Rai Technology University

ENGINEERING MINDS

Operations Research

Subject: OPERATIONS RESEARCH **Credits: 4**

SYLLABUS

Development
Definition, Characteristics and phase of Scientific Method, Types of models. General methods for solving operations research models.

Allocation: Introduction to linear programming formulation, graphical solution, Simplex ethod, artificial variable technique, Duality principle. Sensitivity analysis.

Transportation Problem
Formulation optimal solution. Unbalanced transportation problems, Degeneracy. Assignment problem, Formulation optimal solution, Variation i.e., Non-square (m x n) matrix restrictions.

Sequencing
Introduction, Terminology, notations and assumptions, problems with n-jobs and two machines, optimal sequence algorithm, problems with n-jobs and three machines, problems with n-jobs and m-machines, graphic solutions. Travelling salesman problem.

Replacement
Introduction, Replacement of items that deteriorate with time – value of money unchanging and changing, Replacement of items that fail completely.

Queuing Models
M.M.1 & M.M.S. system cost considerations.

Theory of games
introduction, Two-person zero-sum games, The Maximum –Minimax principle, Games without saddle points – Mixed Strategies, 2 x n and m x 2 Games – Graphical solutions, Dominance property, Use of L.P. to games, Algebraic solutions to rectangular games.

Inventory
Introduction, inventory costs, Independent demand systems: Deterministic models – Fixed order size systems – Economic order quantity (EOQ) – Single items, back ordering, Quantity discounts (all units quantity discounts), Batch – type production systems: Economic production quantity – Single items, Economic production quantity multiple items. Fixed order interval systems: Economic order interval (EOI) –Single items, Economic order interval (EOI) – Multiple items.

Network Analysis
Elements of project scheduling by CPM and PER

Suggested Readings:

Operations Research: an introduction, Hamdy A. Taha, Pearson Education
Operations Research: theory and application, J.K. Sharma, Macmillan Publishers
Introduction to Operations Research: concept and cases, Frederick S. Hillier and Gerald J. Lieberman, Tata McGraw-Hill

COURSE OVERVIEW

Motivation To Study The Subject

The aim of this subject is to provide a student with a broad and in depth knowledge of a range of operation research models and techniques, which can be applied to a variety of industrial applications.

Salient Aspects of the Subject

To achieve the credit of this subject a student must:

1. Investigate various Methods for solving different operations research models
2. Investigate suitable procedures for a given Transportation, Assignment problems
3. Investigate suitable procedures for a given sequencing and queuing problems
4. Investigate various procedures for different games (game theory)
5. Investigate Various Inventory models and elements of scheduling by CPM and PERT

Learning Outcome

To achieve each outcome a student must demonstrate the ability to :

1. Identify different types of OR models

   Identify and select procedures for solving various OR models
2. Identify and select procedures for various Transportation problems

   Identify and select procedures for various assignment problems
3. Identify and select procedures for various sequencing problems

   Identify and select procedures for various queuing problems
4. Identify and select suitable methods for various games

   Apply linear programming to games

   Obtain algebraic solution to games
5. Identify various inventory models

   Investigate network analysis on elements of scheduling by CPM and PERT techniques.

OPERATION RESEARCH

CONTENT

OPERATION RESEARCH

CONTENT

LESSON 1
INTRODUCTION TO OPERATION RESEARCH

Let us throw some light of what we are learning in this lesson:

In this lecture we will be studying:

- What is Operation Research
- Definition
- Scope

Operations Research is the science of rational decision-making and the study, design and integration of complex situations and systems with the goal of predicting system behavior and improving or optimizing system performance.

The formal activities of operation research were initiated in England during World War II to make decisions regarding the best utilization of war material. After the war the ideas advanced in military operations were adapted to improve efficiency and productivity in the civilian sector. That developed to today's dominant and indispensable decision-making tool, Operations research. It encompasses managerial decision making, mathematical and computer modeling and the use of information technology for informed decision-making.

The concepts and methods of Operations Research are pervasive. Students and graduates advise the public and private sectors on energy policy; design and operation of urban emergency systems; defense; health care; water resource planning; the criminal justice system; transportation issues. They also address a wide variety of design and operational issues in communication and data networks; computer operations; marketing; finance; inventory planning; manufacturing; and many areas designed to improve business productivity and efficiency. The subject impacts biology, the internet, the airline system, international banking and finance. It is a subject of beauty, depth, infinite breadth and applicability.

The Meaning of Operations Research

From the historical and philosophical summary just presented, it should be apparent that the term "operations research" has a number of quite distinct variations of meaning. To some, OR is that certain body of problems, techniques, and solutions that has been accumulated under the name of OR over the past 30 years, and we apply OR when we recognize a problem of that certain genre. To others, it is an activity or process-something we do, rather than know-which by its very nature is applied. Perhaps in time the meaning will stabilize, but at this point it would be premature to exclude any of these interpretations. It would also be counterproductive to attempt to make distinctions between "operations research" and the "systems approach." While these terms are sometimes viewed as distinct, they are often conceptualized in such a manner as to defy separation. Any attempt to draw boundaries between them would in practice be arbitrary.

How, then, can we define operations research? The Operational Research Society of Great Britain has adopted the following definition:

Operational research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government, and defense. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management determine its policy and actions scientifically.

Operations research is concerned with scientifically deciding how to best design and operate man-machine systems, usually under conditions requiring the allocation of scarce resources.

Although both of these definitions leave something to be desired, they are about as specific as one would want to be in defining such a broad area. It is noteworthy that both definitions emphasize the *motivation* for the work; namely, to aid decision makers in dealing with complex real-world problems. Even when the methods seem to become so abstract as to lose real-world relevance, the student may take some comfort in the fact that the ultimate goal is always some useful application. Both definitions also mention *methodology*, describing it only very generally as "scientific." That term is perhaps a bit too general, inasmuch as the methods of science are so diverse and varied. A more precise description of the OR methodology would indicate its reliance on "models." Of course, that term would itself require further elaboration, and it is to that task that we now turn our attention.

Operations Research has been defined so far in various ways and still not been defined in an authoritative way. Some important and interesting opinions about the definition of OR which have been changed according to the development of the subject been given below:

OR is a scientific method of providing executive departments with a quantitative basis for decision regarding the operations under their control.

-Morse and Kimbal(9164)

OR is a scientific method of providing executive with an analytical and objective basis for decisions.

- P.M.S.Blacket(1948)

OR is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide these in control of the operations with optimum solutions to the problem."

-Churchman, Acoff, Arnoff (1957)

OR is the art of giving bad answers to problems to which otherwise worse answers are given.

-T. L Saaty (1958)

OR is a management activity pursued in two complementary ways-one half by the free and bold

Exercise of commonsense untrammeled by any routine, and other half by the application of a repertoire of well-established pretreated methods and techniques.

-Jagjit Singh (1968)

OR utilizes the planned approach (updated scientific method) and an inter-disciplinary team in order to represent complex functional relationships as mathematical models for purpose of providing a quantitative basis for decision making and uncovering new problems for quantitative analysis.

- Thieanfand"Klekamp (1975)

From all above opinions, we arrive at the conclusion that whatever else 'OR' may be, it is certainly concerned with optimization problems. *A decision, which taking into account all the present circumstances can be considered the best one, is called an optimal decision.*

There are three main reasons for why most of the definitions of Operations Research are not satisfactory.

First of all. Operations Research is not a science like any well-defined physical, biological, social phenomena. While chemists know about atoms and molecules and have theories about their interactions; and biologists know about living organisms and have theories about vital processes, *operations researchers* do not claim to know or have theories about operations. Operations Research is not a scientific research into the control of operations. It is essentially a collection of mathematical techniques and tools, which in conjunction with a system approach are applied to solve practical decision problems of an *economic* or *engineering* nature. Thus it is very difficult to define Operations Research precisely.

ii. Operations Research is inherently inter-disciplinary in nature with applications not only in military and business but also in medicine, engineering, physics and so on. Operations Research makes use of experience and expertise of people from different disciplines for developing new methods and procedures. Thus, inter-disciplinary approach is an important characteristic of Operations Research, which is not included in most of its definitions. Hence most of the definitions are not satisfactory.

iii. Most of the definitions of Operations Research have been offered at different times of development of 'OR' and hence are bound to emphasise its only one or the other aspect.

For example. 8th of the above definitions is only concerned with war alone. First definition confines 'OR' to be a scientific methodology applied for making operational decisions. It has no concern about the characteristics of different operational decisions and has not described how the scientific methods are applied in complicated situations. Many more definitions have been given by various authors but most of them fail to consider all basic characteristics of 'OR'. However, with further development of 'OR' perhaps more precise definitions should be forthcoming.

Notes:

LESSON 2
MODELS IN OPERATIONS RESEARCH

In this lecture we have to study:

- Modelling in Operations Research
- Principles of Modelling
- General Methods for Solving 'Or' Models

The essence of the operations research activity lies in the construction and use of models. Although modeling must be learned from individual experimentation, we will attempt here to discuss it in broad, almost philosophical terms. This overview is worth having, and setting a proper orientation in advance may help to avoid misconceptions later.

First, one should realize that some of the connotations associated with the word "model" in common English usage are not present in the OR use of the word. A model in the sense intended here is just a simplified representation of something real. This usage does carry with it the implication that a model is always, necessarily, a representation that it is less than perfect.

There are many conceivable reasons why one might prefer to deal with a substitute for the "real thing" rather than with the "thing" itself. Often, the motivation is economic-to save money, time, or some other valuable commodity. Sometimes it is to avoid risks associated with the tampering of a real object. Sometimes the real environment is so complicated that a representative model is needed just to understand it, or to communicate with others about it. Such models are quite prevalent in the life sciences, physical chemistry, and physics.

The first step is construction of the model itself, which is indicated by the line labeled "Formulation." This step requires a set of coordinated decisions as to what aspects of the real system should be incorporated in the model, what aspects can be ignored, what assumptions can and should be made, into what form the model should be cast, and so on. In some instances, formulation may require no particular creative skill, but in most cases-certainly the interesting ones-it is decidedly an art. The election of the essential attributes of the real system and the omission of the irrelevant ones require a kind of selective perception that cannot be defined by any precise algorithm It is apparent, then, that the formulation step is characterized by a certain amount of arbitrariness, in the sense that equally competent researchers, viewing the same real system, could come up with completely different models. At the same time, the freedom to select one's assumptions cannot be taken to imply that one model is any better than another. A discussion of the boundary between reasonable and unreasonable assumptions would take us into philosophical issues, which we could not hope to resolve. In fact, as one ponders the problem of model formulation, one discloses all sorts of metaphysical questions, such as how to define the precise boundary between the model and its referent, how to distinguish between what is real and what is only our perception, the implications of the resultant cause and effects of modeling procedure, and so on. These issues are mentioned only to reemphasize that it is quite meaningless to speak of the "right" way to formulate a mode" The formative stages of the modeling process might be repeated and analyzed many times before the proper course of action becomes readily apparent. Once the problem formulation and definition is agreed upon, a more scientific step in the modeling process is begun.

The step labeled "Deduction" involves techniques that depend on the nature of the model. It may involve solving equations, running a computer program, expressing a sequence of logical statements-whatever is necessary to solve the problem of interest relative to the model. Provided that the assumptions are clearly stated and well defined, this phase of modeling should *not* be subject to differences of opinion. The logic should be valid and the mathematics should be rigorously accurate. All reasonable people should agree that the model conclusions follow from the assumptions, even if they do not all agree with the necessary assumptions. It is simply a matter of abiding by whatever formal rules of manipulation are prescribed by the methods in use. It is a vital part of the modeling process to rationalize, analyze, and conceptualize all components of the deductive process.

The final step, labeled "Interpretation," again involves human judgment. The model conclusions must be translated to real-world conclusions cautiously, in full cognizance of possible discrepancies between the model and its real-world referent. Aspects of the real system that were either deliberately or unintentionally overlooked when the model was formulated may turn out to be important.

To further clarify the modeling approach to problem solving, contrast it to the experimentally based "scientific method" of the natural sciences. Here, the first step is the development of a hypothesis, which is arrived at, generally by induction, following a period of informal observation. At that point an experiment is devised to test the hypothesis. If the experimental results contradict the hypothesis, the hypothesis is revised and retested. The cycle continues until a verified hypothesis, or *theory,* is obtained. The result of the process is something that purports to be "truth," "knowledge," or "a law of nature." In contrast to model conclusions, theories are independently verifiable statements about factual matters. Models are invented; theories are discovered.

It might also be noted that there exist other formalized procedures for reaching conclusions about the real world. The system of trial by jury, as a means to decide between guilt or innocence of accused criminals, is one example. The determination of policy within a group by vote according to personal preference is another. Thus, modeling is a very important but certainly not unique method to assist us in dealing with complicated real-world problems.

Modelling in Operations Research

The essence of the operations research activity lies in the construction and use of models. Although modeling must be learned from individual experimentation, we will attempt here to discuss it in broad, almost philosophical terms. This overview is worth having, and setting a proper orientation in advance may help to avoid misconceptions later.

Definition. *A model in the sense used in 0 R is defined as a representation of an actual object or situation. It shows the relationships (direct or indirect) and inter-relationships of action and reaction in terms of cause and effect.*

Since a model is an abstraction of reality, it thus appears to be less complete than reality itself. For a model to be complete, it must be a representative of those aspects of reality that are being investigated.

The main objective of a model is to provide means for analysing the behaviour of the system for the purpose of improving its performance. Or, if a system is not in existence, then a model defines the ideal structure of this future system indicating the functional relationships among its elements. The reliability of the solution obtained from a model depends on the validity of the model in representing the real systems. A model permits to 'examine the behaviour of a system without interfering with ongoing operations.

Models can be Classified According to Following Characteristics:

1. Classification by Structure

i. *Iconic models. Iconic models* represent the system as it is by scaling it up or down *(i.e.,* by enlarging or reducing the size). In other words, it is an image.

   For example, a toy airplane is an iconic model of a real one. Other common examples of it are: photographs, drawings, maps etc. A model of an atom is scaled up so as to make it visible to the naked eye. In a globe, the diameter of the earth is scaled down, but the globe has approximately the same shape as the earth, and the relative sizes of continents, seas, etc., are approximately correct. The iconic model is usually the simplest to conceive and the most specific and concrete. Its function is generally descriptive rather than explanatory. Accordingly, it cannot be easily used to determine or predict what effects many important changes on the actual system.

ii. *Analogue models.* The models, in which one set of properties is used to represent another set of properties, are called *analogue models.* After the problem is solved, the solution is reinterpreted in terms of the original system.

   For example, graphs are very simple analogues because distance is used to represent the properties such as: *time, number, percent, age, weight,* and many other properties. Contour lines on a map represent the rise and fall of the heights. In general, analogues are less specific, less concrete but easier to manipulate than are iconic models.

iii. *Symbolic (Mathematical) models.* The *symbolic or mathematical model* is one which employs a set of mathematical symbols *(i.e., letters, numbers*, etc.) to represent the decision variables of the system. These variables are related together by means of a mathematical equation or a set of equations to describe the behaviour (or properties) of the system. The solution of the problem is then obtained by applying well-developed mathematical techniques to the model.

   The symbolic model is usually the easiest to manipulate experimentally and it is most general and abstract. Its function is more often explanatory rather than descriptive.

2. Classification by Purpose

Models can also be classified by purpose of its utility. *The purpose of a model may be* descriptive predictive or prescriptive.

i. *Descriptive models.* A descriptive model simply describes some aspects of a situation based on observations, survey. Questionnaire results or other available data. The result of an opinion poll represents a descriptive model.

ii. *Predictive models.* Such models can answer 'what if' type of questions, *i.e.* they can make predictions regarding certain events. For example, based on the survey results, television networks such models attempt to explain and predict the election results before all the votes are actually counted.

iii. *Prescriptive models.* Finally, when a predictive model has been repeatedly successful, it can be used to prescribe a source of action. For example, linear programming is a prescriptive (or normative) model because it prescribes what the managers ought to do.

3. Classification by Nature of Environment

These are mainly of two types:

i. *Deterministic models.* Such models assume conditions of complete certainty and perfect knowledge.

   For example, linear programming, transportation and assignment models are deterministic type of models.

ii. *Probabilistic (or Stochastic) models.* These types of models usually handle such situations in which the consequences or payoff of managerial actions cannot be predicted with certainty. However, it is possible to forecast a pattern of events, based on which managerial decisions can be made. For example, insurance companies are willing to insure against risk of fire, accidents, sickness and so on, because the pattern of events have been compiled in the form of probability distributions.

4. Classification by Behaviour

i. *Static models.* These models do not consider the impact of changes that takes place during the planning horizon, *i.e.* they are independent of time. Also, in a static model only one decision is needed for the duration of a given time period.

ii *Dynamic models.* In these models, *time* is considered as one of the important variables and admits the impact of changes generated by time. Also, in dynamic models, not only one but a series of interdependent' decisions is required during the planning horizon.

5. Classification by Method of Solution

i. *Analytical models.* These models have a specific mathematical structure-and thus can be solved by known analytical or mathematical techniques. For example, a *general linear progra11iming* model as well as the specially structured *transportation* and *assignment models are analytical models.* .

ii. *Simulation models.* They also have a mathematical structure but *they cannot be solved* by purely using the 'tools' and 'techniques' of mathematics. A simulation model is essentially computer-assisted experimentation on a mathematical structure of a real time structure in order to study the system under a variety of assumptions.

Simulation modelling has the advantage of being more flexible than mathematical modelling and hence can be used to represent complex systems which otherwise cannot be formulated mathematically. On the other hand, simulation has the disadvantage of not providing general solutions like those obtained from successful mathematical models.

6. Classification by Use of Digital Computers

The development of the digital computer has led to the introduction of the following types of modelling in OR.

i. *Analogue and Mathematical models combined.* Sometimes analogue models are also expressed in terms of mathematical symbols. Such models may belong to both the types (ii) and (iii) in classification 1 above.

For example, Simulation model is of analogue type but mathematical formulae are also used in it. Managers very frequently use this model to *'simulate'* their decisions by summarizing the activities of industry in a scale-down period.

ii. *Function models.* Such models are grouped on the basis of the function being performed.

For example, a function may serve to acquaint to scientist with such things as-tables, carrying data, a blue-print of layouts, a program representing a sequence of operations (like' in computer programming).

(Hi) *Quantitative models.* Such models are used to measure the observations.

For example, degree of temperature, yardstick, a unit of measurement of length value, etc.

Other examples of quantitative models are: (i) *transformation models* which are useful in converting a measurement of one scale to another *(e.g.,* Centigrade *vs.* Fahrenheit conversion scale), and (ii) the *test models* that act as 'standards' against which measurements are compared *(e.g.,* business dealings, a specified standard production control, the quality of a medicine).

iv. *Heuristic models.* These models are mainly used to explore alternative strategies (courses of action) that were overlooked previously, whereas mathematical models are used to represent systems possessing well-defined strategies. Heuristic models do not claim to find the best solution to the problem.

Principles of Modelling

Let us now outline general principles useful in guiding to formulate the models within the context of 0 R. The model building and their users both should be consciously aware of the following *Ten* principles:

1. *Do not build up a complicated model when simple one will suffice.* Building the strongest possible model is a common guiding principle for mathematicians who are attempting to extend the theory or to develop techniques that have wide applications. However, in the actual practice of building models for specific purposes, the best advice is to "keep it simple".
2. *Beware of molding the problem to fit the technique.* For example, an expert on linear programming techniques may tend to view every problem he encounters as required in a linear programming solutions. In fact, not all optimization problems involve only linear functions. Also, not all OR problems involve optimization. As a matter of fact, not all real-world problems call for *operations research!* Of course, everyone search reality in his own terms, so the field of OR is not unique in this regard. Being human we rely on the methods we are most comfortable in using and have been most successful within the past. We are certainly not able to use techniques in which we have no competence, and we cannot hope to be competent in all techniques. We must divide OR experts into three main categories:

   (i) Technique developers. (ii) Teacher and (iii) Problem solvers.

   In particular one should be ready to tolerate the behaviour "I have found a cure but I am trying to search a disease to fit it" among *technique developers* and *teachers.*
3. *The deduction phase of modelling must be conducted rigorously*. The reason for requiring rigorous deduction is that one wants to be sure that if model conclusions are inconsistent with reality, then the defect lie in the assumptions. One application of this principle is that one must be extremely careful when programming computers. Hidden "bugs" are especially dangerous when they do not prevent the program from running but simply produce results, which are not consistent with the intention of the model.
4. *Models should be validated prior to implementation.* For example, if a model is constructed to forecast the monthly sales of a particular commodity, it could be tested using historical data to compare the forecasts it would have produced to the actual sales. In case, if the model cannot be validated prior to its implementation, then it can be implemented in phases for validation. For example a new model for inventory control may be implemented for a certain selected

group of items while the older system is retained for the majority of remaining items. If the model proves successful, more items can be placed within its range. It is also worth noting that real things change in time. A highly satisfactory model may very well degrade with age. So periodic re-evaluation is necessary.

5. *A model should never be taken too literally.* For example, suppose that one has to construct an elaborate computer model of Indian economy with many competent researchers spending a great deal of time and money in getting all kinds of complicated interactions and relationships. Under such circumstances, it can be easily believed as if the model duplicates itself the real system. This danger continues to increase as the models become larger and more sophisticated, as they must deal with increasingly complicated problems.
6. *A model should neither be pressed to do, nor criticized for failing to do that for which it was never intended.* One example of this error would be the use of forecasting model to predict so far into the future that the data on which the forecasts are based have no relevance. Another example is the use of certain network methods to describe the activities involved in a complex project. A model should not be stretched beyond its capabilities.
7. *Beware of over-selling a model.* This principle is of particular importance for the OR professional because most non-technical benefactors of an operations researcher's work are not likely to understand his methods. The increased technicality of one's methods also increases the burden of responsibility on the OR professional to distinguish clearly between his role as model manipulator and model interpreter. In those cases where the assumptions can be challenged, it would be dishonest to use the model.
8. *Some of the primary benefits of modelling are associated with the process of developing the model.* It is true in general that a model is never as useful to anyone else as it is to those who are involved in building it up. The model itself never contains the full knowledge and understanding of the real system that the builder must acquire in order to successfully model it, and there is no practical way to convey this knowledge and understanding properly. In some cases the sole benefits may occur while the model is being developed. In such cases, the model may have no further value once it is completed, An example of this case might occur when a small group of people attempts to develop a formal plan for some subject. The plan is the final model, but the real problem may be to agree on 'what the objectives ought to be'.
9. A *model cannot be any better than the information that goes into it.* Like a computer program, a model can only manipulate the data provided to it; it cannot recognize and correct for deficiencies in input. Models may *condense* data or con*vert* it to more useful forms, but they do not have the capacity to generate it. In some situations it is always better to gather more information about the system instead of exerting more efforts on modern constructions.
10. *Models cannot replace decision makers.* The purpose of OR models should not be supposed to provide "Optimal solutions" free from human subjectivity and error. OR models can aid decision makers and thereby permit better decisions to be made. However they do not make the job of decision making easier. Definitely, the role of experience, intuition and judgment in decision-making is undiminished.

General Methods for Solving 'OR' Modles

In OR, we do not have a single general technique that solves all mathematical models that arise in practice. Instead, the type and complexity of the mathematical model dictate the nature of the solution method. For example, in Section 1.1 the solution of the tickets problem requires simple ranking of the alternatives based on the total purchasing price, whereas the solution of the rectangle problem utilizes differential calculus to determine the maximum area.

The most prominent OR technique is linear programming. It is designed for models with strict linear objective and constraint functions. Other techniques include integer programming (in which the variables assume integer values), dynamic programming (in which the original model can be decomposed into smaller sub. problems), network programming (in which the problem can be modeled as a network), and nonlinear programming (in which the functions of the model are non. linear). The cited techniques are but a partial list of the large number of available OR tools.

A peculiarity of most OR techniques is that solutions are not generally obtained in (formula-like) closed forms. Instead, they are determined by algorithms. An algorithm provides fixed computational rules that are applied repetitively to the problem with each repetition (called iteration) moving the solution closer to the optimum. Because the computations associated with each iteration are typically tedious and voluminous, it is imperative that these algorithms be executed on the computer. .

Some mathematical models may be so complex that it is impossible to solve them by any of the available optimization algorithms. In such cases, it may be necessary to abandon the search for the *optimal* solution and simply seek a *good* solution using heuristics or *rules of thumb*.

Generally three types of methods are used for solving OR models.

Analytic Method. If the OR model is solved by using all the tools of classical mathematics such as: differential calculus and finite differences available for this task, then such type of solutions are called *analytic solutions.* Solutions of various inventory models are obtained by adopting the so-called analytic procedure.

Iterative Method. If classical methods fail because of complexity of the constraints or of the number of variables, then we are usually forced to adopt an iterative method. Such a procedure starts with a trial solution and a set of rules for improving it. The trial solution is then replaced by the improved solution, and the process is repeated until either no further improvement is possible or the cost of further calculation cannot be justified.

Iterative method can be divided into three groups:

a. After a finite number of repetitions, no further improvement will be possible.

b. Although successive iterations improve the solutions, we are only guaranteed the solution as a limit of an infinite process.

c. Finally we include the *trial* and *error* method, which, however, is likely to be lengthy, tedious, and costly even if electronic computers are used.

The Monte-Carlo Method. The basis of so-called Monte-Carlo technique is random sampling of variable's values from a distribution of that variable. Monte-Carlo refers to the use of sampling methods to estimate the value of non-stochastic variables. The following are the main steps of Monte-Carlo method:

Step 1. In order to have a general idea of the system, we first draw *a flow diagram* of the system.

Step 2. Then we take correct sample observations to select some suitable model for the system. In this step we compute the probability distributions for the variables of our interest.

Step 3. We, then, convert the probability distributions to a cumulative distribution function.

Step 4. A sequence of random numbers is now selected with the help of random number tables.

Step 5. Next we determine the sequence of values of variables of interest with the sequence of random numbers obtained in step 4.

Step 6. Finally we construct some standard mathematical function to the values obtained in *step* 5.*Step* 3. *Step* 4. *Step* 5.

Notes:

LESSON 3
INTRODUCTION TO LINEAR PROGRAMMING

In this lecture we will be studying:

- **Linear Programming-Introduction**
- **Definitions**
- **Formulation**

Introduction

Observe the world and predictable patterns will emerge. When air pressure falls, bad weather often follows. High and low tides relate to the setting and rising of the moon. Hair and eye color pass from parent to child. Patterns emerge from systems as small as the atom and as large as the universe. Mathematics allows us to express these patterns as equations, and thereby to predict future changes. Linear algebra, the mathematics of systems of equations, further allows complex interactions within a system to be expressed as a unified entity representing the entire system all at once: a matrix.

Linear algebra provides the methods necessary to analyze unwieldy systems. Insight into information such as the stresses and strains at multiple locations in a building, harmonic frequencies of an aircraft design, and how to maximize and minimize many individual aspects of an entire system allows the development of larger and more complex designs. Of special interest to industry, even after 40 years of intense research, is the ability to optimize.

Linear Programming

Linear programming is a powerful quantitative technique (or operational research technique) designs to solve allocation problem. The term 'linear programming' consists of the two words 'Linear' and 'Programming'. The word 'Linear' is used to describe the relationship between decision variables, which are directly proportional. For example, if doubling (or tripling) the production of a product will exactly double (or triple) the profit and required resources, then it is linear relationship. The word 'programming' means planning of activities in a manner that achieves some 'optimal' result with available resources. A programme is 'optimal' if it maximises or minimises some measure or criterion of effectiveness such as profit, contribution (i.e. sales-variable cost), sales, and cost. Thus, 'Linear Programming' indicates the planning of decision variables, which are directly proportional, to achieve the 'optimal' result considering the limitations within which the problem is to be solved.

Decision Variables

The decision variables refer to the economic or physical quantities, which are competing with one another for sharing the given limited resources. The relationship among these variables must be linear under linear programming. The numerical values of decision variables indicate the solution of the linear programming problem

Objective Function

The objective function of a linear programming problem is a linear function of the decision variable expressing the objective of the decision maker. For example, maximisation of profit or contribution, minimisation of cost/time.

Constraints

The constraints indicate limitations on the resources, which are to be allocated among various decision variables. These resources may be production capacity, manpower, time, space or machinery. These must be capable of being expressed as linear equation (i.e. =) on inequalities (i.e. $\geq$ or $\leq$; type) in terms of decision variables. Thus, constraints of a linear programming problem are linear equalities or inequalities arising out of practical limitations.

Non-negativity Restriction

Non-negativity restriction indicates that all decision variables must take on values equal to or greater than zero.

Divisibility

Divisibility means that the numerical values of the decision variables are continuous and not limited to integers. In other words, fractional values of the decision variables must be permissible in obtaining optimal solution.

Step Involved in the Formulation of LP Problem

The steps involved in the formation of linear programming problem are as follows:

Step 1: Identify the Decision Variables of interest to the decision maker and express them as XI, *X2, X3*

Step 2: Ascertain the Objective of the decision maker whether he wants to minimize or to maximize.

Step 3: Ascertain the cost (in case of minimization problem) or the profit (in case of

maximization problem) per unit of each of the decision variables.

Step 4: Ascertain the constraints representing the maximum availability or minimum

commitment or equality and represent them as less than or equal to ($\leq$) type inequality or greater than or equal to ($\geq$) type inequality or 'equal to' (=) type equality respectively.

Step 5: Put non-negativity restriction as under:

$X_j \geq 0$; j = 1, 2… n (non-negativity restriction)

Step 6: Now formulate the LP problem as under:

Subject Maximize (or Minimize) $Z = c_1x_1 + c_2x_2 \ldots c_nx_n$

to constraints:

$$a_{11}x_1 + a_{12}x_2 \ldots\ldots a_{1n}x_n \leq b_1 \text{ (Maximum availability)}$$
$$a_{21}x_1 + a_{22}x_2 \ldots\ldots a_{2n}x_n \geq b_2 \text{ (Minimum commitment)}$$
$$a_{31}x_1 + a_{32}x_2 \ldots\ldots a_{3n}x_n = b_3 \text{ (Equality)}$$
$$a_{m1}x_1 + a_{m2}x_2 \ldots\ldots a_{mn}x_n \lesseqgtr b_m$$
$$x_1; x_2 \ldots\ldots x_n \geq 0 \text{ (Non-negativity restriction)}$$

where,

x_j = Decision Variables i.e. quantity of j^{th} variable of interest to the decision maker.

c_j = Constant representing per unit contribution (in case of Maximization Problem) or Cost (in case of Minimization Problem) of the j^{th} decision variable.

a_{ij} = Constant representing exchange coefficients of the j^{th} decision variable in the i^{th} constant.

b_i = Constant representing i^{th} constraint requirement or availability.

Example. The manufacturer of patent medicines is proposed to prepare a production plan for medicines A and B. There are sufficient ingredients available to make 20,000 bottles of medicine A and 40,000 bottles of medicine B, but there are only 45,000 bottles into which either of the medicines can be filled. Further, it takes three hours to prepare enough material to fill 1000 bottles of medicine A and one hour to prepare enough material to fill 1 000 bottles of medicine B, and there are 66 hours available for this operation. The profit is Rs. 8 per bottle for medicine A and Rs. 7 per bottle for medicine B.

i. Formulate this problem as a LP.P. ,

ii. How the manufacturer schedule his production in order to maximize profit.

Formulation. (i) Suppose the manufacturer produces XI and X2 thousand of bottles of medicines A and B, respectively. Since it takes three hours to prepare 1000 bottles of medicine A, the time required to fill XI thousand bottles of medicine A will be 3xI hours. Similarly, the time required to prepare X2 thousand bottles of medicine B will be X2 hours. Therefore, total time required to prepare XI thousand bottles of medicine A and X2 thousand bottles of medicine B will be

$3x_1 + x_2$ hours.

Now since the total time available for this operation is 66 hours, $3x_1 + x_2 \leq 66$.

Since there are only 45 thousand bottles available for filling medicines A and B, $x_1 + x_2 \leq 45$.

There are sufficient ingredients available to make 20 thousand bottles of medicine A and 40 thousand bottles of medicine B, hence $x_1 \leq 20$ and $x_2 \leq 40$.

Number of bottles being non-negative, $x_1 \geq 0$, $x_2 \geq 0$.

At the rate of Rs. 8 per bottle for type A medicine and Rs. 7 per bottle for type B medicine, the total profit on x_1 thousand bottles of medicine A and x_2 thousand bottles of medicine B will become

$$P = 8 \times 1000\, x_1 + 7 \times 1000\, x_2 \quad \text{or} \quad P = 8000\, x_1 + 7000\, x_2.$$

Thus, the linear programming problem is:

Max. $P = 8000\, x_1 + 7000\, x_2$, **subject to the constraints:**

$$3x_1 + x_2 \leq 66,\ x_1 + x_2 \leq 45,\ x_1 \leq 20,\ x_2 \leq 40$$

and $x_1 \geq 0,\ x_2 \geq 0$.

Example 2. A toy company manufactures two types of doll, a basic version-doll A and a deluxe version-doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of Rs. 3.00 and Rs. 5.00 per doll, respectively on doll A and B, then how many of each dolls should be produced per day in order to maximize the total profit. Formulate this problem.

Formulation. Let *XI* and *X2* be the number of dolls produced per day of type *A* and *B*, respectively. Let the doll *A* require *t* hrs so that the doll *B* require *2t* hrs. So the total time to manufacture *XI* and *X2* dolls should not exceed *2,000t* hrs. Therefore, *tXI + 2tx2* < *2000t.* Other constraints are simple. Then the linear programming problem becomes:

Maximize $P = 3x_1 + 5x_2$

subject to the restrictions

$$x_1 + 2x_2 \leq 2000 \quad \text{(time constraint)}$$
$$x_1 + x_2 \leq 1500 \quad \text{(plastic constraint)}$$
$$x_2 \leq 600 \quad \text{(dress constraint)}$$

and non–negativity restrictions

$$x_1 \geq 0,\ x_2 \geq 0.$$

Notes:

LESSON 4
L P GRAPHICAL SOLUTION

In this lecture we have to study:

- Linear Programming Method
- Practical Steps Involved In Solving LPP By Graphical Method

Linearprogramminggraphical Method

Graphical Method is used for solving linear programming problems that involve only two variables.

Closed Half Plane

A linear inequality in two variables is known as a half plane. The corresponding equality or the line is known as the boundary of the half plane. The half plane along with its boundary is called a closed half plane. Thus, a closed half plane is a linear inequality in two variables, which include the value of the variable for which equality is attained.

Feasible Solution

Any non-negative solution which satisfies all the constraints is known as a feasible solution of the problem.

Feasible Region

The collection of all feasible solutions is known as a feasible region.

Convex Set

A set (or region) is convex if only if for any two points on the set, the line segment joining those points lies entirely in the set. Thus, the collection of feasible solutions in a linear programming problem form a convex set. In other words, the feasible region of a linear programming problem is a convex set.

Convex Polygon

A convex polygon is a convex set formed by the intersection of a finite number of closed half planes.

Extreme Points or Vertexes or Corner Points

The extreme points of a convex polygon are the points of intersection of the lines bounding the feasible region. The value of the decision variables, which maximise or minimise the objective function is located on one of the extreme points of the convex polygon. If the maximum or minimum value of a linear function defined over a convex polygon exists, then it must be on one of the extreme points.

Redundant Constraint

Redundant constraint is a constraint, which does not affect the feasible region.

Multiple Solutions

Multiple solutions of a linear programming problem are solutions each of which maximize or minimize the objective function. Under graphical method, the existence of multiple solutions is indicated by a situation under which the objective function line coincides with one of the half planes generated by a constraint. In other words, where both the objective function line and one of constraint lines have the same slope.

Unbounded Solution

An unbounded solution of a linear programming problem is a solution whose objective function is infinite. A linear programming problem is said to have unbounded solution if its solution can be made infinitely large without violating any of the constraints in the problem. Since there is no real applied problem, which has infinite returns, hence an unbounded solution always represents a problem that has been incorrectly formulated.

For example, in a maximization problem at least one of the constraints must be an equality' or 'less than or equal to' ($\leq$) type. If all of the constraints are 'greater than or equal to' ($\geq$) type, then there will be no upper limit on the feasible region. Similarly for minimization problem, at least one of constraints must be an 'equality' or 'a greater than or equal to' type ($\geq$) if a solution is to be bounded.

Under graphical method, the feasible solution region extends indefinitely.

Infeasible Problem

A linear programming problem is said to be infeasible if there is no solution that satisfies all the constraints. It represents a state of inconsistency in the set of constraints.

Practical Steps Involved in Solving LPP By Graphical Method

The practical steps involved in solving linear programming problems by Graphical Method are given below:

Step 1: Consider each inequality constraint as equation.

Step 2: Take one variable (say x) in a given equation equal to zero and find the value of other variable (say y) by solving that equation to get one co-ordinate [say $(0, y)$] for that equation.

Step 3: Take the second variable (say y) as zero in the said equation and find the value of first variable (say x) to get another co-ordinate [say $(x, 0)$] for that equation.

Step 4: Plot both the co-ordinates so obtained [i.e., $(0, y)$ and $(x, 0)$] on the graph and join them by a straight line. This straight line shows that any point on that line satisfies the equality and any point below or above that line shows inequality. Shade the feasible region which may be either convex to the origin in case of less than type of inequality (<) or opposite to the origin in case of more than type of inequality (>).

Step 5: Repeat Steps 2 to 4 for other constraints.

Step 6: Find the common shaded feasible region and mark the co-ordinates of its comer points.

Step 7: Put the co-ordinates of each of such vertex in the Objective Function. Choose that vertex which achieves the most optimal solution (i.e., in the case of Maximisation, the vertex that gives the maximum value of 'Z' & in case of Minimisation the vertex that gives the minimum value of 'Z). Solution can be obtained in the following manner:

Vertex No.	Co-ordinates of vertices of Common shaded feasible region	Value of Z
1	(x_1, y_1)	$Z_1 = \ldots$
2	(x_2, y_2)	$Z_2 = \ldots$
3	(x_3, y_3)	$Z_3 = \ldots$
4 & so on	(x_4, y_4)	$Z_4 = \ldots$

Optimal Solution:

Type of Problem	Optimal Solution
(a) In case of maximisation problem.	The vertex which gives the maximum value of Z is the optimal solution.
(b) In case of minimisation problem.	The vertex which gives the minimum value of Z is the optimal solution.

