Unit-2

Time-Domain Analysis

Time Response of a system is the output of the closed-loop system as a function of time

Time Response is divided into two parts
- Transient Response
- Steady-state Response

Transient Response- whenever a circuit is switched from one condition to another, either by a changed in applied source or a change in the circuit elements, there is a transition period during which the Branch current & voltages changes from their former values to new one. This period is called transient period. It is also called as dynamic Response of the system.

~~steady-state~~ -

steady-state- It is simply that part of the total Response that remains after the transient has died-out. So, the steady-state Response can still vary in a fixed pattern such as a sine wave or Ramp function that increases with time.

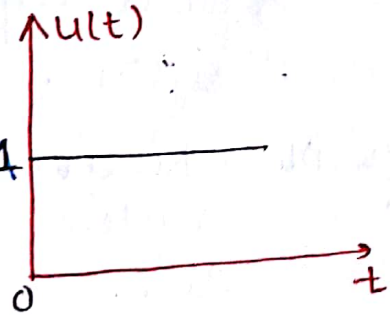
Let $c(t)$ be the total response of the system

$$\therefore \; c(t) = C_t(t) + C_{ss}(t).$$

Where $C_t(t)$ = Transient Response & $C_{ss}(t)$ = steady-state Response.

Standard Test Signals

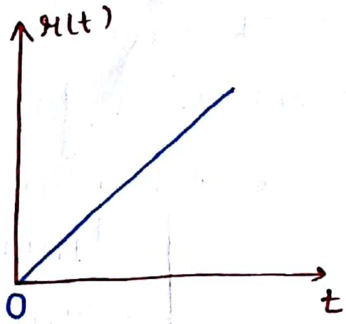
1. **Step signal**- It is a signal whose value changes from one level (usually zero) to another value, A in zero time.



$$u(t) = \begin{cases} A & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

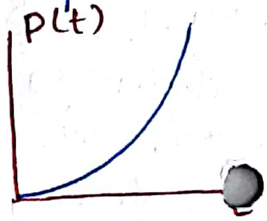
$$\mathcal{L}[u(t)] = \mathcal{L}[A] = \frac{A}{s}$$

2. Ramp Signal - The Ramp is a signal which starts at a value of zero and increases linearly with time.



$$r(t) = \begin{cases} At & ; t \geq 0 \\ 0 & ; t < 0 \end{cases} \qquad R(s) = \frac{A}{s^2}$$

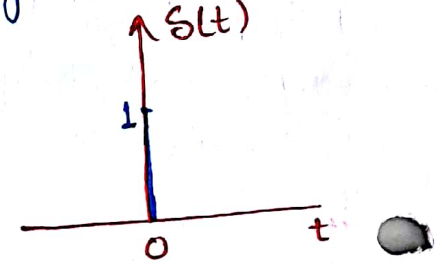
3. Parabolic Signal - is a signal that is one order faster than the Ramp function.



$$p(t) = \begin{cases} \frac{At^2}{2} & ; t \geq 0 \\ 0 & ; t < 0 \end{cases} \qquad \mathcal{L}[p(t)] = P(s) = \frac{A}{s^2}$$

4. Impulse Signal - is defined as a signal which has zero value everywhere except at t=0 where magnitude is infinite.

In order to get finite value, Impulse signal is formed as unit Impulse signal.

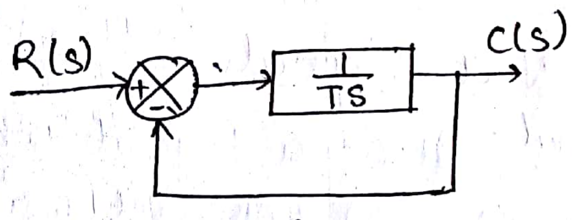


$$\delta(t) = \begin{cases} 1 & ; t = 0 \\ 0 & ; t \neq 0 \end{cases}$$

$$\mathcal{L}[\delta(t)] = 1$$

Time Response of first-order systems

Let us consider a first order unity feedback system.



Example → RC circuit, thermal system or the like.

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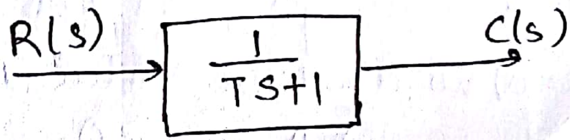

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$$

Unit-step Response of First-order Systems

As for the given system $R(s) = \frac{1}{s}$ (i.e. unit step function)

We know that $\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$

As $R(s) = \frac{1}{s}$; $\therefore C(s) = \frac{1}{s} \cdot \frac{1}{Ts+1}$

Expanding into partial fractions

$1 = A(Ts+1) + Bs \Rightarrow \boxed{A=1} \quad \boxed{B=-T}$.

$$C(s) = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+1/T}$$

Taking inverse laplace transform

$$c(t) = 1 - e^{-\frac{t}{T}} \quad \text{for } t \geq 0.$$

For $t = T$ $\quad c(T) = 1 - e^{-1} = 0.632$

This shows that Response c(t) has reached 63.2% of its total change.

Initial slope of the curve at $t=0$

$$\left.\frac{dc}{dt}\right|_{t=0} = \left.\frac{1}{T} e^{-t/T}\right|_{t=0} = \frac{1}{T}$$ where T is known as the time constant of the system.

output would reach the final value at $t=T$, if it is maintained its initial speed of Response.

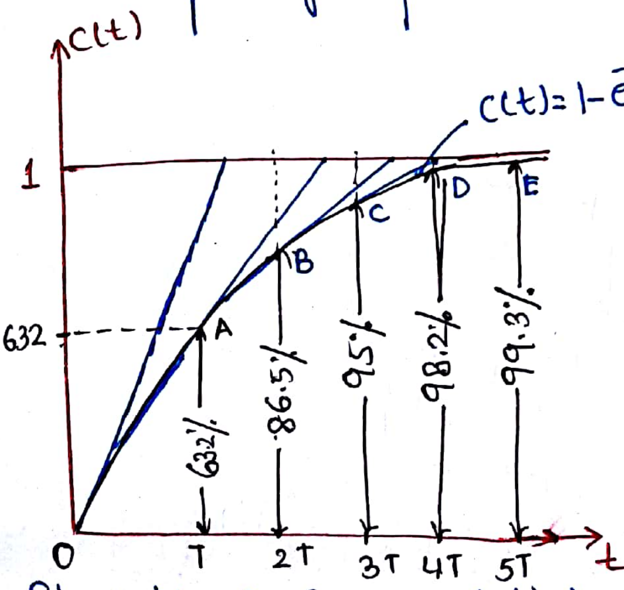

Fig- unit-step Response of first-order system

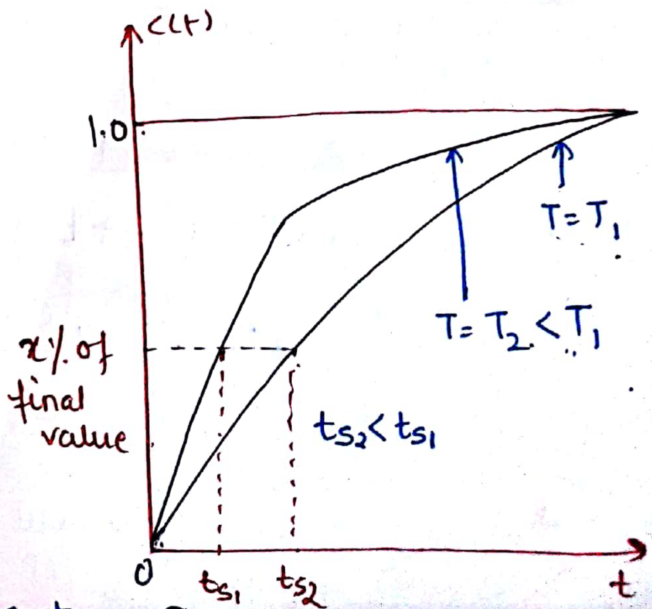

Fig- Effect of time constant on system Response

Here, the time constant, T represents how fast is the system. and tends to Reach the final value.

$\uparrow$ Time constant – sluggish system

$\downarrow$ Time constant – faste Response of system

Error Response of the system is given by

$$e(t) = r(t) - c(t)$$

As $r(t)$ is unit step function & $c(t)$ is Response of system.

$$e(t) = 1 - 1 + e^{-t/T} = e^{-t/T}$$

steady state error is given by

$$e_{ss} = \lim_{t\to\infty} e(t) = 0$$

This means the first order system with unit-step input have zero steady-state error.

Unit-Ramp Response of first-Order system

$$R(s) = \frac{1}{s^2}$$

Now $C(s) = \frac{1}{Ts+1} \cdot R(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{Ts+1}$.

