Z_\epsilon^\phi}{\partial J}\delta J \\ \frac{\partial Z_\epsilon^J}{\partial \phi}\delta\phi & \frac{\partial Z_\epsilon^J}{\partial J}\delta J \end{bmatrix}\right|$$

But, this is

$$\delta A' = \left|\frac{\partial Z_\epsilon^\phi}{\partial \phi}\frac{\partial Z_\epsilon^J}{\partial J}\delta\phi\delta J - \frac{\partial Z_\epsilon^\phi}{\partial J}\frac{\partial Z_\epsilon^J}{\partial \phi}\delta\phi\delta J\right| = \left|\left[\frac{\partial Z_\epsilon^\phi}{\partial \phi}\frac{\partial Z_\epsilon^J}{\partial J} - \frac{\partial Z_\epsilon^\phi}{\partial J}\frac{\partial Z_\epsilon^J}{\partial \phi}\right]\right|\delta\phi\delta J$$

This is all shown in figure 36.4. Notice that what we have written is actually

$$\delta A' = |\det \mathbb{A}|\delta A \tag{36.11}$$

where $\mathbb{A}$ is the same linearized mapping we defined last week, when we were studying the stability of fixed points:

$$\mathbb{A} = \begin{bmatrix} \frac{\partial Z_\epsilon^\phi}{\partial \phi} & \frac{\partial Z_\epsilon^\phi}{\partial J} \\ \\ \frac{\partial Z_\epsilon^J}{\partial \phi} & \frac{\partial Z_\epsilon^J}{\partial J} \end{bmatrix} \tag{36.12}$$

Therefore, the condition that the mapping preserves the area of the tiny infinitesimal region is the same as requiring that the determinant of the linearized mapping has a magnitude equal to one. And of course, if the mapping preserves the area of an infinitesimal region, it will preserve the area of any region, since we can break it down into infinitesimals and then add them up again. Thus we have

$$\text{Area Preserving Map:} \qquad |\det \mathbb{A}| = 1 \tag{36.13}$$

Now all we need to do to show that our map Z_ϵ is area preserving is to calculate the matrix $\mathbb{A}$ and take its determinant. We have

$$Z_\epsilon^\phi(\phi, J) = \phi + J + \epsilon \sin\phi, \qquad Z_\epsilon^J(\phi, J) = J + \epsilon \sin\phi$$

so this means

$$\mathbb{A} = \begin{bmatrix} 1 + \epsilon\cos\phi & 1 \\ \\ \epsilon\cos\phi & 1 \end{bmatrix} \tag{36.14}$$

and

$$\det \mathbb{A} = 1 + \epsilon\cos\phi - \epsilon\cos\phi = 1 \tag{36.15}$$

So, indeed, the mapping preserves areas!

36.3 Fixed Points

Now, as usual, we will think about the fixed points of this mapping. This requires

$$J = Z_\epsilon^J(\phi, J) = \Big(J + \epsilon\sin\phi\Big) \bmod 2\pi$$

and

$$\phi = Z_\epsilon^\phi(\phi, J) = \Big(\phi + J + \epsilon\sin\phi\Big) \bmod 2\pi = \Big(\phi + J\Big) \bmod 2\pi$$

From the second, it is clear that we need $J = 0$ in order to be a fixed point. From the first, we also need

$$\epsilon \sin\phi = 0 \bmod 2\pi$$

If $\epsilon = 0$ (which is when we allow the rigid body to rotate without kicking it), this is true for all ϕ, so that any point in the form $(\phi, 0)$ is a fixed point. This is not surprising: in this case the rigid body is simply stationary for all times, so the angle ϕ never changes. Now, for $0 < \epsilon < 2\pi$, the only way for this to be true is if we have $\phi = 0$ or $\phi = \pm\pi$. For $\epsilon > 2\pi$, there are other possibilities–but $(0, 0)$ and $(\pm\pi, 0)$ are still fixed points.

Now, let's think about what happens near these two fixed points, using the linearized map we have already worked out. But as with the area preservation calculation, we will keep things as general as possible, for as long as possible, so that our results will have greater applicability. What we are going to find is that area preserving maps have a particular structure near their fixed points.

