Communication Systems (ETEC-208) – Credits – 04
Faculty – Dr. Puneet Azad

Course Description

Introduction; Introduction of random variables; Amplitude Modulations; Angle Modulation systems; Noise Theory; Performance of Communication systems

Pre-requisite

1. Signals – types & properties
2. Fourier Series and transforms
3. Linear systems theory
4. Probability

A brief description and revision will be given on Pre-requisites in first lecture

A detailed overview of syllabus along with the practical applications will be summarized.

Syllabus: COMMUNICATION SYSTEMS

Subject Code: ETEC-208 **L T/P C**

UNIT I

Introduction: Overview of Communication system, Communication channels, Mathematical Models for Communication Channels

Introduction of random Variables: Definition of random variables, PDF, CDF and its properties, joint PDF, CDF, Marginalized PDF, CDF, WSS wide stationery, strict sense stationery, non-stationery signals, UDF, GDF, RDF, Binomial distribution, White process, Poisson process, Wiener process. **[T1, T2][No. of Hrs. 11]**

UNIT II

Amplitude Modulation: Need for modulation, Representation of Band Pass signals and systems: Hilbert Transform, In-phase, Quad-phase representations, Power relation, modulation index, Bandwidth efficiency, AM: modulation and demodulation, DSB-SC: Modulation and demodulation, SSB: modulation and demodulation, VSB: modulation and demodulation.

[T1, T2][No. of Hrs. 11]

UNIT III

Angle Modulation Systems: Frequency Modulation, Types of Frequency Modulation, Generation of NBFM, WBFM, Transmission BW of FM Signal, Phase Modulation, Relationship between PM& FM.

Radio Receivers: Functions & Classification of Radio Receivers, Tuned Radio Frequency (TRF) Receiver, Superheterodyne Receiver, Basic Elements, Receiver Characteristics, Frequency Mixers, AGC Characteristics.

[T1, T2][No. of Hrs. 11]

UNIT IV

Noise Theory: Noise, Types of noise, Addition of Noise due to several sources in series and parallel, Generalized Nyquist Theorem for Thermal Noise, Calculation of Thermal Noise for a Single Noise Source, RC Circuits & Multiple Noise sources. Equivalent Noise Bandwidth, Signal to Noise Ratio, Noise-Figure, Noise Temperature, Calculation of Noise Figure

Performance of Communication Systems: Receiver Model, Noise in DSB-SC Receivers, Noise in SSB-SC Receivers, Noise in AM receiver (Using Envelope Detection), Noise in FM Receivers, FM Threshold Effect, Threshold Improvement through Pre-Emphasis and De-Emphasis, Noise in PM system – Comparison of Noise performance in PM and FM, Link budget analysis for radio channels.

[T1, T2][No. of Hrs. 11]

Unit-I (Lecture-1-8)

Lecture 1

Overview of Communication systems, Communication Channels and their Mathematical Models

Introduction / Overview of Communication systems

Electrical Communication systems are designed to send messages or information from a source that generates the messages to one or more destinations. Telephones in our hands, televisions in our living rooms, the computer terminals with access to the Internet in our offices and homes and our newspapers are all capable of providing rapid communications from every corner of the globe. In general, a communication system can be represented by the functional block diagram shown in Fig. 1.1. The information generated by the source may be of the form of voice, a picture, or plain text in some particular language. An essential feature of any source that generates information is that its output is described in probabilistic terms i.e. the output of the source is not deterministic. Otherwise, there would be no need to transmit the message. A transducer is usually required to convert the output of a source into an electrical signal that is suitable for transmission. For example, a microphone serves as the transducer that coverts an acoustic speech into an electrical signals and a video camera convert an image into an electrical signal. At the destination, a similar transducer is required to convert the electrical signal into a form that is suitable for user e.g. acoustic signals, images etc.

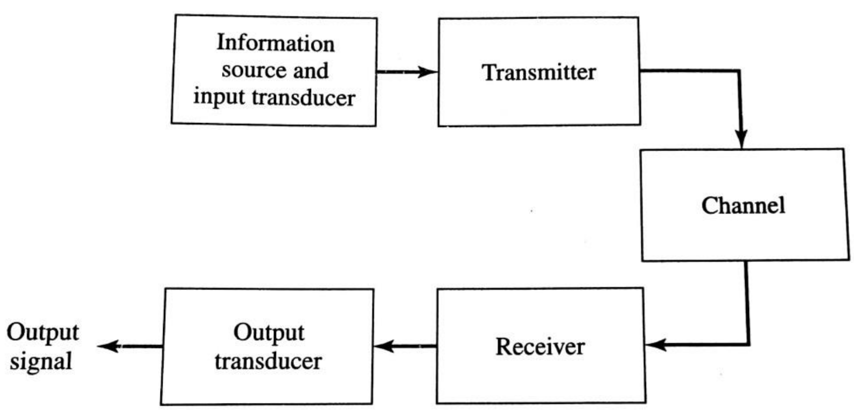


Fig. 1.1Block Diagram of a Communication System

There are three basic parts of a communication systems namely transmitter, channel and receiver.

The Transmitter-*The transmitter converts the electrical signal into a form that is suitable for transmission through the physical channel or transmission medium*.

Govt bodies resp for freq. allocation

a. In radio and TV broadcast, the Federal Communications Commission (FCC) of USA
b. Wireless Planning and Coordination Wing (WPC)-wing of Ministry of communication in India.

Responsibility of a transmitter

1) **Translation of information into freq.-**The transmitter must translate the information signal to be transmitted into the appropriate frequency range that matches the frequency allocation assigned to the transmitter. Thus signals transmitted by multiple radio station do not interfere with one another.

   Modulation - In electronics and telecommunications, **modulation** is the process of varying one or more properties of a periodic waveform, called the *carrier signal*, with a modulating signal that typically contains information to be transmitted.

   In telecommunications, modulation is the process of conveying a message signal, for example a digital bit stream or an analog audio signal, inside another signal that can be physically transmitted. Modulation of a sine waveform transforms a baseband message signal into a passband signal.

   Two types of modulation

   Continuous wave (CW) Modulation (analog) and Pulse Modulation (can be analog and digital) **e**.g. in AM/FM radio broadcast, the information signal that is transmitted is contained in the amplitude/frequency variations of the sinusoidal carrier, which is the center frequency in the frequency band allocated to the radio transmitting station. Similarly for PM.

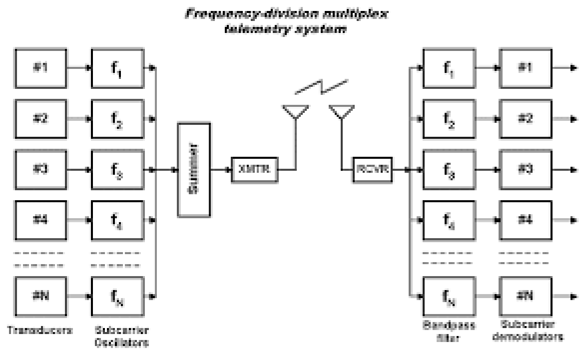
   #Among all the modulation schemes, PCM has emerged as the preferred method for transmission of analog message signals because of Robustness (Regeneration at regular intervals), Flexible operations, Integration of diverse sources into a common format, Security of information

   Choice of Modulation depends on

   a. Amount of bandwidth allocated
   b. Type of Noise and Interference that is encountered while transmission
   c. Electronics devices available for signal amplification

   Multiplexing

d. FDM, in which CW Modulation is used to translate each message signal to reside in a specific frequency slot inside the passband of the channel by assigning it a distinct carrier frequency. A bank of filters is used to separate the different modulated signals and prepare them individually for demodulation.



e. TDM, in which pulse modulation is used to position samples of the different message signals in non-overlapping time slots.

f. CDM, in which each message signal is identified by a distinctive code.

#In FDM, the message signals may overlap at the channel input, hence the system suffer from cross talk. In TDM, the message signal uses the full passband of the channel on a time sharing basis. In CDM, the message signals overlap in both time and frequency across the channel.

Channel

The communication channel is the physical medium that is used to send the signal from the transmitter to the receiver.

e.g.

a) Free Space in case of wireless transmission
b) Wireline, optical fibre, wireless (microwave radio) in case of telephony

Additive Noise – Generated at the front end of receiver, where signal amplification is performed also called thermal noise. Other examples are atmospheric noise like electrical lightening noise and interferences from other users of the channel.

Non-Additive Noise- Multipath propagation in Ionospheric channel used for long/short wave radio transmission. It is signal distortion, which is a time variation in the signal amplitude occurs called fading

Both additive and non-additive noise are random process and their effect must be considered in the design of communication system

2) **The Receiver**– The function of the receiver is to recover the message signal contained in the received signal. The receiver performs demodulation to extract message signal from the sinusoidal carrier. Since the signal demodulation is performed in the presence of additive noise, the demodulated signal is degraded by the presence of these distortion in the received signal.

Advantages of Digital transmission over Analog transmission

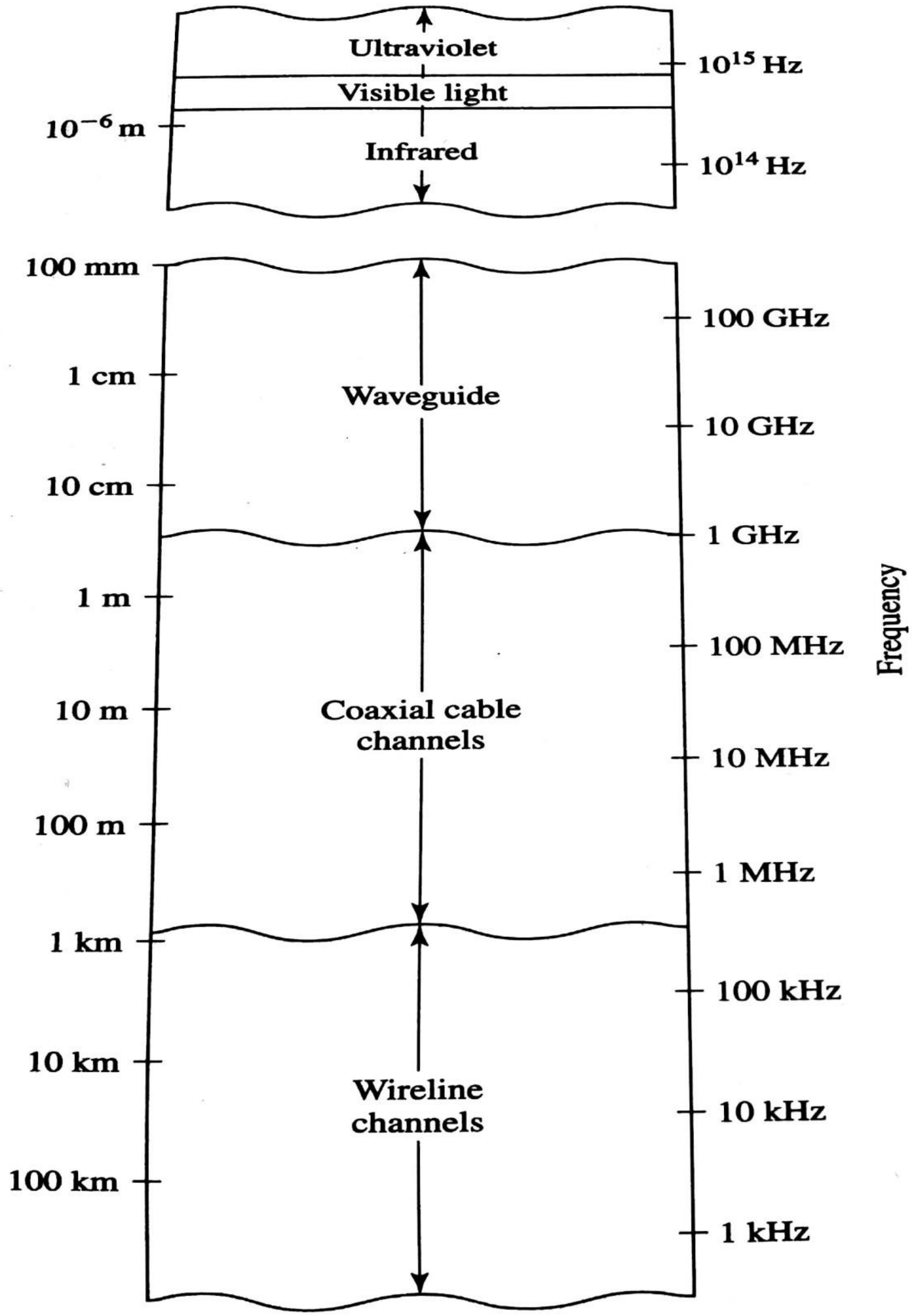
1) Signal fidelity is better in digital transmission(Fidelity also denotes how accurately a copy reproduces its source. "hi-fi" were popularized for equipment and recordings which exhibited more accurate sound reproduction.)
2) Regeneration of signal in long distance transmission eliminating noise at each regeneration point. In analog, noise gets amplified along with signals
3) Digital communication systems are cheaper to implement and redundancy may be removed easily

3) Communication Channels -*Wires / Optical Fibres / Underwater Ocean Channel / Free Space*

Characteristics of several channels

a. *Wireline Channels*

Twisted pair (order of several **Kilo Hertz**) and coaxial channels (order of **Mega Hertz**) are guided electromagnetic channels used for Voice signal transmission. **Both are distorted in amplitude and phase and corrupted by additive noise. Twisted pair wireline channels are also prone to cross talk.**



Frequency range for guided wireline channel

b. *Fibre Optic Channels*

Transmitter: *Information is transmitted by varying the intensity of a Light source-LED or Laser.*

Receiver: *Light intensity is detected by a photo diode, whose output is an electric signal that varies in direct proportion to the power of the light impinging on the photo diode.*

Fibre optic channels offers unique characteristics

i. *Enormous potential bandwidth of the order of $2x10^{13}$ Hz*
ii. *Low transmission losses, as low as 0.1 dB/km*
iii. *Immunity of electromagnetic interference*
iv. *Small Size and weight*
v. *Ruggedness and flexibility*

c. *Wireless electromagnetic channels*

The physical size and configuration of the antenna, which serves as radiator depends on the frequency of operation. To obtain efficient radiation of the electromagnetic energy, the antenna must be no longer than 1/10 of the wavelength.

e.g. AM transmitting 1 MHz requires antenna of 30 meter

Other examples:

1) *In VLF range, where wavelength exceeds 10 KM (10 KHz and below), earth acts as waveguide and comm. signals propagates around the globe.* ***Use****: Navigational aids from shore to ships*
2) *Ground wave propagation (0.3-3 MHz): Use: AM broadcast, Maritime Radio broadcasting*
3) *Sky wave propagation – above 30 MHz (Reflection from Ionosphere) Prob: Signal multipath, signal fading*
4) *LOS Communication –*

$$D = \sqrt{2}h$$

D-Radio Horizon

h= height of antenna tower

e.g D =50 miles, h=1000 ft, for a TV antenna

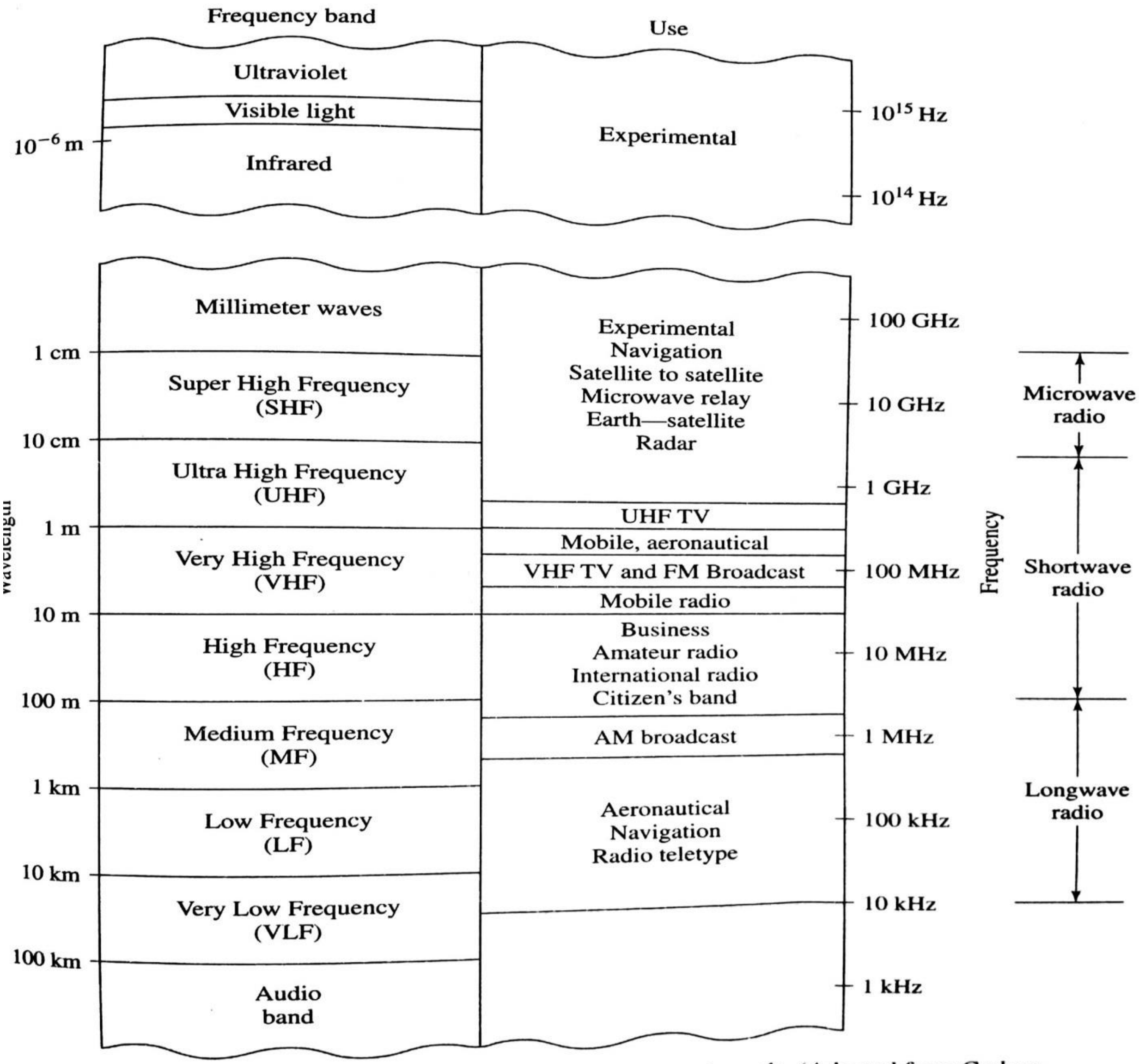


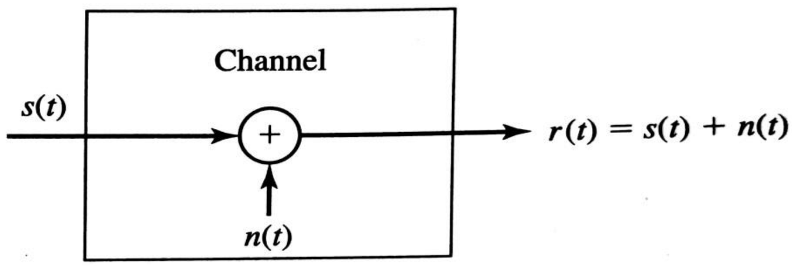
Figure 1.4 Frequency range for wireless electromagnetic channels. (Adapted from Carlson, Sec. Ed.; © 1975 McGraw-Hill. Reprinted with permission of the publisher.)

