02/08/17 (Wednesday) ~~(Tuesday)~~

APPLIED MATHEMATICS – III

DIRICHLET'S CONDITION

A function $f(x)$ satisfies the Dirichlet's condition if -

(1). $f(x)$ is periodic.
(2). $f(x)$ & its integral are finite & single-valued.
(3). $f(x)$ is bounded & can have a finite no. of discontinuities
(4). $f(x)$ has a finite no. of maximas & minimas.

FOURIER SERIES

A function $f(x)$ satisfies the Dirichlet's condition & can be expressed in the trigonometric series, i.e.

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x$$
$$+ b_3 \sin 3x + ---$$

where $a_0, a_1, a_2, ---, b_1, b_2, ---$ are any arbitrary constants

It can also be written as

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

→ Fourier series of function $f(x)$

Orthogonality of Trigonometric Function

The trigonometric functions $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots\}$ are orthogonal on the interval $(c, c+2\pi)$. By definition of orthogonality, this means that the integral of the product of any two different such functions over that interval is zero.

For e.g.- ① $\int_c^{c+2\pi} 1 \cdot \cos x \, dx = 0$

② $\int_c^{c+2\pi} 1 \cdot \sin x \, dx = 0$

③ $\int_c^{c+2\pi} 1 \cdot \cos nx \, dx = 0$

④ $\int_c^{c+2\pi} 1 \cdot \sin nx \, dx = 0$

⑤ $$\int_c^{c+2\pi} \cos nx \cdot \cos mx \, dx = \begin{cases} 0 & ; \text{ if } n \neq m \\ \text{Non-zero} & ; \text{ if } n = m \text{ or } n = m = 0 \end{cases}$$

<u>Case - I</u> Take $n=m$

$$\Rightarrow \int_c^{c+2\pi} \cos^2 mx \, dx = \int_c^{c+2\pi} \left(\frac{1+\cos 2nx}{2}\right) dx$$

$$= \frac{1}{2}\int_c^{c+2\pi} dx + \underbrace{\frac{1}{2}\int_c^{c+2\pi} \cos 2nx \, dx}_{=0}$$

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$$= \frac{1}{2} [x]_c^{c+2\pi} + 0 = \pi = \text{Non-zero}$$

Case:- II $\quad n \neq m$

$$\Rightarrow \int_c^{c+2\pi} \cos nx \cdot \cos mx \, dx = \frac{1}{2} \int_c^{c+2\pi} (\cos(n+m)x + \cos(n-m)x) \, dx$$

$$= \frac{1}{2} \int_c^{c+2\pi} \cos(n+m)x \, dx + \frac{1}{2} \int_c^{c+2\pi} \cos(n-m)x \, dx$$

$$= \frac{1}{2}(0) + \frac{1}{2}(0) = 0$$

Case - III $\quad n = m = 0$

$$\Rightarrow \int_c^{c+2\pi} \cos 0 \cdot \cos 0 \, dx = \int_c^{c+2\pi} 1 \cdot dx = 2\pi = \text{Non-zero}$$

⑥. $\displaystyle\int_c^{c+2\pi} \cos nx \cdot \sin mx \, dx$ $\xrightarrow{\text{3 cases}}$ $n=m$, $n \neq m$, $n=m=0$

Case -I $\quad n = m$

$$\Rightarrow \int_c^{c+2\pi} \sin nx \cos nx \, dx = \frac{1}{2} \int_c^{c+2\pi} \sin 2nx \, dx = 0$$

Case-II $\quad n \neq m$

$$\Rightarrow \int_c^{c+2\pi} \cos nx \sin mx \, dx = \frac{1}{2} \int_c^{c+2\pi} (\sin(m+n)x + \sin(m-n)x) \, dx$$

$$= \frac{1}{2}(0+0) = 0$$

04/08/17 (Friday)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{— ①. in the interval } (c, c+2\pi)$$

Integrating from c to $(c+2\pi)$ w.r.t. x.

$$\int_c^{c+2\pi} f(x)dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \sum_{n=1}^{\infty}\left(a_n \underbrace{\int_c^{c+2\pi} \cos nx\, dx}_{=0} + b_n \underbrace{\int_c^{c+2\pi} \sin n x\, dx}_{=0}\right)$$

$$\Rightarrow \int_c^{c+2\pi} f(x)dx = \frac{a_0}{\cancel{2}} \times \cancel{2}\pi$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\, dx} \quad \text{— ②.}$$

Multiplying eqⁿ ① by $\cos mx$ & then integrating from c to $c+2\pi$ w.r.t. x.

$$\Rightarrow \int_c^{c+2\pi} f(x)\cos mx\, dx = \frac{a_0}{2}\int_c^{c+2\pi} \cos mx\, dx$$

$$+ \sum_{n=1}^{\infty}\left(a_n \underbrace{\int_c^{c+2\pi} \cos nx \cos mx\, dx}_{0 \text{ if } n\neq m \ \& \ \pi \text{ if } n=m} + b_n \underbrace{\int_c^{c+2\pi} \sin nx \cos mx\, dx}_{0 \text{ always}}\right)$$

$$\Rightarrow \int_c^{c+2\pi} f(x)\cos mx\, dx = \frac{a_0}{2}(0) + a_m \pi + 0$$

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$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$; if $f(x)$ is even

$= 0$; if $f(x)$ is odd

$$\Rightarrow \quad a_m = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos mx\,dx$$

or

$$\boxed{a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\,dx}$$

Similarly, $\boxed{b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx}$

Q. Find the fourier series of $f(x) = x - x^2$ in the interval $(-\pi, \pi)$ & show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$.

Ans. We know that $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ — (1).