36.3.1 Eigenvalues, Traces, and Determinants

Suppose we have a general linearized map $\mathbb{A}$ in the vicinity of a fixed point, for a two-dimensional mapping. $\mathbb{A}$ is a 2×2 matrix which we will assume can be diagonalized, and it turns out we can express its eigenvalues just in terms of the trace and determinant of $\mathbb{A}$. Now, when diagonalized $\mathbb{A}$ becomes the matrix $\tilde{\mathbb{A}}$,

$$\tilde{\mathbb{A}} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

which has

$$\mathrm{Tr}\,\tilde{\mathbb{A}} = \lambda_+ + \lambda_-, \qquad \det\tilde{\mathbb{A}} = \lambda_+\lambda_-$$

The interesting thing about this is that traces and determinants are invariant under change of basis, so this means we also have

$$T = \mathrm{Tr}\,\mathbb{A} = \lambda_+ + \lambda_-, \qquad D = \det\mathbb{A} = \lambda_+\lambda_- \tag{36.16}$$

but this is a system of equations we can solve for λ_+ and λ_-. We write

$$\lambda_- = T - \lambda_+$$

then

$$D = \lambda_+(T - \lambda_+), \quad \Rightarrow \quad \lambda_+^2 - T\lambda_+ + D = 0$$

this has two roots. One of them is actually associated with λ_-, so we can write

$$\lambda_\pm = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{T}{2} \pm \sqrt{\left(\frac{T}{2}\right)^2 - D} \tag{36.17}$$

Now, we will write our linearlized mapping as

$$\mathbb{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{36.18}$$

so that

$$T = a_{11} + a_{22}, \qquad D = a_{11}a_{22} - a_{12}a_{21} \tag{36.19}$$

36.3.2 Curves of the Linearized Mapping

Now, if we assume our mapping is area preserving, then we have

$$\lambda_\pm = \frac{T}{2} \pm \sqrt{\left(\frac{T}{2}\right)^2 - 1} \tag{36.20}$$

and

$$1 = a_{11}a_{22} - a_{12}a_{21} \tag{36.21}$$

Suppose we consider a linearized area preserving mapping near a fixed point $(\bar{\phi}, \bar{J})$, so that we map a point

$$(\phi, J) = (\bar{\phi} + \varepsilon^\phi, \bar{J} + \varepsilon^J)$$

to a point

$$(\phi', J') = (\bar{\phi} + \varepsilon'^\phi, \bar{J} + \varepsilon'^J)$$

with

$$\begin{bmatrix} \varepsilon'^\phi \\ \varepsilon'^J \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \varepsilon^\phi \\ \varepsilon^J \end{bmatrix} = \begin{bmatrix} a_{11}\varepsilon^\phi + a_{12}\varepsilon^J \\ a_{21}\varepsilon^\phi + a_{22}\varepsilon^J \end{bmatrix}$$

Now, we know that if $(\tilde{\varepsilon}^\phi, \tilde{\varepsilon}^J)$ is an eigenvector of $\mathbb{A}$, then it satisfies

$$a_{11}\tilde{\varepsilon}^\phi + a_{12}\tilde{\varepsilon}^J = \lambda_\pm \tilde{\varepsilon}^\phi = \left[\frac{T}{2} \pm \sqrt{\left(\frac{T}{2}\right)^2 - 1}\right] \tilde{\varepsilon}^\phi$$

which can be rewritten as

$$\left(a_{11}-\frac{T}{2}\right)\tilde{\varepsilon}^{\phi}+a_{12}\tilde{\varepsilon}^{J}=\pm\tilde{\varepsilon}^{\phi}\sqrt{\left(\frac{T}{2}\right)^{2}-1}$$

Suppose we square this expression: we know that this will eliminate the $\pm$ sign. Then we get

$$\left[\left(a_{11}-\frac{T}{2}\right)\tilde{\varepsilon}^{\phi}+a_{12}\tilde{\varepsilon}^{J}\right]^{2}+\left[1-\left(\frac{T}{2}\right)^{2}\right](\tilde{\varepsilon}^{\phi})^{2}=0 \tag{36.22}$$

or

$$\left(a_{11}-\frac{T}{2}\right)^{2}(\tilde{\varepsilon}^{\phi})^{2}+a_{12}^{2}(\tilde{\varepsilon}^{J})^{2}+2a_{12}\left(a_{11}-\frac{T}{2}\right)\tilde{\varepsilon}^{\phi}\tilde{\varepsilon}^{J}+\left[1-\left(\frac{T}{2}\right)^{2}\right](\tilde{\varepsilon}^{\phi})^{2}=0$$

