2
Chapter

Signals and Waveform Synthesis

Inside This Chapter

2.1. INTRODUCTION

A signal may be considered to be a function of time that represents a physical variable of interest associated with a system. In electrical systems, the excitation (input) and response (output) are given in terms of currents and voltages. Mostly, these currents and voltages are function of time. In general, these functions of time are called signals, *i.e.* signals are also called as the functions.

Signals play an important role in science and technology as communication, aeronautics, bio-medical engineering and speech processing etc.

2.2. CLASSIFICATION OF SIGNALS

Signals may describe a wide variety of physical phenomena, as follows :

2.2.1. Continuous-Time and Discrete-Time Signals

To distinguish between continuous-time and discrete-time signals, we will use the symbol 't' to denote the continuous-time independent variable and 'n' or 'nT' to denote the discrete-time independent variable.

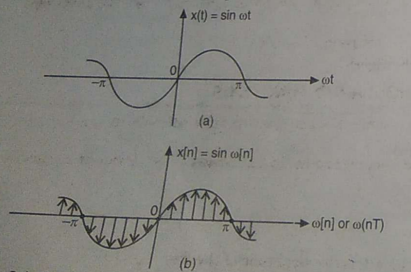


Fig. 2.1. Graphical representations of (a) Continuous-time and (b) discrete-time signals

Illustrations of a continuous-time signal $x(t)$ and a discrete-time signal $x[n]$ or $x(nT)$ are shown in figure 2.1.

2.2.2. Even and Odd Signals

Another set of useful properties of signals relates to their symmetry under time reversal.

A signal $x(t)$ or $x[n]$ is referred to as an even signal if it is identical to its time-reversed counterpart. In continuous-time a signal is *even* if

$$x(-t) = x(t)$$

$x(t) = x(-t)$

while a discrete-time signal is even if

$$x[-n] = x[n]$$

Examples : (*i*) t^n (where n is even) or t^{2n} (where $n \in$ integer) *i.e.* t^2, t^4, ...
(*ii*) $\cos t$, $\sin^2 t$, etc.

A signal is referred to as an odd if the signal is negative of its reflection, i.e.,

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

Examples : (*i*) t^n (where n is odd) or t^{2n+1} (where $n \in$ integer) *i.e.*, t, t^3, ...
(*ii*) $\sin t$, etc.

Note :
There are some functions (signals), which are neither even nor odd.
Examples : e^t, $t^2 + t$ etc.

Theorems :

(*i*) Sum of even functions = even function
(*ii*) Sum of odd functions = odd function
(*iii*) Multiplication of even and even functions = even function
(*iv*) Multiplication of odd and odd functions = even function
(*v*) Multiplication of even and odd functions = odd function
(*vi*) Sum of even and odd functions = Neither even nor odd.

2.2.3. Periodic and Unperiodic Signals

A signal $x(t)$ is *periodic* if and only if

$$x(t + T_0) = x(t), \qquad -\infty < t < \infty \qquad ...(1)$$

where the constant T_0 is the period of $x(t)$. The smallest value of T_0 such that equation (1) is satisifed is referred to as the Time-period. Any signal not satisfying equation (1) is called unperiodic or aperiodic.

Examples : (*i*) $\sin t$, $\cos t$, $\sin {}^{t}/_{2}$, $\cos 4t$ etc. are periodic signals
(*ii*) e^t, t^2, t etc. are unperiodic signals.

2.3. STANDARD SIGNALS OR SINGULARITY FUNCTIONS

In order to simulate any signal, some standard signals which are realisable in the laboratory environment are described in this section.

Singularity function can be obtained from one another by successive differentiation or integration

2.3.1. Step Signal

The step signal $f_s(t)$ is defined by

$$f_s(t) = \begin{cases} 0 & ; \ t < 0 \\ K & ; \ t > 0 \end{cases} \qquad \text{(where } K \text{ is the amplitude of the step signal)}$$

The signal $f_s(t)$ is graphed in figure 2.2(a). Note that the function is undefined and discontinu at $t = 0$.

Scanned by CamScanner

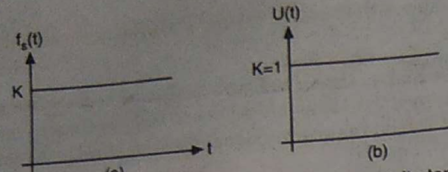


Fig. 2.2. Graphical representation of (a) Step (b) Unit-step signals

If the value of $K = 1$ (unity), then, this step signal $f_s(t)$ is called as $\boxed{\textit{unit step signal}\ U(t)}$, which is shown in figure 2.2(*b*) and defined as

$$U(t) = \begin{cases} 0 \ ; t < 0 \\ 1 \ ; t > 0 \end{cases}$$

It can be observed from above, that the step and unit step signals are zero whenever the argument within the parentheses, namely, t is negative, and they have magnitude K and 1 respectively when the argument is greater than zero. This helps us in defining the shifted or delayed signals as

$$f_s(t-a) = \begin{cases} 0 \ ; t < a \\ K \ ; t > a \end{cases}$$

And,

$$U(t-a) = \begin{cases} 0 \ ; t < a \\ 1 \ ; t > a \end{cases}$$

as shown in figure 2.3(*a*) and (*b*) respectively.