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right)$$

$$= \frac{-2\pi^2}{3} \quad \text{— (2).}$$

$\to 0$ $\because x\cos nx$ is odd f^n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

$$= 0 - \frac{2}{\pi}\int_{0}^{\pi} \underset{\mathrm{I}}{x^2}\,\underset{\mathrm{II}}{\cos nx}\,dx$$

Note : $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

u' → Diff. v_2 → Integrate

$\Rightarrow \cancel{a_n = \frac{2}{\pi}}$

$$\int x^2 \cos nx\,dx = x^2\int \cos nx\,dx - \int\left(2x \int \cos nx\,dx\right)dx$$

$$= \frac{x^2 \sin nx}{n} - 2\int \frac{x \sin nx}{n}\,dx$$

$$= \frac{x^2 \sin nx}{n} - \frac{2}{n}\left[x\int \sin nx\,dx - \int\left(1\cdot\int \sin nx\,dx\right)dx\right]$$

$$= \frac{x^2 \sin nx}{n} - \frac{2}{n}\left[\frac{-x \cancel{\cos x}\ \cos nx}{n} + \frac{\sin nx}{n^2}\right]$$

$$= \frac{x^2 \sin nx}{n} + \frac{2x\cos x}{n^2} - \frac{2\sin x}{n^3}$$

$$\Rightarrow a_n = \frac{-2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^\pi$$

$$\cancel{= \frac{-2}{\pi}\left[0 + \frac{2\pi(-1)}{n^2}\right]}$$

$$\cancel{= \frac{4}{n^2}}$$

$$= \frac{-2}{\pi}\left(0 + \frac{2\pi}{n^2}(-1)^n\right) \qquad \left(\because \cos n\pi = (-1)^n\right)$$

$$= -\frac{4}{n^2}(-1)^n \quad \text{—— (3).}$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{x}_{\text{Odd}}\ \underbrace{\sin nx}_{\text{Odd}}}_{\text{Even}}\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underbrace{x^2}_{\text{Even}}\ \underbrace{\sin nx}_{\text{Odd}}}_{\text{Odd}}\,dx$$

Odd $\Rightarrow 0$

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$$= \frac{2}{\pi} \int_0^\pi x \sin nx \, dx$$

$$\int x \sin nx \, dx = x \int \sin nx \, dx - \int \left(1 \cdot \int \sin nx \, dx\right) dx$$

$$= \frac{-x\cos nx}{n} + \int \frac{\cos nx}{n} \, dx$$

$$= \frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}$$

$$\Rightarrow \quad b_n = \frac{2}{\pi} \left[\frac{\sin nx}{n^2} - \frac{x \cos nx}{n} \right]_0^\pi$$

$$= \frac{2}{\pi} \left[0 - \frac{\pi}{n} \cos n\pi \right]$$

$$= -\frac{2(-1)^n}{n} \quad \text{—} (4).$$

Now, we have the values of all a_0, a_n & b_n. Substituting these in eq^n (1), we get

$$\boxed{f(x) = -\frac{\cancel{2}}{3}\frac{\pi^2}{\cancel{2}} + \sum_{n=1}^{\infty} \left(-\frac{4}{n^2}(-1)^n \cos nx - \frac{2}{n}(-1)^n \sin nx \right)}$$

$\hookrightarrow$ Required Fourier Series

Now, putting $x=0$ in the above eq^n

$$\Rightarrow \quad 0 - 0^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{-4(-1)^n}{n^2} \right)$$

$$\Rightarrow \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

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$$\Rightarrow \boxed{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots}$$

Hence proved.

Q. Find the fourier series of $f(x) = x\cos x$ in the interval $(-\pi, \pi)$.

Ans. $$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x\cos x \, dx = \frac{2}{\pi} \int_0^{\pi} x\cos x \, dx$$

$$\left(\int x\cos x \, dx = x\sin x - \int 1 \cdot \sin x \, dx \right.$$
$$\left. = x\sin x + \cos x \right)$$

$$= \frac{1}{\pi} \left[x\sin x + \cos x \right]_0^{\pi} = \frac{1}{\pi} (0 + 1 - 0 - 1)$$

$$= 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{x}_{\text{Odd}} \underbrace{\cos x}_{\text{Even}} \underbrace{\cos nx}_{\text{Even}}}_{\text{Odd}} \, dx$$

$$= 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x\cos x \sin nx \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x \left(\sin(nx + x) + \sin(nx - x) \right) dx$$

$f(x) = k$
$f(-x) = k = f(x)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x \sin(nx + x)\,dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin(nx - x)\,dx$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos x \sin nx\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x \sin 2x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$\int x \sin 2x\,dx = \frac{-x\cos 2x}{2} + \int 1 \cdot \frac{\cos 2x}{2}\,dx$$
$$= \frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}$$

$$= \frac{1}{\pi}\left[\frac{\sin 2x}{4} - \frac{x\cos 2x}{2}\right]_0^{\pi}$$
$$= \frac{1}{\pi}\left[0 - \frac{x}{2}\right] = \frac{-x}{2\pi}$$

09/08/17

Q. ~~Expand~~ Find the ~~$f(x) = x\cos x$ in~~ fourier series of function

$$f(x) = \begin{cases} \pi + x ; & -\pi < x < 0 \\ \pi - x ; & 0 < x < \pi \end{cases}$$

(Here, $f(x)$ is even)

Imp
(Even f^n is symmetrical about y-axis)

Q. Find the fourier series of function

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$$

and then show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

Ans. $a_0 = a_n = 0$ ($\because f(x)$ is odd).

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2k}{\pi}\int_0^{\pi}\sin nx\,dx$$

($\because f(x)$ is odd)

$$= \frac{-2k}{n\pi}\left[\cos nx\right]_0^{\pi} = \frac{-2k}{\pi}(\cos n\pi - \cos 0)$$

$$= \frac{-2k}{\pi}((-1)^n - 1)$$

$$= \frac{2k}{\pi}(1-(-1)^n)$$

Q. Find the fourier series of $f(x) = |\cos x|$ in $(-\pi, \pi)$.

Ans. $f(x) = |\cos x|$ is even.

