COMPLEMENTARY FUNCTION

Since $F(D, D')$ is *a* polynomial expression of degree n and homogeneous in D and D′, it can be resolved into linear factors. That is, the equation $F(D, D')z = 0$ can be written as ...(3)

$$(D - m_1D')(D - m_2D') \dots (D - m_nD')z = 0 \quad \text{...(4)}$$

where $m_1, m_2, \dots, m_n$ are the roots of the Auxiliary equation $F(m, 1) = 0$ ...(5)

obtained by writing m for D and 1 for D′ in $F(D, D') = 0$.

Thus $(m - m_1)(m - m_2) \dots (m - m_n) = 0$

Equation (4) will be clearly satisfied by the solution of each of the component first order differential equations $(D - m_1D')z = 0$, $(D - m_2D')z = 0$, ..., $(D - m_nD')z = 0$ ...(6)

Each of the equations in (6) is of Lagrange's form.

Let us first find the solution of $(D - mD')z = 0$ ...(7)

or $p - mq = 0$ which is Lagrange's type.

The corresponding auxiliary simultaneous equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{0} \quad \Rightarrow \quad dy + m\,dx = 0 \quad \text{and} \quad dz = 0.$$

Thus, the two solutions of (7) are $y + mx = c_1$ and $z = c_2$

Hence the solution of equation (7) is $z = \phi(y + mx)$ where ϕ is an arbitrary function.

Thus, the combined solution of the component equations in (6), is

$$z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \dots + \phi_n(y + m_nx). \quad \text{...(8)}$$

This is the C.F. of the equation $F(D, D')z = f(x, y)$.

Case of repeated roots. If two roots of (5) are equal, say $m_1 = m_2$ then the corresponding part of the differential equation is $(D - m_1D')(D - m_1D')z = 0$ or $(D - m_1D')^2 z = 0$ ...(9)

Let us write $(D - m_1D')z = t$ ...(10)

$D = \frac{\partial}{\partial x}$

then (9) becomes $(D - m_1D')t = 0$ whose solution is $t = \phi_1(y + m_1x)$.

$D' = \frac{\partial}{\partial y}$

Putting this value of t in (10), we get $(D - m_1D')z = \phi_1(y + m_1x)$

which is again of the Lagrange's form with auxiliary system of simultaneous equations as

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi_1(y + m_1x)}.$$

$\Rightarrow p - m_1q = \phi_1(y + m_1x)$

One solution of the above system is $y + m_1x = c_1$.

Then, for the other solution, we have $\dfrac{dx}{1} = \dfrac{dz}{\phi_1(c_1)}$

which gives $z = x\,\phi_1(c_1) + c_2 = x\,\phi_1(c_1) + \phi_2(c_1)$

or $z = x\,\phi_1(y + m_1x) + \phi_2(y + m_1x)$.

Similarly, if three roots of (5) are equal, say $m_1 = m_2 = m_3$ then the corresponding part in the C.F. is

$$z = x^2\phi_1(y + m_1x) + x\phi_2(y + m_1x) + \phi_3(y + m_1x).$$

This result can be extended to still more number of equal roots.

THE PARTICULAR INTEGRAL: $\dfrac{1}{F(D, D')} f(x, y)$.

As in the case of ordinary differential equations, we define here the operator $\dfrac{1}{F(D, D')}$ by the identity $\dfrac{1}{F(D, D')} F(D, D') f(x, y) = f(x, y) = F(D, D') \dfrac{1}{F(D, D')} f(x, y)$.

Thus, the P.I. $= \dfrac{1}{F(D, D')} f(x, y) = \dfrac{1}{(D - m_1D')(D - m_2D') \dots (D - m_nD')} f(x, y)$.

$$= \frac{1}{D - m_1D'} \cdot \frac{1}{D - m_2D'} \cdots \frac{1}{D - m_nD'} f(x, y).$$

This may be found by solving n equations of first order

$$u_1 = \frac{1}{D - m_nD'} f(x, y), \quad u_2 = \frac{1}{D - m_{n-1}D'} u_1, \quad \dots, \quad z = u_n = \frac{1}{D - mD'} u_{n-1}.$$

To meet the above, let us evaluate $\dfrac{1}{D - mD'} f(x, y)$.

For this, consider y as *a* constant and put $\lambda \equiv mD'$,

then
$$\frac{1}{D - mD'} f(x, y) = \frac{1}{D - \lambda} f(x, y) = e^{\lambda x} \int e^{-\lambda x} f(x, y)\, dx$$
$$= e^{mxD'} \int e^{-mxD'} f(x, y)\, dx = e^{mxD'} \int f(x, y - mx)\, dx.$$

Let this integral, when evaluated, be denoted by $g(x, y)$, then

$$\frac{1}{D - mD'} f(x, y) = e^{mxD'} g(x, y) = g(x, y + mx).$$

Thus, the rule to find $\dfrac{1}{D - mD'} f(x, y)$ can be stated as:

First change y to $y - mx$ in $f(x, y)$, integrate it w.r.t. x considering y as constant, and then in the integral obtained, change y to $y + mx$, The resulting expression is the value of $\dfrac{1}{D - mD'} f(x, y)$.

EXAMPLE 29.1. Solve the second order homogeneous equation

$$(D^2 - DD' - 6D'^2)z = x + y.$$

SOLUTION: The given equation can be written as $(D + 2D')(D - 3D')z = x + y$.

The auxiliary equation is $(m + 2)(m - 3) = 0$ hence $m = -2, 3$.

Therefore, C.F. $= \phi_1(y - 2x) + \phi_2(y + 3x)$

and P.I. $= \dfrac{1}{(D + 2D')(D - 3D')}(x + y) = \dfrac{1}{D + 2D'}\left[\dfrac{1}{D - 3D'}(x + y)\right]$.

Let us set $u = \dfrac{1}{D - 3D'}(x + y)$ and obtain *a* particular integral of the equation

$$(D - 3D')u = x + y$$

$\therefore \quad u = \int (x + y - 3x)\, dx = yx - x^2$. Now, replacing y by $y + 3x$, we get $u = xy + 2x^2$

Hence, in turn, we get $z = \dfrac{1}{D + 2D'} u = \dfrac{1}{D + 2D'} (xy + 2x^2)$

or $\quad z = \int [x(y + 2x) + 2x^2]\, dx = \int (yx + 4x^2)\, dx = \dfrac{yx^2}{2} + \dfrac{4x^3}{3}$

$\therefore \quad z = \dfrac{1}{2}x^2 y + \dfrac{1}{3}x^3$ (after replacing y by $y - 2x$).

Thus, the complete solution is $z = \phi_1 (y - 2x) + \phi_2 (y + 3x) + \dfrac{1}{2}x^2 y + \dfrac{1}{3}x^3$. **Ans.**

THE SHORT-CUT METHODS TO FIND P.I.

Analogus to the short cut methods for finding the Particular Integrals in the case of the ordinary differential equations with constant coefficients, we have the short methods to get the P.I. of the homogeneous linear partial differential equations and these are, as expected, more convenient than the general method given above. We list below the formulae for the short cut methods for obtaining the P.I., without proof.

(*i*) $\dfrac{1}{F(D, D')} e^{ax + by} = \dfrac{1}{F(a, b)} e^{ax + by}$ provided $F(a, b) \neq 0$

(*ii*) $\dfrac{1}{F(D^2, DD', D'^2)} \begin{matrix}\sin\\ \cos\end{matrix} (ax + by) = \dfrac{1}{F(-a^2, -ab, -b^2)} \begin{matrix}\sin\\ \cos\end{matrix} (ax + by)$

provided $F(-a^2, -ab, -b^2) \neq 0$.

(*iii*) $\dfrac{1}{F(D, D')} (x^m y^n) = [F(D, D')]^{-1} (x^m y^n)$

where $[F(D, D')]^{-1}$ is to be expanded in series of powers of D and D'.

In general, in determining $\dfrac{1}{F(D, D')} f(x, y)$ if $f(x, y)$, is *a* function of $ax + by$ we have *a* shorter method for determinating the P.I. as follows.

$D^2 = \dfrac{\partial^2}{\partial x^2}$

It is easy to verify that

$$D^r f(ax + by) = a^r f^{(r)} (ax + by) \quad \text{and} \quad D'^r f(ax + by) = b^r f^{(r)} (ax + by)$$

where $f^{(r)}$ is the r^{th} derivative of f w. r. t. $(ax + by)$.

Since $f(D, D')$ is homogeneous in D and D' of order n,

$$F(D, D') f(ax + by) = F(a, b) f^{(n)} (ax + by)$$

$$\Rightarrow \quad \frac{1}{F(D, D')} f^{(n)} (ax + by) = \frac{1}{F(a, b)} f(ax + by), \text{ provided } F(a, b) \neq 0.$$

Further, let $ax + by = t$. This gives $\dfrac{1}{F(D, D')} f^{(n)}(t) = \dfrac{1}{F(a, b)} f(t)$

Integrating both sides n times w. r. t. t we get

$$\frac{1}{F(D, D')} f(t) = \frac{1}{F(a, b)} \iiint \cdots \int f(t)\, dt \ldots dt \quad \text{where } t = ax + by.$$

There is **a case of failure as well**, when $F(a, b) = 0$. To evaluate $\dfrac{1}{F(D, D')} f(ax + by)$ where $F(a, b) = 0$, we proceed as follows :

Differentiate $F(D, D')$ w.r.t. D partially and multiply the expression by x, then

$$\frac{1}{F(D, D')} f(ax + by) = \frac{x}{\left[\dfrac{\partial}{\partial D} F(D, D')\right]_{D = a,\, D' = b}} f(ax + by).$$

However, if $\left[\dfrac{\partial}{\partial D} F(D, D')\right]_{D = a,\, D' = b}$ is also zero, differentiate $\dfrac{\partial}{\partial D} F(D, D')$ again w.r.t. D partially and multiply the expression by x again, and so on.

EXAMPLE 29.2. Solve the second order partial differential equation

$(r + 2s + t) = e^{2x+3y}$ where symbols have their usual meaning.

SOLUTION: The given equation is $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = e^{2x + 3y}$

or $\quad (D^2 + 2DD' + D'^2)\, z = e^{2x + 3y}$

The auxiliary equation for C.F. is $m^2 + 2m + 1 = 0$ giving $m = -1, -1$.

$\therefore \quad$ C.F. $= \phi_1 (y - x) + x\phi_2 (y - x)$

and $\quad$ P.I. $= \dfrac{1}{D^2 + 2DD' + D'^2} e^{2x + 3y} = \dfrac{1}{2^2 + 2(2)(3) + 3^2} e^{2x + 3y} = \dfrac{1}{25} e^{2x + 3y}$

Therefore, the complete solution of the given equation, is

$$z = \phi_1 (y - x) + x\phi_2 (y - x) + \frac{1}{25} e^{2x + 3y}$$

where ϕ_1 and ϕ_2 are arbitrary functions. **Ans.**

EXAMPLE 29.3. Solve $(D^3 - 4D^2D' + 5\,DD'^2 - 2D'^3)\, z = e^{y-2x} + e^{y+2x} + e^{y+x}$,

where $D \equiv \dfrac{\partial}{\partial x}$, $D' \equiv \dfrac{\partial}{\partial y}$.

SOLUTION: For C.F., the auxiliary equation is $m^3 - 4m^2 + 5m - 2 = 0$ which gives $m = 1, 1, 2$.

Therefore, C.F. $= \phi_1 (y + 2x) + \phi_2 (y + x) + x\phi_3 (y + x)$ where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

Next, P.I. $= \dfrac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} (e^{y-2x} + e^{y+2x} + e^{y+x})$

$$= \frac{1}{(D - D')^2 (D - 2D')} e^{y - 2x} + \frac{1}{(D - D')^2 (D - 2D')} e^{y + 2x} + \frac{1}{(D - D')^2 (D - 2D')} e^{y + x}$$

Here $\quad \dfrac{1}{(D - D')^2 (D - 2D')} e^{y - 2x} = \dfrac{1}{(-2 - 1)^2 (-2 - 2\cdot 1)} e^{y - 2x} = \dfrac{-1}{36} e^{y - 2x}$,

$\quad \dfrac{1}{(D - D')^2 (D - 2D')} e^{y + 2x} = \dfrac{1}{(2 - 1)^2} \cdot \dfrac{1}{(D - 2D')} e^{y + 2x} = 1 \cdot \dfrac{x}{1} \cdot e^{y + 2x}$.

and $\dfrac{1}{(D-D')^2(D-2D')}e^{y+x} = \dfrac{1}{(1-2\cdot 1)}\cdot\dfrac{1}{(D-D')^2}e^{y+x} = (-1)\cdot\dfrac{x}{2(D-D')}e^{y+x}$

$$= (-1)\frac{x\cdot x}{2\cdot 1}e^{y+x} = -\frac{x^2}{2}e^{y+x}.$$

Thus, the complete solution of the given equation, is

$$z = \phi_1(y+2x) + \phi_2(y+x) + x\phi_3(y+x) - \frac{1}{36}e^{y-2x} + x e^{y+2x} - \frac{x^2}{2}e^{y+x}. \qquad \textbf{Ans.}$$

EXAMPLE 29.4. Solve $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$ where $D \equiv \dfrac{\partial}{\partial x}$, $D' \equiv \dfrac{\partial}{\partial y}$

SOLUTION: For C.F. the auxiliary equation is $m^2 - 6m + 9 = 0$ hence $m = 3, 3$.