Example 1

PARLOK Ltd has two products Heaven and Hell. To produce one unit of Heaven, 2 units of material X and 4 units of material Yare required. To produce one unit of Hell, 3 units of material X and 2 units of material Y are required. Only 16 units of material X and 16 units of material Y are available. Material X cost Rs. 2.50 per unit and Material Y cost Rs. 0.25 per unit respectively.

Required: Formulate an LP Model and solve it graphically.

Formulation of LP Model

Calculation of Cost per Unit of each Product

		Heaven		Hell
A	Material X (2.50 × 2)	5.00	(Rs. 2.50 × 3)	7.50
B	Material Y (0.25 × 4)	1.00	(Rs. 0.25 × 2)	0.50
		6.00		8.00

Let x represent the no. of units of product 'Heaven' to produced y represent the no. of units of product 'Hell' to be produced Since the object is to minimise, the objective function is given by -"

Minimise $Z = 6x + 8y$

Subject to constraints

$2x_1 + 3y \leq 16$ [Maximum material X constraint] ....(i)

$4x_1 + 2y \leq 16$ [Maximum material Y constraint] ....(ii)

$x, y \geq 0$ [Non-negativity constraint]

Step 1: Finding the vertex for each constraint by treating the constraint of inequality nature as equality

Constraint (i) in limiting form $2x_1 + 3y = 16$

When $x = 0$, $y = \frac{16}{3}$

When $y = 0$, $x = \frac{16}{2} = 8$

Thus, the vertices are $(0, \frac{16}{3})$ & $(8, 0)$

Constraint (ii) in limiting form $4x_1 + 2y = 16$

When $x = 0$, $y = \frac{16}{2} = 8$

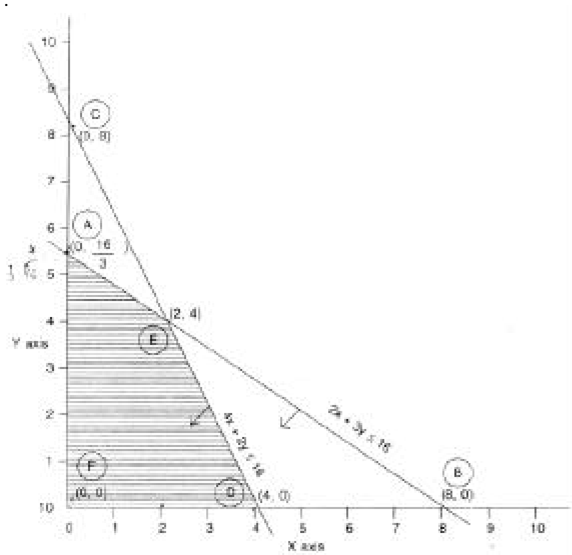
When $y = 0$, $x = \frac{16}{4} = 4$

Thus, the vertices are $(0, 8)$ & $(4, 0)$

Step 2: Plotting both the co-ordinates of the 1st constraint on the graph and joining them by straight line and shading the feasible region, which is convex to origin in case of less than type of inequality. Similarly drawing straight line and shading feasible region for other constraints.

Step 3: Reading the co-ordinates of the vertices of common shaded feasible region and putting the co-ordinates of each of such vertex in the objective function. Selecting those vertices which achieve the most optimal solution (i.e. in case of minimization vertices which give the minimum value of Z)

Set No.	Co-ordinates of Vertices of Common shaded feasible region		Value of Z
1	F	0, 0	$6 \times 0 + 8 \times 0 = 0$
2	A	$0, \frac{16}{3}$	$6 \times 0 + 8 \times \frac{16}{3} = 42.67$
3	E	2, 4	$6 \times 2 + 8 \times 4 = 44.0$
4	D	4, 0	$6 \times 4 + 8 \times 0 = 24.0$



Optimal Solution: Thus set No. 1 gives the minimum value of Z with $x = 0$ and $y=0$.

Example 2

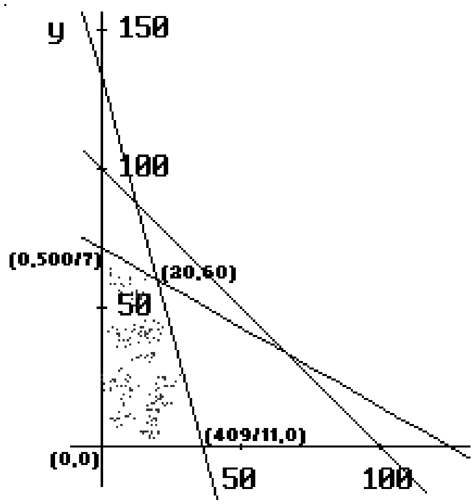
Maximize $f(x, y) = 80x + 70y$ subject to the constraints:

$$\begin{cases} 2x + y \leq 32 \\ x + y \leq 18 \\ x + 3y \leq 36 \\ x, y \geq 0 \end{cases}$$

The corresponding system of linear equations is

$$\begin{cases} y = -2x + 32 \\ y = -x + 18 \\ y = -\frac{1}{3}x + 12 \end{cases}$$

and the polygon which represents the feasible region is



The following table contains the values of the function at the vertices:

(x,y)	f(x,y)
(0,0)	0
(16,0)	1280
(14,4)	1400
(9,9)	1350
(0,12)	840

The maximum value of the function is 1400 and occurs at **(14,4)**.

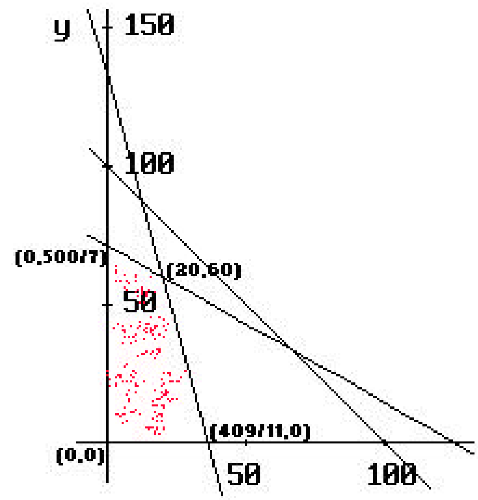
Example 3

Maximize $f(x,y)=143x+60y$ subject to the constraints:

$$\begin{cases} x+y \le 100 \\ 120x+210y \le 15000 \\ 110x+30y \le 4000 \\ x,y \ge 0 \end{cases}$$

Consider the corresponding system of linear equations and the feasible region:

$$\begin{cases} x+y = 100 \\ 120x+210y = 15000 \\ 110x+30y = 4000 \\ x,y = 0 \end{cases}$$



Check the vertices to find that the maximum value is **6460** at **(20,60)**

Problem 2.

A pension fund has Rs.30 million to invest. The money is to be divided among Treasury notes, bonds, and stocks. The rules for administration of the fund require that at least Rs.3 million be invested in each type of investment, at least half the money be invested in Treasury notes and bonds, and the amount invested in bonds not exceed twice the amount invested in Treasury notes. The annual yields for the various investments are 7% for Treasury notes, 8% for bonds, and 9% for stocks. How should the money be allocated among the various investments to produce the largest return?

In millions of rupees, let ***x*****=** the amount in Treasury notes, ***y*****=** the amount in bonds, and **30-(***y*+*y***)=** the amount in stocks. The constraints are: and the objective function to be maximized is:

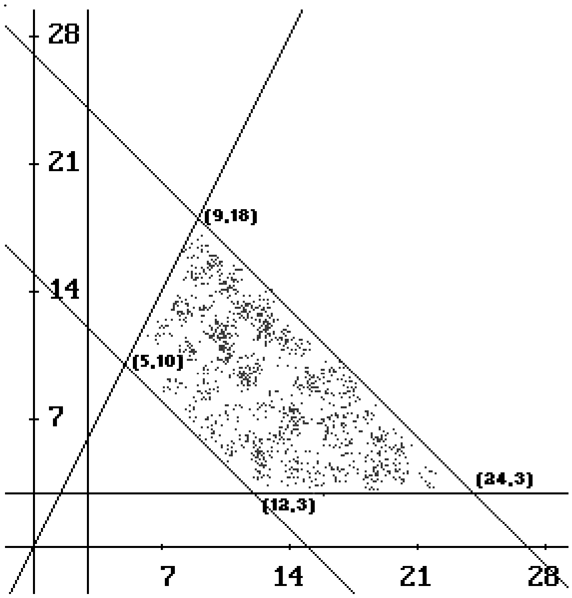
$$\begin{cases} x,y \ge 3 \\ 30-(x+y) \ge 3 \\ x+y \ge 15 \\ y \le 2x \end{cases}$$

$$\begin{aligned} f(x,y) &= .07x+.08y+.09[30-(x+y)] \\ &= 2.7-.02x-.01y \end{aligned}$$

The corresponding system of linear equations is:

$$\begin{cases} x,y = 3 \\ y = -x+27 \\ x+y = 15 \\ y = 2x \end{cases}$$

The feasible region is:



The maximum return is Rs.2.5 million when x**=5** million and y**=10** million.

Notes:

LESSON 5
L P SIMPLEX METHOD

This lesson we are going to study:

Introduction to Simplex Method

Introduction

Observe the world and predictable patterns will emerge. When air pressure falls, bad weather often follows. High and low tides relate to the setting and rising of the moon. Hair and eye color pass from parent to child. Patterns emerge from systems as small as the atom and as large as the universe. Mathematics allows us to express these patterns as equations, and thereby to predict future changes. Linear algebra, the mathematics of systems of equations, further allows complex interactions within a system to be expressed as a unified entity representing the entire system all at once: a matrix.

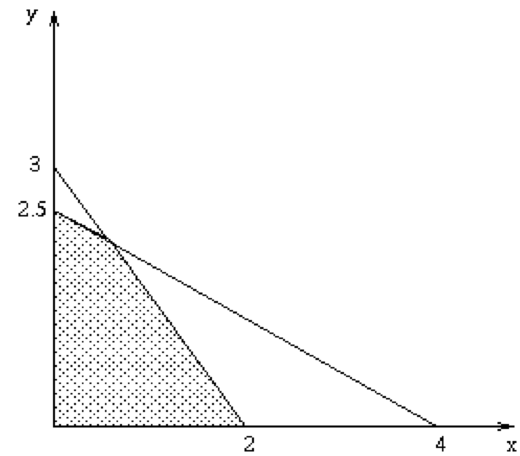
Linear algebra provides the methods necessary to analyze unwieldy systems. Insight into information such as the stresses and strains at multiple locations in a building, harmonic frequencies of an aircraft design, and how to maximize and minimize many individual aspects of an entire system allows the development of larger and more complex designs. Of special interest to industry, even after 40 years of intense research, is the ability to optimize.

Before the 1940s, optimization techniques on systems of equations were essentially heuristics, simply educated guesses. Not until the Simplex Method was first proposed in 1947 by Dantzig did optimization become a defined process resulting in determinate answers. Optimization problems became linear programs, systems of equations that must be satisfied yet are underdetermined and thus allowing an infinite number of solutions.

Before the 1940s, optimization techniques on systems of equations were essentially heuristics, simply educated guesses. Not until the Simplex Method was first proposed in 1947 by Dantzig did optimization become a defined process resulting in determinate answers. Optimization problems became linear programs, systems of equations that must be satisfied yet are underdetermined and thus allowing an infinite number of solutions. The Simplex Method observes that the solution set of such linear programs are convex (see Figures 1 & 2), that is, the solution set of a linear program of *n* variables can be represented as a convex polygon in *n*-space. Furthermore, if a maximum or minimum value of the solution exists, it will exist at a corner of this polygonal region.

Figure 1 The solution set for the linear program:

$$3x + 2y \leq 6$$
$$5x + 8y \leq 20$$



Proving the last, but very important, characteristic is not too difficult. Realizing that each of the variables has a linear relationship with the solution, then each variable has a minimum effect at one extreme and a maximum at the other, with either a non-decreasing or a non-increasing effect in between. Therefore, any solution found away from a corner can be translated along at least one variable, while either increasing or decreasing the solution's value, until the solution set border is reached. The process can always be repeated until a corner of the solution set is encountered.

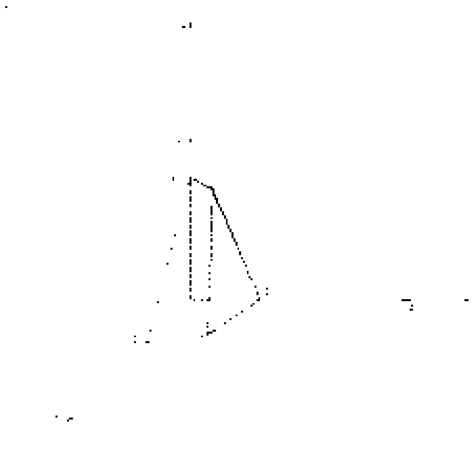


Figure 2

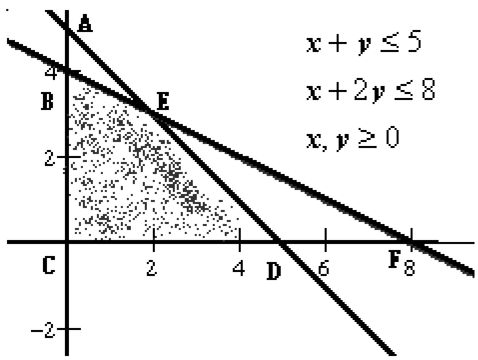
The solution set for the linear program:

$$\begin{aligned} 6x &+ 3y + 2z \le 6 \\ 3x &+ 8y + 12z \le 12 \end{aligned}$$

Thus the Simplex Method must visit corners of the solution set to find the minimum or maximum values. If the solution set involves many variables, this search could take an extended period of time and involve a large number of computations. Thus, variations of the Simplex Method algorithm have been developed to address these specific problems. This study is a comparison between those variations.

Graphical Solution: 2 variables

Def.: Feasible region: set of points that make all linear inequalities in the system true simultaneously.



Def.: Give a feasible region in the plane defined by or constraints. A point of intersection of two boundary lines that is also part of the feasible region is called a corner point (vertex) of the feasible region. In the above, B, C, D, E are vertices for the feasible region shaded in red.

The Fundamental Theorem of Linear Programming: If the feasible region to any linear programming problem has at least one point and is *convex* and if the objective function has a maximum (or minimum) value within the feasible region, then the maximum (or minimum) will always occur at a corner point in that region.

Criteria for the existence of solutions, $f(x,y)=ax+by$ If the feasible region is bounded, then has a maximum and minimum. If the feasible region is unbounded and $a, b>\mathbf{0}$, then f has a minimum; but not a maximum.

The Simplex Method

The Simplex Method is "a systematic procedure for generating and testing candidate vertex solutions to a linear program." (Gill, Murray, and Wright, p. 337) It begins at an arbitrary corner of the solution set. At each iteration, the Simplex Method selects the variable that will produce the largest change towards the minimum (or maximum) solution. That variable replaces one of its compatriots that is most severely restricting it, thus moving the Simplex Method to a different corner of the solution set and closer to the final solution. In addition, the Simplex Method can determine if no solution actually exists. Note that the algorithm is greedy since it selects the best choice at each iteration without needing information from previous or future iterations.

The Simplex Method solves a linear program of the form described in Figure. Here, the coefficients c_j represent the respective weights, or costs, of the variables x_j. The minimized statement is similarly called the cost of the solution. The coefficients of the system of equations are represented by a_{ij}, and any constant values in the system of equations are combined on the right-hand side of the inequality in the variables b_i. Combined, these statements represent a linear program, to which we seek a solution of minimum cost.

$$\begin{aligned} &\text{minimize} && \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} && \sum a_{ij} x_j \le b_i \quad (i = 1,2,\ldots,m < n) \\ & && x_j \ge 0 \quad (j = 1,2,\ldots,n) \end{aligned}$$

this linear program involves solutions of the set of equations. If no solution to the set of equations is yet known, slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$, adding no cost to the solution, are introduced. The initial basic feasible solution (BFS) will be the solution of the linear program where the following holds:

$$\begin{aligned} x_i &= 0 && (i = 1,2,\cdots,n) \\ x_i &= b_{n-i} && (i = n+1, n+2, \cdots, n+m) \end{aligned}$$

Once a solution to the linear program has been found, successive improvements are made to the solution. In particular, one of the non- basic variables (with a value of zero) is chosen to be increased so that the value of the cost function, $\sum_{j=1}^{n} c_j x_j$, decreases. That variable is then increased, maintaining the equality of all the equations while keeping the other non-basic variables at zero, until one of the basic (nonzero) variables is reduced to zero and thus removed from the basis. At this point, a new solution has been determined at a different corner of the solution set.

The process is then repeated with a new variable becoming basic as another becomes non-basic. Eventually, one of three things will happen. First, a solution may occur where no non-basic variable will decrease the cost, in which case the current solution is the optimal solution. Second, a non-basic variable might increase to infinity without causing a basic-variable to become zero, resulting in an unbounded solution. Finally, no solution may actually exist and the Simplex Method must abort.

The Simplex Method

Definitions

Objective Function

The function that is either being minimized or maximized. For example, it may represent the cost that you are trying to minimize.

Optimal Solution

A vector x, which is both feasible (satisfying the constraints) and optimal (obtaining the largest or smallest objective value).

Constraints

A set of equalities and inequalities that the feasible solution must satisfy.

Feasible Solution

A solution vector, x, which satisfies the constraints.

Basic Solution

x of ($Ax=b$) is a *basic solution* if the n components of x can be partitioned into m "basic" and n-m "non-basic" variables in such a way that:

the m columns of A corresponding to the basic variables form a nonsingular basis and the value of each "non-basic" variable is 0.

The constraint matrix A has m rows (constraints) and n columns (variables).

Basis

The set of basic variables.

Basic Variables

A variable in the basic solution (value is not 0).

Non-basic Variables

A variable not in the basic solution (value = 0).

Slack Variable

A variable added to the problem to eliminate less-than constraints.

Surplus Variable

A variable added to the problem to eliminate greater-than constraints.

Artificial Variable

A variable added to a linear program in phase 1 to aid finding a feasible solution.

Unbounded Solution

For some linear programs it is possible to make the objective arbitrarily small (without bound). Such an LP is said to have an unbounded solution.

Simplex Algorithem

Insert the slack variables, and find the slack equations.

Rewrite the objective function to match the format of the slack equations, and adjoin at the bottom.

Write the initial simplex tableau.

Determine the pivot element.

Rules for Selection

i. The most negative indicator in the last row of the tableau determines the pivot column.

ii. The pivot row is selected from among the slack equations by using the smallest-quotient rule.

Consider the augmented matrix of slack equations from a standard maximum-type problem after any number of pivots. If the solution obtained by setting the non-basic variables equal to zero is a corner point of the feasible region and the pivot column has been selected, divide each *positive* element in the column into the corresponding element in the column of constants. Then select the pivot row to be the one corresponding to the *smallest nonnegative quotient* formed.

$$\begin{array}{cccc|c} x & y & s_1 & s_2 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 6 \\ 2 & -1 & 0 & 1 & 6 \end{array}\right]$$

Choose the y-column from which to select a pivot element. The appropriate
quotients are **6** and **- 6**. The first row has the smallest nonnegative quotient. Pivot
on the **1** in the first-row, second-column position:

$$\begin{array}{cccc|c} x & y & s_1 & s_2 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 6 \\ 2 & -1 & 0 & 1 & 6 \end{array}\right] \begin{array}{l} \textbf{Quotient}: 6 \leftarrow \textbf{Pivot row} \\ \textbf{Quotient}: -6 \end{array}$$
$$\begin{array}{c} \uparrow \\ \textbf{Pivot Column} \end{array}$$

iii. The intersection of the pivot column and the pivot row determines the pivot element..

Perform the pivot operation.

Determine whether there is a negative number in the last row to the left of the vertical bar.
If there is a negative number in the last row, then go back to Step 4. If there isn't a negative number is the last row, then the maximum has been reached.

Examples and Problems

Ex. 1.Maximize $P=\mathbf{3}x+\mathbf{4}y+z$ subject to:

$$\begin{aligned} x+2y+z &\le 6 \\ 2x+\quad 2z &\le 4 \\ 3x+y+z &\le 9 \\ x,\ y,\ z &\ge 0 \end{aligned}$$

The slack equations are:

$$\begin{aligned} x+2y+z+s_1 \qquad\qquad &= 6 \\ 2x+\qquad 2z \quad + s_2 \qquad &= 4 \\ 3x+y+z \qquad\qquad + s_3 &= 9 \end{aligned}$$

Rewrite the objective function: $-3x-4y-z+P=0$

The initial simplex tableau:

$$\begin{array}{c} \begin{array}{ccccccccc} x & y & z & s_1 & s_2 & s_3 & P & & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 2 & 1 & 1 & 0 & 0 & 0 & 6 \\ 2 & 0 & 2 & 0 & 1 & 0 & 0 & 4 \\ 3 & 1 & 1 & 0 & 0 & 1 & 0 & 9 \\ \hline -3 & -4 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The most negative indicator in the last row is **- 4** and so the pivot column is the *y*-column:

$$\begin{array}{c} \begin{array}{ccccccccc} x & y & z & s_1 & s_2 & s_3 & P & & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 2 & 1 & 1 & 0 & 0 & 0 & 6 \\ 2 & 0 & 2 & 0 & 1 & 0 & 0 & 4 \\ 3 & 1 & 1 & 0 & 0 & 1 & 0 & 9 \\ \hline -3 & -4 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{ccccccccc} & \uparrow & & & & & & & \end{array} \end{array}$$

pivot column

Applying the smallest-quotient rule to the coefficients of *y* in the slack equations only, gives quotients of **3** (from the first row) and **9** (from the third row). Because **3** is the smallest quotient, the resulting pivot element is **2** Pivoting results in

$$\begin{array}{c} \begin{array}{ccccccccc} x & y & z & s_1 & s_2 & s_3 & P & & \end{array} \\ \left[\begin{array}{ccccccc|c} \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 3 \\ 2 & 0 & 2 & 0 & 1 & 0 & 0 & 4 \\ \frac{5}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 & 6 \\ \hline -1 & 0 & 1 & 2 & 0 & 0 & 1 & 12 \end{array}\right] \end{array}$$

The next pivot column is the *x*-column and the smallest quotient is **2** (from the second row):

$$\begin{array}{c} \begin{array}{ccccccccc} x & y & z & s_1 & s_2 & s_3 & P & & \end{array} \\ \left[\begin{array}{ccccccc|c} \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 3 \\ 2 & 0 & 2 & 0 & 1 & 0 & 0 & 4 \\ \frac{5}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 & 6 \\ \hline -1 & 0 & 1 & 2 & 0 & 0 & 1 & 12 \end{array}\right] \\ \begin{array}{ccccccccc} \uparrow & & & & & & & & \end{array} \end{array}$$

pivot column

Pivoting on that element gives:

No negative indicators exist and so the maximum has been reached. The solution is:

$$z = 0,\ s_1 = 0,\ s_2 = 0,\ x = 2,\ y = 2,\ s_3 = 1,\ P = 14$$

2. Maximize $f(x, y) = 5x + y$ subject to

$$x + 2y \le 6$$
$$4x + 3y \le 120$$
$$x, y \ge 0$$

The slack equations are:

$$x + 2y + s_1 = 6$$
$$4x + 3y + s_2 = 120$$

Rewrite the objective function:

$$P = 5x + y \ \Rightarrow\ -5x - y + P = 0$$

The initial simplex tableau:

$$\begin{array}{c} \begin{array}{ccccccc} x & y & s_1 & s_2 & P & & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 6 \\ 4 & 3 & 0 & 1 & 0 & 120 \\ \hline -5 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The most negative indicator in the last row is **- 5** and so the pivot column is the ***x***-column:

$$\begin{array}{c} \begin{array}{ccccccc} x & y & s_1 & s_2 & P & & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 6 \\ 4 & 3 & 0 & 1 & 0 & 120 \\ \hline -5 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{ccccccc} \uparrow & & & & & & \end{array} \end{array}$$

$$p.c.:\quad 6/1 = 6,\ 120/4 = 30$$

The smallest non-negative quotient is **6**. Pivot about **1** in row 1:

$$\begin{array}{c} \begin{array}{ccccccc} x & y & s_1 & s_2 & P & & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 6 \\ 0 & -5 & -4 & 1 & 0 & 96 \\ \hline 0 & 9 & 5 & 0 & 1 & 30 \end{array}\right] \end{array}$$

No negative indicators exist and so the maximum has been reached. The solution is:

$$x = 6,\ y = 0,\ s_1 = 0,\ s_2 = 96,\ P = 30$$

An investor has up to Rs.150, 000 to invest in any of three areas: stocks, bonds, or a money market. The annual rates of return are estimated to be: 5%, 9%, 8%, respectively. No more than Rs.75, 000 can be invested in any one of these areas. How much should be invested in each area in order to maximize the annual income?

Maximize: $P = .05x + .09y + .08z$

subject to the following constraints:

$$x + y + z \le 150$$
$$x \le 75$$
$$y \le 75$$
$$z \le 75$$

The slack equations are:

$$\begin{aligned} x + y + z + s_1 \quad &= 150 \\ x \quad + s_2 \quad &= 75 \\ y \quad + s_3 \quad &= 75 \\ z \quad + s_4 &= 75 \\ -.05x - .09y - .08z + P &= 0 \end{aligned}$$

The initial tableau is:

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 150 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & -.09 & -.08 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The smallest non-negative quotient is **75** and so pivot about **1** in row 3:

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 150 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & -.09 & -.08 & 0 & 0 & 0 & 0 & 1 & 0 \\ & \uparrow & & & & & & & \\ & p.c. & & & & & & & \end{array}$$

$150/1,\ 75/1$

The resulting tableau is:

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ \hline 1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 75 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & 0 & -.08 & 0 & 0 & .09 & 0 & 1 & 6.75 \end{array}$$

The smallest non-negative quotient is **75** and occurs in rows 1 and 4. Arbitrarily take **1** in row 4 pivoting purposes:

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ \hline 1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 75 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & 0 & -.08 & 0 & 0 & .09 & 0 & 1 & 6.75 \\ & & \uparrow & & & & & & \\ & & p.c. & & & & & & \end{array}$$

$75/1$

The resulting tableau is:

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ \hline 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & 0 & 0 & 0 & 0 & .09 & .08 & 1 & 12.75 \end{array}$$

The smallest non-negative quotient is **0** and occurs in row 1:

$$\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & s_4 & P & \\ \hline 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 75 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\ \hline -.05 & 0 & 0 & 0 & 0 & .09 & .08 & 1 & 12.75 \\ \uparrow & & & & & & & & \\ p.c. & & & & & & & & \end{array}$$

$0/1,\ 75/1$

Pivoting yields the following tableau. Since there are no negative entries in the bottom row, this tableau gives the solution.

$$\begin{array}{c}
\begin{array}{cccccccc} x & y & z & s_1 & s_2 & s_3 & s_4 & P \end{array} \\
\left[\begin{array}{cccccccc|c}
1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 75 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 75 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 75 \\
\hline
0 & 0 & 0 & .05 & 0 & .04 & .03 & 1 & 16.50
\end{array}\right]
\end{array}$$

Max. income is Rs.16,500 **when** $x = 0$ **is in stocks,** $y =$ Rs.75000 **is in bonds, and** $z = \$75,000$ **is in a money market**

Notes:

LESSON 6
SIMPLEX METHOD: ARTIFICIAL VARIABLE TECHNIQUES

Today we will be discussing following points:

Artificial Variable Techniques

Examples

Artificial Variable Techniques

Linear programming problems, in which constraints may also have $\geq$and '=' signs after ensuring that all *bi* are $\geq$ 0, are considered in this section. In such problems, basis matrix is not obtained as an identity matrix in the starting simplex table; therefore we introduce a new type of variable, called, the *artificial variable.* These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. Artificial variables can be eliminated from the simplex table as and when they become zero (non-basic). The process of eliminating artificial variables is performed in *Phase I* of the solution, and *Phase II* is used to get an optimal solution.

Greater Than Problems

The *Standard Simplex Method* requires all constraints to be of the form: $\leq$ Suppose we violate this with one or more constraints of the form: $\geq$

For example:

$$\text{Maximize } f = 5x + 10y \text{ subject to } \begin{cases} x + y \leq 20 \\ 2x - y \geq 10 \\ x, y \geq 0 \end{cases}$$

$$\text{Maximize } f = 5x + 10y \text{ subject to } \begin{cases} x + y \leq 20 \\ 2x - y \geq 10 \\ x, y \geq 0 \end{cases}$$

Change the second constraint to a $\leq$

$$\begin{cases} x + y \leq 20 \\ -2x + y \leq -10 \\ x, y \geq 0 \end{cases}$$

$$\text{Maximize } f = 5x + 10y \text{ subject to } \begin{cases} x + y \leq 20 \\ 2x - y \geq 10 \\ x, y \geq 0 \end{cases}$$

Change the second constraint to a $\leq$

$$\begin{cases} x + y \leq 20 \\ -2x + y \leq -10 \\ x, y \geq 0 \end{cases}$$

Setup the initial simplex tableau

$$\begin{array}{c} \begin{matrix} x & y & s_1 & s_2 & f & \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ -2 & 1 & 0 & 1 & 0 & -10 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \begin{matrix} x & y & s_1 & s_2 & f & \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ -2 & 1 & 0 & 1 & 0 & -10 \\ -5 & -10 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Eliminate the negative entry in the right column:

Look for the "most" negative entry to the left of the right column. $\therefore$ the first column is the pivot column. Since there is a pos. ratio, we pivot about the 1 in row 1.

$$\begin{array}{c} \begin{matrix} x & y & s_1 & s_2 & f & \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ 0 & 3 & 2 & 1 & 0 & 30 \\ 0 & -5 & 5 & 0 & 1 & 100 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \begin{matrix} x & y & s_1 & s_2 & f & \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ 0 & 3 & 2 & 1 & 0 & 30 \\ 0 & -5 & 5 & 0 & 1 & 100 \end{bmatrix} \end{array}$$

The tableau is in standard form. Pivot in the second column and pivot about the 3 in row 2.

$$\begin{array}{c} \begin{matrix} x & y & s_1 & s_2 & f & \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ 0 & 1 & 2/3 & 1/3 & 0 & 10 \\ 0 & -5 & 5 & 0 & 1 & 100 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1/3 & -1/3 & 0 & 10 \\ 0 & 1 & 2/3 & 1/3 & 0 & 10 \\ 0 & 0 & 25/3 & 5/3 & 1 & 150 \end{bmatrix}$$

$$\begin{array}{c} \begin{matrix} x & y & s_1 & s_2 & f & \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 20 \\ 0 & 1 & 2/3 & 1/3 & 0 & 10 \\ 0 & -5 & 5 & 0 & 1 & 100 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \begin{matrix} x & y & s_1 & s_2 & f & \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1/3 & -1/3 & 0 & 10 \\ 0 & 1 & 2/3 & 1/3 & 0 & 10 \\ 0 & 0 & 25/3 & 5/3 & 1 & 150 \end{bmatrix} \end{array}$$

The maximum value is 150 when $x = 10,\ y = 10$

Minimizing an Objective Function

The *Standard Simplex Method* requires the objective function to be maximized. Suppose we wish to minimize an objective function

For example:

The minimization of f can be accomplished by *maximizing* the negative of f, $-f$. We can define $g = -f$. Therefore, our example can be re-stated in the form:

Maximize $g = -f = -3x - 2y$ subject to the constraints:

$$x + y \geq 10$$
$$x - y \leq 15$$

Adjust the constraints to fit the criteria of the standard simplex method:

$$-x - y \leq -10$$
$$x - y \leq 15$$

Introduce the slack variables:

$$-x - y + s_1 = -10$$
$$x - y + s_2 = 15$$
$$3x + 2y + g = 0$$

Setup the initial simplex tableau:

$$\begin{array}{ccccc|c} x & y & s_1 & s_2 & g & \\ \hline -1 & -1 & 1 & 0 & 0 & -10 \\ 1 & -1 & 0 & 1 & 0 & 15 \\ \hline 3 & 2 & 0 & 0 & 1 & 0 \end{array}$$

Determine the pivot element. In row #1 there is a tie. Let us arbitrarily pick the y-column.
There are no "pos/pos" ratios and so the only choice is the -1 in row #1:

$$\begin{array}{ccccc|c} x & y & s_1 & s_2 & g & \\ \hline -1 & \langle -1 \rangle & 1 & 0 & 0 & -10 \\ 1 & -1 & 0 & 1 & 0 & 15 \\ \hline 3 & 2 & 0 & 0 & 1 & 0 \end{array}$$

Pivoting results in:

$$\begin{array}{ccccc|c} x & y & s_1 & s_2 & g & \\ \hline 1 & \langle 1 \rangle & -1 & 0 & 0 & 10 \\ 2 & 0 & -1 & 1 & 0 & 25 \\ \hline 1 & 0 & 2 & 0 & 1 & -20 \end{array}$$

Since there are no negative numbers above the -20 in the last column, the simplex tableau satisfies the criteria for the standard simplex method and so we can proceed as if we had the standard simplex problem. Since there are no negative numbers to the left of the vertical line in the bottom row, we are finished. The solution is: $x=0$, $y=10$, and $g = -20$. But g is the negative of f. Therefore, the minimum of $f = 20$. Notice that it was simply luck that the solution fell out without having to do additional pivoting.

Equality as a Constraint Sometimes a linear programming contains an equality as a constraint. There is more than one way of handling this. The approach that will be used involves solving for one of the variables in the equality and substituting for this variable in the problem.

Example: Maximize: $f = -2y + 5z$ subject to the following constraints:

$$3x - 2y + z \leq 8$$
$$-4x + 3y - z \leq 4$$
$$2x - 3y - 6z \leq 6$$
$$x - y + z \geq 1$$
$$x + y + z = 5$$

Since the last constraint is an equality, we arbitrarily solve for one of the variables, say z:

$$z = 5 - x - y$$

We substitute for z in the remaining constraints. We leave it up to the reader to get the following:

$$2x - 3y \leq 3$$
$$-3x + 4y \leq 9$$
$$8x + 3y \leq 36$$
$$y \leq 2$$

In addition, all variables must be nonnegative. This includes z and results in an additional constraint:

$$z = 5 - x - y \geq 0 \Rightarrow 5 - x - y \geq 0$$

Finally, we make the substitution in the objective function to get:

$$f = -5x - 7y + 25$$

Since we have five constraints, we need five slack variables and get the following initial simplex tableau:

$$\begin{array}{cccccccc|c} x & y & s_1 & s_2 & s_3 & s_4 & s_5 & f & \\ \hline 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 3 \\ -3 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 9 \\ 8 & 3 & 0 & 0 & 1 & 0 & 0 & 0 & 36 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 5 \\ \hline 5 & 7 & 0 & 0 & 0 & 0 & 0 & 1 & 25 \end{array}$$

Since there are no negative numbers above the vertical line in the last column, we now have a standard maximum-type linear programming problem.

We notice that there are no negative entries in the bottom row to the left of the vertical line.

This means that we have a solution without having to do any pivoting!
The solution is $\text{Max} = 25 \ni: x = 0, y = 0, z = 5$

The Dual Problem

For every linear programming problem there is a corresponding linear programming problem called the *dual*. If the original problem is a maximum then its dual is a minimum and if the original problem is a minimum then its dual is a maximum. In either case the final tableau of the dual will contain both the solution to the dual and the solution to the original problem.

Ex. Minimize $f = 32x + 12y$ subject to the constraints:

$$4x + 3y \geq 6$$
$$8x + 2y \geq 5$$

with x and y nonnegative.
Since this is a minimum problem, the dual will be a maximum.

1. Setup the matrix consisting of the constraints and the coefficients of the objective function:

$$P = \begin{bmatrix} 4 & 3 & 6 \\ 8 & 2 & 5 \\ 32 & 12 & 0 \end{bmatrix}$$

2. Compute the transpose of P:

$$P^T = \begin{bmatrix} 4 & 8 & 32 \\ 3 & 2 & 12 \\ 6 & 5 & 0 \end{bmatrix}$$

3. Create the new objective function and new constraints of the maximum problem with the new variables u and v:

$$4u + 8v \leq 32$$
$$3u + 2v \leq 12$$
$$g = 6u + 5v$$

4. Setup the initial simplex tableau:

$$\begin{array}{c} \begin{array}{ccccc} u & v & s_1 & s_2 & g \end{array} \\ \left[\begin{array}{ccccc|c} 4 & 8 & 1 & 0 & 0 & 32 \\ 3 & 2 & 0 & 1 & 0 & 12 \\ \hline -6 & -5 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

5. Solve the problem:

$$\begin{array}{c} \begin{array}{ccccc} u & v & s_1 & s_2 & g \end{array} \\ \left[\begin{array}{ccccc|c} 4 & 8 & 1 & 0 & 0 & 32 \\ \langle 3 \rangle & 2 & 0 & 1 & 0 & 12 \\ \hline -6 & -5 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccccc} u & v & s_1 & s_2 & g \end{array} \\ \left[\begin{array}{ccccc|c} 4 & 8 & 1 & 0 & 0 & 32 \\ \langle 1 \rangle & 2/3 & 0 & 1/3 & 0 & 4 \\ \hline -6 & -5 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccccc} u & v & s_1 & s_2 & g \end{array} \\ \left[\begin{array}{ccccc|c} 0 & \langle 16/3 \rangle & 1 & -4/3 & 0 & 16 \\ 1 & 2/3 & 0 & 1/3 & 0 & 4 \\ \hline 0 & -1 & 0 & 2 & 1 & 24 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccccc} u & v & s_1 & s_2 & g \end{array} \\ \left[\begin{array}{ccccc|c} 0 & \langle 1 \rangle & 3/16 & -1/4 & 0 & 3 \\ 1 & 2/3 & 0 & 1/3 & 0 & 4 \\ \hline 0 & -1 & 0 & 2 & 1 & 24 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{ccccc} u & v & s_1 & s_2 & g \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & 3/16 & -1/4 & 0 & 3 \\ 1 & 0 & -1/8 & 1/2 & 0 & 2 \\ \hline 0 & 0 & 3/16 & 7/4 & 1 & 27 \end{array}\right] \end{array}$$

$\text{Max of } 27 \text{ at } u = 2,\ v = 3$

6. So, the solution to the original problem is read from the feet of the slack variable columns:
   $x = 3/16$, $y = 7/4$ with a minimum of 27. We will leave it up to the student to verify that this
   indeed is the solution.