$\Rightarrow \quad b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\,dx + \int_{\pi/2}^{\pi}(-\cos x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left[\sin x\right]_0^{\pi/2} - \left[\sin x\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[(1-0)-(0-1)\right] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}|\cos x|\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cos nx\,dx + \int_{\pi/2}^{\pi}(-\cos x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\int_0^{\pi/2}(\cos(x+nx)+\cos(x-nx))\,dx - \frac{1}{2}\int_{\pi/2}^{\pi}(\cos(x+nx)+\cos(x-nx))\,dx\right]$$

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$$\frac{4}{\pi} \lim_{n \to 1} \frac{\sin(n\pi/2) \cdot \frac{\pi}{2}}{2n}$$

$$\lim \frac{1}{1} = 1$$

$$= \frac{1}{\pi} \left[\left(\frac{\sin(1+n)x}{1+n} + \frac{\sin(1-n)x}{1-n} \right)_0^{\pi/2} - \left(\frac{\sin(1+n)x}{1+n} + \frac{\sin(1-n)x}{1-n} \right)_{\pi/2}^{\pi} \right]$$

$$= \frac{-4}{\pi(n^2-1)} \cos\left(\frac{n\pi}{2}\right) \quad ; n \neq 1$$

This is not defined for $n=1$. $\therefore$ We will find a_1 (i.e. value of a_n for $n=1$).

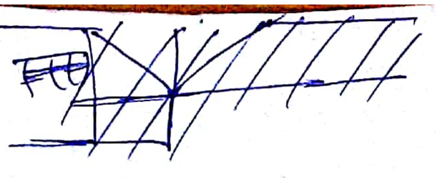
$$\Rightarrow a_1 = \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos x \, dx - \int_{\pi/2}^{\pi} \cos x \cos x \, dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \left(\frac{1+\cos 2x}{2} \right) dx - \int_{\pi/2}^{\pi} \left(\frac{1+\cos 2x}{2} \right) dx \right]$$

$$= \frac{1}{\pi} \left\{ \left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2} - \left[x + \frac{\sin 2x}{2} \right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{1}{\pi} \left[\left(\frac{\pi}{2} + 0 \right) - \left(\pi + 0 - \frac{\pi}{2} - 0 \right) \right]$$

$$= \frac{1}{\pi} (\pi - \pi)$$



11/08/17 (Friday)

HALF RANGE SERIES

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \text{ if } f(x) \text{ is even, } 0 \text{ if } f(x) \text{ is odd.}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \text{ if } f(x) \text{ is even, } 0 \text{ if } f(x) \text{ is odd.}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \text{ if } f(x) \text{ is } \cancel{\text{odd}}\ \text{even},\ \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \text{ if } f(x) \text{ is } \cancel{\text{even}}\ \text{odd.}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \quad (\text{if } f(x) \text{ is even}) \rightarrow \boxed{\text{Cosine Series}}$$

$$= \sum_{n=1}^{\infty} b_n\sin nx \quad (\text{if } f(x) \text{ is odd}) \rightarrow \boxed{\text{Sine Series}}$$

Q. Represent the following f^n by a fourier sine series.

$$f(t) = \begin{cases} t & ; \ 0 < t \le \pi/2 \\ \pi/2 & ; \ \frac{\pi}{2} < t \le \pi \end{cases}$$

Ans.

$$\left.\begin{array}{l} a_0 = 0 \\ a_n = 0 \end{array}\right\} \text{ because } f(x) \text{ is odd.}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\left(\int_0^{\pi/2} t\cdot\sin nt\,dt + \int_{\pi/2}^{\pi}\frac{\pi}{2}\sin nt\,dt\right)$$

$$\cancel{= \frac{2}{\pi}\int}$$

$$\int \underset{I}{t} \underset{II}{\sin nt}\, dt = \frac{-t\cos nt}{n} + \int 1 \cdot \frac{\cos nt}{n}\, dt$$

$$= -\frac{t\cos nt}{n} + \frac{\sin nt}{n^2}$$

$$\Rightarrow \int_0^{\pi/2} t\sin nt\, dt = \left[\frac{t\cos nt}{n} + \frac{\sin nt}{n^2}\right]_0^{\pi/2}$$

$$= \cancel{\cos}\left(-\frac{\frac{\pi}{2}\cos n\pi/2}{n} + \frac{\sin n\pi/2}{n^2}\right) - (0+0)$$

$$= -\frac{\pi}{2}\frac{\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2}$$

$$\Rightarrow b_n = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2} - \frac{\pi}{2n}\left[\cos nt\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\cancel{-\frac{\pi}{2}\frac{\cos(n\pi/2)}{n}} + \frac{\sin(n\pi/2)}{n^2} - \frac{\pi}{2n}(-1)^n + \cancel{\frac{\pi}{2}\frac{\cos(n\pi/2)}{n}}\right]$$

$$= \frac{2\sin(n\pi/2)}{\pi n^2} - \frac{(-1)^n}{n}$$

$$b_1 = \frac{2\sin\pi/2}{\pi} + 1 = \frac{2}{\pi} + 1$$

$0<x<1$
$(0,4), (2,2)$

FOURIER SERIES OF ARBITRARY INTERVAL
$(-c, c)$ or $(0, 2c)$

In $(-c, c)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right)$$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx \quad \text{or} \quad \frac{1}{c}\int_{0}^{2c} f(x)\,dx$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx \quad \text{or} \quad \frac{1}{c}\int_{0}^{2c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx \quad \text{or} \quad \frac{1}{c}\int_{0}^{2c} f(x)\sin\frac{n\pi x}{c}\,dx$$