and then remembering that $T=a_{11}+a_{22}$, this gives

$$\Big(1-a_{11}a_{22}\Big)(\tilde{\varepsilon}^{\phi})^{2}+a_{12}^{2}(\tilde{\varepsilon}^{J})^{2}+a_{12}(a_{11}-a_{22})\tilde{\varepsilon}^{\phi}\tilde{\varepsilon}^{J}=0$$

and using the fact that $D=1=a_{11}a_{22}-a_{12}a_{21}$ this is

$$a_{12}^{2}(\tilde{\varepsilon}^{J})^{2}-a_{12}a_{21}(\tilde{\varepsilon}^{\phi})^{2}+a_{12}(a_{11}-a_{22})\tilde{\varepsilon}^{\phi}\tilde{\varepsilon}^{J}=0$$

and finally

$$a_{12}(\tilde{\varepsilon}^{J})^{2}-a_{21}(\tilde{\varepsilon}^{\phi})^{2}+(a_{11}-a_{22})\tilde{\varepsilon}^{\phi}\tilde{\varepsilon}^{J}=0 \tag{36.23}$$

Now, remember what this equation is: it's a condition that any eigenvector $\tilde{\varepsilon}=(\tilde{\varepsilon}^{\phi},\tilde{\varepsilon}^{J})$ of the matrix $\mathbb{A}$ will satisfy. And we know that under the mapping an eigenvector $\tilde{\varepsilon}$ becomes

$$\tilde{\varepsilon}\qquad\rightarrow\qquad\mathbb{A}\tilde{\varepsilon}=\lambda_{\pm}\tilde{\varepsilon}$$

that is, it is simply *rescaled* by an amount $\lambda_{\pm}$. But a rescaled eigenvector is still an eigenvector–so the mapped point $\lambda_{\pm}\tilde{\varepsilon}$ satisfies the same equation. that is, all points that begin satisfying equation 36.23 also satisfy it after the mapping. What is interesting is that this is just a special case. If we *don't* start out with an eigenvector, then we won't satisfy equation 36.23. But we can write

$$a_{12}(\varepsilon^{J})^{2}-a_{21}(\varepsilon^{\phi})^{2}+(a_{11}-a_{22})\varepsilon^{\phi}\varepsilon^{J}=C \tag{36.24}$$

for some constant C. This defines a curve in the phase space near the fixed point. And, we can show that if a point $(\varepsilon^{\phi},\varepsilon^{J})$ satisfies this equation, then the mapping of this point $(\varepsilon'^{\phi},\varepsilon'^{J})$ will, too:

$$
\begin{aligned}
& a_{12}(\varepsilon'^J)^2 - a_{21}(\varepsilon'^\phi)^2 + (a_{11}-a_{22})\varepsilon'^\phi\varepsilon'^J \\
= \;& a_{12}(a_{21}\varepsilon^\phi + a_{22}\varepsilon^J)^2 - a_{21}(a_{11}\varepsilon^\phi + a_{12}\varepsilon^J)^2 + (a_{11}-a_{22})(a_{11}\varepsilon^\phi + a_{12}\varepsilon^J)(a_{21}\varepsilon^\phi + a_{22}\varepsilon^J) \\
= \;& \Big[a_{12}a_{21}^2 - a_{21}a_{11}^2 + a_{11}a_{21}(a_{11}-a_{22})\Big](\varepsilon^\phi)^2 + \Big[a_{12}a_{22}^2 - a_{21}a_{12}^2 + a_{12}a_{22}(a_{11}-a_{22})\Big](\varepsilon^J)^2 \\
& + \Big[2a_{12}a_{21}a_{22} - 2a_{21}a_{11}a_{12} + (a_{11}-a_{22})(a_{11}a_{22} + a_{12}a_{21})\Big]\varepsilon^\phi\varepsilon^J \\
= \;& a_{21}(a_{12}a_{21} - a_{11}a_{22})(\varepsilon^\phi)^2 + a_{12}(a_{11}a_{22} - a_{12}a_{21})(\varepsilon^J)^2 + (a_{11}-a_{22})(a_{11}a_{22} - a_{12}a_{21})\varepsilon^\phi\varepsilon^J \\
& = a_{12}(\varepsilon^J)^2 - a_{21}(\varepsilon^\phi)^2 + (a_{11}-a_{22})\varepsilon^\phi\varepsilon^J
\end{aligned}
$$

So, the linearized mapping always takes a point near the fixed point to another point lying on the same curve, defined by the equation 36.24.

