Unit-III
POLAR PLOTS

The sinusoidal transfer function $G(j\omega)$ is a complex function & is given by

$$G(j\omega) = Re[G(j\omega)] + j\,Im[G(j\omega)]$$
$$G(j\omega) = |G(j\omega)| \angle G(j\omega) = M\angle\phi .$$

[Note: Locus - a particular position or place where something occurs]

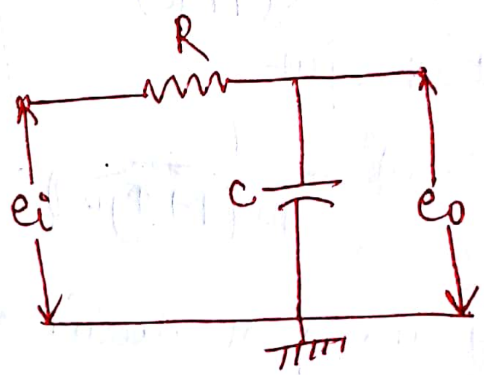
$G(j\omega)$ is represented as phasor of magnitude M & phase angle ϕ.

→ ω is varied from 0 to ∞, so M & ϕ changes

→ Tip of phase $G(j\omega)$ traces a locus in the complex plane.

Polar Plot - The locus traced by the tip of the phasor $G(j\omega)$ as the frequency ω is varied from 0 to ∞ is known as polar plot.

Ex → Let us consider a simple RC filter as shown in figure.



RC filter circuit

$$E_i(s) = R\,I(s) + \frac{1}{Cs} I(s)$$

$$E_i(s) = \left(R + \frac{1}{Cs}\right) I(s)$$

$$E_o(s) = \frac{1}{Cs} I(s).$$

$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{1}{1+RCs} = \frac{1}{1+sT}.$$

T = Time constant ; $T = RC$

~~G(jω)~~ Now put $s = j\omega$

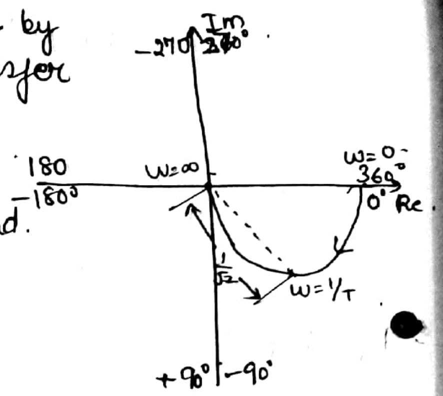
$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+(\omega T)^2}} \angle -\tan^{-1}\omega T = M\angle\phi$$

For $w=0$; $|G(j0)|=1$, $\angle G(j0) = 0^\circ$; $w\uparrow$, $M\downarrow$, $\phi\uparrow$ (negatively

For $w=\infty$; $|G(j\infty)|=0$; $\angle G(j\infty) = -90^\circ$

For $w=\frac{1}{T}$, $M=|G(j\frac{1}{T})| = \frac{1}{\sqrt{2}}$; $\angle G(j\frac{1}{T}) = -45^\circ$.

Polar plot can be drawn very easily by rationalizing the sinusodial transfor function.



Seperating the Real and Imaginary parts & find points $G(j0)$ & $G(j\infty)$ and drawing a smooth curve joining those points.

~~Let the transfor~~ function

$$G(s) = \frac{1}{s(1+Ts)}$$

Put $s=jw$

$$G(jw) = \frac{1}{jw(1+Tjw)}$$

For $w \to \infty$.

$$G(jw) = \frac{1}{1+jwT}.$$

When the plot doesn't cut any axis then put $w=\frac{1}{T}$.

$$M = G\left(j\frac{1}{T}\right) = \frac{1}{\sqrt{2}} \qquad \angle G\left(j\frac{1}{T}\right) = -45^\circ.$$

Steps to find Polar Plot

1) Transfer function, $G(s)$
2) Put $s=jw$
3) Find $|G(jw)|$, $\angle G(jw)$.
4) ~~Initial step~~ $\lim_{w\to 0} |G(jw)|$, $\lim_{w\to\infty} |G(jw)|$.
5) $\lim_{w\to 0} \angle G(jw)$, $\lim_{w\to\infty} \angle G(jw)$.
6) Draw polar plot
7) Real axis plot cuts, $Im=0$
   Imaginary plot cuts, Real part $=0$
8) Magnitude

Type zero System

① $$G(s) = \frac{K}{(1+sT_1)(1+sT_2)} \quad ; \text{Put } s=jw$$

$$G(jw) = \frac{K}{(1+jwT_1)(1+jwT_2)}$$

$$|G(jw)| = \frac{K}{\sqrt{1+(wT_1)^2}\sqrt{1+(wT_2)^2}} \quad ; \angle G(jw) = -\tan^{-1} wT_1 - \tan^{-1} wT_2$$

For magnitude

$$\lim_{w\to 0} |G(jw)| = \frac{K}{\sqrt{1}\sqrt{1}} = K.$$

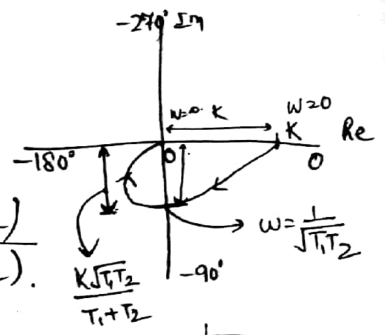
$$\lim_{w\to\infty} |G(jw)| = \frac{K}{\infty\cdot\infty} = 0$$

For phase angle.

$$\lim_{w\to 0} \angle G(jw) = -\tan^{-1} 0 - \tan^{-1} 0 = 0.$$

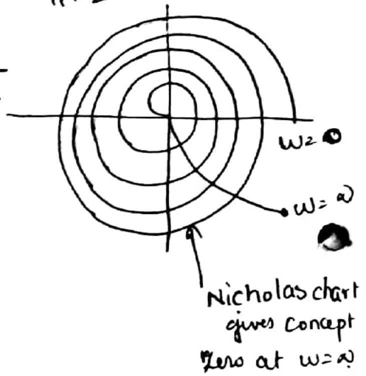
$$\lim_{w\to\infty} \angle G(jw) = -\tan^{-1}(\infty) - \tan^{-1}(\infty) = -90^\circ - 90^\circ = -180^\circ$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	K	0°
∞	0	-180°



$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)} \times \frac{(1-j\omega T_1)(1-j\omega T_2)}{(1-j\omega T_1)(1-j\omega T_2)}.$$

$$= \frac{K(1-\omega^2T_1T_2)}{1+\omega^2T_1^2+\omega^2T_2^2+\omega^4T_1^2T_2^2} - j\,\frac{K\omega(T_1+T_2)}{1+\omega^2T_1^2+\omega^2T_2^2+\omega^4T_1^2T_2^2}$$



Plot cuts at Imaginary axis then put Real part = 0.