Therefore C.F. $= \phi_1(y+3x) + x\phi_2(y+3x)$. $(12x^2+36xy)$

Next, P.I $= \dfrac{1}{D^2 - 6DD' + 9D'^2}(12x^2 + 36xy)$ $\left[D\left(1-\dfrac{3D'}{D}\right)\right]^2$

$$= \frac{1}{(D-3D')^2}(12x^2+36xy) = \frac{1}{D^2}\left[1-\frac{3D'}{D}\right]^{-2}(12x^2+36xy).$$

$$= \frac{1}{D^2}\left[1 + 2\cdot\left(\frac{3D'}{D}\right) + \frac{(-2)(-3)}{2\cdot 1}\cdot\left(\frac{3D'}{D}\right)^2 + \ldots\right](12x^2+36xy) \quad D=\frac{\partial}{\partial x},\ D'=\frac{\partial}{\partial y}$$

$$= \frac{1}{D^2}\left[1 + \frac{6D'}{D} + \frac{27D'^2}{D^2}\right](12x^2+36xy) \qquad \text{(retaining terms upto } D'^2)$$

$$= \frac{1}{D^2}\left[(12x^2+36xy) + \frac{6}{D}(0+36x) + 0\right] = \frac{1}{D^2}(12x^2+36xy) + \frac{1}{D^3}(6\times 36x)$$

$$= x^4 + 6x^3y + 6\cdot(36)\frac{x^4}{2\cdot 3\cdot 4} = 10x^4 + 6x^3y$$

Thus, the complete solution is $z = \phi_1(y+3x) + x\phi_2(y+3x) + 10x^4 + 6x^3y$. **Ans.**

EXAMPLE 29.5. Solve $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{\partial^3 z}{\partial y^3} = x^3y^3$.

SOLUTION: The given equation can be written as: $(D^3 - D'^3)z = (xy)^3$.

For complementary function, the auxiliary equation is $m^3 - 1 = 0$ whose roots are $m = 1, \omega, \omega^2$ where ω and ω^2 are the complex cube roots of unity.

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y+\omega x) + \phi_3(y+\omega^2 x)$

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions and $\omega = \left(-1+i\sqrt{3}\right)/2$ and $\omega^2 = \left(-1-i\sqrt{3}\right)/2$.

Next, P.I. $= \dfrac{1}{D^3 - D'^3}x^3y^3 = \dfrac{1}{D^3}\left[1-\dfrac{D'^3}{D^3}\right]^{-1}(x^3y^3) = \dfrac{1}{D^3}\left[1 + \dfrac{D'^3}{D^3} + \dfrac{D'^6}{D^6} + \ldots\right](x^3y^3)$

$$= \frac{1}{D^3}\left[1+\frac{D'^3}{D^3}\right](x^3y^3) + 0 = \frac{1}{D^3}\left[x^3y^3 + \frac{x^6}{4\cdot 5\cdot 6}(3\cdot 2\cdot 1)\right]$$

$$= \frac{x^6y^3}{4\cdot 5\cdot 6} + \frac{1}{20}\cdot\frac{x^9}{7\cdot 8\cdot 9} = \frac{x^6y^3}{120} + \frac{x^9}{10080}.$$

Therefore, the complete solution is

$$z = \phi_1(y+x) + \phi_2(y+\omega x) + \phi_3(y+\omega^2 x) + \frac{x^6y^3}{120} + \frac{x^9}{10080}. \qquad \textbf{Ans.}$$

EXAMPLE 29.6. Solve the third order homogeneous partial differential equation

$$\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x\partial y^2} - 6\frac{\partial^3 z}{\partial y^3} = \sin(x+2y) + x^2y.$$

SOLUTION: The given equation can be written as

$$(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + x^2y.$$

The auxiliary equation is $m^3 - 7m - 6 = 0$ with roots $-1, -2, 3$.

$\therefore$ C.F. $= \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$ where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

Next, P.I $= \dfrac{1}{(D+D')(D+2D')(D-3D')}\{\sin(x+2y) + x^2y\}$.

Evaluating it term by term, we get

$$\frac{1}{(D+D')(D+2D')(D-3D')}\sin(x+2y) = \frac{1}{(D+D')(D^2 - DD' - 6D'^2)}\sin(x+2y)$$

$$= \frac{1}{(D+D')\{-1+1\cdot 2-6(-4)\}}\sin(x+2y)$$

$$= \frac{1}{25}\cdot\frac{1}{D+D'}\sin(x+2y) = \frac{1}{25}\cdot\frac{D-D'}{D^2-D'^2}\sin(x+2y) = \frac{1}{25}\cdot\left(\frac{(D-D')}{-1-(-4)}\right)\sin(x+2y)$$

$$= \frac{1}{75}[D\sin(x+2y) - D'\sin(x+2y)] = -\frac{1}{75}\cos(x+2y).$$

and $\dfrac{1}{(D+D')(D+2D')(D-3D')}(x^2y) = \dfrac{1}{D^3 - 7DD'^2 - 6D'^3}(x^2y)$

$$= \frac{1}{D^3}\left[1-\left(7\frac{D'^2}{D^2} + 6\frac{D'^3}{D^3}\right)\right]^{-1}(x^2y) = \frac{1}{D^3}\left[1+\left(7\frac{D'^2}{D^2} + 6\frac{D'^3}{D^3} + \ldots\right)\right](x^2y)$$

$$= \frac{1}{D^3}\left[x^2y + 0 + 0 + \ldots\right] = \iiint x^2y\,dx\,dx\,dx = \frac{x^5y}{3\cdot 4\cdot 5}$$

Thus, P.I. $= \dfrac{-1}{75}\cos(x+2y) + \dfrac{x^5y}{60}$.

Therefore, the complete solution is

$$z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) + \frac{x^5y}{60} - \frac{1}{75}\cos(x+2y). \qquad \textbf{Ans.}$$

EXAMPLE 29.7. Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x\cos 2y$ where $D \equiv \dfrac{\partial}{\partial x}$, $D' \equiv \dfrac{\partial}{\partial y}$.

SOLUTION: For C.F., the auxiliary equation is $m^3 + m^2 - m - 1 = 0$ whose roots are $1, -1, -1$.

$\therefore$ C.F. $= \phi_1(y-x) + x\phi_2(y-x) + \phi_3(y+x)$ where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

Next, P.I. $= \dfrac{1}{(D+D')^2(D-D')}e^x\cos 2y$

$$= e^x\frac{1}{(D+1+D')^2(D+1-D')}\cos 2y \qquad \text{(replacing D by D + 1)}$$

$$= e^x\frac{1}{\left[(D+1)^2 - D'^2\right](D+D'+1)}\cos 2y = e^x\frac{1}{(D+D'+1)(D^2 - D'^2 + 2D + 1)}\cos 2y$$

$$= e^x \frac{1}{(D+D'+1)} \cdot \frac{1}{(0-(-4)+2D+1)} \cos 2y = e^x \frac{1}{(D+D'+1)(2D+5)} \cos 2y$$

$$= e^x \frac{1}{2D^2+2DD'+7D+5D'+5} \cos 2y = e^x \frac{1}{0+0+7D+5D'+5} \cos 2y$$

$$= e^x \frac{7D+5D'-5}{(7D+5D')^2-25} \cos 2y = e^x \frac{7D+5D'-5}{49D^2+70DD'+25D'^2-25} \cos 2y$$

$$= e^x \frac{7D+5D'-5}{0+0+25(-4)-25} \cos 2y = e^x \frac{7D+5D'-5}{-125} \cos 2y$$

$$= \frac{e^x}{-125}(0-10\sin 2y - 5\cos 2y) = \frac{e^x}{25}(2\sin 2y + \cos 2y)$$

Thus the complete solution is $z = \phi_1(y-x) + x\phi_2(y-x) + \phi_3(y+x) + \frac{e^x}{25}(2\sin 2y + \cos 2y)$ **Ans.**

EXAMPLE 29.8. Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

[GGSIPU III Sem End Term 2012; IV Sem I Test 2015]

SOLUTION: For C.F. the A.E. is $m^2 - m - 2 = 0$ $\therefore$ $m = -1, 2$

$\therefore$ C.F. $= \phi_1(y+2x) + \phi_2(y-x)$ where ϕ_1 and ϕ_2 are arbitrary functions

and P.I. $= \frac{1}{D^2 - DD' - 2D'^2}(y-1)e^x = \frac{1}{(D-2D')(D+D')}(y-1)e^x$

$$= \frac{1}{3D'}\left[\frac{1}{D-2D'} - \frac{1}{D+D'}\right](y-1)e^x = \frac{1}{3D'}\left[\frac{1}{D-2D'}(y-1)e^x - \frac{1}{D+D'}(y-1)e^x\right]$$

$$= \frac{1}{3D'}\left[\int_{y\text{ constt}} (y-2x-1)e^x dx - \int_{y\text{ constt.}} (y+x-1)e^x dx\right]$$

$$= \frac{1}{3D'}\left[\left\{(y-2x-1)e^x - \int(-2)e^x dx\right\} - \left\{(y+x-1)e^x - \int 1 e^x dx\right\}\right]$$

$$= \frac{1}{3D'}\left[\left\{(y-2x-1)e^x + 2e^x\right\}_{y\to y+2x} - \left\{(y+x-1)e^x - e^x\right\}_{y\to y-x}\right]$$

$$= \frac{1}{3D'}\left[(y+2x-2x-1+2)e^x - (y-x+x-1-1)e^x\right]$$

$$= \frac{1}{3D'}[(y+1)e^x - (y-2)e^x] = \frac{1}{3D'} \cdot 3e^x = y e^x$$

$\therefore$ The solution is $z = \phi_1(y+2x) + \phi_2(y-x) + y e^x$. **Ans.**

EXAMPLE 29.9. Solve $(D^2 + 2DD' - 8D'^2)z = \sqrt{2x+3y}$.

SOLUTION: The auxiliary equation is $m^2 + 2m - 8 = 0$ whose roots are $m = -4$ and 2.

$\therefore$ C.F.$= \phi_1(y+2x) + \phi_2(y-4x)$ where ϕ_1, ϕ_2 are arbitrary functions.

Next, P.I. $= \frac{1}{(D-2D')(D+4D')}(2x+3y)^{\frac{1}{2}} = \frac{1}{(D-2D')}\left[\frac{1}{(D+4D')}(2x+3y)^{\frac{1}{2}}\right]$

$$= \frac{1}{(D-2D')}\int[2x+3\{y-(-4)x\}]^{\frac{1}{2}} dx = \frac{1}{(D-2D')}\int[14x+3y]^{\frac{1}{2}} dx$$

$$= \frac{1}{(D-2D')}\left[\frac{2}{3\cdot 14}(14x+3y)^{\frac{3}{2}}\right]_{(y\to y-4x)}$$

$$= \frac{1}{21}\cdot\frac{1}{(D-2D')}[14x+3(y-4x)]^{\frac{3}{2}} = \frac{1}{21}\cdot\frac{1}{D-2D'}(2x+3y)^{\frac{3}{2}}$$

$$= \frac{1}{21}\int[2x+3(y-2x)]^{\frac{3}{2}} dx = \frac{1}{21}\int(3y-4x)^{\frac{3}{2}} dx = \frac{1}{21}\cdot\left[\frac{2(3y-4x)^{\frac{5}{2}}}{5(-4)}\right]_{(y\to y+2x)}$$

$$= -\frac{1}{210}[3(y+2x)-4x]^{\frac{5}{2}} = -\frac{1}{210}(2x+3y)^{\frac{5}{2}}$$

Therefore the general solution is

$$z = \phi_1(y+2x) + \phi_2(y-4x) - \frac{1}{210}(2x+3y)^{\frac{5}{2}}.$$ **Ans.**

EXAMPLE 29.10. Solve the equation $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$.

[GGSIPU III Sem End Term 2006]

SOLUTION: For complementary function we have the auxiliary equation of the given equation as $m^3 - 7m - 6 = 0$ whose roots are $m = -1, -2, 3$.