36.3.3 Hyperbolas and Ellipses

Consider equation 36.24, which (if we work backwards towards where it came from) can also be written as

$$\left[\left(a_{11} - \frac{T}{2}\right)\tilde{\varepsilon}^\phi + a_{12}\tilde{\varepsilon}^J\right]^2 + \left[1 - \left(\frac{T}{2}\right)^2\right](\tilde{\varepsilon}^\phi)^2 = Ca_{12} \tag{36.25}$$

in this form it is clear what kinds of shapes we have. There are three alternatives, depending on the sign of the second term. If we have

$$1 - \left(\frac{T}{2}\right)^2 > 0$$

then these shapes are ellipses. In this case, the two eigenvalues $\lambda_\pm$ are a pair of complex conjugates. If we have

$$1 - \left(\frac{T}{2}\right)^2 < 0$$

then these shapes are hyperbolas. In this case, the two eigenvalues are both real. Since we have $\lambda_+\lambda_- = 1$, one of these eigenvalues has a magnitude greater than one, and the other has a magnitude less than one. Suppose we say $|\lambda_+| > 1$ and $|\lambda_-| < 1$. Remembering our earlier analysis, we argued before that in this circumstance there would be two lines corresponding to the two eigenvalues. A point on the λ_+ line will move farther away from the fixed point after every mapping, staying on the line. A point on the λ_- line will move closer to the fixed point after every mapping, staying on the line. And a general point may move towards the fixed point for a while, but eventually moves farther away, converging to the λ_+ line. This is exactly the way we describe a hyperbola.

And finally if we have

$$1 - \left(\frac{T}{2}\right)^2 = 0$$

then these shapes are just lines, or pairs of lines. In fact, looking back at equation 36.20, this gives $\lambda_+ = \lambda_- = \pm 1$. Depending on the sign here, we either have all points as fixed points, or we have points flipping back and forth over the fixed point, staying a constant distance from the fixed point.

Now, this pattern of ellipses and hyperbolas near fixed points is very reminiscent of what we got near fixed points for phase diagrams of energy conserving systems–yet our underlying system, the kicked rotator, definitely doesn't conserve energy. What we are seeing here is actually a reflection of the area preservation. If you think about it, we *couldn't* have an attractor fixed point in an area preserving map, or a repulsor fixed point either. And our whole algebraic argument was based on the fact that $\det \mathbb{A} = 1$.

36.3.4 Near Fixed Point (0,0)

Near the fixed point $(0, 0)$, the linearized mapping is given by

$$\mathbb{A} = \begin{bmatrix} 1+\epsilon & 1 \\ \epsilon & 1 \end{bmatrix} \tag{36.26}$$

which means that the eigenvalues are found by solving

$$\det[\mathbb{A} - \lambda\mathbb{I}] = (1+\epsilon-\lambda)(1-\lambda) - \epsilon = \lambda^2 - (2+\epsilon)\lambda + 1 = 0$$

which gives

$$\lambda = \frac{2+\epsilon \pm \sqrt{4+4\epsilon+\epsilon^2-4}}{2} = 1 + \frac{\epsilon \pm \sqrt{\epsilon^2+4\epsilon}}{2} \tag{36.27}$$

Both eigenvalues are clearly real. And for $\epsilon > 0$, clearly one of them will be greater than 1:

$$\lambda_+ = 1 + \frac{\epsilon + \sqrt{\epsilon^2+4\epsilon}}{2} > 1$$

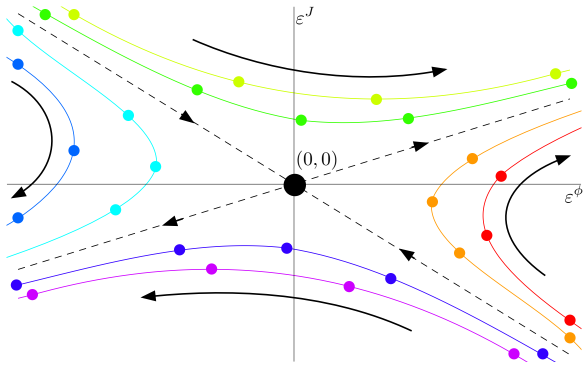

Figure 36.5: The mapping follows hyperbolas in the vicinity of the fixed point $(0, 0)$