$$\frac{K(1-\omega^2T_1T_2)}{1+\omega^2T_1^2+\omega^2T_2^2+\omega^4T_1^2T_2^2} = 0$$

$$1-\omega^2T_1T_2 = 0 \Rightarrow \omega = \frac{1}{\sqrt{T_1T_2}}$$

Now, put $\omega = \frac{1}{\sqrt{T_1T_2}}$ in Imaginary part

$$|G(j\omega)| = -j\,\frac{\cancel{K\omega}\,\{K\cdot\frac{1}{\sqrt{T_1T_2}}(T_1+T_2)}{1+\frac{1}{\cancel{T_1}T_2}\cdot T_1^{\cancel{2}}+\frac{1}{T_1\cancel{T_2}}T_2^{\cancel{2}}+\frac{1}{T_1^2T_2^2}T_1^2T_2^2}$$

$$= \frac{K\cdot\frac{1}{\sqrt{T_1T_2}}(T_1+T_2)}{2+\frac{T_1}{T_2}+\frac{T_2}{T_1}} = \frac{K\cdot\frac{1}{\sqrt{T_1T_2}}(T_1+T_2)}{\frac{T_1^2+T_2^2+2T_1T_2}{T_1T_2}}$$

$$= \frac{K(T_1T_2)\cancel{(T_1+T_2)}}{\sqrt{T_1T_2}\,(T_1+T_2)^{\cancel{2}}} = \frac{K\sqrt{T_1T_2}}{T_1+T_2}$$

② $G(s) = \frac{K}{(1+sT_1)(1+sT_2)(1+sT_3)}$ (Transfer function given)

Step-II :
Put $s = j\omega$.

$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}.$$

Step-III :-

Magnitude.

$$|G(j\omega)| = \frac{K}{\left(\sqrt{1+(\omega T_1)^2}\right)\left(\sqrt{1+(\omega T_2)^2}\right)\left(\sqrt{1+(\omega T_3)^2}\right)}$$

Phase Angle.

$$\angle G(j\omega) = -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$$

Step-IV -

Magnitude

at $\omega\to 0$ $|G(j\omega)| = K$.

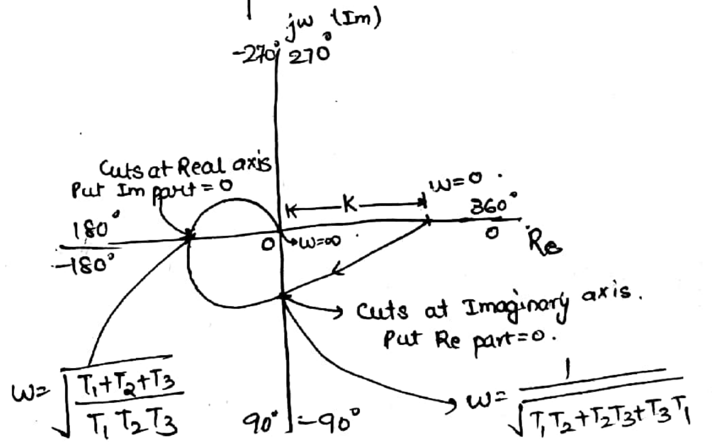
at $\omega\to\infty$ $|G(j\omega)| = 0$

Phase Angle.

at $\omega\to 0$ $\angle G(j\omega) = 0^\circ$

at $\omega\to\infty$ $\angle G(j\omega) = -270^\circ$

Step-V :-



Step VI :- First Rationalize the Transfer function

$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)} \times \frac{(1-j\omega T_1)(1-j\omega T_2)(1-j\omega T_3)}{(1-j\omega T_1)(1-j\omega T_2)(1-j\omega T_3)}$$

$$G(j\omega) = \frac{K(1-j\omega T_3 - j\omega T_2 - \omega^2 T_2 T_3 - j\omega T_1 - \omega^2 T_1 T_3 - \omega^2 T_1 T_2 + j\omega^3 T_1 T_2 T_3)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)(1+\omega^2 T_3^2)}$$

Seperate Real & Imaginary parts

$$G(j\omega) = \frac{K(1-\omega^2 T_2 T_3 - \omega^2 T_1 T_3 - \omega^2 T_1 T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)(1+\omega^2 T_3^2)} - \frac{jK(\omega T_1 + \omega T_2 + \omega T_3 - \omega^3 T_1 T_2 T_3)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)(1+\omega^2 T_3^2)}$$

⇒ For the polar plot cuts at Imaginary axis.

Real part = 0

$$\frac{K(1-\omega^2 T_2 T_3 - \omega^2 T_1 T_3 - \omega^2 T_1 T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)(1+\omega^2 T_3^2)} = 0$$

$$1-\omega^2 T_2 T_3 - \omega^2 T_1 T_3 - \omega^2 T_1 T_2 = 0$$

$$1 - \omega^2(T_1 T_2 + T_2 T_3 + T_3 T_1) = 0$$

$$\boxed{\omega = \frac{1}{\sqrt{T_1 T_2 + T_2 T_3 + T_3 T_1}}}$$

For the polar plot cuts at Real Axis ; Im part = 0.

$$\frac{-jK(\omega T_1 + \omega T_2 + \omega T_3 - \omega^3 T_1 T_2 T_3)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)(1+\omega^2 T_3^2)} = 0$$

$$\omega(T_1 + T_2 + T_3 - \omega^2 T_1 T_2 T_3) = 0$$

$$\boxed{\omega = \sqrt{\frac{T_1 + T_2 + T_3}{T_1 T_2 T_3}}}$$

Q:- Sketch polar plot for given systems. → tranfer function of

a) $G(s) = \frac{1}{(1+s)(1+2s)}$

Step I → Transfer function given

$$G(s) = \frac{1}{(1+s)(1+2s)}$$

Step II : Put $s = j\omega$

$$G(j\omega) = \frac{1}{(1+j\omega)(1+2j\omega)}.$$

Step III : Magnitude

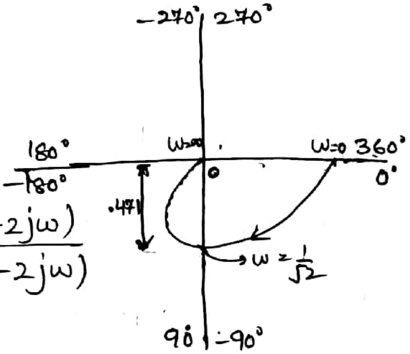
$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}.$$

Phase Angle $\angle G(j\omega) = -\tan^{-1}\omega - \tan^{-1}2\omega$

Step IV - ~~Pnst~~

Magnitude	Phase Angle
$\lim_{\omega\to 0} \lvert G(j\omega)\rvert = 1.$	$\lim_{\omega\to 0} \angle G(j\omega) = 0°.$
$\lim_{\omega\to\infty} \lvert G(j\omega)\rvert = 0$	$\lim_{\omega\to\infty} \angle G(j\omega) = -180°$

Step-V - Sketch polar plot



Step-VI - As polar plot cuts on Imaginary axis, Real part should be zero.

$$G(jw) = \frac{1}{(1+jw)(1+2jw)} \times \frac{(1-jw)(1-2jw)}{(1-jw)(1-2jw)}$$

$$= \frac{1-jw-2jw-2w^2}{(1+w^2)(1+4w^2)}.$$

$$G(jw) = \frac{1-2w^2-3jw}{(1+w^2)(1+4w^2)} = \frac{1-2w^2}{(1+w^2)(1+4w^2)} - j\frac{3w}{(1+w^2)(1+4w^2)}$$

$$Re[G(jw)] = 0.$$

$$\frac{1-2w^2}{(1+w^2)(1+4w^2)} = 0 \Rightarrow \boxed{w=\frac{1}{\sqrt{2}}}$$

Step-VII - Find magnitude.