$\therefore$ C.F. $= \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$ where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

Next, P.I. $= \frac{1}{D^3 - 7DD'^2 - 6D'^3}[\sin(x+2y) + e^{2x+y}]$

$$= \frac{1}{-1^2 D - 7D(-4) - 6(-4)D'}\sin(x+2y) + \frac{1}{2^3 - 7.2.1^2 - 6.1^3}e^{2x+y}$$

$$= \frac{1}{27D + 24D'}\sin(x+2y) - \frac{1}{12}e^{2x+y}$$

$$= \frac{1}{3}\left(\frac{9D-8D'}{81D^2 - 64D'^2}\right)\sin(x+2y) - \frac{e^{2x+y}}{12}$$

$$= \frac{9D-8D'}{3[81(-1)-64(-4)]}\sin(x+2y) - \frac{e^{2x+y}}{12}$$

$$= \frac{9\cos(x+2y) - 16\cos(x+2y)}{3(175)} - \frac{e^{2x+y}}{12} = \frac{-1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$$

$\therefore$ Complete solution is $z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) - \frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$. **Ans.**

NON-HOMOGENEOUS LINEAR EQUATIONS

The linear partial equation $F(D, D')z = f(x, y)$ ...(1)

is non-homogeneous if $F(D, D')$, the polynomial expression in D, D', is not homogeneous.

When $F(D, D')$ is homogeneous it can always be reduced into linear factors but it is not always true when $F(D, D')$ is non-homogeneous. However, we shall undertake only those non-homogeneous equations which can be reduced into linear factors.

For **Complementary Function** we consider

$$F(D, D')z = (a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2) \dots (a_nD + b_nD' + c_n)z = 0 \quad \text{...(2)}$$

where a_i, b_i, c_i $(i = 1, 2, \dots, n)$ are constants.

Any solution of the equation $(a_iD + b_iD' + c_i)z = 0$ ...(3)

is a solution of (2) also.

Equation (3) can also be written as $a_ip + b_iq = -c_iz$ which is of Lagrangian form whose auxiliary equation is

Pdx+Qdy=R

$$\frac{dx}{a_i} = \frac{dy}{b_i} = \frac{dz}{-c_iz} \quad \text{...(4)}$$

The first two factors in (4), give

$$a_i\,dy = b_i\,dx \quad \text{hence} \quad a_iy - b_ix = k_1 \quad \text{...(5)}$$

where k_1 is an arbitrary constant.

$k_2 = \phi(k_1)$

Similarly, the first and the last factor in (4), give

$$\frac{dx}{a_i} = \frac{dz}{-c_iz} \quad \text{hence} \quad \frac{x}{a_i} = \frac{-1}{c_i}\log z + \log k \quad \text{or} \quad z = k_2\, e^{-c_ix/a_i} \quad \text{...(6)}$$

where k_2 is another arbitrary constant.

Combining (5) and (6), we get $z = \phi_i(a_iy - b_ix)\, e^{-c_ix/a_i}$.

Thus, the C.F. $= \sum_{i=1}^{n} \phi_i(a_iy - b_ix)\, e^{-c_ix/a_i}$ $(a_i \neq 0)$ ...(7)

where ϕ_i $(i = 1, 2, \dots, n)$ are arbitrary functions of their arguments.

In case $F(D, D')$ has repeated linear factors, e.g., suppose $a_iD + b_iD' + c_i$ and $a_2D + b_2D' + c_2$ are same, i.e., if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, then the corresponding expression in C.F. for (1) will be

$$e^{-c_ix/a_i}[\phi_1(a_1y - b_1x) + x\phi_2 \cdot (a_1y - b_1x)] \quad \text{which can be easily verified.}$$

Next, **the Particular Integral** of non-homogeneous partial differential equations can be obtained in a way similar to those of ordinary differential equations. Some of the standard cases are listed below:

I. $f(x, y) = e^{ax + by}$

we have $D^r e^{ax+by} = a^r e^{ax+by}$ and $D'^r e^{ax+by} = b^r e^{ax+by}$ hence

$F(D, D')e^{ax+by} = F(a, b)e^{ax+by}$. Operating both sides by $\dfrac{1}{F(D, D')}$ we get

$$e^{ax+by} = \frac{1}{F(D, D')}F(a, b)e^{ax+by} \quad \text{hence} \quad \frac{1}{F(D, D')}e^{ax+by} = \frac{1}{F(a, b)}e^{ax+by}.$$

II. $f(x, y) = \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax + by)$.

we have $D^2 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax + by) = -a^2 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax + by)$ and $D'^2 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax + by) = -b^2 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax + by)$.

From these results one can visualize that

$$\frac{1}{F(D, D')}\begin{matrix}\sin\\ \cos\end{matrix}(ax+by) = \frac{1}{F(D^2, DD', D'^2, D, D')}\begin{matrix}\sin\\ \cos\end{matrix}(ax+by)$$

$$= \frac{1}{F(-a^2, -ab, -b^2, D, D')}\begin{matrix}\sin\\ \cos\end{matrix}(ax+by)$$

after this we rationalise sort of, the denominator and can evaluate that further. It will be more clear when we take up examples.

III. $f(x, y) = x^m y^n$.

Here, as usual, $\frac{1}{F(D, D')}x^m y^n = [F(D, D')]^{-1}x^m y^n$

which can be evaluated after expanding $[F(D, D')]^{-1}$ in powers of D and D′ as done in the case of ordinary differential equations.

IV. $f(x, y) = e^{ax+by}$ V where V is some function of x and y. Here, as done in the ordinary differential equation, we have

$$\frac{1}{[F(D, D')]}\left[e^{ax+by}V(x, y)\right] = e^{ax+by}\frac{1}{F(D+a, D'+b)}V(x, y).$$

EXAMPLE 29.11. Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} - z = \cos(x+2y) + e^y$.

SOLUTION: The given equation can be written as

$$(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y \quad \text{or} \quad (D-1)(D-D'+1)z = \cos(x+2y) + e^y.$$

The C.F. corresponding to the factor $(D-1)$ is $e^x \phi_1(y)$ and corresponding to the factor $(D - D' + 1)$ the C.F. is $e^{-x}\phi_2(y+x)$.

∴ C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y+x)$ where ϕ_1 and ϕ_2 are arbitrary functions.

and P.I. $= \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y) + \frac{1}{D^2 - DD' + D' - 1}e^y$

First consider $\frac{1}{D^2 - DD' + D' - 1}e^y = \frac{1}{D^2 - DD' + D' - 1}e^{0x+1y}$

It is clearly *a* case of failure, since $D^2 - DD' + D' - 1$ becomes 0 on replacing D by 0 and D′ by 1. Hence we have the above expression

$$= x\frac{1}{\frac{\partial}{\partial D}(D^2 - DD' + D' - 1)}e^y = x\frac{1}{2D - D'}e^y = x\frac{1}{2(0)-(1)}e^y = -xe^y.$$

Next, $\frac{1}{D^2 - DD' + D' - 1}\cos(x+2y) = \frac{1}{-1+(1)(2)+D'-1}$

$$= \frac{1}{D'}\cos(x+2y) = \int \cos(x+2y)\,dy = \frac{1}{2}\sin(x+2y).$$

∴ P.I. $= -xe^y + \frac{1}{2}\sin(x+2y)$.

Thus, the complete solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) + \frac{1}{2}\sin(x+2y) - xe^y$. **Ans.**

$D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

EXAMPLE 29.12. Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = xy + e^{x+2y}$.

SOLUTION: Given equation can be written as $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$

or $(D - D')(D + D' - 3)z = xy + e^{x+2y}$ $\quad D - D' = D + (-1)D' + 0$

∴ C.F. $= e^{0x}\phi_1(y+x) + e^{3x}\phi_2(y-x)$.

and P.I. $= \frac{1}{(D-D')(D+D'-3)}(xy) + \frac{1}{(D-D')(D+D'-3)}e^{x+2y}$

Now $\frac{1}{(D-D')(D+D'-3)}e^{x+2y} = e^x\frac{1}{(D+1-D')(D+1+D'-3)}e^{2y}$

$$= e^x\frac{1}{(D+D'-2)}\left[\frac{1}{(D-D'+1)}e^{2y}\right]$$

$$= e^x\frac{1}{(D+D'-2)}\frac{1}{(0-2+1)}e^{2y}$$

$$= -e^x\frac{1}{D+D'-2}e^{2y} \quad \text{(case of failure)}$$

$$= -e^x\frac{y}{\frac{\partial}{\partial D'}(D+D'-2)}e^{2y} = -e^x y\cdot\frac{1}{1}e^{2y} = -y\,e^{x+2y}.$$

and $\frac{1}{(D-D')(D+D'-3)}xy = \frac{-1}{3(D-D')}\left[1 - \frac{D+D'}{3}\right]^{-1}(xy)$

$$= \frac{-1}{3(D-D')}\left[1 + \frac{1}{3}(D+D') + \frac{1}{9}(D+D')^2 + \ldots\right]xy$$

$$= \frac{-1}{3(D-D')}\left[xy + \frac{1}{3}D(xy) + \frac{1}{3}D'(xy) + \frac{D^2}{9}(xy) + \frac{2}{9}DD'(xy) + \frac{D'^2}{9}(xy) + \ldots\right]$$

$$= \frac{-1}{3(D-D')}\left[xy + \frac{y}{3} + \frac{x}{3} + \frac{2}{9}\right] = \frac{-1}{3D}\left(1 - \frac{D'}{D}\right)^{-1}\left[xy + \frac{x}{3} + \frac{y}{3} + \frac{2}{9}\right]$$

$$= \frac{-1}{3D}\left(1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \ldots\right)\left[xy + \frac{x}{3} + \frac{y}{3} + \frac{2}{9}\right]$$

$$= \frac{-1}{3D}\left(xy + \frac{x}{3} + \frac{y}{3} + \frac{2}{9} + \frac{x^2}{2} + \frac{x}{3}\right) = \frac{-1}{3}\left[\frac{x^2}{2}y + \frac{x^2}{6} + \frac{xy}{3} + \frac{2x}{9} + \frac{x^3}{6} + \frac{x^2}{6}\right]$$

$$= -\left[\frac{1}{6}x^2y + \frac{x^2}{9} + \frac{xy}{9} + \frac{2x}{27} + \frac{x^3}{18}\right]$$

$\frac{1}{D} = \int dx = \frac{1}{D} = \int dx$

∴ The complete solution is

$$z = \phi_1(y+x) + e^{2x}\phi_2(y-x) - ye^{x+2y} - \left[\frac{1}{6}x^2y + \frac{x^2}{9} + \frac{xy}{9} + \frac{2x}{27} + \frac{x^3}{18}\right]$$ **Ans.**

EXAMPLE 29.13. (a) Solve the equation $(D^2 - D')z = 2y - x^2$. [GGSIPU IV Sem End Term 2015]

(b) Solve $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + \sin(2x+y) + xy$.

SOLUTION: (a) The equation is non-homogenous and is irreducible.

However, we can construct solution which contains the required number of arbitrary constants. The method is derived from the following result.

$$F(D, D')\, e^{ax+by} = F(a, b)\, e^{ax+by} \qquad \text{...(1)}$$

Which is valid in case of reducible as well a irreducible $F(D, D')$.

In the given case, the equation is irreducible hence solution is $z = e^{ax+by}$, provided $a^2 - b^2 = 0$ or $a = \pm\sqrt{b}$.

Therefore the complementary function is

$$z = C_1 e^{\sqrt{b}x+by} + C_2 e^{-\sqrt{b}x+by}$$

where C_1, C_2 and b are arbitrary constants.

The particular integral is

$$z = \frac{1}{D^2 - D'}(2y - x^2) = \frac{1}{D^2}\left[1 - \frac{D'}{D^2}\right]^{-1}(2y - x^2)$$

$$= \frac{1}{D^2}\left[1 + \frac{D'}{D^2} + \frac{D'^2}{D^4} + \ldots\right](2y - x^2) = \frac{1}{D^2}\left[2y - x^2 + \frac{1}{D^2}(2) + 0\right]$$

$$= \frac{1}{D^2} 2y = yx^2$$

$\therefore$ The solution of the equation is

$$z = C_1 e^{\sqrt{b}x+by} + C_2 e^{-\sqrt{b}x+by} + x^2 y \quad \textbf{Ans.}$$

Note that we could also write

$$PI = -\frac{1}{D'}\left(1 - \frac{D^2}{D'}\right)^{-1}(2y - x^2) = -\frac{1}{D'}\left(1 + \frac{D^2}{D'} + \frac{D^2}{D'^2} + \ldots\right)(2y - x^2)$$

(b) The given equation can be written as

$$(D + D')(D - 2D' + 2)z = e^{2x+3y} + \sin(2x+y) + xy.$$

Hence C.F. $= e^{0x}\phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$.