Using partial fractions,

$C(s) =$

$$\cancel{C(s) = As + Bs^2 + C(Ts+1)}.$$

$$1 \; \cancel{C(s)} = As(Ts+1) + B(Ts+1) + Cs^2$$

$$\boxed{B=1}, \boxed{A=-T}, \boxed{C=T^2}$$

$$C(s) = -\frac{T}{s} + \frac{1}{s^2} + \frac{T^2}{Ts+1} = -\frac{T}{s} + \frac{1}{s^2} + \frac{T}{s+\frac{1}{T}}$$

Taking Inverse laplace, we get

$$c(t) = -T + t + Te^{-t/T} \Rightarrow c(t) = t + T(e^{-t/T} - 1) \text{ for } t \geq 0$$

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The error signal, $e(t) = r(t) - c(t)$

$$= t - (t - T + Te^{-t/T})$$

$$e(t) = T(1 - e^{-t/T})$$

Steady-state error is given by

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} T(1 - e^{-t/T}) = T$$

$\downarrow T \rightarrow$ smaller e_{ss} in ramp I/P.

Reducing the T, not only improves its speed of response but also Reduces its steady-state error to the Ramp I/P.

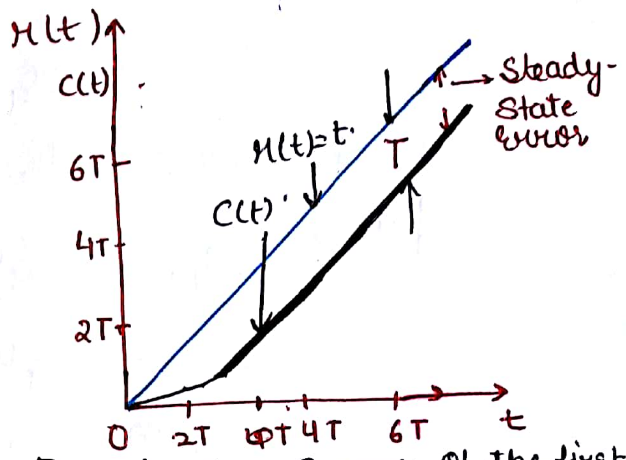


Fig-Unit-Ramp Response of the first-order system

Unit-Impulse Response of First-order Systems

$R(s) = 1$. & output $C(s) = 1 \cdot \frac{1}{Ts+1} = \frac{1}{Ts+1} = \frac{1/T}{s+\frac{1}{T}}$

Take Inverse Laplace Transform, we get

$c(t) = \frac{1}{T} e^{-t/T}$ for $t \geq 0$.

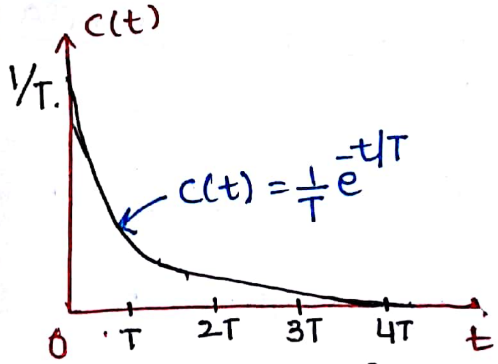


Fig-unit Impulse Response of a first-order system

Second Order System

Response of a Second-order system to the unit-step Input

The closed loop transfer function $\frac{C(s)}{R(s)}$ of the system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

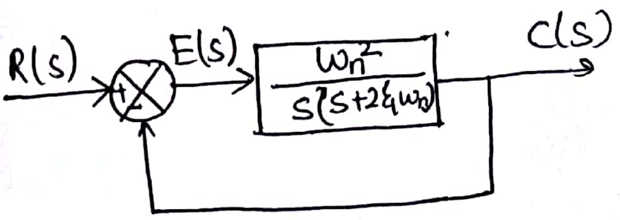


Fig-Second-order System

This is called standard form of second order system.
The dynamic behaviour of a system can be described in terms of two parameters i.e. ξ & ω_n.
The denominator of the polynomial is called characteristics polynomial and $q(s)=0$ is called characteristic equation.

$$\therefore \; s^2+2\xi\omega_n s+\omega_n^2=0$$

Roots of characteristic equation are for $\xi<1$

$$s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$s_1, s_2 = -\xi\omega_n \pm j\omega_d$$

Most of the control system with the exception of Robotic control systems are designed with $\xi<1$, to have high response speed.

whereas ω_n = natural frequency

ξ = Damping Ratio

& ω_d = Damped Natural frequency

On the Basis of Damping Ratio, ξ, the response of a system is as follows:-

Response of an ~~undamped~~ underdamped system – $(0<\xi<1)$.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s+2\xi n+j\omega_d)(s+2\xi\omega_n-j\omega_d)}$$

For unit step I/P $R(s)=\frac{1}{s}$, $\therefore$

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} = \frac{1}{s} - \frac{s+2\xi\omega_n}{(s^2+2\xi\omega_n s+\omega_n)^2}$$

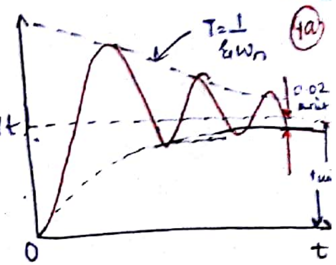
$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{[(s+\xi\omega_n)^2+\omega_n^2-\omega_n^2\xi^2]}$$

$$= \cancel{\frac{1}{s} - \frac{s+\xi\omega_n}{(s^2+2\xi\omega_n s+\omega_n^2)}} = \frac{1}{s} - \cancel{\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}} - \frac{\xi\omega_n}{\omega_d}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi}{\sqrt{1-\xi^2}}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

(1a)



For $\xi<1$ – system is overdamped

$\xi<1$, the time Response presents damped oscillations.

Response settles within 2% of desired value (1 unit) after damping out the oscillations in a time $4T$, where $T=1/\xi\omega_n$.

For $\xi=0$ –
$$C(t) = 1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left[\omega_n\sqrt{1-\xi^2}\,t+\tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

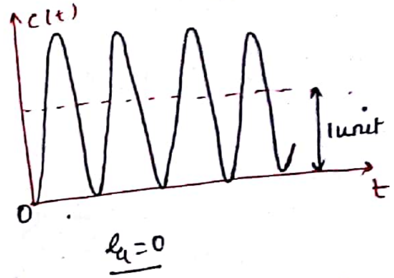
$\xi=0$.

$$c(t) = 1-\frac{e^{0}}{\sqrt{1-0}}\sin\left[\omega_n\sqrt{1-0}\,t+\tan^{-1}\frac{\sqrt{1-0}}{0}\right].$$

$$= 1-\sin(\omega_n t+\tan^{-1}\infty)$$

$$= 1-\sin\left(\omega_n t+\frac{\pi}{2}\right)$$

$$c(t) = (1-\cos\omega_n t)$$



$\xi=0$

For $\xi=1$ –
$$c(t) = \lim_{\xi\to1}\left[1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left(\omega_n\sqrt{1-\xi^2}\,t+\tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right)\right]$$

Put $\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$.

$$c(t) = \lim_{\xi\to1}\left[1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left(\omega_n\sqrt{1-\xi^2}+\phi\right)\right].$$

$\xi=\cos\theta$, $\sqrt{1-\xi^2}=\sin\theta$.

$$c(t) = \lim_{\xi\to1}\left\{1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin(\omega_n\sqrt{1-\xi^2}\,t).\cos\phi + \cos(\omega_n\sqrt{1-\xi^2}\,t)\sin\phi\right]\right\}$$

$$c(t) = \lim_{\xi\to1}\left[1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sin(\omega_n\sqrt{1-\xi^2}\,t)\,\xi + \cos(\omega_n\sqrt{1-\xi^2}\,t).\sqrt{1-\xi^2}\right)\right] \quad -(1)$$

For $\lim_{\xi\to1} \sin\left(\omega_n\sqrt{1-\xi^2}\,t\right)\xi \to \omega_n\sqrt{1-\xi^2}\,t$. — ⓐ

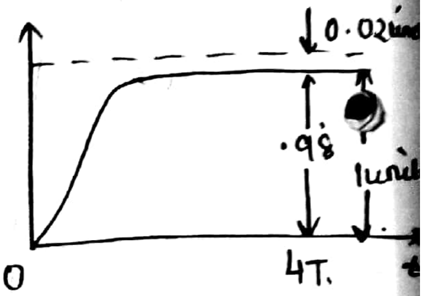
$\lim_{\xi\to1} \cos\left(\omega_n\sqrt{1-\xi^2}\,t\right) \to 1$ — ⓑ

Put ⓐ & ⓑ in eqn ①

$$c(t) = \lim_{\xi\to1}\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\omega_n\sqrt{1-\xi^2}\,t + 1\cdot\sqrt{1-\xi^2}\right)\right].$$

$$= \lim_{\xi\to1}\left[1 - \frac{e^{-\xi\omega_n t}}{\cancel{\sqrt{1-\xi^2}}}\cancel{\sqrt{1-\xi^2}}\,(\omega_n t+1)\right].$$

$$c(t) = \left[1 - e^{-\omega_n t}(\omega_n t+1)\right]$$



d) $\xi > 1$

$$C(s) = \frac{\omega_n^2 \cdot R(s)}{s^2+2\xi\omega_n s+\omega_n^2}$$

$r(t) = 1$.