Rest for magnitude Put $w=\frac{1}{\sqrt{2}}$ in Imaginary part.

$$G\left(j\frac{1}{\sqrt{2}}\right) = -j\frac{3\times\frac{1}{\sqrt{2}}}{\left(\cancel{1}+\frac{1}{2}\right)\left(1+\cancel{4}^{2}\times\frac{1}{\cancel{2}}\right)} = -j\frac{\frac{\cancel{3}}{\sqrt{2}}}{\frac{3}{2}\times\cancel{3}} = -j\,0.471$$

Type I System

$$G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$$

Step-II - Put $s=jw$

$$G(jw) = \frac{K}{jw(1+jwT_1)(1+jwT_2)}$$

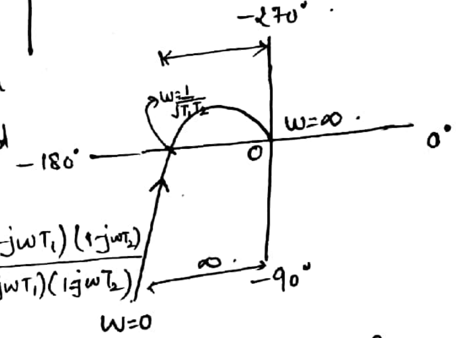
Step-III - Magnitude $\frac{K}{w\sqrt{1+(wT_1)^2}\sqrt{1+(wT_2)^2}} = |G(jw)|$

Phase angle, $\angle G(jw) = -90° - \tan^{-1} wT_1 - \tan^{-1} wT_2$

Step-IV -

Magnitude, $\lvert G(jw)\rvert$	Phase Angle, $\angle G(jw)$
At $w\to 0$ $\lvert G(jw)\rvert = \infty$	At $w\to 0$ $\angle G(jw) = -90°$
At $w\to\infty$ $\lvert G(jw)\rvert = 0$.	At $w\to\infty$ $\angle G(jw) = -270°$

Step V - Sketch polar plot.



Step VI - As polar plot cuts on Real axis, imaginary part should be zero to find w value.

$$|G(jw)| = \frac{K}{jw(1+jwT_1)(1+jwT_2)} \times \frac{-jw(1-jwT_1)(1-jwT_2)}{-jw(1-jwT_1)(1-jwT_2)}$$

$$= \frac{K(-jw-w^2T_1)(1-jwT_2)}{w^2(1+w^2T_1^2)(1+w^2T_2^2)} = \frac{K(-jw-w^2T_2-w^2T_1+jw^3T_1T_2)}{w^2(1+w^2T_1^2)(1+w^2T_2^2)}$$

$$= \frac{-K(w^2T_1+w^2T_2)}{w^2(1+w^2T_1^2)(1+w^2T_2^2)} + j\frac{K(w^3T_1T_2-w)}{w^2(1+w^2T_1^2)(1+w^2T_2^2)}$$

$$\frac{Kw(w^2T_1T_2-1)}{w^2(1+w^2T_1^2)(1+w^2T_2^2)} = 0 \Rightarrow w = \frac{1}{\sqrt{T_1T_2}}$$

Step V - Magnitude.

$$\left|G\left(j\frac{1}{\sqrt{T_1T_2}}\right)\right| = \frac{K\left(\frac{1}{\cancel{T_1}T_2}\times\cancel{T_1}+\frac{1}{T_1\cancel{T_2}}\times\cancel{T_2}\right)}{\frac{1}{T_1T_2}\left(1+\frac{1}{\cancel{T_1}T_2}T_1^{\cancel{2}}\right)\left(1+\frac{1}{T_1\cancel{T_2}}T_2^{2}\right)}.$$

$$= \frac{K\left(\frac{1}{T_1}+\frac{1}{T_2}\right)}{\frac{1}{T_1T_2}\left(\frac{T_2+T_1}{T_2}\right)\left(\frac{T_1+T_2}{T_1}\right)} = \frac{K\left(\frac{\cancel{T_1+T_2}}{\cancel{T_1T_2}}\right)}{\frac{1}{\cancel{T_1T_2}}\left(\frac{T_2+T_1}{T_2}\right)\left(\frac{\cancel{T_1+T_2}}{T_1}\right)}$$

$$= \frac{K(T_1T_2)}{(T_1+T_2)}.$$

$$= \frac{K(T_1 T_2)}{(T_1+T_2)}$$

Type-II System :- $G(s) = \dfrac{K}{s^2(1+sT_1)(1+sT_2)}$

Put $s = j\omega$.

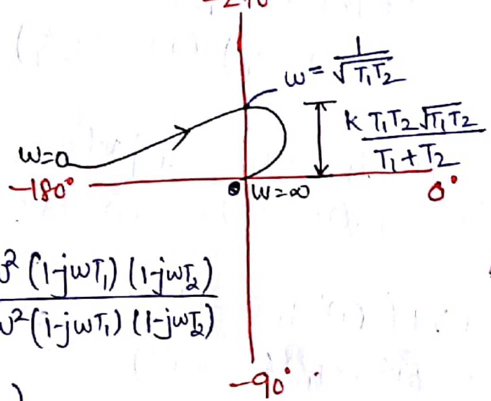
$$G(j\omega) = \frac{K}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)}.$$

$$|G(j\omega)| = \frac{K}{\omega^2\left(\sqrt{1+\omega^2T_1^2}\right)\left(\sqrt{1+\omega^2T_2^2}\right)}.$$

$$\angle G(j\omega) = -180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_1.$$

Magnitude	Phase Angle
$\lvert G(j\omega)\rvert =$ at $\omega\to 0$ $\lvert G(j\omega)\rvert = \infty$	at $\omega\to 0$ $\angle G(j\omega) = -180^\circ$
at $\omega\to\infty$ $\lvert G(j\omega)\rvert = 0$	at $\omega\to\infty$ $\angle G(j\omega) = -360^\circ$

Sketch polar plot.



