Assignment No: 3 Manav_17.

Q.3 Find the eigen values and corresponding linearly independent eigen vectors.

$$\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

→ The characteristic equation is

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

On simplification, we get

$$-(\lambda-4)(\lambda-1)(\lambda+1) = 0$$

$$\therefore \lambda = 4, -1, +1.$$

1. For $\lambda = 1$, $[A - \lambda_1 I]X = 0$ gives

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By solving, we get

$$x_1 = 0, \quad x_2 + x_3 = 0$$

$$\therefore X = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

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→ Hence corresponding to $\lambda = 1$, eigenvector is $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

2. For $\lambda = -1$, $[A - \lambda_2 I]x = 0$, gives

$$\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving we get,

$$x_1 + \frac{6}{7}x_3 = 0, \quad x_2 + \frac{2}{7}x_3 = 0$$

$$\therefore x = t\begin{bmatrix} -6/7 \\ -2/7 \\ 1 \end{bmatrix}$$

Hence, corresponding to $\lambda = -1$, eigenvector is $\begin{bmatrix} -6/7 \\ -2/7 \\ 1 \end{bmatrix}$

3. For $\lambda = 4$ $[A - \lambda_3 I]x = 0$ gives

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving we get

$$x_1 + 3x_3 = 0, \quad x_2 + x_3 = 0$$

$$x = t\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

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Q.3

→ Hence, corresponding to $\lambda = 4$, eigenvector is $\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$

Q.8. Find eigenvalues and eigenvectors of

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 6 & 2 & 4 & -3 \end{bmatrix}$$

→ The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 3 & -3-\lambda & 0 & 0 \\ 1 & 2 & 1-\lambda & 0 \\ 6 & 2 & 4 & -3-\lambda \end{vmatrix} = 0$$

On solving, we get

$$\lambda^4 + 4\lambda^3 - 2\lambda^2 - 12\lambda + 9 = 0$$

i.e. $(\lambda+3)(\lambda-1)(\lambda+3)(\lambda-1) = 0$

$\therefore \lambda = -3, 1, -3, 1$

For $\lambda = -3$, $[A - \lambda_1 I]X = 0$, we get

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Q.8
→

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 2 & 4 & 0 \\ 6 & 2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving we get

$x_1 = 0$, $x_2 + 2x_3 = 0$

$$X = t \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

Hence correspond to $\lambda = -3$, eigen vectors are

$$\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

2. For $\lambda = 1$, $[A - \lambda_2 I]X = 0$, we get

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 6 & 2 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving, we get

$x_1 = 0$, $x_2 = 0$, $x_3 - x_4 = 0$

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Q.8.

→ $x = t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Hence, corresponding to $\lambda = 1$, the eigenvector is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Q.11. Verify Cayley Hamilton and show that $A^{-1} = A^2 - 5A + 9I$
where $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

→ The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

on solving, we get

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

Cayley-Hamilton theorem states that this equation is satisfied by the matrix A,
ie. $A^3 - 5A^2 + 9A - I = 0$ ...(1)

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Q.11

→ Now, $A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$

and $A^3 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$

Now, $A^3 - 5A^2 + 9A - I$

$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

a. Now, multiply the above equation by A^{-1}.

$\cancel{A^2 - 5A}$ $\quad A^2 - 5A + 9I - A^{-1} = 0$

$\therefore A^{-1} = A^2 - 5A + 9I \quad \cdots (2)$

$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

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Q.11. b. To find A^{-2} multiply (2) by A^{-1}.

$\therefore A^{-2} = A - 5I + 9A^{-1}$

$$= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 9\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 20 & 52 \\ 8 & 7 & 18 \\ 18 & 16 & 41 \end{bmatrix}$$

c. To find A^4 multiply (2) by

$\therefore A^4 = 5A^3 - 9A^2 + A$

$$\therefore A^4 = 5\begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 9\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix} -55 & 104 & 24 \\ -20 & 45 & 32 \\ 32 & -42 & 13 \end{bmatrix}$$

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Q.13. Find $\tan A$ for $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$

→ The characteristic equation of A is

$$\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$\therefore \lambda^2 - 9 = 0$. $\therefore \lambda = 3, -3$.

1. For $\lambda = 3$, $[A - \lambda_1 I]X = 0$ gives

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

on solving

$-4x_1 + 4x_2 = 0$ $\quad \therefore x_1 - x_2 = 0$

$x_1 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \therefore$ The eigenvector is $[1, 1]$

2. For $\lambda = -3$, $[A - \lambda_2 I]X = 0$, gives

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

on solving

$2x_1 + 4x_2 = 0$

$\therefore x_2 = t\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad \therefore$ The eigenvector is $[2, -1]$

$\therefore M = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and $|M| = -3$

$\therefore M^{-1} = \dfrac{\text{adj}.M}{|M|} = -\dfrac{1}{3}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$

Q.13

→ Now, $D = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

$\because f(A) = \tan A$ $\quad f(D) = \begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix} = \begin{bmatrix} \tan 3 & 0 \\ 0 & \tan 3 \end{bmatrix}$

$\therefore \tan A = M f(D) \cdot M^{-1}$

$\therefore \tan A = \frac{-1}{3} \begin{bmatrix} \tan 3 & -4\tan 3 \\ -2\tan 3 & -\tan 3 \end{bmatrix}$ //

Q.16. Diagonalise the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

using an orthogonal matrix.

→ The characteristic equation is

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

on solving, we get

$(2-\lambda)(\lambda-2)(\lambda-8) = 0$

$\therefore \lambda = 2, 2, 8$

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1. For $\lambda = 2$, $[A - \lambda_1 I]X = 0$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

on solving

$2x_1 - x_2 + x_3 = 0$ ... (1)

we see rank of matrix is 1 and no. of variables is 3. Hence, there are $3-1=2$ linearly independent solutions.

The vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ are linearly independent

$\therefore$ For $\lambda = 2$, eigen vectors are $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

2. For $\lambda = 8$, $[A - \lambda_2 I]X = 0$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & 5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

on solving

$x_1 + x_2 - x_3 = 0$ and $x_2 + x_3 = 0$

$X_2 = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ $\therefore$ eigen vector is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Q.16

→ since we want an orthogonal matrix let
$X_3 = [x \; y \; z]$

This vector X_3 must be orthogonal to X_2 and must satisfy (1)

$\therefore 2x - y + z = 0$ and $x + 2y + 0z = 0$

By Cramer's Rule

$\frac{x}{0-2} = \frac{-y}{0-1} = \frac{z}{4+1} = t$

$\therefore x = -2t, \quad y = t, \quad z = 5t$

$\therefore X_3 = t\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$

Now, we normalise vectors X_1, X_2, X_3. Thus, we get normalised vectors S_1, S_2, S_3

The norms of these vectors are

$\|X_1\| = \sqrt{6}, \quad \|X_2\| = \sqrt{5}, \quad \|X_3\| = \sqrt{30}$

$\therefore S_1 = \frac{X_1}{\sqrt{6}}, \quad S_2 = \frac{X_2}{\sqrt{5}}, \quad S_3 = \frac{X_3}{\sqrt{30}}$

if we write $P = [S_1 \; S_2 \; S_3]$

$P = \begin{bmatrix} 2\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ -1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{bmatrix}$

then P is an orthogonal matrix. and

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Q.16
→ $P^{-1}AP = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ where $P^{-1} = P'$.

$\therefore$ Required matrix is $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$