$R(s) = 1/s$.

$$C(s) = \frac{\omega_n^2}{s\left(s^2+2\xi\omega_n s+\omega_n^2\right)}$$

$$C(s) = \frac{1}{s}\cdot\frac{\omega_n^2}{(s+\xi\omega_n)^2-\left(\omega_n^2(\xi^2-1)\right)}$$

$$C(s) = \frac{1}{s}\cdot\frac{\omega_n^2}{\left[s+\left(\xi+\sqrt{\xi^2-1}\right)\omega_n\right]\left[s+\left(\xi-\sqrt{\xi^2-1}\right)\omega_n\right]}$$

Solving By using partial fractions.

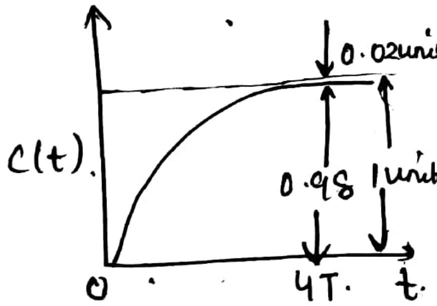
$$C(s) = \frac{1}{s} - \frac{1}{2\sqrt{\xi^2-1}\left(\xi-\sqrt{\xi^2-1}\right)\left[s+\left(\xi-\sqrt{\xi^2-1}\right)\omega_n\right]}$$

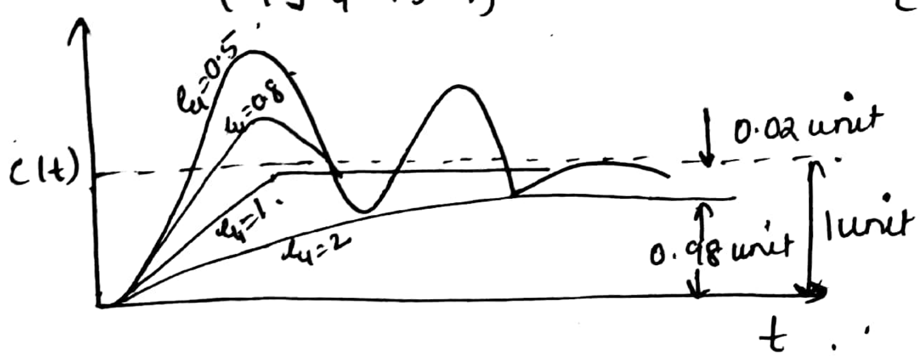
$$+ \frac{1}{2\sqrt{\xi^2-1}\left(\xi+\sqrt{\xi^2-1}\right)\left[s+\left(\xi+\sqrt{\xi^2-1}\right)\omega_n\right]}$$

Taking Inverse Laplace Transform on both sides, we get (46)

$$c(t) = 1 - \frac{e^{-(\xi_1 - \sqrt{\xi_1^2-1})\omega_n t}}{2\sqrt{\xi_1^2-1}\left(\xi_1 - \sqrt{\xi_1^2-1}\right)} + \frac{e^{-(\xi_1+\sqrt{\xi_1^2-1})\omega_n t}}{2\sqrt{\xi_1^2-1}\left(\xi_1+\sqrt{\xi_1^2-1}\right)}$$

$$T = \frac{1}{(\xi_1 - \sqrt{\xi_1^2-1})\omega_n}$$





⇒ Derivation for expression for Rise time, peak time, delay time, peak overshoot, Settling time and steady-State Error.

I) **Rise Time, t_r** – output of a 2^{nd}-order underdamped system excited by a unit-step ~~wait~~ input is given by –

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) \quad \text{—①}$$

As Rise time is the time taken by the o/p to rise from 0 to 100% of final value.

At $t = t_r$ $\quad c(t_r) = 1$

So eq^n ① becomes,

$$c(t_r) = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta)$$

$$1 = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta).$$

$$\Rightarrow \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 0.$$

As $\dfrac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \neq 0.$

$$\therefore \sin(\omega_d t_r + \theta) = 0. = \sin\pi.$$

$$\omega_d t_r + \theta = \pi \Rightarrow \omega_d t_r = \pi - \theta.$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d}} \text{ seconds.}$$

where $\theta = \tan^{-1} \dfrac{\sqrt{1-\xi^2}}{\xi}.$

For $\downarrow t_r$, $\uparrow \omega_d$.

Peak Time, t_p – As o/p of a unit step I/p is

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

As peak time is defined as the time at which the max^m value of magnitude occurs

$\therefore$ $t = t_p$, slope of $c(t)$ must be zero.

$$\left.\frac{dc(t)}{dt}\right|_{t=t_p} = -\frac{e^{-\xi\omega_n t}(-\xi\omega_n)\sin(\omega_d t+\theta)}{\sqrt{1-\xi^2}} - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\cos(\omega_d t+\theta)\cdot\omega_d$$

$$0 = \left.\frac{e^{-\xi\omega_n t}(\xi\omega_n)\sin(\omega_d t+\theta)}{\sqrt{1-\xi^2}} - \frac{\omega_d e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\cos(\omega_d t+\theta)\right|_{t=t_p}$$

$$\frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}}\left[\xi\omega_n \sin(\omega_d t_p+\theta) - \omega_d \cos(\omega_d t_p+\theta)\right] = 0.$$

$$\xi\omega_n \sin(\omega_d t_p+\theta) - \omega_d \cos(\omega_d t_p+\theta) = 0.$$

$$\cancel{\xi\omega_n \sin(\omega_d t_p+\theta) = \omega_d \cos(\omega_d t_p+\theta)}$$

$$\cancel{\tan(\omega_d t_p+\theta) = \frac{\omega_d}{\xi\omega_n} = \frac{\omega_n\sqrt{1-\xi^2}}{\xi\omega_n}}$$

$$\cancel{\omega_d t_p + \theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}$$

$$\xi\omega_n \sin(\omega_d t_p+\theta) - \omega_n\sqrt{1-\xi^2}\cos(\omega_d t_p+\theta) = 0.$$

As $\xi = \cos\theta$; $\sin\theta = \sqrt{1-\xi^2}$.

$$\omega_n\cos\theta \sin(\omega_d t_p+\theta) - \omega_n \sin\theta\cos(\omega_d t_p+\theta) = \sin\pi$$

$$\omega_n \sin(\omega_d t_p + \theta - \theta) = 0 \Rightarrow \sin(\omega_d t_p) = \sin\pi$$

$$\omega_d t_p = \pi \Rightarrow \boxed{t_p = \frac{\pi}{\omega_d}}$$

Relation Between ξ & θ.

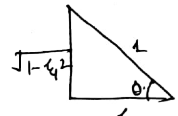


Fig- Relation B/W θ & ξ

First under shoot occurs at $t = \frac{2\pi}{\omega_d}$

Second undershoot occurs at $t = \frac{3\pi}{\omega_d}$.

and so on.

Peak Overshoot, M_p :- O/P of a system

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta)$$

Peak overshoot is the difference between the peak value & the Reference Input.

$$\therefore M_p = c(t_p) - 1 = \left[1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}}\sin(\omega_d t_p+\theta)\right] - 1$$

$$= -\frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}}\sin(\omega_d t_p+\theta)$$

Substituting $t_p = \frac{\pi}{\omega_d}$.

$$M_p = -\frac{e^{-\xi\omega_n\cdot\frac{\pi}{\omega_d}}}{\sqrt{1-\xi^2}}\sin\left(\omega_d\cdot\frac{\pi}{\omega_d}+\theta\right).$$

$$= -\frac{e^{-\frac{\xi\omega_n\cdot\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}}\sin(\pi-\theta).$$

$$M_p = -\frac{e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}(-\sin\theta)}{\sqrt{1-\xi^2}} = \frac{e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}\sin\theta}{\sqrt{1-\xi^2}}.$$

As $\sin\theta = \sqrt{1-\xi^2}$.

$$M_p = \frac{e^{-\pi\xi/\sqrt{1-\xi^2}} \times \cancel{\sqrt{1-\xi^2}}}{\cancel{\sqrt{1-\xi^2}}}$$

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$$M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}.$$

$$\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100.$$

Settling Time, t_s – The settling time is given by.

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\xi\omega_n} \quad (2\% \text{ criterion})$$

$$t_s = 3T = \cancel{4}\,\frac{3}{\sigma} = \frac{3}{\xi\omega_n} \quad (5\% \text{ criterion})$$

Steady state error, e_{ss} – O/P of a system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta).$$

$$e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{t\to\infty}{Lt}\ [r(t) - c(t)]$$

As $r(t) = 1$

$$e_{ss} = \underset{t\to\infty}{Lt}\ [1 - c(t)] = \underset{t\to\infty}{Lt}\ \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta).$$

$$e_{ss} = 0$$

For Ramp input, $r(t) = t$, $R(s) = 1/s^2$.

$$e_{ss} = \underset{t\to\infty}{Lt}\ [r(t) - c(t)].$$

$$c(t) = L^{-1}\left[\frac{\omega_n^2}{s^2(s^2 + 2\xi\omega_n s + \omega_n^2)}\right] = t - \frac{2\xi}{\omega_n} + \frac{e^{-\omega_n\xi t}}{\omega_n\sqrt{1-\xi^2}} \sin\left[\omega_n\sqrt{1-\xi^2}\, t + \theta\right]$$

$$e_{ss} = \underset{t\to\infty}{Lt}\ [r(t) - c(t)] = \frac{2\xi}{\omega_n} = \frac{1}{K_v}$$

Take Inverse Laplace transform,

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\omega_d t$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right)$$

As $\xi = \cos\theta$ & $\sqrt{1-\xi^2} = \sin\theta$.