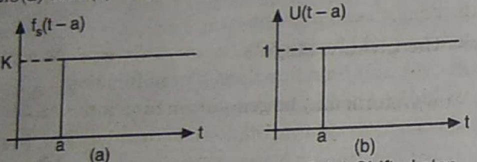


Fig. 2.3. Graphical representation of (a) Shifted step (b) Shifted unit-step signals.

EXAMPLE 2.1 **Express the given waveform as shown in figure 2.4 in terms of step signal.**

Solution :

$$f(t) = K\,U[t-(-t_1)]$$
$$f(t) = K\,U(t+t_1)$$

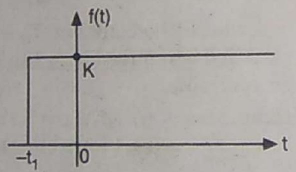


Fig. 2.4.

2.3.2. Ramp Signal

The ramp signal $f_r(t)$ is defined by

$$f_r(t) = \begin{cases} 0 \ ; t < 0 \\ Kt \ ; t \geq 0 \end{cases}$$

(where K is the slope of the ramp signal)

The signal $f_r(t)$ is graphed in figure 2.5(*a*). If the value of slope $K = 1$, then, this ramp signal $f_r(t)$ is called as *unit ramp signal* $r(t)$ which is shown in figure 2.5(*b*) and defined as

$$r(t) = \begin{cases} 0 \ ; t < 0 \\ t \ ; t \geq 0 \end{cases}$$

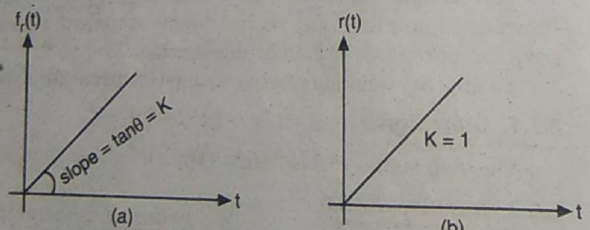


Fig. 2.5. Graphical representation of (a) Ramp (b) Unit-ramp signals

And, shifted ramp signal as shown in figure 2.6(*a*) and is described as

$$f_r(t-a) = \begin{cases} 0 & ; t < a \\ K(t-a) \ ; & t \geq a \end{cases}$$

And, shifted unit ramp signal as shown in figure 2.6(*b*) and is described as

$$r(t-a) = \begin{cases} 0 & ; t < a \\ (t-a) \ ; & t \geq a \end{cases}$$

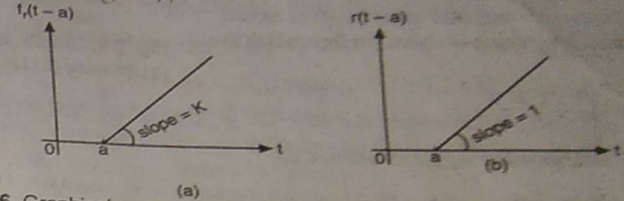


Fig. 2.6. Graphical representation of (a) Shifted ramp (b) Shifted unit-ramp signals

Note :

Unit ramp signal;

$$r(t) = t\,U(t)$$
$$\boxed{r(t-a) = (t-a)\,U(t-a)}$$

And ramp signal,

$$f_r(t) = K\,r(t) = K\,t\,U(t)$$
$$f_r(t-a) = K\,r(t-a) = K\,(t-a)\,U(t-a)$$

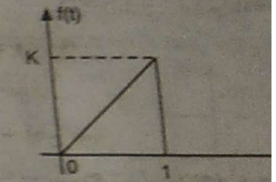


Fig. 2.7.

EXAMPLE 2.2 **Express the given triangular waveform as shown in figure 2.7, in ter[ms of] ramp and step signals.**

Solution : This triangular waveform may be generated using three signals as shown in figure (*b*) and (*c*).

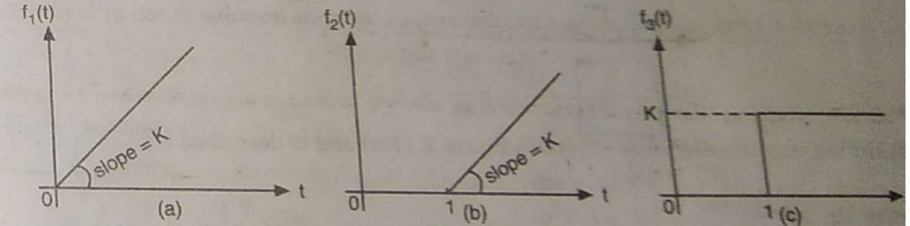


Fig. 2.8.

From the above representation,

$$\begin{aligned} f(t) &= f_1(t) - f_2(t) - f_3(t) \\ &= K\,r(t) - K\,r(t-1) - K\,U(t-1) \\ &= K\,t\,U(t) - K(t-1)\,U(t-1) - K\,U(t-1) \\ &= K[t\,U(t) - (t-1+1)\,U(t-1)] \\ &= K[t\,U(t) - t\,U(t-1)] \\ &= K\,t[U(t) - U(t-1)] \end{aligned}$$

Kr(t) – Kr(t

– U(t

Kt U(t) – Kt

– U

Kt{U(t) –

2.3.3. Impulse Signal

(It is also known as Dirac delta signal)

The impulse signal $f_\delta(t)$ is defined by

Scanned by CamScanner

$$f_\delta(t) = \begin{cases} 0 \; ; \; t \neq 0 \\ A \; ; \; t = 0 \end{cases}$$