~~In $(-c, c)$, $2c$ is length of variable x.~~

Q. A periodic function of length 4 is defined as $f(x) = |x|$, $-2 < x < 2$.

Ans. $\left(\begin{matrix} 2c = 4 \\ c = 2 \end{matrix}\right)$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} |x|\,dx = \frac{1}{2}\left[\int_{-2}^{0} -x\,dx + \int_{0}^{2} x\,dx\right]$$

$$= \frac{1}{2}\left[-\left[\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2}\right] = \frac{1}{2}\left[-(0-2) + 2\right]$$

$$= 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-2}^{2} |x|\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{-1}{2}\int_{-2}^{0} x\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx$$

$$\left[\int \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{\cos}\frac{n\pi x}{2}\,dx = x\,\frac{\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)} - \int 1.\,\frac{\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)}\,dx \right.$$

$$\left. = \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right]$$

$$\int_{-2}^{0} x\cos\frac{n\pi x}{2}\,dx = \left(0+\frac{4}{n^2\pi^2}\right) - \left(0+\frac{4}{n^2\pi^2}\cos n\pi\right)$$

$$= \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}(-1)^n = \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$\int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx = \left[0+\frac{4}{n^2\pi^2}(-1)^n\right] - \left[0+\frac{4}{n^2\pi^2}\right]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\Rightarrow a_n = \frac{-1}{\cancel{2}}\times\frac{\cancel{4}^{\,2}}{n^2\pi^2}\left(1-(-1)^n\right) + \frac{1}{\cancel{2}}\times\frac{\cancel{4}^{\,2}}{n^2\pi^2}\left((-1)^n-1\right)$$

$$= \frac{2}{n^2\pi^2}\left(-1+(-1)^n+(-1)^n-1\right)$$

$$= \frac{4}{n^2\pi^2}\left((-1)^n-1\right)$$

16/08/17 (Wednesday)

$$\boxed{\int_a^b f(x)\,dx = \sum \frac{b-a}{p} f(x_i)} \qquad i=0,1,---,p$$

HARMONIC ANALYSIS

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \sum_{i=0}^{p} \frac{(\pi-(-\pi))}{p} f(x_i)$$

$$= \frac{2}{p} \sum_{i=0}^{p} f(x_i)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \sum_{n=1}^{p} \frac{2\pi}{p} f(x_i) \cos nx_i$$

$$= \frac{2}{p} \sum_{n=1}^{p} f(x_i) \cos nx_i$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx = \frac{1}{\pi} \sum_{n=1}^{p} \frac{2\pi}{p} f(x_i) \cancel{\cos nx_i} \sin nx_i$$

$$= \frac{2}{p} \sum_{n=1}^{p} f(x_i) \sin nx_i$$

π rad $\to$ 180
1 rad $\to \frac{180}{\pi}$

$(a_1 \cos x + b_1 \sin x)$ is called first harmonic
$(a_2 \cos 2x + b_2 \sin 2x)$ " " second harmonic
⋮
$(a_n \cos nx + b_n \sin nx)$ " " n^{th} harmonic

Q. Given that y is a function of x (in degrees) in the interval $[0°, 360°]$ the experimental values of y corresponding to values of x are given in the following table:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°
y	298	356	373	337	254	155	80	51	60

x	270°	300°	330°	360°
y	93	147	221	298

Expand y upto first 3 fourier harmonics.

Ans. $P = 12$ (Count the total no. of values & subtract 1).

$$a_0 = \frac{2}{P} \sum_{n=1}^{P} f(x_i)$$

$$= \frac{2}{12}(2425) = 404.167 \simeq 404.17$$

$$a_n = \frac{2}{12} \sum_{n=1}^{P} y_i \cos nx_i$$

$$\Rightarrow a_1 = \frac{2}{12} \sum_{1}^{12} y_i \cos x_i \quad , \quad a_2 = \frac{2}{12} \sum_{1}^{12} y_i \cos 2x_i \quad , \quad a_3 = \frac{2}{12} \sum_{1}^{12} y_i \cos 3x_i$$

$$\Rightarrow \quad a_1 = \frac{2}{12}\left(y_1 \cos x_1 + y_2 \cos x_2 + \cdots + y_{12} \cos x_{12}\right)$$
$$= 107.048$$

$$a_2 = \frac{2}{12}\left(y_1 \cos 2x_1 + y_2 \cos 2x_2 + \cdots + y_{12} \cos 2x_{12}\right)$$
$$= \cancel{121.203} \quad -13$$

$$a_3 = \frac{2}{12}\left(y_1 \cos 3x_1 + y_3 \cos 3x_2 + \cdots + y_{12} \cos 3x_{12}\right)$$
$$= 2.0$$

Similarly, $b_1 = 121.203$

$b_2 = 9.093$

$b_3 = -1$

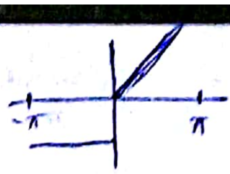
$$y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x)$$

Fourier Integral

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right)\right)$$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(t)\,dt \qquad b_n = \frac{1}{c}\int_{-c}^{c} f(t) \sin \frac{n\pi t}{c}\,dt$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(t) \cos \frac{n\pi t}{c}\,dt$$



$$\boxed{f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos u(t-x)\, du\, dt}$$ where $u = \frac{n\pi}{c}$

Fourier sine & cosine series

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin ux\, dx \int_0^\infty f(t) \sin ut\, dt$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos ux\, du \int_0^\infty f(t) \cos ut\, dt$$

18/08/17 (Friday)

Note

1. If $f(x)$ is continuous in the range of definition, then fourier series converges to $f(x)$.
2. If $f(x)$ has ordinary discontinuity at x, then series converges to $\frac{1}{2}[f(x^+) + f(x^-)]$.