Next, the P.I. corresponding to e^{2x+3y}, is

$$= \frac{1}{(D + D')(D - 2D' + 2)} e^{2x+3y} = \frac{1}{(2+3)(2-6+2)} e^{2x+3y} = -\frac{1}{10} e^{2x+3y}.$$

and P.I. corresponding to $\sin(2x+y)$, is

$$= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'}\sin(2x+y) = \frac{1}{-4 + 2(1) - 2(-1) + 2D + 2D'}\sin(2x+y)$$

$$= \frac{1}{2(D + D')}\sin(2x+y) = \frac{1}{2}\frac{D - D'}{D^2 - D'^2}\sin(2x+y)$$

$$= \frac{1}{2}\frac{(D - D')}{(-4+1)}\sin(2x+y) = -\frac{1}{6}[2\cos(2x+y) - \cos(2x+y)] = -\frac{1}{6}\cos(2x+y).$$

and finally, P.I. corresponding to xy, is

$$= \frac{1}{(D + D')(D - 2D' + 2)}(xy) = \frac{1}{2D}\left(1 + \frac{D'}{D}\right)^{-1}\left(1 + \frac{1}{2}D - D'\right)^{-1}(xy)$$

$$= \frac{1}{2D}\left(1 + \frac{D'}{D}\right)^{-1}\left[1 - \frac{D}{2} + D' + \left(\frac{1}{4}D^2 + D'^2 - DD'\right)\ldots\right](xy)$$

$$= \frac{1}{2D}\left(1 + \frac{D'}{D}\right)^{-1}\left[xy - \frac{y}{2} + x - 1\right] = \frac{1}{2D}\left[1 - \frac{D'}{D} + \frac{D'^2}{D^2}\ldots\right]\left(xy - \frac{y}{2} + x - 1\right)$$

$$= \frac{1}{2D}\left(1 - \frac{D'}{D}\right)\left(xy - \frac{y}{2} + x - 1\right) = \frac{1}{2D}\left[xy - \frac{y}{2} + x - 1 - \frac{x^2}{2} + \frac{x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^2 y}{2} - \frac{xy}{2} + \frac{x^2}{2} - x - \frac{x^3}{6} + \frac{x^2}{4}\right] = \frac{1}{2}\left[\frac{1}{2}x^2 y - \frac{1}{2}xy - \frac{x^3}{6} + \frac{3}{4}x^2 - x\right]$$

$\therefore$ P. I. $= \frac{-1}{10}e^{2x+3y} - \frac{1}{6}\cos(2x+y) + \frac{1}{2}\left[\frac{1}{2}x^2 y - \frac{1}{2}xy - \frac{x^3}{6} + \frac{3}{4}x^2 - x\right]$

$\therefore$ Complete solution is $z = \text{C.F.} + \text{P.I.}$ **Ans.**

PHYSICAL APPLICATIONS

The linear partial differential equations of order two have maximum applications. Major thrust areas of applications in Science and Engineering are vibration of strings and membranes, gravitational and electric potential problems and heat flow problems.

For example,

(*i*) The diffusion equation in one dimension $\dfrac{\partial u}{dt} = c^2 \dfrac{\partial^2 u}{dx^2}$ $\qquad \dfrac{\partial^2 u}{\partial x^2}$

(*ii*) Laplace equation in two dimensions $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ $\qquad \nabla^2 u = 0$

in the conduction of heat in a plate in steady state.

The equation is also valid in electrostatic potential problems.

(*iii*) Wave equation : $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, [*GGSIPU III Sem II Term 2011*]

which takes place in problems of vibration of string, rod, etc.

We introduce here the **METHOD OF SEPARATION OF VARIABLES.**

In this method the function, say u, of independent variables x and y, is taken as a product of two functions one of x and the other of y alone. Let us illustrate it with the help of examples.

EXAMPLE 29.14. **(a)** Use the method of separation of variables to solve the partial differential equation $\quad 3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0, \quad u(x, 0) = 4e^{-x}$.

[*GGSIPU III Sem End Term 2006*]

(b) Solve $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial t} + u$, given that $u(x, 0) = 3e^{-4x}$.

[*GGSIPU III Sem II Term 2013*]

SOLUTION: (a) By the method of separation of variables let $u = X(x)Y(y)$ then the given equation becomes

$$3\frac{dX}{dx}Y + 2X\frac{dY}{dy} = 0 \qquad \text{or} \qquad 3X'Y + 2XY' = 0$$

or $\qquad \dfrac{3X'}{X} = \dfrac{-2Y'}{Y} = K$ (a pure constant).

$\Rightarrow \qquad \dfrac{3X'}{X} = K$, which gives $X = e^{\frac{Kx}{3}}$

and $\qquad \dfrac{Y'}{Y} = \dfrac{-K}{2}$, which gives $Y = e^{-Ky/2}$.

Therefore, we have $\qquad u = XY = ce^{\frac{Kx}{3}} \cdot e^{-\frac{Ky}{2}}$ where c is some constant of integration.

$u = u(x, y) = \qquad u'(x, 0) \ldots y = 0$

The use of the initial condition $u(x, 0) = 4e^{-x}$ gives $ce^{\frac{Kx}{3}} = 4e^{-x}$. Hence $c = 4$ and $K = -3$.

Therefore $u = 4e^{-x} \cdot e^{\frac{3y}{2}}$ and this is the required solution. **Ans.**

(b) Let $u(x, t) = X(x)\,T(t)$

then the given equation becomes $\dfrac{dX}{dx}T = X\dfrac{dT}{dt} + XT \qquad$ or $\qquad X'T = XT' + XT$

$$\Rightarrow \quad \frac{X'}{X} = \frac{T'}{T} + 1 = k, \text{ a constant.}$$

$\therefore$ We have $\frac{X'}{X} = k$ and $\frac{T'}{T} = k - 1$.

Their solutions are $\log X = kx + \log a$ or $X = ae^{kx}$

and $\log T = (k-1)\,t + \log b$ or $T = b\,e^{(k-1)t}$

$\therefore$ $u(x, t) = XT = ae^{kx} b\, e^{(k-1)t} = C\, e^{kx + (k-1)t}$ where C is an arbitrary constant.

We are given that $u(x, 0) = 3e^{-4x}$ $\therefore$ $3e^{-4x} = C\,e^{kx}$ $\Rightarrow$ $C = 3$ and $k = -4$

$\therefore$ The solution is $u(x, t) = 3e^{-4x}\, e^{-5t}$ **Ans.**

CLASSIFICATION OF LINEAR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

The general form of a second order partial differential equation in two variables x and y with constant coefficients, is

$$a\frac{\partial^2 z}{\partial x^2} + 2h\frac{\partial^2 z}{\partial x \partial y} + b\frac{\partial^2 z}{\partial y^2} + 2f\frac{\partial z}{\partial x} + 2g\frac{\partial z}{\partial y} + cz = f(x, y) \quad \text{...(1)}$$

where a, b, h, f, g and c are constants. It can also be written as

$$F(D, D')\,z = [aD^2 + 2hDD' + bD'^2 + 2fD + 2gD' + c]\,z = f(x, y) \text{ where } D \equiv \frac{\partial}{\partial x} \text{ and } D' \equiv \frac{\partial}{\partial x}$$

$\frac{\partial}{\partial y}$

Let us now recall here that the general equation of the conic in x and y

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is said to be elliptic if $h^2 < ab$ or $h^2 - ab < 0$

parabolic if $h^2 = ab$ or $h^2 - ab = 0$

hyperbolic if $h^2 > ab$ or $h^2 - ab > 0$.

In a similar fashion, the above equation (1) is called

ELLIPTIC if $h^2 - ab < 0$

PARABOLIC if $h^2 - ab = 0$

HYPERBOLIC if $h^2 - ab > 0$.

For example, the Laplace equation in two variables

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ is elliptic since $a = b = 1,\ h = 0$, hence $h^2 < ab$,

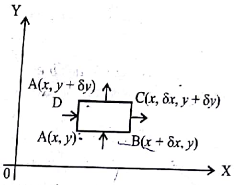
the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ is hyperbolic since $a = -b = 1,\ h = 0$ hence $h^2 > ab$

and the equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ is parabolic since $a = c^2,\ b = 0,\ h = 0$ hence $h^2 = ab$.

LAPLACE EQUATION IN TWO-DIMENSIONAL STEADY STATE HEAT FLOW

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The two dimensional heat equation $\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ reduces to Laplace equation or potential equation in two dimensions given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ when the heat flow is in steady state, i.e. $\frac{\partial u}{\partial t} = 0$.



This partial differential equation (elliptic) is one of the most important equtions of applied Mathematics. It arises in dealing with the problems of velocity potential, the stream functions of an irrotational incompressible fluid and the electrostatic potential. Laplace equation is also encountered while dealing with the conduction of heat in a plate in steady state.

Let us consider a rectangular element ABCD of a plate, with sides δx, δy and of unit breadth as shown in the adjoining figure. If the temperature at any point depends only on x, y and time t and is independent of the z-coordinate, the heat flow is called two-dimensional and is in the XY-plane.

The amount of heat entering the plate per second from the side AB, is equal to $-K\left(\frac{\partial u}{\partial y}\right)_y \delta x$ and that from the side AD, is equal to $-K\left(\frac{\partial u}{\partial x}\right)\delta y$, whereas the amount of heat flowing out per second from the surface CD is $-K\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} \delta x$ and that from the surface BC is $-K\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} \delta y$. Therefore the total gain of heat by the plate per second is the difference of the outflowing heat and the inflowing heat, equal to

$$K\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right]\delta x + K\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)\right]\delta y \quad \text{...(1)}$$

Using Taylor's expansion, we have

$$\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} = \frac{\partial u}{\partial y} + \delta y\frac{\partial^2 u}{\partial y^2} + \ldots \quad \text{and} \quad \left(\frac{\partial u}{\partial x}\right)_{x+\delta x} = \frac{\partial u}{\partial x} + \delta x\frac{\partial^2 u}{\partial x^2} + \ldots$$

Therefore the total gain of heat by the plate per second becomes $K\delta x\delta y\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ on retaining terms upto second degree only. The rate of gain of heat by the plate is also given by $s\,\rho\,\delta x\,\delta y\,\frac{\partial u}{\partial t}$ where s is the specific heat and ρ the density of the plate material. ...(2)

Equating the above mentioned two expressions for the gain of heat by the plate per second, we get

$$K\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = s\rho\frac{\partial u}{\partial t}.$$

or
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = a^2\frac{du}{dt}, \quad \text{where } a^2=\frac{s\rho}{K}. \qquad ...(3)$$

The equation (3) gives the temperature distribution of the plate in the transit state. However, with steady state, u is independent of time t hence $\frac{\partial u}{\partial t}=0$, and the equation of temperature distribution of the plate in the steady state is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$ $\Rightarrow \nabla^2 u = 0$

SOLVING THE LAPLACE EQUATION BY THE METHOD OF SEPARATION OF VARIABLES

In this method the dependent variable, say u, is assumed to be the product of two functions as $u = XY$ where X is a function of x alone and Y is the function of y alone.

Let us consider the solution of the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ ...(1)

which, on writing $u = X(x)\,Y(y)$, becomes

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0 \quad \text{or} \quad YX''+XY''=0$$

which can be written as $\frac{X''}{X}=-\frac{Y''}{Y}$. ...(2)

Since the L.H.S. is a function of x alone and the R.H.S. is a function of y alone, the relation (2) is possible only when each side is equal to some constant K, say. Thus

$$\frac{X''}{X}=-\frac{Y''}{Y}=K. \qquad ...(3)$$

$\Rightarrow \dfrac{\frac{d^2X}{dx^2}}{X} = \dfrac{-\frac{d^2Y}{dy^2}}{Y} = K$

Now, three cases arise as follows.

CASE I: K is positive, we can write $k=p^2$.

then from (3), we get $\frac{d^2X}{dx^2}-p^2X=0$ and $\frac{d^2Y}{dy^2}+p^2Y=0.$

These yield the solutions $X=c_1e^{px}+c_2e^{-px}$ and $Y=c_3\cos py+c_4\sin py$

$\therefore \quad u=(c_1e^{px}+c_2e^{-px})(c_3\cos py+c_4\sin py)$. This is not a feasible solution.

CASE II: K = 0

In this case we have $X''=0,\ Y''=0$ which yield $X=c_1x+c_2$ and $Y=c_3y+c_4$

$\therefore \quad u=(c_1x+c_2)(c_3y+c_4)$. But this also is not feasible.

CASE III: K is negative, we can write $K=-p^2$.

then from (3), we get $\frac{d^2X}{dx^2}+p^2X=0$ and $\frac{d^2Y}{dy^2}-p^2Y=0$

These yield the solutions $X=c_1\cos px+c_2\sin px$, $Y=c_3e^{py}+c_4e^{-py}$

$\therefore \quad u=(c_1\cos px+c_2\sin px)(c_3e^{py}+c_4e^{-py})$. This is a feasible solution.

EXAMPLE 29.15. Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ subject to the boundary conditions

(a) $u(0,y)=\sin y$ (b) $u\to 0$ as $x\to\infty$.

SOLUTION: From the given boundary conditions it is clear that it falls under case I mentioned above so that the solution is

$$u=(c_1e^{px}+c_2e^{-px})(c_3\sin py+c_4\cos py) \qquad ...(1)$$

Using the boundary condition (b), we get $c_1(c_3\sin py+c_4\cos py)=0 \ \therefore\ c_1=0.$

Hence we have $u=e^{-px}\ (c'_3\sin py+c'_4\cos py)$. ...(2)

Putting $x=0$ and using the boundary condition (a), we get

$c'_3=1,\ c'_4=0$ and $p=1$

Therefore the solution of the given equation is $u=e^{-x}\sin y$. **Ans.**

EXAMPLE 29.16. Find the solution of the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ which satisfies the boundary conditions:

Imp

(i) $u\to 0$ as $y\to\infty$ for all x

(ii) $u=0$ at $x=0$ for all y

(iii) $u=0$ at $x=l$ for all y

(iv) $u=lx-x^2$ if $y=0$ for all x between $(0,l)$.