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t\right]$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_n\sqrt{1-\xi^2} + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) \quad \text{for } t \geq 0$$

Error signal, $e(t) = r(t) - c(t)$

$$e(t) = e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right), \text{ for } t \geq 0$$

The error signal exhibits a damped sinusoidal oscillation.

At steady state

$$e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = 0$$

No error exists b/w o/p & I/P.

If $\xi = 0$, Response Becomes undamped & oscillations continue indefinitely. Response $c(t) = 1 - \cos\omega_n t$ for $t \geq 0$

$\omega_n \to$ undamped natural frequency

$\omega_d = \omega_n$ for $\xi = 0$

$\omega_d < \omega_n$ for $\xi > 0$.

Time Response of an undamped ($\xi > 1$), the second-order systemis

A ξ is increased, Response becomes progressively less oscillatory till it becomes critically damped for $\xi = 1$..

for $\xi > 1$, overdamped

$\Rightarrow$ For $0 < \xi < 1$, poles are complex conjugate pair.

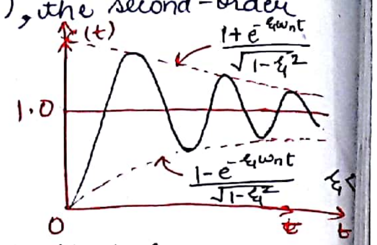
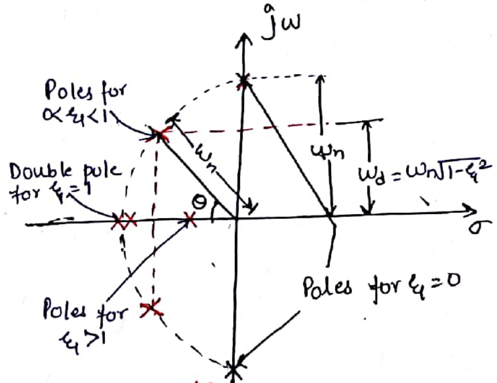


Fig - Unit-step Response of a second-order underdamped system

$$s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$
$$= -\xi\omega_n \pm j\omega_d$$



TIME RESPONSE SPECIFICATIONS

Control systems are generally designed with damping less than one, i.e. oscillator step Response.

→ High-order control system usually have a pair of complex conjugate poles with damping less than one which dominate over other poles.

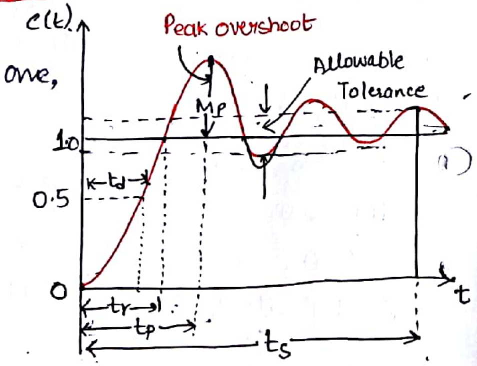


Fig - Time Response specifications

→ Therefore, the time response of second- and higher-order control systems to a step input is generally of damped oscillatory nature as shown in fig.

The transient Response of a system to a unit-step input depends on the initial conditions. (5)

1. Delay Time, t_d:- The delay time is the time required for the Response to reach 50% of the final value the very first time.

$$t_d = \frac{1+0.7\xi}{\omega_n} \text{ sec}$$

2. Rise time, t_r - It is the time required for the response to rise from 0 to 100% of the final value for underdamped systems and from 10% to 90% of the final value for overdamped systems.

$$t_r = \frac{\pi - \theta}{\omega_d}$$

where $\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$, $\omega_d = \omega_n\sqrt{1-\xi^2}$

3. Peak time, t_p:- The peak time is the time required for the response to reach the first peak of the overshoot.

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

4. Peak overshoot, M_p:- The peak or max^m overshoot is the max^m peak value of the response curve measured from unity. If steady-state value of the Response differs from unity, the it is common to use the max^m % overshoot.

$$M_p\% = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

The amount of max^m (percent) overshoot directly indicates the relative stability of the system.

5. Settling time, t_s - The settling time is the time required for the response curve to reach and steady within a particular tolerance band (usually 2% or 5% of its final value).

For 2% criterion $t_s = 4T = \frac{4}{\xi\omega_n}$

For 5% criterion $t_s = 3T = \frac{3}{\xi\omega_n}$

6. **Steady-state Error, e_{ss}** :- It indicates the error between the actual output and desired output as t tends to infinity, i.e.

$$e_{ss} = \underset{t\to\infty}{Lt}\,[r(t) - c(t)] = \underset{t\to\infty}{Lt}\, e(t)$$

Q1:- For the system with the following transfer function, determine the type and order of the system:

(a) $G(s)H(s) = \frac{K(s+3)}{s^2(s+2)(s+5)}$

No of poles at origin decide the type of system & highest degree decides the order of system.

No of Poles at origin $= 2$

Degree $= 4$. ; order $= 4$.

(b) $G(s)H(s) = \frac{K}{s(s+1)(s^2+2s+3)}$

Type $= 1$ as one pole at origin

Order $= 4$ highest degree

(c) $G(s)H(s) = \frac{s+2}{(s-3)(s+0.1)}$

Type $= 0$ no pole at origin

Order $= 2$ highest degree

(d) $G(s)H(s) = \frac{20}{s^3(s^4+8s^2+16)}$

Type $= 3$; Order $= 7$

Q:- Determine the type of damping in systems.

a) $\frac{C(s)}{R(s)} = \frac{8}{s^2+3s+8}$ Characteristic equation $s^2+3s+8 = 0$.

Comparing with standard equation

$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

$\omega_n = \sqrt{8} = 2.82$

$2\xi\omega_n = 3 \Rightarrow 2\xi \times 2.82 = 3 \Rightarrow \xi = 0.53$

Underdamped system.

b) $\frac{C(s)}{R(s)} = \frac{2}{s^2+4s+2}$.

- Characteristic Equation is $s^2+4s+2 = 0$

$\omega_n = \sqrt{2} = 1.414$. $\Rightarrow 2\xi\omega_n = 4$

$2\xi \times 1.414 = 4$

$\Rightarrow \xi = 1.41$

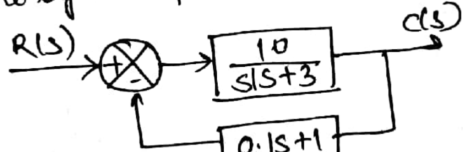
$\xi > 1$; overdamped system

(c) $\frac{C(s)}{R(s)} = \frac{2}{s^2+4}$ Characteristic equation is $s^2+4=0$

$\omega_n = 2$, $\xi = 0$

- System is undamped.

Q. A position control system with velocity feedback is shown in fig. What is the response of the system for a unit-step Input?



Sol $\frac{C(s)}{R(s)} = \frac{\frac{10}{s(s+3)}}{1 + \frac{10(0.1s+1)}{s(s+3)}}$

$\frac{C(s)}{R(s)} = \frac{10}{s^2+4s+10}$. $\Rightarrow C(s) = \frac{10R(s)}{s^2+4s+10}$

As R(s) is unit step I/P.

$$C(s) = \frac{10}{S(S^2+4S+10)} \approx \frac{10}{S}$$

Using partial fraction method

$$C(s) = \frac{A}{S} + \frac{Bs+C}{S^2+4S+10}.$$

$A = 1$, $B = -1$, $C = -4$

$$C(s) = \frac{1}{S} + \frac{(-S-4)}{S^2+4S+10} = \frac{1}{S} - \frac{(S+4)}{S^2+4S+10}$$

$$= \frac{1}{S} - \frac{S+4}{(S+2)^2+(\sqrt{6})^2} = \frac{1}{S} - \frac{S+2}{(S+2)^2+(\sqrt{6})^2} - \frac{2}{(S+2)^2+(\sqrt{6})^2}$$

$$C(s) = \frac{1}{S} - \frac{S+2}{(S+2)^2+(\sqrt{6})^2} - \frac{2}{\sqrt{6}}\frac{\sqrt{6}}{(S+2)^2+(\sqrt{6})^2}.$$

Taking Inverse Laplace Transform,

$$c(t) = 1 - e^{-2t}\sin\sqrt{6}t - \frac{2}{\sqrt{6}}e^{-2t}\cos\sqrt{6}t$$

Q: The open loop transfer function of a unity feedback is

$$G(s) = \frac{4}{S(S+1)}$$

Determine the nature of Response of the closed-loop system for unit-step input. Also determine the rise time, peak time, overshoot & settling time.

Sol:

$$\frac{C(s)}{R(s)} = \frac{4}{S^2+S+4}.$$

Characteristic equation is $S^2+S+4=0$

$\omega_n = 2$, $2\xi\omega_n = 1 \Rightarrow 2\xi \times 2 = 1 \Rightarrow \xi = 0.25$

$\xi < 1$, underdamped system

$\omega_d = \omega_n\sqrt{1-\xi^2} = 2\sqrt{1-(.25)^2} = 1.936$ rad/sec

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\frac{\sqrt{1-(.25)^2}}{.25} = 1.310 \text{ rad}.$$

Rise time, $t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi - 1.310}{1.936} = 0.945$ s

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{1.936} = 1.622$ s.