(where A is the area of the impulse signal and some times called the strength of the impulse)

The signal $f_\delta(t)$ is graphed in figure 2.9(a). If the value of the area $A = 1$, then, this impulse signal $f_\delta(t)$ is called as *unit impulse signal* $\delta(t)$, which is shown in figure 2.9(b) and defined as

$$\delta(t) = \begin{cases} 0 \; ; \; t \neq 0 \\ 1 \; ; \; t = 0 \end{cases}$$

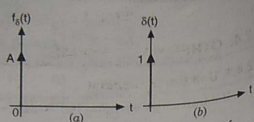


Fig. 2.9. Graphical representation of (a) Impulse (b) Unit-Impulse signals

The area of the unit impulse signal is defined as

$$\text{Area} = \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Therefore, unit impulse signal is also graphed as a sequence of small values $\in_1, \in_2, \in_3$ (where $\in_1 > \in_2 > \in_3$) as shown in figure 2.10.

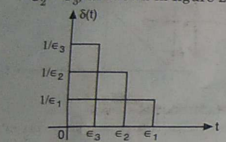


Fig. 2.10.

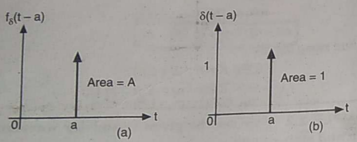


Fig. 2.11. Graphical representation of (a) shifted impulse (b) Shifted unit impulse signals

It is clear as $\in$ tends to zero, even then the area of the unit impulse function is always one, as

Area

$$= \boxed{\frac{1}{\in_3} \cdot \in_3} = \frac{1}{\in_2} \cdot \in_2 = \frac{1}{\in_1} \cdot \in_1 = 1$$

$\int \delta(t) dt = \frac{1}{\varepsilon} \times \varepsilon = 1$

And, shifted impulse signal as shown in figure 2.11(a) and is described as

$$f_\delta(t-a) = \begin{cases} 0 \; ; \; t \neq a \\ A \; ; \; t = a \end{cases}$$

And, shifted unit impulse signal as shown in figure 2.11(b) and is described as

$$\delta(t-a) = \begin{cases} 0 \; ; \; t \neq a \\ 1 \; ; \; t = a \end{cases}$$

2.3.4. Relationship Between Standard Signals

Derivative of step signal = Impulse signal

Derivative of ramp signal = Step signal

Step ramp impulse Integral

In other words, we can say :

Integral of impulse signal = Step signal

Integral of step signal = Ramp signal

Mathematically,

$$\frac{d}{dt}[f_s(t)] = f_\delta(t) \quad \text{or} \quad \int f_\delta(t)\, dt = f_s(t)$$

$$\frac{d}{dt}[f_r(t)] = f_s(t) \quad \text{or} \quad \int f_s(t)\, dt = f_r(t)$$

2.4. OTHER BASIC SIGNALS

2.4.1. Unit Doublet Signal

If a unit impulse signal $\delta(t)$ is differentiated with respect to t, we get

$$\delta'(t) = \frac{d}{dt}[\delta(t)] = +\infty \text{ and } -\infty; \quad t = 0$$
$$= 0 \quad ; \quad t \neq 0$$

This signal is called unit doublet signal $\delta'(t)$ which is shown in figure 2.12(a).
And shifted unit doublet signal $\delta'(t-a)$ as shown in figure 2.12(b) and is described as

$$\delta'(t-a) = +\infty \text{ and } -\infty; t = a$$
$$\delta'(t) = 0 \quad ; \quad t \neq a$$

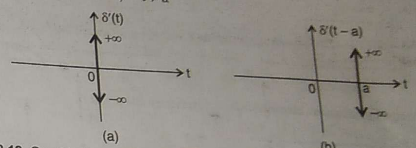


Fig. 2.12. Graphical representation of (a) Unit doublet (b) Shifted unit doublet signals

Note :

Derivative of unit impulse signal = unit doublet signal

or

Integral of unit doublet signal = unit impulse signal

Mathematically,

$$\frac{d}{dt}[\delta(t)] = \delta'(t) \quad \text{or} \quad \int \delta'(t)\, dt = \delta(t)$$

2.4.2. Exponential Signal

Exponential signal $f(t)$ as shown in figure 2.13 and is described as

$$f(t) = \begin{cases} 0 & ; t \leq 0 \\ Ke^{-at}; & t \geq 0 \end{cases} \qquad \text{(where } a \text{ and } K \text{ are real constants)}$$

The inverse of a has the dimension of time and is called as the *time constant*. $\tau = {}^1/_a$. This is the time taken to reach 63.2% of the total change from initial to final.

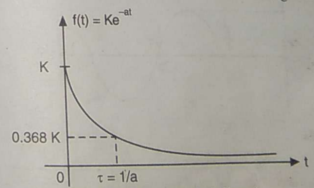


Fig. 2.13. Exponential Signal

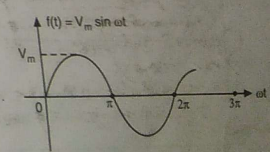


Fig. 2.14. Sinusoidal Signal

Scanned by CamScanner

2.4.3. Sinusoidal Signal

Sinusoidal signal $f(t)$ as shown in figure 2.14 and is described as

$$f(t) = \begin{cases} 0 & ; t < 0 \\ V_m \sin \omega t & ; t \geq 0 \end{cases}$$

(where V_m is peak amplitude, ω is angular frequency in rad/sec.)