The other falls between 0 and 1: At $\epsilon = 0$ we have

$$\lambda_- = 1 + \frac{\epsilon - \sqrt{\epsilon^2 + 4\epsilon}}{2} = 1$$

and as $\epsilon \to \infty$ we get

$$\lambda_- = 1 + \frac{\epsilon - \sqrt{\epsilon^2 + 4\epsilon}}{2} = 1 + \frac{\epsilon}{2}\left[1 - \sqrt{1 + \frac{4}{\epsilon}}\right] \to 1 + \frac{\epsilon}{2}\left[1 - 1 - \frac{2}{\epsilon}\right] = 0$$

This is the situation where we have hyperbolas near the fixed point, which is shown in figure 36.5.

36.3.5 Near Fixed Point $(\pm\pi, 0)$

Now let's consider the mapping near the other fixed point, $(\pm\pi, 0)$. Near this, the linearized mapping is

$$\mathbb{A} = \begin{bmatrix} 1-\epsilon & 1 \\ -\epsilon & 1 \end{bmatrix} \tag{36.28}$$

This is identical to equation 36.7, except with $\epsilon \to -\epsilon$. Therefore, we can immediately say the eigenvalues will be

$$\lambda = 1 - \frac{\epsilon \pm \sqrt{\epsilon^2 - 4\epsilon}}{2} \tag{36.29}$$

What we notice here is that for these eigenvalues to be real we require $\epsilon^2 - 4\epsilon > 0$, which means $\epsilon > 4$. Therefore, something special must happen at $\epsilon = 4$.

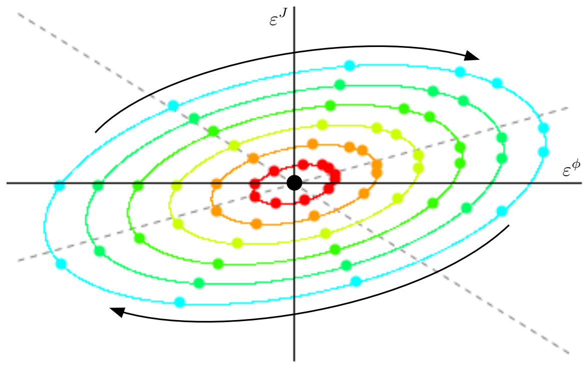

Figure 36.6: The mapping follows ellipses in the vicinity of the fixed point $(\pm\pi, 0)$

Small Impulse Force: $\epsilon \leq 4$

For $\epsilon \leq 4$, we have a pair of complex eigenvalues, and we know that this means the points moves along ellipses near the fixed point under the mapping. This is shown in figure 36.6.

Large Impulse Force: $\epsilon > 4$

As we increase the size of the impulse force, however, at some point we have $\epsilon > 4$. Both our eigenvalues, given by equation 36.29, are now real. At $\epsilon = 4$, we have $\lambda_+ = \lambda_- = -1$, and as we increase ϵ past this point, one of these increases towards zero: $-1 < \lambda_+ < 0$, and the other decreases towards negative infinity: $\lambda_- < -1$. The negative signs indicate that the mapping will flip points from one side of the fixed point to the other. But as they do this, they move along hyperbolas. The key insight here is that at $\epsilon = 4$ we transition from having a stable fixed point at $(\pm\pi, 0)$ to having an unstable one.

36.4 Transition to Chaos

Suppose we look at what happens to the mapping graphically as we change the value of ϵ. What we will do is consider a set of 70 initial points, with random ϕ and J spaced out evenly on the range $[-\pi, \pi]$. Then we will create a series of 400 points from each of these initial points, and plot them all on the range $[-\pi, \pi]$. These, for increasing values of ϵ, are shown in figures 36.7 through 36.12.