Notes:

LESSON 7
SENSITIVITY ANALYSIS USING THE DUAL SIMPLEX METHOD

Sensitivity Analysis

Problems

Sensitivity Analysis using the Dual Simplex Method

I will use as an example the following linear programming problem:

$$
\begin{array}{ll}
\text{maximize} & 2x_1 + 2x_2 + x_3 - 3x_4 \\
\text{subject to} & 3x_1 + x_2 - x_4 \le 1 \\
& x_1 + x_2 + x_3 + x_4 \le 2 \\
& -3x_1 + 2x_3 + 5x_4 \le 6 \\
\text{all variables} & \ge 0
\end{array}
$$

The optimal tableau is as follows:

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	2	0	0	2	1	1	0	3	=	z
0	−2	0	1	1	−1	1	0	1	=	x_3
0	3	1	0	−1	1	0	0	1	=	x_2
0	1	0	0	3	2	−2	1	4	=	s_3

Thus we have

$$B^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & -2 & 1 \end{pmatrix} \begin{matrix} x_3 \\ x_2 \,. \\ s_3 \end{matrix}$$

Each of the examples below will start from this basis (i.e. the changes made in one section will not affect later sections).

(1) Change in b

Suppose we change b to $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$. The new $\beta = B^{-1}\mathrm{b} = \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} \begin{matrix} x_3 \\ x_2 \\ s_3 \end{matrix}$. The new value of

z in the basic solution is y Tb = 5, but the basic solution is not feasible. We need a dual simplex pivot: x3 leaves, and (in a tie for minimum ratio) x1 enters.

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	0	1	3	0	2	0	4	=	z
0	1	0	−1/2	−1/2	1/2	−1/2	0	1/2	=	x_1
0	0	1	3/2	1/2	−1/2	3/2	0	3/2	=	x_2
0	0	0	1/2	7/2	3/2	−3/2	1	11/2	=	s_3

This is now feasible and thus optimal.

2. Parametric programming

We will see how the optimal solution depends on b_2. With $\mathrm{b} = \begin{pmatrix} 1 \\ p \\ 6 \end{pmatrix}$, $\beta = \Bigg(\begin{matrix} -1\text{ -} \\ 1 \\ 8\, - \end{matrix}$

so our basis x_3, x_2, s_3 is optimal for $1 \le p \le 4$. The z value is $1 + p$, and the tablea

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	2	0	0	2	1	1	0	$1+p$	=	z
0	−2	0	1	1	−1	1	0	$-1+p$	=	x_3
0	3	1	0	−1	1	0	0	1	=	x_2
0	1	0	0	3	2	−2	1	$8-2p$	=	s_3

If p < 1, the tableau is not feasible: x3 leaves and x1 enters. The resulting tableau is

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	0	1	3	0	2	0	$2p$	=	z
0	1	0	−1/2	−1/2	1/2	−1/2	0	$(1-p)/2$	=	x_1
0	0	1	3/2	1/2	−1/2	3/2	0	$(-1+3p)/2$	=	x_2
0	0	0	1/2	7/2	3/2	−3/2	1	$(15-3p)/2$	=	s_3

This is feasible (and thus optimal) if 1=3 _ p _ 1. If p < 1=3, x2 leaves and s1 enters:

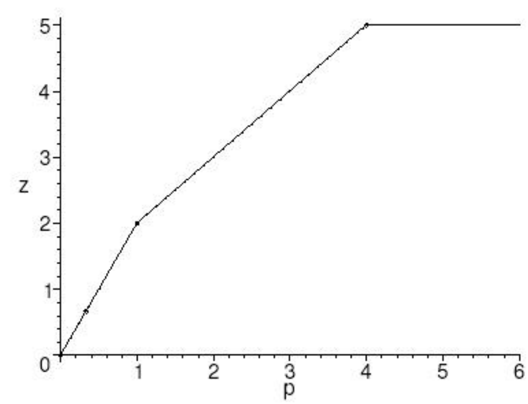
z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	0	1	3	0	2	0	$2p$	=	z
0	1	1	1	0	0	1	0	p	=	x_1
0	0	−2	−3	−1	1	−3	0	$1-3p$	=	s_1
0	0	3	5	5	0	3	1	$6+3p$	=	s_3

This is feasible, and thus optimal, if 0 _ p _ 1=3. If p < 0, x1 would have to leave but no variable could enter, and the problem would be infeasible.

Now on the other side, returning to the tableau that was optimal for 1 _ p _ 4, if p > 4 we must let S_3 leave and S_2 enters.

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	5/2	0	0	7/2	2	0	1/2	5	=	z
0	−3/2	0	1	5/2	0	0	1/2	3	=	x_3
0	3	1	0	−1	1	0	0	1	=	x_2
0	−1/2	0	0	−3/2	−1	1	−1/2	$-4+p$	=	s_2

This is optimal if p _ 4. Plotting the optimal z as a function of p, we have



3. Deleting a Variable

Deleting a variable means requiring it to be 0. In effect, it becomes an artificial variable. If the variable is already non-basic, nothing needs to be done. If it is basic, however, it must leave the basis. This can be done with a \sign-reversed Dual Simplex pivot". It's \sign-reversed" because the variable starts o_ with a positive value, rather than a negative value which is usual in the Dual Simplex method. We can make it look like an ordinary Dual Simplex pivot if we multiply the departing variable's row (except for the 1 for that basic variable) by _1 After the first pivot, we can remove the deleted variable from the problem. More Dual Simplex pivots may be necessary until the basic solution is feasible for the primal problem.

In our example, suppose we want to delete the variable x2, which is basic and has the value 1 in the optimal solution of the original problem. Here's the tableau with the x2 row multiplied by _1:

z	x_1	$-x_2$	x_3	x_4	s_1	s_2	s_3	rhs		
1	2	0	0	2	1	1	0	3	=	z
0	-2	0	1	1	-1	1	0	1	=	x_3
0	-3	1	0	1	-1	0	0	-1	=	$-x_2$
0	1	0	0	3	2	-2	1	4	=	s_3

$-x_2$ leaves, and x_1 enters.

z	x_1	$-x_2$	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	2/3	0	8/3	1/3	1	0	7/3	=	z
0	0	$-2/3$	1	1/3	$-1/3$	1	0	5/3	=	x_3
0	1	$-1/3$	0	$-1/3$	1/3	0	0	1/3	=	x_1
0	0	1/3	0	10/3	5/3	-2	1	11/3	=	s_3

This is feasible, so it is the new optimal solution.

4. Deleting a Constraint

Deleting a constraint means that we no longer care about the value of its slack variable.

In effect, that slack variable has become sign-free. If the slack variable was already basic, no pivoting is necessary, but if it is non-basic (with a nonzero entry in the z row) we will want it to enter the basis: increasing if the entry was negative (which would only happen if the constraint was an equality), or decreasing if the entry was positive. Recall that if a variable enters decreasing, we calculate ratios in the rows where the entry of the entering variable is negative. After the first pivot, we can delete the equation for the deleted constraint's slack variable from the tableau. More pivots may be necessary until the basic solution is feasible for the primal problem.

In our example, suppose we delete the first constraint. Then s1, which is non-basic and has entry 1 in the z row, enters decreasing. The only ratio to be calculated is for x3, so x3 leaves the basis. The next tableau is

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	0	1	3	0	2	0	4	=	z
0	2	0	-1	-1	1	-1	0	-1	=	s_1
0	1	1	1	0	0	1	0	2	=	x_2
0	-3	0	2	5	0	0	1	6	=	s_3

We can delete the s1 equation. The solution is now optimal.

5. Adding a Constraint

The new constraint has a new slack variable, and a new row of the tableau in which this slack variable is basic. That row comes from the equation for the new slack variable, but we need to substitute in the expressions for the basic variables so that it's expressed in terms of the non-basic variables. If the value of the new slack variable is negative (or if the new constraint is an equality and the value is nonzero), a Dual Simplex pivot will be needed.

In our example, we add the constraint x1 +2x2 +2x3 _ 3, or x1 +2x2 +2x3 +s4 = 3.

Substituting in the values of the basic variables x3, x2 and s3, we get the new row of the tableau:

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	rhs		
1	2	0	0	2	1	1	0	0	3	=	z
0	-2	0	1	1	-1	1	0	0	1	=	x_3
0	3	1	0	-1	1	0	0	0	1	=	x_2
0	1	0	0	3	2	-2	1	0	4	=	s_3
0	-1	0	0	0	0	-2	0	1	-1	=	s_4

We need a dual simplex pivot, with s_4 leaving, and s_2 entering.

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	rhs		
1	3/2	0	0	2	1	0	0	1/2	5/2	=	z
0	$-5/2$	0	1	1	-1	0	0	1/2	1/2	=	x_3
0	3	1	0	-1	1	0	0	0	1	=	x_2
0	2	0	0	3	2	0	1	-1	5	=	s_3
0	1/2	0	0	0	0	1	0	$-1/2$	1/2	=	s_2

This is optimal.

6. Changing entries in A

There are two easy cases here, and one difficult case, depending on whether the con-straints and variables where the entries are changed are basic or non-basic.

a. Suppose the changes are only in the coefficients of one or more non-basic variables.

Then it is as if the old versions of these variables were removed and new versions added. If the entries for the new versions in the z row are negative, pivots will be necessary. In our example, suppose we change the coefficient of x5 in the first constraint from _1

to -2. Think of this as introducing a new x_5. We calculate $r_5 = (1,1,0)\begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} - (-3) = 1 \geq 0$, so the optimal solution is unchanged.

b. Suppose the changes are only in a constraint whose slack variable is basic. Just as in\Adding a constraint", we express the new version of this slack variable in terms of the non-basic variables. If the value of this slack variable is negative, one or more Dual Simplex pivots will be needed.

In our example, suppose we change the coefficient of x2 in the third constraint from 0 to 2. The slack variable s3 for the original version of the constraint was 4 in the basic solution, and x2 was 1; the new slack variable s0

3 = s3 _ 2x2 will be 4 _ 2 _ 1 = 2.

This is still feasible, so no pivoting is necessary.

c. Suppose the changes are for a basic variable and involve one or more constraints with non-basic slack variables. Then the situation is more complicated. It is even possible that the B matrix may not be invertible (so that the current basis can no longer be used as a basis). Or if the basis can still be used, the solution may or may not be feasible for either the primal or the dual. One strategy that often works well is first to add a new version of the variable, and then delete the old version.

In our example, suppose we change the coefficient of x2 in the second constraint from 1 to 2. We first add the new version of x_2, call it x_{02}, with coefficients 1, 2, 0 in

the three constraints and 2 in the objective. Its η value is $(1,1,0)\begin{pmatrix}1\\2\\0\end{pmatrix} - 2 = 1 \geq 0$, so x_2' doesn't enter the basis. However, we need to put it into the tableau. We have $d = B^{-1}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$, so the tableau is

z	x_1	x_2	x_2'	x_3	x_4	s_1	s_2	s_3	rhs		
1	2	0	1	0	3	1	1	0	3	=	z
0	−2	0	1	1	2	−1	1	0	1	=	x_3
0	3	1	1	0	−1	1	0	0	1	=	x_2
0	1	0	−2	0	1	2	−2	1	4	=	s_3

Now we delete the old x_2 in a "sign-reversed Dual Simplex" pivot. After changing signs in the x_2 equation, we make $-x_2$ leave. The ratios are 2/3 for x_1, 1/1 for x_2', 1/1 for s_1, so x_1 enters. The new tableau is

z	x_1	$-x_2$	x_2'	x_3	x_4	s_1	s_2	s_3	rhs		
1	0	2/3	1/3	0	11/3	1/3	1	0	7/3	=	z
0	0	−2/3	5/3	1	4/3	−1/3	1	0	5/3	=	x_3
0	1	−1/3	1/3	0	−1/3	1/3	0	0	1/3	=	x_1
0	0	1/3	−7/3	0	4/3	5/3	−2	1	11/3	=	s_3

We delete the old x2, and we have an optimal solution.

In some cases it would be useful to know if making the change will increase or decrease the objective value. Again, consider first adding the new version of the variable, and then deleting the old version. Note that primal simplex pivots increase the objective (or in the case of degeneracy may keep it the same, but never decrease it), while dual simplex pivots usually decrease the objective and never increase it.

If the new version of the variable will not enter the basis when it's added, because its t value is not negative, then there is no primal simplex pivot but there will be at least one dual simplex pivot. So the change can't increase the objective, and will probably decrease it. This is what occurred in the example above, where the change decreased the objective from 3 to 7=3.

if the new version of the variable does enter the basis because its t value is negative, the primal simplex pivots (if not degenerate) will increase the objective. If in this process the old version of the variable will leave the basis, there will be no need for dual simplex pivots to remove the old version, and so the objective can't decrease.

For example, suppose at the same time as we change the coefficient of x2 in the second constraint from 1 to 2 we also increase c2 (for the new version of the variable) from 2 to 4. Then _0

2 decreases from 1 to _1, so the tableau is

.

z	x_1	x_2	x_2'	x_3	x_4	s_1	s_2	s_3	rhs		
1	2	0	−1	0	3	1	1	0	3	=	z
0	−2	0	1	1	2	−1	1	0	1	=	x_3
0	3	1	1	0	−1	1	0	0	1	=	x_2
0	1	0	−2	0	1	2	−2	1	4	=	s_3

This time x0

2 will enter the basis. Since this won't be a degenerate pivot, the objective will increase. There is a tie for minimum ratio between x3 and x2; we may as well choose X2 to leave (since it will be leaving anyway). At this point we know that no dual simplex pivots will be needed to make x2 leave, so the change will increase the objective. In fact the next tableau is

.

z	x_1	x_2	x_2'	x_3	x_4	s_1	s_2	s_3	rhs		
1	5	1	0	0	2	2	1	0	4	=	z
0	−5	−1	0	1	3	−2	1	0	0	=	x_3
0	3	1	1	0	−1	1	0	0	1	=	x_2'
0	7	2	0	0	−1	4	−2	1	6	=	s_3

which is optimal. We can then delete the column for the old x2. The objective has increased from 3 to 4.

Notes:

LESSON 8
GROUP DISCUSSION ON THE TOPICS COVERED TILL DATE

Notes:

LESSON 9
TRANSPORTATION PROBLEM

This lecture will cover the following topics:

- **Introduction**
- **Practical steps involved in solving T P**

Transportation Problem

The transportation problem is a special type of linear programming problem where the 'objective is to minimise the cost of distributing a product from a number of sources or origins to a number of destinations. Because of its special structure the usual simplex method is not suitable for solving transportation problems. These problems require a special method of solution. The origin of a transportation problem is the location from which shipments are dispatched. The destination of a transportation problem is the location to which shipments are transported. The unit transportation cost is the cost of transporting one unit of the consignment from an origin to a destination.

In the most general form, a transportation problem has a number of origins and a number of destinations. A certain amount of a particular consignment is available in each origin. Likewise, each destination has a certain requirement. The transportation problem indicates the amount of consignment to be transported from various origins to different destinations so that the total transportation cost is minimised without violating the availability constraints and the requirement constraints. The decision variables X_{ij} of a transportation problem indicate the amount to be transported from the *ith* origin to the *jth* destination. Two subscripts are necessary to describe these decision variables. A transportation problem can be formulated as a linear programming problem using decision variables with two subscripts.

Example: A manager has four Factories (i.e. origins) and four Warehouses (i.e. destinations). The quantities of goods available in each factory, the requirements of goods in each warehouse and the costs of transportation of a product from each factory to each warehouse are given. His objective is to ascertain the quantity to be transported from various factories to different warehouses in such a way that the total transportation cost is minimised.

Balanced Transportation Problem

Balanced Transportation Problem is a transportation problem where the total availability at the origins is equal to the total requirements at the destinations. For example, in case the total production of 4 factories is 1000 units and total requirements of 4 warehouses is also 1000 units, the transportation problem is said to be a balanced one.

Unbalanced Transportation Problem

Unbalanced transportation problem is a transportation problem where the total availability at the origins is not equal to the total requirements at the destinations. For example, in case the total production of 4 factories is 1000 units and total requirements of 4 warehouses is 900 units or 1,100 units, the transportation problem is said to be an unbalanced one. To make an unbalanced transportation problem, a balanced one, a dummy origins) or a dummy destination (s) (as the case may be) is introduced with zero transportation cost per unit.

Dummy Origin/Destination

A dummy origin or destination is an imaginary origin or destination with zero cost introduced to make an unbalanced transportation problem balanced. If the total supply is more than the total demand we introduce an additional column which will indicate the surplus supply with transportation cost zero. Likewise, if the total demand is more than the total supply, an additional row is introduced in the Table, which represents unsatisfied demand with transportation cost zero.

Practical Steps Involved In Solving Transportation Problems Of Minimization Type

The practical steps involved in solving transportation problems of minimization type are given below:

Step 1- See whether Total Requirements are equal to Total Availability; if yes, go to Step 2; if not, Introduce a Dummy Origin / Destination, as the case may be, to make the problem a balanced one Taking Transportation Cost per unit as zero for each Cell of Dummy OriginlDestination or as otherwise indicated.

Step 2- Find Initial Feasible Solution by following either the Least Cost Method(or LCM) or North-West Comer Method (or NWCM) or Vogel's Approximation Method (or VAM)

Step 3- After obtaining the Initial Feasible Solution Table, see whether Total Number of Allocations are equal to "$m + n - 1$",; if yes, go to Step 4; if not, introduce an infinitely small quantity Independent Cell. (i.e., for which no Loop can be formed).

Note: Introduce as many number of '*e*' as the total number of allocated cells falls below "$m + n - 1$".

Step 4-

Optimality Test: Carry out the Optimality Test on the Initial Solution Table to find out the optimal solution.

Step 5 – Calculate the Total Minimum Cost = S (X ij x C ij),

where, X = Units Allocated to a Cell;

C = Shipping Cost per Unit of a Cell;

i = Row Number;

j = Column Number

Practical Steps Involved In Solving Transportation Problems of Maximization Type

The practical steps involved in solving transportation problems of maximization type are given below:

Step 1- 1Derive Profit Matrix by calculating the Profit by the following equation: .

Profit = Selling Price - Production Cost - Transportation Cost

Step 2 -1 See whether Total Requirements are equal to Total Availability; if yes, go to Step 3; if not, introduce a Dummy Origin/Destination, as the case may be, to make the problem a balanced one, taking Profit per unit as zero for each Cell of Dummy Origin/Destination or as otherwise indicated.

Step 3 -1 Derive Loss Matrix by deducting each element from the largest element in order to use minimization technique.

Step 4- After obtaining the Initial Feasible Solution Table, see whether Total Number of Allocations are equal to "*m* + *n* - 1",; if yes, go to Step 5; if not, introduce an infinitely small quantity' *e*' to the Least Cost Independent Cell. (i.e., for which no Loop can be formed).

Note:Introduce as many number of '*e*' as the total number of allocated cells falls below "*m* + *n* -1".

Step 5 -1 *Optimality Test:* Carry out the Optimality Test on the Initial Solution Table to find out the optimal solution.

Step 6 - Calculate the Maximum Profit = Zigma (Xij x Pij)

where, X = Units Allocated to a Cell;

P = Profit per Unit of a Cell; i = Row Number;

j = Column Number

Example: 1

Minimisation Problem

Build the mathematical model for the following transportation problem:

	Cost Matrix				Supply
	W_1	W_2	W_3	W_4	
F_1	1	2	4	4	6
F_2	4	3	2	0	8
F_3	0	2	2	1	10
Demand	4	5	8	6	

$W_i \rightarrow$ Warehouse, $F_i \rightarrow$ Factory and cell entries are unit/costs.

Solution

Step 1 - Introducing dummy warehouse with zero cost per unit as the total demand is not equal to total supply in order to make the problem balanced one.

Let Xjj represent the quantity transported from Fj to Wj

Formulation of the mathematical model for the transportation problems.

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	1 (x_{11})	2 (x_{12})	4 (x_{13})	4 (x_{14})	0 (x_{15})	6
F_2	4 (x_{21})	3 (x_{22})	2 (x_{23})	0 (x_{24})	0 (x_{25})	8
F_3	0 (x_{31})	2 (x_{32})	2 (x_{33})	1 (x_{34})	0 (x_{35})	10
Demand	4	5	8	6	1	

$$
\begin{aligned}
\text{Minimise } Z = \; & 1x_{11} + 2x_{12} + 4x_{13} + 4x_{14} + 0x_{15} + 4x_{21} + 3x_{22} + 2x_{23} + 0x_{24} + 0x_{25} \\
& 0x_{31} + 2x_{32} + 2x_{33} + 1x_{34} + 0x_{35} \\
\text{Subject to } \; & x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 6 \\
& x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 8 \\
& x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 10 \\
& x_{11} + x_{21} + x_{31} = 4 \\
& x_{12} + x_{22} + x_{32} = 5 \\
& x_{13} + x_{23} + x_{33} = 8 \\
& x_{14} + x_{24} + x_{34} = 6 \\
& x_{15} + x_{25} + x_{35} = 1 \\
& x_{ij} \geq 0
\end{aligned}
$$

Maximisation Problem

A multi-plant company has three manufacturing plants, A, B and C, and two markets X and Y.

Production Cost of A, B and C is Rs. 1500, 1600 and 1700 per piece respectively. Selling price in X and Y are Rs. 4400 and Rs. 4700 respectively. Demand in X and Y3500 and 3600 pieces respectively.

Production capacity at A, Band C is 2000, 3000 and 4000 pieces respectively.

Transportation costs are as follows:

From/To	*X*	*Y*
A	1000	1500
B	2000	3000
C	1500	2500

Build a mathematical model.

Solution

Step 1: Introducing dummy market with zero profit as the total demand is not equal to total supply in order to make the problem balanced one.

Calculating the profit as follows:

Profit = Selling Price - Cost of Production - Transportation Cost

Let Xij represent the quantity transported from factory i to market j.

		X	Y	Z	Supply
	A	1900 (x_{11})	1700 (x_{12})	0 (x_{13})	2000
From	B	800 (x_{21})	100 (x_{22})	0 (x_{23})	3000
	C	1200 (x_{31})	500 (x_{32})	0 (x_{33})	4000
Demand		3500	3600	1900	

This mathematical model is formulated below from the above matrix.

$$
\begin{aligned}
\text{Maximise } Z = \; & 1900x_{11} + 1700x_{12} + 0x_{13} + 800x_{21} + 100x_{22} + 0x_{23} + 1200x_{31} + 500x_{32} + 0x_{33} \\
\text{Subject to } \; & x_{11} + x_{12} + x_{13} = 2{,}000 \\
& x_{21} + x_{22} + x_{23} = 3{,}000 \\
& x_{31} + x_{32} + x_{33} = 4{,}000 \\
& x_{11} + x_{21} + x_{31} = 3{,}500 \\
& x_{12} + x_{22} + x_{32} = 3{,}600 \\
& x_{13} + x_{23} + x_{33} = 1{,}900 \\
& x_{ij} \geq 0
\end{aligned}
$$

LESSON 10
TRANSPORTATION PROBLEM (CONTD..)

This lecture we will be learning:

- **Least Cost Method**
- **North-west Corner Method**
- **Vogel's Approximation Method**

Finding Initial (or Basic) Feasible Solution

In general, any basic feasible solution of a transportation problem with m origins (such as factories) and n destinations (such as warehouses) should have '$m + n - 1$' non zero basic variables.

A transport problem is said to be a degenerate transport problem if it has a basic feasible solution with number of non-zero basic variables less than $m + n - 1$.

According to Must fit, "A degenerate basic feasible solution in a transportation problem exists if and only if some partial sum of availabilities (row) is equal to a partial sum of requirements (column)".

Initial feasible solution can be obtained by any of the following three methods:

Method I : Least Cost Method (or LCM)

Method II : Northwest Comer Method (or NWCM)

Method III : Vogel's Approximation Method (or VAM)

Let us discuss these methods one by one as under:

Method I: Least Cost Method (or LCM)

The practical steps involved in the Least Cost Method are given below:

Step 1: Make maximum possible Allocation to the Least. Cost Cell depending upon the demand/supply for the Column Row containing that Cell. In case of Tie in the Least Cost Cells, make allocation to the Cell by which maximum demand or capacity is exhausted.

Step 2: Make allocation to the Second Lowest Cost Cell depending upon the remaining demand/supply for the Row/ Column containing that Cell.

Step 3: Repeat the above Steps till all Rim Requirements are exhausted, Le., entire demand and supply is exhausted.

Example

Find the initial basic feasible solution by Least Cost Method.

	W_1	W_2	W_3	W_4	Supplies
F_1	48	60	56	58	140
F_2	45	55	53	60	260
F_3	50	65	60	62	360
F_4	52	64	55	61	220
Demand	200	320	250	210	

$W_j \rightarrow$ Warehouse, $F_i \rightarrow$ Factory and cell entries are unit costs

Solution

Initial Feasible Solution by Least Cost Method (LCM)

Method II: North-west Corner Method (or NWCM)

The practical steps involved in the North-West Comer Method are given below:

Step 1:Make maximum possible allocation to the Upper-Left Comer Cell (also known as North-West Comer Cell) in the First Row depending upon the availability of supply for that Row and demand requirement for the Column containing that Cell.

Note: Unit transportation cost is completely ignored.

Step 2:Move to the Next Cell of the First Row depending upon remaining supply for that Row and the demand requirement for the next Column. Go on till the Row total is exhausted.

Step 3:Move to the next Row and make allocation to the Cell below the Cell of the preceding Row in which the last allocation was made and follow Steps I and 2.

Step 4:Follow Steps I to 3 till all Rim requirements are exhausted, i.e., the entire demand and supply is exhausted.

Example

Find initial feasible solution by North West Comer Method in Problem.

	W_1	W_2	W_3	W_4	Supplies
F_1	48	60	56	58	140
F_2	45	55	53	60	260
F_3	50	65	60	62	360
Demand	200	320	250	210	

Solution

Initial Feasible Solution by North West Corner Method (NWCM)

	W_1	W_2	W_3	W_4	Supplies
F_1	17 [140]	5	9	7	140
F_2	20 [60]	10 [200]	12	5	260
F_3	15	0 [120]	5 [240]	3	360
F_4	65	65	65 [10]	65 [210]	220
Demand	200	320	250	210	

Method III: Vogel's Approximation Method (or VAM)

The practical steps involved in Vogel's Approximation Method *(or VAM)* are given below:

Step 1:Row Difference: Find the difference between Smallest and Second Smallest element of each Row, representing the. Opportunity Cost of not making the allocation to the Smallest Element Cell, and write the difference on the right-hand side of the concerned Row. In case of tie between two smallest elements, the difference should be taken as zero.

Step 2:Column Difference: Find the difference between Smallest and Second Smallest element of each column, representing the Opportunity Cost of not making the allocation to the Smallest Element Cell, and write the difference below the concerned Column. In case of tie between two smallest elements, the difference should be taken as zero.

Step 3:Make the Largest Difference amongst all Differences by an arrow indicating the allocation to be made to the row/ column having largest difference. Allocate maximum possible quantity to the Least Cost Cell of the Selected row/column depending upon the quantity available. In case of tie between the Differences, select the row or column having least cost cell. However, in case of tie even in case of Least Cost, make allocation to that Cell by which maximum requirements are exhausted.

Step 4: Shade the Row/Column whose availability or requirement is exhausted so that it shall not be considered for any further allocation.

Step 5:Repeat Step 3 and 4 till entire demand and supply is exhausted.

Step 6:Draw the Initial Feasible Solution Table obtained after the above steps.

Example

Find the initial feasible solution by Vogel's Approximation Method.

	W_1	W_2	W_3	Supplies
F_1	48	60	56	140
F_2	45	55	53	260
F_3	50	65	60	360
F_4	52	64	55	220
Demand	200	320	250	

Note: Cell entries are the unit transportation costs

Solution

Introducing a Dummy warehouse with zero cost per unit as the total demand is not equal to total supply in order to make the problem balanced one.

Initial Feasible Solution by Vogel's Approximation Method (VAM)

	W_1	W_2	W_3	Dummy	Supply	D_1	D_2	D_3	D_4	D_5
F_1	48	60 [60]	56 [80]	0	140	48	8	4	4	4
F_2	45	55 [260]	53	0	260	45	8	2	2	—
F_3	50 [200]	65	60 [160]	0	360	50	10	5	5	5
F_4	52	64	55 [10]	0 [210]	220	52	3	9	—	—
Dem.	200	320	250	210						
D_1	3	5	2	0						
D_2	3	5	2	—						
D_3	—	5	2	—						
D_4	—	5	3	—						
D_5	—	5	4	—						

Application of Optimality Test

The practical steps involved in Optimality Test are given below:

Compute "U_i" and "V_j" for all Rows and Columns respectively on the basis of Allocated Cells such that $C_{ij} = U_i + V_j$ after taking any U_i or $V_j = 0$,

where C_{ij} = Shipping Cost per unit of Occupied Cell;
i = Row Number;
j = Column Number;
U_i = Shipping Cost per unit of Supplying Station; and
V_j = Shipping Cost per unit of Receiving Station

Note: While taking any *Ui* or Vj=0, that row or column which is having maximum allocated cells, should preferably be selected.

Compute Opportunity Cost, or say OC, for Unallocated Cells where

OC =Cij-*(Uij + Vij.)*

If 'OC' of each cell is *either* positive or zero, Initial Feasible Solution is the Optimal Solution. However, if 'OC' for any Cell is negative, Initial Feasible Solution is not optimal. In that case, Find Closed Loop for the Cell having negative 'OC' and transfer entire quantity from the Allocated Cell having minimum quantity, that is covered by that Loop amongst all Allocated

Cells covered by that Loop, to the Unallocated Cell having negative 'OC'

Note: The above procedure will be followed even in case 'OC' of any Unallocated Cell is "zero" and 'OC' of other Unallocated Cells is positive to get Alternate Solution.

Step 4: See whether total number of allocated cells after Step 3 is equal to "$m + n - 1$"; if yes, go to step 5 if not introduce an infinitely small quantity '*e*' to the Least Cost Independent Cell, Le., *for* which no Loop can be formed.

Note: Introduce as many number of '*e*' as the total number of Allocated Cells falls below "$m + n - 1$".

Step 5: Repeat Steps 1 to 4 till 'OC' of all Unallocated Cells is, positive.

Example

When the number of allocation Equal to "M + N - 1"

Test the following initial solution *for* optimality.

Solution

Step 1 → *Calculation of U_i and V_j on the basis of costs of allocated cells*

	W_1	W_2	W_3	W_4	U_i
F_1		60	56	58	$U_1 = 0$
F_2		55			$U_2 = -5$
F_3	50			62	$U_3 = 4$
F_4			55		$U_4 = -1$
V_j	$V_1 = 46$	$V_2 = 60$	$V_3 = 56$	$V_4 = 58$	

Step 2 → *Writing C_{ij} Matrix of Costs for unallocated cells*

	W_1	W_2	W_3	W_4
F_1	48			
F_2	45		53	60
F_3		65	60	
F_4	52	64		61

Step 3 → *$U_i + V_j$ Matrix for unallocated Cells*

	W_1	W_2	W_3	W_4
F_1	46			
F_2	41		51	53
F_3		64	50	
F_4	45	59		57

Step 4 → *Δ_{ij} Matrix where $\Delta_{ij} = C_{ij} - (U_i + V_j)$*

	W_1	W_2	W_3	W_4
F_1	2			
F_2	4		2	7
F_3		1	0	
F_4	7	5		4

[When the Number of Allocation is Less than 'M + N - **1']**

Test the following initial solution for optimality

Solution

Step 1: *Calculation of Uj and Vj on the basis of costs of allocated cells*

	W_1	W_2	W_3	W_4	U_i
F_1	1	2			$U_1 = 0$
F_2			2	0	$U_2 = -1$
F_3	0		2		$U_3 = -1$
V_j	$V_1 = 1$	$V_2 = 2$	$V_3 = 3$	$V_4 = 1$	

Step 2 → *C_{ij} Matrix of Costs for unallocated cells*

	W_1	W_2	W_3	W_4
F_1			4	4
F_2	4	3		
F_3		2		1

Step 3 → *$U_i + V_j$ Matrix for unallocated Cells*

	W_1	W_2	W_3	W_4
F_1			3	1
F_2	0	1		
F_3		1		0

Step 4 → *Δ_{ij} Matrix where $\Delta_{ij} = C_{ij} - (U_i + V_j)$*

	W_1	W_2	W_3	W_4
F_1			1	3
F_2	4	2		
F_3		1		1

Step 5 → *Since all Δ_{ij} are positive the above solution is optimal. The optimal solution is given below:*

Factory	Warehouse	Qty.	Cost per Unit	Total cost
F_1	W_1	6	2	12
F_2	W_3	2	2	4
F_2	W_4	6	0	0
F_3	W_3	4	0	0
F_3	W_3	6	2	12
				28

Notes:

LESSON 11
DEGENERACY

Today's class we will see:

- **Degeneracy in T P**
- **Examples**

Degeneracy In Transportation Problems

The solution procedure for non-degenerate basic feasible solution with exactly $m + n - 1$ strictly positive allocations in *independent positions* has been discussed so far. However, sometimes it is not possible to get such initial feasible solution to start with. Thus degeneracy occurs in the transportation problem whenever a number of occupied cells is less than $m + n - 1$.

We recall that a basic feasible solution to an m-origin and n-destination transportation problem can have at most $m + n -$ 1number of positive (non- zero) basic variables. If this number is exactly $m + n$ - I, the BPS is said to *be non-degenerate;* and if less than $m + n - 1$ the basic solution degenerates. It follows that whenever the number of basic cells is less than $m + n$ - I, the transportation problem is a degenerate one.

Degeneracy in transportation problems can occur in two ways

1. Basic feasible solutions may be degenerate from the initial stage onward.
2. They may become degenerate at any intermediate stage.

Resolution of Degeneracy During the Initial Stage

To resolve degeneracy, allocate an extremely small amount of goods (close to zero) to *one* or *more* of the empty cells so that a number of occupied cells becomes $m + n$ - I. The cell containing this extremely small allocation is, of course, considered to be an occupied cell.

Rule: The extremely small quantity usually denoted by the Greek letter ~ (delta) [also sometimes by E (epsilon)] is introduced in the *least cost* independent cell subject to the following assumptions. If necessary, two or more ∇ ' s can be introduced in the least and second least cost independent cells.

1. $\Delta < x_{ij}$ for $x_{ij} > 0$. 2. $x_{ij} + \Delta = x_{ij} - \Delta$, $x_{ij} > 0$. 3. $\Delta + 0 = \Delta$.

*4. If there are more than one Δ's in the solution, $\Delta < \Delta'$, whenever Δ is *above* Δ'. If Δ and Δ' are in the same row, $\Delta < \Delta'$ when Δ *is to the left of* Δ'.

Example 14. *A company has three plants A, Band* C *and three warehouses* X, *Y and* Z. *Number of units available at the plants is 60, 70 and 80, respectively. Demands at* X, *Y and* Z *are 50, 80 and 80, respectively.*

Unit Costs of transportation are as follows:

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

Solution. Step 1. First, write the given cost-requirement *Table* in the following manner:

	X	Y	Z	Available
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
Requirement	50	80	80	210 (Total)

Step 2.

Using either the *"Lowest Cost Entry Method"* or *"Vogel's Approximation Method*, obtain the initial solution.

Step 3:

Since the number of occupied cells in *Table* is 4, which does not equal to $m + n - 1$ (that is 5), the problem is degenerate at the very beginning, and so the attempt to assign *Uj* and *Vj* values to *Table* will not succeed. However, it is possible to resolve this degeneracy by addition of to DELTA some suitable cell.

Step3.

				Available
			60(3)	60
	50(3)		20(9)	70
		80(3)		80
Requirement	50	80	80	

Generally, it is not possible to add to any empty cell. But, must be added to one of those empty cells which make possible the determination of a unique set of *Uj* and *Vj*. So choose such empty cell with careful judgment, for if the empty cell (1,1) is made an occupied cell (by addition ofÑ) it will not be possible to assign *Uj* and *Vj* values. The allocations in cell (1, 1), (1, 3), (2, 1), and (2,3) will become in non-independent (rather than independent) positions.

On the other hand, the addition of d to any of the cells (1, 2), (2, 2), (3, 1) and (3, 3) will enable us to resolve the degeneracy and allow to determine a unique set of *Uj* and Vj values. So proceed to resolve the degeneracy by allocating d to least cost independent empty cell (3, 3) as shown in

Table

			60(3)	60
	50(3)		20(9)	70
		80(3)	Δ (5)	80 + Δ = 80
Requirement	50	80	80 + Δ = 80	

Now proceed to test this solution for optimality.

Step 4. Values of *Uj* and Vj will be obtained as shown in *Table*

Once a unique set of *Uj* and Vj values has been determined, various steps of the transportation algorithm can be applied in a routine manner to obtain an optimal solution.

			u_i
		• (3)	− 2
• (3)		• (9)	4
	• (3)	• (5)	0
v_j −1	3	5	

Matrix $[c_{ij}]$ for empty cells only

(8)	(7)	•
•	(8)	•
(11)	•	•

Matrix $[u_i + v_j]$ for empty cells

− 3	1	•
•	7	•
− 1	•	•

Since all cell-evaluations in *Table* are positive, the solution under test is optimal. The real total cost of subsequent solutions did not happen to change in the example after d was introduced. In general, this will not be the case. In as much as the infinitesimal quantity d plays only an auxiliary role and has no significance, it is removed when the optimal solution is obtained. Hence, the final answer is given in *Table*

Matrix $[c_{ij} - (u_i + v_j)]$

11	6	•
•	1	•
12	•	•

Table 12.64

		60(3)
50(3)		**20**(9)
	80(3)	

Thus minimum cost is : 180 + 150 + 180 + 240 = Rs. 750.

Resolution of Degeneracy During the Solution Stages

The transportation problem may also become degenerate during the solution stages. This happens when most favorable quantity is allocated to the empty cell having the largest negative cell-evaluation resulting in simultaneous vacation of two or more of currently occupied cells. To resolve degeneracy, allocate d to one or more of recently vacated cells so that the number of occupied cells is m + n - 1 in the new solution.