Lecture 2

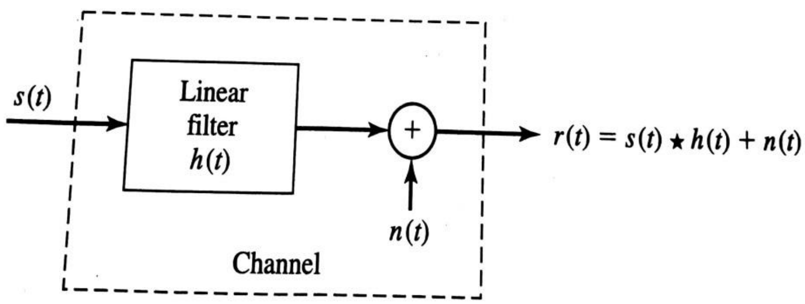
Mathematical Models for Communication channels

1) The Additive Noise Channel

The transmitter signal s(t) is corrupted by additive random noise n(t). Noise introduced through electronic components and amplifiers is also called thermal noise characterized statistically as a Gaussian noise process.



2) Linear Filter Channel



In some physical channels such as telephone channels, Filters are used to ensure that the transmitted signals do not exceed specified bandwidth limitations and thus do not interfere with each other. Thus, they are characterized as Linear Filter channels with additive noise.

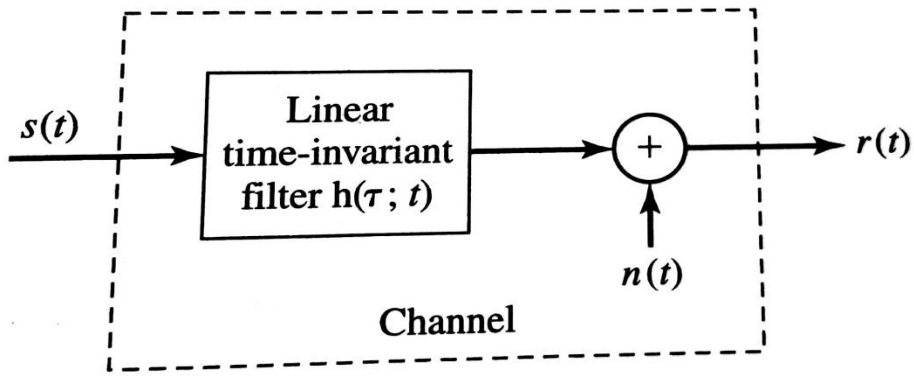
$$r(t) = s(t) * h(t) + n(t)$$

Where h(t) is the impulse response of the linear filter and * is the convolution

3) Linear Time Variant Filter Channel

Physical channels such as under water acoustic channels and ionospheric radio channels, which result in time-variant multipath propagation of the transmitted signal may be characterized as time-variant linear filters.

$$r(t) = s(t) * h(T; t) + n(t)$$


s(t)
Linear
time-invariant
filter h(τ ; t)
+
r(t)
n(t)
Channel

Lecture 3

Random Variables-Definition and basic concept

Random Variables

The outcome of an experiment may be a real number (rolling of a die), or non-numerical quantity (head or tail in tossing a coin). From a mathematical point of view, it is simpler to have numerical values for all the outcomes. Thus a real number may be assigned to each sample point as per some rule. **A term random variable is used to signify a rule by which a real number is assigned to each possible outcome of an experiment.** Thus, X(.) is a function that maps sample points $\boldsymbol{\lambda_1, \lambda_2, \ldots \lambda_m}$ into real numbers $x_1, x_2, \ldots x_n$.

Let the possible outcomes be $\boldsymbol{\lambda i}$**,** which may not be numbers. Thus, the rule or functional relationshipby which we assign real numbers $X(\lambda_i)$ to each possible outcome is called a Random variable.

Discrete or Continuous

If in any finite interval, $X(\boldsymbol{\lambda})$ assumesonly a finite number of distinct values, then the Random variable is discrete e.g. tossing a die.

If $X(\boldsymbol{\lambda})$ can assume any value within an interval, the Random variable is continuous.e.g. if we fire a bullet, the bullet may miss its mark and the magnitude of miss will be continuous random variable.

Cumulative Distribution Function

The CDF of a random variable 'X' may be defined as the probability that a random variable 'X' takes a value less than or equal to x. Thus, CDF provides the probabilistic description of a random variable. CDF is the probability that the outcome of an experiment will be one of those outcome for which $X(\lambda_i) \leq x$, where x is any given number

$$F_{\mathbf{x}}(x) = P(X \leq x)$$

A CDF $F_X(x)$ has the following properties

1. $0 \leq F_{\mathbf{x}}(x) \leq 1$ …(2.1)
2. $F_{\mathbf{x}}(\infty) = 1$ follows from the fact that F_X contains all possible events …(2.2)
   $(F_{\mathbf{x}}(\infty) = P(X \leq \infty))$
3. $F_{\mathbf{x}}(-\infty) = 0$follows from the fact that F_X contains no possible events …(2.3)
   $(F_{\mathbf{x}}(-\infty) = P(X \leq -\infty))$
4. $F_{\mathbf{x}}(x)$is a non-decreasing function i.e.

$$F_{\mathbf{x}}(x_1) \leq F_{\mathbf{x}}(x_2) \;\; for x_1 \leq x_2 \qquad \ldots(2.4)$$

Proof: $F_{\mathbf{x}}(x_2) = P(X \leq x_2)$

$= P[(X \leq x_1) U (x_1 < X \leq x_2)]$

$= P(X \leq x_1) + P(x_1 < X \leq x_2)$ (because the sets are disjoint)

$= F_{\mathbf{x}}(x_1) + P(x_1 < X \leq x_2)$ …(2.5)

Because $P(x_1 < X \leq x_2)$ is non-negative, the result follows

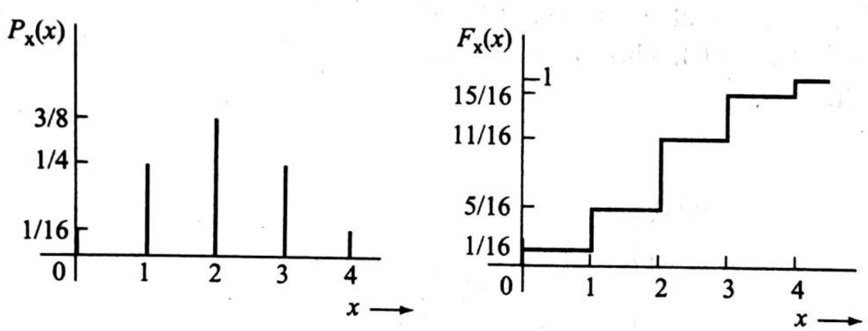
Example:

5 **In an experiment, a trial consists of four successive tosses of a coin. If we define an RV x a: the number of heads appearing in a trial, determine $P_{\mathrm{x}}(x)$ and $F_{\mathrm{x}}(x)$.**

A total of 16 distinct equiprobable outcomes are listed in Example 8.4. Various probabilitie: can be readily determined by counting the outcomes pertaining to a given value of x. Fo example, only one outcome maps into x=0, whereas six outcomes map into x=2. Hence $P_{\mathrm{x}}(0) = 1/16$ and $P_{\mathrm{x}}(2) = 6/16$. In the same way, we find

$$P_{\mathrm{x}}(0) = P_{\mathrm{x}}(4) = 1/16$$
$$P_{\mathrm{x}}(1) = P_{\mathrm{x}}(3) = 4/16 = 1/4$$
$$P_{\mathrm{x}}(2) = 6/16 = 3/8$$

The probabilities $P_{\mathrm{x}}(x_i)$ and the corresponding CDF $F_{\mathrm{x}}(x_i)$ are shown in Fig. 8.8.



(a) (b)

Sample space of 16 points

HHHH	TTTT
HHHT	TTTH
HHTH	TTHT
HHTT	TTHH
HTHH	THTT
HTHT	THTH

HTTH THHT
HTTT THHH

Lecture 4

Probability Density Function

From eqn 2.5,

$$\mathrm{F}\mathbf{x}(x+\Delta x) = \mathrm{F}\mathbf{x}(\mathrm{x}) + P(x < X \leq x + \Delta x) \qquad \ldots(2.6)$$

If$\Delta x \to 0$, then we can also express $\mathrm{F}\mathbf{x}(x+\Delta x)$ via taylor expression

$$\mathrm{F}\mathbf{x}(x+\Delta x) \approx \mathrm{F}\mathbf{x}(x) + \frac{d\mathrm{F}\mathbf{x}(x)}{dx}\Delta x \qquad \ldots(2.7)$$

Thus, $\frac{d\mathrm{F}\mathbf{x}(x)}{dx}\Delta x = P(x < X \leq x + \Delta x)$ …(2.8)

We designate the derivative of $\mathrm{F}\mathbf{x}(x)$w.r.t. x by $\mathrm{p}\mathbf{x}(x)$

$$\frac{d\mathrm{F}\mathbf{x}(x)}{dx} = \mathrm{p}\mathbf{x}(x) \qquad \ldots(2.9)$$

The function $\mathrm{p}\mathbf{x}(x)$ is called the Probability density function of the random variable X. PDF is a more convenient way of describing a continuous random variable. It is defined as the derivative of cumulative distribution function.
It follows from eqn. 2.8 that the probability of observing the Random Variable X in the interval $(x, x+\Delta x)$ is $\mathrm{p}\mathbf{x}(x)\Delta x$ $(\Delta x \to 0)$. This is the area under the PDF $\mathrm{p}\mathbf{x}(x)$ over the interval Δx.

Properties of PDF

1) PDF must be non-negative

   $p_x(x) \geq 0$ for all x

   Since CDF is a monotone increasing function and PDF is the derivative of CDF, so the derivative of a monotone increasing function will always be positive.

   It is true that the probability of an impossible event is zero and that of a certain event is 1, the converse is not true. An event whose probability is 0 is not necessarily an impossible event and an event with a probability of 1 is not necessarily a certain event.
   Temp of a city=5 to 50^{o}C
   Prob that T=34.56 is 0. But this is not an impossible event
   Prob that T=34.56 is 1, although is not a certain event.
2) CDF can be derived from PDF by integrating it

$$\mathrm{F}\mathbf{x}(\mathrm{x}) = \int_{-\infty}^{x} p_X(x)dx = \int_{-\infty}^{x} f_X(x)dx$$

3) Area under the PDF curve is always equal to unity

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

This is seen from the fact that

$$\int_{-\infty}^{\infty} f_X(x)dx = F(\infty) - F(-\infty) = 1 - 0 = 1$$

Example – Consider an experiment of rolling 2 die together. There are 36 outcomes represented by 36$\boldsymbol{\lambda}_{\mathbf{ij}}$, where i is the number on 1st die and j is the number on 2nd die.

P(1)=0; P(2)=P(12)=1/36;

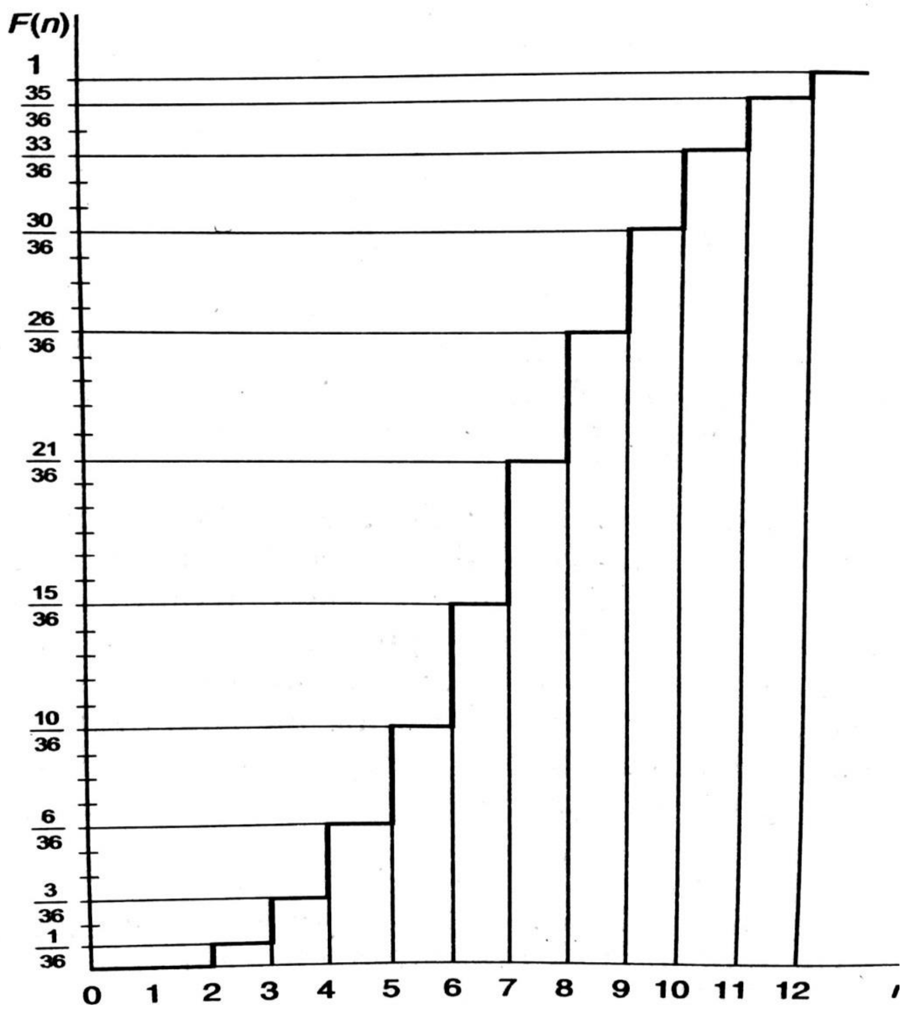
P(3)=P(11)=2/36; P(4)=P(10)=3/36;

P(5)=P(9)=4/36; P(6)=P(8)=5/36;

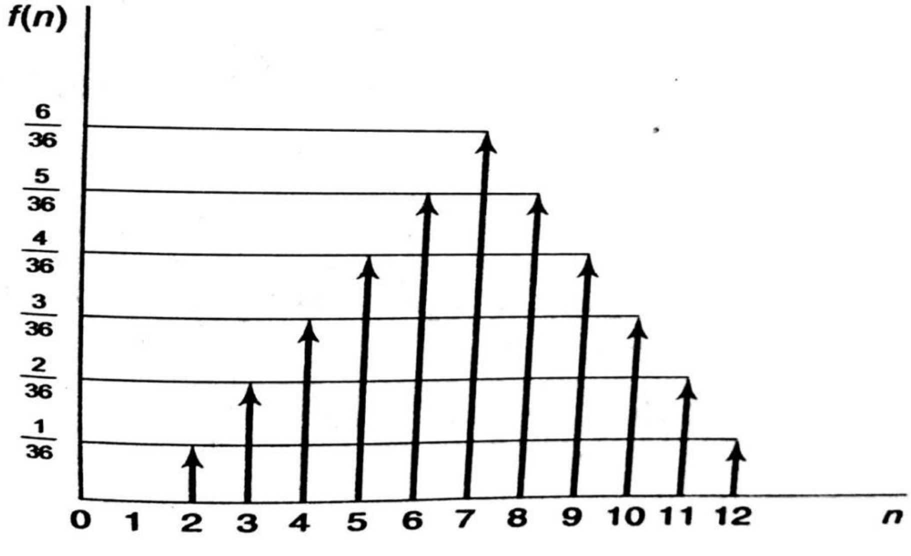
P(7)=6/36

The CDF for n=3 is

$$\begin{aligned} \mathrm{F}(3) &= \mathrm{P}(\mathrm{n} \leq 3) \\ &= \mathrm{P}(1) + \mathrm{P}(2) + \mathrm{P}(3) = 0 + \frac{1}{36} + \frac{2}{36} = \frac{3}{36} \end{aligned}$$



CDF



PDF

Derivative of a step of Amplitude A is an impulse of strength I=A, we have $f(2) = \frac{1}{36}\delta(n-2), f(3) = \delta(n-3)$ .... (Height of the impulse is proportional to the strength of impulse)

EXAMPLE 5.13. The PDF of a random variable is given by,

$$f_X(x) = \begin{cases} k, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

where, k is a constant

(i) Determine the value of k

(ii) Let a = −1 and b = 2 calculate $P(|X| \le c)$ for c = 1/2

Solution: (i) The value of k

From the property 2 of PDF, k must be a positive constant. From the property 3 of PDF, the area under PDF is equal to 1.