Q. $f(x) = \begin{cases} -\pi & ; -\pi < x < 0 \\ x & ; 0 < x < \pi \end{cases}$

Ans. $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\, dx + \int_0^\pi x\, dx\right]$

$= \frac{1}{\pi}\left(-\pi(0+\pi) + \frac{1}{2}\pi^2\right) = \frac{1}{\pi}\left(-\frac{1}{2}\pi^2\right)$

$= -\frac{\pi}{2}$

$$\int x\cos nx\,dx = \frac{x\sin nx}{n} - \int 1\cdot\frac{\sin nx}{n}\,dx = \frac{x\sin nx}{n} + \frac{\cos nx}{n^2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$\cancel{= \frac{1}{\pi}\ \pi(\cos 0 - \cos n\pi) +}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}\left[\sin nx\right]_{-\pi}^{0} + \left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}(0) + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{(-1)^n - 1}{\pi n^2} = \frac{\cos n\pi - 1}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{n}(1 - 2\cos n\pi)$$

$$\Rightarrow f(x) = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{1-2\cos n\pi}{n}\sin nx$$

$f(x)$ is discontinuous at $x=0$ ($\because f(0) = -\pi$ & 0).

$\therefore$ Series converges to $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$. at $x=0$

$$\therefore \frac{1}{2}(-\pi + 0) = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n^2}\cos 0 + \sum_{n=1}^{\infty}\frac{1-2\cos n\pi}{n}\sin 0$$

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n^2}$$

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$\frac{1}{3}\left(0.8 + 0.3 - 0.2 - 0.7 - \frac{0.9}{2} + \frac{1.1}{2}\right)$ 0.288 4.2

$= \frac{1}{3}\left(0.2 + 0.2\right) = \frac{0.3}{3} = 0.1$ $6 \overline{)1.73}$ $\cos(360° - 60°)$ $\cos 60°$

Q. Find first two harmonics for the functions $f(x)$ given by the following table :

$\theta°$	0°	60°	120°	180°	240°	300°	360°
$f(\theta°)$	0.8	0.6	0.4	0.7	0.9	1.1	0.8

Ans. $p = 6$

$$a_0 = \frac{2}{p} \sum_{n=1}^{6} f(x)$$

$$= \frac{2}{6} \sum_{n=1}^{6} f(x) = \frac{2}{6_3} \times 4.5 = 1.5$$

$$a_n = \frac{2}{p} \sum_{n=1}^{p} f(x_i) \cos n x_i$$

$$\Rightarrow a_1 = \frac{2}{6} \sum_{1}^{6} f(x_i) \cos x_i$$

$$= \frac{2}{6}\left(0.8 \cos 0° + 0.6 \cos 60° + 0.4 \cos 120° + --- + 1.1 \cos 300°\right)$$

$$= 0.1$$

$$a_2 = \frac{2}{6}\left(0.8 \cos 0° + 0.6 \cos 120° + 0.4 \cos 240° + --- + 1.1 \cos 600°\right)$$

Similarly find b_1, b_2.

$a_2 =$

$b_1 = -0.29$

$b_2 =$

(Now, put all in series.)

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30/08/17

Imp. Q. Find the fourier cosine transformation of e^{-x^2}.

Ans. $F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$\Rightarrow F_c(e^{-x^2}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos sx \, dx = \overline{f_c}(s)$ —— ①.

Differentiating ① w.r.t. "s".

$\Rightarrow \frac{d\overline{f_c}(s)}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{e^{-x^2}}_{II} (\underbrace{-x \sin sx}_{I}) \, dx$

$-\int \underbrace{x e^{-x^2}}_{II} \underbrace{\sin sx}_{I} \, dx = \sin sx \int -x e^{-x^2} dx - \int \left(s \cos sx \cdot \int -x e^{-x^2} dx \right) dx$

Put $-x^2 = t \Rightarrow -2x\,dx = dt$

$= \frac{\sin sx}{2} \int e^{t} dt - \frac{1}{2} \int \left(s \cos sx \cdot \int e^{t} dt \right) dx$

$= \frac{\sin sx}{2} e^{t} - \frac{s}{2} \int \cos sx \cdot e^{t} \, dx$

$= \frac{\sin sx}{2} e^{-x^2} - \frac{s}{2} \int \cos sx \cdot e^{-x^2} \, dx$

$\Rightarrow \frac{d\overline{f_c}(s)}{ds} = \sqrt{\frac{2}{\pi}} \left[\left(\frac{\sin sx \cdot e^{-x^2}}{2} \right)_0^\infty - \int_0^\infty \frac{s \cos sx \cdot e^{-x^2}}{2} dx \right]$

$\hookrightarrow 0$

$= -\sqrt{\frac{2}{\pi}} \cdot \frac{s}{2} \int_0^\infty \cos sx \cdot e^{-x^2} dx$

$= -\frac{s}{2} \overline{f_c}(s)$ (From ①).

$$\cancel{L\{\cos sx\}} = \int_{-s}^{\infty} e^{-ax} \cos sx \, dx$$

Now, integrating above eqn.

$$\int \frac{d\,\overline{f_c(s)}}{\overline{f_c(s)}} = \int -\frac{s}{2}\, ds$$

Q. $\int_0^{\infty} f(x) \cos sx \, dx = e^{-s} = \overline{f_c(s)} \; (s \geq 0)$

find $f(x)$.

Ans. This is the fourier cosine transform of $f(x)$.