2.4.4. Gate Signal (or Gate Function)

A rectangular pulse of unit height (*i.e.* amplitude is one), starting at $t = a$ and ending at $t = b$ as shown in figure 2.15, and represented as

$$G_{a,b}(t) = U(t-a) - U(t-b)$$

is called as a gate function.

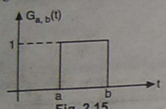


Fig. 2.15.

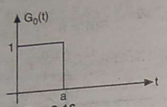


Fig. 2.16.

If this gate function starting at origin as shown in figure 2.16, then,

$$\boxed{G_{0,a}(t) = U(t) - U(t-a)}$$

Application of Gate Function :

If any function (signal) multiplied by a gate function, then that function will have zero value outside the duration of the gate, and the value of the function will be unchanged within the duration of the gate.

EXAMPLE 2.3 **Synthesize the given a single half-sine waveform as shown in figure 2.17.**

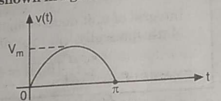


Fig. 2.17.

Solution : Given single half-sine waveform can be constructed from the sum of two functions $v_1(t)$ and $v_2(t)$, as shown in figure 2.18(a) and (b), as

$$v(t) = v_1(t) + v_2(t)$$

$$v_1(t) = V_m \sin \omega t \cdot U(t)$$

$$v_2(t) = V_m \sin \omega\left(t - \frac{T}{2}\right) \cdot U\left(t - \frac{T}{2}\right)$$

$$v(t) = V_m\left[\sin \omega t \cdot U(t) + \sin \omega\left(t - \frac{T}{2}\right) \cdot U\left(t - \frac{T}{2}\right)\right]$$

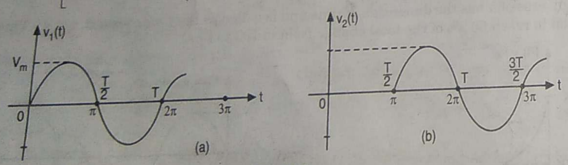


Fig. 2.18.

Note : *Gate function is quite useful for finding out Laplace transform of periodic functions.*

This same problem can be solved by using gate function as follows :

$$v(t) = V_m \sin \omega t \cdot \left[U(t) - U\left(t - \frac{T}{2}\right)\right] = V_m\left[\sin \omega t \cdot U(t) - \sin \omega t \cdot U\left(t - \frac{T}{2}\right)\right]$$

$$= V_m\left[\sin \omega t \cdot U(t) - \sin \frac{2\pi}{T}\left(t - \frac{T}{2} + \frac{T}{2}\right) \cdot U\left(t - \frac{T}{2}\right)\right]$$

$$= V_m\left[\sin \omega t \cdot U(t) - \sin\left\{\frac{2\pi}{T}\left(t - \frac{T}{2}\right) + \pi\right\} \cdot U\left(t - \frac{T}{2}\right)\right]$$

$$= V_m\left[\sin \omega t \cdot U(t) + \sin\left\{\frac{2\pi}{T}\left(t - \frac{T}{2}\right)\right\} \cdot U\left(t - \frac{T}{2}\right)\right]$$

$$= V_m\left[\sin \omega t \cdot U(t) + \sin \omega\left(t - \frac{T}{2}\right) \cdot U\left(t - \frac{T}{2}\right)\right]$$

2.5. DIRECT FORMULA (OR K.M. FORMULA)

If a function is a combination of various gate functions, then we can develop a formula to represent that function directly in terms of step functions. This formula can be called as Direct formula or K.M. formula. It is given by

$$\boxed{f(t) = \sum_{T=-\infty}^{\infty} (A_f - A_i) U(t - T)}$$

where T is the time instant at which function $f(t)$ changes its values, A_f and A_i are the final and initial values at the corresponding time instant respectively.

If the function $f(t)$ exist (define) only for $t \geq 0$, then Direct formula reduces to

$$f(t) = \sum_{T=0}^{\infty} (A_f - A_i) U(t - T)$$

To illustrate the use of the Direct formula, let us consider several examples.

EXAMPLE 2.4 **Synthesize the waveform as shown in figure 2.19 using standard signal.** *(I.P. Univ., 2001)*

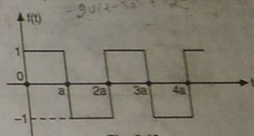


Fig. 2.19.

Solution : The function $f(t)$ can be written as the sum of the gate functions, as

$$f(t) = G_{0,a}(t) + (-1)\, G_{a,2a}(t) + G_{2a,3a}(t) + (-1) G_{3a,4a}(t) + \ldots$$

$$f(t) = 1.[U(t) - U(t-a)] + (-1)[U(t-a) - U(t-2a)] + 1[U(t-2a) - U(t-3a)] + (-1)[U(t-3a) - U(t-4a) + \ldots$$

$$= U(t) - 2U(t-a) + 2U(t-2a) - 2U(t-3a) + \ldots$$

Alternative ways : (Using Direct formula)

$$f(t) = \sum_{T=0}^{\infty} (A_f - A_i) U(t - T)$$

(where T is the time at which function changes its values).