Example. *The cost-requirement table for the transportation problem* is *given* as *below:*

	W_1	W_2	W_3	W_4	W_5	Available
F_1	4	3	1	2	6	40
F_2	5	2	3	4	5	30
F_3	3	5	6	3	2	20
F_4	2	4	4	5	3	10
Required	30	30	15	20	5	

Solution. By '*North- West-Corner Rule*', the non-degenerate initial solution is obtained in Table

	W_1	W_2	W_3	W_4	W_5	Available
F_1	**30**(4)	**10**(3)				40
F_2		**20**(2)	**10**(3)			30
F_3			5(6)	**15**(3)		20
F_4				5(5)	5(3)	10
Required	30	30	15	20	5	

Now, test this solution for optimality to get the following table in usual manner.

Matrix for set of u_i and v_j

					u_i
4	3				0
	2	3			−1
		6	3		2
			5	3	4
v_j 4	3	4	1	− 1	

Matrix $[c_{ij}]$ for empty cells

•	•	(1)	(2)	(6)
(5)	•	•	(4)	(5)
(3)	(5)	•	•	(2)
(2)	(4)	(4)	•	•

Matrix $(u_i + v_j)$ for empty cells

•	•	4	1	− 1
3	•	•	0	− 2
6	5	•	•	1
8	7	8	•	•

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

•	•	− 3	1	7
2	•	•	4	7
− 3	0	•	•	1
− 6 √	− 3	− 4	•	•

Since, the largest negative cell evaluation is *d41*=- 6 (marked *...J),* allocate as much as possible to this cell (4, I). This necessitates shifting of 5 units to this cell (4, 1) as directed by the closed loop in *Table*

						Available
	30 − θ (4)	10 + θ (3)				40
		20 − θ (2)	10 + θ (3)			30
			5 − θ (6)	15 + θ (3)		20
	+ θ √			5 − θ (5)	5	10
Required	30	30	15	20	5	

Here maximum possible value of θ is obtained by usual rule :

$$\min.\,[30-\theta,\ 20-\theta,\ 5-\theta,\ 5-\theta] = 0 \text{ i.e., } \qquad 5-\theta = 0 \text{ or } \theta = 5 \text{ units.}$$

The revised solution becomes:

						Available
	25(4)	15(3)				40
		15(2)	15(3)			30
			0*(6)	20(3)		20
	5(2)			0*(5)	5(3)	10
Required	30	30	15	20	5	

In this solution, the number of allocations becomes less than
$m + n$ - I" on account of simultaneous vacation of two cells [(3, 3), (4,4), as indicated by *]. Hence this is a degenerate solution.

Now, this degeneracy may resolve by adding to one of the recently vacated cells [(3, 3) or (4, 4)]. But in minimization problem, add to recently vacated cell (4,4) only, because it has the lowest shipping cost of Rs 5 per unit.

The rest of the procedure will be exactly the same as explained earlier. This way, the optimal solution can be obtained.

						Available
	5(4)		15(1)	20(2)		40
		30(2)				30
	15(3)				5(2)	20
	10(2)					10
Required	30	30	15	20	5	

Example. *Solve the following transportation problem:*

		D_1	D_2	D_3	D_4	D_5	D_6	Available
	O_1	9	12	9	6	9	10	5
	O_2	7	3	7	7	5	5	6
From	O_3	6	5	9	12	3	11	2
	O_4	6	8	11	2	2	10	9
		4	4	6	2	4	2	22 (Total)

Solution. Using 'V AM' the initial basic feasible solution having the transportation cost Rs. 112 is given below:

Initial BFS

		5(9)				5
	4(3)	Δ (7)			2(5)	6 + Δ = 6
1(6)		1(9)				2
3(6)			2(2)	4(2)		9
4	4	6 + Δ = 6	2	4	2	

Since the number of allocations (= 8) is less than $m + n$ - 1 (= 9), a very small quantity may be introduced in the independent cell (2, 3), although least cost independent cell is (2, 5).

Optimum Table

						u_i
(9) 6	(12) 5	5 (9)	(6) 2	(9) 2	(10) 7	2
(7) 4	4 (3)	Δ (7)	(7) 0	(5) 0	2 (3)	0
1 (6)	(9) 5	1 (9)	(12) 2	(3) 2	(11) 2	2
3 (6)	(8) 5	(11) 9	2 (2)	4 (2)	(10) 7	2
v_j: 4	3	7	0	0	5	

Since all the net evaluations are non-negative, the current solution is an optimum one. Hence the optimum solution is: XI3=5, $X22$ =4, $X26$ =2, $X31$ = 1,$X41$=3, $X44$ =2,$X45$ =4, $X33$ =1.

The optimum transportation cost is given by z=5(9) + 4(3) + L \ (7) + 2(5) + 1(6) + 1(9) + 3(6) + 2(2) + 4(2) =Rs. 112 (since L \ ®0).

Note. In above optimum table, may also be introduced in least cost independent cell (2,5).

Notes:

LESSON 12
ASSIGNMENT PROBLEM

- **Introduction**
- **Steps Involved in Solving T P**
- **Examples**

What Is Assignment Problem

Assignment Problem is a special type of linear programming problem where the objective is to minimise the cost or time of completing a number of jobs by a number of persons.

The assignment problem in the general form can be stated as follows:

"Given *n* facilities, *n* jobs and the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job in such a way that the measure of effectiveness is optimised (Maximised or Minimised)."Several problems of management have a structure identical with the assignment problem.

Example I: A manager has four persons (i.e. facilities) available for four separate jobs (i.e. jobs) and the cost of assigning (i.e. effectiveness) each job to each person is given. His objective is to assign each person to one and only one job in such a way that the total cost of assignment is minimised.

Example II: A manager has four operators for four separate jobs and the time of completion of each job by each operator is given. His objective is to assign each operator to one and only one job in such a way that the total time of completion is minimised.

Example III A tourist car operator has four cars in each of the four cities and four customers in four different cities. The distance between different cities is given. His objective is to assign each car to one and only one customer in such a way that the total distance covered is minimized.

Hungarian Method

Although an assignment problem can be formulated as a linear programming problem, it is solved by a special method known as Hungarian Method because of its special structure. If the time of completion or the costs corresponding to every assignment is written down in a matrix form, it is referred to as a Cost matrix. The Hungarian Method is based on the principle that if a constant is added to every element of a row and/or a column of cost matrix, the optimum solution of the resulting assignment problem is the same as the original problem and vice versa. The original cost matrix can be reduced to another cost matrix by adding constants to the elements of rows and columns where the total cost or the total completion time of an assignment is zero. Since the optimum solution remains unchanged after this reduction, this assignment is also the optimum solution of the original problem. If the objective is to maximise the effectiveness through Assignment, Hungarian Method can be applied to a revised cost matrix obtained from the original matrix.

Balanced Assignment Problem

Balanced Assignment Problem is an assignment problem where the number of facilities is equal to the number of jobs.

Unbalanced Assignment Problem

Unbalanced Assignment problem is an assignment problem where the number of facilities is not equal to the number of jobs. To make unbalanced assignment problem, a balanced one, a dummy facility(s) or a dummy job(s) (as the case may be) is introduced with zero cost or time.

Dummy Job/Facility

A dummy job or facility is an imaginary job/facility with zero cost or time introduced to make an unbalanced assignment problem balanced.

An Infeasible Assignment

An Infeasible Assignment occurs in the cell (i, j) of the assignment cost matrix if *ith* person is unable to perform *jth* job. .

It is sometimes possible that a particular person is incapable of doing certain work or a specific job cannot be performed on a particular machine. The solution of the assignment problem should take into account these restrictions so that the infeasible assignments can be avoided. This can be achieved by assigning a very high cost to the cells where assignments are prohibited.

Practical Steps Involved In Solving Minimisation Problems

Step 1: See whether Number of Rows are equal to Number of Column. If yes, problem is balanced one; if not, then add a Dummy Row or Column to make the problem a balanced one by allotting zero value or specific value (if any given) to each cell of the Dummy Row or Column, as the case may be.

Step 2: Row *Subtraction:* Subtract the minimum element of each row from all elements of that row.

Note: If there is zero in each row, there is no need for row subtraction.

Step 3: Column *Subtraction:* Subtract the minimum element of each column from all elements of that column.

Note: If there is zero in each column, there is no need for column subtraction.

Step 4: Draw minimum number of Horizontal and/or Vertical Lines to cover all zeros.

To draw minimum number of lines the following procedure may be followed:

1. Select a row containing exactly one uncovered zero and draw a vertical line through the column containing this zero and repeat the process till no such row is left.

2. Select a column containing exactly one uncovered zero and draw a horizontal line through the row containing the zero and repeat the process till no such column is left.

Step 5: If the total lines covering all zeros are equal to the size of the matrix of the Table, we have got the optimal solution; if not, subtract the minimum uncovered element from all uncovered elements and add this element to all elements at the intersection point of the lines covering zeros.

Step 6: Repeat Steps 4 and 5 till minimum number of lines covering all zeros is equal to the size of the matrix of the Table.

Step 7: *Assignment:* Select a row containing exactly one unmarked zero and surround it by , 'and draw a vertical line through the column containing this zero. Repeat this process till no such row is left; then select a column containing exactly one unmarked zero and surround it by, and draw a horizontal line through the row containing this zero and repeat this process till no such column is left.

Note: If there is more than one unmarked zero in any row or column, it indicates that an alternative solution exists. In this case, select anyone arbitrarily and pass two lines horizontally and vertically.

Step 8: Add up the value attributable to the allocation, which shall be the minimum value.

Step 9 : *Alternate Solution:* If there are more than one unmarked zero in any row or column, select the other one (i.e., other than the one selected in Step 7) and pass two lines horizontally and vertically. Add up the value attributable to the allocation, which shall be the minimum value.

Practical Steps Problems Involved Solving In Maximisation

Step 1: See whether Number of Rows is equal to Number of Columns. If yes, problem is a balanced one; if not, then adds a Dummy Row or Column to make the problem a balanced one by allotting zero value or specific value (if any given) to each cell of the Dummy Row or Column, as the case may be.

Step 2: Derive Profit Matrix by deducting cost from revenue.

Step 3: Derive Loss Matrix by deducting all elements from the largest element.

Step 4: Follow the same Steps 2 to 9 as involved in solving Minimisation Problems.

Minimization Problem

Problem

Carefree Corporation has four plants each of which can manufacture anyone of the four products. Product costs differ from one plant to another as follow:

	Product			
Plant	***1***	***2***	***3***	***4***
A	33	40	43	32
B	45	28	31	23
C	42	29	36	29
D	27	42	44	38

You are required:

a. to obtain which product each plant should produce to minimise cost,

b. to build a Linear Programming Model.

Solution

Part (a)

Step 1 : *Row Deduction:* Subtracting the minimum element of each row from all the elements of that row:

1	8	11	0
22	5	8	0
13	0	7	0
0	15	17	11

Step 2 : *Column Deduction:* Subtracting the minimum element of each column of above matrix from all the elements of that column and then drawing the minimum number of lines (whether horizontal/vertical) to cover all the zeros:

1	8	4	0
22	5	1	0
13	0	0	0
0	15	10	11

Since number of lines drawn =3 and order of matrix =4, we will have to take the step to increase the number of zeros.

Step 3 ; Subtracting the minimum uncovered element (1 in this case) from all the uncovered elements and adding to the elements at intersection points, and then drawing the minimum numbers of lines to cover all zeros.

0	7	3	0
21	4	0	0
13	0	0	1
0	15	10	12

Since the number of lines drawn (4) = order of matrix (4), the above matrix will provide the optimal solution.

Step 4 : *Assignment:* Selecting a row containing exactly one unmarked zero and surrounding it by '0' and draw a vertical line thorough the column containing this zero.

Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '0' and draw a horizontal line through the row containing this zero and repeating this process till no such column is left.

0	7	3	$\boxed{0}$
21	4	$\boxed{0}$	0
13	$\boxed{0}$	0	11
$\boxed{0}$	15	10	12

Step 5 → Computing the minimum

Thus, the optimal assignment pattern is as follows :

Plant	*Product*	*Cost*
A →	4	Rs. 32
B →	3	Rs. 31
C →	2	Rs. 29
D →	1	Rs. 27
		Rs. 119

Part (b)

Formulation of the LP model for the given assignment problem:

33 x_{11}	40 x_{12}	43 x_{13}	32 x_{14}
45 x_{21}	28 x_{22}	31 x_{23}	23 x_{24}
42 x_{31}	29 x_{32}	36 x_{33}	29 x_{34}
27 x_{41}	42 x_{42}	44 x_{43}	38 x_{44}

Minimize $Z = 33x_{11} + 40x_{12} + 43x_{13} + 32x_{14} + 45x_{21} + 28x_{12} + 31x_{23} + 23x_{24} + 42x_{31} + 29x_{32} + 36x_{33} + 29x_{34} + 27x_{41} + 42x_{42} + 44x_{43} + 38x_{44}$

Subject to
$$x_{11} + x_{12} + x_{13} + x_{14} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$
$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$
$$x_{13} + x_{23} + x_{33} + x_{43} = 1$$
$$x_{14} + x_{24} + x_{34} + x_{44} = 1$$
$$\& \ xij \geq 0$$

Notes:

LESSON 13
VARIATIONS IN THE ASSIGNMENT PROBLEM

This Lecture Will Look Into

- **Maximal Assignment Problem**

In this s ection, we shall discuss two variations of the assignment problem.

The Maximal Assignment Problem

Sometimes, the assignment problem deals with the maximization of an objective function rather than to, minimize it. For example, it may be required to assign persons to jobs in such a way that the expected profit is maximum. Such problem may be solved easily by first converting it to a minimization problem and then applying the usual procedure of assignment algorithm. This conversion can be very easily done by subtracting from the highest element, all the elements of the given profit matrix; or equivalently, by placing minus sign before each element of the profit-matrix in order to make it cost-matrix.

Following examples will make the procedure clear.

Example 1. *(Maximization Problem). A company has* 5*jobs to be done. The following matrix shows the return in rupees on assigning I th (i* = 1,2,3,4,5) *machine to the jth job* (j =A, B, C, D, E). *Assign the five jobs to the five machines so as to maximize the total expected profit*

		Jobs				
		A	B	C	D	E
Machines	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

Solution. *Step* 1. *Converting from Maximization to Minimization:*

Since the highest element is]4, so subtracting all the elements from]4, the following reduced cost (opportunity loss of maximum profit) matrix is obtained.

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	2

Step 2. Now following the usual procedure of solving an assignment problem, an optimal assignment is obtained in the following table:

1	~~0~~	$\boxed{0}$	~~0~~	5
~~0~~	13	~~0~~	5	$\boxed{0}$
5	1	7	$\boxed{0}$	5
3	$\boxed{0}$	9	4	5
$\boxed{0}$	3	3	1	5

This table gives the optimum assignment as : $1 \rightarrow C$, $2 \rightarrow E$, $3 \rightarrow D$, $4 \rightarrow B$, $5 \rightarrow A$;
with maximum profit of Rs. 50.

Example 2. (Maximization Problem). A company has four territories open, and four salesmen available for assignment. The territories are not equally rich in their sales potential.. It is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territory	: I	II	III	IV
Annual sales (Rs.)	: 60,000	50,000	40,000	30,000

Four salesmen are also considered to differ in their ability: it is estimated that, working under the same conditions, their yearly sales would be proportionately as follows:

Salesman	:	*A*	*B*	*C*	*D*
Proportion	:	7	5	5	4

If the criterion is maximum expected total sales, then intuitive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest, and so on. Verify this answer by the assignment technique.

Solution. Step 1. To construct the effectiveness matrix:

In order to avoid the fractional values of annual sales of each salesman in each territory, it will be rather convenient to consider the sales for 21 years (the sum of proportions: 7 + 5 + 5 + 4 =21), taking Rs. 10,000 as one unit. Divide the individual sales in each territory by 21, if the annual sales by salesman are required.

Thus, the sales matrix for maximization is obtained as follows:

Sales in 10 thousand of rupees

Sales proportion ↓	6 I	5 II	4 III	3 IV
7 A	42	35	28	21
5 B	30	25	20	15
5 C	30	25	20	15
4 D	24	20	16	12

Step 2. (To convert 'the maximum sales matrix' to 'minimum sales matrix'.)

The problem of 'maximization' can be converted to 'minimization' one, by simply multiplying each element of given matrix (Table) by -I. Thus resulting matrix becomes:

	I	II	III	IV
A	– 42	– 35	– 28	– 21
B	– 30	– 25	– 20	– 15
C	– 30	– 25	– 20	– 15
D	– 24	– 20	– 16	– 12

Step 3. Subtracting the smallest element in each row from every element in that row, we get the reduced matrix

0	7	14	21
0	5	10	15
0	5	10	15
0	4	8	12

Step 4. Subtract the smallest element in each column from every element in that column to get the second reduced matrix

Step5.

Since all zeros in Table can be covered by minimum number of lines.(L1 ,L2) , which is less than 4 (the number of rows in the matrix), the optimal assignment is not possible at this stage.

In Table select the minimum element' l' among all uncovered elements. Then subtract this value 1 from each uncovered element, and add 1 at the intersection of two lines L1,L2 . Thus, the revised matrix is obtained as Table

L_2	L_3		
0	2	5	8
0	0	1	2
0	0	1	2
0	0	0	0

Step 6. Again repeat Step 5. Since the minimum number of lines (L1, L2) in Table to cover all zeros is less than 4 (the number of rows/columns), subtract the min. element 1 from all uncovered elements and add 1 at the intersection of lines (L1 ,L2) and (L1 ,L2). Then find the optimal assignment as explained in Step 7.

Step 7. To find an optimal assignment. Since there is a single zero element in row 1and column 4 only, make the zero assignment by putting '0 ' around these two zeros and cross-out other zeros in column 1 and row 4. Other zero-assignments are quite obvious from the following tables:

	I	II	III	IV
A	$\boxed{0}$	2	4	7
B	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1
C	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
D	2	1	$\cancel{0}$	$\boxed{0}$

	I	II	III	IV
A	$\boxed{0}$	2	4	7
B	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
C	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1
D	2	1	$\cancel{0}$	$\boxed{0}$

Thus, two possible solutions are: (i) A-I, B-II, C-III, D-IV; (ii) A-I, B-III, C-II, D-IV.

Both the solutions show that the best salesman A is assigned to the richest territory I, the worst salesman D to the poorest territory IV. Salesman B and C being equally good, so they may be assigned to either II or III. This verifies the answer.

Notes:

LESSON 14

GROUP DISCUSSION/QUIZ ON UNIT 2 PORTIONS

Notes:

LESSON 15
SEQUENCING INTRODUCTION

We will be discussing following topics today

- **Sequencing – Introduction**
- **Examples**

Introduction

Suppose there are *n* jobs to perform, each of which requires processing on some or all of *m* different machines. The effectiveness *(i.e.* cost, time or mileage, etc.) can be measured for any given sequence of jobs at each machine, and the most suitable sequence is to be selected (which optimizes the effectiveness measure) among all *(n!)m* theoretically possible sequences. Although, theoretically, it is a1ways possible to select the best sequence by testing each one, but it is practically impossible because of large number of computations.

In particular, if $m = 5$ and $n = 5$, the total number of possible sequences will be (5 !)5 = 25,000,000,000. Hence the effectiveness for each of (5 !)5 sequences is to be computed before selecting the most suitable one. But, this approach is practically impossible to adopt. So easier methods of dealing with such problems are needed.

Before proceeding to our actual discussion we should explain what the sequencing problem is. The problem of sequencing may be defined as follows:

Definition. Suppose there are n jobs (1, 2, 3,..., n), each of which has to be processed one at a time at each of m machines A, *B,* C, ... The order of processing each job through machines is given (for example, job I is processed through machines A, C, B-in this order). The time that each job must require on each machine is known. The problem is to find a sequence among *(n !)m* number of all possible sequences (or combinations) (or order) for processing the jobs so that the total elapsed time for all the jobs will be minimum. Mathematically, let

Ai = time for job *i* on machine A,

Bi = time for job *i* on machine *B,* etc.

T = time from start of first job to completion of the last job.

Then, the problem is to determine for each machine a sequence of jobs *i1, i2, i3, ..., in-* where

(i1,i2, i3, ..., in) is the permutation of the integers which will minimize *T*.

Terminology and Notations

The following *terminology* and notations will be used in this chapter.

1. Number of Machines. It means the service facilities through which. a job must pass before it is completed. .

   For example, a book to be published has to be processed through composing, printing, binding, etc. In this example, the book constitutes the *job* and the different processes constitute the *number of machines.* .

2. Processing Order. It refers to the order in which various machines are required for completing the job.
3. Processing Time. It means the time required by each job on each machine. The notation *Tij* will denote the processing time required for Ith job b y*jth* machine *(i* = 1,2,..., *n;j* = 1, 2, ...,*m).*
4. Idle Time on a Machine. This is the time for which a machine remains idle during the total elapsed time. The notation *Xij* shall be (used to denote the idle time of machine *j* between the *end* of the (i - 1)th job and the start of the ith job.
5. Total Elapsed Time. This is the time between starting the first job and completing the last job. This also includes *idle time,* if exists. It will be denoted by the symbol *T*.
6. No Passing Rule. This rule means that P1lssing is not allowed, *i.e.* the same order of jobs is maintained over each machine. If each of the n-jobs is to be processed through two machines *A* and *B* in the order *AB,* then this rule means that each job will go to machine *A* first and then to *B.*

Principal Assumptions

1. No machine can process more than one operation at a time.
2. Each operation, once started, must be performed till completion.
3. A job is an entity, *i.e.* even though the job represents a lot of individual parts, no lot may be processed by more than one machine at a time.
4. Each operation must be completed before any other operation, which it must precede, can begin.
5. Time intervals for processing are independent of the order in which operations are performed.
6. There is only one of each type of machine.
7. A job is processed as soon as possible subject to ordering requirements.
8. All jobs are known and are ready to start processing before the period under consideration begins.
9. The time required to transfer jobs between machines is negligible.

Solution of Sequencing Problem

At present, solutions of following cases are available:

1. n jobs and two machines A and B, all jobs processed in the order AB.
2. n jobs and three machines A, Band C, all jobs processed in the order ABC.,
3. Two jobs and m machines. Each job is to be processed through the machines in a prescribed order (which is not necessarily the same for both the jobs).
4. Problems with n jobs and m-machines.

Notes:

LESSON 16
SEQUENCING INTRODUCTION (CONTD...)

This lesson we will be discussing following topics today

- **Processing n jobs and 2 Machines**
- **Processing n jobs and 3 Machines**
- **Processing n jobs and m Machines**

Processing n Jobs Through Two Machines

The problem can be described as: (i) only two machines *A* and *B* are involved, *(ii)* each job is processed in the order *AB,* and *(iii)* the exact or expected processing times *A), Az, A3, ..., An; B), B2, B3, ..., Bn* are known

Processing Times	Job (i)				
	1	2	3	...	n
A_i	A_1	A_2	A_3	...	A_n
B_i	B_1	B_2	B_3	...	B_n

The problem is to sequence (order) the jobs so as to minimize the total elapsed time *T.*

The solution procedure adopted by *Johnson* (1954) is given below.

Solution Procedure

Step 1. Select, the least processing time occurring in the list *A* I, *Az, A3,...,* Ar and *Bt, B2, B3' ..., BII'* If there is a tie, either of the smallest processing time should be selected.

Step 2. If the least processing time is *Ar* select *r th* job *first.* If it is *Bs,* do the s th job *last* (as the given order *is AB).*

Step 3. There are now *n* - I jobs left to be ordered. Again repeat steps I and n for the reduced set of processing times obtained by deleting processing times for both the machines corresponding to the job already assigned. .

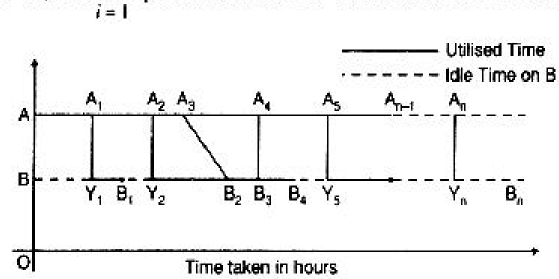
Continue till all jobs have been ordered. The resulting ordering will minimize the elapsed time T.

Proof. Since passing is not allowed, all *n* jobs must be processed on machine *A* without any *idle time* for it. On the other hand, machine *B is* subject to its remaining idle time at various stages. Let *Yj* be the time for which m machine B remains idle after completing *(i* - l)th job and before starting processing the ith job *(i=*1,2, ..., *n).* Hence, the total elapsed time T is given by

$$T = \sum_{i=1}^{n} B_i + \sum_{i=1}^{n} Y_i$$

where some of the B_i's may be zero.

Now we wish to minimize T. However, since $\sum_{i=1}^{n} B_i$ is the total time for which machine B has to work and thus being constant, it does not form a part of minimizing T. So the problem is reduced to that of minimizing $\sum_{i=1}^{n} Y_i$. A very convenient procedure for obtaining a sequence of performing jobs so as to minimize $\sum_{i=1}^{n} Y_i$ is well



explained by the following *Gantt Chart*.

From this chart, it is clear that

$$Y_1 = A_1$$

$$Y_2 = \begin{cases} A_1 + A_2 - Y_1 - B_1, & \text{if } A_1 + A_2 > Y_1 + B_1 \\ 0 & \text{otherwise} \end{cases}$$

The expression for Y_2 may be rewritten as

$$Y_2 = \max\{A_1 + A_2 - Y_1 - B_1, 0\}$$

Thus, $\quad Y_1 + Y_2 = \max\{A_1 + A_2 - B_1, A_1\}$, since $Y_1 = A_1$.

Similarly, $\quad Y_3 = \max\{A_1 + A_2 + A_3 - B_1 - B_2 - Y_1 - Y_2, 0\}$

$$\therefore \quad Y_1 + Y_2 + Y_3 = \max\left\{\left(\sum_{i=1}^{3} A_i - \sum_{i=1}^{2} B_i\right), \sum_{i=1}^{2} Y_i\right\}$$

$$= \max\left\{\left(\sum_{i=1}^{3} A_i - \sum_{i=1}^{2} B_i\right), \left(\sum_{i=1}^{2} A_i - B_i\right), A_1\right\}$$

In general, we get

$$\sum_{i=1}^{n} Y_i = \max\left\{\left(\sum_{i=1}^{n} A_i - \sum_{i=1}^{n-1} B_i\right), \left(\sum_{i=1}^{n-1} A_i - \sum_{i=1}^{n-2} B_i\right), \ldots, A_1\right\} = \max_{1 \le r \le n}\left\{\sum_{i=1}^{r} A_i - \sum_{i=1}^{r-1} B_i\right\}$$

Now, if we denote $\sum_{i=1}^{n} Y_i$ by $D_n(S)$, then the problem becomes that of finding the sequence $< S^* >$ for processing the jobs 1, 2,..., n so as to have the inequality $D_n(S^*) \le D_n(S_0)$ for any sequence $< S_0 >$ other than

$< S^* >$. In other words, we have to find the optimal sequence $< S >$ so as to minimize $D_n(S)$. This can be done iteratively by successively interchanging the consecutive jobs. Each such interchange of jobs gives a value of $D_n(S)$ less than or equal to its value before the change.

Johnson's Algorithm for n Jobs 2 Machines

The Johnson's iterative procedure for determining the optimal sequence for an n-job 2-machine sequencing problem can be outlined as follows:

Step 1. Examine the A;'s and B;'s for i = 1,2, ..., n and find out min *[Ai,Bi]*

Step2.

i. If this minimum be Ak for some $i = k$, do (process) the kth job first of all.

ii. If this minimum be *Br for* some $i = r$, do (process) the rth job last of all.

Step 3.

i. If there is a tie for minima $Ak = Bn$ process the kth job first of all and rth job in the last.

ii. If the tie for the minimum occurs among the A;'s, select the job corresponding to the minimum of *B;*'s and process it first of all.

iii. If the tie for minimum occurs among the *B;*'s, select the job corresponding to the minimum of *Ai's* and process it in the last. Go to next step.

Step4. Cross-out the jobs already assigned and repeat steps 1 to 3 arranging the jobs next to first or next to last, until all the jobs have been assigned.

Example 1. There are five jobs each of which must go through the two machines A and B in the order AB. Processing times are given below:

Processing time (hours)					
Job	1	2	3	4	5
Time for *A*	5	1	9	3	10
Time for *B*	2	6	7	8	4

Determine a sequence for five jobs that will minimize the elapsed time T. Calculate the total idle time for the machines in this period.

Solution. Apply steps I and II of solution procedure. It is seen that the smallest processing time is one hour for job 2 on the machine A. So list the job 2 at first place as shown below.

2				

Now, the reduced list of processing times becomes

Job	A	B
1	5	$\boxed{2}$
3	9	7
4	3	8
5	10	4

Again, the smallest processing time in the reduced list is 2 for job 1 on the machine B. So place job 1 last.

2				1

Continuing in the like manner, the next reduced list is obtained

Job	A	B
3	9	7
4	$\boxed{3}$	8
5	10	4

leading to sequence

2	4			1

and the list

Job	A	B
3	9	$\boxed{7}$
5	10	4

gives rise to sequence

2	4		5	1

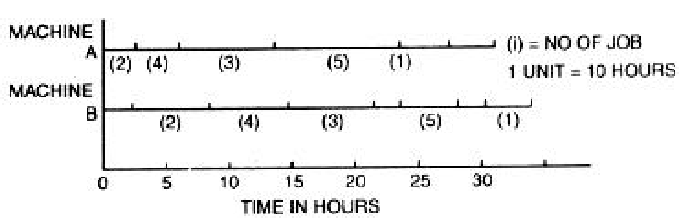
Finally, the optimal sequence is obtained,

2	4	3	5	1

Further, it is also possible to calculate the minimum elapsed time corresponding to the optimal sequencing, using the individual processing time given in the statement of the problem. The details are given in Table

Job sequence	Machine A		Machine B	
	Time in	Time out	Time in	Time out
2	0	1	1	7
4	1	4	7	15
3	4	13	15	22
5	13	23	23	27
1	23	28	28	$\boxed{30}$

Thus, the minimum time, i.e. the time for starting of job 2 to completion of the last job 1, is 30 hrs only. During this time, the machine A remains idle for 2 hrs (from 28 to 30 hrs) and the machine B remains idle for 3 hrs only (from 0-1,22-23, and 27-28 hrs). The total elapsed time can also be calculated by using Gantt chart as follows:



From the Fig it can be seen that the total elapsed time is 30 hrs, and the idle time of the machine B is 3 hrs. In this problem, it is observed that job may be held in inventory before going to the machine. For example, 4th job will be free on machine A after 4th hour and will start on machine B after 7th hr. Therefore, it will be kept in inventory for 3 hrs. Here it is assumed that the storage space is available and the cost of holding the inventory for each job is either same or negligible. For short duration process problems, it is negligible. Second general assumption is that the order of completion of jobs has no significance, i.e. no job claims the priority.

Processing n Jobs Through Three Machines

The problem can be described as: *(i)* Only three machines *A*, *B* and C are involved, *(ii)* each job is processed in the prescribed order *ABC*, *(iii)* transfer of jobs is not permitted, *i.e.* adhere strictly the order over each machine, and *(iv)* exact or expected processing times are given in Table

Job	Machine A	Machine B	Machine C
1	A_1	B_1	C_1
2	A_2	B_2	C_2
3	A_3	B_3	C_3
:	:	:	:
n	A_n	B_n	C_n

Optimal Solution. So far no general procedure is available for obtaining an optimal sequence in this case.

However, the earlier method adopted by *Johnson* (1954) can be extended to cover the special cases where *either one or both* of the following conditions hold:

i. The minimum time on machine *A* the maximum time on machine *B*.

ii. The minimum time on machine C the maximum time on machine *B*.

The procedure explained here (without proof) is to replace the problem with an equivalent problem, involving *n* jobs and two fictitious machines denoted by G and *H*, and corresponding time *Gj* and *Hj* are defined by

$$G_i = A_i + B_i, \;\; H_i = B_i + C_i.$$

If this problem with prescribed ordering *GH* is solved, the resulting optimal sequence will also be optimal for the original problem.

Rules for deleting the programs which cannot be optimal.

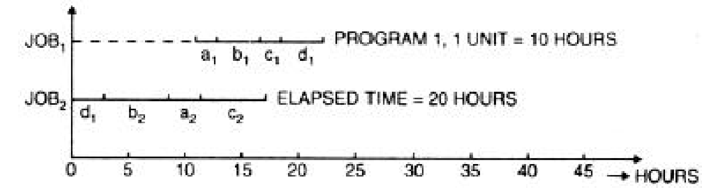
Rule no.	Technological orderings for		Delete programs containing
	Job 1	Job 2	
I	X...Y	Y...	xy
II	X...Y...	...XY	$\bar{x}y$
III	...X...Y	...XY...	$\bar{x}y$
IV	...XY...	X...Y...	xy
V	...XY...Z...	...X...YZ...	$x\bar{y}z$
VI	...X...YZ...	...XY...Z	$\bar{x}y\bar{z}$

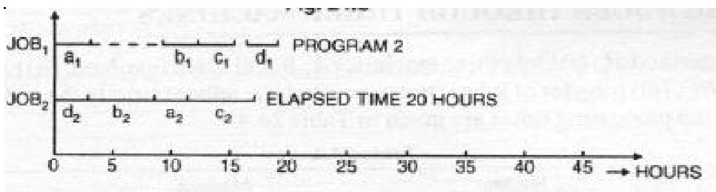
Here... stands for other machines, if any. Applying the rules to this example, it is observed by taking *A* as X, and D as *Y* (I rule), that delete the programs containing *ad.* Such a program is 16thonly. Again by n rule taking *A* as X and C as *Y*, all those programs are deleted which contain *lie, i.e.*, the 5th program. Other rules are not applicable to our problem. Thus we have only following five programs.

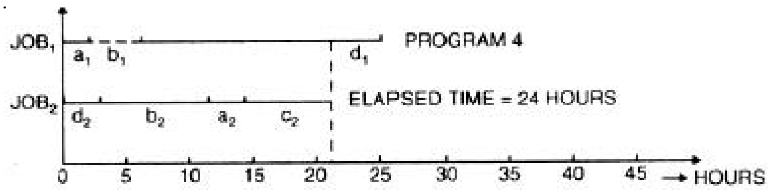
Program No. 1	2	4	6	8
$\bar{a}$	a	a	a	a
$\bar{b}$	$\bar{b}$	b	$\bar{b}$	b
$\bar{c}$	$\bar{c}$	$\bar{c}$	c	c
$\bar{d}$	$\bar{d}$	$\bar{d}$	$\bar{d}$	d

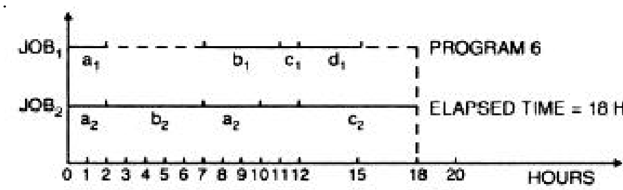
Now finally we enumerate all these programs one by one using Gantt Chart as shown below:

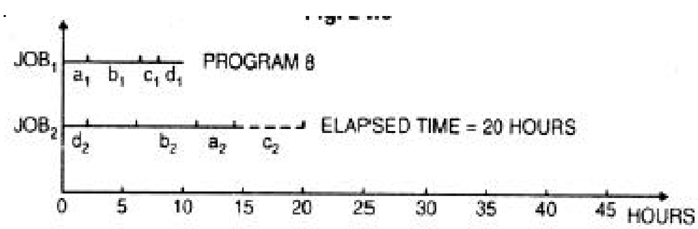
'From these charts it is clear that optimum program is 6th and the minimum elapsed time is 18 hours.











Example. *There are five jobs, each of which must go through machines A, Band C in the order ABC. Processing times are given in Table*

Job i	Processing Times		
	A_i	B_i	C_i
1	8	5	4
2	10	6	9
3	6	2	8
4	7	3	6
5	11	4	5

Determine a sequence for five jobs that will minimize the elapsed time T.

Solution. Here min $Ai = 6$, max $Bi = 6$, min $Ci = 4$.

Since one of two conditions is satisfied by min Ai =max Bi, so the procedure adopted in *Example 1* can be followed.

The equivalent problem, involving five jobs and two fictitious machine G and *H*, becomes:

Job i	Processing Times	
	$G_i (= A_i + B_i)$	$H_i (= B_i + C_i)$
1	13	9
2	16	15
3	8	10
4	10	9
5	15	9

This new problem can be solved by the procedure described earlier. Because of ties, possible optimal sequences are :

(*i*) 3 | 2 | 1 | 4 | 5 (*ii*) 3 | 2 | 4 | 1 | 5

(*iii*) 3 | 2 | 4 | 5 | 1 (*iv*) 3 | 2 | 5 | 4 | 1

(*v*) 3 | 2 | 1 | 5 | 4 (*vi*) 3 | 2 | 5 | 1 | 4

It is possible to calculate the minimum elapsed time for first sequence as shown in Table

Job	Machine *A*		Machine *B*		Machine *C*	
	Time in	Time out	Time in	Time out	Time in	Time out
3	0	6	6	8	8	16
2	6	16	16	22	22	31
1	16	24	24	29	31	35
4	24	31	31	34	35	41
5	31	42	42	46	46	51

Thus, any of the sequences from *(i)* to *(vi)* may be used to order the jobs through machines *A*, *B* and C. and they all will give a minimum elapsed time of 51 hrs. Idle time for machine *A* is 9 hrs, for *B* 31 hrs, for C 19hrs.

Processing n Jobs Through m Machines

Let each of the *n* jobs be processed through *m* machines, say *M*I, *M2, M3, ..., Mm*in the order *MI,M2,M3 ... Mm,* and *Tij* denote the time taken by the ith machine to complete the jth job. The step-by-step procedure for obtaining an optimal sequence is as follows:

Step 1. First find, (*i*) $\min_j (T_{1j})$, (*ii*) $\min_j (T_{mj})$, and (*iii*) $\max_j (T_{2j}, T_{3j}, ..., T_{(m-1)j})$ for $j = 1, 2, ..., n$.

Step 2. Then check whether
(*i*) $\min_i (T_{1j}) \geq \max_j (T_{ij})$ for $i = 2, 3, ..., m-1$, or(*ii*) $\min_j (T_{mj}) \geq \max_j (T_{ij})$ for $i = 2, 3, ..., m-1$.

Step 3. If inequalities of *step* 2 are not satisfied, this method fails. Hence go to next step.