$\therefore$ The fourier cosine inverse transformation of $\overline{f_c(s)}$ is (Overall multiplication should be $2/\pi$).

$$F_c^{-1}(\overline{f_c(s)}) = \frac{2}{\pi} \int_0^{\infty} e^{-s} \cos sx \, ds$$

$$\Rightarrow f(x) = \frac{2}{\pi} \left[\frac{e^{-s}}{(-1)^2 + x^2} \left(-\cos sx + x \sin sx \right) \right]_0^{\infty}$$

$$\Rightarrow f(x) = \frac{2}{\pi} \left[0 - \frac{1}{1+x^2} (-1) \right] = \frac{2}{\pi} \cdot \frac{1}{1+x^2}$$

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$f_c(f(x))$ $\cancel{\int (1-s)\cos sx\,ds = \frac{(1-s)}{x}[\sin sx]_0^1 - \frac{1}{x}\int (-1)[\sin sx]_0^1 ds}$

$\cancel{= (1-s)\frac{\sin x}{x} + \frac{1}{x}\int \sin x\,ds}$

$\cancel{= \frac{1-s}{x}\sin x - \frac{\sin x}{x}}$

Q. $\int_0^\infty f(x)\cos sx\,dx = 1-s \quad ; \quad 0 \le s \le 1$
$\qquad = 0 \quad ; \quad s > 1$ $\Big\} = \overline{f_c}(s)$

Hence show that $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$.

Ans. ∴ Fourier cosine inverse transformation of $\overline{f_c}(s)$ is

$$F_c^{-1}(\overline{f_c}(s)) = \frac{2}{\pi}\int_0^\infty \overline{f_c}(s)\cos sx\,ds$$

$$= \frac{2}{\pi}\left[\int_0^1 (1-s)\cos sx\,ds + \int_1^\infty 0\cdot\cos sx\,ds\right]$$

$$\cancel{= \frac{2}{\pi}\left(\left[\frac{\sin sx}{x}\right]_0^1 - \int_0^1 \underset{\mathrm{I}}{s}\,\underset{\mathrm{II}}{\cos sx}\,ds\right)}$$

$$\cancel{= \frac{2}{\pi}\left[\frac{\sin x}{x} - \frac{s}{x}[\sin sx]_0^1 + \frac{1}{x}\int_0^1 1\cdot[\sin sx]_0^1 ds\right]}$$

$$\cancel{= \frac{2}{\pi}\left[\frac{\sin x}{x} - \frac{s}{x}\sin x - \frac{1}{x^2}[\cos sx]_0^1\right]}$$

$$\cancel{= \frac{2}{\pi}\left[\frac{\sin x}{x} - \frac{s\sin x}{x} - \frac{1}{x^2}(\cos x - 1)\right]}$$

$$= \frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right) = f(x).$$

Now, the fourier cosine transformation of $f(x)$ is

$$\overline{f_c}(s) = \int_0^\infty \frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right)\cos sx\,dx = 1-s \quad ; \quad 0 \le s \le 1$$
$$= 0 \quad ; \quad s > 1$$

Put $s=0$.

$$\Rightarrow \int_0^\infty \frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right) dx = 1$$

$$\int_0^\infty \frac{2}{\pi} \times \left(\frac{2\sin^2 x/2}{x^2}\right) dx = 1$$

Put $\frac{x}{2} = t \Rightarrow dx = 2\,dt$ | If $x=0$, $t=0$; If $x=\infty$, $t=\infty$

$$\Rightarrow \int_0^\infty \frac{\cancel{2}}{\pi} \times \frac{\cancel{2}\sin^2 t}{4t^2} \cdot 2\,dt = 1$$

$$\Rightarrow \boxed{\int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}}$$

Hence proved.

01/09/17

Fourier Transformation of Partial Derivative functions

If $u = u(x,y)$ is a f^n of of two independent variables x & y and if the fourier transformation of $u(x,y)$ w.r.t. x is $\bar{u}(s,y)$ where $\bar{u}(s,y) = \int_{-\infty}^{\infty} u(x,y)\, e^{-isx}\, dx$

$= F(u(x,y))$ w.r.t. x.

Q1. Find the fourier transformation of $\frac{\partial u}{\partial x}$ w.r.t. x, where $u = u(x,t)$ & when $x \to \pm\infty$, $u \to 0$.

Ans. $$F\left(\frac{\partial u}{\partial x}\right) = \int_{-\infty}^{\infty} \underbrace{\frac{\partial u}{\partial x}}_{II}\, \underbrace{e^{-isx}}_{I}\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \underbrace{\frac{\partial u}{\partial x}}_{II} \underbrace{e^{-isx}}_{I} dx = \left[e^{-isx} \cdot u\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -ise^{-isx} \cdot u \, dx$$

$$= \underbrace{(e^{-\infty} \cdot u - e^{\infty} \cdot u)}_{\downarrow \; 0} + is \int_{-\infty}^{\infty} u \cdot e^{-isx} dx$$

($\because$ When $x \to \infty$, $y \to 0$)

$$= 0 + is \cdot \bar{u}(s,t)$$

$$= i \cdot s \cdot \bar{u}(s,t)$$

Q2. Find the fourier transformation of $\frac{\partial^2 u}{\partial x^2}$ w.r.t. x.

~~When~~ $u \to 0$ & $\frac{\partial u}{\partial x} \to 0$ when $x \to \pm\infty$.

Ans. $$F\left(\frac{\partial^2 u}{\partial x^2}\right) = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-isx} dx$$

$$\int \underbrace{\frac{\partial^2 u}{\partial x^2}}_{II} \cdot \underbrace{e^{-isx}}_{I} dx = e^{-isx} \cdot \frac{\partial u}{\partial x} - \int -is \underbrace{e^{-isx}}_{I} \cdot \underbrace{\frac{\partial u}{\partial x}}_{II} dx$$

$$= e^{-isx} \cdot \frac{\partial u}{\partial x} + is \int e^{-isx} \cdot \frac{\partial u}{\partial x} dx$$

$$\cancel{= e^{-isx} \cdot \partial u}$$

$$\Rightarrow F\left(\frac{\partial^2 u}{\partial x^2}\right) = \left[e^{-isx} \cdot \frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} + is \underbrace{\int_{-\infty}^{\infty} e^{-isx} \cdot \frac{\partial u}{\partial x} dx}_{= is\bar{u}(s,t)}$$

(Found in above ques).

$$= 0 + (is)(is)\bar{u}(s,t)$$

$$= (is)^2 \bar{u}(s,t)$$

$\Rightarrow$ $\boxed{\text{When } u \to 0,\ \frac{\partial u}{\partial x} \to 0 \text{ when } x \to \pm\infty, \text{ then } F\left(\frac{d^n u}{dx^n}\right) = (is)^n \cdot \bar{u}(s,t)}$

where $u = u(x,t)$

Q3. Find $F\left(\frac{\partial u}{\partial t}\right)$ w.r.t. x.