Delay time, $t_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7\times0.25}{2} = 05875$ sec

Peak overshoot, $M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.4326$

$\% M_p = 43.26\ \%$.

Settling time, t_s (5% error) $t_s = \frac{3}{\xi\omega_n} = 6$s.

(2% error), $t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.25\times2} = 8$ s.

Q: Measurements conducted on a Servo mechanism shows the system response to be $c(t) = 1+0.2e^{-60t} - 1.2e^{-10t}$ when subjected to a unit-step Input.

a) Obtain expression for closed loop transfer function.

b) Determine the undamped natural frequency & damping Ratio of the System.

Sol: $c(t) = 1+0.2e^{-60t} - 1.2e^{-10t}$

Take Laplace Transform,

$$C(s) = \frac{1}{S} + \frac{0.2}{S+60} - \frac{1.2}{S+10} = \frac{(S+60)(S+10)+0.2S(S+10)-12S(S+60)}{S(S+10)(S+60)}.$$

$$C(s) = \frac{600}{S(S^2+70S+600)} = \frac{600}{S(S+60)(S+10)}.$$

$$C(s) = \frac{600\cdot\frac{1}{S}}{(S+60)(S+10)}$$

$$\frac{C(s)}{R(s)} = \frac{600}{(S+10)(S+60)}$$

b) Calculation of ξ & ω_n.

$$\frac{C(s)}{R(s)} = \frac{600}{s^2+70s+600}$$

Characteristic equation is $s^2+70s+600=0$.

$$\omega_n = \sqrt{600} = 24.49 \text{ rad/s}.$$

$$2\xi\omega_n = 70 \Rightarrow 2\xi \times 24.49 = 70,$$

$$\Rightarrow \boxed{\xi = 1.43}$$

system is overdamped as $\xi > 1$

Q:- A unity feedback system is characterized by an open-loop transfer function $G(s) = \frac{K}{s(s+10)}$. Determine K so that system will have $\xi = 0.5$. For value of K, find t_s, M_p, time to peak overshoot for unit step I/P.

Sol.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K}{s^2+10s+K},$$

Characteristic equation is $s^2+10s+K=0$.

$2\xi\omega_n = 10$ As $\xi = 0.5$, $\omega_n = \sqrt{K}$.

$$2 \times 0.5\sqrt{K} = 10 \Rightarrow \sqrt{K} = 10 \Rightarrow \boxed{K = 100}.$$

$$\omega_n = \sqrt{100} \Rightarrow \boxed{\omega_n = 10 \text{ rad/sec}}$$

$$t_s \text{ for } (2\% \text{ criterion}) = \frac{4}{\xi\omega_n} = \frac{4}{0.5 \times 10} = 0.8s$$

$$t_s \text{ for } (5\% \text{ criterion}) = \frac{3}{\xi\omega_n} = \frac{3}{0.5 \times 10} = 0.6s.$$

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = e^{-\frac{\pi \times 0.5}{\sqrt{1-.25}}} \times 100 = 16.3\%.$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{0.5\sqrt{1-.25}} = 0.363s$$

Q:- A unity feedback system is characterized by an open-loop transfer function $G(s) = \frac{K}{s(s+10)}$. Determine the gain K so that the system will have a damping ratio of 0.5. For this value of K, determine the settling time, peak overshoot & time to peak overshoot for a unit-step input.

Sol

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \qquad H(s) = 1.$$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K}{s^2+10s+K}.$$

Characteristic equation is $s^2+10s+K=0$

$\omega_n = \sqrt{K}$, $\xi = 0.5$ $\quad 2\xi\omega_n = 10.$

$2 \times 0.5\sqrt{K} = 10.$

$\sqrt{K} = 10 \Rightarrow \boxed{K = 100}.$

$\boxed{\omega_n = 10}$

$$\frac{C(s)}{R(s)} = \frac{100}{s^2+10s+100}$$

settling time, $t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.5 \times 10} = 0.8s.$ (2%)

For 5%, $t_s = \frac{3}{\xi\omega_n} = \frac{3}{0.5 \times 10} = 0.6s.$

Peak overshoot, $M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = e^{\frac{-\pi \times 0.5}{\sqrt{1-0.25}}} = 0.163$

$\% M_p = 16.3\%.$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{3.14}{10 \times \sqrt{1-.25}} = 0.363s$$

Q. The open loop transfer function of unity feedback control system is given by $G(s) = \frac{K}{s(1+sT)}$.

a) By what factor, the amplifier gain K should be multiplied so that damping ratio increased from 0.2 to 0.8?

b) By what factor, the time constant, T should be multiplied so that the damping ratio is reduced from 0.9 to 0.3?

Sol: $\frac{C(s)}{R(s)} = \frac{K/T}{s^2 + \frac{s}{T} + \frac{K}{T}}$

Characteristic equation is $s^2 + \frac{s}{T} + \frac{K}{T} = 0$

$2\xi\omega_n = \frac{1}{T}$, $\omega_n = \sqrt{\frac{K}{T}}$

$2\xi\sqrt{\frac{K}{T}} = \frac{1}{T} \Rightarrow \xi = \frac{1}{2\sqrt{KT}}$.

a) when $\xi_1 = 0.2$; K_1 ; $\xi_2 = 0.8$; K_2

$$\frac{\xi_1}{\xi_2} = \frac{\frac{1}{2\sqrt{K_1 T}}}{\frac{1}{2\sqrt{K_2 T}}} = \sqrt{\frac{K_2}{K_1}} \Rightarrow \frac{K_2}{K_1} = \left(\frac{\xi_1}{\xi_2}\right)^2$$

$$\frac{K_2}{K_1} = \left(\frac{0.2}{0.8}\right)^2 = \frac{1}{16} \Rightarrow K_2 = \frac{1}{16}K_1$$

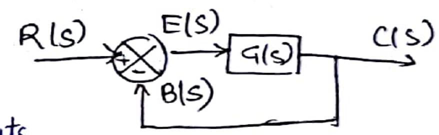
K_1 at $\xi_1 = 0.2$ should be multiplied by 1/16 to increase damping from 0.2 to 0.8.

(b) when $\xi_1 = 0.9$; T_1 ; $\xi_2 = 0.3$, T_2

$$\frac{T_2}{T_1} = \left(\frac{\xi_1}{\xi_2}\right)^2 \Rightarrow T_2 = \left(\frac{0.9}{0.3}\right)^2 T_1 \Rightarrow T_2 = 9T_1$$

T_1 should be multiplied by 9 to reduce the damping ratio from 0.9 to 0.3

STEADY-STATE ERRORS AND ERROR CONSTANTS



Steady-state error is a measure of system accuracy. These errors arise from the nature of the inputs, type of system and from non-linearities of system components such as static friction etc. These generally due to amplifier drifts, ageing or deterioration.

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Error signal is $E(s) = \frac{C(s)}{G(s)}$. $C(s) = E(s).G(s)$.

$$\therefore E(s) = \frac{C(s)}{G(s)}$$

$$E(s) = \frac{\cancel{G(s)}.R(s)}{\cancel{G(s)}(1+G(s))} = \frac{R(s)}{1+G(s)}.$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$$

steady-state error $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$

i) Static Position Error Constant, K_p

The steady-state error of the system for a unit-step input $[r(t) = 1 ; R(s) = 1/s^2]$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s.\frac{R(s)}{1+G(s)} = \lim_{s\to 0} \frac{s.\frac{1}{s}}{1+G(s)}.$$

$e_{ss} = \frac{1}{\ldots} = \frac{1}{\ldots}$

$\therefore K_p = \underset{s\to 0}{Lt}\ G(s)$ is defined as position error constant.