Therefore, $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

or $$\int_a^b k\,dx = 1$$

or $$k[b-a] = 1$$

Therefore, $$k = \frac{1}{(b-a)}$$ **Ans.**

Hence, $$f_X(x) = \begin{cases} \dfrac{1}{(b-a)} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

This PDF has been plotted in figure 5.9 which is called as uniform PDF.

(ii) Value of $P(|X| \le c)$

We are expected to obtain the probability of $(-1/2 \le X \le 1/2)$.

According to property 4, we have

$$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f_X(x)\,dx$$

Thus, $$P(-1/2 \le X \le 1/2) = \int_{-1/2}^{1/2} \frac{1}{b-a}\,dx$$

or, $$P(-1/2 \le X \le 1/2) = \int_{-1/2}^{1/2} (1/3)\,dx = (1/3)(1/2 + 1/2) = 1/3$$

This has been shown by the shaded portion shown in figure 5.9.

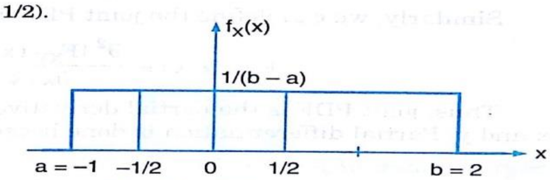


Fig. 5.9. PDF for example 5.13.

Lecture 5, Joint PDF

5.2.1 Joint Probability Density Function (PDF)

Here, we will define jointly continuous random variables. Basically, two random variables are jointly continuous if they have a joint probability density function as defined below.

Definition 5.1

Two random variables X and Y are jointly continuous if there exists a nonnegative function $f_{XY}: \mathbb{R}^2 \to \mathbb{R}$, such that, for any set $A \in \mathbb{R}^2$, we have

$$P((X,Y) \in A) = \iint_A f_{XY}(x,y)dxdy \qquad (5.15)$$

The function $f_{XY}(x,y)$ is called the joint probability density function (PDF) of X and Y.

In the above definition, the domain of $f_{XY}(x,y)$ is the entire $\mathbb{R}^2$. We may define the range of (X,Y) as

$$R_{XY} = \{(x,y) | f_{X,Y}(x,y) > 0\}.$$

The above double integral (Equation 5.15) exists for all sets A of practical interest. If we choose $A = \mathbb{R}^2$, then the probability of $(X,Y) \in A$ must be one, so we must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$$

The intuition behind the joint density $f_{XY}(x,y)$ is similar to that of the PDF of a single random variable. In particular, remember that for a random variable X and small positive δ, we have

$$P(x < X \leq x+\delta) \approx f_X(x)\delta.$$

Similarly, for small positive δ_x and δ_y, we can write

$$P(x < X \leq x+\delta_x, y \leq Y \leq y+\delta_y) \approx f_{XY}(x,y)\delta_x\delta_y.$$

Example 5.15

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x+cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant c.
b. Find $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$.

- Solution
  - a. To find c, we use

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1.$$

Thus, we have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_0^1\int_0^1 x+cy^2 \; dxdy \\ &= \int_0^1 \left[\frac{1}{2}x^2 + cy^2x\right]_{x=0}^{x=1} dy \\ &= \int^1 \frac{1}{2} + cy^2 \; dy \end{aligned}$$

$$= \left[\frac{1}{2}y + \frac{1}{3}cy^3\right]_{y=0}^{y=1}$$
$$= \frac{1}{2} + \frac{1}{3}c.$$

Therefore, we obtain $c = \frac{3}{2}$.

b. To find $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$, we can write

$$P((X,Y) \in A) = \iint_A f_{XY}(x,y)dxdy, \quad \text{for } A = \{(x,y)|0 \leq x, y \leq 1\}.$$

Thus,

$$P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2}) = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \left(x + \frac{3}{2}y^2\right) dxdy$$
$$= \int_0^{\frac{1}{2}} \left[\frac{1}{2}x^2 + \frac{3}{2}y^2x\right]_0^{\frac{1}{2}} dy$$
$$= \int_0^{\frac{1}{2}} \left(\frac{1}{8} + \frac{3}{4}y^2\right) dy$$
$$= \frac{3}{32}.$$

We can find marginal PDFs of X and Y from their joint PDF. This is exactly analogous to what we saw in the discrete case. In particular, by integrating over all y's, we obtain $f_X(x)$. We have

Marginal PDFs

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy, \quad \text{for all } x,$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx, \quad \text{for all } y.$$

Example 5.16

In Example 5.15 find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

- Solution
  - For $0 \leq x \leq 1$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$
$$= \int_0^1 \left(x + \frac{3}{2}y^2\right) dy$$
$$= \left[xy + \frac{1}{2}y^3\right]_0^1$$
$$= x + \frac{1}{2}.$$

Thus,

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for $0 \leq y \leq 1$, we have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$
$$= \int_0^1 \left(x + \frac{3}{2}y^2\right) dx$$
$$= \left[\frac{1}{2}x^2 + \frac{3}{2}y^2x\right]_0^1$$
$$= \frac{3}{2}y^2 + \frac{1}{2}.$$

Thus,

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2 + \frac{1}{2} & 0 \leq y \leq 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

Example 5.17
Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} cx^2y & 0 \leq y \leq x \leq 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

a. Find R_{XY} and show it in the $x-y$ plane.
b. Find the constant c.
c. Find marginal PDFs, $f_X(x)$ and $f_Y(y)$.
d. Find $P(Y \leq \frac{X}{2})$.
e. Find $P(Y \leq \frac{X}{4} | Y \leq \frac{X}{2})$.

- Solution
  - a. From the joint PDF, we find that

$$R_{XY} = \{(x,y) \in \mathbb{R}^2 | 0 \leq y \leq x \leq 1\}.$$

Figure 5.6 shows R_{XY} in the $x-y$ plane.

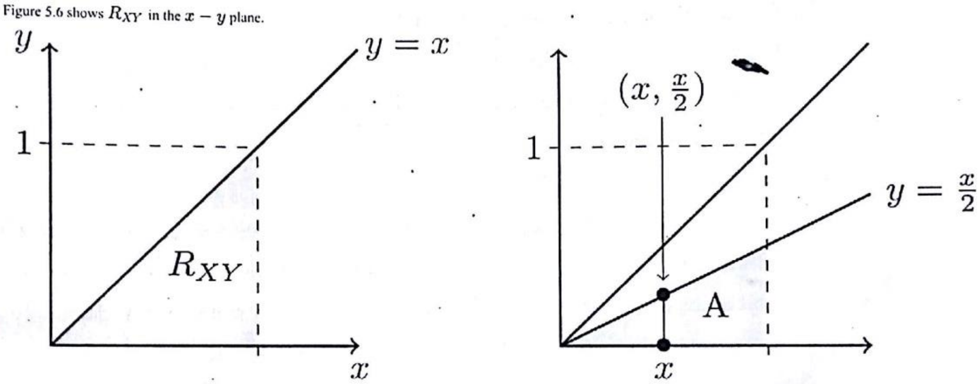


Figure 5.6: Figure shows R_{XY} as well as integration region for finding $P(Y \leq \frac{X}{2})$.

b. To find the constant c, we can write

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_0^1\int_0^x cx^2y \; dydx \\ &= \int_0^1 \frac{c}{2}x^4 dx \\ &= \frac{c}{10}. \end{aligned}$$

Thus, $c = 10$.
c. To find the marginal PDFs, first note that $R_X = R_Y = [0,1]$. For $0 \leq x \leq 1$, we can write

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^x 10x^2ydy \\ &= 5x^4. \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 5x^4 & 0 \leq x \leq 1 \\ & \\ 0 & \text{otherwise} \end{cases}$$

For $0 \leq y \leq 1$, we can write

$$f_Y(y) = \int^{\infty} f_{XY}(x,y)dx$$

$$f_Y(y) = \int_{-\infty} f_{XY}(x,y)dx$$
$$= \int_y^1 10x^2y\,dx$$
$$= \frac{10}{3}y(1-y^3).$$

Thus,

$$f_Y(y) = \begin{cases} \frac{10}{3}y(1-y^3) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

d. To find $P(Y \leq \frac{X}{2})$, we need to integrate $f_{XY}(x,y)$ over region A shown in Figure 5.6. In particular, we have

$$P\left(Y \leq \frac{X}{2}\right) = \int_{-\infty}^{\infty}\int_0^{\frac{x}{2}} f_{XY}(x,y)dydx$$
$$= \int_0^1\int_0^{\frac{x}{2}} 10x^2y\ dydx$$
$$= \int_0^1 \frac{5}{4}x^4dx$$
$$= \frac{1}{4}.$$

e. To find $P(Y \leq \frac{X}{4}|Y \leq \frac{X}{2})$, we have

$$P\left(Y \leq \frac{X}{4}|Y \leq \frac{X}{2}\right) = \frac{P\left(Y \leq \frac{X}{4}, Y \leq \frac{X}{2}\right)}{P\left(Y \leq \frac{X}{2}\right)}$$
$$= 4P\left(Y \leq \frac{X}{4}\right)$$
$$= 4\int_0^1\int_0^{\frac{x}{4}} 10x^2y\ dydx$$
$$= 4\int_0^1 \frac{5}{16}x^4dx$$
$$= \frac{1}{4}.$$

← previous
next →

If X and Y are independent,

$$P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) = \left[\int_{x1}^{x2} p_X(x)dx\right]\left[\int_{y1}^{y2} p_Y(y)dx\right]$$

Therefore,

$$p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x,y)dy$$

And

$$p_Y(y) = \int_{-\infty}^{\infty} p_{XY}(x,y)dx$$

EXAMPLE 5.17. The joint PDF of the random variables X and Y is given by,

$$f_{XY}(x, y) = C \cdot e^{-(ax + by)}\, u(x) \cdot u(y)$$

where, a and b are constants. Find the value of C.

Solution: The joint CDF $F_{XY}(\infty, \infty) = P(X \le \infty, Y \le \infty)$. This will be equal to 1 because the probability of $X \le \infty$ and $Y \le \infty$ covers all the possible values of random variables X and Y.

Therefore, $F_{XY}(\infty, \infty) = 1$

According to the definition of CDF, we have

$$F_{XY}(\infty, \infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x.y)\,dx\,dy = 1$$

Substituting the value of joint PDF, we get

$$F_{XY}(\infty, \infty) = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(ax+by)}u(x)\,u(y)\,dx\,dy$$

or, $$1 = C\int_0^{\infty} e^{-(ax)}\,dx \cdot \int_0^{\infty} e^{-by}\,dy$$

or, $$1 = \frac{C}{ab}\left[e^{-ax}\right]_0^{\infty} \cdot \left[e^{-by}\right]_0^{\infty}$$

or, $$1 = \frac{C}{ab}[e^{-\infty} - e^{-0}][e^{-\infty} - e^{0}]$$

or, $$1 = \frac{C}{ab}[-1][-1]$$

Hence, $C = ab$ **Ans.**

Example 6.4

The joint probability density of the random variables X and Y is

$$f(x, y) = \frac{1}{4}e^{-|x|-|y|} \quad -\infty < x < \infty, -\infty < y < \infty$$

(a) Are X and Y statistically independent random variables?

(b) Calculate the probability that $X \le 1$ and $Y \le 0$.

Solution

(a) Since $f(x, y)$ can be written as

$$f(x, y) = \frac{1}{2}e^{-|x|}\frac{1}{2}e^{-|y|} = f(x)f(y)$$

X and Y are statistically independent.

(b) $$P(X \le 1, Y \le 0) = \int_{-\infty}^{1} dx \int_{-\infty}^{0} dy f(x, y)$$

$$= \int_{-\infty}^{1} \frac{1}{2}e^{-|x|}\,dx \int_{-\infty}^{0} \frac{1}{2}e^{-|y|}$$

$$= \left(\frac{2 - e^{-1}}{2}\right)\frac{1}{2} = \frac{1}{4}(2 - e^{-1})$$

Example 6.3

Consider the probability density $f(x) = ae^{-b|x|}$, where X is a random variable whose allowable values range from $x = -\infty$ to $x = +\infty$. Find (a) the cumulative distribution function $F(x)$, (b) the relationship between a and b, and (c) the probability that the outcome X lies between 1 and 2.

Solution

(a) The cumulative distribution function is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} ae^{-b|x|}\,dx$$

$$= \begin{cases} \dfrac{a}{b}e^{bx} & x \le 0 \\ \dfrac{a}{b}(2 - e^{-bx}) & x \ge 0 \end{cases}$$

(b) In order that $f(x)$ be a probability density, it is necessary that

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} ae^{-b|x|}\,dx = \frac{2a}{b} = 1$$

so that $a/b = \dfrac{1}{2}$.

(c) The probability that X lies in the range between 1 and 2 is

$$P(1 \le X \le 2) = \frac{b}{2}\int_{1}^{2} e^{-b|x|}\,dx = \frac{1}{2}(e^{-b} - e^{-2b})$$

Lecture 6

Types of Distributions

Gaussian Random Variables / Gaussian Distribution (Normal Distribution)

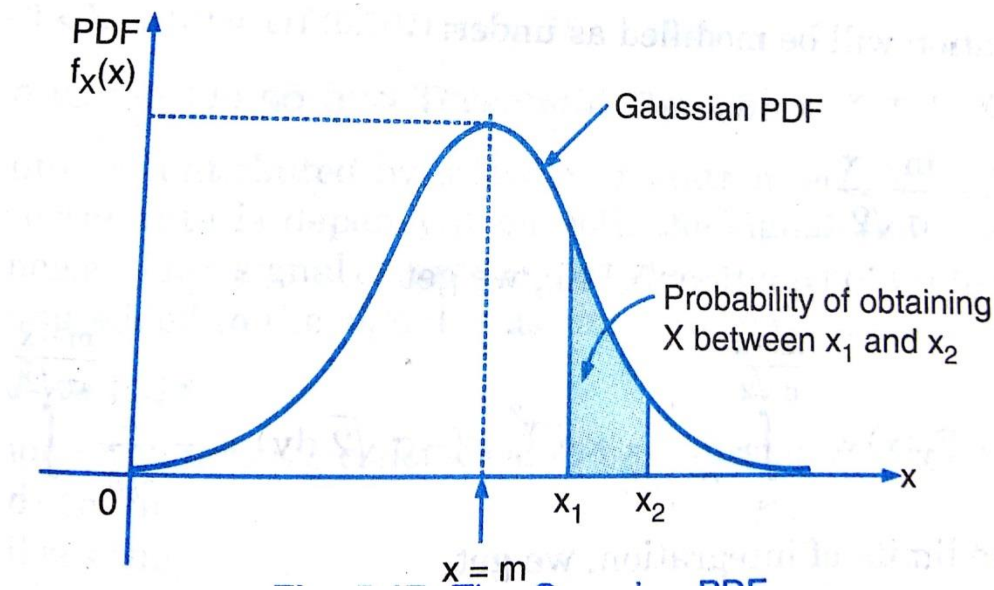
The Gaussian (also called Normal) probability density function is of the greatest importance because many naturally occurring experiment are characterzied by random variables with a Gaussian density. The majority of noise processess observed in practiseare Gaussian and many naturally occurring experiments are characterized by continuous random variables with Gaussian PDF.