$$= (1-0) \cdot U(t-0) + (-1-1) \cdot U(t-a) + [1-(-1)] \cdot U(t-2a) + (-1-1) \cdot U(t-3a) + \ldots$$

$$= U(t) - 2U(t-a) + 2U(t-2a) - 2U(t-3a) + \ldots$$

Scanned by CamScanner

EXAMPLE 2.5 **Synthesize the waveform as shown in figure 2.20 using step functions.**

Solution : (*i*) Using gate functions

$f(t) = G_{0,2}(t) + (-2)\cdot G_{2,3}(t) + 2\cdot G_{3,5}(t)$

$f(t) = 1\cdot[U(t) - U(t-2)] + (-2)\cdot$

$\quad [U(t-2) - U(t-3)] + 2\cdot [U(t-3) - U(t-5)]$

$\quad = U(t) - 3U(t-2) + 4U(t-3) - 2U(t-5)$

(*ii*) Using Direct formula :

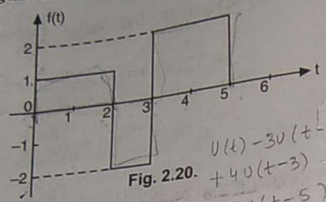


Fig. 2.20.

$U(t) - 3U(t-2) + 4U(t-3) - 2U(t-5)$

$$f(t) = \sum_{T=0}^{\infty}(A_f - A_i)U(t-T)$$

$f(t) = (1-0)\cdot U(t-0) + (-2-1)\cdot U(t-2) + [2-(-2)]\cdot U(t-3) + (0-2)\cdot U(t-5)$

$\quad = U(t) - 3U(t-2) + 4U(t-3) - 2U(t-5)$

EXAMPLE 2.6 **Synthesize the waveform as shown in figure 2.21.**

Solution : (*i*) Using Gate functions :

$f(t) = G_{1,2}(t) + 2G_{2,3}(t) + G_{3,4}(t)$

$\quad = 1\cdot[U(t-1) - U(t-2)] + 2\cdot[U(t-2)$

$\quad - U(t-3)] + 1\cdot[U(t-3) - U(t-4)]$

$\quad = U(t-1) + U(t-2) - U(t-3) - U(t-4)$

(*ii*) Using Direct formula :

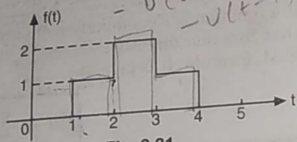


Fig. 2.21.

$U(t-1) + U(t-2) - U(t-3) - U(t-4)$

$$f(t) = \sum_{T=0}^{\infty}(A_f - A_i)U(t-T)$$

$\quad = (1-0)\cdot U(t-1) + (2-1)\cdot U(t-2) + (1-2)\cdot U(t-3) + (0-1)\cdot U(t-4)$

$\quad = U(t-1) + U(t-2) - U(t-3) - U(t-4)$

EXAMPLE 2.7 **Synthesize the following waveforms as shown in figure 2.22. Using gate functions.**

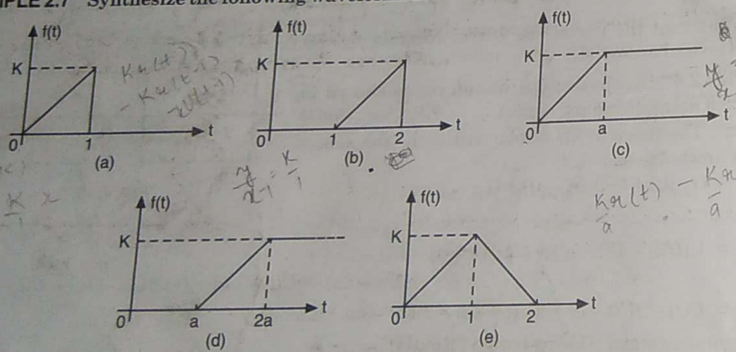


Fig. 2.22.

Solution :

(*a*) $f(t) = Kt\cdot[U(t) - U(t-1)]$

(*b*) $f(t) = K(t-1)\cdot[U(t-1) - U(t-2)]$ (since $K(t-1)$ is the expression of the line)

(*c*) This function $f(t)$ can be represented as the sum of two functions, one is from 0 to a and other is a to infinity. As,

$$f(t) = \frac{K}{a}\cdot t[U(t) - U(t-a)] + KU(t-a) = \frac{K}{a}tU(t) - \frac{K}{a}(t-a)U(t-a)$$

(*d*) Similar to part (*c*)

$f(t) = \frac{K}{a}(t-a)[U(t-a) - U(t-2a)] + KU(t-2a)$

$\quad = \frac{K}{a}(t-a)U(t-a) - \frac{K}{a}(t-2a)U(t-2a)$

(*e*) Given waveform can break into two functions. One is from 0 to 1 and other from 1 to 2. As,

$f(t) = Kt\cdot[U(t) - U(t-1)] - K(t-2)\cdot[U(t-1) - U(t-2)]$

$\quad = Kt\,U(t) - 2Kt\,U(t-1) + 2K\,U(t-1) + Kt\,U(t-2) - 2K\,U(t-2)$

$\quad = Kt\,U(t) - 2K(t-1)\,U(t-1) + K(t-2)\,U(t-2)$

EXAMPLE 2.8 **Synthesize the given waveform as shown in figure 2.23.**

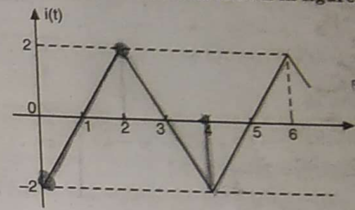


Fig. 2.23.