As plot cuts on Imaginary axis, then put Real part is equal to zero.

$$G(j\omega) = \frac{K}{-\omega^2(1+j\omega T_1)(1+j\omega T_2)} \times \frac{-\omega^2(1-j\omega T_1)(1-j\omega T_2)}{-\omega^2(1-j\omega T_1)(1-j\omega T_2)}$$

$$= \frac{-K\omega^2(1-j\omega T_2 - j\omega T_1 - \omega^2 T_1T_2)}{\omega^4(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

$$= \frac{-K\omega^2(1-\omega^2T_1T_2)}{\omega^4(1+\omega^2T_1^2)(1+\omega^2T_2^2)} + \frac{jK\omega^3(T_1+T_2)}{\omega^4(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

For the value of ω cut at polar plot.

$$\frac{-K\omega^2(1-\omega^2T_1T_2)}{\omega^4(1+\omega^2T_1^2)(1+\omega^2T_2^2)} = 0 \Rightarrow \omega = \frac{1}{\sqrt{T_1T_2}}$$

Magnitude

$$G\left(j\frac{1}{\sqrt{T_1T_2}}\right) = \frac{j\,K\left(\frac{1}{\sqrt{T_1T_2}}\right)^3(T_1+T_2)}{\left(\frac{1}{\sqrt{T_1T_2}}\right)^4\left[1+\frac{1}{\cancel{T_1}T_2}T_1^{\cancel{2}}\right]\left[1+\frac{1}{T_1\cancel{T_2}}T_2^{\cancel{2}}\right]}.$$

$$= \frac{jK(T_1+T_2)}{\frac{1}{\sqrt{T_1T_2}}\cdot\left(\frac{T_1+T_2}{T_2}\right)\left(\frac{T_1+T_2}{T_1}\right)} = jK\frac{T_1T_2\sqrt{T_1T_2}}{T_1+T_2}.$$

Q:- Sketch polar plot for $G(s) = \dfrac{1}{s^2(1+s)(1+2s)}$

Sol. Put $s = j\omega$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega)(1+2j\omega)}$$

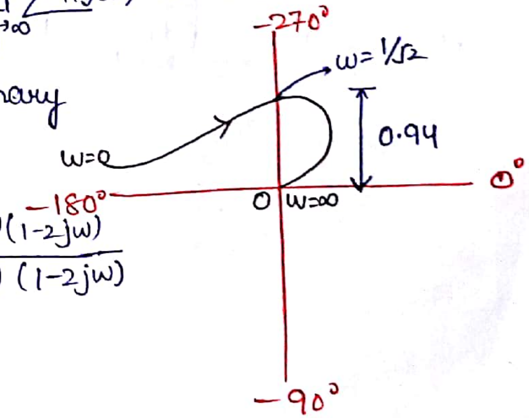
$$|G(j\omega)| = \frac{1}{\omega^2\sqrt{1+\omega^2}\,\sqrt{1+4\omega^2}}$$

$$\angle G(j\omega) = -180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

to

Magnitude, $\lvert G(j\omega)\rvert$	Phase Angle, $\angle G(j\omega)$.
at $\omega\to 0$ $\lvert G(j\omega)\rvert = \infty$	at $\omega\to 0$ $\angle G(j\omega) = -180^\circ$
at $\omega\to\infty$ $\lvert G(j\omega)\rvert = 0$	at $\omega\to\infty$ $\angle G(j\omega) = -360^\circ$

As polar plot cuts at Imaginary axis, so Real part is equal to zero.



$$G(j\omega) = \frac{1}{-\omega^2(1+j\omega)(1+2j\omega)} \times \frac{(1-j\omega)(1-2j\omega)}{(1-j\omega)(1-2j\omega)}$$

$$G(j\omega) = \frac{(1-3j\omega-4\omega^2)}{-\omega^2(1+\omega^2)(1+4\omega^2)}$$

$$G(j\omega) = \frac{1-4\omega^2}{-\omega^2(1+\omega^2)(1+4\omega^2)} + j\frac{3\omega}{\omega^2(1+\omega^2)(1+4\omega^2)}$$

~~Repo~~ Real part = 0

$$\frac{1-4\omega^2}{-\omega^2(1+\omega^2)(1+4\omega^2)} = 0 \Rightarrow \omega = \frac{1}{\sqrt{2}}$$

Magnitude.

$$G\left(j\frac{1}{\sqrt{2}}\right) = j\frac{3\frac{1}{\sqrt{2}}}{\frac{1}{2}\left(1+\frac{1}{2}\right)\left(1+4\times\frac{1}{2}\right)} = \frac{j\frac{3}{\sqrt{2}}}{\frac{1}{2}\times\frac{3}{2}\times 3}$$

$$= \frac{4}{\sqrt{2}\times 3} = \frac{2\sqrt{2}}{3} = .942$$

Inverse Polar Plot – The inverse polar plot of $G(j\omega)$ is a graph of $1/G(j\omega)$ as a function of ω.

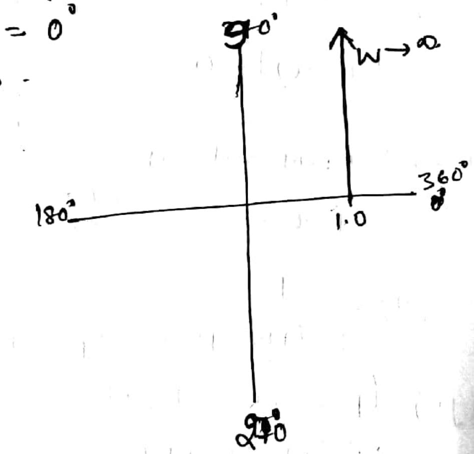
For Example $G(j\omega) = \frac{1}{1+j\omega T}$.

Inverse $\frac{1}{G(j\omega)} = G^{-1}(j\omega) = 1+j\omega T$

Magnitude $|G^{-1}(j\omega)| = \sqrt{1+\omega^2T^2}$

Phase Angle $\angle G^{-1}(j\omega) = \tan^{-1}\omega T$.

Magnitude	Phase Angle
$\lim_{\omega\to 0} \lvert G(j\omega)\rvert = 1$	$\lim_{\omega\to 0} \angle G(j\omega) = 0^\circ$
$\lim_{\omega\to\infty} \lvert G(j\omega)\rvert = \infty$	$\lim_{\omega\to\infty} = 90^\circ$



Polar Plot

Polar Plot – The polar plot of a sinusoidal transfer function $G(j\omega)$ is a plot of magnitude of $G(j\omega)$ versus phase angle of $G(j\omega)$ on polar coordinates as 'ω' is varied from zero to infinity.

∴ Polar Plot is the locus of vectors $|G(j\omega)| \angle G(j\omega)$ as ω is varied from zero to infinity. In frequency Response, we have

Magnitude, $M = |G(j\omega)H(j\omega)|$

Phase angle, $\phi = \angle G(j\omega)H(j\omega)$

Advantage – It depicts the frequency response characteristics of a system over the entire frequency range in a single plot.

Disadvantage – Plot doesnot indicate the contribution of each individual factor of open loop transfer function.

Procedure to sketch the Polar Plot –

Step-I – Determine the open loop transfer function $G(s)H(s)$ of the system.

Step-II – Put $s=j\omega$ in $G(s)H(s)$ to obtain $G(j\omega)H(j\omega)$

Step III – At $\omega=0$ and $\omega=\infty$, calculate $|G(j\omega)|$ by $\lim_{\omega\to 0}|G(j\omega)|$ and $\lim_{\omega\to\infty}|G(j\omega)|$.