2) <u>Static Velocity Error Constant, K_v</u>

The steady-state of the system for a unit ramp input

$r(t) = t,\quad R(s) = \frac{1}{s^2}$

$$e_{ss} = \underset{s\to 0}{Lt}\ sE(s) = \underset{s\to 0}{Lt}\ \frac{s.R(s)}{1+G(s)} = \underset{s\to 0}{Lt}\ \frac{s.\,{}^{1}/_{s^2}}{1+G(s)}$$

$$= \underset{s\to 0}{Lt}\ \frac{1}{s+sG(s)} = \frac{1}{\underset{s\to 0}{Lt}\ sG(s)}$$

$$K_v = \underset{s\to 0}{Lt}\ sG(s)$$

$$\boxed{e_{ss} = \frac{1}{K_v}}$$

3) <u>Static Acceleration Error Constant, K_a</u>

Steady state error for a unit parabolic input

$r(t) = \frac{t^2}{2};\quad R(s) = \frac{1}{s^3}$

$$e_{ss} = \underset{s\to 0}{Lt}\ sE(s) = \underset{s\to 0}{Lt}\ \frac{sR(s)}{1+G(s)} = \underset{s\to 0}{Lt}\ \frac{s.\,{}^{1}/_{s^3}}{1+G(s)}$$

$$= \underset{s\to 0}{Lt}\ \frac{1}{s^2+s^2G(s)} = \frac{1}{\underset{s\to 0}{Lt}\ s^2G(s)}$$

$$K_a = \underset{s\to 0}{Lt}\ s^2G(s)$$

$$\boxed{e_{ss} = \frac{1}{K_a}}$$

Q: For control system with open-loop transfer functions given below, explain what type of input signal gives rise to a constant steady-state error & calculate their values. (10)

(a) $G(s) = \frac{20}{(s+1)(s+4)}$

$K_p = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \frac{20}{(s+1)(s+4)} = \frac{20}{4} = 5$ $\qquad e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+5} = \frac{1}{6}$

$K_v = \underset{s\to 0}{Lt}\ sG(s) = \underset{s\to 0}{Lt}\ s.\frac{20}{(s+1)(s+4)} = 0$

$K_a = \underset{s\to 0}{Lt}\ s^2G(s) = \underset{s\to 0}{Lt}\ s^2.\frac{20}{(s+1)(s+4)} = 0$

(b) $G(s) = \frac{10(s+4)}{s(s+1)(s+2)}$

$K_p = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \frac{10(s+4)}{s(s+1)(s+2)} = \cancel{10\times 4}\ \infty$

$K_v = \underset{s\to 0}{Lt}\ sG(s) = \underset{s\to 0}{Lt}\ \frac{\cancel{s}10(s+4)}{\cancel{s}(s+1)(s+2)} = \frac{10\times 4}{1\times 2} = 20$

$K_a = \underset{s\to 0}{Lt}\ s^2G(s) = \underset{s\to 0}{Lt}\ \frac{s^{\cancel{2}}\times 10(s+4)}{\cancel{s}(s+1)(s+2)} = 0$

$e_{ss} = \frac{1}{K_v} = \frac{1}{20} = 0.05$

(c) $G(s) = \frac{20}{s^2(s+1)(s+4)}$

$K_p = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \frac{20}{s^2(s+1)(s+4)} = \infty$

$K_v = \underset{s\to 0}{Lt}\ sG(s) = \underset{s\to 0}{Lt}\ \frac{20s}{s^2(s+1)(s+4)} = \infty$

$K_a = \underset{s\to 0}{Lt}\ s^2G(s) = \underset{s\to 0}{Lt}\ \frac{20\cancel{s^2}}{\cancel{s^2}(s+1)(s+4)} = 5$

$e_{ss} = \frac{1}{K_a} = \frac{1}{5} = 0.2$

Types of control system

Open-loop transfer function can be written in two forms:-

1) Pole-zero form.

$$G(s) = \frac{K(s+z_1)(s+z_2)----}{s^n(s+P_1)(s+P_2)----}$$

2) Time-constant form

$$G(s) = \frac{K(1+T_{z_1}s)(1+T_{z_2}s)---}{s^n(1+T_{P_1}s)(1+T_{P_2}s)---}$$

I) Stedy-State Error : Type-0 system

For type 0 system – no pole at origin

$$G(s) = \frac{K(1+T_{z_1}s)(1+T_{z_2}s)---}{(1+T_{P_1}s)(1+T_{P_2}s)----}$$

$$K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{K(1+T_{z_1}s)(1+T_{z_2}s)---}{(1+T_{P_1}s)(1+T_{P_2}s)---} = K.$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{K+1} = \text{finite value}.$$

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{s\cdot K(1+T_{z_1}s)(1+T_{z_2}s)--}{(1+T_{P_1}s)(1+T_{P_2}s)---}$$

$$K_v = 0 \quad , e_{ss}(t) = \frac{1}{K_v} = \frac{1}{0} = \infty$$

$$K_a = \lim_{s\to 0} s^2G(s) = \lim_{s\to 0} \frac{s^2\cdot K(1+T_{z_1}s)(1+T_{z_2}s)---}{(1+T_{P_1}s)(1+T_{P_2}s)----}$$

$$K_a = 0$$

$$e_{ss}(t) = \frac{1}{K_a} = \frac{1}{0} = \infty$$

II) Steady-State Error : Type-1 System

For type-1 system – 1 pole at origin

$$G(s) = \frac{K(1+T_{z_1}s)(1+T_{z_2}s)---}{s(1+T_{P_1}s)(1+T_{P_2}s)---}$$

$$K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{K(1+T_{z_1}s)(1+T_{z_2}s)---}{s(1+T_{P_1}s)(1+T_{P_2}s)---} = \infty.$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{\cancel{s}\cdot K(1+T_{z_1}s)(1+T_{z_2}s)---}{\cancel{s}(1+T_{P_1}s)(1+T_{P_2}s)---}$$

$$K_v = K$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K} = \text{finite value}.$$

$$K_a = \lim_{s\to 0} s^2G(s) = \lim_{s\to 0} \frac{s^{\cancel{2}}K(1+T_{z_1}s)(1+T_{z_2}s)---}{\cancel{s}(1+T_{P_1}s)(1+T_{P_2}s)---} = 0.$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty.$$

III) Steady-Steady Error : Type-2 system

For type-2 system – 2 poles at origin

$$G(s) = \frac{K(1+T_{z_1}s)(1+T_{z_2}s)---}{s^2(1+T_{P_1}s)(1+T_{P_2}s)---}$$

$$K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{K(1+T_{z_1}s)(1+T_{z_2}s)---}{s^2(1+T_{P_1}s)(1+T_{P_2}s)---}$$

$$K_p = \infty.$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0.$$

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{\cancel{s}\cdot K(1+T_{z_1}s)(1+T_{z_2}s)---}{s^{\cancel{2}}(1+T_{P_1}s)(1+T_{P_2}s)---}.$$

$$K_v = \infty$$

$$e_{ss} = \frac{1}{K_v} = 0$$

$$K_a = \lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} \frac{\cancel{s^2}\, K(1+T_{z_1}s)(1+T_{z_2}s)\ldots}{\cancel{s^2}(1+T_{P_1}s)(1+T_{P_2}s)\ldots} = K$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{K} = \text{finite value}$$

⇒ Response with P, PI, PD and PID controllers

I) Proportional Control

In P~~I~~ control, the actuating Signal, $E_a(s)$ is proportional to error signal $E(s)$.

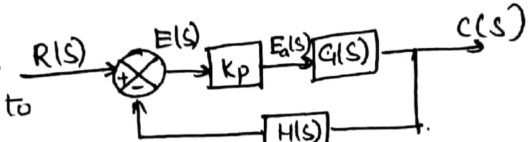


Fig- Block diagram of proportional control system

→ For quick Response, it is necessary that control system should be underdamped.

→ underdamped system has exponentially decaying oscillations in the o/p time Response during transient period.

→ overdamped system has sluggish response which can be made faster by increasing the forward path gain of the system.

→ ↑ in forward path gain, ↓ Steady-state error, but ↑ in peak overshoot

→ For a satisfactory performance of a control system, a convenient adjustment has to be made b/w max^m overshoot & steady state error.

This can be done ~~by~~ without scarifying the steady-state accuracy, max^m overshoot by modifying the actual signal

$$\frac{E_a(s)}{E(s)} = K_p$$

K_p = Proportional gain

Derivative control- The actuating signal consists of proportional error signal & derivative of error signal i.e.

$$e_a(t) = e(t) + T_d \frac{d}{dt} e(t).$$

Taking Laplace transform, we get

$$E_a(s) = E(s) + T_d\, s\, E(s) = (1+sT_d)\, E(s).$$

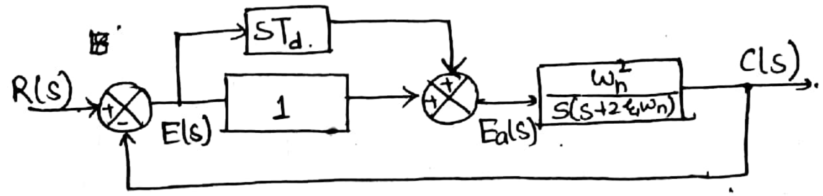


Fig- Block diagram of a control system with derivative control

Overall transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{(1+sT_d)\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}{1+\dfrac{\omega_n^2(1+sT_d)}{s(s+2\xi\omega_n)}}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2(1+sT_d)}{s^2+2\xi\omega_n s+\omega_n^2+\omega_n^2 T_d s}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2(1+sT_d)}{s^2+(2\xi\omega_n+\omega_n^2 T_d)s+\omega_n^2}$$

Characteristic equation is

$$s^2+(2\xi\omega_n+\omega_n^2 T_d)s+\omega_n^2 = 0. \quad \text{—(1)}$$

ξ' is effective damping Ratio of second-order system with derivative control, its characteristic equation is

$$s^2+2\xi'\omega_n s+\omega_n^2 = 0 \quad \text{—(2)}$$

Comparing eq's (1) & (2)

$$2\xi'\omega_n = 2\xi\omega_n + \omega_n^2 T_d$$

$$\xi_1' = \frac{2\xi_1\omega_n + \omega_n^2 T_d}{2\omega_n} = \xi_1 + \frac{T_d\omega_n}{2}$$