Solution : Using gate functions, we can represent as,

$i(t) = (2t-2)[U(t) - U(t-2)] + (-2t+6)[U(t-2) - U(t-4)] + (2t-10)[U(t-4) - U(t-6)] + ...$

$\quad = (2t-2)U(t) + (-2t+2-2t+6)\,U(t-2) + (2t-6+2t-10)U(t-4) + ...$

$\quad = (2t-2)\,U(t) + (-4t+8)\,U(t-2) + (4t-16)\,U(t-4) + ...$

$\quad = 2(t-1)\,U(t) - 4(t-2)\,U(t-2) + 4(t-4)\,U(t-4) + ...$

EXAMPLE 2.9 **Express the given waveform as shown in figure 2.24 using standard signal.**

Solution : The function $v(t)$ can be written as the sum of a ramp and a step functions, as

$v(t) = 2[r(t) - r(t-1)] - 2U(t-3)$

Putting

$r(t) = t\,U(t)$

and $r(t-1) = (t-1)\,U(t-1)$

$v(t) = 2t\,U(t) - 2(t-1)\,U(t-1) - 2U(t-3)$

Alternative way : (using gate functions) :

$v(t) = 2t\cdot G_{0,1}(t) + 2\cdot G_{1,3}(t)$

$\quad = 2t[U(t) - U(t-1)] + 2\cdot[U(t-1) - U(t-3)]$

$\quad = 2t\,U(t) - 2t\,U(t-1) + 2U(t-1) - 2U(t-3)$

$\quad = 2t\,U(t) - 2(t-1)\,U(t-1) - 2U(t-3)$

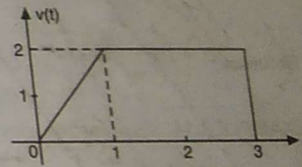


Fig. 2.24.

EXAMPLE 2.10 **Express the waveform shown in figure 2.25 in terms of delayed functions.**

Solution : (Using gate functions)

$f(t) = t\cdot G_{0,1}(t) + (t-1)\,G_{1,2}(t)$

$\quad = t[U(t) - U(t-1)] + (t-1)\,[U(t-1) - U(t-2)]$

$\quad = t\,U(t) - U(t-1) - (t-1)\,U(t-2)$

$\quad = t\,U(t) - U(t-1) - (t-2)\,U(t-2) - U(t-2)$

$\quad = r(t) - U(t-1) - r(t-2) - U(t-2)$

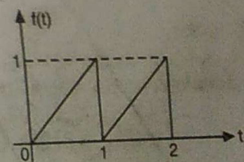


Fig. 2.25.

EXAMPLE 2.11 **Synthesize the given waveform as shown in figure 2.26 using step and ramp signals.**

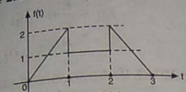


Fig. 2.26.

Solution : (Using gate functions)

$$f(t) = 2t\,G_{0,\,1}(t) + G_{1,\,2}(t) + (-2t + 6)\;G_{2,\,3}(t)$$
$$= 2t[U(t) - U(t-1)] + [U(t-1) - U(t-2)] + (-2t+6)[U(t-2) - U(t-3)]$$
$$= 2t \cdot U(t) - (2t-1)\;U(t-1) - (2t-5)\;U(t-2) + (2t-6)\;U(t-3)$$

$2r(t) - 2r(t-1) - u(t-1) + u(t-2)$

$\frac{y-2}{x-2} = \frac{0-2}{1}$

$y - 2 = -2x + 4$

$-2x + 6$

EXAMPLE 2.12 **Sketch the waveform from the expression**

$$v(t) = U(t) + \sum_{K=1}^{\infty} (-1)^K . 3U(t-K)$$

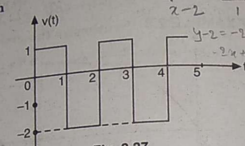


Fig. 2.27.

Solution : $v(t) = U(t) + \sum_{K=1}^{\infty} (-1)^K . 3U(t-K)$

Expanding given expression, as

$$v(t) = U(t) - 3U(t-1) + 3U(t-2) - 3U(t-3) + 3U(t-4) - \ldots$$
$$= [U(t) - U(t-1)] - 2[U(t-1) - U(t-2)] + [U(t-2) - U(t-3)] - 2[U(t-3) - U(t-4)] + \ldots$$

$U(t) - 3U(t-1) + 3U(t-2)$

Waveform for above expression is shown in figure 2.27.

EXAMPLE 2.13 **Synthesize the given waveform as shown in figure 2.28 using gate signals.**

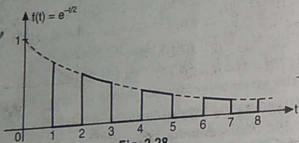


Fig. 2.28.

Solution : $f(t) = e^{-t/2}\,[U(t) - U(t-1)] + e^{-t/2}\,[U(t-2) - U(t-3)] + e^{-t/2}\,[U(t-4) - U(t-5)] + \ldots$

$$= e^{-t/2}\;[U(t) - U(t-1) + U(t-2) - U(t-3) + U(t-4) - \ldots]$$

EXAMPLE 2.14 **Express the given waveform as shown in figure 2.29 using ramp signals.**

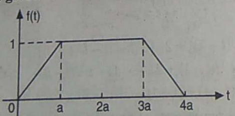


Fig. 2.29.