Step-IV – calculate the phase angle of $\angle G(j\omega)$ at $\omega=0$ & $\omega=\infty$ by $\lim_{\omega\to 0}\angle G(j\omega)$ and $\lim_{\omega\to\infty}\angle G(j\omega)$

Step-V – Rationalize the function and seperate Real and imaginary parts.

Step-VI – Equate the imaginary part $Im|G(j\omega)|$ to zero if polar plot cuts the Real axis ~~called point~~ and calculate point of intersection. Put this point of intersection in $G(j\omega)$ to find the magnitude of curve.

Equate the Real part $Re|G(j\omega)|$ to zero if polar plots cuts the imaginary axis and calculate point of intersection. Put this point in the $G(j\omega)$ to get magnitude of polar plot.

Type 'zero' System

$$G(s) = \frac{K}{(1+sT_1)(1+sT_2)}$$

Step-I :- Put $s = j\omega$, $G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$

$$|G(j\omega)| = \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}}$$

$$\angle G(j\omega) = -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

Step-II :- Taking limit of $|G(j\omega)|$

$$\lim_{\omega\to 0} |G(j\omega)| = \lim_{\omega\to 0} \frac{K}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}} = K$$

$$\lim_{\omega\to\infty} |G(j\omega)| = 0$$

Step-III :- Taking limit of $\angle G(j\omega)$

$$\lim_{\omega\to 0} \angle G(j\omega) = \lim_{\omega\to 0} -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 = 0^\circ$$

$$\lim_{\omega\to 0} \angle G(j\omega) = \lim_{\omega\to\infty}\left(-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2\right) = -90^\circ - 90^\circ = -180^\circ$$

Step-IV - Separating the Real and Imaginary parts.

$$G(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)} \times \frac{(1-j\omega T_1)(1-j\omega T_2)}{(1-j\omega T_1)(1-j\omega T_2)}$$

$$= \frac{K(1 - j\omega T_1 - j\omega T_2 - \omega^2 T_1 T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}$$

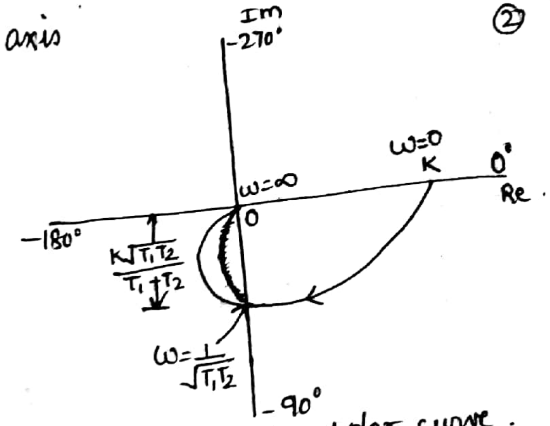
$$G(j\omega) = \frac{K(1-\omega^2 T_1 T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)} - \frac{j\omega K(T_1+T_2)}{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}$$

As plot cuts the Imaginary axis the $Re\, G(j\omega) = 0$

$$\frac{K(1-\omega^2 T_1 T_2)}{(1+(\omega T_1)^2)\,[1+(\omega T_2)^2]} = 0$$

$$1 - \omega^2 T_1 T_2 = 0$$

$$\boxed{\omega = \frac{1}{\sqrt{T_1 T_2}}}$$



Put $\omega = \frac{1}{\sqrt{T_1 T_2}}$ in $G(j\omega)$ to find the magnitude polar curve.

$$G\left(j\frac{1}{\sqrt{T_1T_2}}\right) = 0 - \frac{j\frac{K}{\sqrt{T_1T_2}}(T_1+T_2)}{\left(1+\frac{1}{T_1T_2}\cdot T_1^2\right)\left(1+\frac{1}{T_1T_2}T_2^2\right)}$$

$$= -\frac{j\, K\frac{(T_1+T_2)}{\sqrt{T_1T_2}}}{\left(1+\frac{T_1}{T_2}\right)\left(1+\frac{T_2}{T_1}\right)} = \frac{-jK\frac{(T_1+T_2)}{\sqrt{T_1T_2}}}{\frac{(T_1+T_2)^2}{T_1T_2}}$$

$$G\left(j\frac{1}{\sqrt{T_1T_2}}\right) = \frac{-jK\sqrt{T_1T_2}}{T_1+T_2}$$

$$|G(j\omega)| = \frac{K\sqrt{T_1T_2}}{T_1+T_2}$$

Type '~~Two~~ one' System - $G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$

Step-I - Put $s = j\omega$; $G(j\omega) = \frac{K}{(j\omega)(1+j\omega T_1)(1+j\omega T_2)}$

$$|G(j\omega)| = \frac{K}{\omega\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}} \;;\; \angle G(j\omega) = -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

Step-II :- Taking limit of $|G(j\omega)|$

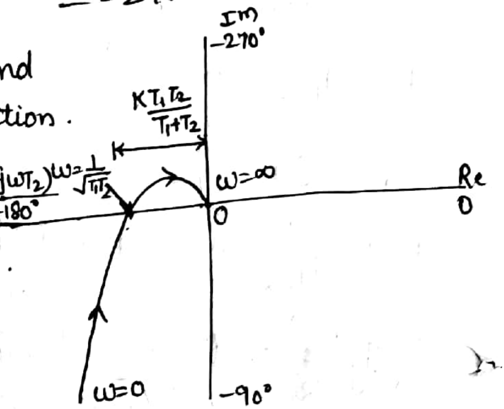
$$\lim_{\omega\to 0} |G(j\omega)| = \lim_{\omega\to 0} \frac{K}{\omega\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}} = \infty$$

$\lim_{\omega\to\infty} |G(j\omega)| = 0$

Step-III :- Taking the limit of $\angle G(j\omega)$

$\lim_{\omega\to 0} \angle G(j\omega) = -90°$; $\lim_{\omega\to\infty} \angle G(j\omega) = \lim_{\omega\to\infty}(-90° - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$
$= -270°$

Step-IV - seperating the Real and imaginary parts of transfer function.