SOLUTION: We apply the method of separation of variables. The given boundary conditions suggest that the appropriate form of solution is

$$u=(c_1e^{py}+c_2e^{-py})(c_3\cos px+c_4\sin px) \qquad ...(1)$$

From boundary condition (i) $u\to 0$ as $y\to\infty$ for all x, gives $c_1=0.$

$\therefore \quad u=c_2e^{-py}(c_3\cos px+c_4\sin px)$ or $u=e^{-py}(c_5\cos px+c_6\sin px)$ ...(2)

where c_5 and c_6 are arbitrary constants.

Now from boundary condition (ii) $0=e^{-py}(c_5+0) \Rightarrow c_5=0$

$\therefore$ we have $u=c_6e^{-py}\sin px.$ where c_6 is an arbitrary constant.

$\sin\theta = 0 \Rightarrow \theta = n\pi$

From boundary condition (iii), we have

$e^{-py}\sin pl=0 \Rightarrow \sin pl=0 \ \therefore\ pl=n\pi$ or $p=n\pi/l$ where n is any integer.

$\therefore \quad u=c_6e^{-n\pi y/l}\sin\frac{n\pi x}{l}$ where c_6 is an arbitrary constant.

Now we confine ourselves to n being positive integers., that is, taking $n=1,2,3,\ldots$ and varying the constants c_6 also for each n, we get the general solutions as

$$u=\sum_{n=1}^{\infty}b_n\,e^{-n\pi y/l}\sin(n\pi x/l) \qquad ...(4)$$

where $b_1, b_2, b_3, \ldots$ are constants to be chosen so as to satisfy the boundary condition (iv), that is $u(x,0)=lx-x^2$ for all x in $0<x<l$.

$$\therefore \quad lx-x^2=\sum_{n=1}^{\infty}b_n\sin(n\pi x/l) \qquad ...(5)$$

For obtaining the constants $b_1, b_2, b_3, \ldots$, the relation (5) itself suggests that we should consider the Fourier half-range sine expansion of the function $f(x)=lx-x^2$ for $0<x<l$.

The half-range Fourier sine series for $f(x)$ in $(0, l)$ is

$$lx - x^2 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[-(lx - x^2)\cos\left(\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi}\right]_0^l - \frac{2}{l}\int_0^l -(l-2x)\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}dx$$

$$= 0 + \frac{2}{n\pi}\int_0^l (l-2x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{n\pi}\left[(l-2x)\sin\left(\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi}\right]_0^l - \frac{2}{n\pi}\int_0^l (-2)\sin\left(\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi}dx$$

$$= 0 - \frac{4l}{n^2\pi^2}\left[\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right]_0^l = -\frac{4l^2}{n^3\pi^3}(\cos n\pi - 1)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{8l^2}{n^3\pi^3} & \text{if } n \text{ is odd} \end{cases}$$

Thus $b_2 = b_4 = b_6 = \ldots = 0$ and $b_1 = \dfrac{8l^2}{1^3\pi^3}$, $b_3 = \dfrac{8l^2}{3^3\pi^3}$, ...

$$\therefore \quad u = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l}\cdot e^{-\pi y/l} + \frac{1}{3^3}\sin\frac{3\pi x}{l}\cdot e^{-3\pi y/l} + \frac{1}{5^3}\sin\frac{5\pi x}{l}\cdot e^{-5\pi y/l} + \ldots\right]$$

$$= \frac{8l^2}{\pi^3}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^3}e^{\frac{-(2r-1)\pi y}{l}}\sin\frac{(2r-1)\pi x}{l}.$$

The above result can also be written as

$$u = \frac{8l^2}{\pi^3}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^3}e^{\frac{-(2r+1)\pi y}{l}}\sin\frac{(2r+1)\pi x}{l}.$$

EXAMPLE 29.17. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in the interval $0 \le x \le \pi$, subject to the boundary conditions:

(i) $u(0, y) = 0$ (ii) $u(\pi, y) = 0$

(iii) $u(x, 0) = 1$ (iv) $u(x, y) \to 0$ as $y \to \infty$ for all x.

SOLUTION: Here we apply the method of separation of variables and choose

$$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}), \quad p > 0 \qquad \ldots(1)$$

as appropriate solution suiting the given boundary conditions. From the boundary condition (iv) $u(x, y) \to 0$ as $y \to \infty$ for all x, it follows that $c_3 = 0$.

Therefore we can write $u = e^{-py}(c'_1 \cos px + c'_2 \sin px)$. ...(2)

From the boundary condition (i) $u(0, y) = 0$, we get $c'_1 = 0$

hence (2) becomes $u = c'_2 \sin px\, e^{-py}$. ...(3)

Next, using the boundary condition (ii) $u(\pi, y) = 0$, we get

$$c'_2 \sin(\pi p)\, e^{-py} = 0 \quad \text{hence} \quad \sin \pi p = 0$$

which implies that p must be an integer, say n.

Thus, the solution is $u = c'_2 \sin nx\, e^{-ny}$, $n \in$ I.

A more general solution will be $u = \sum_{n=1}^{\infty} b_n \sin nx\; e^{-ny}$ ...(4)

Lastly, the boundary condition (iii) $u(x, 0) = 1$, yields $1 = \sum_{n=1}^{\infty} b_n \sin nx$.

To evaluate b_n, $n \in$ I we write down the half-range Fourier sine series for unity, for which

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \sin nx\, dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{2}{n\pi}[1 - \cos n\pi]$$

$$\Rightarrow \quad b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Thus, the required solution is

$$u = \frac{4}{\pi}\sin x\, e^{-y} + \frac{4}{3\pi}\sin 3x\, e^{-3y} + \frac{4}{5}\sin 5x\, e^{-5y} + \ldots$$

or
$$u = \frac{4}{\pi}\left[\sin x\, e^{-y} + \frac{1}{3}\sin 3x\, e^{-3y} + \frac{1}{5}\sin 5x\, e^{-5y} + \ldots\right] \qquad \textbf{Ans.}$$

EXAMPLE 29.18. An infinitely long metal plate of width 1 with insulated surfaces has its temperature zero along both the long edges $y = 0$ and $y = 1$ at infinity. If the edge $x = 0$ is kept at fixed temperatire T_0. find the temperature T at any point (x, y) of the plate in steady state.

Solution: We know that in steady state the temperature T at any point (x, y) satisfies the

Laplace equation $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0$ ...(1)

The boundary conditions can be stated as

(i) $T(x, 0) = 0$ (ii) $T(x, l) = 0$ (iii) $T(\infty, y) = 0$ (iv) $T(0, y) = T_0$.

These boundary conditions suggest that the solution of (1) has to be chosen as

$$T = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py). \qquad \ldots(2)$$

The condition $T(x, 0) = 0$, yields $c_3 = 0$

hence (2) can be written as $T = \sin py\,(c'_1 e^{px} + c'_2 e^{-px})$ where c'_1, c'_2 are arbitrary constants. ...(3)

The condition $T(x, l) = 0$ for all x, yields

$$\sin pl = 0 \quad \text{hence} \quad pl = n\pi \quad \text{or} \quad p = n\pi/l, \quad n \in I.$$

then (3) becomes $T = \sin\frac{n\pi y}{l}\left(c'_1 e^{n\pi x/l} + c'_2 e^{-n\pi x/l}\right)$ where c'_1, c'_2 are arbitrary constants.

From the condition $T(\infty, y) = 0$, we get $c'_1 = 0$

$\therefore \quad T = c'_2 e^{-n\pi x/l} \sin(n\pi y/l)$ where n is any integer.

The most general solution is of the form

$$T(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{l} e^{-\frac{n\pi x}{l}}.$$

According to the condition (*iv*) $T(0, y) = T_0$, we have

$$T_0 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{l}.$$

To find $b_1, b_2, b_3, \ldots$ we need to obtain the Fourier half-range sine series for $T = T_0$ in $(0, l)$.

Accordingly, $\quad T = \sum_{1}^{\infty} b_n \sin\frac{n\pi y}{l}$

where $b_n = \frac{2}{l}\int_0^l T_0 \sin\frac{n\pi y}{l}\,dy = \frac{2T_0}{l}\left[-\cos\frac{n\pi y}{l}\left(\frac{l}{n\pi}\right)\right]_0^l = \frac{2T_0}{n\pi}\left(1 - (-1)^n\right).$

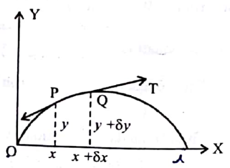
$$\therefore \quad T = \sum_{1}^{\infty} \frac{2T_0}{n\pi}[1 - (-1)^n] e^{-n\pi x/l} \sin\frac{n\pi y}{l}$$

$$= \frac{4T_0}{\pi}\left[\frac{1}{1}\sin\frac{\pi y}{l} e^{-\pi x/l} + \frac{1}{3}\sin\frac{3\pi y}{l} e^{-3\pi x/l} + \frac{1}{5}\sin\frac{5\pi y}{l} e^{-5\pi x/l} + \ldots\right] \qquad \textbf{Ans.}$$

THE WAVE EQUATION IN VIBRATION OF STRING $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Let us consider the vibration of an elastic string of length l in the vertical plane, the ends of which are fixed. For convenience we take the origin at one fixed end, the X-axis along the length of the string (when undisturbed) and Y-axis, naturally perpendicular to the X-axis. The displacement y of any point of the string is a function of two variables x, its distance from O, and the time t.

To find the relation between y, x and t we take a small element PQ ($= \delta x$) of the string at *a* distance x from O. The transverse displacement y of any point of the string is *a* function of two variables x and t and the transverse acceleration of this element is $\frac{\partial^2 y}{\partial t^2}$. Let m be the mass per unit length of the string therefore the mass of the element δx of the string is $m\delta x$. If ψ and $\psi + \delta\psi$ are the angles made by the tangents at P and Q with the X-axis (the tension T acts at each point of the string along the tangent to the curve), then the vertical component of the force, which this element is subjected to, is



$$T\sin(\psi + d\psi) - T\sin\psi.$$

This can be written as $T\tan(\psi + \delta\psi) - T\tan\psi$ simply because ψ is small (as the vibrations are small).

Since $\tan\psi = \frac{dy}{dx}$ hence, equating the forces, we have

$$T\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\} = m\delta x \frac{\partial^2 y}{\partial t^2} \quad (= \text{mass} \times \text{acceleration})$$

or
$$m\frac{\partial^2 y}{\partial t^2} = \frac{T\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\}}{\delta x}$$

Taking limit as $\delta x \to 0$, we have $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ where $c^2 = \frac{T}{m}$.

This partial differential equation is homogeneous of order two and is of hyperbolic type. It gives the vibrations of an elastic string placed along the X-axis stretched to length l between two fixed points $x = 0$ and $x = l$. Let $y(x, t)$ represent the deflection (displacement from the equilibrium position. Then the small transverse vibrations of the string is governed by the one dimension wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$

with boundary conditions $y(0, t) = 0$, $y(l, t) = 0$. ...(2)

The form of motion of the string will depend on the initial displacement (at $t = 0$), $y(x, 0) = f(x)$ ...(3)

and the initial velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$. ...(4)

To solve (1) we use the method of separation of variables.

Assume that $\quad y(x, t) = X(x)\,T(t)$ ...(5)

Putting (5) in (1) gives $XT'' = a^2 T X''$ or $\frac{X''}{X} = \frac{T''}{a^2 T}$. ...(6)

In (6) L.H.S. is a function of x only and R.H.S. is a function of t only and this is possible only when each of these is equal to some constant k. Therefore, we have two differential equations as

$$X'' - kX = 0 \quad \text{and} \quad T'' - ka^2 T = 0.$$

Three cases arise here

Case I: $k = 0$ then we have $X'' = 0$ and $T'' = 0$ $\therefore$ $X = c_1 x + c_2$ and $T = c_3 t + c_4$.

hence $y = (c_1 x + c_2)(c_3 t + c_4)$ which does not suit the boundary conditions and initial conditions.

Case II: $k > 0$, let $k = p^2$, then we have $X'' - p^2 X = 0$ and $T'' - a^2 p^2 T = 0$

$\Rightarrow X = (c_1 e^{px} + c_2 e^{-px})$ and $T = (c_3 e^{apt} + c_4 e^{-apt})$.

Hence $y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{apt} + c_4 e^{-apt})$.

which again does not suit the boundary and initial conditions.

Case III: $k < 0$ let $k = -p^2$ then we have

$X'' + p^2 X = 0$ and $T'' + a^2 p^2 T = 0$

$\Rightarrow X = (c_1 \cos px + c_2 \sin px)$ and $T = (c_3 \cos apt + c_4 \sin apt)$

which definitely suits the given conditions so $y = (c_1 \cos px + c_2 \sin px)(c_3 \cos apt + c_4 \sin apt)$.