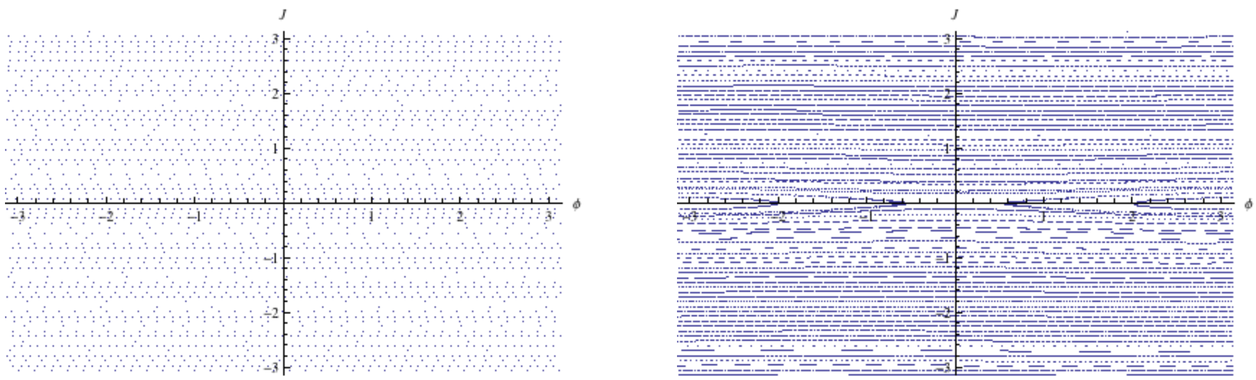

Figure 36.7: $\epsilon = 0$ on the left, $\epsilon = 0.01$ on the right.

With $\epsilon = 0$, points do not change their angular momentum value J because there is no torque acting on the system. The mapping therefore creates horizontal lines of dots in the figure–though really we should remember that the left and right sides need to be periodically identified so these dots lie along circles, with the mapping going around and around. This is clear in figure 36.7 on the left. Now, on the right in the same figure we have $\epsilon = 0.01$. This is a very small perturbation of the system, so it naturally looks very similar. Most of the lines of dots seem unchanged, or at most slightly distorted. However, we can just barely see the beginnings of ellipses forming around the fixed point $(\pm\pi, 0)$.

As we continue to increase ϵ by small amounts, with $\epsilon = 0.05$ and $\epsilon = 0.1$ in figure 36.8, we see the hyperbolic structure near $(0, 0)$ and the elliptical structure near $(\pm\pi, 0)$ grow, though this change is still relatively localized. Notice that the ellipses are representing motion of the rigid body that rocks back and forth around the equilibrium point $\phi = \pi$. (Remember the impulse force acts in the direction upward from $\phi = 0$ to $\phi = \pi$. So it makes sense that so long as the force is not too large, motion around this equilibrium point is a stable rocking back and force.) The lines, on the other hand, represent motion where the rigid body rotates around and around.

In figure 36.9 we see values $\epsilon = 0.25$ and $\epsilon = 0.5$. The ellipses and hyperbolas are now quite clear and these structures are large. They are also clearly at some slant, which warps the entire look of the mapping. We can also see structures that look like ellipses forming elsewhere in the mapping, around the points $(0, \pi)$ and (π, π). These actually correspond to fixed points of Z_ϵ^2, in a story that is by now quite familiar to us. (So the mapping switches back and forth from one ellipse to the other.) We also see the very beginning of something different emerging at the point $(0, 0)$ in the $\epsilon = 0.5$ mapping.

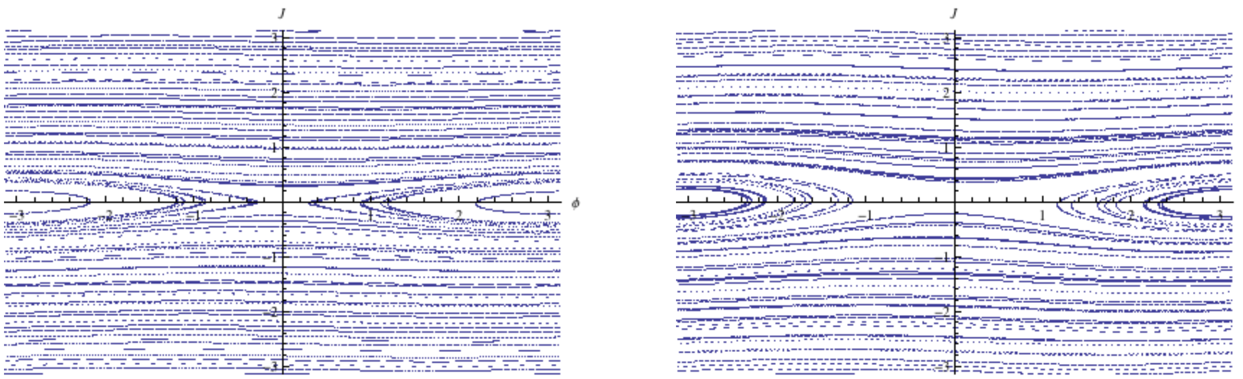

Figure 36.8: $\epsilon = 0.05$ on the left, $\epsilon = 0.1$ on the right.