Ans. $F\left(\frac{\partial u}{\partial t}\right) = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} \cdot e^{-isx} \cdot dx = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u \cdot e^{-isx}\, dx$

$$\boxed{F\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t} \bar{u}(s,t)}$$

Similarly, $\boxed{F\left(\frac{\partial^n u}{\partial t^n}\right) = \frac{\partial^n}{\partial t^n} \bar{u}(s,t)}$

Fourier Sine & Cosine Transformation of Partial Derivative Functions

If u is a function of two independent variables x & t & if the fourier cosine transformation of u w.r.t. x denoted by $\bar{u}_c(s,t)$, then $\bar{u}_c(s,t) = \int_0^{\infty} u(x,t) \cos sx\, dx$.

& similarly, the fourier sine transformation of u w.r.t x is $\bar{u}_s(s,t) = \int_0^{\infty} u(x,t) \sin sx\, dx$

Q1. Find the fourier cosine transformation of $\frac{\partial u}{\partial x}$ w.r.t x.

$u \to 0$ when $x \to +\infty$.

Ans. $$F_c\left(\frac{\partial u}{\partial x}\right) = \int_0^\infty \underbrace{\frac{\partial u}{\partial x}}_{II} \underbrace{\cos sx}_{I}\, dx$$

$$= \left[\cos sx \cdot u\right]_0^\infty + \int_0^\infty s \sin sx \cdot u \, dx$$

$$= [\underbrace{\cos\infty \cdot 0}_{(\because \text{ When } x\to\infty,\ u\to 0)} - \cos 0 \cdot u(0,t)] + s \underbrace{\int_0^\infty u \sin sx \, dx}_{\bar{u}_s(s,t)}$$

$$\boxed{F_c\left(\frac{\partial u}{\partial x}\right) = -u(0,t) + s\bar{u}_s(s,t)}$$

Q2. Find the fourier cosine transformation of $\frac{\partial^2 u}{\partial x^2}$ w.r.t x.

$u \to 0$ & $\frac{\partial u}{\partial x} \to 0$ when $x \to +\infty$.

Ans. $$F_c\left(\frac{\partial^2 u}{\partial x^2}\right) = \int_0^\infty \underbrace{\frac{\partial^2 u}{\partial x^2}}_{II} \underbrace{\cos sx}_{I}\, dx$$

$$= \left[\cos sx \cdot \frac{\partial u}{\partial x}\right]_0^\infty + \int_0^\infty s \sin sx \cdot \frac{\partial u}{\partial x} \, dx$$

$$= 0 - \cos 0 \cdot \left(\frac{\partial u}{\partial x}\right)_{x=0} + s \int_0^\infty \underbrace{\frac{\partial u}{\partial x}}_{II} \cdot \underbrace{\sin sx}_{I}\, dx$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s\left[[\sin sx . u]_0^\infty - \int_0^\infty s\cos sx . u \, dx \right]$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s\left[(0-0) - s \underbrace{\int_0^\infty u . \cos sx \, dx}_{\bar{u}_c(s,t)} \right]$$

$$\Rightarrow \boxed{F_c\left(\frac{\partial^2 u}{\partial x^2}\right) = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2 \bar{u}_c(s,t)}$$

Note : If the given boundary conditions are :

1. $0 < x < \infty$
2. $\frac{\partial u}{\partial x} = 0$ at $x = 0$

then we use the fourier cosine transformation formula calculated above $\left(\text{i.e } F_c\left(\frac{\partial^2 u}{\partial x^2}\right) = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2 \bar{u}_c(s,t) \right)$.

Q3. Calculate the fourier sine transformation of $\frac{\partial u}{\partial x}$ w.r.t. x.

$u \to 0$ when $x \to \infty$.

Ans. $$F_s\left(\frac{\partial u}{\partial x}\right) = \int_0^\infty \frac{\partial u}{\partial x} . \sin sx \, dx = [\sin sx . u]_0^\infty - \int_0^\infty s\cos sx . u \, dx$$

$$= (0-0) - s \underbrace{\int_0^\infty u \cos sx \, dx}_{\bar{u}_c(s,t)}$$

$$\Rightarrow \boxed{F_s\left(\frac{\partial u}{\partial x}\right) = -s\bar{u}_c(x,t)}$$

Q4. Calculate the fourier sine transformation of $\frac{\partial^2 u}{\partial x^2}$ w.r.t. x.

$u \to 0, \frac{\partial u}{\partial x} \to 0$ when $x \to \infty$.

Ans. $F_s\left(\frac{\partial^2 u}{\partial x^2}\right) = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cdot \sin sx \, dx$

$$= \left[\sin sx \cdot \frac{\partial u}{\partial x}\right]_0^\infty - \int_0^\infty s \cos sx \cdot \frac{\partial u}{\partial x} \, dx$$

$$= 0 - s \underbrace{\left[\int_0^\infty \cos sx \cdot \frac{\partial u}{\partial x} \, dx\right]}$$

$\hookrightarrow = -u(0,t) + s\bar{u}_s(s,t)$

(Calculated in ques ①).

$$\Rightarrow \boxed{F_s\left(\frac{\partial^2 u}{\partial x^2}\right) = s u(0,t) - s^2 \bar{u}_s(s,t)}$$

Note: If the given boundary conditions are:

① $0 < x < \infty$

② $u(x,t) = 0$ at $x = 0$

then we use fourier sine transformation formula calculated above.