Step 4. Convert the *m*-machine problem into 2-machine problem considering two fictitious machines *G* and *H*, so that

$$T_{Gj} = T_{1j} + T_{2j} + \dots + T_{(m-1)j} \text{ and } T_{Hj} = T_{2j} + T_{3j} + \dots + T_{mj}.$$

Now determine the optimal sequence of *n* jobs through 2 machines by using the optimal sequence algorithm.

Step 5. In addition to conditions given in *step* 4, if $T_{2j} + T_{3j} + \dots + T_{(m-1)j} = C$ (*a fixed positive constant*) for all $j = 1, 2, ..., n$, then determine the optimal sequence for *n* jobs and two machines M_1 and M_m in the order M_1M_m by using the optimal sequence algorithm.

Example 4. *Solve the following sequencing problem giving an optimal solution when passing is not allowed.*

		Job (j)				
		A	B	C	D	E
Machine (i)	M_1	11	13	9	16	16
	M_2	4	3	5	2	6
	M_3	6	7	5	8	4
	M_4	15	8	13	9	11

Solution. In this example,

$$\min_j T_{1j} = 9 = T_{13},\ \min_j T_{4j} = 8 = T_{42}\ ;\ \max_j T_{2j} = 6 = T_{25},\ \max_j T_{3j} = 8 = T_{34}.$$

Since the conditions

$$(\min_j T_{1j} \geq \max_i T_{ij},\ i = 2, 3) \text{ and } (\min_j T_{4j} \geq \max_j T_{ij},\ i = 2, 3)$$

are satisfied, convert this problem into five jobs and two-machine problem.

Also, $M_{2j} + M_{3j} = 10$, a fixed positive constant for all $j = 1, 2, ..., 5$, therefore the problem will reduce to an optimal sequence for five jobs and two machines M_1 and M_4 in the order M_1M_4 meaning thereby M_2 and M_3 have no effect on the optimality of the sequences. Following the usual optimal sequence algorithm, obtain the optimal sequence C → A → E → D → B. Therefore, the total elapsed time may be calculated as follows :

		Machine			
		M_1	M_2	M_3	M_4
Job	C	0–9	9–14	14–19	19–32
	A	9–20	20–24	24–30	32–45
	E	20–36	36–42	42–46	46–57
	D	36–52	52–54	54–62	62–71
	B	52–65	65–68	68–75	75–83

This table shows that the total elapsed time is 83 hrs.

Example 5. There are 4 jobs each of which has to go through the machines Mi, i = 1, 2, ..., 6 in the order M.M2...M6. Processing times are given.

		Machine (i)					
		M_1	M_2	M_3	M_4	M_5	M_6
Job (j)	A	20	10	9	4	12	27
	B	19	8	11	8	10	21
	C	13	7	10	7	9	17
	D	22	6	5	6	10	14

Determine a sequence of these four jobs, which minimizes the total elapsed time T.

Solution. In this example,

$$(\min_j T_{1j} = 13,\ \min_j T_{6j} = 14) \text{ and } (\max_j T_{2j} = 10,\ \max_j T_{3j} = 11,\ \max_j T_{4j} = 8,\ \max_j T_{5j} = 12).$$

Since the conditions :

$$(\min_j T_{1j} \geq \max_j T_{ij}) \text{ and } (\min_j T_{6j} \geq \max_j T_{ij}) \text{ for } i = 2, 3, 4, 5$$

are satisfied, the problem can be converted into 4-job and 2-machine problem.

Thus if G and H are two fictitious machines such that

$$T_{Gj} = \sum_{i=1}^{m-1} T_{ij} \text{ and } T_{Hj} = \sum_{i=2}^{m} T_{ij},$$

then the problem can be reformulated as 4-job and 2-machine problem :

		Jobs			
		A	B	C	D
Machines	G	55	56	46	49
	H	62	58	50	41

Using the optimal sequence algorithm, an optical sequence is obtained as C ~ A ~ B ~ D. The total elapsed time may be calculated as:

		Machines					
		M_1	M_2	M_3	M_4	M_5	M_6
Jobs	C	0–13	13–20	20–30	30–37	37–46	46–63
	A	13–33	33–43	43–52	52–56	56–68	68–95
	B	33–52	52–60	60–71	71–79	79–89	95–116
	D	52–74	74–80	80–85	91–101	91–101	116–130

This table shows that the total elapsed time is 130 hrs.

Graphical Method

In the two job m-machine problem, there is a graphical procedure, which is rather simple to apply and usually provides good (though not necessarily optimal) results. The following example will make the graphical procedure clear.

Example 3. Use graphical method to minimize the time needed to process the following jobs on the machines shown below, i.e. for each machine find the job, which should be done first. Also calculate the total time needed to complete both the jobs.

Job 1	Sequence of Machines	:	A	B	C	D	E
	Time	:	2	3	4	6	2
Job 2	Sequence of Machines	:	C	A	D	E	B
	Time	:	4	5	3	2	6

Solution.

Step 1. First, draw a set of axes, where the horizontal axis represents processing time on job 1 and the vertical axis represents processing time on job 2

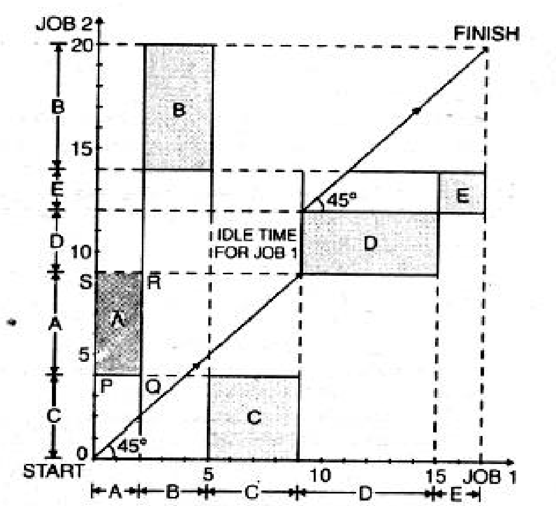
Step 2. Layout the machine time for two jobs on the corresponding axes in the given technological order Machine A takes 2 hrs for job 1 and 5 hrs for job 2. Construct the rectangle PQRS for the machine A. Similarly, other rectangles for machines B, C, D and E are constructed as shown.

Step 3. Make programme by starting from the origin 0 and moving through various states of completion (points) until the point marked

Graphical solution for the 2-job 5-machine sequencing problem. 'finish' is obtained. Physical interpretation of the path thus chosen involves the series of segments, which are horizontal or vertical or diagonal making an angle of 45° with the horizontal. Moving to the right means that job 1is proceeding while job 2 is idle, and moving upward means that job 2 is proceeding while job 1 is idle, and moving diagonally means that both the jobs are proceeding simultaneously. Further, both the jobs cannot be processed simultaneously on the same machine. Graphically, diagonal movement through the blocked-out (shaded) area is not allowed, and similarly for other machines too.

Step4.To find an optimal path. An optimal path (programme) is one that minimizes idle time for job (horizontal movement). Similarly, an optimal path is one that minimizes idle time for job 2 (vertical movement). Choose such a path on which diagonal movement is as much as possible. According to this, choose a good path by inspection as shown by arrows.

Step5. To find the elapsed time. The elapsed time is obtained by adding the idle time for either of the job to the processing time for that job. In this problem, the idle time for the chosen path is seen to be 3 hrs. for the job I, and zero for the job 2. Thus, the total elapsed time, 17 + 3 = 20 hrs is obtained.



LESSON 17
REPLACEMENT

In this lecture we will be discussing:

- **Replacement introduction**
- **Replacement of Items that Deteriorate**
- **Examples**

Introduction: The Replacement Problem

The replacement problems are concerned with the situations that arise when some items such as men, machines, electric-light bulbs, etc. need replacement due to their decreased efficiency, failure or breakdown. Such decreased efficiency or complete breakdown may either be gradual or all of a sudden.

The replacement problem arises because of the following factors:

1. *The old item has become in worse condition and work badly or requires expensive maintenance.*
2. *The old item has failed due to accident or otherwise and does not work at alt, or the old item is*

   Expected to fail shortly.
3. *A better or more efficient design of machine or equipment has become available in the market.*

In the case of items whose efficiency go on decreasing according to their age, it requires to spend more money on account of increased operating cost, increased repair cost, increased scrap, etc. So in such cases, the replacement of an old item with new one is the only alternative to prevent such increased expenses.

Thus the problem of replacement is to decide best policy to determine an age at which the replacement is most economical instead of continuing at increased cost. The need for replacement arises in many situations so that different type of decisions may have to be taken. For example,

i. We may decide whether to wait for complete failure of the item (which might cause some loss), or to replace earlier at the expense of higher cost of the item.

ii. The expensive items may be considered individually to decide whether we should replace now or, if not, when it should be reconsidered for replacement.

iii. It may be decided whether we should replace by the same type of item or by different type (latest model) of item.

The problem of replacement is encountered in the case of both men and machines. Using probability it is possible to estimate the chance of death (or failure) at various ages.

The main objective of replacement is to direct the organization for maximizing its profit (or minimizing the cost).

Failure Mechanism of Items

The term 'failure' has a wider meaning in *business* than what it has in our daily life. There are *two* kinds of failure.

1. Gradual Failure. The mechanism under this category is progressive. That is, as the life of an item increases, its efficiency deteriorates, causing:

   i. Increased expenditure for operating costs,

   ii. decreased productivity of the equipments,

   iii. Decrease in the value of the equipment, *i.e.*, the resale of saving value decreases.

For example, mechanical items like *pistons, bearings, rings* etc. Another example is '*Automobile tyres* '.

2. Sudden Failure. This type of failure is applicable to those items that do not deteriorate markedly with service but which ultimately fail after some period of using. The period between installation and failure is not constant for any particular type of equipment but will follow some frequency distribution, which may be progressive, retrogressive or random in nature.

i. Progressive failure: Under this mechanism, probability of failure increases with the increase in the life of an item. For example, *electric light bulbs, automobile tubes*, etc.

ii. Retrogressive failure: Certain items have more probability of failure in the beginning of their life, and as the time passes the chances of failure become less. That is, the ability of the unit to survive in the initial period of life increases its expected life. Industrial equipments with this type of distribution of life span are exemplified by aircraft engines.

iii. Random failure: Under this failure, constant probability of failure is associated with items that fail from random causes such as *physical shocks,* not related to age. In such a case, virtually all items fail before aging has any effect. For example, vacuum tubes in air-borne equipment have been shown to fail at a rate independent of the age of the tube.

The replacement situations may be placed into *four* categories:

1. Replacement of capital equipment that becomes worse with time, *e.g. machines tools, buses in a transport organization, planes*, etc.
2. Group replacement of items that fail completely, *e.g., light bulbs, radio tubes*, etc.
3. Problems of mortality and staffing.
4. Miscellaneous Problems.

Replacement of Items that Deteriorate

Costs to be Considered

In general, the costs to be included in considering replacement decisions are all those costs that depend upon the choice or age of machine. In some special problems, certain costs need not be included in the calculations. For example, in considering the optimum decision of replacement for a particular machine, the costs that do not change with the age of the machine need not be considered.

When The Replacement Is Justified?

This question can easily be answered by considering a case of truck owner whose problem is to find the 'best' time at which he should replace the old truck by new one. The truck owner wants to transport goods as cheaply as possible. The associated costs are:

(i) The running costs, and *(ii)* the capital costs of purchasing a truck.

These associated costs can be expressed as average cost per month. Now the truck owner will observe that the average monthly cost will go on decreasing, longer the replacement is postponed. However, there will come an age at which the rate of increase of running costs more than compensates the saving in average capital costs. Thus, at this age the replacement is justified.

Replacement Policy for Items Whose Maintenance Cost

Increases With Time, and Money Value is Constant

Theorem 22.1. *The cost of maintenance of a machine is given, as a function increasing with time and its scrap value is constant.*

a. If time is measured continuously, then the average annual cost will be minimized by replacing the machine when the average cost to date becomes equal to the current maintenance cost.

b. If time is measured in discrete units, then the average annual cost will be minimized by replacing the machine when the next period's maintenance cost becomes greater than the current average cost.

Proof. (a)-When time *'t'* is a continuous variable.

Let Rt = maintenance cost at time t, C = the capital cost of the item, S = the scrap value of the item.

Obviously, annual cost of the item at any time $t = Rt$ + C- S.

Since the maintenance cost incurred during 'n' years becomes $= \int_0^n R_t\,dt$, the total cost incurred on the item will become :

$$P(n) = \int_0^n R_t\,dt + C - S.$$

Hence average total cost is given by $$F(n) = \frac{P(n)}{n} = \frac{1}{n}\int_0^n R_t\,dt + \frac{C-S}{n}$$

Now, we have to find such time n for which $F(n)$ is minimum. Therefore, differentiating $F(n)$ w.r.t. 'n',

$$\frac{dF(n)}{dn} = \frac{1}{n}R_n + \left(-\frac{1}{n^2}\right)\int_0^n R_t\,dt - \frac{C-S}{n^2} = 0, \text{ for minimum of } F(n),$$

which gives $$R_n = \frac{1}{n}\int_0^n R_t\,dt + \frac{C-S}{n} = \frac{P(n)}{n}\text{, by virtue of equation}$$

Hence, maintenance cost at time n = average cost in time n.

(b) When time 't' is a discrete variable.

Since the time is considered in discrete units, the cost equation (22.1) can be written as

$$F(n) = \frac{P(n)}{n} = \sum_{t=1}^{n}\frac{R_t}{n} + \frac{C-S}{n}.$$

By using finite differences, $F(n)$ will be minimum if the following relationship is satisfied. :

$$\Delta F(n-1) < 0 < \Delta F(n)$$

Now, differencing (22.4) under the summation sign by definition of first difference.

$$\begin{aligned}\Delta F(n) &= F(n+1) - F(n)\\ &= \left[\sum_{t=1}^{n+1}\frac{R_t}{n+1} + \frac{C-S}{n+1}\right] - \left[\sum_{t=1}^{n}\frac{R_t}{n} + \frac{C-S}{n}\right]\\ &= \left(\frac{R_{n+1}}{n+1} + \sum_{t=1}^{n}\frac{R_t}{n+1}\right) - \sum_{t=1}^{n}\frac{R_t}{n} + (C-S)\left[\frac{1}{n+1} - \frac{1}{n}\right]\\ &= \frac{R_{n+1}}{n+1} + \sum_{t=1}^{n}R_t\left(\frac{1}{n+1} - \frac{1}{n}\right) + (C-S)\left[\frac{1}{n+1} - \frac{1}{n}\right]\\ &= \frac{R_{n+1}}{n+1} - \sum_{t=1}^{n}\frac{R_t}{n(n+1)} - \frac{C-S}{n(n+1)}.\end{aligned}$$

Since $\Delta F(n) > 0$ for minimum of $F(n)$, so

$$\frac{R_{n+1}}{n+1} > \sum_{t=1}^{n}\frac{R_t}{n(n+1)} + \frac{C-S}{n(n+1)} \quad\text{or}\quad R_{n+1} > \sum_{t=1}^{n}\frac{R_t}{n} + \frac{C-S}{n}$$

or $R_{n+1} > P(n)/n$, by virtue of equation (22.4).

Similarly, it can be shown that $R_n < P(n)/n$, by virtue of $\Delta F(n-1) < 0$

Hence $R_{n+1} > (P(n)/n) > R_n$.

This completes the proof.

Example 1. *The cost of a machine is Rs. 6100 and its scrap value is only Rs. 100. The maintenance costs are found from experience to be:*

Year	:	1	2	3	4	5	6	7	8
Maintenance cost in Rs.	:	100	250	400	600	900	1250	1600	2000

When should machine be replaced?

Solution. First, find an average cost per year during the life of the machine as follows:

Total cost in first year =Maintenance cost in first year + loss in purchase price =100 + (6100 - 100)=Rs. 6100

:. Average cost in first year =Rs. 6100.

Total cost up to two years =Maintenance cost up to two years + loss in purchase price

=(100 + 250) + 6000 =Rs. 6350.

:. Average cost per year during first two years =Rs. 3175.

In a similar fashion, average cost per year during first three years =6750/3 =Rs. 2250.00,

average cost per year during first four years = 7350/4:: Rs. 1837.50,

average cost per year during first five years =8250/5 =Rs. 1650.00,

average cost per year during first six years =9500/6 =Rs. 1583.33,

average cost per year during first seven years =11100/7 =Rs. 1585.71*(Note).*

These computations may be summarized in the following tabular form.

Replace at the end of year (n)	Maintanence cost (R_n)	Total maintenance cost (ΣR_n)	Difference between Price and Resale price $(C-S)$	Total cost $P(n)$	Average cost $\frac{P(n)}{n}$
(1)	(2)	(3)	(4)	(5) = (3) + (4)	(6)
1	100	100	6000	6100	6100
2	250	350	6000	6350	3175
3	400	750	6000	6750	2250
4	600	1350	6000	7350	1837
5	900	2250	6000	8250	1650
→6	1250	3500	6000	9500	$\boxed{1583}$
7	$\boxed{1600}$	5100	6000	11100	1586
8	2000	7100	6000	13100	1638

Here it is observed that the maintenance cost in the 7th year becomes greater than the average cost for 6 years *[i.e.* $R7 > P(6)/6$*].* Hence the machine should be replaced at the end of 6th year.

Alternatively, last column of above table shows that the average cost starts increasing in the 7th year, so the machine should be replaced before the beginning of 7th year, *i.e.* at the end of 6th year.

LESSON 18
REPLACEMENT (CONTD...)

This lecture will be covering:

- **Money Value, Present Worth Factor *(PWF)*, And Discount Rate**
- **Replacement Policy When Maintenance Cost Increases With Time And Money Value Changes With Constant Rate**
- **How To Select The Best Machine?**

Money Value, Present Worth Factor (PWF), And Discount Rate

Money value. Since money has a value over time, we often speak: *'money is worth 10% per year'.* This can be explained in the following two alternative ways:

i. *In one way*, spending Rs. 100 today would be equivalent to spending Rs. 110 in a year's time: In other words, if we plan to spend Rs. 110 after a year from now, we could equivalently spend Rs. 100 today which would be of worth Rs. 110 next year.

ii. *Alternatively*, if we borrow Rs. 100at the rate of interest 10%per year and spend this amount today, then we have to pay Rs. 110 after one year. '

Thus we conclude that Rs. 100 today will be equivalent to Rs. 110 after a year from now. Consequently, one rupee after a year from now is equivalent to {1.1) raised to power -1 rupee today.

Discount rate (*Depreciation value*). The *present worth factor* of unit amount to be spent after one year is given by $v = (1 + r)^{-1}$, where r is the interest rate. Then, v is called ***discount rate*** (technically known as the ***depreci- ation value***). Following example will explain the effect of considering money value with prescribed rate of interest.

Example 7. *Let the value of money be assumed to be 10% per year and suppose that machine A is replaced after every 3 years whereas machine B is replaced after every six years. The yearly costs of both the machines are given as under:*

Year :	1	2	3	4	5	6
Machine A :	1,000	200	400	1,000	200	400
Machine B :	1,700	100	200	300	400	500

Determine which machine should be purchased.

Solution. Present worth factor is given by $v = \frac{100}{100+10} = \frac{10}{11}$.

$\therefore$ Total discount cost (present worth) of A for 3 years = Rs.$\left[1{,}000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2\right]$

= Rs. 1,512 (nearly).

Again the total discount cost of B for six years

$$= \text{Rs.}\left[1{,}700 + 100 \times \frac{10}{11} + 200 \times \left(\frac{10}{11}\right)^2 + 300 \times \left(\frac{10}{11}\right)^3 + 400 \times \left(\frac{10}{11}\right)^4 + 500 \times \left(\frac{10}{11}\right)^5\right]$$

= Rs. 2765.

Average yearly cost of A = 1,512/3 = Rs.504

and average yearly cost of B = 2,765/6 = Rs. 461.

Although, this shows the apparent advantage with B, but the comparison is unfair because the periods of consideration are different.

So, if we consider 6 years period for machine A also, then the total discount of A will be

$$= 1{,}000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2 + 1{,}000 \times \left(\frac{10}{11}\right)^3 + 200 \times \left(\frac{10}{11}\right)^4 + 400 \times \left(\frac{10}{11}\right)^5 = \text{Rs. } 2647,$$

which is Rs. 118 less costlier than machine B over the same period.

Hence machine A should be purchased.

Replacement Policy When Maintenance Cost Increases With Time

And Money Value Changes With Constant Rate

As already explained in the preceding section, the money value can be interpreted in two different ways.

Accordingly, the optimal replacement policy can be determined by the following two methods:

i. The maintenance cost increases with time and the money value decreases with constant rate *i.e.* depreciation value is given.

ii. The amount to be spent is borrowed at a given rate of interest under the condition of repaying it in pre-decided number of installments.

Theorem. *The maintenance cost increases with time and the money value decreases with constant rate i.e. depreciation value is given. Then replacement policies will be-*

i. *Replace if the next period's cost is greater than the weighted average of previous costs.*

ii. *Do not replace if the next period's cost is less than the weighted average of previous costs.*

Proof. *First Method.* Suppose that the item (which may be a machine or equipment etc.) is available for use over a series of time periods of equal intervals (say, one year).

Let C= purchase price of the item to be replaced

Rj = running (or maintenance) cost incurred at the beginning of year

r =rate of interest

$v = 1/(1+r)$ is the present worth of a rupee to be spent a year hence.

The process can be divided into two major steps:

Step 1. *To find the present worth of total expenditure.*

Let the item be replaced at the end of every nth year. The year wise present worth of expenditure on the item in the successive cycles of n years can be calculated as follows:

Year	1	2	...	n	$n+1$	$n+2$	.....	$2n$	$2n+1$	...
Present worth	$C+R_1$	R_2v	...	R_nv^{n-1}	$(C+R_1)v^n$	R_2v^{n+1}	.....	R_nv^{2n-1}	$(C+R_1)v^{2n}$	...

Assuming that the item has no resale price at the time of replacement, the present worth of all future discounted costs associated with the policy of replacing the item at the end of every n years will be given by

$$P(n) = [(C+R_1) + R_2v + \ldots + R_nv^{n-1}] + [(C+R_1)v^n + R_2v^{n+1} + \ldots + R_nv^{2n-1}]$$
$$+ [(C+R_1)v^{2n} + R_2v^{2n+1} + \ldots + R_nv^{3n-1}] + \ldots \text{and so on.}$$

Summing up the right-hand side, we get

$$P(n) = (C+R_1)(1 + v^n + v^{2n} + \ldots) + R_2v(1 + v^n + v^{2n} + \ldots) + \ldots + R_nv^{n-1}(1 + v^n + v^{2n} + \ldots)$$
$$= (C + R_1 + R_2v + \ldots\ldots + R_nv^{n-1})(1 + v^n + v^{2n} + \ldots)$$

$$= (C + R_1 + R_2v + \ldots + R_nv^{n-1}).\frac{1}{1-v^n} \quad [\because v < 1, \text{the sum of infinite G.P. is } 1/(1-v^n)] .$$

$$\therefore \quad P(n) = \frac{F(n)}{1-v^n}, \quad P(n+1) = \frac{F(n+1)}{1-v^{n+1}}$$

where, for simplicity, $F(n) = C + R_1 + \ldots + R_nv^{n-1}$.

Step 2. To determine replacement policy so that P (n) is minimum. "
Since n is measured in discrete units, we shall use the method of finite differences in order to minimize the present worth expenditure P(n).

Obviously, if $P(n+1) > P(n) > P(n-1)$, *i.e.* $\Delta P(n) > 0 > \Delta P(n-1)$, then $P(n)$ will be minimum.
So by definition of first difference

$$\Delta P(n) = P(n+1) - P(n) = \frac{F(n+1)}{1-v^{n+1}} - \frac{F(n)}{1-v^{n}} \quad \text{[from the equation (22.8b)]}$$

$$= \frac{F(n+1)(1-v^n) - F(n)(1-v^{n+1})}{(1-v^{n+1})(1-v^n)} \quad \left(\frac{N'}{D'} \text{ form}\right)$$

For convenience, we first simplify the N' of $\Delta P(n)$ only. That is,

$$N' = F(n+1)(1-v^n) - F(n)(1-v^{n+1})$$
$$= [F(n+1) - F(n)] + v^{n+1} F(n) - v^n F(n+1)$$
$$= R_{n+1}v^n + v^{n+1} F(n) - v^n [F(n) + v^n R_{n+1}] \qquad [\because F(n+1) = F(n) + R_{n+1}v^n]$$
$$= v^n(1-v^n)R_{n+1} - v^n(1-v)F(n)$$

$$\therefore \quad \Delta P(n) = \frac{v^n(1-v^n)R_{n+1} - v^n(1-v)F(n)}{(1-v^{n+1})(1-v^n)} = \frac{v^n(1-v)}{(1-v^{n+1})(1-v^n)}\left[\frac{1-v^n}{1-v}R_{n+1} - F(n)\right]$$

Simply setting $n-1$ for n in (22.9),

$$\Delta P(n-1) = \frac{v^{n-1}(1-v)}{(1-v^n)(1-v^{n-1})}\left[\frac{1-v^{n-1}}{1-v}R_n - F(n-1)\right]$$

After little simplification of R.H.S. (*see foot-note*)*

$$\Delta P(n-1) = \frac{v^{n-1}(1-v)}{(1-v^n)(1-v^{n-1})}\left[\frac{1-v^n}{1-v}R_n - F(n)\right]$$

The quantity $v^n(1-v)/(1-v^{n+1})(1-v^n)$ in equation (22.9) is always positive, since $|v| < 1$. Thus, $\Delta P(n)$ has the same sign as the quantity under bracket [...] in (22.9), with similar explanation for $\Delta P(n-1)$ in (22.10) also.

Hence the condition, $\Delta P(n-1) < 0 < \Delta P(n)$, for minimum present worth expenditure becomes

$$\frac{1-v^n}{1-v}R_n - F(n) < 0 < \frac{1-v^n}{1-v}R_{n+1} - F(n)$$

or
$$\frac{1-v^n}{1-v}R_n < F(n) < \frac{1-v^n}{1-v}R_{n+1}$$

or
$$R_n < \frac{C + R_1 + R_2 v + \ldots + R_n v^{n-1}}{1 + v + v^2 + \ldots + v^{n-1}} < R_{n+1}$$

or
$$R_n < \frac{F(n)}{\Sigma v^{n-1}} < R_{n+1}.$$

The expression between R_n and R_{n+1} in (22.12) above is called the **'*weighted average cost*'** of previous n years with weights $1, v, v^2, \ldots, v^{n-1}$, respectively.

The value of n satisfying the relationship (22.11) or (22.12) will be the best replacement age of the item.

This proves the theorem.

How to Select the Best Machine?

In the problem of choosing a best machine (or item), the costs that are constant over time for each given machine will still have to be taken into account, although these costs may differ for each machine. Only those costs that are same for the machines under comparison can be excluded.

Suppose two machines MI and *M2* are at our choice. The data required for determining the best replacement age of each type of machine is also given from past experience. Thus, a best selection can be done by adopting the following outlined procedure:

Step 1. First find the best replacement age for machine *M*I and *M2*both by using the relationship:

$$R_{n+1} > \frac{F(n)}{\Sigma v^{n-1}} > R_n$$

Suppose the optimum replacement age for machines *MI* and *M2*comes out be *nl* and *n2'* respectively.

Step 2. Compute the fixed annual payment (or weighted average cost) for each machine by using the formula:

$$x = \frac{C + R_1 + R_2 + \ldots + R_n v^{n-1}}{1 + v + v^2 + \ldots + v^{n-1}} = \frac{F(n)}{\Sigma v^{n-1}}$$

and substituting in this formula $n = n1$ for machine *M1* and $n = n2$ for machine *M2*. Let it be *XI* and *X2* for machines *M*I and *M2*, respectively.

Step 3.

i. If XI < *X2'*then choose machine *MI*,

ii. If *XI* > *X2*, then choose machine *M2*.

iii. If *XI* =*X2'* then both machines are equally good.

Notes:

LESSON 19
QUEUING THEORY

This lesson discussion will be on following portions

- **Queuing Theory Introduction**
- **MIM/1 Queuing Model**

Introduction

In everyday life, it is seen that a number of people arrive at a cinema ticket window. If the people arrive "too frequently" they will have to wait for getting their tickets or sometimes do without it. Under such circumstances, the only alternative is to form a queue, called the *waiting line,* in order to maintain a proper discipline. Occasionally, it also happens that the person issuing tickets will have to wait, *(i.e.* remains idle), until additional people arrive. Here the arriving people are called the *customers* and the person issuing the tickets is called a *server.*

Another example is represented by letters arriving at a typist's desk. Again, the letters represent the *customers* and the typist represents the *server.* A third example is illustrated by a machine breakdown situation. A broken machine represents a *customer* calling for the service of a repairman. These examples show that the term *customer* may be interpreted in various numbers of ways. It is also noticed that a service may be performed either by moving the *server* to the *customer* or the *customer* to the *server.*

Thus, it is concluded that waiting lines are not only the lines of human beings but also the aero planes seeking to land at busy airport, ships to be unloaded, machine parts to be assembled, cars waiting for traffic lights to turn green, customers waiting for attention in a shop or supermarket, calls arriving at a telephone switch-board, jobs waiting for processing by a computer, or anything else that require work done on and for it are also the examples of costly and critical delay situations. Further, it is also observed that arriving units may form one line and be serviced through only one station (as in a . . .. doctor's clinic), may form one line and be served through several stations (as in a barber shop), may form several lines and be served through as many stations {*e.g.* at check out counters of supermarket). Servers may be in parallel or in series. When in parallel, he arriving customers may form a single queue as shown in *Fig.* Or individual queues in front of each server as is common in big post-offices. Service times may be constant or variable and customers may be served singly or in batches (like passengers boarding a bus).

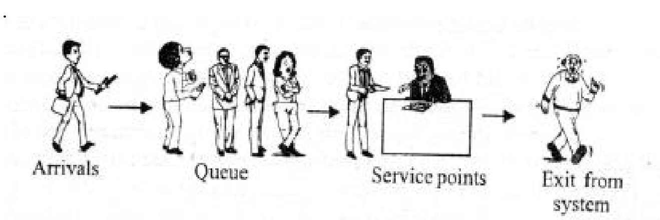


Fig (a). Queuing system with single queue and single service station..

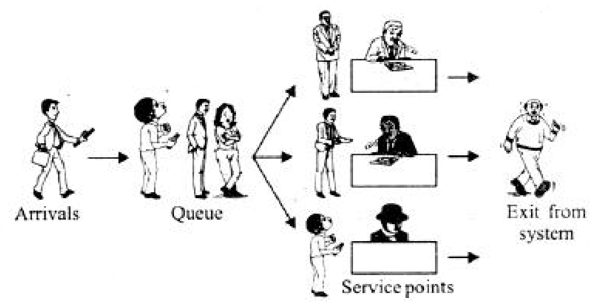


Fig. (b). Queuing system with single queue and several service stations.

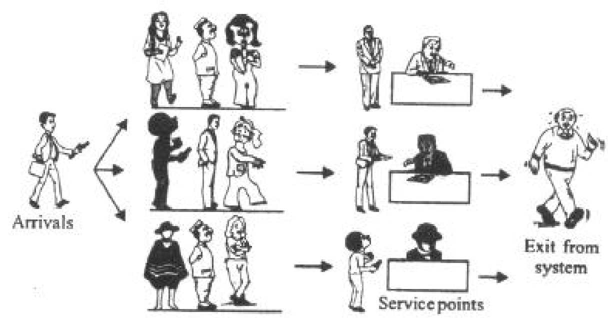


Fig. (c) Queuing system with several queues and several queues

Fig. Illustrates how a machine shop may be thought of as a system of queues forming in front of a number of service centres, the arrows between the centers indicating possible routes for jobs processed in the shop. Arrivals at a service centre are either new jobs coming into the system or jobs, partially processed, from some other service centre. Departures from a service centre may become the arrivals at another service centre or may leave the system entirely, when processing on these items is complete.

Queuing theory is concerned with the statistical description of the behaviour of queues with finding, *e.g.,* the probability distribution of the number in the queue from which the mean and variance of queue length and the probability distribution of waiting time for a customer, or the distribution of a server's busy periods can be found. In operational research problems

Involving queues, Investigators must measure the existing system to make an objective assessment of its characteristics and must determine how changes may be made to the system, what effects of various kinds of changes in the system's characteristics would be, and whether, in the light of the costs incurred in the systems, changes should be made to it. A model of the queuing system under study must be constructed in this kind of analysis and the results of queuing theory are required to obtain the characteristics of the model and to assess the effects of changes, such as the addition of an extra server or a reduction in mean service time.

Perhaps the most important general fact emerging from the theory is that the degree of congestion in a queuing system (measured by mean wait in the queue or mean queue length) is very much dependent on the amount of irregularity in the system. Thus congestion depends not just on mean rates at which customers arrive and are served and may be reduced without altering mean rates by regularizing arrivals or service times, or both where this can be achieved.

Meaning of a Queueing Model

A Queuing Model is a suitable model to represent a service-oriented problem where customers arrive randomly to receive some service, the service time being also a random variable.

Objective of a Queueing Model

The objective of a queuing model is to find out the optimum service rate and the number of servers so that the average cost of being in queuing system and the cost of service are minimised.

The queuing problem is identified by the presence of a group of customers who arrive randomly to receive some service. The customer upon arrival may be served immediately or if willing may have to wait until the server is free.

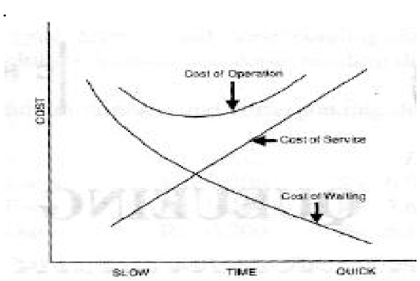
Application of a Queueing Model

The queuing models are basically relevant to service oriented organisations and suggest ways and means to improve the efficiency of the service. This model can be applied in the field of business (banks, booking counters), industries (servicing of machines), government (railway or post-office counters), transportation (air port, harbour) and everyday life (elevators, restaurants, doctor's clinic).

Relationship Between Service and Cost

Improvement of service level is always possible by increasing the number of employees. Apart from increasing the cost an immediate consequence of such a step unutilised or idle time of the servers. In addition, it is unrealistic to assume that a large-scale increase in staff is possible in an organisation. Queuing methodology indicates the optimal usage of existing manpower and other resources to improve the service. It can also indicate the cost implication if the existing service facility has to be improved by adding *more* servers.

The relationship between queuing and service rates can be diagrammatically illustrated using the cost curves as shown in following figure.



At a slow service rate, queues build up and the cost of queuing increases. An ideal service unit will minimise the operating cost of the entire system.

Arrival

The statistical pattern of the arrival can be indicated -through -

i. the probability distribution of the number of arrivals in a specific period of time, or

ii. the probability distribution of the time between two successive arrival *(known as interarrival time)* number of arrivals is a discrete variable whereas the interarrival times are continuous random and variable. A remarkable result in this context is that if the number of arrivals follows a 'Poisson Distribution', the corresponding interarrival time follows an 'Exponential Distribution'. This property is frequently used to derive elegant results on queuing problems.

Service

The time taken by a server to complete service is known as service time. The service time is a statistical variable and can be studied *either* as the number of services completed in a given period of time *or* the time taken to complete the service. The data on actual service time should be analysed to find out the probability distribution of service time. The number of services completed is a discrete random variable while the service time is a continuous random variable.

Server

A server is a person or a mechanism through which service is offered. The service may be offered through a single server such as a ticket counter or through several channels such as a train arriving in a station with several platforms. Sometimes the service is to be carried out sequentially through several phases known as multiphase service. In government, the papers move through a number of phases in terms of official hierarchy till they arrive at the appropriate level where a decision can be taken.

Time Spent in the Queueing System

The time spent by a customer in a queuing system is the sum of waiting time before service and the service time. The waiting time of a customer is the time spent by a customer in a queuing system before the service starts. The probability distribution of waiting time depends upon the probability distribution of interarrival time and service time.

Queue Discipline

The queue discipline indicates the order in which members of the queue are selected *for* service. It is most frequently assumed that the customers are served on a first come first serve basis. This is commonly referred to as FIFO (first in, first out) system. Occasionally, a certain group of customers receive priority in service over others even if they arrive late. This is commonly referred to as priority queue. The queue discipline does not always take into account the order of arrival. The server chooses one of the customers to offer service at random. Such a system is known as service in random order (SIRO). While allotting an item with high demand and limited supply such as a test match ticket or share of a public limited company, SIRO system is the only possible way of offering service when it is not possible to identify the order of arrival.

Kendall's Notation

Kendall's Notation is a system of notation according to which the various characteristics of a queuing model are identified. Kendall (Kendall, 1951) has introduced a set of notations, which have become standard in the literature of queuing models. A general queuing system is denoted by (a/b/c) :(d/e) where

a = probability distribution of the inter arrival time.

b = probability distribution of the service time.

c = number of servers in the system.

d = maximum number of customers allowed in the system.

e = queue discipline

In addition, the size of the population is important *for* certain types of queuing problem although not explicitly mentioned in the Kendall's notation. Traditionally, the exponential distribution in queuing problems is denoted by M. Thus *(MIMI!):* (¥/ FIFO) indicates a queuing system when the inter arrival times and service times are exponentially distributed having one server in the system with first in first out discipline and the number of customers allowed in the system can be infinite.

State of Queueing System

The transient state of a queuing system is the state where the probability of the number of customers in the system depends upon time. The steady state of a queuing system is the state where the probability of the number of customers in the system is independent of *t*.

Let Pn (t) indicate the probability of having n customers in the system at time *t.* Then if Pn(t) depends upon *t*, the queuing system is said to be in the transient state. After the queuing system has become operative *for* a considerable period of time, the probability Pn(t) may become independent of t. The probabilities are then known as steady state probabilities

Poisson Process

When the number of arrivals in a time interval of length *t* follows a Poisson distribution with parameter *(At),* which is the product of the arrival rate (A-)and the length of the interval *t,* the arrivals are said to follow a poison process.

Relationship Between Poisson Process And Exponential

Probability Distribution

1. If the number of arrivals in a time interval of length *(t)* for How's a Poisson Process, then corresponding inter arrival time follows an 'Exponential Distribution'.
2. If the interarrival times are independently, identically distributed random variables with an exponential probability distribution, then the number of arrivals in a time interval of length *(t)* follows a Poisson Process with arrival rate identical with parameter of the Exponential Distribution.