$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)} \times \frac{(-j\omega)(1-j\omega T_1)(1-j\omega T_2)}{(-j\omega)(1-j\omega T_1)(1-j\omega T_2)}$$

$$G(j\omega) = \frac{-K\omega^2(T_1+T_2)}{\omega^2(1+\omega^2T_1^2)(1+\omega^2T_2^2)} - \frac{jK\omega(\omega^2T_1T_2-1)}{\omega^2(1+\omega^2T_1^2)(1+\omega^2T_2^2)}$$

As polar plot crosses the Real Axis, then $Im\, G(j\omega) = 0$

$$\frac{-jK\omega(\omega^2T_1T_2-1)}{\omega^2(1+\omega^2T_1^2)(1+\omega^2T_2^2)} = 0$$

$$\boxed{\omega = \frac{1}{\sqrt{T_1T_2}}}$$

Magnitude of polar curve

$$G(j\omega) = \frac{-K\frac{1}{T_1T_2}(T_1+T_2)}{\frac{1}{T_1T_2}\left(1+\frac{1}{T_1T_2}T_1^2\right)\left(1+\frac{1}{T_1T_2}T_2^2\right)} - 0$$

$$= \frac{\frac{-K(T_1+T_2)}{T_1T_2}}{\frac{1}{T_1T_2}\left(1+\frac{T_1}{T_2}\right)\left(1+\frac{T_2}{T_1}\right)} = -\frac{KT_1T_2}{T_1+T_2}$$

$$\boxed{|G(j\omega)| = \frac{KT_1T_2}{T_1+T_2}}$$

3. Type 'Two' system - $G(s) = \frac{K}{s^2(1+sT_1)(1+sT_2)}$ (3)

Step-I - Put $s=j\omega$; $G(j\omega) = \frac{K}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)}$

$|G(j\omega)| = \frac{K}{\omega^2\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}}$; $\angle G(j\omega) = -180° - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$

Step-II - Taking limit for $|G(j\omega)|$

$\lim_{\omega\to 0} |G(j\omega)| = \infty$; $\lim_{\omega\to\infty} |G(j\omega)| = 0$

Step-III - Taking limit for $\angle G(j\omega)$

$\lim_{\omega\to 0} \angle G(j\omega) = -180°$; $\lim_{\omega\to\infty} |G(j\omega)| = -360°$

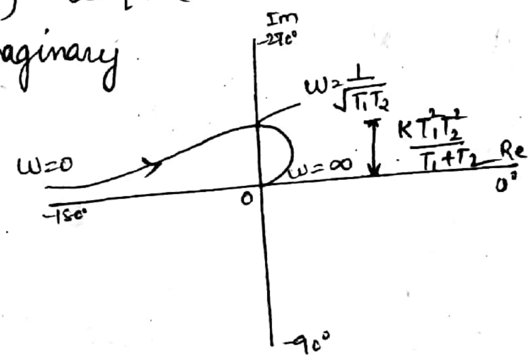
Step-IV - seperating the Real and Imaginary axis

$$G(j\omega) = \frac{K}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)} \times \frac{(-j\omega)^2(1-j\omega T_1)(1-j\omega T_2)}{(-j\omega)^2(1-j\omega T_1)(1-j\omega T_2)}$$

$$G(j\omega) = \frac{-\omega^2K(1-\omega^2T_1T_2)}{\omega^4(1+(\omega T_1)^2)(1+(\omega T_2)^2)} + \frac{jK\omega^2(T_1+T_2)}{\omega^4[1+(\omega T_1)^2][1+(\omega T_2)^2]}$$

As polar plot crosses the imaginary axis $Re\, G(j\omega) = 0$.

$$\frac{-\omega^2K(1-\omega^2T_1T_2)}{\omega^4(1+\omega^2T_1^2)(1+\omega^2T_2^2)} = 0$$



$$\boxed{\omega = \frac{1}{\sqrt{T_1T_2}}}$$

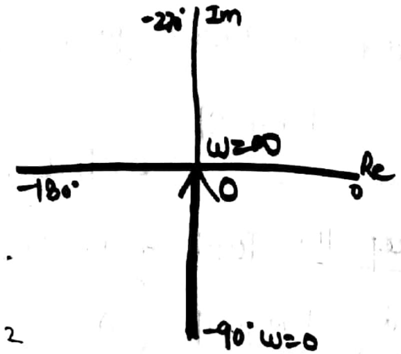
Magnitude of polar curve is

$$G(j\omega) = 0 + \frac{jK\frac{1}{T_1T_2}(T_1+T_2)}{\frac{1}{T_1^2T_2^2}\left[1+\frac{T_1^2}{T_1T_2}\right]\left[1+\frac{T_2^2}{T_1T_2}\right]} = \frac{j\frac{K(T_1+T_2)}{T_1T_2}}{\frac{(T_1+T_2)^2}{T_1^2T_2^2}}$$

$$G(j\omega) = \frac{jKT_1^2T_2^2}{T_1+T_2} \qquad |G(j\omega)| = \frac{KT_1^2T_2^2}{T_1+T_2}$$

Polar Plots of some standard Type function

1. $G(s) = \frac{1}{s}$ Put $s = j\omega$; $G(j\omega) = \frac{1}{j\omega}$

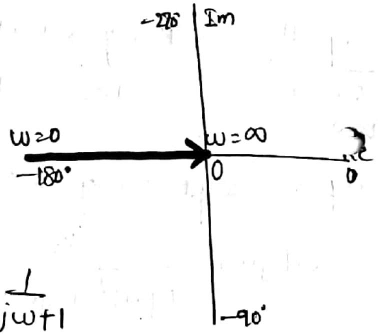


$|G(j\omega)| = \frac{1}{\omega}$; $\angle G(j\omega) = -90^\circ$

$\underset{\omega\to 0}{\text{At}} |G(j\omega)| = \infty$; $\underset{\omega\to\infty}{\text{At}} |G(j\omega)| = 0$

$\underset{\omega\to 0}{\text{At}} \angle G(j\omega) = -90^\circ$; $\underset{\omega\to\infty}{\text{At}} \angle G(j\omega) = -90^\circ$

2. $G(s) = \frac{1}{s^2}$ Put $s = j\omega$; $G(j\omega) = \frac{1}{(j\omega)^2}$

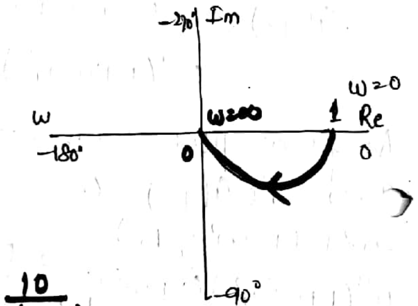


$|G(j\omega)| = \frac{1}{-\omega^2}$; $\angle G(j\omega) = -180^\circ$

$\underset{\omega\to 0}{\text{At}} |G(j\omega)| = \infty$; $\underset{\omega\to\infty}{\text{At}} |G(j\omega)| = 0$

$\underset{\omega\to 0}{\text{At}} \angle G(j\omega) = -180^\circ$; $\underset{\omega\to\infty}{\text{At}} |G(j\omega)| = -180^\circ$

3. $G(s) = \frac{1}{s+1}$; Put $s = j\omega$; $G(j\omega) = \frac{1}{j\omega+1}$

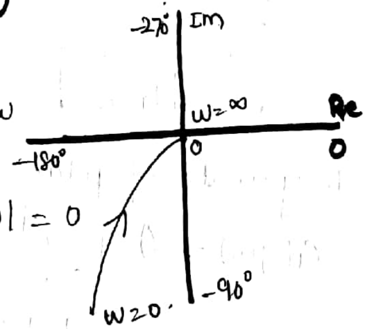


$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$; $\angle G(j\omega) = -\tan^{-1}\omega$

$\underset{\omega\to 0}{\text{At}} |G(j\omega)| = 1$; $\underset{\omega\to\infty}{\text{At}} |G(j\omega)| = 0$

$\underset{\omega\to 0}{\text{At}} \angle G(j\omega) = 0^\circ$; $\underset{\omega\to\infty}{\text{At}} \angle G(j\omega) = -90^\circ$