Solution : $f(t) = \frac{1}{a}\cdot t[U(t) - U(t-a)] + [U(t-a) - U(t-3a)] + \left[\left(-\frac{1}{a}\right)t + 4\right][U(t-3a) - U(t-4a)]$

$$f(t) = \frac{1}{a}\,t\;U(t) - \frac{1}{a}\,t\;U(t-a) + U(t-a) - U(t-3a) - \frac{1}{a}\,t\;U(t-3a) + 4U(t-3a) + \frac{1}{a}\,t\;U(t-4a) - 4U(t-4a)$$

$$= \frac{1}{a}\,t\;U(t) - \frac{1}{a}\,(t-a)\;U(t-a) - \frac{1}{a}(t-3a)\;U(t-3a) + \frac{1}{a}(t-4a)\;U(t-4a)$$
$$= \frac{1}{a}\,r(t) - \frac{1}{a}\,r(t-a) - \frac{1}{a}\,r(t-3a) + \frac{1}{a}\,r(t-4a)$$

EXAMPLE 2.15 **Sketch the waveform**

$$i(t) = 1.5(1 - e^{-4t})\;U(t) - 1.5[1 - e^{-4(t-0.1)}]\,U(t-0.1)$$

Solution : If $i_1(t) = 1.5(1 - e^{-4t})\;U(t)$ as shown in figure 2.30(a)

and $i_2(t) = 1.5[1 - e^{-4(t-0.1)}]\,U(t-0.1)$ as shown in figure 2.30(b).

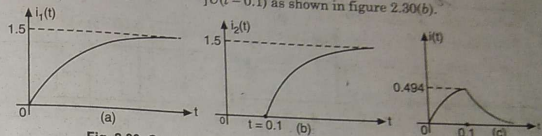


Fig. 2.30. Graphical representations of (a) $i_1(t)$, (b) $i_2(t)$ and (c) $i(t)$

Therefore, $i(t) = i_1(t) - i_2(t)$; as shown in figure 2.30(c).

EXAMPLE 2.16 **Write an equation for the parabolic pulse $v(t)$ shown in figure 2.31 using delayed step functions.**

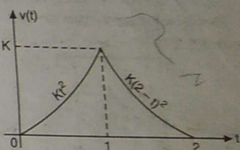


Fig. 2.31.

Solution : (Using gate functions)

$$v(t) = Kt^2 \cdot G_{0,\,1}(t) + K(2-t)^2 \cdot G_{1,\,2}(t)$$
$$= K[t^2 \,.\, U(t) + (-t^2 + 4 + t^2 - 4t)\;U(t-1) - (2-t)^2\;U(t-2)]$$
$$= K[t^2 \,.\, U(t) - 4(t-1)\;U(t-1) - (t-2)^2\;U(t-2)]$$

EXAMPLE 2.17 **The accompany figure 2.32 shows a waveform made up of straight line segments. For this waveform, write an equation for the $v(t)$ in terms of steps, ramps and other related function as needed.**

(I.P. Univ., 2000)

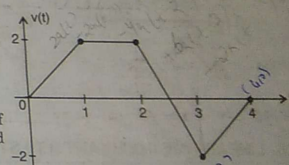


Fig. 2.32.

Solution : $v(t) = v_1(t) + v_2(t) + v_3(t)$

Where, $v_1(t)$, $v_2(t)$ and $v_3(t)$ are the three parts of the given waveform from $t = 0$ to $t = 2$, $t = 2$ to $t = 3$ and $t = 3$ to $t = 4$ respectively.

$$v_1(t) = 2r(t) - 2r(t-1) - 2U(t-2)$$
$$= 2t\;U(t) - 2(t-1)\;U(t-1) - 2U(t-2)$$
$$v_2(t) = (-4t + 10)\;[U(t-2) - U(t-3)]$$
$$v_3(t) = (2t-8)\;[U(t-3) - U(t-4)]$$

Therefore,

$$v(t) = 2t\;U(t) - 2\;(t-1)\;U(t-1) + (-2 - 4t + 10)\;U(t-2) + (4t - 10 + 2t - 8)\;U(t-3) - (2t-8)\;U(t-4)$$
$$= 2t\;U(t) - 2(t-1)\;U(t-1) - 4(t-2)\;U(t-2) + 6(t-3)\;U(t-3) - 2(t-4)\;U(t-4)$$

or $v(t) = 2r(t) - 2r(t-1) - 4\,r(t-2) + 6r(t-3) - 2r(t-4)$

Scanned by CamScanner

EXAMPLE 2.18 Sketch the following signals :
(i) $U(t-5) - U(t-8)$ (ii) $U(t-6) + U(t-4)$
(iii) $t^2 [U(t-1) - U(t-3)]$ (iv) $(t-4) [U(t-1) - U(t-4)]$

Solution :