Mim/1 Queueing Model

The M/M/1 queuing model is a queuing model where the arrivals follow a Poisson process, service times are exponentially distributed and there is one server.

The assumption of M/M/1 queuing model are as follows:

1. The number of customers arriving in a time interval *t* follows a Poisson Process with parameter A.
2. The interval between any two successive arrivals is exponentially distributed with parameter A.
3. The time taken to complete a single service is exponentially distributed with parameter 1.1
4. The number of server is one.
5. Although not explicitly stated both the population and the queue size can be infinity.
6. The order of service is assumed to be FIFO.

If $\frac{\lambda}{\mu} < 1$, the steady state probabilities exist and P_n the number of customers in the system follows a geometric distribution with parameter $\frac{\lambda}{\mu}$ **(also known as traffic intensity).**

The probabilities are :

P_n = P (No. of customers in the sytem = n)

$$= \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right); n = 1, 2, \ldots\ldots$$

$$P_0 = 1 - \frac{\lambda}{\mu}$$

The time spent by a customer in the system taking into account both waiting and service time is an **exponential distribution** with parameter $\mu - \lambda$. the probability distribution of the waiting time before service can also be derived in an identical manner. The expected number of customers in the system is given by –

$$L_s = E(n) = \sum_{n=1}^{\infty} nP_n = \sum_{n=1}^{\infty} n\left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$= \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho}$$

The expected number of customers in the queue is given by –

$$L_q = \sum_{n=1}^{\infty} (n-1)P_n = \sum_{n=1}^{\infty} nP_n = \sum_{n=1}^{\infty} P_n$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{1 - \rho}$$

Average waiting time of a customer in the system $W_s = \frac{1}{\mu - \lambda}$.

Average waiting time of a customer in the queue $W_q = \frac{\lambda}{\mu(\mu - \lambda)}$.

Optimum Value of Service Rate (μ)

The study of queuing models helps us to find a cost model, which minimises the sum of costs of service and of waiting time per unit time. Generally, it is assumed that the cost of waiting is directly proportional to the total time that the customers spend in the system, both waiting and in service. The service can usually be ascertained from the various available records.

Assuming a single server model with an arrival rate A and service rate 11and that the service rate 11is controllable, optimum value of service rate 11can be determined as follows:

Let

C_1 = cost per unit increase in μ per unit time.

C_2 = cost per unit time per person of waiting.

$T(\mu)$ = The total cost of average waiting and service per unit time where

$T(\mu) = C_1\mu + C_2L_s$

In an M/M/1 : ∞/FIFO system it follows that –

$$T(\mu) = C_1\mu + \frac{C_2\lambda}{\mu - \lambda}$$

The optimum value of μ is obtained by minimising $T(\mu)$ with respect to μ by calculus methods. This value is given by

$$\mu = \lambda + \left(\frac{C_2\lambda}{C_1}\right)^{1/2}$$

Fundamental Process Component Elements of a Queueing

The fundamental components of a queuing process are listed below:

1. The input process or the arrivals
2. Service mechanism
3. Queue discipline

We now give a brief description of each of the above components:

1. The Input Process: Customers arrive at a service station for service. They do not come at regular intervals but arrivals into the system occur according to some chance mechanism. Often an arrival occurs at random and is independent of what has previously occurred. Customers may arrive for service individually or in groups. Single arrivals are illustrated by customers visiting a bank. On the other hand, families visiting a restaurant, is an example of bulk or group arrival. Arrivals may occur at a constant rate or may be in accordance with some probability distribution such as Poisson distribution or normal distribution etc. Frequently the population of the units or customers requiring service may be infinite e.g. passengers waiting across the booking counters but there are situations where the population may be limited such as the number of particular equipment breaking down and requiring service from the factory's maintenance crew. Thus, in connection with the input process, the following information is usually called for and is considered relevant:
   i. arrival distribution
   ii. inter-arrival distribution
   iii. mean arrival rate; (or the average number of customers arriving in one unit of time) and
   iv. mean time between arrivals.
2. Service Mechanism: Analogous to the input process, there are probability distribution of service times and number of customers served in an interval. Service time can either be fixed (as in the case of a vending machine) or distributed in accordance with some probability distribution. Service facilities can be anyone of the following types:
   i. Single channel facility or one queue-one service station facility – This means that there is only one queue in which the customer waits till the service point is ready to take him for servicing. A library counter is an example of this.
   ii. One queue-several service station facilities - In this case customers wait in a single queue until one of the service stations is ready to take them for servicing. Booking at a service station that has several mechanics each handling one vehicle, illustrates this type of model.
   iii. Several queues-one service stations - In such a situation, there are several queues and the customer can join anyone of these but the service station is only one.
   iv. Multi-channel facility - In this model, each of the servers has a different queue. Different cash counters in an Electricity Board Office where the customers can make payment in respect of their electricity bills provides an example of this model. Booking counters at railway station provide another example.
   v. Multi-stage multi-channel facilities - In this case, customers require several types of service and different service stations are there. Each station provides a specialised service and the customer passes through each of the several stations before leaving the system.

For example, machining of a certain steel item may consist of cutting, turning, knurling, drilling, grinding and packaging etc., each of which is performed by a single server in a series.

In connection with the service mechanism the following information is often obtained from the point of view of the queuing theory:

i. Distribution of number of customers serviced
ii. Distribution of time taken to service customers
iii. Average number of customers being serviced in one unit of time at a service station.
iv. Average time taken to service a customer.

3. Queue Discipline: Queue discipline may refer to many things. One of such things is the order in which the service station selects the next customer from the waiting line to be served. In this context, queue discipline may be like first in first out or last in first out or may be on the basis of certain other criteria. For example, it may be random when a teacher picks up the students for recitation. Sometimes the customer may be given a priority basis for service as on the basis of ladies first. Another aspect of queue discipline is whether a customer in a queue can move to a shorter queue in the multi-channel system. Queue discipline also refers to the manner in which the customers form into queue and the manner in which they behave in the queue. For example, a customer may get impatient and leave the queue.

Conditions For Single Channel Queueing Model

The single channel queuing model can be fitted in situations where the following seven conditions are fulfilled:

1. The number of arrivals per unit of time is described by Poisson distribution. The mean arrival rate is denoted by μ.
2. The service time has exponential distribution. The average service rate is denoted by μ.
3. Arrivals are from infinite population.
4. The queue discipline is FIFO, that is, the customers are served on a first come first serve basis.
5. There is only a single service station.
6. The mean arrival rate is less than the mean service rate i.e. μ < 11.
7. The waiting space available for customers in the queue is infinite.

Limitations of Single Channel Queueing Model

The single channel queuing model referred above is the simplest model, which is based on the above-mentioned assumptions. However, in reality, there are several limitations of this model in its applications. One obvious limitation is the possibility that the waiting space may in fact be limited. Another possibility is that arrival rate is state dependent. That is, potential customers are discouraged from entering the queue if they observe a long line at the time they arrive. Another practical limitation of the model is that the arrival process is not stationary. It is quite possible that the service station would experience peak periods and slack periods during which the arrival rate is higher and lower respectively than the overall average. These could occur at particular times during a day or a week or particular weeks during a year. There is *not* a great deal one can do to account *for* stationary without complicating the mathematics enormously. The population *of* customers served may be finite, the queue discipline may not be first come first serve. In general, the validity *of* these models depends on stringent assumptions that are often unrealistic in practice.

Even when the model assumptions are realistic, there is another limitation *of* queuing theory that is often overlooked. Queuing models give steady state solution, that is, the models tell us what will happen after queuing system has been in operation long enough *to* eliminate the effects *of* starting with an empty queue at the beginning *of* each business day. In some applications, the queuing system never reaches a steady state, so the model solutions are *of* little value.

Notes:

LESSON 20
APPLICABILITY OF QUEUEING MODEL TO INVENTORY PROBLEMS

This lesson we will be learning:

- **Practical Formulae Involved in Queuing Theory**
- **M M S Queuing Model**

Queues are common feature in inventory problems. We are confronted with queue-like situations in stores *for* spare parts in which machines wait for components and spare parts in service station. We can also look at the flow *of* materials as inventory queues in which demands wait in lines, conversely materials also wait in queues for demands *to* be served. If there is a waiting line *of* demands, inventory state tends *to* be higher than necessary. Also, if there is a negative state *of* inventories, then demands form a queue and remain unfulfilled. Thus, the management is faced with the problem *of* choosing a combination *of* controllable quantities that minimise losses resulting from the delay *of* some units in the queue and the occasional waste *of* service capacity in idleness. An increase in the potential service capacity will reduce the intensity *of* congestion, but at the same time, it will also increase the expense due *to* idle facilities in periods *of* 'NO demand'. Therefore, the ultimate goal is to achieve an economic balance between the cost *of* service and the cost associated with the waiting *of* that service. Queuing theory contributes vital information required for such a decision by predicting various characteristics *of* the waiting line, such as average queue length. Based on probability theory, it attempts to minimise the extent and duration *of* queue with minimum *of* investment in inventory and service facilities. Further, it gives the estimated average time and intervals under sampling methods, and helps in decision *of* optimum capacity so that the cost of investment is minimum keeping the amount *of* queue within tolerance limits.

PRACTICAL FORMULAE INVOLVED IN QUEUEING THEORY

1.	**Arrival Rate per hour**	$= \lambda$
2.	**Service Rate per hour**	$= \mu$
3.	**Average Utilisation Rate (*or* Utilisation Factor), ρ**	$= \frac{\lambda}{\mu}$
4.	**Average Waiting Time in the System, (waiting and servicing Time) W_s**	$= \frac{1}{(\mu-\lambda)}$
5.	**Average Waiting Time in the Queue, W_q**	$= \frac{\lambda}{\mu(\mu-\lambda)}$
6.	**Average Number of Customers (including the one who is being served) in the System, L_s**	$= \frac{\lambda}{(\mu-\lambda)}$
7.	**Average Number of Customers (excluding the one who is being served) in the Queue, L_q**	$= \frac{\lambda^2}{\mu(\mu-\lambda)}$
8.	**Average Number of Customers in Non-Empty Queue that forms from time to time**	$= \frac{\mu}{(\mu-\lambda)}$
9.	**Probability of no Customer in the System, or, System is Idle or Idlemen factor P_0**	$= 1-(\lambda/\mu)$ or $1-\rho$ $= 1 -$ utilization factor
10.	**Probability of no customer in queue and a customer is being served P_1**	$= \left[\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\right]$
11.	**Probability that there is no customer to be served**	$= P_0 + P_1$
12.	**Probability of having '*n*' customers in the System**	$= \left[\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n\right]$
13.	**Probability of having '*n*' customers in the queue**	$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{n+1}$
14.	**Probability of having more than 'n' customers in the System**	$= \left(\frac{\lambda}{\mu}\right)^{n+1}$
15.	**Probability of having less than n customers in the system or probability that an arrival will not have to wait outside the indicated space**	$= 1-\left(\frac{\lambda}{\mu}\right)^n$
16.	**Probability of having *n* or more customers in the system *or*, probability that an arrival will have to wait outside the indicated space**	$= \left(\frac{\lambda}{\mu}\right)^n$
17.	**Probability that a customer will wait for more than '*t*' hours in the queue**	$= \rho \times e^{-t(\mu-\lambda)}$ $= \frac{\lambda}{\mu} \times e^{-t(\mu-\lambda)}$
18.	**Total Costs associated with System**	= Average No. of customers in system × Opportunity cost of customer + cost of serving deptt.
19.	**Total Costs associated with Queue**	= Average No. of customers in Queue× opportunity cost of customer + Cost of Serving Deptt.

Problem [Case of Reservation Counter]

An airlines organisation has one reservation clerk on duty in its local branch at any given time. The clerk handles information regarding passenger reservations and flight timings. Assume that the number of customers arriving during any given period is Poisson distributed with an arrival rate of eight per hour and that the reservation clerk can service a customer in six minutes on an average, with an exponentially distributed service time.

i. What is the probability that the system is busy?
ii. What is the average time a customer spends in the system?
iii. What is the average length of the queue and what is the average number of customers in the system?

SOLUTION

The arrival rate $= \lambda = 8$ customers per hour

Service rate $= \mu = \frac{60}{6} = 10$ customers per hour

(i) The probability that the system is busy *or* probability of having 1 or more customer $= \left(\frac{\lambda}{\mu}\right)^1 = \left(\frac{8}{10}\right)^1 = 0.8$

(ii) The average time a customer spends in the system:

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{10-8} = \frac{1}{2} = 0.5 \text{ hour} = 30 \text{ minutes}$$

(iii) The average length of the queue:

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(8)^2}{10(10-8)} = \frac{64}{10 \times 2} = 3.2 \text{ customer}$$

The average number of customers in the systems:

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{8}{10-8} = 4 \text{ customer}$$

MODEL IV (A) : $(M \mid M \mid S) : (\infty \mid FCFS)$

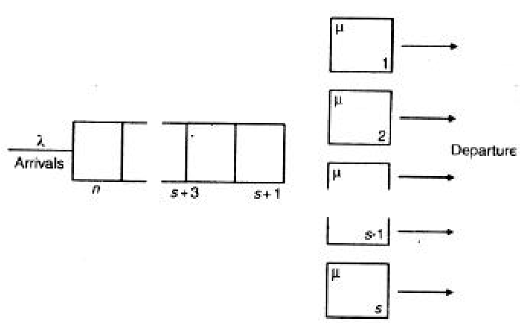
In this more realistic queuing system the customers arrive in a Poisson fashion with mean arrival rate A. There are *s* (fixed) number of counters. (service stations) arranged in parallel, and a customer can go to any of the free counters for his service, where the service time at each counter is *identical* and follows the same exponential distribution law. The mean service rate per busy server ism. Therefore, over all service rate, when there are *n* units in the system, may be obtained in the following two situations:

i. If $n \le s$, all the customers may be served simultaneously. There will be no queue, $(s - n)$ number of servers may remain idle, and then

$$\mu_n = n\mu, \; n = 0, 1, 2, \ldots, s;$$

ii. If $n \ge s$, all the servers are busy, maximum number of customers waiting in queue will be $(n - s)$,

then $\mu_n = s\mu$.



$$P_n'(t) = -(\lambda + n\mu)\, P_n(t) + \lambda P_{n-1}(t) + (n+1)\,\mu P_{n+1}(t), \quad \text{for } n = 1, 2, \ldots, s-1$$
$$P_n'(t) = -(\lambda + s\mu)\, P_n(t) + \lambda P_{n-1}(t) + s\mu P_{n+1}(t), \text{ for } n \geq s.$$

Considering the case of steady state, *i.e.* when $t \to \infty$, $P_n(t) \to P_n$ (independent of t) and hence $P_n'(t) \to 0$ for all n, above equations become

Now, this model may also be considered as a special case of Model II with

$$\lambda_n = \lambda \ (\text{for } n = 0, 1, 2, \ldots)$$
$$\mu_n = \begin{cases} n\mu \ (\text{for } n = 0, 1, 2, \ldots, s) \\ s\mu \ (\text{for } n \geq s) \end{cases}$$

Consequently, the steady state results are obtained as follows :

(a) To obtain the system of steady state equations.

Following the similar arguments as for equations (23.53) and (23.51) in Model I, we get

$$P_0(t + \Delta t) = P_0(t)\,[1 - \lambda \Delta t] + P_1(t)\,\mu \Delta t + O(\Delta t), \text{ for } n = 0,$$
$$P_n(t + \Delta t) = P_n(t)\,[1 - (\lambda + n\mu)\,\Delta t] + P_{n-1}(t)\,.\,\lambda \Delta t + P_{n+1}(t)\,(n+1)\,\mu \Delta t + O(\Delta t),$$
$$\text{for } n = 1, 2, 3, \ldots, s-1;$$

and $$P_n(t + \Delta t) = P_n(t)\,[1 - (\lambda + s\mu)\,\Delta t] + P_{n-1}(t)\,\lambda \Delta t + P_{n+1}(t)\,s\mu \Delta t + O(\Delta t),$$
$$\text{for } n = s, s+1, s+2, \ldots$$

Now dividing these equations by Δt and taking limit as $\Delta t \to 0$, the governing *differential-equations* are

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t), \quad \text{for } n = 0$$

$$0 = -\lambda P_0 + \mu P_1, \text{ for } n = 0$$
$$0 = -(\lambda + n\mu)\, P_n + \lambda P_{n-1} + (n+1)\,\mu P_{n+1}, \text{ for } 0 < n < s$$
$$0 = -(\lambda + s\mu)\, P_n + \lambda P_{n-1} + s\mu P_{n+1}, \text{ for } n \geq s.$$

This is the system of steady state *difference equations.*

(b) To solve the system of difference equations (23.93b), (23.94b) and (23.95b).

Here $P_0 = P_0$ (initially),

$$P_1 = \frac{\lambda}{\mu} P_0 \text{ [from (23.93b)]},$$

$$P_2 = \frac{\lambda}{2\mu} P_1 = \frac{\lambda^2}{2!\,\mu^2} P_0 \quad \text{[putting } n = \text{ lin (23.94b) and then substituting for } P_1\text{]},$$

$$P_3 = \frac{\lambda}{3\mu} P_2 = \frac{1}{3!}\frac{\lambda^3}{\mu^3} P_0 \quad \text{[putting } n = 2\text{n (23.94b) and then using the value of } P_2\text{]},$$

$$\ldots \qquad \ldots$$

In general, $$\boldsymbol{P_n = \frac{\lambda}{n\mu} P_{n-1} = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, \text{ for } 1 \leq n \leq s}$$

$$\ldots \qquad \ldots$$

$$P_s = \frac{\lambda}{s\mu} P_{s-1} = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s P_0$$

$$P_{s+1} = \frac{\lambda}{s\mu} P_s = \frac{1}{s} \cdot \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^{s+1} P_0 \quad \textbf{[note carefully]}$$

$$P_{s+2} = \frac{\lambda}{s\mu} P_{s+1} = \frac{1}{s^2} \cdot \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^{s+2} P_0$$

Again, in general,

$$\boldsymbol{P_n = P_{s+(n-s)} = \frac{1}{s^{n-s}} \cdot \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^n P_0, \text{ for } n \geq s.}$$

Now, find P_n using the fact $\sum_{n=0}^{\infty} P_n = 1.$

This can be broken as $$\underbrace{\sum_{n=0}^{s-1} P_n}_{(1 \leq n \leq s-1)} + \underbrace{\sum_{n=s}^{\infty} P_n}_{(n \geq s)} = 1.$$

or $$\sum_{n=0}^{s-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0\right] + \sum_{n=s}^{\infty}\left[\frac{1}{s^{n-s}} \cdot \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^n P_0\right] = 1.$$

or $$P_0\left[\sum_{n=0}^{s-1} \frac{s^n}{n!}\left(\frac{\lambda}{s\mu}\right)^n + \frac{1}{s!}\sum_{n=s}^{\infty} \frac{s^n}{s^{n-s}} \cdot \left(\frac{\lambda}{s\mu}\right)^n\right] = 1.$$

or $$P_0\left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!}\sum_{n=s}^{\infty} \rho^n\right] = 1, \text{ where } \rho = \left(\frac{\lambda}{s\mu}\right)$$

or $$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!}\left(\frac{\rho^s}{1-\rho}\right)^*\right]^{-1} \quad \textbf{(see foot note)}$$

* $\sum_{n=s}^{\infty} \rho^n = \rho^s + \rho^{s+1} + \rho^{s+2} \ldots \infty = \rho^s/(1-\rho)$ (sum of a G.P. of infinite terms, where $\rho < 1$).

LESSON 21
GROUP DISCUSSION/QUIZ ON UNIT 3 PORTIONS

Notes:

LESSON 22
GAME THEORY

This lecture covers the following points:

- Game Theory Introduction
- Minimax criterion of optimality
- Games without saddle point

Introduction

Life is full of struggle and competitions. A great variety of competitive situations is commonly seen in everyday life. For example, candidates fighting an *election* have their conflicting interests, because each candidate is interested to secure more votes than those secured by all others. Besides such pleasurable activities in competitive situations, we come across much more earnest competitive situations, of military battles, advertising and marketing campaigns by competing business firms, etc.

What should be the bid to win a big Government contract in the pace of competition from several contractors? Game must be thought of, in a broad sense, not as a kind of sport but as competitive situation, a kind of conflict in which somebody must *win* and somebody must *lose.*

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcomes. The game theory has only been capable of analysing very simple competitive situations. Thus, there has been a great gap between what the theory can handle and most actual competitive situations in industry and elsewhere. So the primary contribution of game theory has been its concepts rather than its formal application to solving real problems.

Game is defined as an activity between two or more persons involving activities by each person according to a set of rule at the end of which each person receives some benefit or satisfaction or suffers loss (negative benefit). The set of rules defines the game. Going through the set of rules once by the participants defines a play.

Characteristicsof Game Theory

There can be various types of games. They can be classified on the basis of the following characteristics.

i. *Chance of strategy:* If in a game, activities are determined by skill, it is said to be *a game of strategy;* if they are determined by chance, it is a *game of chance.* In general, a game may involve game of strategy as well as a game of chance. In this chapter, simplest models of games of strategy will be considered.

ii. *Number of persons:* A game is called an n-person game if the number of persons playing is 11.The person means an individual or a group aiming at a particular objective.

iii. *Number of activities:* These may *be finite* or *infinite.*

iv. *Number of alternatives (choices) available to each person* in a particular activity may also be finite or infinite. A *finite* game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is said to be *infinite.* .

v. *Information to the players about the past activities of other players* is completely available, partly available, or not available at all.

vi. *Payoff:* A quantitative measure of satisfaction a person gets at the end of each play is called *a payoff.* It is a real-valued function of variables in the game. Let *vi* be the payoff to the player *Pi*,

$1 \leq i \leq n$, in an n person game. If $\sum_{i=1}^{n} v_i = 0$, then the game is said to be a *zero-sum game.*

In this lesson, we shall discuss *rectangular games (also called two-person zero-sum).*

Basic Definitions

1. Competitive Game. A competitive situation is called a *competitive game* if it has the following four properties:
   i. There are finite number *(n)* of competitors (called players) such that *n* ~ 2. In case *n* = 2, it is called a two-person game and in case *n* > 2, it is referred t9 as an n-person game.
   ii. Each player has a list of finite number of possible activities (the list may not be same for each player).
   iii. A play is said to *occur* when each player chooses one of his activities. The choices are assumed to be made simultaneously, *i.e.* no player knows the choice of the other until he has decided on his own.
   iv. Every combination of activities determines an outcome (which may be points, money or any thing else whatsoever) which results in again of payments (+ ve, - ve orzero) to each player, provided each player is playing uncompromisingly to get as much as possible. Negative gain implies the loss of same amount.
2. Zero-sum and Non-zero-sum Games. Competitive games are classified according to the number of players involved, *i.e.* as a *two person game.. three person game*, etc. Another important distinction is between *zero-sum games* and *nonzero-sum* games. If the players make payments only to each other, *i.e.* the loss of one is the gain of others, and nothing comes from outside, the competitive game is said to be *zero-sum.*

Mathematically, suppose an n-person game is played by *n* players *P1, P2* '..., *P*" whose respective pay-offs at the end of a play of the game are *VI* , *V2*, ... *V*"then, the game will be called zero-sum if $\sum_{i=1}^{n} v_i = 0$ at each play of the game. A game which is not zero-sum is called a *nonzero-sum game.* Most of the competitive games are zero-sum games. An example of a nonzero-sum game is the 'poker' game in which a certain part of the pot is removed from the 'house' before the final payoff.

3. Strategy. A strategy of a player has been loosely defined as a rule for decision-making in advance of all the plays by which he decides the activities he should adopt. In other words, a strategy for a given player is a set of rules (programmes) that specifies which of the available course of action he should make at each play.

   This strategy may be of two kinds:

i. *Pure Strategy*. : If a player knows exactly what the other player is going to do, a *deterministic* situation is obtained and objective function is to maximize the gain. *Therefore, the pure strategy is a decision rule always to select a particular course of action.* A pure strategy is usually represented by a number with which the course of action is associated.

ii. *Mixed Strategy*. [Agra92; Kerala (Stat.) 83]: If a player is guessing as to which activity is to be selected by the other on any particular occasion, a *probabilistic* situation is obtained and objective function is to maximize the *expected gain.*

   Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

   Mathematically, a mixed strategy for a player with *m* (:2:2) possible courses of action, is denoted by the set S of *m* non-negative real numbers whose sum is unity, representing probabilities with which each course of action is chosen. If Xi*(i* =I, 2, 3, ..., *m)* is the probability of choosing the course *i, then*

$$S = (x_1, x_2, x_3, \ldots, x_m)$$

subject to the conditions $\quad x_1 + x_2 + x_3 + \ldots + x_m = 1$

and $\quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \ldots, x_m \geq 0.$

4. Two-Person, Zero-Sum *(Rectangular)* Games. A game with only two players (say, *Player A* and *Player B)* is called a *'two-person, zero-sum game'* if the losses of one player are equivalent to the gains of the other, so that the sum of their net gains is zero.

   Two-person, zero-sum games are also called *rectangular games* as these are usually represented by a payoff matrix in rectangular form.

5. Payoff Matrix. Suppose the player *A* has *m* activities and the player *B* has *n* activities. Then a payoff matrix can be formed by adopting the following rules:

   i. Row designations for each matrix are activities available to player A.

   ii. Column designations for each matrix are activities available to player *B.*

   iii. Cell entry *'vij* , is the payment to player *A* in *A's* payoff matrix when *A* chooses the activity *i* and *B* chooses the activity.

   iv. With a 'zero-sum, two person game', the cell entry in the player *B'* s payoff matrix will be negative of the corresponding cell entry 'Vi}, in the player *A's* payoff matrix so that sum of payoff matrices for player *A* and player *B* is ultimately zero.

The player *A*'s payoff matrix

The player *B*'s payoff matrix

		Player B					
		1	2	...	*j*	...	*n*
	1	v_{11}	v_{12}	...	v_{1j}	...	v_{1n}
	2	v_{21}	v_{22}	...	v_{2j}	...	v_{2n}
Player A	:	:	:		:		:
	i	v_{i1}	v_{i2}	...	v_{ij}	...	v_{in}
	:	:	:		:		:
	m	v_{m1}	v_{m2}	...	v_{mj}	...	v_{mn}

		Player B					
		1	2	...	*j*	...	*n*
	1	$-v_{11}$	$-v_{12}$	...	$-v_{1j}$	...	$-v_{1n}$
	2	$-v_{21}$	$-v_{22}$	...	$-v_{2j}$	...	$-v_{2n}$
Player A	:	:	:		:		:
	i	$-v_{i1}$	$-v_{i2}$	...	$-v_{ij}$	...	$-v_{in}$
	:	:	:		:		:
	m	$-v_{m1}$	$-v_{m2}$	...	$-v_{mj}$	...	$-v_{mn}$

In order to make the above concepts a clear, consider the coin matching game involving two players only. Each player selects either a head *H* or a tail *T.* If the outcomes match *(H, H* or *T, T), A* wins Re 1from *B* , otherwise *B* wins Re 1from *A* . This game is a two-person zero-sum game. since the winning of one player is taken as losses for the other. Each has his choices between two *pure* strategies *(H* or *T).* This yields the following (2 x 2) payoff matrix to player *A*.

It will be shown later that the optimal solution to such games requires each player to play one pure strategy or a mixture of pure strategies.

		B	
		H	T
A	*H*	+1	−1
	T	−1	+1

Minimax (Maximin) Criterion And Optimal Strategy

The 'minimax criterion of optimality' states that if a player lists the worst possible outcomes of all his potential strategies, he will choose that strategy to be most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an optimal strategy.

Example 1. *Consider (two-person, zero-sum) game matrix, which represents payoff to the player A. Find, the optimal strategy, if any.*(See *Table)*

Solution.

The player A wishes to obtain the largest possible *'Vij,* by choosing one of his activities (I, II, III), while the player *B* is determined to make A's gain the minimum possible by choice of activities from his list (I, II, III). The player A is called the *maximizing player* and *B* the *minimizing player*. If the player A chooses the 1stactivity, then it could happen that the player *B* also chooses his 1st activity. In this case the player *B* can guarantee a gain of at least - 3 to player A , i.e.

min {- 3 , - 2 , 6} =(-3).

Similarly, *for* other choices of the player A , i.e. II and III activities, *B* can force the player A to get only 0 and-4, respectively, by his proper choices from(I,II, III), *i.e.* min{2,0,2}= 0 and

min{5,-2,-4}= (-4)

The minimum value in each row guarantee by the player A is indicate~ 'row minimum' in *Table.*

The best choice *for* the player A is to maximize his least gains -3, 0, -4 and opt II strategy which

assures at most the gain 0, i.e. $\max\{-3, \ 0, \ -4\} = \boxed{0}$

In general, the player A should try to maximize his least games or to find out "$\max_i \min_j v_{ij}$"

Player *B,* on the other hand, can argue similarly to keep A's game the minimum. He realizes that if he plays his 1st pure strategy, he can loose no *more* than 5 =max {- 3, 2. 5} regardless of A's selections. Similar arguments can be applied *for* remaining strategies II and III. Corresponding results are indicated in *Table J9-4* by *'column maximum '*. The player *B* will then select the strategy that minimizes his maximum losses. This is given by the strategy II and his corresponding loss is given by

The player A's selection is called the *maximin strategy* and his corresponding gain is called the *maximin value* or *lower value* (.!::) of the game. The player *B'* s selection is called the *minimax value* or *upper value* (*v)* of the game. The selections made by player A and *B* are based on the so called *minimax* (or *maximin) criterion.* It is seen from the governing conditions that the minimax (upper) value *v* is greater than or equal to the maximin (lower) value.!::(see *Theorem J9. J).* In the

case where equality holds *i.e.,* $\max_i \min_j v_{ij} = \min_j \max_i v_{ij}$ or $\underline{v} = \bar{v}$, the correspond-

ing pure strategies are called the *'optimal'* strategies and the game is said to have a *saddle point.*

Example 2. *Consider the following game:*

	i \ *j* (B)	1	2	3
	1	3	−4	8
A	2	−8	5	−6
	3	6	−7	6

As discussed in ***Example 1***, $\max_i \min_j v_{ij} = 4$, $\min_j \max_i v_{ij} = 5$.

Also, $\max_i \min_j v_{ij} < \min_j \max_i v_{ij}$

Such games are said to be the games ***without saddle point.***

Example 3. *Find the range of values of p and q which will render the entry* (2, 2) *a saddle point for the game:*

Solution. First ignoring the values of p and q determine the maximin and minimax values of the payoff matrix as below :

	Player B		
Player A	2	4	5
	10	7	q
	4	p	6

Since the entry (2, 2) is a saddle point, maximin value $\underline{v} = 7$, minimax value $\overline{v} = 7$.

This imposes the condition on p as $p \leq 7$ and on q as $q \geq 7$. Hence the range of p and q will be $p \leq 7$, $q \geq 7$.

Theorem 19·1. ***Let*** $[v_{ij}]$ ***be the payoff matrix for a two-person zero-sum game. If*** $\underline{v}$ ***denotes the maximin value and*** $\overline{v}$ ***the minimax value of the game, then*** $\overline{v} \geq \underline{v}$. ***That is,*** $\min_j [\max_i \{v_{ij}\}] \geq \max_i [\min_j \{v_{ij}\}]$ **[Meerut (Stat.) 90]**

		Player B			
		B_1	B_2	B_3	Row Min.
	A_1	2	4	5	2
Player A	A_2	$\boxed{10}$	7	q	7
	A_3	4	p	$\boxed{6}$	4
Column Max		10	7	6	

Proof. We have, $\max_i \{v_{ij}\} \geq v_{ij}$ for any j, and $\min_j \{v_{ij}\} \leq v_{ij}$ for any i.

Let the above maximum be attained at $i = i^*$ and the minimum be attained at $j = j^*$. So

$$v_{i^*j} \geq v_{ij} \geq v_{ij^*} \quad \text{for any } i \text{ and } j.$$

This implies that

$$\min_j [v_{i^*j}] \geq v_{ij} \geq \max_i \{v_{ij^*}\} \quad \text{for any } i \text{ and } j.$$

Hence

$$\min_j [\max_i \{v_{ij}\}] \geq \max_i [\min_j \{v_{ij}\}] \quad \text{or} \quad \overline{v} \geq \underline{v}.$$

Saddle Point, Optimal Strategies and Value of The Game

Definitions

Saddle Point. *A saddle point of a payoff matrix is the position of such an element in the payoff matrix, which is minimum in its row and maximum in its column.*

Mathematically, *if a payoff matrix* $\{v_{ij}\}$ *is such that* $\max_i [\min_j \{v_{ij}\}] = \min_j [\max_i \{v_{ij}\}] = v_{rs}$ (say), *then the matrix is said to have a saddle point* (r, s).

Optimal Strategies. *If the payoff matrix* $\{v_{ij}\}$ *has the saddle point* (r, s), *then the players (A and B) are said to have* ***rth and sth optimal strategies****, respectively.*

Value of Game. *The payoff* (v_{rs}) *at the saddle point* (r, s) *is called the* ***value of game*** *and it is obviously equal to the maximin* ($\underline{v}$) *and minimax value* ($\overline{v}$) *of the game.*

A game is said to be a ***fair game*** if $\overline{v} = \underline{v} = 0$. A game is said to be ***strictly determinable*** if $\overline{v} = v = \underline{v}$.

Note. A saddle point of a payoff matrix is, sometimes, called the equilibrium point of the payoff matrix.

In *Example 1*, $\underline{v} = \overline{v} = 0$. This implies that the game has a saddle point given by the entry (2, 2) of payoff matrix. The value of the game is thus equal to zero and both players select their strategy as the optimal strategy.

In this example, it is also seen that no player can improve his position by other strategy.

In general, a matrix need not have a saddle point as defined above. Thus, these definitions of optimal strategy and value of the game are not adequate to cover all cases so need to be generalized. The definition of a saddle point of a function of several variables and some theorems connected with it form the basis of such generalization.

Rules for Determining a Saddle Point:

1. *Select the minimum element of each row of the payoff matrix and mark them by '0'.*
2. *Select the greatest element of each column of the payoff matrix and mark them by* '0 '.
3. *there appears an element in the payoff matrix marked by '0' and* 'D' *both, the position of that element is a saddle point of the payoff matrix.*

Solution of Games With Saddle Points

To obtain a solution of a rectangular game, it is feasible to find out:

i. the best strategy for player *A*

ii. the best strategy for player *B,* and

iii. the value of the game *(Vr,s).*

It is already seen that the best strategies for players *A* and *B* will be those, which correspond to the row and column, respectively, through the saddle point. The value of the game to the player *A* is the element at the saddle point, and the value to the player *B* will be its negative.

Example4. *Player A can choose his strategies from* {AI, *A2, A3} only, while B can choose from the set (B1, B2) only. The rules of the game state that the payments should be made in accordance with the selection of strategies:*

Strategy Pair Selected	*Payments to be Made*	*Strategy Pair Selected*	*Payments to be Made*
(A_1, B_1)	*Payer A Pays Re. 1 to player B*	(A_2, B_2)	*Player B pays Rs 4 to player A*
(A_1, B_2)	*Player B pays Rs. 6 to player A*	(A_3, B_1)	*Player A pays Rs 2 to player B*
(A_2, B_1)	*Player B pays Rs 2 to player A*	(A_3, B_2)	*Player A pays Rs. 6 to player B*

What strategies should A and B play in order to get the optimum benefit of the play?

Solution. With the help of above rules the following payoff matrix is constructed :

The payoffs marked 'O' represent the minimum payoff in each row and those marked '☐' represent the maximum payoff in each column of the payoff matrix.

Obviously, the matrix has a saddle point at position (2, 1) and the value of the game is 2.

Thus, the optimum solution to the game is given by :

		Player B	
		B_1	B_2
Player A	A_1	-1	$\boxed{6}$
	A_2	$\boxed{2}$	4
	A_3	-2	-6

i. The optimum strategy for player *A* is *A2;*

ii. The optimum strategy for player *B* is *B*I ; and

iii. The value of the game is Rs. 2 for player A and Rs. (- 2) for player B.

Also, since *v not equals* 0, the game is not fair, although it is strictly determinable.

Rectangular Games Without Saddle Point

As discussed earlier, if the payoff matrix *{vii}* has a saddle point *(r, s),* then *i = r,j = s* are the optimal strategies of the game and the payoff *Vrs(=v)* is the value of the game. On the other hand, if the given matrix has no saddle point, the game has no optimal strategies. The concept of optimal strategies can be extended to all matrix games by introducing game probability with choice and mathematical expectation with pay off.

Let player A choose a particular activity i such that $1 \le i \le m$ with probability x_i. This can also be interpreted as the relative frequency with which A chooses activity i from number of activities of the game. Then set $\mathbf{x} = \{x_i, 1 \le i \le m\}$ of probabilities constitute the strategy of A. Similarly, $\mathbf{y} = \{y_j, 1 \le j \le n\}$ defines the strategy of the player B.

Thus, the vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ of non-negative numbers satisfying $x_1 + x_2 + \ldots + x_m = 1$ is called the *mixed strategy of the player A*. Similarly, the vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ of non-negative numbers satisfying $y_1 + y_2 + \ldots + y_n = 1$ is called the *mixed strategy of the player B*.

Consider the symbol S_m which denotes the set of ordered m-tuples of non-negative numbers whose sum is unity and $\mathbf{x} \in S_m$. Similarly, $\mathbf{y} \in S_n$. Unless otherwise stated, assume that $\mathbf{x} \in S_m$ and $\mathbf{y} \in S_n$, where $\mathbf{x}$ and $\mathbf{y}$ are mixed strategies of player A and B, respectively.

The mathematical expectation of the payoff function $E(\mathbf{x}, \mathbf{y})$ in a game whose payoff matrix is $\{v_{ij}\}$ is defined by

$$E(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i v_{ij}) y_j \qquad = \mathbf{x}^T \mathbf{v}\, \mathbf{y} \quad (\textit{in matrix form})$$

where $\mathbf{x}$ *and* $\mathbf{y}$ *are the mixed strategies of players A and B*, respectively,

Thus the player A should choose $\mathbf{x}$ so as to maximize his minimum expectation and the player B should choose $\mathbf{y}$ so as to minimize the player A's greatest expectation. In other words, the player

A tries for $\max_{\mathbf{x}} \min_{\mathbf{y}} E(\mathbf{x}, \mathbf{y})$ and B tries for $\min_{\mathbf{y}} \max_{\mathbf{x}} E(\mathbf{x}, \mathbf{y})$.