Now damping Ratio is increased using derivative control, but max^m overshoot is reduced. ∴, transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2 T_d\left(s+\frac{1}{T_d}\right)}{s^2+2\xi_1'\omega_n s+\omega_n^2}$$

Error function is given by

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} = \frac{1}{1+\frac{\omega_n^2(1+sT_d)}{s(s+2\xi_1\omega_n)}}$$

$$= \frac{s(s+2\xi_1\omega_n)}{s^2+(2\xi_1\omega_n+\omega_n^2T_d)s+\omega_n^2}$$

For Ramp input, $r(t)=t$

$R(s)=\frac{1}{s^2}$

$$E(s) = \frac{s(s+2\xi_1\omega_n)\cdot\frac{1}{s^2}}{s^2+(2\xi_1\omega_n+\omega_n^2T_d)s+\omega_n^2}$$

steady-state error

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s^2(s+2\xi_1\omega_n)\cdot\frac{1}{s^2}}{s^2+(2\xi_1\omega_n+\omega_n^2T_d)s+\omega_n^2}$$

$$e_{ss} = \frac{2\xi_1\omega_n}{\omega_n^2} = \frac{2\xi_1}{\omega_n}$$

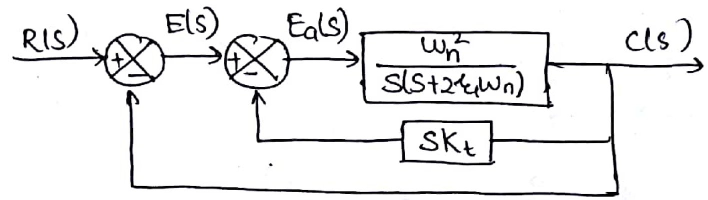
Comparing with steady-state error of a 2nd order system without derivative control, both values are same.
∴ steady-state is not affected by derivative control.

→ Comparison of transfer functions of second-order system with & without derivative control, ω_n is unchanged but zero has been added at $s=-1/T_d$, results t_r is reduced.

<u>Derivative Feedback Control</u> – The derivative feedback control or rate feedback control is known as tachometer feedback control.

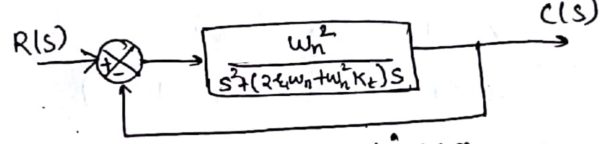
→ Actuating signal is the difference of the proportional error signal & the derivative of output signal.

Actuating signal is $e_a(t) = e(t) - K_t\frac{dc(t)}{dt}$



Taking Laplace transform, we get

$E_a(s) = E(s) - K_t s C(s)$



<u>Simplified Block diagram</u>

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+(2\xi_1\omega_n+\omega_n^2K_t)s+\omega_n^2}$$

Characteristic equation is

$$s^2+(2\xi_1\omega_n+\omega_n^2K_t)s+\omega_n^2=0.$$

ξ_1' is damping Ratio of the system with derivative feedback control, its characteristic equation is

$$s^2+2\xi_1'\omega_n s+\omega_n^2=0.$$

$$2\xi_1'\omega_n = 2\xi_1\omega_n+\omega_n^2K_t$$

$$\xi_1' = \xi_1+\frac{\omega_n^2K_t}{2\xi_1}$$

This Shows that by using derivative feedback control damping Ratio increases, & max^n overshoot is Reduced.

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} = \frac{1}{1+\frac{\omega_n^2}{s^2+(2\xi\omega_n+\omega_n^2K_t)s}}$$

$$= \frac{s^2+(2\xi\omega_n+\omega_n^2K_t)s}{s^2+(2\xi\omega_n+\omega_n^2K_t)s+\omega_n^2}$$

For Ramp input $r(t)=t$, $R(s)=\frac{1}{s^2}$

$$\therefore \frac{E(s)}{R(s)} = \frac{\frac{1}{s^{\cancel{2}}}\cdot\cancel{s}\,[s+(2\xi\omega_n+\omega_n^2K_t)]}{s^2+(2\xi\omega_n+\omega_n^2K_t)s+\omega_n^2}$$

$$E(s) = \frac{(s+(2\xi\omega_n+\omega_n^2K_t))}{s(s^2+(2\xi\omega_n+\omega_n^2K_t)s+\omega_n^2)}.$$

Steady state error

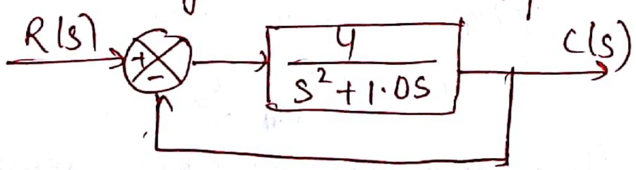
$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{\cancel{s}\,(s+(2\xi\omega_n+\omega_n^2K_t))}{\cancel{s}\,(s^2+(2\xi\omega_n+\omega_n^2K_t)s+\omega_n^2)}$$

$$e_{ss} = \frac{2\xi\omega_n+\omega_n^2K_t}{\omega_n^2} = \frac{2\xi}{\omega_n}+K_t.$$

$\therefore$ This shows by using derivative feedback control the steady-state error is increased.

Q:- A closed loop control system with unity feedback is shown in figure. By using derivative control, the damping Ratio is to be made 0.75. Determine the value of T_d. Also determine the rise time, peak time and peak overshoot without derivative control and with derivative control. The input to the system is a unit-step.

Sol.



Sol.
$$\frac{C(s)}{R(s)} = \frac{4}{s^2+1.0s+4} = \frac{4}{s^2+s+4}.$$

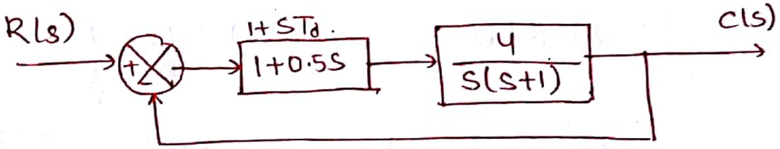
Characteristic equation is $s^2+s+4=0$.

$\omega_n = 2$ rad/sec $2\xi\omega_n=1 \Rightarrow \xi = \frac{1}{2\times 2} = 0.25$

Damping Ratio with derivative feedb control

$$\xi' = \xi + \frac{\omega_n T_d}{2}.$$

$$0.75 = 0.25 + \frac{\cancel{2}\times T_d}{\cancel{2}} \Rightarrow T_d = 0.5$$



$$\frac{C(s)}{R(s)} = \frac{\frac{4(1+0.5s)}{s(s+1)}}{1+\frac{4(1+0.5s)}{s(s+1)}} = \frac{4(1+0.5s)}{s^2+3s+4}.$$

$$\frac{C(s)}{R(s)} = \frac{2(s+2)}{s^2+3s+4}.$$

Without derivative control.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2+s+4}.$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = 1.3181 \qquad \omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\omega_d = 2\sqrt{1-(.25)^2} = 1.936.$$

$$t_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7(0.25)}{2} = 0.5875\,s.$$

$$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.3181}{1.936} = .942s$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{1.936} = 1.622s.$$

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = 0.4443.$$

$$M_p\% = 44.43\%$$

With derivative feedback control

$$\frac{C(s)}{R(s)} = \frac{2(s+2)}{s^2+3s+4}.$$

$$C(s) = \frac{2(s+2)R(s)}{s^2+3s+4}.$$ $r(t) = u(t) = 1$, $R(s) = \frac{1}{s}$.

$$C(s) = \frac{2(s+2)}{s(s^2+3s+4)} = \frac{A}{s} + \frac{Bs+c}{s^2+3s+4}.$$

$$A(s^2+3s+4) + (Bs+c)s = 2(s+2).$$

For $s=0$; $\boxed{A=1}$

Comparing the coeff. of s^2 terms

$A+B=0 \Rightarrow B=-A \Rightarrow \boxed{B=-1}$.

Comparing the coeff of s terms.