The Gaussian PDF is defined as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$$

Where

m= Mean of the random variable
σ^2=Variance of the random variable
Gaussian PDF is also known as Normal PDF. The shape of the Gaussian PDF is bell type as shown in Fig



Important

1) The Gaussian PDF is a bell shaped function with a peak at x=m i.e., corresponding to the mean value of the random variable X.
2) The Gaussian PDF has an even symmetry about the peak.
   Therefore, $f(x = m - \sigma) = f(x = m + \sigma)$
3) Probability of obtaining 'X' above and below the mean value is equal is equal i.e.1/2
   Thus, $P(X \le m) = P(X > m) = 1/2$
4) Area under the Gaussian PDF is 1
5) Probability of observing 'X' between x_1 and $x_2$can be obtained by integrating the Gaussian PDF between the limits x_1 and x_2

$$P(x_1 < X \le x_2) = \int_{x1}^{x2} f_X(x)dx$$

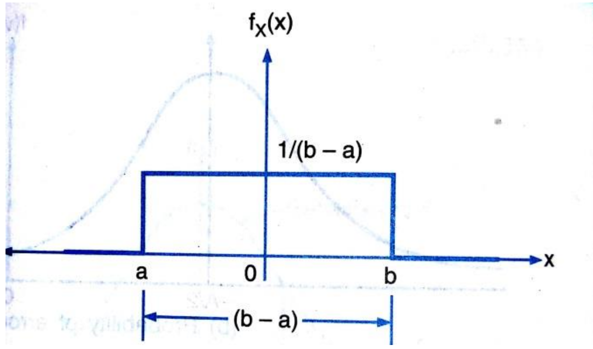
$$=\int_{x1}^{x2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} dx$$

Uniform Distribution

If a continuous random variable 'X' is equally likely to be observed in a finite range and is likely to have a zero value outside the finite range, then the random variable is said to have a uniform distribution. The PDF of a random variable is given by

$$f_X(x) = \begin{Bmatrix} \frac{1}{(b-a)} & for\ a \le x \le b \\ 0 & elsewhere \end{Bmatrix}$$

For example, when the phase of a sinusoid is random, it is usually modeled as a uniform random variable between 0 and 2π.



Rayleigh's Distribution

In mobile radio channels, the Rayleigh distribution is commonly used to describe the statistical time varying nature of the received envelope of a flat fading signal, or the envelope of an individual multipath component. It is well known that the envelope of the sum of two quadrature Gaussian noise signals obeys a Rayleigh distribution.

The Rayleigh's distribution is used for continuous random variables. It describes a continuous random variables produced from two Gaussian random variables. Let X and Y be two independent Gaussian random variables having

$$m_x = m_y = m \text{ and } \sigma_x = \sigma_y = \sigma$$

The Rayleigh continuous random variable R is related to X and Y by the transformation shown as in fig 5.26 shown in next page

The Gaussian random variables X and Y are related to the Rayleigh's random variable R and φ as

$$R = \sqrt{x^2 + y^2}$$

and $$\varphi = \tan^{-1}[Y/X]$$

The Rayleigh density is characterized by

$$f_R(r) = \begin{Bmatrix} \frac{r}{\sigma^2} e^{\frac{-r^2}{2\sigma^2}} & r \geq 0 \\ 0 & r < 0 \end{Bmatrix}$$

The PDF curve for the Rayleigh PDF is shown in figure

$$f_R(r) = 0 \text{ at r=0}$$

The above equation can be derived from two independent Gaussian RVs as follows. Let x and y be independent Gaussian variables with identical PDFs

$$p_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x)^2/2\sigma^2}$$

and $$p_y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y)^2/2\sigma^2}$$

Then

$$p_{xy}(x, y) = p_x(x)p_y(y) \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$$

The points in the (x,y) plane can also be described in polar coordinates as (r, φ)

random
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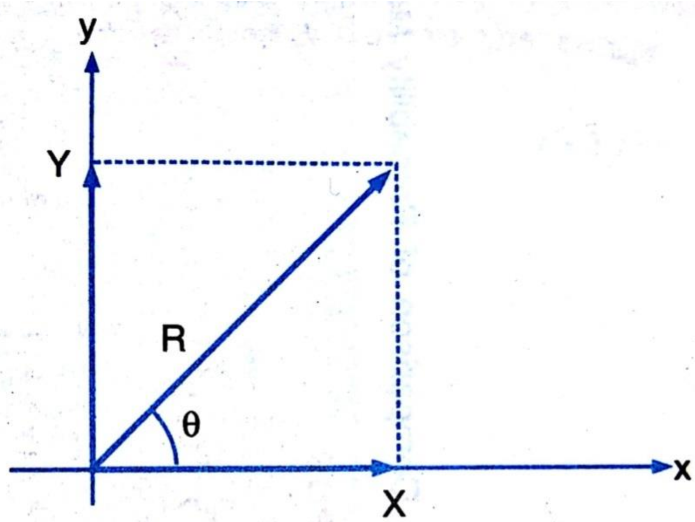


Fig. 5.26. Rectangular to polar conversion

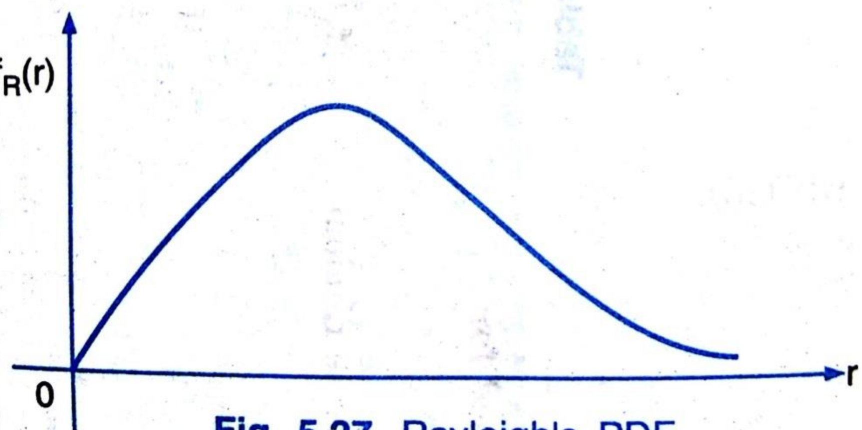


Fig. 5.27. Rayleigh's PDF

Binomial Probability Distribution

To understand binomial distributions and binomial probability, it helps to understand binomial experiments and some associated notation; so we cover those topics first.

Binomial Experiment

A **binomial experiment** is a statistical experiment that has the following properties:

- The experiment consists of *n* repeated trials.
- Each trial can result in just two possible outcomes. We call one of these outcomes a success and the other, a failure.
- The probability of success, denoted by *P*, is the same on every trial.
- The trials are independent; that is, the outcome on one trial does not affect the outcome on other trials.

Consider the following statistical experiment. You flip a coin 2 times and count the number of times the coin lands on heads. This is a binomial experiment because:

- The experiment consists of repeated trials. We flip a coin 2 times.
- Each trial can result in just two possible outcomes - heads or tails.
- The probability of success is constant - 0.5 on every trial.
- The trials are independent; that is, getting heads on one trial does not affect whether we get heads on other trials.

Notation

The following notation is helpful, when we talk about binomial probability.

- x: The number of successes that result from the binomial experiment.
- n: The number of trials in the binomial experiment.
- P: The probability of success on an individual trial.
- Q: The probability of failure on an individual trial. (This is equal to 1 - P.)
- $n!$: The factorial of n (also known as n factorial).
- $b(x; n, P)$: Binomial probability - the probability that an n-trial binomial experiment results in exactly x successes, when the probability of success on an individual trial is P.
- ${}_nC_r$: The number of combinations of n things, taken r at a time.

Binomial Distribution

A **binomial random variable** is the number of successes *x* in *n* repeated trials of a binomial experiment. The probability distribution of a binomial random variable is called a **binomial distribution**.

Suppose we flip a coin two times and count the number of heads (successes). The binomial random variable is the number of heads, which can take on values of 0, 1, or 2. The binomial distribution is presented below.

Number of heads	Probability
0	0.25
1	0.50
2	0.25

The binomial distribution has the following properties:

- The mean of the distribution (μ_x) is equal to $n * P$.
- The variance (σ^2_x) is $n * P * (1 - P)$.
- The standard deviation (σ_x) is sqrt[$n * P * (1 - P)$].

Binomial Formula and Binomial Probability

The **binomial probability** refers to the probability that a binomial experiment results in exactly *x* successes. For example, in the above table, we see that the binomial probability of getting exactly one head in two coin flips is 0.50.

Given *x*, *n*, and *P*, we can compute the binomial probability based on the binomial formula:

Binomial Formula. Suppose a binomial experiment consists of *n* trials and results in *x* successes. If the probability of success on an individual trial is *P*, then the binomial probability is:

$$b(x; n, P) = {}_nC_x * P^x * (1 - P)^{n-x}$$

or

$$b(x; n, P) = \{ n! / [x! (n - x)!] \} * P^x * (1 - P)^{n-x}$$

Example 1

Suppose a die is tossed 5 times. What is the probability of getting exactly 2 fours?

Solution: This is a binomial experiment in which the number of trials is equal to 5, the number of successes is equal to 2, and the probability of success on a single trial is 1/6 or about 0.167. Therefore, the binomial probability is:

$$b(2; 5, 0.167) = {}_5C_2 * (0.167)^2 * (0.833)^3$$
$$b(2; 5, 0.167) = 0.161$$

Cumulative Binomial Probability

A **cumulative binomial probability** refers to the probability that the binomial random variable falls within a specified range (e.g., is greater than or equal to a stated lower limit and less than or equal to a stated upper limit).

For example, we might be interested in the cumulative binomial probability of obtaining 45 or fewer heads in 100 tosses of a coin (see Example 1 below). This would be the sum of all these individual binomial probabilities.

$$b(x \leq 45; 100, 0.5) =$$
$$b(x = 0; 100, 0.5) + b(x = 1; 100, 0.5) + \ldots + b(x = 44; 100, 0.5) + b(x = 45; 100, 0.5)$$

Binomial Calculator

As you may have noticed, the binomial formula requires many time-consuming computations. The Binomial Calculator can do this work for you - quickly, easily, and error-free. Use the Binomial Calculator to compute binomial probabilities and cumulative binomial probabilities. The calculator is free. It can be found under the Stat Tables tab, which appears in the header of every Stat Trek web page.

Binomial Calculator

Example 1

What is the probability of obtaining 45 or fewer heads in 100 tosses of a coin?

Solution: To solve this problem, we compute 46 individual probabilities, using the binomial formula. The sum of all these probabilities is the answer we seek. Thus,

$$b(x \leq 45; 100, 0.5) = b(x = 0; 100, 0.5) + b(x = 1; 100, 0.5) + \ldots + b(x = 45; 100, 0.5)$$
$$b(x \leq 45; 100, 0.5) = 0.184$$

Example 2

The probability that a student is accepted to a prestigious college is 0.3. If 5 students from the same school apply, what is the probability that at most 2 are accepted?

Solution: To solve this problem, we compute 3 individual probabilities, using the binomial formula. The sum of all these probabilities is the answer we seek. Thus,

$$b(x \leq 2; 5, 0.3) = b(x = 0; 5, 0.3) + b(x = 1; 5, 0.3) + b(x = 2; 5, 0.3)$$
$$b(x \leq 2; 5, 0.3) = 0.1681 + 0.3601 + 0.3087$$
$$b(x \leq 2; 5, 0.3) = 0.8369$$

Example 3

What is the probability that the world series will last 4 games? 5 games? 6 games? 7 games? Assume that the teams are evenly matched.

Solution: This is a very tricky application of the binomial distribution. If you can follow the logic of this solution, you have a good understanding of the material covered in the tutorial, to this point.

In the world series, there are two baseball teams. The series ends when the winning team wins 4 games. Therefore, we define a success as a win by the team that ultimately becomes the world series champion.

For the purpose of this analysis, we assume that the teams are evenly matched. Therefore, the probability that a particular team wins a particular game is 0.5.

Let's look first at the simplest case. What is the probability that the series lasts only 4 games. This can occur if one team wins the first 4 games. The probability of the National League team winning 4 games in a row is:

$$b(4; 4, 0.5) = {}_4C_4 * (0.5)^4 * (0.5)^0 = 0.0625$$

Similarly, when we compute the probability of the American League team winning 4 games in a row, we find that it is also 0.0625. Therefore, probability that the series ends in four games would be 0.0625 + 0.0625 = 0.125; since the series would end if either the American or National League team won 4 games in a row.

Now let's tackle the question of finding probability that the world series ends in 5 games. The trick in finding this solution is to recognize that the series can only end in 5 games, if one team has won 3 out of the first 4 games. So let's first find the probability that the American League team wins exactly 3 of the first 4 games.

$$b(3; 4, 0.5) = {}_4C_3 * (0.5)^3 * (0.5)^1 = 0.25$$

Okay, here comes some more tricky stuff, so listen up. Given that the American League team has won 3 of the first 4 games, the American League team has a 50/50 chance of winning the fifth game to end the series. Therefore, the probability of the American League team winning the series in 5 games is 0.25 * 0.50 = 0.125. Since the National League team could also win the series in 5 games, the probability that the series ends in 5 games would be 0.125 + 0.125 = 0.25.

The rest of the problem would be solved in the same way. You should find that the probability of the series ending in 6 games is 0.3125; and the probability of the series ending in 7 games is also 0.3125.

EXAMPLE 5.22. Assume that 8 digit binary words are being transmitted over a noisy channel, with a per digit error probability of 0.01. Calculate the probability that 3 digits out of 8 are in error. Also obtain the values of mean and variance for a random variable representing the number of errors. Use binomial distribution.

Solution:

(i) Let the word length = n digits = 8

Let the number of digits in error = k = 3

Let the number of digits with no error = (n – k) = 5

Let the probability of error per digit = p = 0.01

Let the probability of correct digit = 1 – p = 0.99

(ii) Now we have,

$$\begin{bmatrix}\text{Probability of n bits words} \\ \text{with k errors}\end{bmatrix} = P(X = k) = {}^nC_k . p^k (1-p)^{n-k}$$

Substituting the values, we get

$$P(X = k) = {}^8C_3.(0.01)^3.(0.99)^{8-3}$$

$$= \frac{8!}{(8-3)!\,3!}(0.01)^3.(0.99)^5 = 5.32 \times 10^{-5} \quad \textbf{Ans.}$$

Thus, the probability that 3 digits out of 8 are in error is 5.32×10^{-5}.

The mean value of the random variable representing the errors is given by,

$$m_x = np = 8 \times 0.01 = 0.08 \quad \textbf{Ans.}$$

The variance is given by

$$\sigma_x^2 = np(1-p)$$

Therefore,

$$\sigma_x^2 = 0.08\,(0.99) = 0.0792 \quad \textbf{Ans.}$$

5.18.3. Poisson Distribution

This is another standard probability distribution used for the discrete random variables. As the number 'n' increases, the binomial distribution becomes difficult to handle. If 'n' is very large, probability 'p' is very small and the mean value 'np' is finite, then the binomial distribution can be approximated by the Poisson distribution. Poisson distribution is thus the limiting case of binomial distribution.

The probability of the random variable having Poisson distribution is given by:

$$P(X = k) = \frac{m^k \cdot e^{-m}}{k!} \qquad ...(5.135)$$

where, 'm' is the mean value and m = np. Substituting this value of m in equation (5.135), we get,

$$P(X = k) = \frac{(np)^k \cdot e^{-np}}{k!} \qquad ...(5.136)$$

The mean value of Poisson distribution is given by:

$$m_x = np \qquad ...(5.137)$$

and the variance of the Poisson distribution is given by:

$$\sigma_x^2 = np \qquad ...(5.138)$$

Thus, for the Poisson distribution, mean and variance are equal.

Thus, the standard deviation $\sigma_x = \sqrt{np}$ ...(5.139)

EXAMPLE 5.24. Suppose 10,000 digits are transmitted over a noisy transmission channel having error probability per digit equal to 5 × 10^{-5}. Estimate the probability of getting two digits in errors. Use the Poisson's distribution.

Solution: Let us define a random variable such that the number of errors is upto 2

Therefore, the probability of getting upto two errors is given by,

$$P(X = k) = \frac{(np)^k \cdot e^{-np}}{k!} \quad \text{using equation (5.135)}$$

Here, n = Number of digits = 10000

k = Number of digits in error = 2

p = Probability of error per digit = 5×10^{-5}

Substituting these values, we get,

$$P(X = k) = \frac{(10{,}000 \times 5 \times 10^{-5})^2 \cdot e^{-(10{,}000 \times 5 \times 10^{-5})}}{2!} = \frac{(0.5)^2 \cdot e^{-0.5}}{2}$$

Hence, $P(X = k) = 0.0758$ **Ans.**

Lecture 7

Random Process

Using a random variable X, we can measure the temp of a city and record values of X at noon over many days. From this data, we can determine $p_X(x)$, the PDF of the RV X.

But since the temp is a function of time and it may have an exactly different distribution at different time of the day and have entirely different distribution from the temp at different times. Still the two temp may be related via a Joint pdf. Thus this RV x is a function of time and can be expressed as x(t) defined for a time in

terval$t \in [t_a, t_b]$. A RV that is a function of time and is random for every instant $[t_a, t_b]$. A RV that is a function of time is called a Random Process or stochastic process. Thus, a random process is a collection of infinite number of RVs. Communication signals as well as noises, typically random and varying with time, as well characteristic by random variable.