Using the boundary condition $y(0, t) = 0$ we get $c_1 = 0$ hence

$$y(x, t) = c_2 \sin px\,(c_3 \cos apt + c_4 \sin apt) = \sin px\,(c'_3 \cos apt + c'_4 \sin apt).$$

where c'_3 and c'_4 are arbitrary constants.

Next, using the boundary condition $y(l, t) = 0$, we get

$$\sin pl = 0, \text{ then } pl = n\pi \text{ or } p = \frac{n\pi}{\lambda} \text{ for } n = 1, 2, 3, \ldots$$

Therefore, we have $y(x, t) = \sin\frac{n\pi x}{l}\left(c_3' \cos\frac{an\pi t}{l} + c_4' \sin\frac{an\pi t}{l}\right), n = 1, 2, 3, \ldots$ ...(7)

This actually, gives infinitely many solutions for $n \in N$. As such the general solution can be written as

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[A_n \cos\frac{an\pi t}{l} + B_n \sin\frac{an\pi t}{l}\right] \ldots \quad \text{...(8)}$$

The unknown constants A_n and B_n are determined using the initial conditions.

CASE I: Initial deflection is given as $y(x, 0) = f(x)$.

Putting $t = 0$ in (8), we get $\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$.

Clearly A_n's are Fourier coefficients in the half-range Fourier sine series of $f(x)$ in the interval $(0, l)$.

$$\therefore \quad A_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \,;\; n = 1, 2, \ldots \quad \text{...(8)}$$

CASE II: Initial velocity is given as $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$.

Differentiating (8) partially w.r.t. t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[-\frac{an\pi}{l} A_n \sin\frac{an\pi t}{l} + \frac{an\pi}{l} B_n \cos\frac{an\pi t}{l}\right]$$

$$\therefore \quad \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\cdot\left(\frac{an\pi}{l}\right)B_n\Big] = g(x).$$

which is the Fourier half-range sine series of $g(x)$ where $B_n = \frac{2}{an\pi}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$ ...(9)

Thus, the general solution of the one-dimensional wave equation (1) with boundary condition (2) and initial conditions (3) and (4), is

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[A_n \cos\left(\frac{an\pi t}{l}\right) + B_n \sin\left(\frac{n\pi at}{l}\right)\right]$$

where A_n and B_n are given by (8) and (9).

Corollary 1: When only deflection is given, i.e., $f(x) \neq 0$ and $g(x) = 0$ then $B_n = 0$ and $y = \sum_1^{\infty} A_n \cos\frac{an\pi t}{l} \sin\left(\frac{n\pi x}{l}\right)$ where A_n is given by (8).

Corollary 2: When only initial velocity is given, i.e., $g(x) \neq 0$ and $f(x) = 0$ then the solution is

$$y = \sum_1^{\infty} B_n \sin\left(\frac{an\pi t}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \quad \text{where } B_n \text{ is given by (9).}$$

EXAMPLE 29.19. (a) Solve the partial differential equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ representing the vibration of a string of length l, fixed at both ends, subject to the boundary conditions $y(0, t) = 0$, $y(l, t) = 0$ and initial conditions $y = y_0 \sin\frac{\pi x}{l}$ and $\frac{\partial y}{\partial t} = 0$ at $t = 0$. [GGSIPU III Sem End Term 2010]

(b) Solve the vibrating string problem with

(i) $u(0, t) = 0 = u(l, t)$

(ii) $u(x, 0) = \begin{cases} x & , \; 0 < x < l/2 \\ l - x, & l/2 < x < l \end{cases}$

(iii) $u_t(x, 0) = x(l - x), \; 0 < x < l$. [GGSIPU III Sem End Term 2013]

SOLUTION: (a) We use the method of separation of variables here.

Let us take $u = X(x) \cdot T(t)$ then the boundary conditions $y(0, t) = y(l, t) = 0$ suggest the form of solution to be $y = (c_1 \cos px + c_2 \sin px)(c_3 \cos apt + c_4 \sin apt)$ ...(1)

Applying the boundary condition $y(0, t) = 0$, gives

$$0 = (c_1 + 0)(c_3 \cos apt + c_4 \sin apt) \Rightarrow c_1 = 0.$$

$\therefore$ (1) becomes $y = \sin px\,(c_3' \cos apt + c_4' \sin apt)$ where c_3' and c_4' are arbitrary constants. ...(2)

Now using the boundary condition $y(l, t) = 0$, we get

$$0 = \sin pl\,(c_3' \cos apt + c_4' \sin apt)$$

which implies that $\sin pl = 0$ hence $p = n\pi/l$ where n is any integer.

Thus (2) becomes $y = \sin\frac{n\pi x}{l}\left(c_3' \cos\frac{an\pi t}{l} + c_4' \sin\frac{an\pi t}{l}\right)$. ...(3)

From (3) we can write $\frac{\partial y}{\partial t} = \sin\frac{n\pi x}{l}\left[-\frac{an\pi}{l} c_3' \sin\frac{an\pi t}{l} + \frac{an\pi}{l} c_4' \cos\frac{an\pi t}{l}\right]$

Since $\frac{\partial y}{\partial t} = 0$ at $t = 0$, we get $0 = c_4' \frac{an\pi}{l} \sin\frac{n\pi x}{l} \Rightarrow c_4' = 0$

$\therefore$ (3) becomes $y = c_3' \sin\frac{n\pi x}{l} \cos\frac{an\pi t}{l}$.

From the initial condition $y = y_0 \sin\frac{\pi x}{l}$ at $t = 0$, we have

$y_0 \sin\frac{\pi x}{l} = c_3' \sin\frac{n\pi x}{l}$. It implies that $c_3' = y_0$ and $n = 1$.

Therefore, the solution is $y = y_0 \sin\frac{\pi x}{l} \cos\frac{a\pi t}{l}$. **Ans.**

(b) The vibration of the string follows the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } u(x, t) \text{ is the displacement function.}$$

Here $u(0, t) = 0$ and $u(l, t) = 0$ and the initial displacement $u(x, 0) = f(x) = \begin{cases} x, & 0 < x < l/2 \\ l - x, & l/2 < x < l \end{cases}$

and the initial velocity $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) = x(l - x),\ 0 < x < l.$

The solution is $u(x, t) = \sum_{1}^{\infty}\left[A_n \cos\frac{an\pi t}{l} + B_n \sin\frac{an\pi t}{l}\right]\sin\frac{n\pi x}{l}$,

where
$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2}{l}\left[\left\{-x\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)\right\}_0^{l/2} - \int_0^{l/2} -1\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)dx\right]$$

$$+\frac{2}{l}\left[\left\{-(l-x)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)\right\}_{l/2}^{l} - \int_{l/2}^{l}(+1)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)dx\right]$$

$$= \frac{2}{l}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\left\{\sin\frac{n\pi x}{l}\right\}_0^{l/2}\right] + \frac{2}{l}\left[\frac{+l^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left\{\sin\frac{n\pi x}{l}\right\}_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] + \frac{2}{l}\left[\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

and
$$B_n = \frac{l}{an\pi}\cdot\frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}dx = \frac{2}{an\pi}\int_0^l x(l-x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{an\pi}\left[\left\{-x(l-x)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)\right\}_0^l - \int_0^l (2x-l)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)dx\right]$$

$$= \frac{2}{an\pi}\left[0 - \left\{(2x-l)\sin\frac{n\pi x}{l}\left(\frac{l^2}{n^2\pi^2}\right)\right\}_0^l + \int_0^l 2\sin\frac{n\pi x}{l}\cdot\left(\frac{l^2}{n^2\pi^2}\right)dx\right]$$

$$= \frac{2}{an\pi}\left[-\frac{l^3}{n^2\pi^2}(\sin n\pi - 0) - 2\frac{l^3}{n^3\pi^3}\left(\cos\frac{n\pi x}{l}\right)_0^l\right]$$

$$= \frac{-4l^3}{ax^4\pi^4}(\cos n\pi - 1) = \begin{cases}\dfrac{8l^3}{an^4\pi^4}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even}\end{cases}$$

∴ The desired solution is

$$u(x, t) = \frac{4l}{\pi^2}\sum_{1}^{\infty}\left[\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{an\pi t}{l}\sin\frac{n\pi x}{l}\right] + \frac{8l^3}{a\pi^4}\sum_{1}^{\infty}\left[\frac{1}{(2n-1)^4}\sin\frac{(2n-1)a\pi t}{l}\sin\frac{(2n-1)\pi x}{l}\right] \quad \textbf{Ans.}$$

EXAMPLE 29.20. **(a)** A thin uniform tightly stretched vibrating string, fixed at the points $x = 0$ and $x = l$, satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{subject to the initial condition } y(x, 0) = y_0\sin^3\frac{\pi x}{l}.$$

Find the displacement $y(x, t)$ at any x and any time t.

[*GGSIPU III Sem End Term 2009; IV Sem End Term 2015*]

(b) The vibrations of an elastic string is governed by the partial differential equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$. The length of the string is π and the ends are fixed. The initial velocity is zero and the initial deflection is $u(x, 0) = 2(\sin x + \sin 3x)$. Find the deflection $u(x, t)$ of the vibrating string for $t > 0$.

[*GGSIPU III Sem II Term 2010*]

SOLUTION: (a) Here the boundary conditions and the initial conditions can be specified as

(*i*) $y(0, t) = 0$, (*ii*) $y(l, t) = 0$

(*iii*) $\frac{\partial y}{\partial t} = 0$ at $t = 0$ and (*iv*) $y(x, 0) = y_0\sin^3(\pi x/l)$.

The solution of the given equation which satisfies the conditions (*i*), (*ii*) and (*iii*) as obtained in Example 29.19 above, is

$$y = c'_3\sin\frac{n\pi x}{l}\cos\frac{cn\pi t}{l}, \quad n = 1, 2, 3, \ldots \qquad \text{...(1)}$$

Since c'_3 depends on n, the more general solution is

$$y = \sum_{n=1}^{\infty} a_n\sin\frac{n\pi x}{l}\cos\frac{cn\pi t}{l} \qquad \text{...(2)}$$

Now applying the condition (*iv*) in (2), gives

$$y_0\sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} a_n\sin\frac{n\pi x}{l}$$

Using $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$ here, we get

$$\frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) = a_1\sin\frac{\pi x}{l} + a_2\sin\frac{2\pi x}{l} + a_3\sin\frac{3\pi x}{l} + a_4\sin\frac{4\pi x}{l} + \ldots$$

This will be satisfied if $a_1 = \frac{3y_0}{4}$, $a_2 = 0$, $a_3 = -\frac{y_0}{4}$, $a_4 = a_5 = \ldots = 0$.

Thus, the final solution of the given equation, is

$$y = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l}\cos\frac{c\pi t}{l} - \sin\frac{3\pi x}{l}\cos\frac{3\pi tc}{l}\right). \qquad \textbf{Ans.}$$

(b) The suitable solution is $u = (c_1\cos mx + c_2\sin mx)(c_3\cos mt + c_4\sin mt)$ ...(1)

The initial and boundary conditions are

$$u(0, t) = 0,\ u(\pi, t) = 0,\ u(x, 0) = 2(\sin x + \sin 3x) \text{ and } \frac{\partial u}{\partial t}(x, 0) = 0.$$

Since $u(0, t) = 0$ we get $c_1 = 0$; and since $u(\pi, t) = 0$, we have

$$c_2 \sin m\pi\,(c_3 \cos mt + c_4 \sin mt) = 0 \;\Rightarrow\; \sin m\pi = 0 \quad \therefore \quad m = n \text{ where } n \in I.$$

Thus, $u = \sin nx\,(c_3 \cos nt + c_4 \sin nt)$

$$\therefore \quad \frac{\partial u}{\partial t} = \sin nx\,(-c_3 \cdot n \sin nt + c_4 \cdot n \cos nt)$$

and since $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = 0$, we have $c_4 = 0$.

Therefore, $u = c_3 \sin nx \cos nt = b_n \sin nx \cos nt.$

Most general solution is $u = \sum_{n=1}^{\infty} b_n \sin nx \cos nt.$

Since $u(x, 0) = 2 \sin x + \sin 3x$, we have

$$2 \sin x + 2 \sin 3x = \sum b_n \sin nx \;\Rightarrow\; b_1 = 2,\ b_2 = 0,\ b_3 = 2,\ b_4 = b_5 = \ldots = 0.$$

Therefore, the solution is $y = 2 \sin x \cos t + 2 \sin 3x \cos 3t.$ **Ans.**

EXAMPLE 29.21. (a) Solve the boundary value problem $\dfrac{\partial^2 y}{\partial t^2} = 4\dfrac{\partial^2 y}{\partial x^2}$, given that $y(0, t) = 0$, $y(5, t) = 0$, $y(x, 0) = 0$ and $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 5 \sin \pi x.$ [*GGSIPU II Sem II Term 2007*]

(b) Solve the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, $0 < x < l$, $0 < t < 4$ with boundary conditions and initial condition $u(x, 0) = f(x)$ and $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = g(x)$, $0 \le x \le l$.