Q1:- Sketch polar plot for $G(s) = \frac{10}{s(s+1)}$



Sol: Step-I: Put $s = j\omega$; $G(j\omega) = \frac{10}{j\omega(j\omega+1)}$

$|G(j\omega)| = \frac{10}{\omega\sqrt{1+\omega^2}}$; $\angle G(j\omega) = -90^\circ - \tan^{-1}\omega$

Step-II: Taking the limit of $|G(j\omega)|$

$\underset{\omega\to 0}{\text{At}} |G(j\omega)| = \underset{\omega\to 0}{\text{At}} \frac{10}{\omega\sqrt{1+\omega^2}} = \infty$; $\underset{\omega\to\infty}{\text{At}} |G(j\omega)| = 0$

Step-III: Taking the limit of $\angle G(j\omega)$

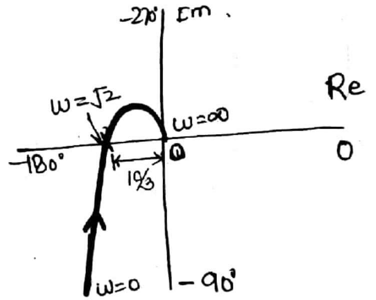
$\underset{\omega\to 0}{\text{At}} \angle G(j\omega) = -90^\circ$; $\underset{\omega\to\infty}{\text{At}} \angle G(j\omega) = -180^\circ$

(4)

Q:- Sketch the polar plot for $G(s) = \frac{20}{s(s+1)(s+2)}$.

Sol: Step-I: Put $s = j\omega$; $G(j\omega) = \frac{20}{j\omega(j\omega+1)(j\omega+2)}$

$|G(j\omega)| = \frac{20}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}}$; $\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}\left(\frac{\omega}{2}\right)$



Step-II: Taking limit of $|G(j\omega)|$

$\underset{\omega\to 0}{\text{At}} |G(j\omega)| = \infty$; $\underset{\omega\to\infty}{\text{At}} |G(j\omega)| = 0$

Step-III: Taking limit of $\angle G(j\omega)$

$\underset{\omega\to 0}{\text{At}} \angle G(j\omega) = -90^\circ$; $\underset{\omega\to\infty}{\text{At}} \angle G(j\omega) = -270^\circ$

Step-IV: Separating the Real and Imaginary parts

$$G(j\omega) = \frac{20}{(j\omega)(j\omega+1)(j\omega+2)} \times \frac{(-j\omega)(1-j\omega)(2-j\omega)}{(-j\omega)(1-j\omega)(2-j\omega)}$$

$$G(j\omega) = \frac{-j20\omega(2-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)} - \frac{60\omega^2}{\omega^2(1+\omega^2)(4+\omega^2)}$$

As polar plot cuts Real axis, so equate Imaginary axis is equal to zero.

$$\frac{-20\omega(2-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)} = 0 \Rightarrow \boxed{\omega = \sqrt{2} \text{ rad/s}}$$

Put $\omega = \sqrt{2}$ in $G(j\omega)$ to get magnitude.

$$G(j\omega) = 0 - \frac{60(\sqrt{2})^2}{(\sqrt{2})^2(1+(\sqrt{2})^2)(4+2)} = \frac{-10}{3}$$

$$|G(j\omega)| = \frac{10}{3}$$

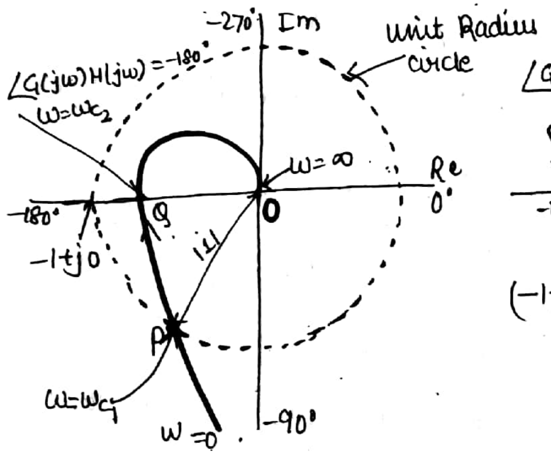
Phase Margin, Gain Margin and stability on Polar Plot

Gain Margin, $\boxed{GM = -20\log_{10}|G(j\omega)H(j\omega)|_{\omega=\omega_{c_2}}}$

ω_{c_2} = phase cross-over frequency or point of intersection of polar plot with negative Real axis.

Phase Margin, $PM = 180^\circ + \underline{/G(j\omega)H(j\omega)}\,\big|_{\omega=\omega_{c_1}}$

ω_{c_1} = gain cross-over frequency or point of intersection of polar plot with ~~+ve~~ Real point on unit Radius circle $(-1+j0)$.

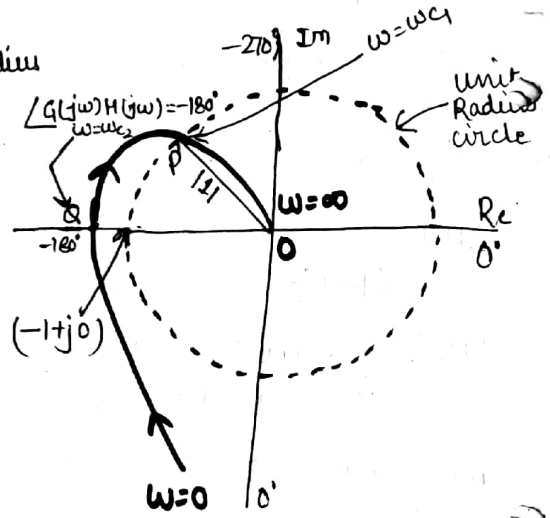


~~(a) $\omega_{pc} > \omega_{gc}$~~

(a) $\omega_{c_1} > \omega_{c_2}$

At point P, cuts unit circle of Radius 1 unit is $\omega=\omega_{c_1}$ (gain cross-over frequency)

At point Q, cuts the Real axis at $\omega=\omega_{c_2}$ is phase cross-over frequency.