$3A + C = 2 \Rightarrow 3+C=2 \Rightarrow \boxed{C=-1}$

$$C(s) = \frac{1}{s} - \frac{s+1}{s^2+3s+4} = \frac{1}{s} - \frac{s+1}{(s+1.5)^2+(\sqrt{1.75})^2}.$$

$$= \frac{1}{s} - \frac{s+1.5-0.5}{(s+1.5)^2+1.32^2} = \frac{1}{s} - \frac{s+1.5}{(s+1.5)^2+(1.32)^2} + \frac{0.5}{1.32}\frac{1.32}{(s+1.5)^2+(1.32)^2}$$

Taking inverse laplace transform we get

$$c(t) = 1 - e^{-1.5t}\cos 1.32t + 0.378\, e^{-1.5t}\sin 1.32t$$

For rise time, $t = t_r$

$c(t) = 1 \Rightarrow c(t_r) = 1.$

$$c(t_r) = 1 - e^{-1.5t_r}(\cos 1.32t_r - 0.378\sin 1.32t_r)$$

$$\cancel{1} = \cancel{1} - e^{-1.5t_r}(\cos 1.32t_r - 0.378\sin 1.32t_r).$$

$$e^{-1.5t_r} \neq 0.$$

$$\cos 1.32t_r = 0.378\sin 1.32t_r.$$

~~tan 1.32tr = 0.378~~.

$$\frac{\sin 1.32t_r}{\cos 1.32t_r} = \frac{1}{0.378} \Rightarrow \tan 1.32t_r = 2.645$$

$$1.32t_r = \tan^{-1}(2.645)$$

$$t_r = \frac{\tan^{-1}(2.645)}{1.32} = .916s$$

For peak time $t = t_p$; $\frac{d}{dt}c(t_p) = 0$.

$$\frac{dc(t)}{dt} = (-1.5)(e^{-1.5t})(\cos 1.32t_p - 0.378\sin 1.32t).$$
$$- e^{-1.5t}((-\sin 1.32t)1.32 - 0.378 \times 1.32\cos 1.32t) = 0$$

$e^{-1.5t}$ { ~~cos 1.32t − 0.378~~ .

$$-e^{-1.5t}[-1.5\cos 1.32t_p + 0.378 \times 1.5\sin 1.32t_p - \sin 1.32\sin 1.3$$
$$- .498\cos 1.32t_p] = 0.$$

~~1.998 cos 1.32t~~ −

$$-1.5\cos 1.32t_p + 0.567\sin 1.32t_p - 1.32\sin 1.32t_p.$$
$$- .498\cos 1.32t_p = 0$$

$$-1.998\cos 1.32t_p - .753\sin 1.32t_p.$$

$$1.998\cos 1.32t_p = 0.753\sin 1.32t_p.$$

$$-2.653 = \tan 1.32t_p.$$

$$1.32t_p = \tan^{-1}(-2.653) = \pi - \tan^{-1}(2.653)$$

$$t_p = \frac{\pi - \tan^{-1}(2.653)}{1.32} = 1.46s$$

For Maximum overshoot.

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

$$c(t_p = 1.46) = 1 - e^{-1.5 \times 1.46}(\cos 1.32 \times 1.46 - 0.378 \sin 1.32 \times 1.46)$$

$$= 1 - e^{-2.19}(\cos 1.927 - 0.378 \sin 1.927).$$

$$\cancel{= 1 - 0.49 (\cos 11}$$

$$c(t_p) = 1 - .112(-.348 - .354) = 1.08$$

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} = \frac{1.08 - 1}{1} = 0.08$$

$M_p\% = 8\%$.

⇒ INTEGRAL CONTROL

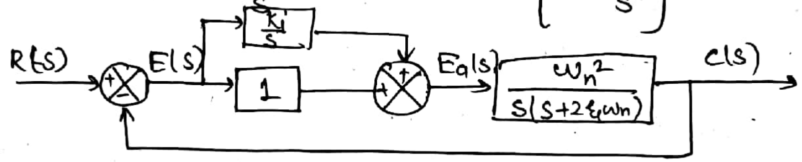
For Integral control action, the actuating signal consists of proportional error signal added with integral error signal. i.e.

$$e_a(t) = e(t) + k_i \int e(t)$$

where k_i is constant

Taking Laplace transform, we get

$$E_a(s) = E(s) + \frac{k_i}{s} E(s) = E(s)\left[1 + \frac{k_i}{s}\right]$$



$$\frac{C(s)}{R(s)} = \frac{\left(1+\frac{k_i}{s}\right)\left(\frac{\omega_n^2}{s^2+2\xi\omega_n s}\right)}{1 + \left(1+\frac{k_i}{s}\right)\left(\frac{\omega_n^2}{s^2+2\xi\omega_n s}\right)}.$$

$$\frac{C(s)}{R(s)} = \frac{(s+k_i)\omega_n^2}{s^3 + 2\xi\omega_n s + \omega_n^2 s + k_i\omega_n^2}$$

Characteristic equation is $s^3 + 2\xi\omega_n s^2 + \omega_n^2 s + k_i\omega_n^2 = 0$. is of third order

$$G(s) = \frac{(s+k_i)\omega_n^2}{s^2(s+2\xi\omega_n)}. \qquad H(s) = 1.$$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} = \frac{1}{1 + \frac{(s+k_i)\omega_n^2}{s^2(s+2\xi\omega_n)}}$$

$$\frac{E(s)}{R(s)} = \frac{s^2(s+2\xi\omega_n)}{s^2 + 2\xi\omega_n s^2 + \omega_n^2 s + k_i\omega_n^2}$$

$$E(s) = \frac{s^2(s+2\xi\omega_n) \cdot R(s)}{s^2 + 2\xi\omega_n s^2 + \omega_n^2 s + k_i\omega_n^2}$$

- For Ramp I/P, $r(t) = t$, $R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \cdot \frac{1}{\cancel{s^2}} \frac{\cancel{s^2}(s+2\xi\omega_n)}{\cancel{s^2}+2\xi\omega_n s^2 + \omega_n^2 s + k_i\omega_n^2}$$

$e_{ss} = 0$.

For unit parabolic I/p $r(t) = \frac{t^2}{2}$, $R(s) = \frac{1}{s^3}$

- $$e_{ss} = \lim_{s \to 0} s \cdot \frac{1}{s^3} \frac{s^2(s+2\xi\omega_n)}{s^2 + 2\xi\omega_n s^2 + \omega_n^2 s + k_i\omega_n^2}$$

$$e_{ss} = \frac{2\xi\omega_n}{k_i\omega_n^2} = \frac{2\xi}{k_i\omega_n}$$

⇒ Proportional-plus-Integral Plus Derivative Control (PID Control)

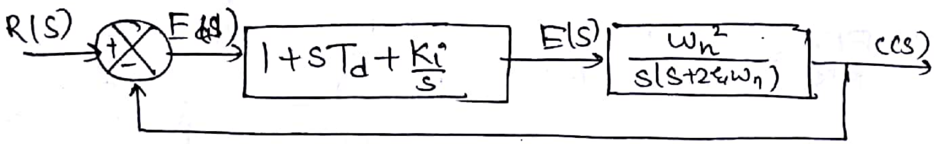
Actuating signal consists of proportional error signal added with integral and derivative of error signal.

$$e_a(t) = e(t) + T_d \frac{de(t)}{dt} + k_i \int e(t)\,dt$$

Taking Laplace Transform, we get

$$E_a(s) = E(s) + sT_d E(s) + \frac{K_i}{s} E(s)$$

$$E_a(s) = E(s)\left[1 + sT_d + \frac{K_i}{s}\right]$$

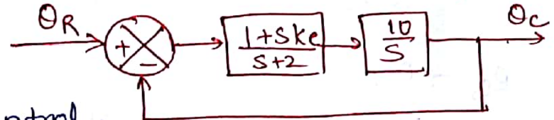


90% control are PI type.

Q: The control system shown in figure employs proportional plus error rate control. Determine the value of error Rate constant K_e so that the damping Ratio is 0.6.

a) Determine the value of settling time and maximum overshoot. Find the steady-state error if the input is a unit-Ramp.

(b) What will be those values without error Rate control?



Sol. (a) With error Rate Control

$$G(s) = \frac{10(1+sK_e)}{s(s+2)}.$$

$$\frac{C(s)}{R(s)} = \frac{10(1+sK_e)}{s^2 + 2s(2+10K_e)+10}$$

Characteristic equation is $s^2 + (2+10K_e)s + 10 = 0$.

$\omega_n = \sqrt{10} = 3.16$ rad/s.

$10K_e + 2 = 2\xi \times 3.16$, $\Rightarrow \xi = \frac{10K_e+2}{6.32} = \frac{5K_e+1}{3.16}$

$10K_e + 2 = 2\xi\omega_n$. $\Rightarrow K_e = \frac{2\xi\omega_n - 2}{10} = \frac{\xi\omega_n - 1}{5}$.

$$K_e = \frac{2 \times 0.6 \times 3.16 - 2}{10} = 0.18$$

settling time, $t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.6 \times 3.16} = 2.11\,s$.

$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = e^{\frac{-\pi \times 0.6}{\sqrt{1-(0.6)^2}}} = 0.0949$

$M_p\% = 9.49\%$.

$e_{ss} = \frac{r(t)}{K_v} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{\lim_{s\to 0} \cancel{s}\frac{10(1+s\times 0.18)}{\cancel{s}(s+2)}}$

$e_{ss} = \frac{2}{10} = \frac{1}{5} = 0.2$ rad

(b) Without error Rate Error

When $K_e = 0$ $\quad G(s) = \frac{10}{s(s+2)}$.

$\frac{C(s)}{R(s)} = \frac{10}{s^2+2s+10}$ $\quad \omega_n = \sqrt{10} = 3.16$ rad/sec.

$\cancel{2}\xi\omega_n = \cancel{2} \Rightarrow \xi = \frac{1}{3.16} = 0.32$

$t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.32 \times 3.16} = 4\,s$

$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = e^{\frac{-\pi \times 0.32}{\sqrt{1-(0.32)^2}}} = 0.351$

$M_p\% = 35.1\%$.

$e_{ss}(t) = \frac{1}{K_v} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{\lim_{s\to 0} \cancel{s}\frac{10}{\cancel{s}(s+2)}} = \frac{1}{5} = 0.2$ rad