[*GGSIPU III Sem End Term 2011*]

SOLUTION: (a) Applying the method of separation of variables to the wave equation $\dfrac{\partial^2 y}{\partial t^2} = 2^2 \dfrac{\partial^2 y}{\partial x^2}$.
The suitable solution is $y = (c_1 \cos px + c_2 \sin px)\ (c_3 \cos 2pt + c_4 \sin 2pt)$

Applying the initial condition $y(x, 0) = 0$, we have $c_3 (c_1 \cos px + c_2 \sin px) \Rightarrow c_3 = 0$

$$\therefore \quad y = c_4 (c_1 \cos px + c_2 \sin px) \sin 2pt = (c_1' \cos px + c_2' \sin px) \sin 2pt$$

where c_1' and c_2' are arbitrary constants.

Now using $y(0, t) = 0$ we get $0 = c_1' \sin 2pt \Rightarrow c_1' = 0$

$$\therefore \quad y = c_2' \sin 2pt \sin px.$$

Further, since $y(5, t) = 0$ we have $c_2' \sin 2pt \sin 5p = 0 \Rightarrow \sin 5p = 0 = \sin n\pi$

thus, we have $p = n\pi/5$, $n = 1, 2, 3, \ldots$

Therefore $\quad y = c_2' \sin (n\pi 2t/5) \sin (n\pi x/5).$

Also the boundary condition $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 5 \sin \pi x$, gives

$$c_2' \frac{n\pi 2}{5} \cos\left(\frac{n\pi 2t}{5}\right)_{t=0} \sin\left(\frac{n\pi x}{5}\right) = 5 \sin \pi x$$

$$\Rightarrow \quad n = 5 \quad \text{and} \quad c_2' = \frac{5}{2\pi}$$

Therefore, we have $y = \dfrac{5}{2\pi} \sin \pi x \sin 2\pi t.$ **Ans.**

(b) The suitable solution is $u = (C_1 \cos px + C_2 \sin px)\ (C_3 \cos cpt + C_4 \sin cpt)$

Since $u(0, t) = 0$ we have $C_1 = 0.$

and since $u(l, t) = 0$, we have $0 = (C_2 \sin pl)\ (C_3 \cos cpt + C_4 \sin cpt)$

$$\Rightarrow \sin pl = 0 \quad \therefore \quad p = \frac{n\pi}{l},\ n \in I.$$

$$\therefore \quad u = C_2 \sin \frac{n\pi x}{l}\left[C_3 \cos \frac{cn\pi t}{l} + C_4 \sin \frac{n\pi ct}{l}\right] = \sin \frac{n\pi x}{l}\left(c_3' \cos \frac{cn\pi t}{l} + c_4' \sin \frac{cn\pi t}{l}\right).$$

Next $u(x, 0) = f(x)$ gives $c_3' \sin \dfrac{n\pi x}{l} = f(x)$, hence $c_3' = \dfrac{f(x)}{\sin \frac{n\pi x}{l}}.$

and $\quad u = f(x) \cos \dfrac{n\pi ct}{l} + c_4' \sin \dfrac{n\pi x}{l} \sin \dfrac{n\pi ct}{l}.$

$$\therefore \quad \frac{\partial u}{\partial t} = -f(x) \sin \frac{n\pi ct}{l} \cdot \left(\frac{n\pi c}{l}\right) + c_4' \sin \frac{n\pi x}{l} \cdot \cos\left(\frac{n\pi ct}{l}\right)\left(\frac{n\pi c}{l}\right).$$

Since $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = g(x)$ we have

$$g(x) = 0 + c_4' \sin \frac{n\pi x}{l} \cdot \left(\frac{n\pi c}{l}\right) \quad \therefore \quad c_4' = \frac{l}{n\pi c} \frac{g(x)}{\sin \frac{n\pi x}{l}}.$$

$$\therefore \quad u = f(x) \cos \frac{n\pi ct}{l} + \frac{l}{n\pi c}\, g(x) \sin \frac{n\pi ct}{l}. \quad \textbf{Ans.}$$

EXAMPLE 29.22. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in equilibrium state. If it is set vibrating by giving to each of its points a velocity $\mu x(l - x)$, find the displacement of the string at any point x from one end at any point of time t.

SOLUTION: The partial differential equation for the vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \ldots(1)$$

As per the boundary conditions provided, the form of solution of (1), is

$$y = (c_1 \cos px + c_2 \sin px)\ (c_3 \cos apt + c_4 \sin apt) \quad \ldots(2)$$

Here $y(0, t) = 0$, gives $0 = (c_1 + 0)\ (c_3 \cos apt + c_4 \sin apt) \Rightarrow c_1 = 0.$

Then (2) becomes $y = c_2 \sin px\ (c_3 \cos apt + c_4 \sin apt)$

or $\quad y = \sin px\ (c'_3 \cos apt + c'_4 \sin apt)$ where c'_3 and c'_4 are arbitrary constants. ...(3)

Further, since the string is initially at rest, $y(x, 0) = 0$

$\therefore$ (3) gives $0 = \sin px\ (c'_3 + 0) \Rightarrow c'_3 = 0$.

Therefore, (3) becomes $y = c'_4 \sin px \sin apt$. ...(4)

Also, from the condition $y(l, t) = 0$, we get $0 = c'_4 \sin pl \sin apt$

which gives $\sin pl = 0. \Rightarrow pl = n\pi$ or $p = \frac{n\pi}{l}$, $n = 1, 2, 3, \ldots$

$\therefore$ (4) becomes $y = c'_4 \sin \frac{n\pi x}{l} \sin \frac{an\pi t}{l}$, $n = 1, 2, 3, \ldots$

Thus, the most general solution can be written as

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sin \frac{an\pi t}{l}. \qquad \ldots(5)$$

$$\therefore \quad \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{an\pi t}{l} \left(\frac{an\pi}{l}\right).$$

Using the boundary condition $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \mu x(l - x)$, we get

$$\mu x(l - x) = \sum_{n=1}^{\infty} b_n \left(\frac{an\pi}{l}\right) \sin \frac{n\pi x}{l}. \qquad \ldots(6)$$

To determine b_n we expand $\mu x(l - x)$ as a half range Fourier sine series in $(0, l)$, to get

$$\mu x(l - x) = \sum_{n=1}^{\infty} b'_n \sin \frac{n\pi x}{l} \qquad \ldots(7)$$

where $b'_n = \frac{2}{l}\int_0^l \mu x(l - x) \sin \frac{n\pi x}{l} dx = \frac{4\mu l^2}{\pi^3 n^3}[1 - (-1)^n]$ (on integrating by parts twice)

Comparing (6) and (7), yields $\frac{an\pi}{l} b_n = b'_n = \frac{4\mu l^2}{\pi^3 n^3}[1 - (-1)^n]$

$$\therefore \quad b_n = \frac{l}{an\pi} \cdot \frac{4\mu l^2}{\pi^3 n^3}[1 - (-1)^n] = \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{8\mu l^3}{an^4\pi^4} & \text{when } n \text{ is odd} \end{cases}.$$

Thus, the required solution is $y(x, t) = \sum_{n=1,3,5,\ldots} \frac{8\mu l^3}{an^4\pi^4} \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}$

or
$$y = \frac{8\mu l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi x}{l} \sin \frac{(2n-1)a\pi t}{l}. \qquad \textbf{Ans.}$$

EXAMPLE 29.23. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ under the conditions $u = 0$ when $x = 0$ and when $x = \pi$, $\frac{\partial u}{\partial t} = 0$ when $t = 0$ and $u(x, 0) = x$, $0 < x < \pi$.

[*GGSIPU III Sem End Term 2012*]

SOLUTION: The possible solution suiting the boundary and initial conditions is

$$u = (C_1 \cos px + C_2 \sin px)\ (C_3 \cos apt + C_4 \sin apt).$$

Since $u = 0$ at $x = 0$ we have $C_1(C_3 \cos apt + C_4 \sin apt) = 0$ hence $C_1 = 0$.

Therefore $u = C_2 \sin px\ (C_3 \cos apt + C_4 \sin apt) = \sin px(C_3' \cos apt + C_4' \sin apt)$.

Further since $u = 0$ at $x = \pi$ we get $\sin \pi p = 0$ hence $p\pi = n\pi$, $n \in N$

$p = n$, $n \in I$, hence $u = \sin nx(C_3' \cos ant + C_4' \sin ant)$.

Also, $\frac{\partial u}{\partial t} = 0$ when $t = 0$, hence $C_4' = 0$

$\therefore \quad u = \sin nx\ (C_3' \cos ant)$ where $n = 0, 1, 2, \ldots$

In general we can write $u(x, t) = \sum_{n=0}^{\infty} C_n \sin nx \cos ant$.

$$\therefore \quad u(x, 0) = \sum_{0}^{\infty} C_n \sin nx$$

Given that $u(x, 0) = x$, $0 < x < \pi$, we can write half-range Fourier series for x, $0 < x < \pi$, as

$$u(x, 0) = \sum b_n \sin nx$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx = \frac{2}{\pi}\left[-\frac{x \cos nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} \frac{1(-\cos nx)}{n} dx$$

$$= -\frac{2}{n}\cos n\pi + \frac{2}{n\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = -\frac{2}{n}(-1)^n + 0$$

$$\therefore \quad u(x, 0) = \sum_{1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

But $u(x, 0) = \sum_{0}^{\infty} C_n \sin nx$ hence $C_n = \frac{2}{n}(-1)^{n+1}$.

Thus, $u(x, t) = \sum_{n=0}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx \cos ant$ **Ans.**

EXAMPLE 29.24. The points of trisection of a string are pulled aside through the same distance d on opposite position of equilibrium and the string is released from rest. Find the expression for the displacement of the string at subsequent time and show that the mid point of the string always remains at rest.

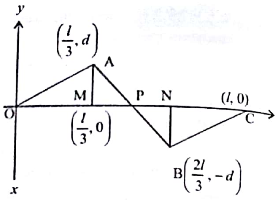
[*GGSIPU III Sem End Term 2009, II Term 2013*]

SOLUTION: The adjoining figure depicts the initial position of the string in parts OA, AB and BC.

Here OM = MN = NC = $l/3$

and AM = NB = d, OP = PC = $l/2$.

Equation of the part OA of the string, is $y = \left(\frac{3d}{l}\right)x$,



equation of the part AB of the string, is $y = \frac{3d}{l}(l - 2x)$

and the equation of the part BC of the string, is $y = \frac{3d}{l}(x - l)$.

The vibrations of the string are given by the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ...(1)

and the boundary conditions can be listed as

(i) $y(0, t) = 0$ (ii) $\frac{\partial y}{\partial t} = 0$ at $t = 0$, (iii) $y(l, t) = 0$

and (iv) $y(x, 0) = \begin{cases} \frac{3dx}{l}, & 0 < x < l/3 \\ \frac{3d}{l}(l - 2x), & l/3 < x < 2l/3. \\ \frac{3d}{l}(x - l), & 2l/3 < x < l \end{cases}$

As in Example 29.20(a) the solution of (1) satisfying the boundary conditions (i), (ii) and (iii), is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{an\pi t}{l} \quad ...(2)$$

so that $$y(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}. \quad ...(3)$$

This is also the Fourier half range sine series of $y(x, 0)$ where

$$b_n = \frac{2}{l}\int_0^l y \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^{l/3}\left(\frac{3d}{l}\right)x\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_{l/3}^{2l/3}\frac{3d}{l}(l - 2x)\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_{2l/3}^{l}\frac{3d}{l}(x - l)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{6d}{l^2}\left[-\left\{x\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)\right\}_0^{l/3} - \int_0^{l/3} -\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)dx + \left\{(l - 2x)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)\right\}_{l/3}^{2l/3}\right.$$

$$\left. - \int_{l/3}^{2l/3}(-2)\left(-\cos\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)dx + \left\{-(x - l)\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)\right\}_{2l/3}^{l} - \int_{2l/3}^{l} -\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)dx\right]$$

$$= \frac{6d}{l^2}\cdot\frac{l}{n\pi}\left[-\frac{l}{3}\cos\frac{n\pi}{3} + \left\{\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right\}_0^{l/3} - \left(\frac{-l}{3}\cos\frac{2n\pi}{3} - \frac{l}{3}\cos\frac{n\pi}{3}\right)\right.$$

$$\left. - 2\left\{\sin\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right\}_{l/3}^{2l/3} - \frac{l}{3}\cos\frac{2n\pi}{3} + \frac{l}{n\pi}\left\{\sin\frac{n\pi x}{l}\right\}_{2l/3}^{l}\right]$$

$$= \frac{6d}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right) = -\frac{12d}{n^2\pi^2}\cos\frac{n\pi}{2}\sin\frac{n\pi}{6}.$$

Therefore the required solution is

$$y(x, t) = -\frac{12d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{2}\sin\frac{n\pi}{6}\cdot\sin\frac{n\pi x}{l}\cos\frac{n\pi t}{l}$$

At the midpoint P of the string $x = l/2$, we have

$$y\left(\frac{l}{2}, t\right) = -\frac{12d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{2}\sin\frac{n\pi}{6}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{l}$$

$$= -\frac{6d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin n\pi\sin\frac{n\pi}{6}\cos\frac{n\pi t}{l} = 0 \quad \text{as} \quad \sin n\pi = 0,\ n \in I.$$

Therefore, the mid point of the string always remains at rest.