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04/09/17

Q1. Solve the P.D.E. $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$; $0<x<\infty$, $t>0$.

The boundary conditions are (1). $u(x,t)=0$ at $x=0$, $0<x<\infty$, $t>0$

(2). $u(x,0)=e^{-x}$, $t>0$, $0<x<\infty$

(3). $u\to 0$ & $\frac{\partial u}{\partial x}\to 0$ at $x\to\infty$

Ans. If x lies b/w $-\infty$ & $+\infty$, we use fourier transformation.

But here, x lies b/w 0 & ∞. $\therefore$ We will use either sine or cosine fourier transformation depending on other boundary conditions

Applying Fourier sine transformation on given P.D.E.

$$\Rightarrow \int_0^\infty \frac{\partial u}{\partial t}\sin sx\, dx = 2\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin sx\, dx$$

$$\Rightarrow \frac{d\bar{u}_s(s,t)}{dt} = 2\left[s\underbrace{u(0,t)}_{0} - s^2\bar{u}_s(s,t)\right]$$

$$\Rightarrow \frac{d\bar{u}_s(s,t)}{dt} = -2s^2\bar{u}_s(s,t)$$

$$\Rightarrow \frac{d\bar{u}_s(s,t)}{\bar{u}_s(s,t)} = -2s^2 dt$$

$$\Rightarrow \log\bar{u}_s(s,t) = -2s^2t + \log k$$

$$\Rightarrow \bar{u}_s(s,t) = ke^{-2s^2t} \quad \text{—} (1)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$\log_e 0 = -\infty$
$\log_e \infty = \infty$
$\log_e 1 = 0$

Now, applying fourier sine transformation on 2nd boundary condition.

$$\Rightarrow \int_0^\infty u(x,0)\sin sx\,dx = \int_0^\infty e^{-x}\sin sx\,dx$$

$$\Rightarrow \bar{u}_s(s,0) = \left[\frac{e^{-x}}{1+s^2}(-\sin sx - s\cos sx)\right]_0^\infty$$

$$\left(\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right)$$

$$\Rightarrow \bar{u}_s(s,0) = \left[0 - \frac{1}{1+s^2}(-0 - s\cdot 1)\right]$$

$$\Rightarrow \bar{u}_s(s,0) = \frac{s}{1+s^2} \quad \text{—— (2)}$$

Put $t=0$ in eq^n (1).

$$\Rightarrow \bar{u}_s(s,0) = ke^0 = k \quad \text{—— (3)}$$

From (2) & (3), $\boxed{k = \frac{s}{s^2+1}}$

$$\Rightarrow \boxed{\bar{u}_s(s,t) = \frac{se^{-2s^2t}}{s^2+1}} \quad \text{—— (4)}$$

Now applying fourier sine inverse transformation on eq^n (4).

$$\Rightarrow \boxed{u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}e^{-2s^2t}\sin sx\,ds}$$

Q. Solve the P.D.E. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0<x<\infty$, $t>0$,

for the given boundary conditions :

① $\frac{\partial u}{\partial x} = 0$ at $x=0$.

② $u(x,0) = x$ when $0<x<1$
$= 0$ when $x>1$

③ $|u(x,t)| < m$, i.e $u(x,t)$ is bounded

Ans - Here, we can consider $u \to 0$ & $\frac{\partial u}{\partial x} \to 0$ at $x \to \infty$.

($\because$ $u(x,t)$ is bounded).

Now, applying fourier cosine transformation on given P.D.E

$$\int_0^\infty \frac{\partial u}{\partial t} \cos sx \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos sx \, dx$$

$$\Rightarrow \frac{d}{dt} \bar{u}_c(s,t) = \underbrace{-\left(\frac{\partial u}{\partial x}\right)_{x=0}}_{\to 0} - s^2 \bar{u}_c(s,t)$$

$$\Rightarrow \frac{d\bar{u}_c(s,t)}{\bar{u}_c(s,t)} = -s^2 \, dt$$

$$\Rightarrow \log \bar{u}_c(s,t) = -s^2 t + \log k$$

$$\Rightarrow \bar{u}_c(s,t) = ke^{-s^2 t} \quad \text{---} ①.$$

$\int u \cdot v \cdot dx = uv^1 - u_1 v^2 + u_2 v^3 - - - - .$

Now, applying fourier cosine transformation on 2nd boundary condition.

$$\Rightarrow \int_0^\infty u(x,0) \cos sx \, dx = \int_0^1 \underset{I}{x} \underset{II}{\cos sx} \, dx + 0$$

$$\Rightarrow \bar{u}_c(s,0) = \left[\frac{x \sin sx}{s}\right]_0^1 - \int_0^1 1 \cdot \frac{\sin sx}{s} dx$$

$$= \left(\frac{\sin s}{s} - 0\right) + \frac{1}{s^2}\left[\cos sx\right]_0^1$$

$$= \frac{\sin s}{s} + \frac{1}{s^2}(\cos s - 1) \quad \text{—— (2).}$$

Put $t=0$ in eq^n (1).

$$\Rightarrow \bar{u}_c(s,0) = ke^0 = k \quad \text{—— (3).}$$

From (2) & (3),

$$\boxed{k = \frac{\sin s}{s} + \frac{\cos s - 1}{s^2}}$$

$$\Rightarrow \bar{u}_c(s,t) = \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right) e^{-s^2 t} \quad \text{—— (4).}$$

Now, ~~appline~~ applying Fourier cosine inverse transformation on eq^n (4).

$$\Rightarrow \boxed{u(x,t) = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right) e^{-s^2 t} \cos sx \, ds}$$