From RV to Random process

To specify the Random ProcessX(t), we need to record daily temperatures for each value of t (for each time of the day). This can be done by recording temperatures of every instant of the day, which gives a waveform $X(t, \delta_1)$, where $\delta_1$indicates the day for which the record was taken. The process is repeated for a large no of days. The collection of all possible days is called **ensemble**of therandom processX(t), an example is shown below

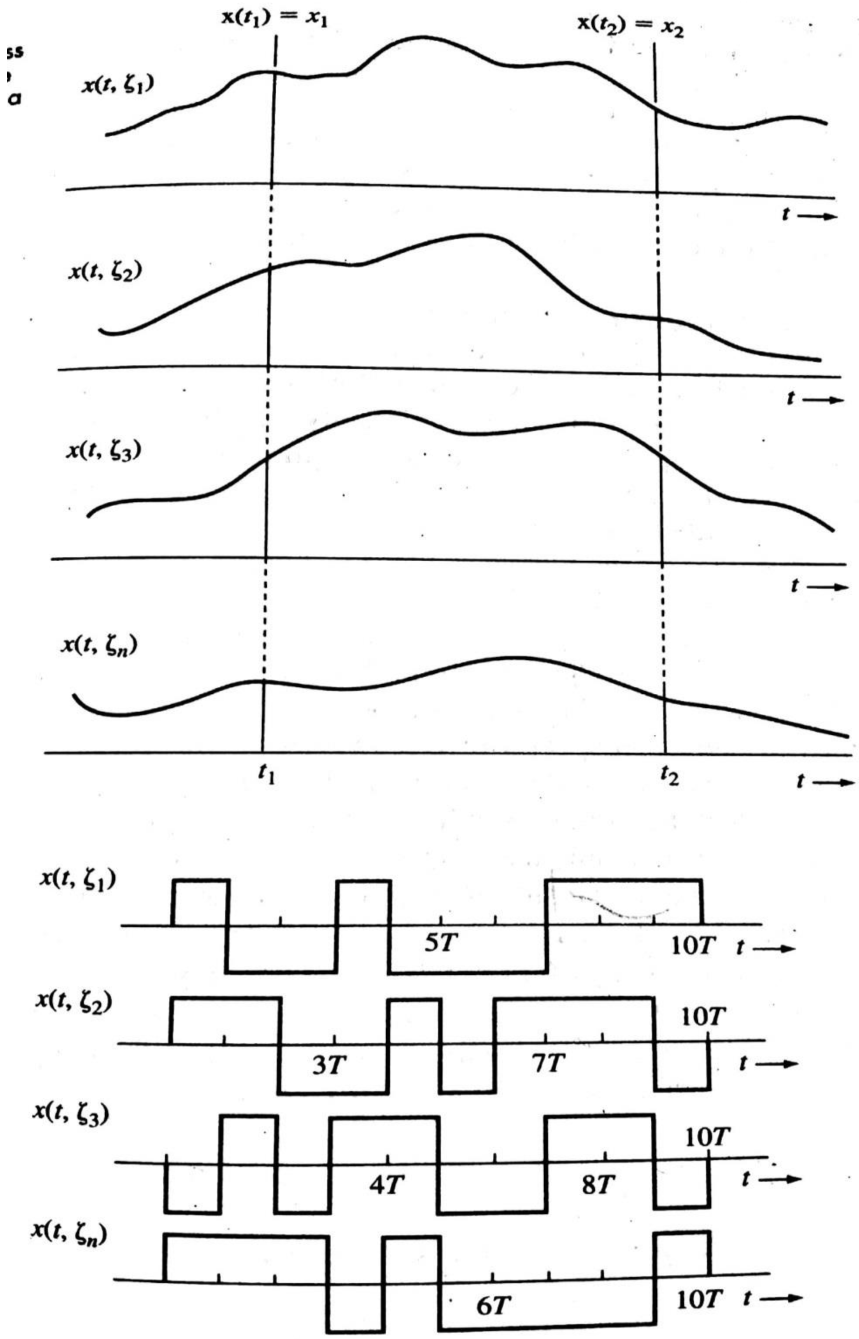


Fig: Random process to represent temp of a city , Ensemble with a finite no of sample fn (o/p of binary data generator over the period 0-10T)

Ensemble may be finite or infinite. Above fig, In 1st case, ensemble has infinite waveforms. In 2nd case, the waveforms are limited over the interval.

Other definition: Random process is the outcome of an experiment, where the outcome of each trial is a waveform (a sample function) that is a function of time.

##Waveforms in the ensemble are not random, but deterministic.

Randomness is associated not with the waveform but with the uncertainty as to which waveform would occur in a given trial. This is analogues to the case of RV of tossing a coin.

Autocorrelation Function of a Random Process

The spectral component of a process depends on the rapidity of the amplitude change with time. This can be measured by correlating amplitudes at $t1$and $t1 + \tau$. Fig shows the two slow and running processes. For $x(t)$, the amplitudes at $t1$ and $t1 + \tau$ are similar, thus have stronger correlation. For $y(t)$, the amplitudes at $t1$ and $t1 + \tau$ have little resemblance, thus weaker correlation. **The Correlation is a measure of the similarity of two RV.** If the RVs $x(t_1)$ and $x(t_2)$are denoted by $x_1$and x_2, then the autocorrelation function $R_X(t_{1,}t_2)$ is defined as

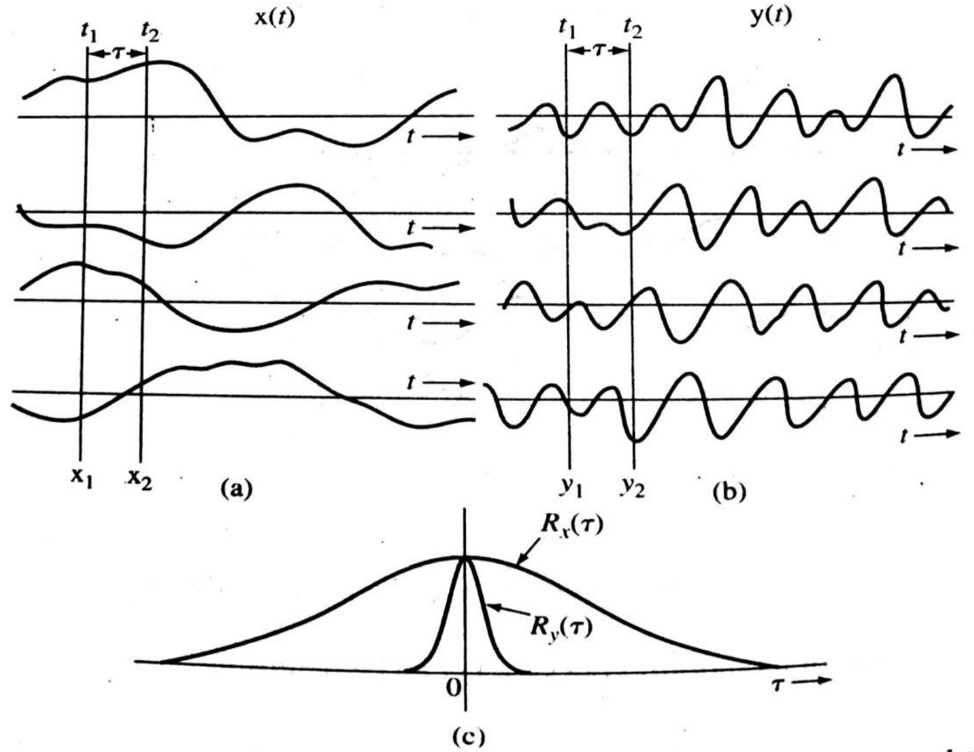
$$R_X(t_{1,}t_2) = \overline{x(t_1)x(t_2)} = \overline{x_1x_2}$$

It is computed by multiplying amplitudes at t_1 and t_2 of a sample function and the averaging the product over the ensemble. The product $x_1x_2$will be positive for most sample function of $x(t)$, but y_1y_2 remains is equally likely positive or negative. Hence, $\overline{x_1x_2}$ will be larger than $\overline{y_1y_2}$ and x_1 and x_2 will show correlation for larger values of τ and y_1 and y_2 will lose correlation quickly.

Thus, $R_X(t_{1,}t_2)$, the autocorrelation function of $x(t)$ provides valuable information about the frequency content of the process. PSD of autocorrelation function is given by

$$R_X(t_{1,}t_2) = \overline{x_1x_2} = \iint_{-\infty}^{\infty} x_1x_2p_x(x_{1,}\, x_{2;}t_{1,}t_2)dx_1dx_2$$

Fig.



Lecture 8

Stationary (Strict sense stationary) and Non-Stationary Process

A Random process whose statistical characteristics do not change with time in classified as a **Stationary random process**, thus a shift in time origin is not possible to detect. Thus, the pdf of x at $t_1$and at $t_1 + t_0$ must be same. This is possible only if $f_x(x;t)$ is independent of t.

n^th order density or higher order pdf can be written as

$$f_X(x_1x_2 \dots \dots x_n; t_1, t_2 \dots \dots t_n) = f_X(x_1x_2 \dots \dots x_n; t_1 + T, t_2 + T \dots \dots t_n + T)$$

Where $f_X(x_1x_2 \dots \dots x_n; t_1, t_2 \dots \dots t_n) = \frac{\partial^n F_X(x_1x_2 \dots\dots x_n; t_1, t_2 \dots\dots t_n)}{\partial x_1\, \partial x_2 \dots \partial x_n}$

Thus,for SSS, first order density $f_x(x_1;t) = f_X(x_1)$ and so on…

Also, for a real stationary process,

$$R_X(t_{1,}t_2) = R_X(t_2 - t_1)$$

The random process x(t) representing the temperature of a city is an example of a non-stationary process because the temperature statistics (e.g. mean value) depend on the time of the day.
Noise process is stationary because its statistics (mean, mean square value) do not change with time.

Wide sense Stationary

A process that is not stationary in that strict sense, may yet have a mean value and autocorrelation function that are independent of the shift of time origin.
i.e.$E[X(t)] = m\ (constant)$
i.e.
Autocorrelation depends only on time difference.
E$[X(t)X(t+\tau)] =\ R_X(t_2 - t_1) = R_X(\tau)$
Putting $\tau = 0$, we get

$$\mathrm{E}[X^2(t)] = R_X(0)$$

i.e. average power of WSS is constant and independent of time.

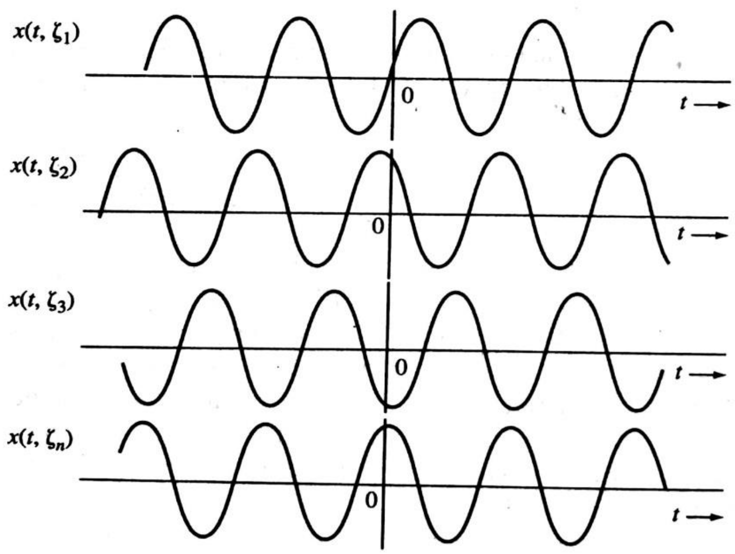
Show that the random process

$$x(t) = A\cos(\omega_c t + \Theta)$$

where Θ is an RV uniformly distributed in the range $(0, 2\pi)$, is a wide-sense stationary process.

The ensemble (Fig. 9.5) consists of sinusoids of constant amplitude A and constant frequency ω_c, but the phase Θ is random. For any sample function, the phase is equally likely to have any value in the range $(0, 2\pi)$. Because Θ is an RV uniformly distributed over the range $(0, 2\pi)$, one can determine[1] $p_x(x, t)$ and, hence, $\overline{x(t)}$, as in Eq. (9.2). For this particular case, however, $\overline{x(t)}$ can be determined directly as a function of random variable Θ:

$$\overline{x(t)} = \overline{A\cos(\omega_c t + \Theta)} = A\,\overline{\cos(\omega_c t + \Theta)}$$



Because $\cos(\omega_c t + \Theta)$ is a function of an RV Θ, we have [see Eq. (8.61b)]

$$\overline{\cos(\omega_c t + \Theta)} = \int_0^{2\pi} \cos(\omega_c t + \theta) p_\Theta(\theta)\, d\theta$$

Because $p_\Theta(\theta) = 1/2\pi$ over $(0,\ 2\pi)$ and 0 outside this range,

$$\overline{\cos\,(\omega_c t + \Theta)} = \frac{1}{2\pi}\int_0^{2\pi} \cos\,(\omega_c t + \theta)\,d\theta = 0$$

Hence,

$$\overline{\mathrm{x}(t)} = 0 \tag{9.7a}$$

Thus, the ensemble mean of sample function amplitudes at any instant t is zero.

The autocorrelation function $R_\mathrm{x}(t_1,\ t_2)$ for this process also can be determined directly from Eq. (9.3a),

$$\begin{aligned}
R_\mathrm{x}(t_1,\ t_2) &= \overline{A^2 \cos\,(\omega_c t_1 + \Theta)\cos\,(\omega_c t_2 + \Theta)} \\
&= A^2\,\overline{\cos\,(\omega_c t_1 + \Theta)\cos\,(\omega_c t_2 + \Theta)} \\
&= \frac{A^2}{2}\left\{\overline{\cos\,[\omega_c(t_2 - t_1)]} + \overline{\cos\,[\omega_c(t_2 + t_1) + 2\Theta]}\right\}
\end{aligned}$$

The first term on the right-hand side contains no RV. Hence, $\overline{\cos[\omega_c(t_2 - t_1)]}$ is $\cos\,[\omega_c(t_2 - t_1)]$ itself. The second term is a function of the uniform RV Θ, and its mean is

$$\overline{\cos\,[\omega_c(t_2 + t_1) + 2\Theta]} = \frac{1}{2\pi}\int_0^{2\pi} \cos\,[\omega_c(t_2 + t_1) + 2\theta]\,d\theta = 0$$

Hence,

$$R_\mathrm{x}(t_1,\ t_2) = \frac{A^2}{2}\cos\,[\omega_c(t_2 - t_1)] \tag{9.7b}$$

or

$$R_\mathrm{x}(\tau) = \frac{A^2}{2}\cos\,\omega_c\tau \qquad \tau = t_2 - t_1 \tag{9.7c}$$

From Eqs. (9.7a) and (9.7b) it is clear that $\mathrm{x}(t)$ is a wide-sense stationary process.

Example 6.18

Consider a random binary process X, synchronous with clock assumes any of the two values +1 or −1 with equal probability. At any clock trigger, arriving at an interval of T, the transition probabilities are also equal. Find autocorrelation function $R_X(\tau)$.

Solution

From Eq. (6.158),

$$\text{the PSD } G_X(f) = V_b^2 T_b \left(\frac{\sin \pi f T_b}{\pi f T_b}\right)^2 \Bigg|_{V_b=1, T_b=T}$$

$$= T\left(\frac{\sin \pi f T}{\pi f T}\right)^2$$

Thus $R_X(\tau) = F^{-1}[G_X(f)]$

$$= F^{-1}\left[T\left(\frac{\sin \pi f T}{\pi f T}\right)^2\right] = \begin{cases} 1-|\tau|/T & |\tau| < T \\ 0 & |\tau| > T \end{cases}$$

[From Additional Problem 16 of Chapter 1]

Example 6.19

Consider a random process, $X(t) = A\cos(\omega t + \theta)$ where θ is a uniform random variable in the range $[-\pi, \pi]$ and A, ω are constant. Find if x(t) is WSS.

Solution

Since θ is uniformly distributed, its pdf, $f_\theta(\theta) = \dfrac{1}{2\pi}$ for $-\pi \le \theta \le \pi$ and 0 elsewhere.

Then, mean $m = E[X(t)] = \int_{-\infty}^{\infty} A\cos(\omega t + \theta) f_\theta(\theta) d\theta$

$$= \frac{A}{2\pi}\cos(\omega t + \theta) d\theta = 0$$

And autocorrelation

$$R(\tau) = E[X(t)\, X(t+\tau)]$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\cos[\omega(t+\tau) + (\theta)]d\theta$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}[\cos \omega\tau + \cos(2\omega t + 2\theta + \omega\tau)d\theta$$

$$= \frac{A^2}{2}\cos \omega\tau$$

Since, mean is constant and autocorrelation depends on time difference only, $x(t)$ is WSS.

Example 6.20

Given, a WSS random process X(t) with mean $E[X(t)] = m$. This is applied as input to an LTI system with impulse response $h(t) = e^{-at}u(t)$. Find mean of the output.

Solution

From Example 1.18 of Chapter 1, $H(\omega) = \dfrac{1}{a + j\omega}$

If $Y(t)$ is output of LTI system, required mean $E[y(t)] = mH(0) = m/a$

Example 6.21

The message, a random process M(t) is mixed with a white channel noise N(t). If $S_M(\omega) = \dfrac{1}{1+\omega^2}$ and $S_M(\omega) = 0.2$, find optimal filter that maximizes output SNR. Is the filter realizable?