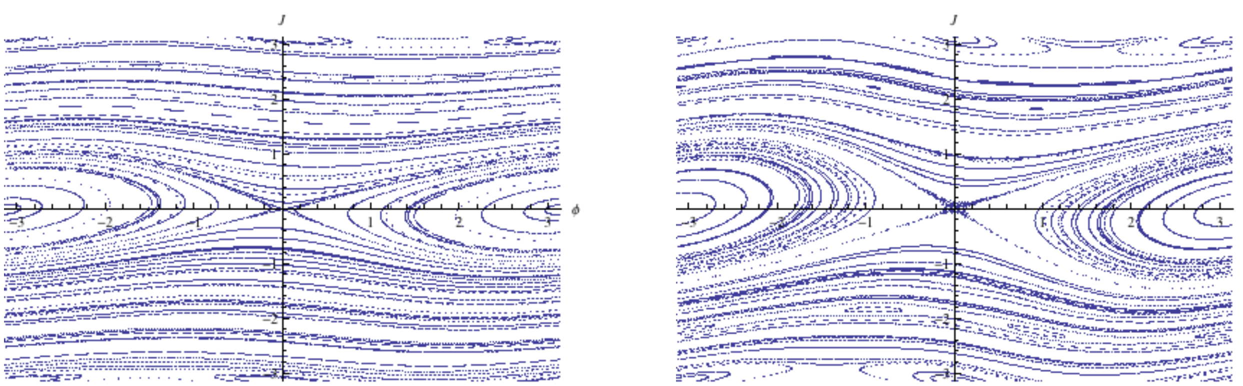

Figure 36.9: $\epsilon = 0.25$ on the left, $\epsilon = 0.5$ on the right.

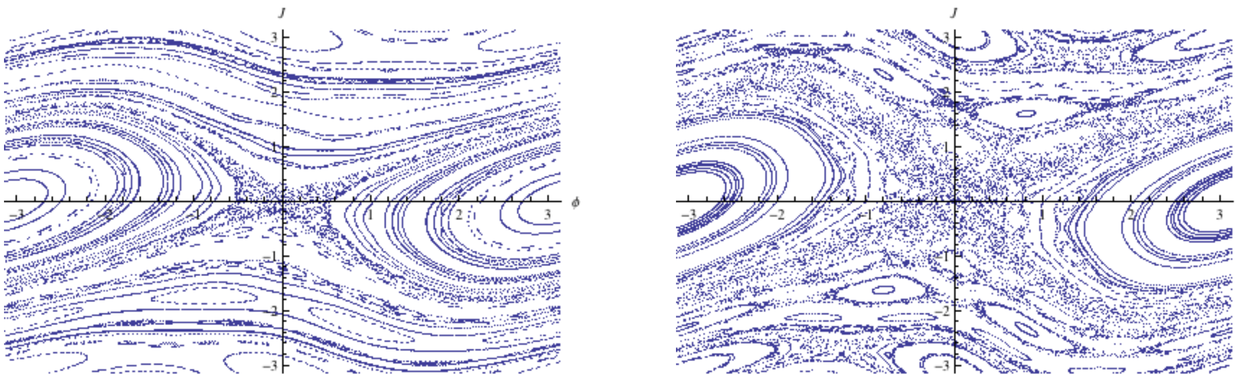

Figure 36.10: $\epsilon = 0.75$ on the left, $\epsilon = 1.0$ on the right.

In figure 36.10 we have the values $\epsilon = 0.75$ and $\epsilon = 1.0$. Here we can definitely see that, starting at the center $(0, 0)$ and moving outwards as we increase ϵ we have a region where points no longer seem to lie along definite curves–but rather they form random distributions. What we are seeing here is a form of chaos–the rigid body no longer evolves in a predictable manner in this region. However, in most of phase space the motion is still not chaotic.