At this stage it is possible to define the *strategic saddle point* of the game with mixed strategies.

Strategic Saddle Point. *Definition.* *If* $\min_{\mathbf{y}} \max_{\mathbf{x}} E(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}_0, \mathbf{y}_0) = \max_{\mathbf{x}} \min_{\mathbf{y}} E(\mathbf{x}, \mathbf{y})$, *then* $(\mathbf{x}_0, \mathbf{y}_0)$ *is called the* ***strategic saddle point*** *of the game where* $\mathbf{x}_0$ *and* $\mathbf{y}_0$ *define the optimal strategies, and* $v = E(\mathbf{x}_0, \mathbf{y}_0)$ *is the value of the game.*

According to the *minimax theorem* a strategic saddle point will always exist.

Example. *In a game of matching coins with two players, suppose one player wins Rs.* 2 *when there are two heads and wins nothing when there are two tails; and losses Re.* 1 *when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game.*

Solution. The payoff matrix (for the player A) is given by

$$\begin{array}{cc|cc|l} & & \multicolumn{2}{c|}{\text{Player } B} & \\ & & H & T & \text{Row Min.} \\ \hline \text{Player } A & H & 2 & -1 & -1 \\ & T & -1 & 0 & -1 \quad \text{Maximin (lower) value } (\underline{v}) \\ \hline & \text{Column Max:} & 2 & 0 & \\ \end{array}$$

$\uparrow$ Minimax (upper) value ($\bar{v}$)

Here, maximin value $(\underline{v}) = -1 \neq$ minimax value $(\bar{v}) = 2$.

So the matrix is whithout saddle point.

Now, let us outline here how one finds the best strategies for such games and the expected amounts to be gained or lost by the players.

Let the player A plays H with probability x and T with probability $1 - x$ so that $x + (1 - x) = 1$. Then, if the player B plays H all the time, A's expected gain will be

$$E(A, H) = x.2 + (1 - x)(-1) = 3x - 1.$$

Similarly, if the player B plays T all the time, A's expected gain will be

$$E(A, T) = x(-1) + (1 - x)\, 0 = -x.$$

It can be shown mathematically that if the player A chooses x such that

$$E(A, H) = E(A, T) = E(A), \text{ say},$$

then this will determine the best strategy for him.

Thus, $\qquad 3x - 1 = -x$ or $x = 1/4$

Therefore, best strategy for the player A is to play H and T with probability 1/4 and 3/4 , respectively. Since this is a mixed strategy, it is usually denoted by the set {1/4 , 3/4}. So expected gain for the player A is given by

$$E(A) = \frac{1}{4} \cdot 2 + \frac{3}{4}(-1) = -\frac{1}{4}$$

Now, whatever be the set $\{y, 1-y\}$ of probabilities with which the player B plays either H or T, A's expected gain will always remain equal to $-1/4$. To verify this,

$$E(A, y, 1-y) = y\left[\frac{1}{4} \cdot 2 + \frac{3}{4}(-1)\right] + (1-y)\left[\frac{1}{4}(-1) + \frac{3}{4}0\right]$$

$$= y\left(-\frac{1}{4}\right) + (1-y)\left(-\frac{1}{4}\right) = -\frac{1}{4}.$$

The same procedure can be applied for the player B .Let the probability of the choice of H be denoted by y and that of T by $(1 - y)$. For best strategy of the player B,

$$E(B, H) = E(B, T) = E(B), \text{ say}$$

or $\quad y \cdot 2 + (1-y)(-1) = y(-1) + (1-y)\,0$

or $\quad 4y = 1$

or $\quad y = 1/4$ and therefore $1 - y = 3/4$.

Therfore, $E(B) = \frac{1}{4} \cdot 2 + \frac{3}{4}(-1) = -\frac{1}{4}$.

Here, $E(A) = E(B) = -1/4$. Thus, the complete solution of the game is :

(i) The player A should play H and T with probabilities 1/4 and 3/4 , respectively. Thus, A's optimal strategy is $\mathbf{x}_0 = (1/4, 3/4)$.

(ii) The player B should play H and T with probabilities 1/4 and 3/4, respectively. Thus, B's optimal strategy is $\mathbf{y}_0 = (1/4, 3/4)$.

(iii) The expected value of the game is $-1/4$ to the player A. Here $(\mathbf{x}_0, \mathbf{y}_0)$ is the strategic saddle point of this game.

LESSON 23
GAME THEORY GRAPHICAL METHOD

This Lecture Describes

- **Game Theory Graphical Method**

Graphical Method For (2 X *n)* AND *(mx*2) Games

The optimal strategies for a (2 x *n)* or *(m x* 2) matrix game can be located easily by a simple *graphical method.* This method enables us to reduce the 2 x *n* or *m* x 2 matrix game to 2 x 2 game that could be easily solved by the earlier methods.

If the graphical method is used for a particular problem, then the same reasoning can be used to solve any I game with mixed strategies that have only two undominated pure strategies for one of the players.

Optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. It is clear that if one player has only two strategies, the other will also use two strategies. Hence, graphical method can be used to find two strategies of the player. The method can be applied to 3 x *n* or *m'x*3 Games also by carefully drawing three dimensional diagram.

Graphical Method for 2 x n Games

Consider the (2 x *n)* game, assuming that the game does not have a saddle point.

Since the player *A* has two strategies, it follows that *X2*=1 - XI , XI ;::: 0 , *X2* ;::: O. Thus, for each of the pure strategies available to the player *B.* the expected payoff for the player *A*. would be as follows:

$$\begin{array}{ccc|ccccc|}
 & & & & & B & & \\
 & & & y_1 & y_2 & y_3 & \cdots & y_n \\
 & & & B_1 & B_2 & B_3 & \cdots & B_n \\
\hline
A & x_1 & A_1 & v_{11} & v_{12} & v_{13} & \cdots & v_{1n} \\
 & 1-x_1 & A_2 & v_{21} & v_{22} & v_{23} & \cdots & v_{2n} \\
\hline
\end{array}$$

B's Pure Strategies	A's Expected Payoff $E_j(x_1)$
B_1	$v_{11}x_1 + v_{21}(1-x_1) = (v_{11} - v_{21})x_1 + v_{21}$
B_2	$v_{12}x_1 + v_{22}(1-x_1) = (v_{12} - v_{22})x_1 + v_{22}$
$\vdots$	$\vdots \quad \vdots \quad \vdots$
B_n	$v_{1n}x_1 + v_{2n}(1-x_1) = (v_{1n} - v_{2n})x_1 + v_{2n}$

This shows that the player *A's* expected payoff varies Linearly with *XI.* According to the maximin criterion for mixed strategy games; the player *A* should select the value of XI so as to maximize his minimum expected payoff. This may be done by plotting the following

$$\begin{aligned}
E_1(x_1) &= (v_{11} - v_{21})\,x_1 + v_{21} \\
E_2(x_1) &= (v_{12} - v_{22})\,x_1 + v_{22} \\
&\vdots \\
E_n(x_1) &= (v_{1n} - v_{2n})\,x_1 + v_{2n}
\end{aligned}$$

as functions of XI .The lowest boundary of these lines will give the minimum expected payoff as function of XI .The highest point on this lowest boundary would then give the *maximin* expected payoff and the optimum value of Xl (=Xl *).

Now determine only two strategies for player *B* corresponding to those two lines which pass through the *Maximin* point *P*.This way, it is possible to reduce the game to 2 x 2 which can be easily solved either by using formulae or by *arithmetic method.*

Outlines of Graphical Method :

To determine maximin value v, we take different values of x_1 on the horizontal line and values of $E(x_1)$ on the vertical axis. Since $0 \le x_1 \le 1$, the straight line $E_j(x_1)$ must pass through the points $\{0, E_j(0)\}$ and $\{1, E_j(1)\}$, where $E_j(0) = v_{2j}$ and $E_j(1) = v_{1j}$. Thus the lines $E_j(x_1) = (v_{1j} - v_{2j})\, x_1 + v_{2j}$ for $j = 1, 2, ..., n$ can be drawn as follows :

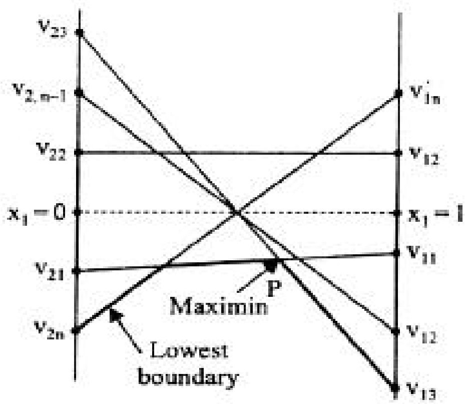
Step 1. Construct two vertical axes, *axis* I at the point *x)* =0 and *axis* 2 at the point *x)* :;: 1.

Step 2. Represent the payoffs *V2j'j* = 1. 2, ..., *n* on *axis* I and payoff line vlj ,*j* =I, 2, ..., *n* on *axis* 2.

Step 3. Join the point representing *vij* on *Axis* 2 to the point representing *V2j*on *axis* I. The resulting straight-line is the expected payoff line $E_j(x_1)$, $j = 1, 2, \ldots, n$.

Step 4. Mark the lowest boundary of the lines *E.i(x)* so plotted, by *thick line* segments. The *highest point* on this *lowest boundary* gives the *maximin* point *P* and identifies the two critical. moves of player *B.*

If there are more than two lines passing through the maximin point *P.* there are ties for the optimum mixed strategies for player *B.* Thus any two such lines with *opposite* sign slopes will define an alternative optimum for



Graphical Solution of m x 2 Games

The *(m* x 2) games are also treated in the like manner except that the *minimax* point *P* is the *lowest point* on the *uppermost boundary* instead of highest point on the lowest boundary.

From this discussion, it is concluded that any (2 x *n)* or *(m* X2) game is basically equivalent to a (2 x 2) game.

Now each point of the discussion is explained by solving numerical examples for (2 x *n)* and

(m x 2) games.

Example 1. *Solve the following* (2 x 3) *game graphically.*

			y_1 I	y_2 II	y_3 III
A	x_1	I	1	3	11
	$1 - x_1$	II	8	5	2

Solution. This game does not have a saddle point. Thus the player *A's* expected payoff corresponding to the player *B's* pure strategies are given. Three expected payoff lines are:

$$E(x_1) = -7x_1 + 8,\ E(x_1) = -2x_1 + 5 \text{ and } E(x_1) = 9x_1 + 2$$

and can be plotted on a graph as follows

B's Pure Strategies	*A*'s Expected Payoff $E(X_1)$
I	$E(x_1) = 1.x_1 + 8(1 - x_1) = -7x_1 + 8$
II	$E(x_1) = 3.x_1 + 5.(1 - x_1) = -2x_1 + 5$
III	$E(x_1) = 11x_1 + 2(1 - x_1) = 9x_1 + 2$

First, draw two parallel lines one unit apart and mark a scale on each. These two lines will represent two strategies available to the player *A*. Then draw lines to represent each of players *B*' s strategies.

For example, to represent the player *B's* 1st strategy, join mark 1on scale I to mark 8 on scale II; to represent the player *B's* second strategy, join mark 3 on scale! to mark 5 on scale II, and so on. Since the expected payoff *E (X1)* is the function of *xi* alone, these three expected payoff lines can be drawn by taking *Xi* as x-axis and *E(x,)* as *y*- axis. .

Points *A*, *P*, *B*, C on the lowest boundary (shown by a thick line in Fig. represent the lowest possible expected gain to the player *A* for any value of *Xl* between 0 and 1. According to the *maximin* criterion, the player *A* chooses the best of these worst outcomes. Clearly, the *highest* point *P* on the *lowest* boundary will given the largest expected gain *PN* to *A*. So best strategies for the player *B* are those, which pass through the point *P*. Thus, the game is reduced to 2 x 2 *Table*.

Now, by solving the simultaneous equations

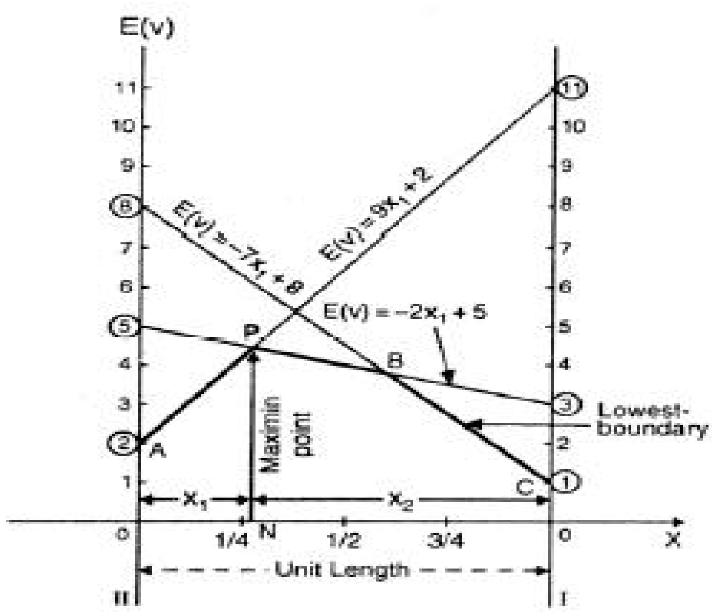
$$3x_1 + 5x_2 = v,\ 11x_1 + 2x_2 = v,\ x_1 + x_2 = 1\ (For\ player\ A)$$
$$3y_2 + 11y_3 = v,\ 5y_2 + 2y_3 = v,\ y_2 + y_3 = 1\ (For\ player\ B)$$

The solution of the game is obtained as follows:

(i) The player *A* chooses the optimal mixed strategy $(x_1, x_2) = (3/11,\ 8/11)$,

(ii) The player *B* chooses the optimal mixed strategy $(y_1, y_2, y_3) = (0, 2/11, 9/11)$,

(iii) The value of the game to the player *A* is $v = 49/11$.



LESSON 24
GAME THEORY

This Lesson Will Explain

- Solution of mxn Game by Linear Programming
- Algebraic Method

Solution of m x *n* Games by Linear Programming

Following example of (3 x 3) game will make the computational procedure clear.

Example. *Solve* (3 X3) *game by the simplex method of linear programming whose payoff matrix is given below.*

		Player B 1	Player B 2	Player B 3
Player A	1	3	– 1	– 3
	2	– 3	3	– 1
	3	– 4	– 3	3

Solution. First apply *minimax (maximin)* criterion to find the minimax (v) and maximin (y) value of the game. Thus, the following matrix is obtained. Since, maximin value is - 3, it is possible that the value of the game (v) may be negative or zero because - 3 < v < 3. Thus, a constant c is added to all elements of the matrix, which is *at least* equal to the - ve of the maximin value,

i.e. c $\geq$ 3. Let c =5. The matrix is shown in *Table* 1.9. Now, following the reasoning of the player *B*' s linear programming problem is:

Maximize $y_0 = Y_1 + Y_2 + Y_3$

subject to the constraints :

$$8Y_1 + 4Y_2 + 2Y_3 \leq 1\ ,\ 2Y_1 + 8Y_2 + 4Y_3 \leq 1\ ,\ 1Y_1 + 2Y_2 + 8Y_3 \leq 1\ ,\ Y_1 \geq 0,\ Y_2 \geq 0,\ Y_3 \geq 0$$

Introducing slack variables, the constraint equations become :

$$\left.\begin{aligned} 8Y_1 + 4Y_2 + 2Y_3 + Y_4 \qquad\qquad &= 1 \\ 2Y_1 + 8Y_2 + 4Y_3 \qquad + Y_5 \qquad &= 1 \\ 1Y_1 + 2Y_2 + 8Y_3 \qquad\qquad + Y_6 &= 1 \end{aligned}\right\}$$

$$Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5,\ Y_6 \geq 0.$$

Now the following simplex table is formed.

B	C_B	Y_B	α_1	α_2	α_3	α_4 (β_1)	α_5 (β_2)	α_6 (β_3)	Min. Ratio (Y_B/α_k)
	$c_j \rightarrow$		1	1	1	0	0	0	
α_4	0	1	$\leftarrow \boxed{8}$ ↑	4	2	1	0	0	1/8 ←
α_5	0	1	2	8	4	0	1	0	1/2
α_6	0	1	1	2	8	0	0	1	1/1
	$y_0 = C_B Y_B = 0$		$(-1)^*$ ↑	−1	−1	0 ↓	0	0	$\leftarrow \Delta_j = C_B\alpha_j - c_j$
α_1	1	1/8	1	1/2	1/4	1/8	0	0	1/2
α_5	0	3/4	0	7	7/2	−1/4	1	0	3/14
α_6	0	7/8	0	3/2	$\leftarrow \boxed{31/4}$ ↑	−1/8	0	1	7/62 ←
	$y_0 = 1/8$		0	−1/2	$(-3/4)^*$ ↑	1/8	0	0 ↓	$\leftarrow \Delta_j$
α_1	1	3/31	1	14/31	0	4/31	0	−1/31	3/14
α_5	0	11/31	0	$\leftarrow \boxed{196/31}$ ↑	0	−6/31	1	14/31	11/196 ←
α_3	1	7/62	0	6/31	1	−1/62	0	4/31	7/12
	$y_0 = 13/62$		0	$(-11/31)^*$ ↑	0	7/62	0 ↓	3/31	$\leftarrow \Delta_j$
α_1	1	1/14	1	0	0	1/7	1/14	0	
α_2	1	11/196	0	1	0	−3/98	31/196	−1/14	
α_3	1	5/49	0	0	1	−1/98	−3/98	1/7	
	$y_0 = 45/196$		0	0	0	5/49	11/196	1/14	← all $\Delta_j \geq 0$

Thus, the solution for *B*'s original problem is obtained as :

$$y_1^* = \frac{Y_1}{y_0} = \frac{1/14}{45/196} = \frac{14}{45}, \quad y_2^* = \frac{Y_2}{y_0} = \frac{11/196}{45/196} = \frac{11}{45}.$$

$$y_3^* = \frac{Y_3}{y_0} = \frac{5/49}{45/196} = \frac{20}{45}, \quad v^* = \frac{1}{y_0} - c = \frac{196}{45} - 5 = -\frac{29}{45}.$$

The optimal strategies for the player *A* are obtained from the final table of the above problem. This is given by duality rules :

$$x_0 = y_0 = \frac{45}{196}, \quad X_1 = \Delta_4 = \frac{5}{49}, \quad X_2 = \Delta_5 = \frac{11}{196}, \quad X_3 = \Delta_6 = \frac{1}{14}.$$

Hence,

$$x_1^* = \frac{X_1}{x_0} = \frac{20}{45}, \quad x_2^* = \frac{X_2}{x_0} = \frac{11}{45}, \quad x_3^* = \frac{X_3}{x_0} = \frac{14}{45}, \quad v^* = \frac{29}{45}.$$

Algebraic Method for the Solution of a General Game

The algebraic method is a direct attempt to solve unknowns although this method becomes quite lengthy when there are more strategies (courses of action) for players. Such large games can be solved first by transforming the problem into a linear programming problem and then solving it by the *simplex method* on an electronic computer.

First, suppose that all inequalities given hold as equations. Then solve these equations for unknowns. Sometimes equations are not consistent. In such cases, one or more of the inequalities must hold as strict inequalities (with '>' or '<' signs). Hence, there will be no alternative except to rely on trial-and-error method for solving such games. Following important theorems will be helpful in making the computations easier.

Theorem

If for any j $(j = 1, 2, 3, \ldots, n)$ $v_{1j}x_1 + v_{2j}x_2 + \ldots + v_{mj}\,x_m > v$, *then* $y_j = 0$,

and similarly, if for any i $(i = 1, 2, 3, \ldots, m)$ $v_{i1}y_1 + v_{i2}y_2 + \ldots + v_{in}y_n < v$, *then* $x_i = 0$.

Alternative Statement : *Let v be the value of an* $m \times n$ *game. If for an optimum strategy* $\mathbf{x}^* \in S_m$, $E(\mathbf{x}^*, \mathbf{e}_j) > v$ *for some* $\mathbf{e}_j \in S_n$, *then every strategy* $\mathbf{y}^* \in S_n$ *has* $y_j^* = 0$,

Similarly, if every optimal strategy $\mathbf{y}^* \in S_n$, *then every optimal strategy* $\mathbf{x}^* \in S_m$ *has* $x_i^* = 0$.

Proof. We know that for any optimal strategy $\mathbf{x}^* \in S_m$, we always have

$$E(\mathbf{x}^*, \mathbf{e}_j) \geq v \text{ for all } \mathbf{e}_j \in S_n \quad \ldots(1)$$

We are given that $E(\mathbf{x}^*, \mathbf{e}_j) > v$ for some $\mathbf{e}_j \in S_n$. If possible let us suppose $y_j^* \neq 0$ (*i.e.* $y_j^* > 0$).

Then $\quad y_j^* E(\mathbf{x}^*, \mathbf{e}_j) > y_j^* v$. $\quad \ldots(2)$

Now $\quad E(\mathbf{x}^*, \mathbf{y}^*) = \Sigma\, y_k^* E(\mathbf{x}^*, \mathbf{e}_k) \quad = \sum_{k \neq j} y_k^* E(\mathbf{x}^*, \mathbf{e}_k) + y_j^* E(\mathbf{x}^*, \mathbf{e}_j)$

or $\quad v > \sum_{k \neq j} y_k^* v + y_j^* v$ or $v > (\sum_{k \neq j} y_k^* + y_j^*)v$ or $v > \Sigma\, y_k^* v$ [from (1) and (2)]

or $\quad v > v$ (since $\Sigma\, y_k^* = 1$) which is a contradiction.

Hence our assumption is wrong; and therefore, we must have $y_j^* = 0$.

Similarly, the second part of the theorem can be proved.

Example. *Find the value and optimal strategies for two players of the rectangular game whose payoff matrix is given by*

Solution. First, it is seen that this game does not have a saddle point. Also, this game cannot be reduced to *A* Xz (2 x 2) by the property of dominance. Hence, this game can be solved by the algebraic method.

			B: y_1 I	y_2 II	y_3 III
A	x_1	I	1	-1	-1
	x_2	II	-1	-1	3
	x_3	III	-1	2	-1

Step 1. Let (x_1, x_2, x_3) and (y_1, y_2, y_3) denote the optimal mixed strategies for players *A* and *B*, respectively, and *v* be the vlaue of the game to the player *A*.

Now, for the player *A*, following relationships (*as obtained in Sec. 19·10*) can be established :

$$1x_1 + (-1)\,x_2 + (-1)\,x_3 \geq v$$
$$-1\,x_1 + (-1)\,x_2 + 2\,x_3 \geq v$$
$$-1\,x_1 + 3x_2 + (-1)\,x_3 \geq v.$$

Similarly, for the player *B*,

$$1.y_1 + (-1)\,y_2 + (-1)\,y_3 \leq v$$
$$-1.y_1 + (-1)\,y_2 + 3\,y_3 \leq v$$
$$-1.y_1 + 2y_2 + (-1)\,y_3 \leq v.$$

Additional relationship required to ensure that x_1, x_2, x_3 and y_1, y_2, y_3 are probabilities, are :

$$x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0 \text{ and } y_1 + y_2 + y_3 = 1,\ y_1, y_2, y_3 \geq 0.$$

Now the values of seven unknowns x_1, x_2, x_3; y_1, y_2, y_3 and v satisfying above relationships are to be evaluated.

Step 2. Suppose all inequalities hold as equations, then

$$x_1 - x_2 - x_3 = v \quad ...(i) \qquad\qquad y_1 - y_2 - y_3 = v \quad ...(v)$$

$$-x_1 - x_2 + 2x_3 = v \quad ...(ii) \qquad\qquad -y_1 - y_2 + 3y_3 = v \quad ...(vi)$$

$$-x_1 + 3x_2 - x_3 = v \quad ...(iii) \qquad\qquad -y_1 + 2y_2 - y_3 = v \quad ...(vii)$$

$$x_1 + x_2 + x_3 = 1 \quad ...(iv) \qquad\qquad y_1 + y_2 + y_3 = 1. \quad ...(viii)$$

Now with the help of the equation (iv), equations (i), (ii) and (iii) give us

$$x_1 = \frac{v+1}{2},\ x_2 = \frac{v+1}{4},\ x_3 = \frac{v+1}{3}$$

and substituting these values of x_1, x_2, x_3 in equation (iv), we get

$$\frac{v+1}{2} + \frac{v+1}{4} + \frac{v+1}{3} = 1.$$

Therefore, $v + 1 = 12/13$ or $v = -1/13$.

Hence $x_1 = 6/13$, $x_2 = 3/13$, $x_3 = 4/13$.

Again with the help of equation (viii), equations (v), (vi) and (vii) give us

$$y_1 = \frac{v+1}{2},\ y_2 = \frac{v+1}{3},\ y_3 = \frac{v+1}{4}$$

and substituting these values of y_1, y_2, y_3 in equation (viii),

$$\frac{v+1}{2} + \frac{v+1}{4} + \frac{v+1}{3} = 1.$$

Therefore, $v + 1 = 12/13$ or $v = -1/13$, which was expected also by minimax theorem.

Thus, $y_1 = 6/13$, $y_2 = 4/13$ and $y_3 = 3/13$.

Hence, the solution of the game is :

(i) Optimal mixed strategy for the player A is (6/13 , 3/13 , 4/13)

(ii) Optimal mixed strategy for the player B is (6/13, 4/13, 3/13)

(iii) The value of the game to the player A is $= -1/13$.

LESSON 25
GROUP DISCUSSION/QUIZ ON UNIT 4 PORTIONS

Notes:

LESSON 26
INVENTORY CONTROL

This Lecture Will Teach you the Following

- **Inventory Introduction**
- **Inventory Model**

Introduction

Definition

The function of directing the movement of goods through the entire manufacturing cycle from the requisitioning of raw materials to the inventory of finishes goods orderly mannered to meet the objectives of maximum customer-service with minimum investment and efficient (low-cost) plant operation.

What Is Inventory?

In broad sense, inventory may be defined as the stock of goods, commodities or other economic resources that are stored or reserved in order to ensure smooth and efficient running of business affairs.

The inventory or stock of goods may be kept in any of the following forms:

i. Raw material inventory, i.e. raw materials which are kept in stock for using in the production of\ goods.

ii. Work-in-process inventory, i.e. semi finished goods or goods in process, which are stored during the production process.

iii. Finished goods inventory, i.e. finished goods awaiting shipment from the factory.

iv. Inventory also include: furniture, machinery, fixtures, etc.

The term inventory may be classified in two main categories.

1- Direct Inventories

The items which playa direct role in the manufacture and become an integral part of finished goods are included in the category of direct inventories. These may be further classified into four main groups:

a. Raw material inventories are provided:
   - i. for economical bulk purchasing,
   - ii. to enable production rate changes
   - iii. to provide production buffer against delays in transportation,
   - iv. for seasonal fluctuations.

b. Work-in-process inventories are provided:
   - i. to enable economical lot production,
   - ii. to cater to the variety of products
   - iii. for replacement of wastages,
   - iv. to maintain uniform production even if amount of sales may vary.

c. Finished-goods inventories are provided:
   - i. for maintaining off-self delivery,
   - ii. to allow stabilization of the production level
   - iii. for sales promotion.

d. Spare parts.

Indirect Inventories

Indirect inventories include those items, which are necessarily required for manufacturing but do not become the component of finished production, like: oil, grease, lubricants, petrol, and office-material maintenance material, etc.

Types of Inventory Models

Basically, there are five types of inventory models:

i. Fluctuation Inventories. These have to be carried because sales and production times cannot be predicted accurately. In real-life problems, there are fluctuations in the demand and lead-times that affect the production of items. Such types of reserve stocks or safety stocks are called fluctuation inventories.

ii. Anticipation Inventories. These are built up in advance for the season of large sales, a promotion programme or a plant shutdown period. In fact, anticipation inventories store the men and machine hours for future requirements.

iii. Cycle lot-size) inventories. In practical situations, it seldom happens that the rate of consumption is the same as the rate of production or purchasing. So the items are procured in larger quantities than they are required. This results in cycle (or lot-size) inventories.

iv. Transportation Inventories. Such inventories exist because the materials are required to move from one place to another. When the transportation time is long, the items under transport cannot be served to customers. These inventories exist solely because of transportation time.

v. Decoupling Inventories. Such inventories are needed for meeting out the demands during the decoupling period of manufacturing or purchasing.

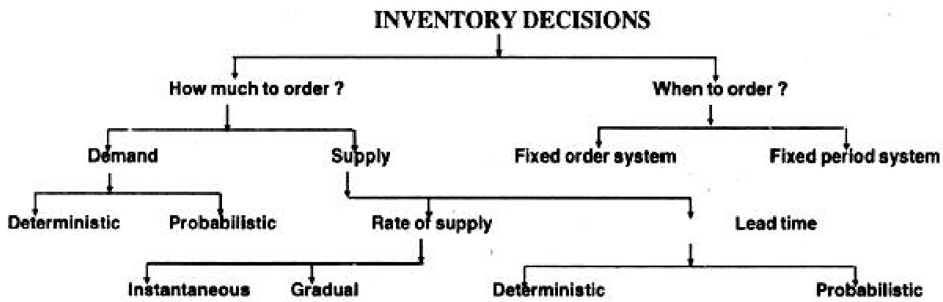
Inventory Decisions

The managers must take two basic decisions in order to accomplish the functions of inventory. The decisions made for every item in the inventory are:

i. How much amount of an item should be ordered when the inventory of that item is to be replenished?

ii. When to replenish the inventory of that item?

Inventory decisions may be classified as follows:



Before taking inventory decisions, it is necessary to develop an inventory model.

How to Develop an Inventory Model?

As explained earlier, inventory models are concerned with two main decisions: how much to order at a time and when to order so as to minimize the total cost? The sequence of basic steps required for developing an inventory model may be organized as follows:

Step 1. First take the physical stock of all the inventory items in an organization.

Step2. Then, classify the stock of items into various categories. Although several methods are available to classify the inventories; but the selected method must serve the objectives of inventory management.

For example, inventory items may be classified as raw materials, work-in process. Purchased components, consumable stores and maintenance spares, and finished goods, etc.

Step 3. Each of above classifications may be further divided into several groups. For example, consumable stores and maintenance spares can be further divided into the following groups:

(i) building materials. (ii) hardware items, (iii) lubricants and oils, (iv) textiles and fibres, (v) electric spares, (vi) mechanical spares, (vii) stationary items, etc.

Step 4. After classification of inventories, each item should be assigned a suitable code. Coding system' should be flexible so that new items may also be permitted for inclusion.

Step 5. Since the number of items in an organization is very large, separate inventory management model should be developed for each category of items.

Step 6. Use A-B-C or V-E-D classification (as discussed in the next chapter) which provide a basis for a selective control of inventories through formulation of suitable inventory policies for each category.

Step 7. Now decide about the inventory model to be developed. For example, fixed-order-quantity system may be developed for' A' class and high valued 'B' class items, whereas periodic review system may be developed for low valued 'B' class and 'C' class items. .

Step 8. For this, collect data relevant to determine ordering cost, shortage cost, inventory carrying cost, etc.

Step 9. Then, make an estimate of annual demand for each inventory item and their prevailing market price.

Step 10. Estimate lead-time safety stock and reorder level, if supply is not instantaneous. Also, decide about the service-level to be provided to the customers.

Step 11. Now develop the inventory mode.

Step 12. Finally, review the position and make suitable alterations, if required, due to current situations or constraints.

Before we proceed to discuss inventory models, it is very desirable to consider briefly the costs involved in the inventory decisions.

Inventory Costs

1. Holding Cost *(C]* or *Ch)'* The cost associated with carrying or holding the goods in stock is known as *holding* or *carrying* cost which is usually denoted by C. or *Ch* per unit of goods for a unit of time. Holding cost is assumed to vary directly with the size of inventory as well as the time for which the item is held in stock. The following components constitute the holding cost:,
   i. *Invested Capital Cost.* This is the interest charge over the capital investment. Since this is the most important component, a careful investigation is required to determine its rate.
   ii. *Record Keeping and Administrative Cost.* This signifies the need of keeping funds for maintaining the records and necessary administration.
   iii. *Handling Costs.* These include all costs associated with movement of stock such as: *cost of labour over-head cranes, gantries* and *other machinery* required for this purpose.
   iv. *Storage Costs.* These involve the rent of storage space or depreciation and interest even if the own space is used.
   v. *Depreciation, Deterioration and Obsolescence Costs.* Such costs arise due to the items in stock being out of fashion or the items undergoing chemical changes during storage *(e.g.* rusting in steel).
   vi. *Taxes and Insurance Costs.* All these costs require careful study and generally amounts to I% to 2% of the invested capital.
   vii. *Purchase Price or Production Costs.* Purchase price per-unit item is affected by the quantity

purchased due to *quantity discounts* or *price-breaks*. Production cost per unit item depends upon the length of production runs. For long smooth production runs this cost in lower due to more efficiency of men and machines. So the order quantity must be suitably modified to take the advantage of these price discounts. If P is the purchase price of an item and I is the stock holding cost per unit item expressed as a fraction of stock value (in rupees), then the holding cost C. $= IP$.

viii. *Salvage Costs or Selling Price.* When the demand for an item is affected by its quantity in stock, the decision model of the problem depends upon the profit maximization criterion and includes the revenue (sales tax etc.) from the sale of the item. Generally, salvage costs are combined with the storage costs and not considered independently.

2. Shortage Costs or Stock-out Costs *(C2* or *C,).* The penalty costs that are incurred as a result of unning out of stock *(i.e.*, shortage) are known as *shortage* or *stock-out* costs. These are denoted by C2 or *Cs* per unit of goods (or a specified period.

   These costs arise due to shortage of goods, sales may be lost, and good will may be lost either by a delay in meeting the demand or being quite unable to meet the demand at all. In the case where the unfilled demand for the goods can be satisfied at a latter date (backlog case), these costs are usually assumed to vary directly with the shortage quantity and the delaying time both. On the other hand, if the unfilled demand is lost (no backlog case), shortage costs become proportional to shortage quantity only.

3. Set-up Costs *(C3* or *Co),* these include the fixed cost associated with obtaining goods through *placing of an order* or *purchasing* or *manufacturing* or *setting up a machinery* before starting production. So they include costs of purchase, requisition, follow-up, receiving the goods, quality control, etc. These are also called *order costs* or replenishment costs, usually denoted by C3 or *Co* per production run (cycle). They are assumed to be independent of the quantity ordered or produced.

Why Inventory is Maintained?

As we are aware of the fact that the inventory is maintained for efficient and smooth running of business affairs. If a manufacturer has no stock of goods at all, on receiving a sale-order he has to place an order for purchase of raw materials, wait for their receipt and then start his production. Thus, the customers will have to wait for a long time for the delivery of the goods and may turn to other suppliers. This results in a heavy loss of business. So it becomes necessary to maintain an inventory because of the following reasons:

i. Inventory helps in smooth and efficient running of business.

ii. Inventory provides service to the customers immediately or at a short notice.

iii. Due to absence of stock, the company may have to pay high prices because of piece-wise purchasing. Maintaining of inventory may earn price discount because of bulk-purchasing.

iv. Inventory also acts as a buffer stock when raw materials are received late and so many sale-orders are likely to be rejected.

v. Inventory also reduces product costs because there is an additional advantage of batching and long smooth running production runs.

vi. Inventory helps in maintaining the economy by absorbing some of the fluctuations when the demand for an item fluctuates or is seasonal.

   i. Firstly, demand for the item is at a constant rate and is known to the decision maker in advance.

   ii. Secondly, the lead-time (which is the elapsed time between the placement of the order and its receipt into inventory) or the time required for acquiring an item is also known. Although above two assumptions are rarely valid for inventory problems in the business world, they do allow us to develop a simple model into which more realistic, complicating factors can be introduced.

Let q be the order size under the preceding assumptions, it can be shown from the Fig. that the

number of units in inventory is equal to q when each new order is practically received into inventory and that the inventory is gradually decreased until it becomes zero just at the point when the next order is received.

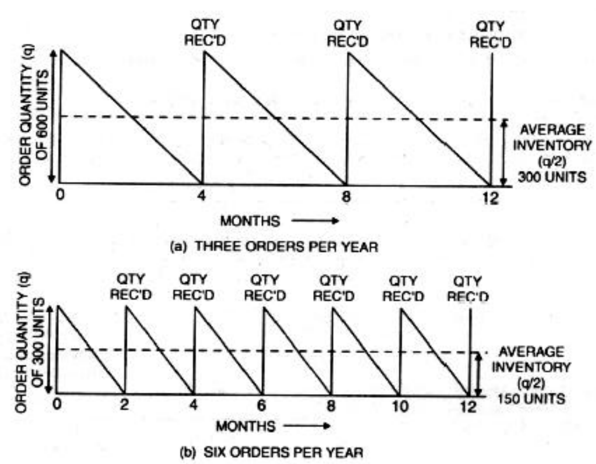


Fig. *(a)* Three orders per year, *(b)* Sixorders per year

LESSON 27
ECONOMIC ORDERING QUANTITY (EOQ)

In This Lecture We Are Going To Learn

- **Economic Order Quantity (EOQ)**
- **Graphical Metod**

Concept of Economic Ordering Quantity (EOQ)

The concept of *economic ordering quantity* was first developed by *F. Harris* in 1916. The concept is that management is confronted with a set of opposing costs-as the lot size *(q)* increases, the carrying charges *(C)* will increase while the ordering costs *(C3)* will decrease. On the other hand, as the lot size *(q)* decreases, the carrying cost *(C)* will decrease but the ordering costs will increase (assuming that only minor deviations from these trends may occur). Thus, *economic ordering quantity (EOQ) is that size of order which minimizes total annual (or other time period as determined by individual firms) cost of carrying inventory and cost of ordering under the assumed conditions of certainty and that annual demands are known.* The concept of EOQ applies to items, which are replenished periodically into inventory in lots covering several periods' need. The EOQ concept is applicable under the following conditions:

a. The item is replenished in lots or batches, either by purchasing or by manufacturing.

b. Consumption of items (or sales or usage rate) is uniform and continuous.