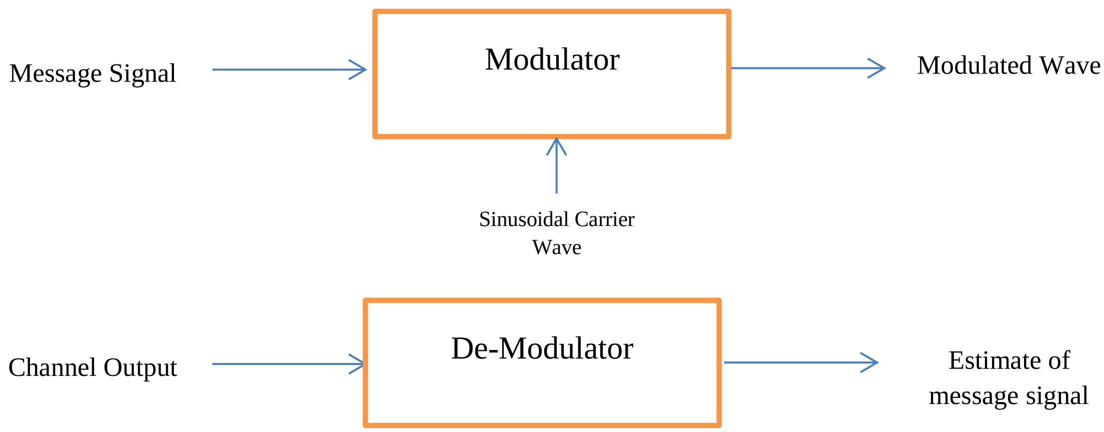
Do only 6.19 & 6.20

Unit-II (Amplitude Modulation) (Lecture-9,10)

Lecture 9

Introduction

The purpose of communication system is to transmit information bearing signals through a communication signal separating the transmitter from the receiver. Information bearing signals also called baseband signals designate the band of frequencies representing the original signal as delivered by a source of information. The proper use of the communication channel requires a shift of the range of baseband frequencies into other frequency ranges suitable for transmission and corresponding shift back to original frequency range after reception. A shift of the range of frequencies in a signal is accomplished by using Modulation, which is defined as the process by which some characteristics of a carrier is varied in accordance with a modulating wave.
Define terms: Modulating Wave, Modulated Wave, De-modulation



Above Fig shows the process of Modulator and Demodulator. The signal received from the transmitter, the receiver input includes channel noise. The degradation in receiver performance due to channel noise is determined by the type of modulation used.

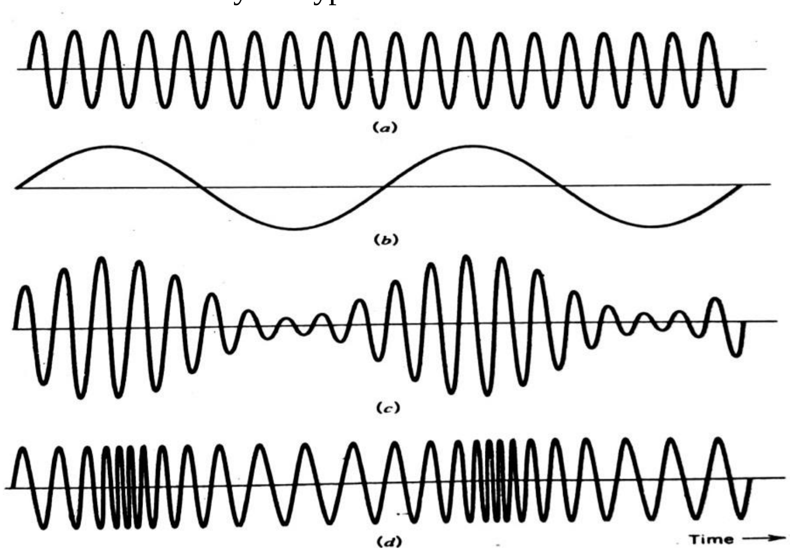


Fig 2.1 (a) Carrier (b) Sinusoidal Modulating Signal (c) Amplitude Modulated signal (d) Frequency Modulated signal

Need for Modulation / Frequency Translation

Frequency Translation serves the following purposes

1. **Frequency Multiplexing**- Transmission of multiple signals along a single transmission channel demands signals recovery in a distinguishable manner. Such multiple transmission i.e. multiplexing may be achieved by translating each one of the original signals to a different frequency range. If one signal is translated to a frequency range f_1' and f_2' and the second signal is translated to a frequency range f_1'' and f_2'' and so on, . If these new frequency ranges do not overlap, then the signal may be separated at the receiver end by appropriate bandpass filters.

   Give example via figure

2. **Frequency Translation**
   With the help of this method, it is easy to adjust the low frequency signal into higher frequency range suitable to be transmitted over the allocated frequency range for a particular application. Like in AM, a 1 KHz signal is translated to 1 MHz using a carrier of high frequency.

3. **Practicability of Antennas**
   When free space is the communication channel, antenna radiate and receive the signal. It turns out that antennas operate effectively only when their dimensions are of the order of magnitude of the wavelength of the signal being transmitted. A signal of freq1KHz (audio tone length may be reduced to the point of practicability by translating the audio tone to a higher frequency
4. **Narrowbanding**
   Suppose a signal is of the range 50 to 10^4 Hz. Ratio of highest to lowest audio frequency is 200. Therefore an antenna suitable for use at one end of the range would be too short and at other end would be too long. Suppose the signal is translated between (10^6+50) to (10^6+10^4). The ratio of highest to lowest frequency is now 1.01. This is called conversion of wideband signal to narrowband signal which may be well conveniently presented. Here wide/narrow band means fractional change in frequency from one band edge to the other
5. **Common Processing**
   It may happen that we may have to process, a number of signals similar in general character but occupying different spectral ranges. It will be necessary, as we go from signal to signal, to adjust the frequency range of our processing apparatus to correspond to the frequency range of the signal to be processed. If the processing apparatus is rather elaborate, it may well be wiser to leave the processing range of each signal in turn to correspond to this fixed frequency range.

Method of Frequency Translation

Method of Frequency Translation →

A signal may be translated to a new spectral range by multiplying the signal with an auxilary signal.

Let the signal be

$$V_m(t) = A_m \cos \omega_m t = A_m \cos 2\pi f_m t$$

$$= \frac{A_m}{2}\left(e^{j\omega_m t} + e^{-j\omega_m t}\right)$$

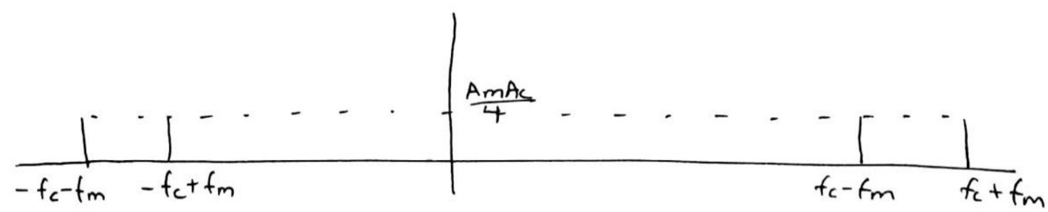
$$= \frac{A_m}{2}\left(e^{j2\pi f_m t} + e^{-j2\pi f_m t}\right)$$

Consider now the result of multiplication of $V_m(t)$ with an auxiliary signal

$$V_c(t) = A_c \cos \omega_c t = \frac{A_c}{2}\left(e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right)$$

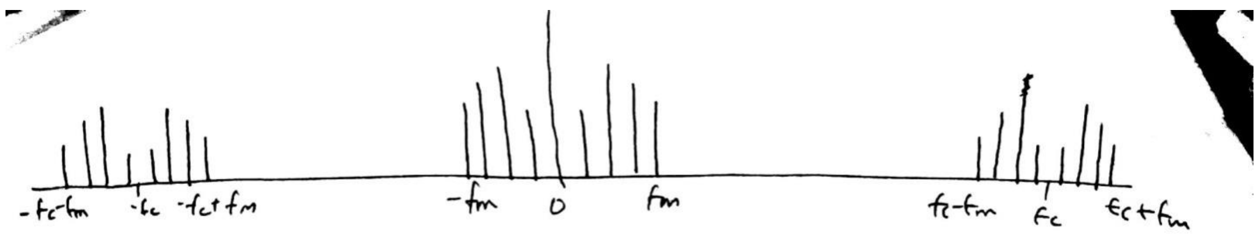
in which A_c is the constant amplitude and f_c is the frequency.

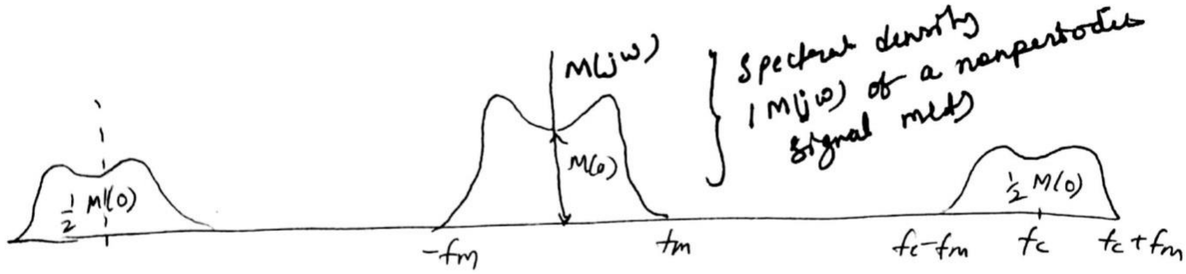
$$V_m(t)\,V_c(t) = \frac{A_m A_c}{4}\left[e^{j(\omega_c+\omega_m)t} + e^{-j(\omega_c+\omega_m)t} + e^{j(\omega_c-\omega_m)t} + e^{-j(\omega_c-\omega_m)t}\right]$$



A generalization of fig ① is shown in ② where a signal is chosen which consists of a superposition of four sinusoidal signals.

→





Spectral density of $m(t)\cos 2\pi f_c t$

The operation of multiplying a signal with an auxiliary sinusoidal signal is called <u>mixing or hetrodyning</u>. In the translated signal, the part of the signal which consists of spectral component above the auxiliary signal, in the range f_c to f_c+f_m is called upper-sideband signal. The part of the signal which consists of the spectral components below the auxiliary signal, in the range f_c-f_m to f_c is called lower-sideband signal.

Lecture 10

Amplitude Modulation

A frequency translated signal from which the baseband signal is easily recoverable is generated by adding to the product of baseband and carrier, the carrier signal itself and is named as Amplitude Modulation. The presence of carrier gives it another name Double Side band with carrier (DSB-C). Thus AM is defined as a process in which the amplitude of the carrier wave is varied about a mean value, linearly with the baseband signal $m(t)$.

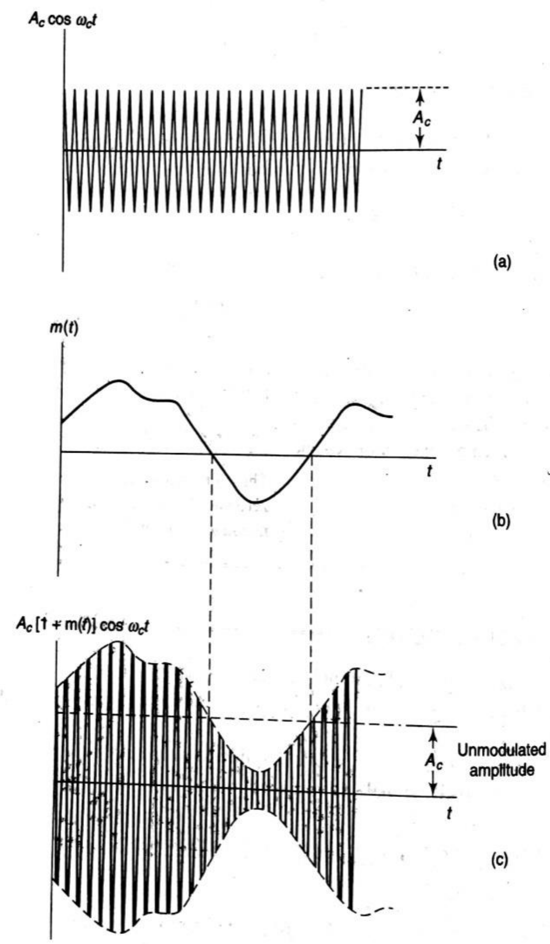


Fig 2.2 AM process

Let the baseband signal be

$$m(\mathrm{t}) = A_m \cos(w_m t)$$

And the carrier signal be

$$c(\mathrm{t}) = A_c \cos(w_c t)$$

The amplitude of the amplitude modulated waveis given by

$$A = A_c + A_m \cos(w_m t) = A_c + mA_c \cos(w_m t)$$
$$= A_c(1 + m\cos(w_m t))$$

Where $m = \frac{A_m}{A_c}$ called modulation index

The instantaneous voltage of the resulting amplitude modulated wave is

$$v(t) = A\cos(w_c t) = A_c[1 + m\cos(w_m t)]\cos(w_c t)$$

$$=[A_c + m(t)]\cos(w_c t) \qquad \text{(Ref:T\&S)}$$

$$v(t) = A_c\cos(w_c t) + mA_c\cos(w_m t)\cos(w_c t)$$

$$= A_c\cos(w_c t) + mA_c[\cos(w_c - w_m)t) + \cos(w_c + w_m)t)]$$

Thus, the equation of AM contains carrier and two sidebands of frequency $(f_c - f_m)$ (LSB) and $(f_c + f_m)$ (USB). Thus the Bandwidth requirement for AM is twice the frequency of the modulating signal. Thus in AM broadcasting, the bandwidth required is twice the highest modulating frequency

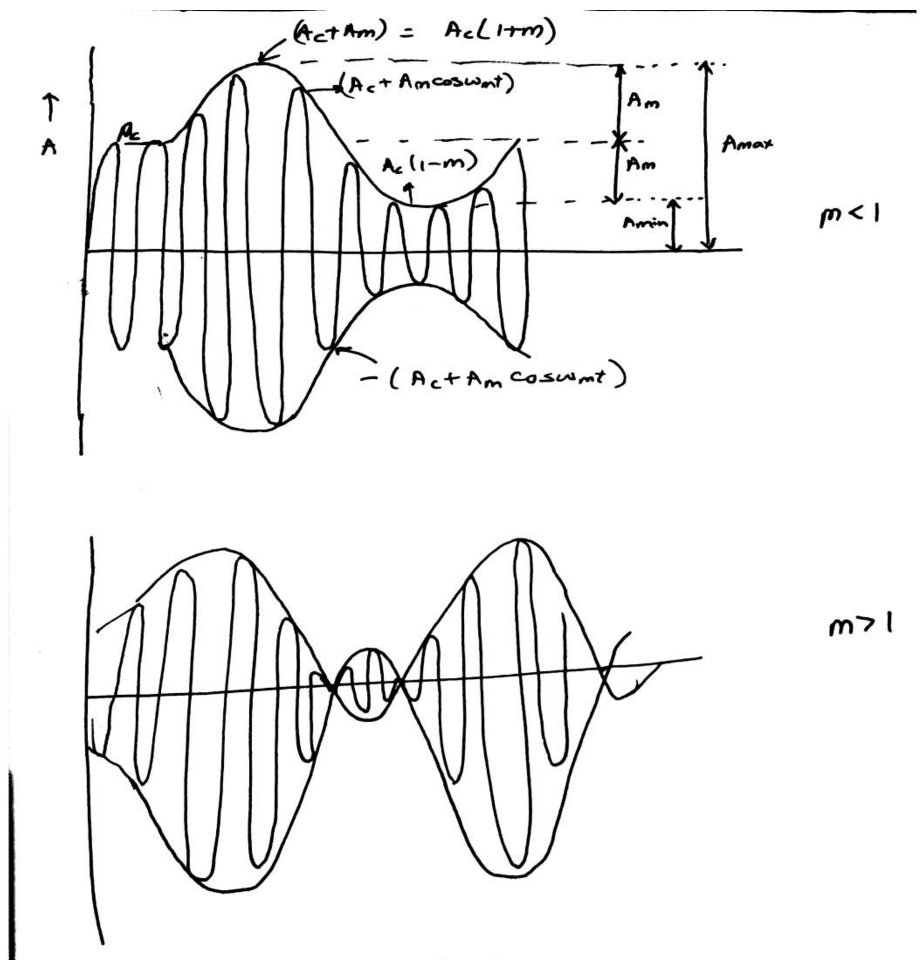


Fig 2.3. Amplitude Modulated wave (a) m<1 (b) m>1

For m>1, the carrier wave becomes overmodulated, resulting in carrier phase reversals whenever the factor m crosses zero. In this case, amplitude of a baseband signal exceeds maximum carrier amplitude i.e.