As we increase ϵ still further, in figure 36.11 to $\epsilon = 2.0$ and $\epsilon = 4.0$, the region of chaotic behavior grows, and that of organized motion decreases. We know that at $\epsilon = 4.0$ the character of the fixed point at $(\pm\pi, 0)$ should change. This is not immediately apparent in the $\epsilon = 4.0$ graph. However, in figure 36.12 we look just past this on the left, with $\epsilon = 4.25$. We see that the ellipse structure that used to be at $(\pm\pi, 0)$ has now moved away from there, splitting into two separate ellipse structures on either side, and we can see a bit of hyperbolic structure between them. We now have small regions of ordered motion surrounded by a sea of chaos. Finally, on the right in figure 36.12 we have $\epsilon = 9.0$, at which point there is no ordered motion left–the system is entirely chaotic.

The central facts here are that the larger the impulse force is, the greater the region of phase space (which we can think of as a set of possible initial conditions) where the motion is chaotic. In addition, the chaotic areas arise first in the vicinity of unstable fixed points, and grow outward from there.

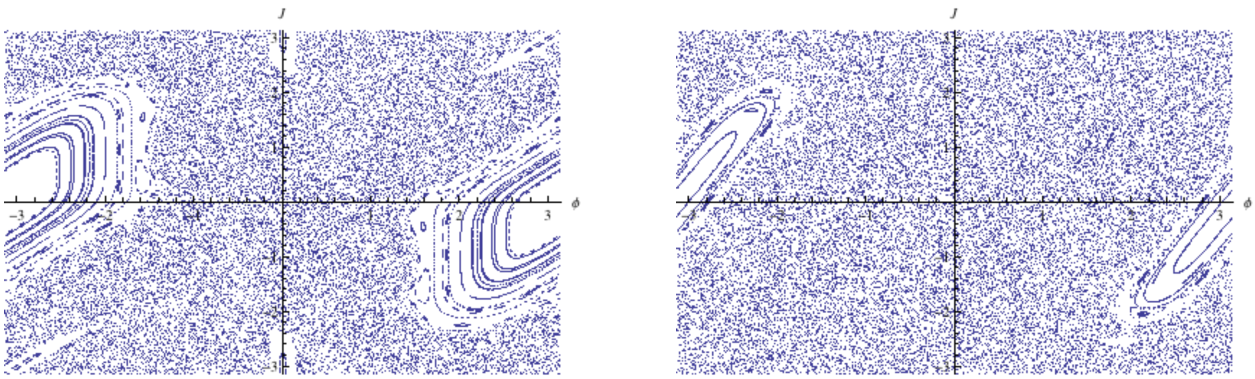

Figure 36.11: $\epsilon = 2.0$ on the left, $\epsilon = 4.0$ on the right.

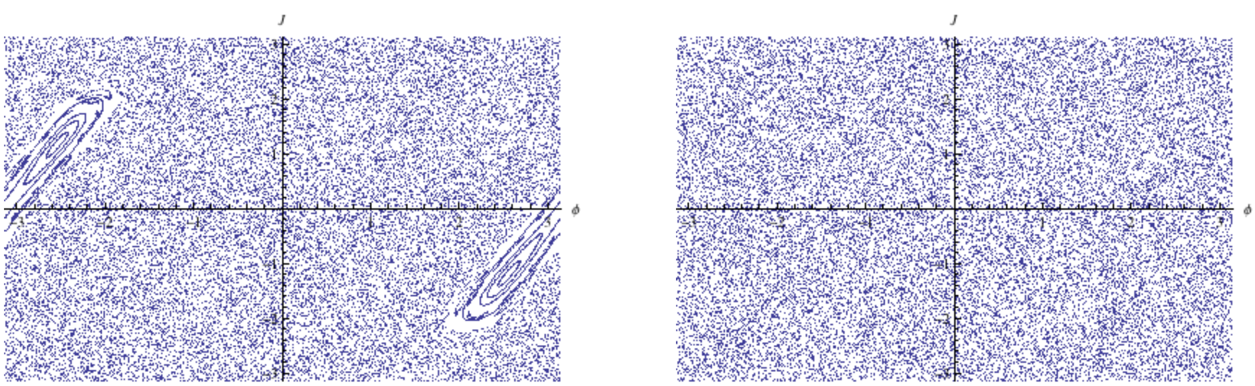

Figure 36.12: $\epsilon = 4.25$ on the left, $\epsilon = 9.0$ on the right.