ONE DIMENSIONAL HEAT FLOW EQUATION; $\frac{\partial u}{\partial t} = A^2\frac{\partial^2 u}{\partial x^2}$

[GGSIPU III Sem II Term 2006]

Consider a thin homogeneous rod (in the form of a straight line) insulated except at the ends by a metal impervious to heat. When heat is flowing uniformly, the amount of heat flowing across any portion of the rod, is directly proportional to the difference of temperatures at the end points of that portion and to the area of cross section and also to the time of flow of heat and is inversely proportional to the length of the portion under consideration.

The quantity of heat Q_1 flowing across any section of the rod at a distance x from some fixed point of the rod, is given by

$$Q_1 = -K\left(\frac{\partial u}{\partial x}\right)_x \cdot A \text{ per second,}$$

where K is the coefficient of conductivity, u the temperature at a distance x from some fixed point on the rod, A is the area of cross section (– ve sign is there because the heat flows from higher to lower temperature). Further, the quantity of heat Q_2 flowing across a section of the rod at a distance $x + \delta x$, is given by

$$Q_2 = -K\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \cdot A \text{ per second.}$$

Thus, the quantity of heat gained by this portion of the rod per second, is

$$Q_1 - Q_2 = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right].$$

But the gain otherwise is $s\rho A\left(\frac{\partial u}{\partial t}\right)\delta x$ where s is the specific heat, ρ the density of the material of the rod and $\frac{\partial u}{\partial t}$ represents the rate of increase of temperature.

Equating the two gains, we get

$$s\,\rho\,\frac{\partial u}{\partial t}=\frac{K\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_{x}\right]}{\delta x}$$

taking limit here as $\delta x \to 0$, we get

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}\quad \text{where}\quad a^2=\frac{K}{s\rho}.$$

This is called the equation of one dimensional heat flow, also known as conduction of heat along a thin uniform rod but without radiation. Here a^2 is material constant.

To find the solution of the above equation we again employ **the method of separation of variables.** Let $u = X(x)\ T(t)$ then

$$X\frac{dT}{dt}=a^2T\frac{d^2X}{dx^2}\quad \text{or}\quad \frac{\frac{dT}{dt}}{a^2T}=\frac{\frac{d^2X}{dx^2}}{X}=\lambda,\ \text{a constant.}$$

Case I. When $\lambda = 0$.

Here we get $\frac{dT}{dt}=0$ and $\frac{d^2X}{dx^2}=0$

Hence $T=\text{constant}$ and $\frac{dX}{dx}=\text{constant}$

$\therefore \quad u = c_1x + c_2$ where c_1 and c_2 are arbitrary constants.

Case II: $\lambda<0$, let $\lambda=-m^2$, say, then

$$\frac{dT}{dt}+a^2m^2T=0 \quad\Rightarrow\quad T=c_1e^{-m^2a^2t}$$

and $$\frac{d^2X}{dx^2}+m^2X=0 \quad\Rightarrow\quad X=c_2\cos mx + c_3\sin mx.$$

$\therefore \quad u=e^{-a^2m^2t}(c'_1\cos mx + c'_2\sin mx).$ where c'_1, c'_2 are arbitrary constants.

Case III: $\lambda>0$, let $\lambda=m^2$, say

then $\frac{dT}{dt}-a^2m^2T=0$ and $\frac{d^2X}{dx^2}-m^2X=0$

$\Rightarrow \quad T=c_1e^{a^2m^2t}$ and $X=c_2e^{mx}+c_3e^{-mx}$

$\therefore \quad u=e^{a^2m^2t}(c'_1e^{mx}+c'_2e^{-mx})$ where c'_1 and c'_2 are arbitrary constants.

EXAMPLE 29.25. (a) Find the temperature in a this metal root of length l with both ends insulated and with initial temperature in the rod $\sin(\pi x/l)$.

[GGSIPU III Sem End Term 2013]

(b) The equation for heat conduction along a bar of length l is $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ neglecting the radiation. Find an expression for $u(x, t)$ if the ends of the bar are maintained at zero temperature and if initially the temperature is T at the centre of the bar and falls uniformly to zero with time.

[GGSIPU IIIrd Sem. End Term 2004]

SOLUTION: (a) Temperature function $u(x, t)$ in the rod follows $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$.

Its suitable solution is $u=(c_1\cos px + c_2\sin px)\,e^{-c^2p^2t}$.
Both the ends of the rod are insulated hence $u(0, t) = 0$ and $u(l, t) = 0$

it implies that $c_1 = 0$ and $\sin pl = 0$ then $pl = n\pi$ or $p=\frac{n\pi}{l}$, $n = 1, 2, 3, \ldots$

Thus, we have $\quad u=c_2\sin\frac{n\pi x}{l}e^{-c^2n^2\pi^2t/l^2}$, $n = 1, 2, 3, \ldots$

Since c_2 depends on the value of n, the general solution can be written as

$$u=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}e^{-c^2n^2\pi^2t/l^2}.$$

Therefore, $u(x,0)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\cdot 1$

But we are given that $u(x, 0)=\sin\frac{\pi x}{l}$, hence $\sum_{1}^{\infty}b_n\sin\frac{n\pi x}{l}=\sin\frac{\pi x}{l}$.

Thus, we have $b_1 = 1$ and $b_2 = b_3 = \ldots = 0$.

$\therefore \quad u=b_1\sin\frac{\pi x}{l}e^{-c^2\pi^2t/l^2} \quad\Rightarrow\quad u=\sin\frac{\pi x}{l}e^{-c^2\pi^2t/l^2}$. **Ans.**

(b) The boundary conditions are $u(0, t) = 0$, $u(l, t) = 0$,
and the initial condition are

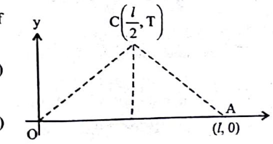
$$\left.\begin{aligned} u(x,0)&=\frac{2T}{l}x && \text{for } 0\le x\le l/2\\ &=\frac{2T}{l}(l-x) && \text{for } l/2\le x\le l.\end{aligned}\right\}\quad \ldots(1)$$

Using the method of separation of variables, the choice of solution under the given boundary conditions, is

$$u=(c_1\cos mx + c_2\sin mx)e^{-a^2m^2t} \quad \ldots(2)$$

The condition $u(0, t) = 0$ gives, $c_1 = 0$, hence

$$u=c_2\sin mx\ e^{-a^2m^2t} \quad \ldots(3)$$



The condition $u(l, t) = 0$ gives $\sin ml = 0 \Rightarrow ml = n\pi$ or $m = \dfrac{n\pi}{l}$ where $n \in N$.

hence $u = c_2 \sin \dfrac{n\pi x}{l}\, e^{-a^2 n^2 \pi^2 t/l^2}$ ...(4)

$\therefore$ the most general solution is $u(x, t) = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l}\, e^{-a^2 n^2 \pi^2 t/l^2}$. ...(5)

Applying the initial condition (1) here, we have $u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l}$.

where
$$b_n = \frac{2}{l}\int_0^l u(x,0) \sin \frac{n\pi x}{l}\, dx$$
$$= \frac{2}{l}\int_0^{l/2} \frac{2T}{l} x \sin \frac{n\pi x}{l}\, dx + \frac{2}{l}\int_{l/2}^{l} \frac{2T}{l}(l - x) \sin \frac{n\pi x}{l}\, dx$$
$$= \frac{4T}{l^2}\left[\left\{-x \cos \frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)\right\}_0^{l/2} - \int_0^{l/2}(-1)\cos \frac{n\pi x}{l}\cdot\frac{l}{n\pi}\, dx\right.$$
$$\left. + \left\{-1(l - x) \cos \frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)\right\}_{l/2}^{l} - \int_{l/2}^{l} \cos \frac{n\pi l}{l}\cdot\frac{l\, dx}{n\pi}\right]$$
$$= \frac{4T}{l^2}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\left\{\sin\frac{n\pi x}{l}\right\}_0^{l/2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{l^2}{n\pi^2}\left\{\sin\frac{n\pi x}{l}\right\}_{l/2}^{l}\right]$$
$$= \frac{4T}{l^2}\left(\frac{l^2}{n^2\pi^2}\right)\left(\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right) = \frac{8T}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Substituting this value of b_n in (4), we get

$$u(x, t) = \frac{8T}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\, e^{-a^2 n^2 \pi^2 t/l^2}$$ which is the required solution. **Ans.**

EXAMPLE 29.26. A bar of length l with insulated sides is initially at 0°C temperature throughout. The end $x = 0$ is kept at 0°C for all time and the heat is suddenly applied such that $\dfrac{\partial u}{\partial x} = 10$ at $x = l$ for all time. Find the temperature function $u(x, t)$.

SOLUTION: The temperature function $u(x, t)$ satisfies the partial differential equation.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

in the present case. The method of separation of variables is to be used here.

The possible solutions of (1) are

$$u = (c_1 \cos nx + c_2 \sin nx)\, e^{-a^2 n^2 t} \quad ...(2)$$
$$u = (c_1 e^{nx} + c_2 e^{-nx})\, e^{-a^2 n^2 t} \quad ...(3)$$
$$u = (c_1 x + c_2) \quad ...(4)$$

The given boundary conditions are $u(0, t) = 0$ ...(5)

and $\left(\dfrac{\partial u}{\partial x}\right)_{x=l} = 10$ for all t ...(6)

and $u(x, 0) = 0.$ ...(7)

The boundary condition (6) is such that none of the above mentioned solutions (2), (3) and (4) satisfies it, therefore we shall use a combination of (2) and (4) to satisfy all the given conditions of the problem. Let the solution be $u = c_1 x + c_2 + (c_3 \cos nx + c_4 \sin nx)\, e^{-a^2 n^2 t}$ ...(8)

The condition $u(0, t) = 0$ when applied to (8), gives

$$0 = c_2 + c_3\, e^{-a^2 n^2 t} \quad \text{for all } t. \quad \therefore \quad c_2 = 0 \text{ and } c_3 = 0.$$

Therefore, (8) becomes $u = c_1 x + c_4 \sin nx\, e^{-a^2 n^2 t}$ ...(9)

$$\therefore \quad \frac{\partial u}{\partial x} = c_1 + n\cdot c_4 \cos nx\, e^{-a^2 n^2 t}$$

Using the condition (6), gives $10 = c_1 + n\cdot c_4 \cos nl e^{-a^2 n^2 t}$

$\Rightarrow \quad c_1 = 10$ and $\cos nl = 0$ as $c_4 \neq 0.$

$\therefore \quad nl = m\pi + \pi/2$ or $n = (2m + 1)\,\pi/(2l)$ where $m \in I$

Therefore, (9) becomes

$$u = 10x + c_4 \sin\frac{(2m+1)\pi x}{2l}\, e^{-a^2 \frac{(2m+1)^2 \pi^2 t}{4l^2}}, \quad m = 0,1,2,3,...$$

The most general solution can be written as

$$u = 10x + \sum_{m=0}^{\infty} b_m \sin\frac{(2m+1)\pi x}{2l}\, e^{-a^2 \frac{(2m+1)^2 \pi^2 t}{4l^2}}$$

Using here the condition (7), we get

$$\sum_{m=0}^{\infty} b_m \sin\left(\frac{(2m+1)\pi x}{2l}\right)\cdot 1 = -10x \quad ...(10)$$

The left hand side in (10) should be the half range Fourier sine series for $-10x$ for which

$$b_m = \frac{2}{l}\int_0^l -10x \sin\left(\frac{(2m+1)\pi x}{2l}\right) dx$$
$$= \frac{20}{l}\left[x \cos\left(\frac{(2m+1)\pi x}{2l}\right)\cdot\frac{2l}{(2m+1)\pi}\right]_0^l - \frac{20}{l}\int_0^l 1\cdot\cos\left(\frac{(2m+1)\pi x}{2l}\right)\cdot\frac{2l}{(2m+1)\pi}\, dx$$
$$= 0 - \frac{40}{(2m+1)\pi}\left[\sin\left(\frac{(2m+1)\pi x}{2\cdot l}\right)\left\{\frac{2l}{(2m+1)\pi}\right\}\right]_0^l = -\frac{80\, l}{(2m+1)^2\pi^2}\sin\left(\frac{(2m+1)\pi}{2}\right)$$

Thus, the final solution is

$$u = 10x - \frac{80\, l}{\pi^2}\sum_{m=0}^{\infty}\frac{\sin(2m+1)\pi x/2}{(2m+1)^2}\cdot e^{\frac{-a^2(2m+1)^2\pi^2 t}{4l^2}}.$$ **Ans.**