$|m(t)| > A_c$. Here m > 1

Modulation Index

$$A_m = \frac{A_{max} - A_{min}}{2}$$

and

$$A_c = A_{max} - A_{m=}A_{max} - \frac{A_{max} - A_{min}}{2} = \frac{A_{max} + A_{min}}{2}$$

From the above 2 equations,

$$m = \frac{A_m}{A_c} = \frac{A_{max} - A_{min}}{A_{max} + A_{min}}$$

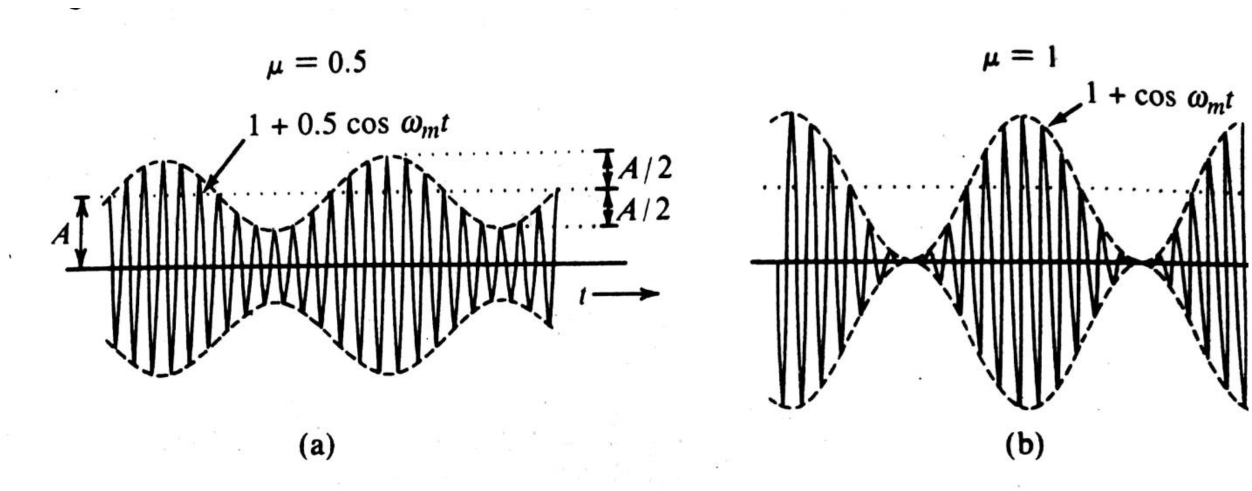
Example 1- The tuned circuit of the oscillator is an simple AM transmitter employs a 50 microhenry (50 μH) coil and a 1 nanofarad (nF) capacitor. If the oscillator output is modulated by audio frequencies upto 10 kHz, what is the frequency range occupied by the sidebands

Sol:

$$fc = \frac{1}{2\pi\sqrt{LC}} = 712\ kHz$$

Since the highest modulating frequency is 10 kHz, the frequency range occupied by the sidebands are at 722 kHz and 702 kHz.

Example 2-, For AM wave for m=0.5 and m=1, sketch the AM Wave



Power Content in an AM Wave

... of the AM wave may be drawn.

1.3 Power Relations in the AM Wave

It has been shown that the carrier component of the modulated wave has the same amplitude as the unmodulated carrier. That is, the amplitude of the carrier is unchanged; energy is either added or subtracted. The modulated wave contains extra energy in the two sideband components. Therefore, the modulated wave contains more power than the carrier had before modulation took place. Since the amplitude of the sidebands depends on the modulation index V_m/V_c, it is anticipated that the total power in the modulated wave will depend on the modulation index also. This relation may now be derived.

The total power in the modulated wave will be

$$P_t = \frac{V_{\text{carr}}^2}{R} + \frac{V_{\text{LSB}}^2}{R} + \frac{V_{\text{USB}}^2}{R} \text{ (rms)} \qquad \textbf{(3-11)}$$

where all three voltages are (rms) values ($\sqrt{2}$ converted to peak), and R is the resistance, (e.g., antenna resistance), in which the power is dissipated. The first term of Equation (3-11) is the unmodulated carrier power and is given by

$$P_c = \frac{V_{\text{carr}}^2}{R} = \frac{(V_c/\sqrt{2})^2}{R}$$

$$= \frac{V_c^2}{2R} \qquad \textbf{(3-12)}$$

Similarly,

$$P_{LSB} = P_{USB} = \frac{V_{SB}^2}{R} = \left(\frac{mV_c/2}{\sqrt{2}}\right)^2 \div R = \frac{m^2V_c^2}{8R}$$

$$= \frac{m^2}{4}\frac{V_c^2}{2R} \tag{3-13}$$

Substituting Equations (3-12) and (3-13) into (3-11), we have

$$P_t = \frac{V_c^2}{2R} + \frac{m^2}{4}\frac{V_c^2}{2R} + \frac{m^2}{4}\frac{V_c^2}{2R} = P_c + \frac{m^2}{4}P_c + \frac{m^2}{4}P_c$$

ations

$$\boxed{\frac{P_t}{P_c} = 1 + \frac{m^2}{2}} \tag{3-14}$$

Equation (3-14) relates the total power in the amplitude-modulated wave to the unmodulated carrier power. This is the equation which must be used to determine, among other quantities, the modulation index in instances not covered by Equation (3-10) of the preceding section. The methods of doing this, as well as solutions to other problems, will be shown in exercises to follow.

It is interesting to note from Equation (3-14) that the maximum power in the AM wave is $P_t = 1.5P_c$ when $m = 1$. This is important, because it is the maximum power that relevant amplifiers must be capable of handling without distortion.

EXAMPLE 3-2 A 400-watt (400-W) carrier is modulated to a depth of 75 percent. Calculate the total power in the modulated wave.

SOLUTION

$$P_t = P_c\left(1 + \frac{m^2}{2}\right) = 400\left(1 + \frac{0.75^2}{2}\right) = 400 \times 1.281$$

$$= 512.5 \text{ W}$$

EXAMPLE 3-3 A broadcast radio transmitter radiates 10 kilowatts (10 kW) when the modulation percentage is 60. How much of this is carrier power?

SOLUTION

$$P_c = \frac{P_t}{1 + m^2/2} = \frac{10}{1 + 0.6^2/2} = \frac{10}{1.18} = 8.47 \text{ kW}$$

Current calculations The situation which very often arises in AM is that the modulated and unmodulated currents are easily measurable, and it is then necessary to calculate the modulation index from them. This occurs when the antenna current of the transmitter is metered, and the problem may be resolved as follows. Let I_c be the unmodulated current and I_t the total, or modulated, current of an AM transmitter, both being rms values. If R is the resistance in which these currents flow, then

$$\frac{P_t}{P_c} = \frac{I_t^2 R}{I_c^2 R} = \left(\frac{I_t}{I_c}\right)^2 = 1 + \frac{m^2}{2}$$

nt
ulation $\Rightarrow$

$$\boxed{\frac{I_t}{I_c} = \sqrt{1 + \frac{m^2}{2}}} \quad \text{or} \quad I_t = I_c \sqrt{1 + \frac{m^2}{2}} \qquad \textbf{(3-15)}$$

EXAMPLE 3-4 The antenna current of an AM transmitter is 8 amperes (8 A) when only the carrier is sent, but it increases to 8.93 A when the carrier is modulated by a single sine wave. Find the percentage modulation. Determine the antenna current when the percent of modulation changes to 0.8.

SOLUTION

$$\left(\frac{I_t}{I_c}\right)^2 = 1 + \frac{m^2}{2}$$

$$\frac{m^2}{2} = \left(\frac{I_t}{I_c}\right)^2 - 1$$

$$m = \sqrt{2\left[\left(\frac{I_t}{I_c}\right)^2 - 1\right]} \qquad \textbf{(3-16)}$$

Here

$$m = \sqrt{2\left[\left(\frac{8.93}{8}\right)^2 - 1\right]} = \sqrt{2[(1.116)^2 - 1]}$$

$$= \sqrt{2(1.246 - 1)} = \sqrt{0.492} = 0.701 = 70.1\%$$

For the second part we have

$$I_t = I_c\sqrt{1 + \frac{m^2}{2}} = 8\sqrt{1 + \frac{0.8^2}{2}} = 8\sqrt{1 + \frac{0.64}{2}}$$

$$= 8\sqrt{1.32} = 8 \times 1.149 = 9.19\text{A}$$

Although Equation (3-16) is merely (3-15) in reverse, it will be found useful in other problems.

Modulation by several sine waves In practice, modulation of a carrier by several sine waves simultaneously is the rule rather than the exception. Accordingly, a way has to be found to calculate the resulting power conditions. The procedure consists of calculating the total modulation index and then substituting it into Equation (3-14), from which the total power may be calculated as before. There are two methods of calculating the total modulation index.

1. Let V_1, V_2, V_3, etc., be the simultaneous modulation voltages. Then the total modulating voltage V_t will be equal to the square root of the sum of the squares of the individual voltages; that is,

   $$\boxed{V_t = \sqrt{V_1^2 + V_2^2 + V_3^2 + \cdots}}$$

   Dividing both sides by V_c, we get

$$\frac{V_t}{V_c} = \frac{\sqrt{V_1^2 + V_2^2 + V_3^2 + \cdots}}{V_c}$$

$$= \sqrt{\frac{V_1^2}{V_c^2} + \frac{V_2^2}{V_c^2} + \frac{V_3^2}{V_c^2} + \cdots}$$

that is,

$$m_t = \sqrt{m_1^2 + m_2^2 + m_3^2 + \cdots} \tag{3-17}$$

2. Equation (3-14) may be rewritten to emphasize that the total power in an AM wave consists of carrier power and sideband power. This yields

$$P_t = P_c\left(1 + \frac{m^2}{2}\right) = P_c + \frac{P_c m^2}{2} = P_c + P_{SB}$$

where P_{SB} is the total sideband power and is given by

$$P_{SB} = \frac{P_c m^2}{2} \tag{3-18}$$

If several sine waves simultaneously modulate the carrier, the carrier power will be unaffected, but the total sideband power will now be the sum of the individual sideband powers. We have

$$P_{SB_T} = P_{SB_1} + P_{SB_2} + P_{SB_3} + \cdots$$

Substitution gives

$$\frac{P_c m_t^2}{2} = \frac{P_c m_1^2}{2} + \frac{P_c m_2^2}{2} + \frac{P_c m_3^2}{2} + \cdots$$

$$m_t^2 = m_1^2 + m_2^2 + m_3^2 + \cdots$$

If the square root of both sides is now taken, Equation (3-17) will once again be the result.

It is seen that the two approaches both yield the same result. To calculate the total modulation index, *take the square root of the sum of the squares of the individual modulation indices*. Note also that this total modulation index must still not exceed unity, or distortion will result as with overmodulation by a single sine wave. Whether modulation is by one or many sine waves, the output of the modulated amplifier will be zero during part of the negative modulating voltage peak if overmodulation is taking place. This point is discussed further in Chapter 6, in conjunction with distortion in AM demodulators.

EXAMPLE 3-5 A certain transmitter radiates 9 kW with the carrier unmodulated, and 10.125 kW when the carrier is sinusoidally modulated. Calculate the modulation index, percent of modulation. If another sine wave, corresponding to 40 percent modulation, is transmitted simultaneously, determine the total radiated power.

SOLUTION

$$\frac{m^2}{2} = \frac{P_t}{P_c} - 1 = \frac{10.125}{9} - 1 = 1.125 - 1 = 0.125$$

$$m^2 = 0.125 \times 2 = 0.250$$

$$m = \sqrt{0.25} = 0.50$$

For the second part, the total modulation index will be

$$m_t = \sqrt{m_1^2 + m_2^2} = \sqrt{0.5^2 + 0.4^2} = \sqrt{0.25 + 0.16} = \sqrt{0.41} = 0.64$$

$$P_t = P_c\left(1 + \frac{m_t^2}{2}\right) = 9\left(1 + \frac{0.64^2}{2}\right) = 9(1 + 0.205) = 10.84 \text{ kW}$$

EXAMPLE 3-6 The antenna current of an AM broadcast transmitter, modulated to a depth of 40 percent by an audio sine wave, is 11 A. It increases to 12 A as a result of simultaneous modulation by another audio sine wave. What is the modulation index due to this second wave?

SOLUTION

From Equation (3-15) we have

$$I_c = \frac{I_t}{\sqrt{1 + m^2/2}} = \frac{11}{\sqrt{1 + 0.4^2/2}} = \frac{11}{\sqrt{1 + 0.08}} = 10.58 \text{ A}$$

Using Equation (3-16) and bearing in mind that here the modulation index is the total modulation index m_t, we obtain

$$m_t = \sqrt{2\left[\left(\frac{I_T}{I_c}\right)^2 - 1\right]} = \sqrt{2\left[\left(\frac{12}{10.58}\right)^2 - 1\right]} = \sqrt{2(1.286 - 1)}$$

$$= \sqrt{2 \times 0.286} = 0.757$$

From Equation (3-17), we obtain

$$m^2 = \sqrt{m_t^2 - m_1^2} = \sqrt{0.757^2 - 0.4^2} = \sqrt{0.573 - 0.16} = \sqrt{0.413}$$

$$= 0.643$$

AM Modulation

DSB-C Modulator

A multiplier is a device that yields an output a signal which is the product of the two input signals. Actually no simple physical device now exists which yields the product alone. On the contrary, all such devices yield at a minimum, not only the product but the input signals themselves. Thus, if inputs are $m(t)$and $\cos(w_c t)$, the device output contain the product $m(t)\cos(w_c t)$ and the input signals also.

e.g. if the baseband signal extends from zero frequency to 1000 Hz, while $f_c = 1000\ MHz$, then the sideband are 1,001,000 Hz and 999,000 Hz and the baseband signal is easily removed by a filter.

Thus, devices available for multiplication yield an output carrier and the lower/upper sidebands. Such output is called amplitude modulated signal, which is double side band with carrier. If we want only product alone, carrier must be suppressed.

2.3.1 DSB-C Modulator

The DSB-C modulator can be generated as discussed in Sec. 2.2.1 by using a mixer (multiplier) that generates the sidebands as well as carrier signal and the baseband message signal. For DSB-C modulator only the baseband message is to be filtered out. Also in any DSB-SC generator if carrier is added by a summer we get DSB-C modulation. However, for switching modulator (Example 2.1) a simpler circuit will do.

Consider the simple switching modulator circuits with only one diode as shown in Fig. 2.11. The BPF passes frequency $\omega_c \pm \omega_m$ where ω_m is maximum frequency of message signal; carrier is represented by $A\cos\omega_c t$. Consider the diode is ideal and carrier is stronger than message. The diode conducts when the combined signal (message plus carrier) is positive. Since, carrier is stronger than message signal the switching of diode is regulated by carrier only. The switching action can be approximated by a pulse train (Prob. 1.59) as

$$s(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \frac{1}{7}\cos 7\omega_c t + \ldots\right) \tag{2.9}$$

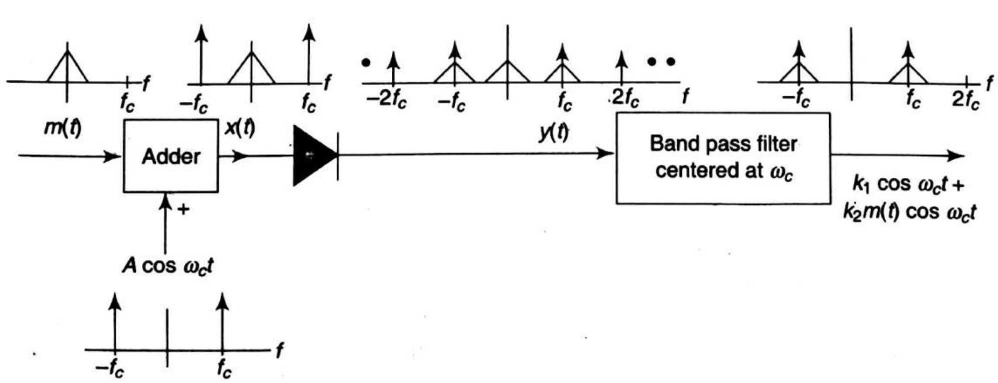
Now the combined signal message $m(t) + A\cos\omega_c t$ will appear at the output when the diode is switched on and otherwise not. Mathematically, diode output can be written as

$$y(t) = [m(t) + A\cos\ \omega_c t]s(t)$$

$$= [m(t) + A\cos\ \omega_c t]\left[\frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \frac{1}{7}\cos 7\omega_c t + \ldots\right)\right]$$

or
$$y(t) = \frac{m(t)}{2} + \frac{A}{2}\cos\ \omega_c t + \frac{2}{\pi}m(t)\cos\ \omega_c t + \frac{2A}{\pi}\cos\omega_c^2 t - \ldots$$

Thus $y(t)$ has baseband signal(1st), a dc term (from 4th), carrier (2nd), and higher harmonics of carrier. The BPF filter that passes $\omega_c \pm \omega_m$ will bring out $\frac{2}{\pi}m(t)\cos\omega_c(t)$ and the carrier component $\frac{A}{2}\cos\omega_c(t)$ which clearly is an AM or DSB-C signal.



AM Demodulation

Envelop Detector

The output of an envelop detector follows the envelop of the modulated signal as shown in fig. On the positive cycle of the input signal, the input grows and may exceed the charged value on the capacity $v_c(t)$, turning on the diode and allowing the capacitor C to charge up to the peak voltage of the input signal cycle. As the input signal falls below this peak value, it falls quickly below the capacitor voltage (which is nearly the peak voltage), thus causing the diode to open. The capacitor now discharges through the resistor R at a slow rate (with a time constant RC). During the next positive cycle, the same procedure repeats. As the input signal rises above the capacitor voltage, the diode conducts again. The capacitor again charges to the peak value of this new cycle. The capacitor discharges slowly during the cutoff period.