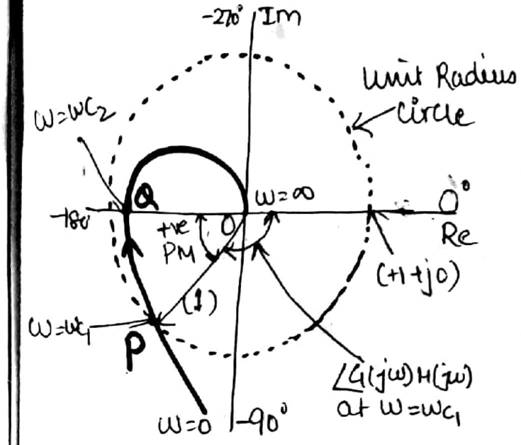


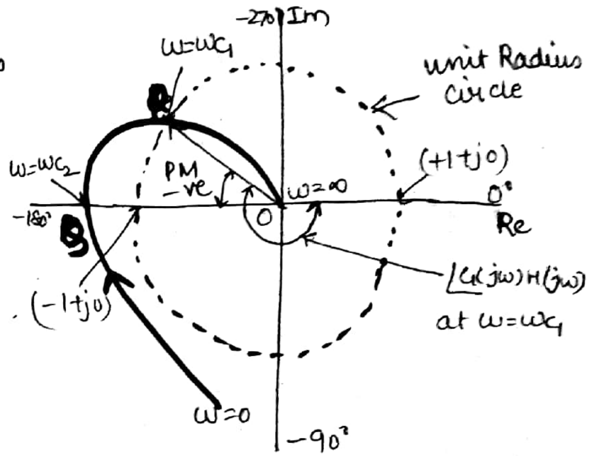
(b) $\omega_{c_2} > \omega_{c_1}$

At point P, cuts unit radius circle gives gain cross-over frequency, $\omega=\omega_{c_1}$.

At point Q, cuts Real axis at $\omega=\omega_{c_2}$, phase cross-over frequency.

Stability of a polar plot





- $\omega_{c_1} < \omega_{c_2}$
  Stable system
  GM & PM are +ve

$\omega_{c_1} > \omega_{c_2}$
unstable system
GM & PM are -ve.

Q:- The open loop transfer function with unity feedback system is given by $G(s) = \dfrac{1}{s(s+1)(2s+1)}$. Sketch the polar plot and determine the Gain Margin & P.M.

Sol: Step-I - Put $s=j\omega$; $G(j\omega) = \dfrac{1}{j\omega(1+j\omega)(1+2j\omega)}$

- $|G(j\omega)| = \dfrac{1}{\omega\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$; $\underline{/G(j\omega)} = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$

Step-II - Taking limit of $|G(j\omega)|$

$\lim_{\omega\to 0} |G(j\omega)| = \infty$; $\lim_{\omega\to\infty} |G(j\omega)| = 0$

Step III:- Taking limit of $\underline{/G(j\omega)}$

$\lim_{\omega\to 0} \underline{/G(j\omega)} = -90^\circ$; $\lim_{\omega\to\infty} \underline{/G(j\omega)} = -270^\circ$

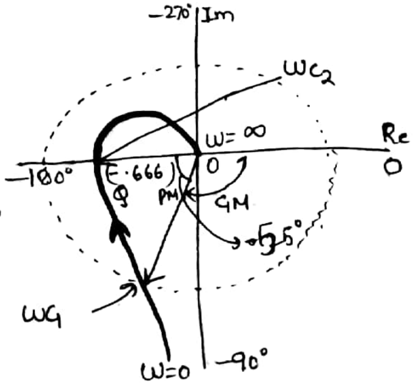
Step-IV - Seperating the Real and Imaginary parts

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)} \times \frac{-j\omega(1-j\omega)(1-2j\omega)}{(-j\omega)(1-j\omega)(1-2j\omega)}$$

$$G(j\omega) = \frac{-3\omega^2}{\omega^2(1+\omega^2)(4+\omega^2)} - \frac{j\omega(1-2\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)}$$

Polar plot cuts ~~at~~ the Real axis so equate the imaginary axis to zero.

$$\frac{-\omega(1-2\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)} = 0 \; ; \Rightarrow \omega = .7071 \text{ rad/s}$$



~~$G(j\omega) = \frac{-3\times(.7071)}{2(3)(6)} =$~~

$$G(j\omega) = \frac{-3(.7071)^2}{(.7071)^2(1+(.7071)^2)(4+(.7071)^2)} = -0.6666$$

$PM = 180° + \angle G(j\omega)|_{\omega_{c_1}} = $ ~~[illegible]~~

$= 180° - 125° = 55°$

$GM = -20 \log_{10} |G(j\omega)H(j\omega)|_{\omega=\omega_{c_2}}$

$GM = -20 \log_{10}(0.666) = 3.53\,dB$

PM and GM are +ve means, system is stable.

Inverse Polar Plots – If the transfer function of the system is $G(j\omega)$ in the frequency domain, then the graph of $[G(j\omega)]^{-1}$ as a function of ω is called inverse polar plot.

i.e. graph of $\left[\frac{1}{G(j\omega)}\right]$ as ω is varied from 0 to ∞ is called inverse polar plot.

If $G(j\omega) = M\angle+\phi$ then $\frac{1}{G(j\omega)} = \frac{1}{M}\angle-\phi$

(6)

Q:- Sketch Inverse polar plot of

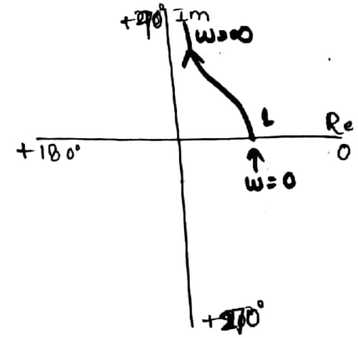
(i) $G(s) = \frac{10}{(s+10)}$ $G(j\omega) = \frac{10}{(j\omega+10)} = \frac{1}{1+0.1j\omega}$

$\frac{1}{G(j\omega)} = 1 + 0.1j\omega = M\angle\phi$

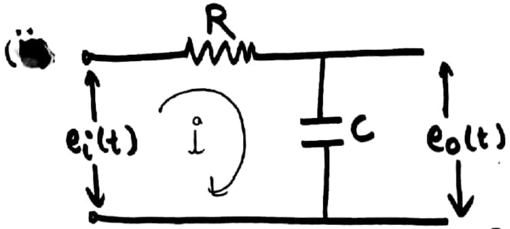
$M = \sqrt{1+(0.1\omega)^2}$, $\angle G(j\omega)^{-1} = \tan^{-1}(.1\omega)$

at $\omega\to 0$ $|G(j\omega)^{-1}| = 1$; at $\omega\to\infty$ $|G(j\omega)^{-1}| = \infty$

at $\omega\to 0$ $\angle G(j\omega)^{-1} = 0°$; at $\omega\to\infty$ $\angle G(j\omega)^{-1} = 90°$



(ii)



$E_i(s) = RI(s) + \frac{1}{sC} I(s) = \left[\frac{1+sRC}{sC}\right] I(s)$

$E_o(s) = \frac{1}{sC} I(s)$

$\frac{E_o(s)}{E_i(s)} = \frac{1}{1+sRC} = T(s)$

Put $s=j\omega$; $T(j\omega) = \frac{1}{1+j\omega RC}$

To obtain inverse polar plot $\frac{1}{T(j\omega)} = 1+j\omega RC$

$|T(j\omega)^{-1}| = \sqrt{1+(\omega RC)^2}$; $\angle T(j\omega)^{-1} = \tan^{-1}\omega RC$

at $\omega\to 0$ $|T(j\omega)^{-1}| = 1$; at $\omega\to\infty$ $|T(j\omega)^{-1}| = \infty$

at $\omega\to 0$ $\angle T(j\omega)^{-1} = 0°$; at $\omega\to\infty$ $\angle T(j\omega)^{-1} = 90°$.