EOQ is that order quantity or optimal order size which minimises the total cost. The model is described under the following situations:

i. Planning period is one year.

ii. Demand is deterministic and indicated by parameter D units per year.

iii. Cost of purchases, or of one unit is C.

iv. Cost of ordering (or procurement cost of replenishment cost) is C3or *Co* 'For manufacturing goods, it is known as set-up cost.

v. Cost of holding stock (also known as inventory carrying cost) is C1or *Ch* per unit per year expressed either in items of cost per unit per period or in terms of percentage charge of the purchase price.

vi. Shortage cost (mostly it is back order cost) is C2or *Cs* per unit per year.

vii. Lead time is L, expressed in unit of time.

viii. Cycle period in replenishment is *t*.

ix. Order size is Q.

Determination of EOQ by Trial and Error Method (or Tabular Method)

This method involves the following steps:

Step 1. Select a number of possible lot sizes to purchase.

Step 2. Determine total cost for each lot size chosen.

Step 3. Finally, select the ordering quantity, which minimizes total cost.

Following illustrative example will make the procedure clear.

For example, suppose annual demand *(D)* equals 8000 units, ordering cost *(C3)* per order is Rs. 12.50, the carrying cost of average inventory is 20% per year, and the cost per unit is Re. 1.00. The following table is computed.

No. of Orders per Year	Lot Size	Average Inventory	Carrying Charges 20% per Year	Ordering Costs Rs. 12.50 per Order	Total Cost per Year
(1)	(2)	(3)	(4)	(5)	(6) = (4) + (5)
			(Rs.)	(Rs.)	(Rs.)
1	8,000	4,000	800.00	12.50	812.50
2	4,000	2,000	400.00	25.00	425.00
4	2,000	1,000	200.00	50.00	250.00
→ 8	1,000	500	100.00	100.00	200.00
12	667	333	66.00	150.00	$\boxed{216.00}$
16	500	250	50.00	200.00	250.00
32	50	125	25.00	400.00	425.00

This table indicates that an order size of 1000 units will give us the lowest total cost among all the seven alternatives calculated in the table. Also, it is important to note that this minimum total cost occurs when:

Carrying Costs =Ordering Costs.

Example 1. *Novelty Ltd. carries a wide assortment of items for its customers. One item, Gay look, is very popular. Desirous of keeping its inventory under control, a decision is taken to order only the optimal economic quantity, for this item, each time. You have the following information. Make your recommendations: Annual Demand:* 1,60,000 *units; Price per unit: Rs.* 20; *Carrying Cost: Re.* 1*per unit or* 5% *per rupee of inventory value; Cost per order: Rs.* 50. *Determine the optimal economic quantity by developing the following table.*

No. of Orders	Size of Orders	Average Inventory	Carrying Costs	Ordering Costs	Total Cost
1	...	...	...	...	...
10	...	...	...	...	...
20	...	...	...	...	...
40	...	...	...	...	...
80	...	...	...	...	...
100	...	...	...	...	...

Solution:

Tabular Method to Find EOQ

Orders per Year	Lot Size	Average Inventory	Carrying Cost (Re. 1) (Rs.)	Ordering Cost (Rs. 50 per order) (Rs.)	Total Cost per Year (Rs.)
1	1,60,000	80,000	80,000	50	80,050
10	16,000	8,000	8,000	500	8,500
20	8,000	4,000	4,000	1,000	5,000
40	4,000	2,000	2,000	2,000	4,000
80	2,000	1,000	1,000	4,000	5,000
100	1,600	800	800	5,000	5,800

Disadvantage of trial & error (or tabular) method.

In above example, we were fortunate enough in finding the lowest possible cost. But, suppose the computation for 8 orders per year had not been made.

Then, we could choose only among the six remaining alternatives for the lowest cost solution. This imposes a serious limitation of this method. So, a relatively large number of alternatives must be computed before the best possible least cost solution is obtained. In this situation, the following *graphical method* may be advantageous.

Graphical Method

The data calculated in the above table can be graphed as below to demonstrate the nature of the opposing costs involved in an EOQ model. This graph shows that annual total costs of inventory, carrying costs and ordering costs first decrease, then hit a lowest point where *inventory carrying costs* equal *ordering costs*, and finally increases as the ordering quantity increases. Our main objective is to find a numerical value for EOQ that will minimize the total variable costs on the graph.

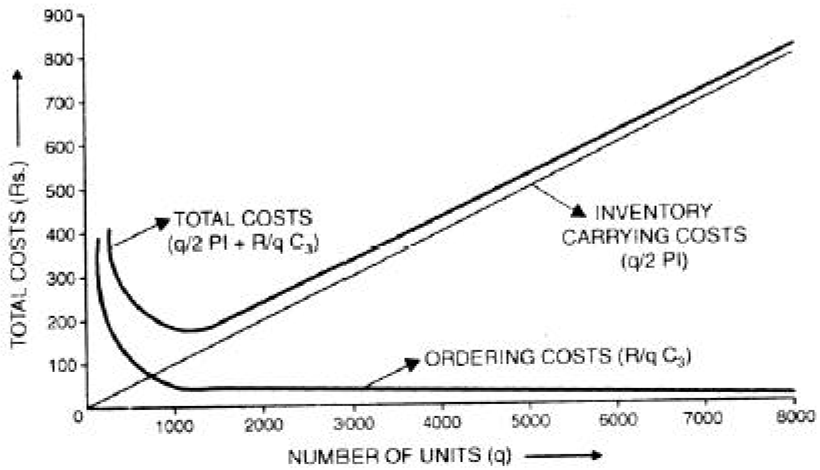


Fig. Economic ordering quantity graph

Disadvantage of graphical Method.

Without specific costs and values an accurate plotting of the carrying costs, ordering costs, and Iota I costs is not feasible, we can solve the EOQ models by the following two more accurate methods.

Algebraic method. This method is based on the fact that the most economical point in terms of total Inventory cost is where the *inventory carrying cost* equals *ordering cost.*

Calculus method. This method is based on the technique of finding the minimum total cost by utilizing the differentiation. This is the best method since it does not suffer from the limitations like previous methods.

LESSON 28
ECONOMIC PRODUCTION QUANTITY AND ECONOMIC ORDER INTERVAL

This Lesson Covers

- **Economic Production Quantity**
- **Economic Order Interval.**

Economic Production Quantity

The EPQ model uses the same assumptions as the simple or basic EOQ model, except that it use a finite replenishment rate. The assumptions are that demand is known and constant, all costs (holding, ordering, purchase) are known and constant, no quantity discounts apply, and no instantaneous replenishment (the entire order is produced or delivered over time).

Top of Form

Generally, in deriving the solution of economic production quantity (EPQ) inventory model, we consider the demand rate and deterioration rate as constant quantity. But in case of real life problems, the demand rate and deterioration rate are not actually constant but slightly disturbed from their original crisp value. The motivation of this paper is to consider a more realistic EPQ inventory model with finite production rate, fuzzy demand rate and fuzzy deterioration rate. The effects of the loss in production quantity due to faulty/old machine have also been taken into consideration. The methodology to obtain the optimum value of the fuzzy total cost is derived and a numerical example is used to illustrate the computation procedure. A sensitivity analysis is also carried out to get the sensitiveness of the tolerance of different input parameters.

One of the unrealistic assumptions of classic finite production model is that it assumes manufacturing facility functions perfectly at all times. However, in most practical settings, due to process deterioration or other factors, defective items are generated. In this paper, random defective rate is considered and all defective items are assumed not repairable. Because of the random scrap rate, it follows that the cycle length of production is not a constant. The renewal reward theorem is employed to cope with the variable cycle length. Disposal cost per scrap item is included in cost analysis, an optimal production quantity that minimizes the expected overall costs for the imperfect quality EPQ model is derived, where backlog is permitted.

Two extended models are developed to consider different inventory situations. One is that when backorder is not permitted versus our main model, which allows backlogging. The other considers that when all defective items are reworked and repaired for the EPQ model with shortage not permitted. Sensitivity analysis in our study includes judgment on whether to allow backorders or not, and determination on whether to rework the defective items or not, in terms of minimizing overall inventory costs.

One of the unrealistic assumptions of classic finite production model is that it assumes manufacturing facility functions perfectly at all times. However, in most practical settings, due to process deterioration or other factors, defective items are generated. In this paper, random defective rate is considered and all defective items are assumed not repairable. Because of the random scrap rate, it follows that the cycle length of production is not a constant. The renewal reward theorem is employed to cope with the variable cycle length. Disposal cost per scrap item is included in cost analysis, an optimal production quantity that minimizes the expected overall costs for the imperfect quality EPQ model is derived, where backlog is permitted.

Two extended models are developed to consider different inventory situations. One is that when backorder is not permitted versus our main model, which allows backlogging. The other considers that when all defective items are reworked and repaired for the EPQ model with shortage not permitted. Sensitivity analysis in our study includes judgment on whether to allow backorders or not, and determination on whether to rework the defective items or not, in terms of minimizing overall inventory costs. A convenient way of determining optimal order quantity with a single production is the basic economic production quantity (EPQ) model. As a result of production rate has often larger than demand rate, there has longer idle time when the equipment produces a single production. A more efficient way is to make Multi-Item production on a single production facility, so used Rotation Cycle method. Although the total cost solution of Rotation Cycle Model is feasible but not is optimum.

Economic Order Interval.

1. Stationary Economic Order Interval. Suppose that there is a constant demand rate $r > 0$ per unit time for a single product. There is a fixed setup cost $K > 0$ for placing an order, a holding cost $h > 0$ per unit stored per unit time and a penalty cost $p > 0$ per unit backordered demand per unit time. Each time the net inventory level (i.e., inventories less backorders) reaches s (< 0), an order for D $(> -s)$ units is placed that brings the net inventory on hand immediately up to $S \equiv s + D$.

(a) Optimal Ordering Policy. Show that the values of s, S and D that minimize the long-run average cost per unit time are given by

$$D = \sqrt{\frac{2Kr}{h} \cdot \frac{h+p}{p}},$$

$$S = \frac{p}{h+p} D \text{ and}$$

$$s = -\frac{h}{h+p} D.$$

(b) Limiting Optimal Ordering Policy as Penalty Cost Increases to Infinity. Find the limiting values of s, S and D as $p \to \infty$. Show that the limiting value of D agrees with the Harris square-root formula given in §1.2 of *Lectures on Supply-Chain Optimization.*

(c) Independence of Optimal Fraction Backordered and the Demand Rate and Setup Cost. What fraction f of the time will there exist no backorders under the policy in part (a)? Show that f does not depend on the demand rate or setup cost. Explain this fact by showing that for any fixed order interval, the same fraction f is optimal.

2. Extreme Flows in Single-Source Networks. Consider the minimum-additive-concave-cost network-flow problem on the graph $(\mathcal{N}, \mathcal{A})$ with demand vector $r = (r_i)$ in which the flows are required to be nonnegative and there is a single source $\sigma \in \mathcal{N}$, i.e., $r_\sigma < 0$ and $r_i \geq 0$ for $i \in \mathcal{N} \setminus \{\sigma\}$. Use a graph-theoretic argument to establish the equivalence of 1°-3° below about a nonnegative flow $x = (x_{ij})$. Also show that 1° implies the second assertion of 3° using Leontief substitution theory from problem 1(a) of Homework 1.

1° x is an extreme flow.

2° The subgraph induced by x is a tree with all arcs directed away from σ.

3° The subgraph induced by x is connected and contains no arc whose head is σ, and $x_{ij}x_{kj} = 0$ for all arcs $(i, j), (k, j) \in \mathcal{A}$ for which $i \neq k$.

3. Cyclic Economic Order Interval. Consider the problem of scheduling orders, inventories and backorders of a single product over periods $1, 2, \ldots$ so as to minimize the (long-run) average-cost per period of satisfying given demands $r_1, r_2, \ldots$ for the product in periods $1, 2, \ldots$. There is a real-valued concave cost $c_i(x_i)$ of ordering $x_i \geq 0$, $h_i(y_i)$ of storing $y_i \geq 0$ and $b_i(z_i)$ of backordering $z_i \geq 0$ in each period $i \geq 1$. Assume that $c_i(0) = h_i(0) = b_i(0) = 0$ for each $i \geq 1$ and that the data are *n-periodic*, i.e.,

$$(c_i, h_i, b_i, r_i) = (c_{i+n}, h_{i+n}, b_{i+n}, r_{i+n}) \text{ for } i = 1, 2, \ldots.$$

Clearly, there is a feasible schedule if and only if $\sum_1^n r_i \geq 0$. Assume this is so in the sequel.

(a) Reduction to Network-Flow Problem on a Wheel. Show that the problem of finding an ordering, inventory and backorder schedule that minimizes the average-cost in the class of n-periodic schedules is a minimum-additive-concave-cost uncapacitated nearly-1-planar network-flow problem. Show also that apart from duplicate arcs between pairs of nodes, the graph is a wheel with the hub node labeled 0 and the nodes associated with demands in periods $1, \ldots, n$ labeled by those periods cyclically around the hub. Thus, the node that immediately follows n in the cyclic order is, of course, node 1. Denote by $(i, k]$ the interval of periods in the cyclic order that begins with the node following i and ends with k for $1 \leq i, k \leq n$.

(b) Existence of Optimal Periodic Schedules. Give explicit necessary and sufficient conditions for the existence of a minimum-average-cost n-periodic schedule in terms of the derivatives at infinity of the cost functions (c_i, h_i, b_i) for $1 \leq i \leq n$.

(c) Extreme Periodic Schedules. Show that a feasible n-periodic schedule $(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$ is an extreme point of the set of such schedules if and only if

- inventories and backorders do not occur in the same period, i.e., $y_i z_i = 0$ for $i = 1, \ldots, n$,
- there is a period $1 \leq l \leq n$ with no inventories or backorders, i.e., $y_l = z_l = 0$, and
- between any two distinct periods $1 \leq i, k \leq n$ of positive production, there is a period j in the interval $[i, k)$ with no inventories or backorders, i.e., $y_j = z_j = 0$.

Show that if also the demands are all nonnegative, then $y_{i-1} x_i = z_i x_i = y_{i-1} z_i = 0$, for $1 \leq i \leq n$ where $y_0 \equiv y_n$.

(d) An $O(n^3)$ Running-Time Algorithm. Use the second condition of (c) to show that if there is a minimum-average-cost n-periodic schedule, then such a schedule can be found with at most $\frac{3}{2}n^3 + O(n^2)$ additions and $n^3 + O(n^2)$ comparisons by solving n n-period economic-order-interval problems. This implementation improves upon the $O(n^4)$ running time of straightforward application of the send-and-split method. [*Hint*: First show how to compute the minimum costs c_{ik} incurred in the interval $(i, k]$ with zero inventories and backorders in periods i and k, and with at most one order placed in the interval $(i, k]$ for all $1 \leq i, k \leq n$ in $O(n^3)$ time.]

(e) Constant-Factor Reduction in Running Time with Nonnnegative Demands and No Backorders. Show that if also the demands are nonnegative and no backorders are allowed, then the running time in (d) can be reduced to $\frac{1}{2}n^3 + O(n^2)$ additions and a like number of comparisons.

(f) An $O(n)$ Running-Time Algorithm with Nonnegative Demands and Linear Costs. Suppose that the demands are nonnegative and the cost functions are linear, i.e., there are constants c_i, h_i and b_i such that $c_i(z) = c_i z$, $h_i(z) = h_i z$ and $b_i(z) = b_i z$ for all $z \geq 0$ and $1 \leq i \leq n$. Show that a minimum-average-cost n-periodic schedule can be found with $4n$ additions and $4n$ comparisons. [*Hint*: Consider the following two-step algorithm. The first step is to determine an optimal period i in which to produce to satisfy the demand in any fixed period j. This can be done by choosing the cheapest of the solutions to two n-period problems. One finds an optimal period in which to order and store until period j, and the other finds an optimal period in which to order after backordering since period j. The second step is to solve the problem under the assumption that one orders in period i, in which case one can delete the inventory and backorder arcs whose head is node i.]

LESSON 29
NETWORK ANALYSIS-CPM

This Lecture is Regarding:

- **Critical Path Method**

Meaning of CPM

A network is a graphical representation of a project, depicting the flow as well as the sequence of well-defined activities and events. A network path consists of a set of activities that connects the networking beginning event to the network terminal event. The longest path through the network is called the critical path and its length determines the minimum duration in which the said project can be completed.

Usefulness of CPM

CPM plays an important role in project planning and control.

1. Network indicates the specific activities required to complete a project.
2 Network indicates the interdependence and sequence of specific activities.
3. It indicates the start and finish time of each activity of the project.
4. It indicates the critical path.
5. lt indicates the duration of critical path.
6. It indicates those activities for which extra effort would not be beneficial in order to shorten the project duration.
7. It indicates those activities for which extra effort would be beneficial in order to shorten the project duration.
8. It enables the project manager to deploy resources from nod-critical activities to critical activities without delaying the overall project duration.
9. It enables the project manager to assign responsibilities for each specific activity.
10. It enables the project manager to allocate resources for each specific activity
11. It can be used as a controlling device to monitor activities of the project by comparing the actual progress against planned progress.
12. It can be used to determine possible alternative solutions.
13. It can be used to determine various cost variances for initiating corrective action by comparing the actual costs against budgeted costs of the project.
14. It enables the project manager to determine a revised plan and schedule of the project by re- computing the project's critical path and the slack of non-critical activities using the actual duration of activities already finished and the revised estimated duration of activities not yet finished.

Assumptions of CPM/PERT

1. A project can be sub-divided into a set of predicable, independent activities.
2. The precedence relationships of project activities can be completely represented by a non-cyclical network graph in which each activity connects directly into its immediate successors.
3. Activity times may be estimated either as single point estimates or as three point estimates (i.e. optimistic, pessimistic or most likely) and are independent of each other.
4. Activity duration is assured to follow the beta distribution, the standard deviation of the distribution is assumed to be .1I6thof its range.

   $$\text{Mean } (t_e) \text{ is assumed to be } = \frac{t_0 + 4t_m + t_p}{6}.$$

   Variances in the length of a project is assumed to be = sum of variances of activities on the critical path.
5. The duration of an activity is linearly related to the cost of resources applied to the activity.

Activity (OR TASK OR JOB)

An activity or task or job is any portion of a project, which consumes time or resources and has a definable beginning and ending. For example, "laying of foundation of a building is an activity which requires the use of resources. In network an activity is represented by an arrow, which may be straight or bent but not broken. An activity must commence from some event. It may be noted that no activity can start until all previous activities in the same chain are completed.

Predecessor Activities - The activities, which immediately come before another activity without any intervening activities are predecessor activities.

Successor Activities - The activities, which follow another activity without any intervening activities are called successor activities to that activity.

Event (Or Node Or Connector)

The starting and finishing point of an activity or a group of activities are called events. In a network, an event is generally represented by a numbered circle (e.g. (1),(2)

Tail Event - The starting point of an activity is called a tail event because it is connected to the tail of an activity. In a network, tail event is represented by symbol "i".

Head Event - The finishing (or terminal) point of an activity is called a head event because it is connected to the head of an activity. In a network, head event is represented by symbol "j".

Note: Head events should always have a number higher than that of the tail events.

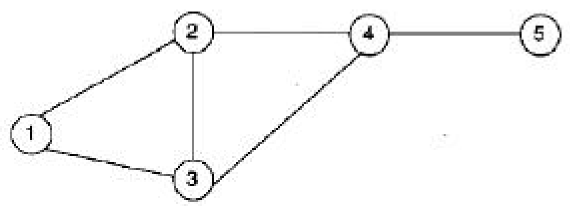
Merged Event - Merge event is an event, which represents the joint completion of more than one activity.

Burst Event - Burst event is an event, which represents the joint starting of more than one activity.

Network (or Arrow Diagram)

A network is a graphical representation of inter-relationship of the various activities of a project. While drawing network, the following rules must be kept in mind.

1. An activity cannot occur until all activities leading to it are complete.
2 No activity can start untill its tail event is reached.



Activity	Head Event	Tail Event
1 - 2	2 (Burst Event)	1 (Burst Event)
1 - 3	3 (Merge Event)	1 (Burst Event)
2 - 3	3 (Merge Event)	2 (Burst Event)
2 - 4	4 (Merge Event)	2 (Burst Event)
3 - 4	4 (Merge Event)	3
4 - 5	5	4

The aforesaid graphical representation showing the inter-relationships of the various Activities of a project are called 'Network'.

Working Methodology of Critical Path Analysis

The working methodology of Critical Path Analysis (CPA) which includes both CPM and PERT, consists of following five steps:

1. Analyse and breakdown the project in terms of specific activities and/or events.
2 Determine the interdependence and sequence of specific activities and prepare a network.
3. Assign estimate of time, cost or both to all the activities of the network.
4. Identify the longest or critical path through the network.
5. Monitor, evaluate and control the progress of the project by re-planning, rescheduling and reassignment of resources.

Conventions Followed In Drawing Network

The conventions followed in drawing a network are as follows:

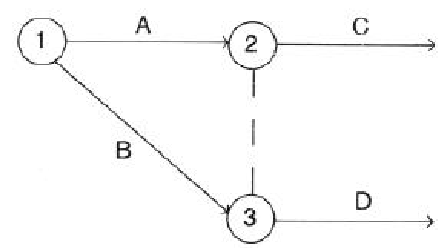
1. Draw the arrow directing from left to right that is, time and progress of the project flows from left to right.
2 As far as possible avoid drawing arrows that cross each other. Wherever crossing of arrows is unavoidable, bridging may be done.
3. An activity is always bounded by two events, called the start event and the end event. No event should hang loosely on the network.
4. Each node should be numbered without ambiguity. The numbers are assigned to events in such a way that the number assigned to the ending event of an activity is greater than the number assigned to the beginning event of that activity.

Dummy Activity/Arrow

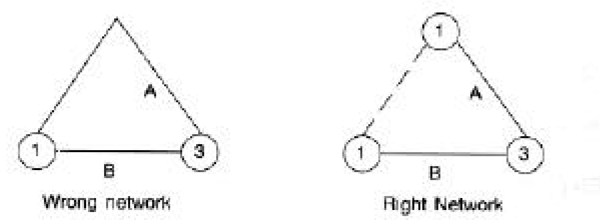
Dummy activity is a hypothetical activity, which requires zero time and zero resources for completion. Dummy arrow represents an activity with zero duration. It is represented by dotted line and is inserted in the network to clarify activity pattern under following situations:

i. It is created to make activities with common starting and finishing events distinguishable, and.
ii. To identify and maintain the proper precedence relationship between activities that are not connected by events.

Consider a situation where A and B are concurrent activities, C is dependent on A and D is dependent on both A and B. Such a situation can be handled by use of a dummy activity as follows:



Avoiding duplication of designation - In the figure below, two activities A and Bare being designated as 1 - 3 in the network given on left hand side. This anamoly has been set right in the network given on right hand side by employing a dummy.



Can a Critical Path Change During the Course of a Period?

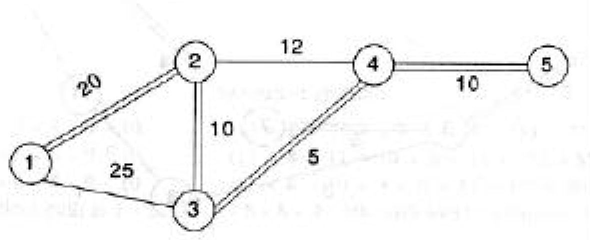
The critical path of a project will not remain static throughout its life; it can change during the course of project completion. Unforeseen circumstances sometimes may cause estimated duration of one or more activities to change. Due to variation in time duration of the project activities, if some activities take more or less time than their expected time, this may render some of the non-critical activities as critical and vice versa. Hence a non- critical path may become a critical path or alternatively a critical path may become non-critical.

For this reason, the project leader is required to closely monitor the project's activities by comparing the planned progress to the actual progress. Such monitoring will enable him to quickly recognise and react to unpredictable events that may significantly alter the estimated duration of some activities. Using the actual duration of activities already finished and the revised estimated duration of activities not yet finished, the project leader can determine a revised plan and schedule of the project by recomputing the project's critical path, the slack of non-critical activities and then rescheduling the starting and finishing times of every activity not yet finished.

Problem 6.1

Draw the network for the following activities and find critical path and total duration of project.

Activity	*Duration (Days)*
1-2	20
1-3	25
2-3	10
2-4	12
3-4	5
4-5	10



Various Paths	*Duration of Paths*
1 - 2 - 4 - 5	20 + 12 + 10 = 42
1 - 2 - 3 - 4 - 5	20 + 10 + 5 + 10 = 45
1 - 3 - 4 - 5	25 + 5 + 10 = 40

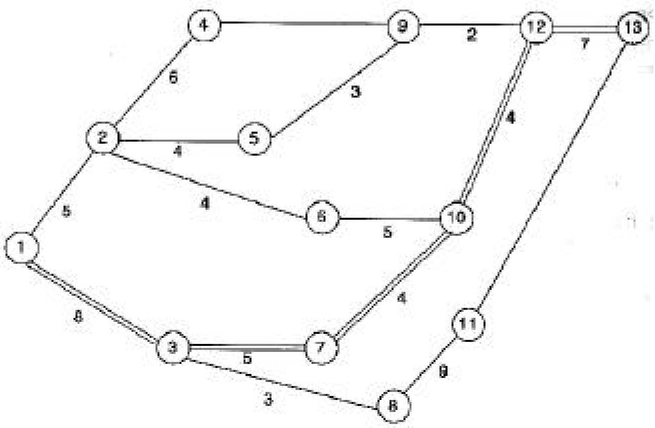
Hence the critical path is 1 - 2 - 3 - 4 - 5 with project duration of 45 days.

Problem 6.2

Draw a network from the following activities and find critical path and duration of project.

Activity	*Duration (Days)*	*Activity*	*Duration (Days)*
1-2	5	5-9	3
1-3	8	6-10	5
2-4	6	7-10	4
2-5	4	8-11	9
2-6	4	9-12	2
3-7	5	10-12	4
3-8	3	11-13	1
4-9	1	12-13	1

Solution



Various Paths	*Duration of Paths*
1 - 2 - 4 - 9 - 12 - 13	(5 + 6 + 1 + 2 + 7) = 21
1 - 2 - 5 - 9 - 12 - 13	(5 + 4 + 3 + 2 + 7) = 21
1 - 2 - 6 - 10 - 12 - 13	(5 + 4 + 5 + 4 + 7) = 25
1 - 3 - 7 - 10 - 12 - 13	(8 + 5 + 4 + 4 + 7) = 28
1 - 3 - 8 - 11 - 13	(8 + 3 + 9 + 1) = 21

Hence the critical path is 1 - 3 - 7 - 10 - 12 - 13 with duration of 28 days.

Notes:

LESSON 30
NETWORK ANALYSIS-PERT

In this lecture we will be discussing:

- **Pert**

Meaning of Pert

PERT stands for Programme Evaluation and Review Technique. PERT is event oriented.

PERT is a probabilistic model i.e. it takes into account uncertainties involved in the estimation of time of a job or an activity. It uses three estimates of the activity time - optimistic, Pessimistic and Most Likely. Thus, the expected duration of each activity is probabilistic and expected duration indicates that there is 50% probability of getting the job done within that time.

PERT is primarily concerned with project time. It enables the project manager to schedule and co-ordinate the various activities so that the project can be completed on scheduled time. PERT is generally used for those projects, which are not of repetitive in nature (e.g. Research and Development Projects) and where time required to complete

Various activities are not known in advance.

The process of PERT analysis includes:

1. Identification of the activities of the project.
2. Estimation of activity time.
3. Defining inter-dependence relationships between the
4. Drawing the net work.
5. Using the network to obtain the scheduling data activities.

Distinction between CPM and PERT

CPM and PERT differ in the various respects:

Basis of Distinction	CPM	PERT
1. Orientation	CPM is **activity** oriented i.e. CPM network is built on the basis of activities.	PERT is **event** oriented.
2. Nature of Model	CPM is a **deterministic** model i.e. it does not take into account the uncertainties involved in the estimation of time for execution of a job or an activity.	PERT is a **probabilistic** model i.e. It takes into account uncertainties involved in the estimation of time of a job or an activity. It uses three estimates of the activity time - optimistic, pessimistic and most likely. Thus, the expected duration of each activity is probabilitistic and expected duration indicates that there is 50% probability of getting the job done within that time.
3. Emphasis	CPM places dual emphasis **on time and cost** and evaluates the trade off between project cost and project time. It enables the project manager to manipulate project duration within certain limits by deploying additional resources so that project duration can be shortened at an optimum cost.	PERT is primarily concerned with project **time**. It enables the project manager to schedule and co-ordinate various activities so that the project can be completed on scheduled time.

Usefulness of Pert

1. Facilitates Planning - PERT facilitates planning. Planning involves the formulation of objectives and goals that are subsequently translated into specific plans and projects. The planning phase of the project is initiated by PERT because it requires the establishment of project objectives and specifications, and then, it provides a realistic and disciplined basis for determination how to attain these objectives, considering pertinent time and resource constraints. Planners are required to specify not only all the activities necessary to complete a project but also their technological dependencies. PERT calculations then layout clearly the implications of these interdependencies and aid the planners in finding problems that might be over looked in large complex problems. PERT provides a realistic way of carrying out more long range and detailed planning of projects including their co-ordination at all the levels of management. In developing a detailed and comprehensive project network, users often make significant improvements over their original ideas; they do a better job of early co-ordination with supplier, engineers, manager's contractors and all other groups associated with the project.
2. Facilitates Controlling - PERT facilitates controlling. The function of control is to institute a mechanism that trigger a warning signal if actual performance is deviating (in terms of time, cost or some other measures of effectiveness) from the plan. PERT is unique in its emphasis on the control phase of project management because it uses statistical analysis along with probability calculations concerning the project completion by a certain period. As a result, it is easier under PERT to revise the plan each time changes are introduced in the network. With the information from PERT, the manager is in a better position to know where troubles may occur, supervision may be needed and where resources may be shifted to keep the project on schedule. It also helps in controlling the project by checking off progress against the schedule and by assigning the schedule manpower and equipments. It enables the managers to analyse the effects of delays and to revise the network in case any changes are required.
3. Facilitates the Application of MBE Principle - PERT facilitates the application of the principle of management by exception by identifying the most critical elements in the plan, focusing management attention on the 10 to 20 per cent of the project activities that are most constraining on the schedule. It continually defines new schedules and illustrates the effects of technical and procedural changes on the overall schedule. Thus, the manager can control the project in a better way.
4. Acts as Project Management Technique - Programme Evaluation is a project management technique. The project to be planned is broken into inter-dependent activities and a network of arrows is constructed to depict these dependency relationships, each activity being represented by one arrow. PERT, however is particularly suited to innovational projects (viz. oil exploration or introducing a new product) for the activities of which it is not possible to estimate the timings precisely.
5. Provides Feedback Information - PERT provides valuable feedback information about the status of the project. The management is provided with a convenient yardstick against which progress of various activities especially those, which are critical, may be measured. This information is extremely important for control. For example, any delay beyond the schedule completion time of an activity will be brought to management's attention. The cause of the delay can be investigated and *remedial action taken. This increases* the likelihood *of completing the project*.

However the success of the technique depends on the management's response to the feedback *information provided.* If *this information is merely field and* no *remedial action* is taken, the purpose of the technique *is* defeated. It *is important* to recognise that there is no self-correcting mechanism built into the model that will cause automatic remedial action. Such action has to be initiated by the persons using the technique. Another problem that may destroy the success of PERT model is the tendency of many activity managers to delay in the reporting of the current status of progress. This causes delays in the up dating of time schedule and the management begins to operate with outdated formation, which could be disastrous. To resolve this problem, it is crucial for the project manager to inform all activity managers about the relative importance of their activity's pact on the total project and where they 'fit in' in the overall project's success.

In short, there is nothing wrong with the PERT network technique. If the feedback information provided by it is properly utilised by the management, it will act only as wall decorates in a business enterprise. However, if proper remedial action initiated for various information provided by it, PERT can prove to be a dynamic tool for planning, Controlling and scheduling a project.

How To Incorporate Uncertainty In Pert Model

Uncertainty can be incorporated in PERT network by assuming that the activity time has a beta distribution. This enables us to calculate the expected activity time and its standard deviation. For this purpose we have the following three different estimates for he completion time of an activity.

Optimistic Time (To) Optimistic time for an activity is the minimum time required to complete an activity if everything goes all right. i.e. under ideal conditions. Normal Time (or Most Likely Time) (Tm) Normal time is the most probable time, which an activity will take. This is the time, which lies between the optimistic time and the pessimistic time.

Pessimistic Time Pessimistic time is the best guess-estimate of the maximum time that would be required to complete an activity if bad luck were encountered at every turn. The estimate does not take into account such natural catastrophes as flood etc. Expected Time Expected time is the average time that an activity will take if it was to be repeated on large number of times and is based on the assumption that the activity time follows beta distribution. It is given by the relation -

$$t_e = \frac{t_0 + 4t_m + t_p}{6}$$

where to' tm, tp are the optimistic, most likely (or Normal) and pessimistic times. Activity Variance Activity variance is the square of l/6th of the difference between the pessimistic and optimistic guess-estimates, i.e.

$$\sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2$$

Project Variance Project variance is the variance of the critical path duration which, in turn, is the sum of variances of the activities on it. From central limit therein it follows that critical path duration is normally distributed. As such variance can be put to use for finding the probability of completing the project by a given date. The physical interpretation of this term is that if the project were to be repeated on myriads of occasions its duration follows a normal distribution with the variance explained above. The formula presumes beta distribution of activity time.

Thus, Project Variance = Sum of Variances of Critical Activities

Practical Steps Involved In Solving Pert Problems

The practical steps involved in solving PERT problems are given below:

Step 1: Prepare a table showing Expected Duration and Variance for each activity of the project:

Activity	*Estimated Duration*				*Standard Deviation*	*Variance*
i - j	*Optmistic* t_o	*Pessimistic* t_p	*Most Likely* t_m	*Expected* t_e	σ	σ^2

Where, Expected Duration $(t_e) = \frac{(t_0 + t_p + 4t_m)}{6}$

Standard Deviation $(\sigma) = \frac{(t_p - t_o)}{6}$

Variances (σ^2) = Square of standard deviation

Step 2→ Draw a Project Network of Activities (*or Jobs*) based upon Expected Duration of the Activities.

Step 3→ Find out the Critical Path.

Step 4→ Find out the Total Expected Duration of the Project based upon the Network.

This is also the Average Duration (*or* $\overline{X}$) of the Project.

Step 5→ Find out the Standard Deviation of the Project by following formula:

$\sigma = \sqrt{\text{(Sum of Variances of Critical Path Activities)}}$

Step 6→ Calculate the Probability of completing the Project, *if required*, within a particular time period, *say X*, as follows:

(i) *Calculate the value of Z as follows:*

$$Z = \frac{X - \overline{X}}{\sigma}$$

(ii) *Find out the Probability for the value of Z from the Normal Variate Table as follows:*

If the value of Z is positive	P = 0.50 + Probability for Z from the Table.
If the value of Z is negative	P = 0.50 - Probability for Z from the Table.

Step 7→ Calculate the Expected Time of completion of the Project at a given probability, *if required*, as follows:

(i) Find out the value of Z from the Normal Variate Table for the given Probability after deducting 0.50 from the Probability.

ii. Put the value of Z in the formula given in Step 6 and find out the value of X, i.e. the Expected Time of completion of the Project at the given probability.

Problem 1

A project has the following activities and characteristics:

Activity	*Estimated duration in days*		
	Optimistic	***Most Likely***	***Pessimistic***
1 - 2	2	5	8
1 - 3	4	10	16
1 - 4	1	7	13
2 - 5	5	8	11
3 - 5	2	8	14
4 - 6	6	9	12
5 - 6	4	7	10

Required

a. Find expected duration of each activity

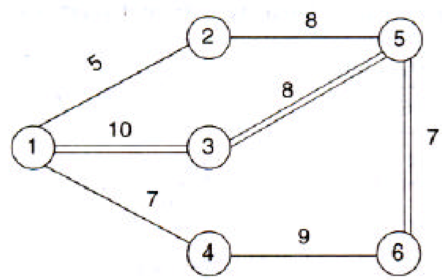
b. Draw the project network and expected duration of the project

c. Find variances of activities on critical path and its standard deviation

Solution

a. Expected duration of each activity and their variances

Activity	*Expected Duration* $t_e = \frac{t_0 + 4t_m + t_p}{6}$	*Variance* $= \sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2$
1 - 2	$\frac{2+4\times5+8}{6} = 5$	$\left(\frac{8-2}{6}\right)^2 = 1$
1 - 3	$\frac{4+4\times10+16}{6} = 10$	$\left(\frac{16-4}{6}\right)^2 = 4$
1 - 4	$\frac{1+4\times7+13}{6} = 7$	$\left(\frac{13-1}{6}\right)^2 = 4$
2 - 5	$\frac{5+4\times8+11}{6} = 8$	$\left(\frac{11-5}{6}\right)^2 = 1$
3 - 5	$\frac{2+4\times8+14}{6} = 8$	$\left(\frac{14-2}{6}\right)^2 = 4$
4 - 6	$\frac{6+4\times9+12}{6} = 9$	$\left(\frac{12-6}{6}\right)^2 = 1$
5 - 6	$\frac{4+4\times7+10}{6} = 7$	$\left(\frac{10-4}{6}\right)^2 = 1$



Various Paths	*Duration of Paths*
1 - 2 - 5 - 6	5 + 8 + 7 = 20
1 - 3 - 5 -6	10 + 8 + 7 = 25
1 - 4 - 6	7 + 9 = 16

Hence the critical path is 1 - 3 - 5 - 6 with expected project duration of 25 days.

(c) Activities lying on critical path

1 - 3, 3 - 5, 5 - 6

Variance = 4 + 4 + 1 = 9

Standard deviation = $\sqrt{\text{Variance}}$

Standard deviation = $\sqrt{9} = 3$

LESSON 31
GROUP DISCUSSION/QUIZ ON UNIT 5 PORTIONS

Notes:

“The lesson content has been compiled from various sources in public domain including but not limited to the internet for the convenience of the users. The university has no proprietary right on the same.”

Rai Technology University

ENGINEERING MINDS

Rai Technology University Campus
Dhodballapur Nelmangala Road, SH -74, Off Highway 207, Dhodballapur Taluk, Bangalore - 561204
E-mail: info@raitechuniversity.in | Web: www.raitechuniversity.in