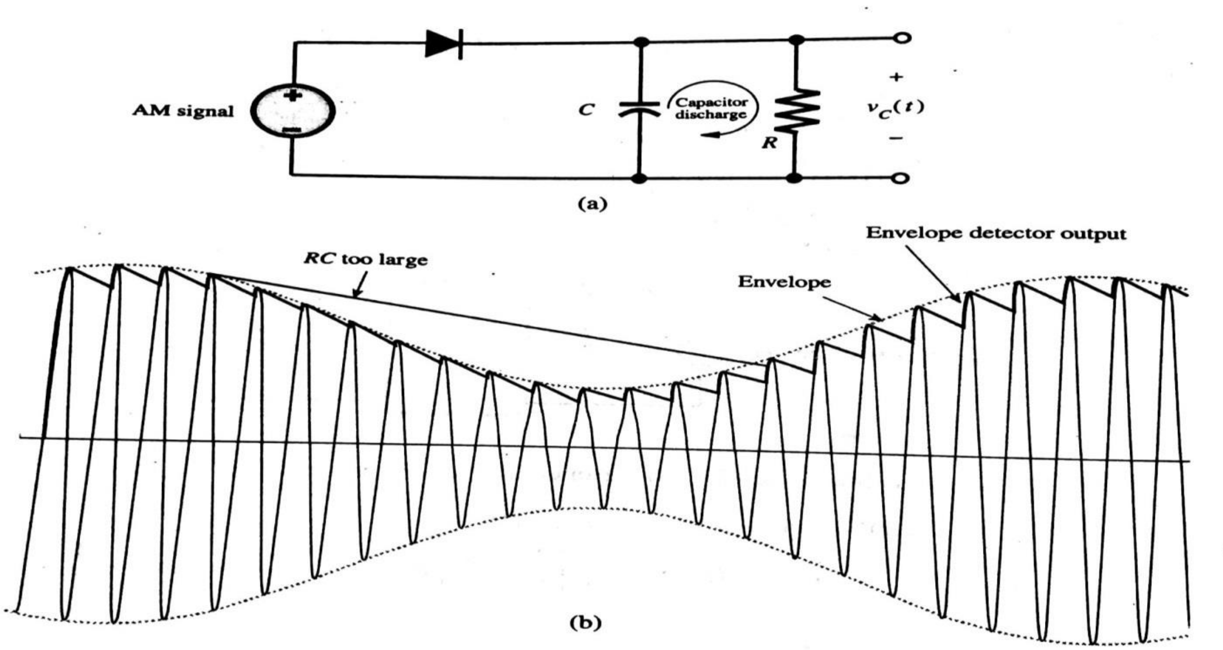


Fig. 2.4-Envelop Detector of AM

During each positive cycle, the capacitor charges up to the peak voltage of the input signal and then decays slowly until the next positive cycle as shown in Fig. 2.4. The output voltage $v_c(t)$ closely follows the (rising) envelop of the input AM cycle, Equally important, the slow capacity discharge via the resistor R allows the capacity voltage to follow a declining envelop. Capacitor discharge between positive peaks causes a ripple signal of frequency $w_c$in the output. This ripple can be reduced by choosing a larger time constant RC so the capacitor discharges very little between the positive peaks $(RC \gg \frac{1}{w_c})$. Picking RC to large would make it impossible for the capacitor voltage to follow a fast declining envelop (Fig 2.4b). Because the maximum rate of AM envelop decline is dominated by the bandwidth B of the message signal m(t), the design criterion of RC should be

$$\frac{1}{w_c} \ll RC \ll \frac{1}{(2\pi B)} \qquad \text{(B is same as } f_m)$$

The envelop detector output is $v_c(t) = A + m(t)$. The dc term can be blocked by a capacitor or a high pass filter. The ripple (saw tooth waveform) may further be reduced by another RC filter.

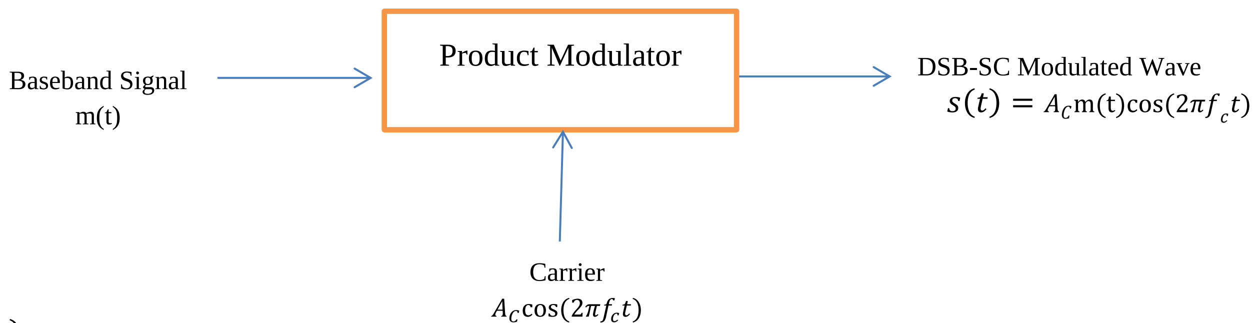
Linear Modulation scheme is defined by

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

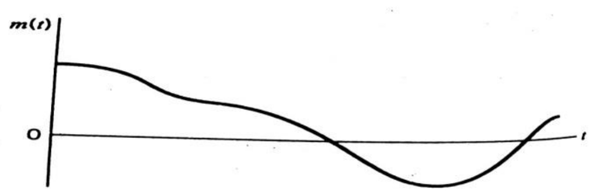
Where $s_I(t)$ is in-phase component of the modulated wave $s(t)$ and $s_Q(t)$is its quadrature component. The above equation is the canonical representation of a narrow band signal. There are 3 types of linear modulation involving a single message signal.

1) Double Side Band Suppressed Carrier (DSB-SC)- Only USB and LSB are transmitted
2) Single Side band (SSB) – Only One side band is transmitted (LSB or USB)
3) Vestigial Side Band (VSB)- Only a vestige (trace) of one of the side bands and a correspondingly modified version of the other side band are transmitted

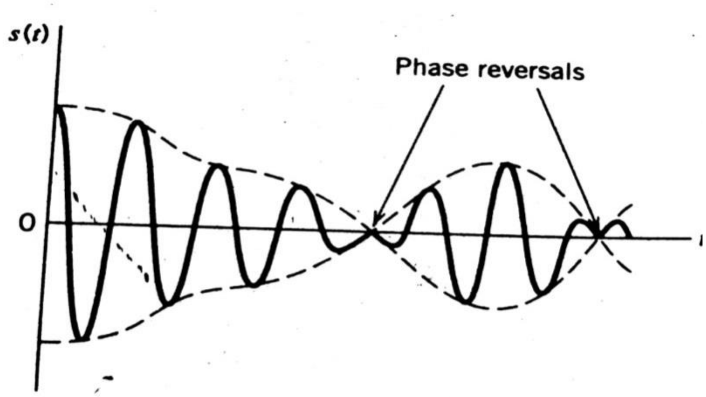
DSB-SC



$s(t) = A_c \mathrm{m(t)} \cos(2\pi f_c t)$



Baseband Signal



DSB-SC modulated signal

A DSB-SC signal $s(t)$is obtained is obtained by simply multiplying modulating signal $m(t)$ and carrier $\cos(w_c t)$. This is achieved by a product modulator as shown in the above figure. The modulated signal $m(t)$ undergoes a phase reversal whenever the message signal $m(t)$ crosses zero.

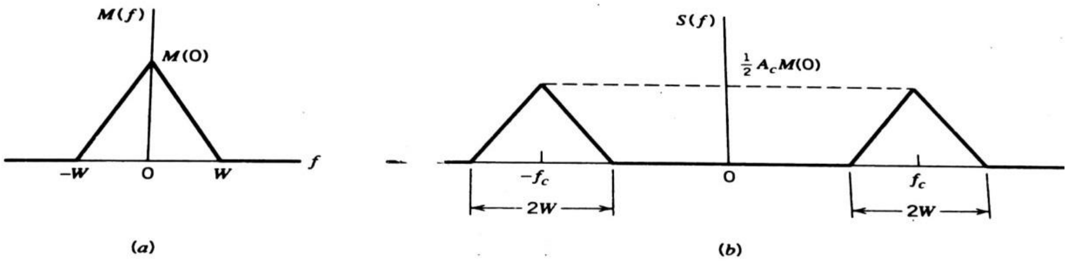


Fig. (a) Spectrum of Baseband Signal (b) Spectrum of DSB-SC modulated signal

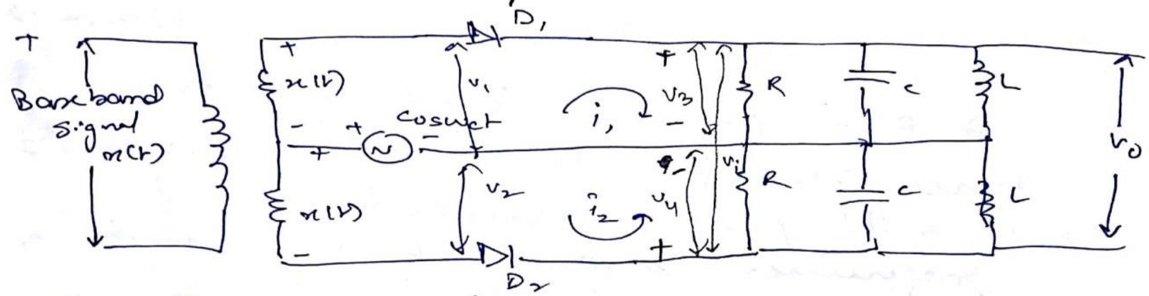
Modulation using Non-Linear Device (Balanced Modulator)

Generation of DSB-SC

M③

1) Balanced Modulator

If two non linear device such as diodes, transistors etc are connected in a balanced mode so as to supress the carriers of each other, then only side bands are left.



S/p voltage across diodes,

$v_1 = \cos w_c t + x(t)$ ①

$v_2 = \cos w_c t - x(t)$ ②

For D_1, the non-linear v-i relationship becomes

$i_1 = a v_1 + b v_1^2$ ③

for D_2, $i_2 = a v_2 + b v_2^2$ ④

Put ① in ③,

$i_1 = a[\cos w_c t + x(t)] + b[\cos w_c t + x(t)]^2$

$= a\cos w_c t + a x(t) + b[\cos^2 w_c t + x^2(t) + 2x(t)\cos w_c t]$

or $= a\cos w_c t + a x(t) + b\cos^2 w_c t + b x^2(t) + 2b\,x(t)\cos w_c t$

Similarly $i_2 = a[\cos w_c t - x(t)] + b[\cos w_c t - x(t)]^2$

$= a\cos w_c t - a x(t) + b\cos^2 w_c t + b x^2(t) - 2b\,x(t)\cos w_c t$

at the i/p of BPF
the net i/p voltage ~~across~~ BPF is

$$V_i = V_3 - V_4$$

Also $V_3 = i_1 R$

$V_4 = i_2 R$

and $V_i = R(i_1 - i_2)$

$$= R[2ax(t) + 4bx(t)\cos w_c(t)]$$

$$= 2R[ax(t) + 2bx(t)\cos w_c(t)]$$

A BPF passes a band of freq.
Since the BPF is centered around $\pm w_c$, it will pass a narrow band of frequencies centered at $\pm w_c$ with a BW of $2w_m$

$\therefore$ o/p of BPF

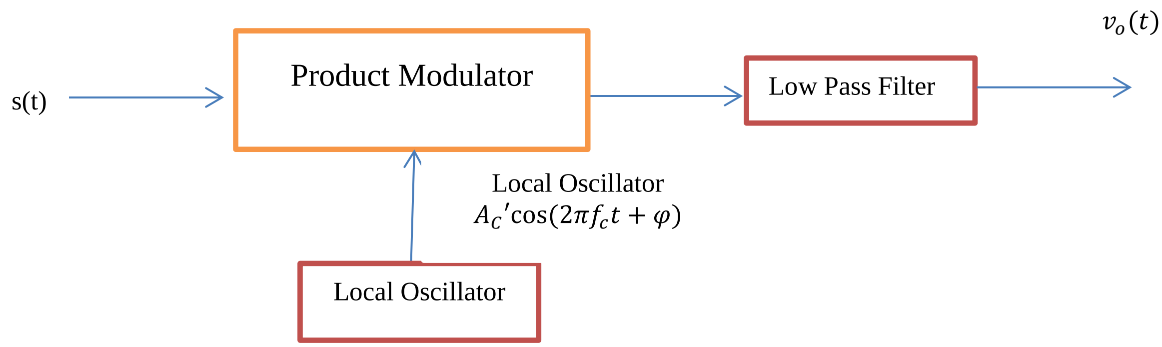
$$v_0 = 4aRx(t)\cos w_c t$$

$$= kx(t)\cos w_c t$$

which is DSB-SC

Coherent Detection of DSB-SC

The Baseband signal $m(t)$ can be uniquely recovered from a DSB-SC wave $s(t)$ with a locally generated sinusoidal wave and then low pass filtering the product as shown in figure below. It is assumed that the local oscillator signal is exactly coherent or synchronized, in both frequency and phase, with the carrier wave $c(t)$ used in the product modulator to generate $s(t)$. This method of generation is known as coherent detection or synchronous demodulation.



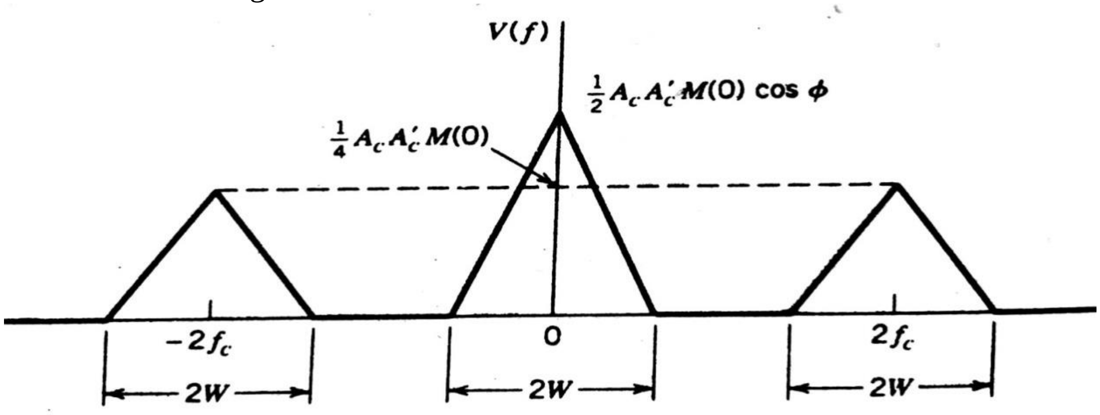
The output of the product modulator is

$$v(t) = A_C{}'\cos(2\pi f_c t + \varphi)s(t)$$

$$= A_C A_C' \cos(2\pi f_c t)\cos(2\pi f_c t + \varphi)\, m(t)$$

$$= \frac{1}{2} A_C A_C' \cos(4\pi f_c t + \varphi)\, m(t) + \frac{1}{2} A_C A_C' \cos(\varphi)\, m(t)$$

The first term in the above equation represents a DSB-SC modulated signal with a carrier frequency $2f_c$, whereas the second term is proportional to the baseband signal $m(t)$. The spectrumis as shown in fig below



Spectrum of product modulator output with a DSB-SC modulated wave as input

Hilbert Transform

Hilbert Transform

We now introduce for later use a new tool known as the **Hilbert transform**. We use $x_h(t)$ and $\mathcal{H}\{x(t)\}$ to denote the Hilbert transform of signal $x(t)$

$$x_h(t) = \mathcal{H}\{x(t)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\alpha)}{t-\alpha}\, d\alpha \tag{4.15}$$

Observe that the right-hand side of Eq. (4.15) has the form of a convolution

$$x(t) * \frac{1}{\pi t}$$

Now, application of the duality property to pair 12 of Table 3.1 yields $1/\pi t \Longleftrightarrow -j\,\mathrm{sgn}\,(f)$. Hence, application of the time convolution property to the convolution (of Eq. (4.15) yields

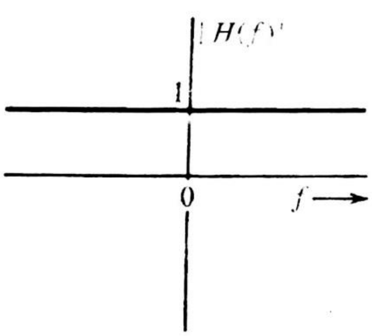
$$X_h(f) = -jX(f)\,\mathrm{sgn}\,(f) \tag{4.16}$$

From Eq. (4.16), it follows that if $m(t)$ passes through a transfer function $H(f) = -j\,\mathrm{sgn}\,(f)$, then the output is $m_h(t)$, the Hilbert transform of $m(t)$. Because

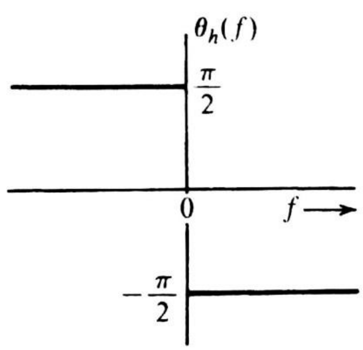
$$H(f) = -j\,\mathrm{sgn}\,(f) \tag{4.17}$$

$$= \begin{cases} -j = 1 \cdot e^{-j\pi/2} & f > 0 \\ j = 1 \cdot e^{j\pi/2} & f < 0 \end{cases} \tag{4.18}$$

it follows that $|H(f)| = 1$ and that $\theta_h(f) = -\pi/2$ for $f > 0$ and $\pi/2$ for $f < 0$, as shown in Fig. 4.14. Thus, if we change the phase of every component of $m(t)$ by $\pi/2$ (without changing its amplitude), the resulting signal is $m_h(t)$, the Hilbert transform of $m(t)$. Therefore, a Hilbert transformer is an ideal phase shifter that shifts the phase of every spectral component by $-\pi/2$.



(a)



(b)