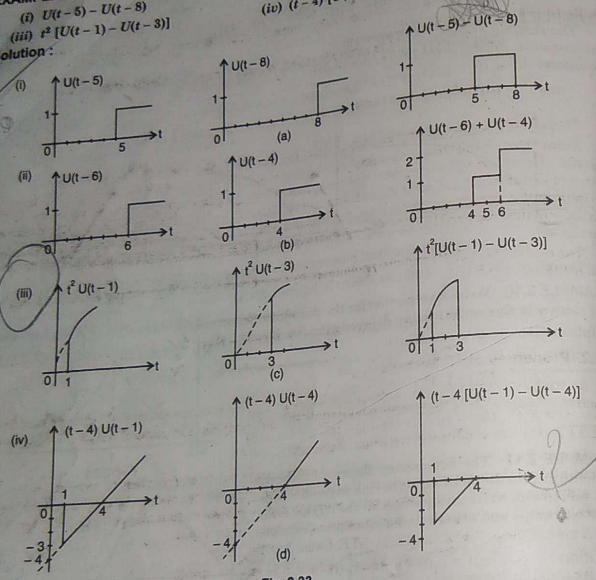


Fig. 2.33.

2.6. LINEAR TIME INVARIANT (LTI) SYSTEMS

A number of basic system properties has introduced and discussed in previous article 1.4. Two of these, linearity and time invariance, play an important role in signal and system analysis for two major reasons. First, many physical systems possess these properties and thus can be modeled as LTI systems. And second, as we know that the LTI systems possess the super-position property as a consequence, if we can represent the input or excitation or cause to an LTI system in terms of a linear combination of a set of basic signals, we can then use super-position to calculate the output or response or effect of the system in terms of its response to these basic signals.

2.6.1. Analysis of LTI Systems

If we take the product of unit impulse signal $\delta(t)$ and any other signal $x(t)$, then this product will provide the signal $x(t)$ existing only at $t = 0$, since $\delta(t)$ exists only at $t = 0$. Mathematically,

$$\boxed{\int_{-\infty}^{\infty} x(t).\delta(t)\, dt = x(t)\big|_{t=0} = x(0)}$$

Significance

Where $x(0)$ is the value of the signal $x(t)$ at $t = 0$.

The above equation is known as *shifting property* of the impulse signal because the impulse shifts the value of $x(t)$ at $t = 0$.

The shifting may also be done at any instant say at $t = t_0$, if we define the impulse signal at the instant t_0, Mathematically,

$$\int_{-\infty}^{\infty} x(t).\delta(t-t_0)\,dt = x(t_0)$$

$x(t)$ /

Therefore, the shifted unit impulse signal $\delta(t-t_0)$ shifts the value of $x(t)$ at $t = t_0$.

From above discussion, any signal can be expressed as linear combination of shifted impulse signals. Using linearity property, the response of the system to any input can be evaluated in terms of linear combination of shifted impulse signals. Thus an LTI system can be characterized by its impulse response. Such a representation, referred to as the convolution integral in the continuous-time case and the convolution sum in discrete-time. Here we will discuss only continuous-time case.

The convolution of $x(t)$ and $h(t)$ is denoted by a special notation [by putting a star between $x(t)$ and $h(t)$] as

$$\boxed{y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

$h(t)$ $y(t)$

2.6.2. Properties of LTI Systems

As we have seen that the characteristics of an LTI system are completely determined by its impulse response. It is important to note that this property hold in general only for LTI systems.

A. LTI Systems with and without Memory

As specified in article 1.4.4, a system is instantaneous or memoryless or zero memory if its output at any time depends only on the value of the input at the same time otherwise dynamic *i.e.*, a dynamic system or system with memory is one whose output depends on past or future values of the input or past output in addition to the present input.

A memoryless continuous-time LTI system has the form

$\boxed{y(t) = K\,x(t)}$ (where K is any constant)

113931 9/01/15

BHARATI VIDYAPEETH
COLLEGE OF ENGINEERING
NEW DELHI-110063

Its impulse response $\boxed{h(t) = K\,\delta(t)}$

Note that if $K = 1$, then this system becomes identity system, with output equal to the input and with unit impulse response equal to the unit impulse.

or $h(t) = 0$ for $t \neq 0$.

If a continuous-time LTI system has an impulse response $h(t)$ that is not identically zero for $t \neq 0$, then the system is dynamic or has memory.

B. Causality of LTI Systems

As specified in article 1.4.5, a system is causal or non-anticipative if the output of the system at any time depends only on values of the input at the present time and in the past otherwise non-causal *i.e.* a non-causal system is the system whose output depends (or anticipates) future values

Note : The unit impulse response of a non-linear system doesnot completely characterize the behaviour of the system.

Scanned by CamScanner

of the input. Specifically, a continuous-time LTI system to be causal, $y(t)$ must not depend $x(\tau)$ for $\tau > t$. This results

$$h(t) = 0 \quad \text{for} \quad t < 0$$

i.e. the impulse response of a causal LTI system must be zero before the impulse occurs. More generally, causality for an LTI system is equivalent to the condition of initial rest. And the system output (using convolution integral) is

$$y(t) = \int_0^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

Note :
The causality is a property of systems, it is common terminology to refer to a signal as being causal if it is zero for $t < 0$. Causality of an LTI system is equivalent to its impulse response being a causal signal.

C. Invertibility of LTI Systems

As discussed in article 1.4.6, a system is invertible only if an inverse system exists that, when cascaded with the original system, yields an output equal to the input to the first system. Here, if an LTI system is invertible, then it has an LTI inverse. If we have a system with impulse response $h(t)$. The inverse system with impulse response $g(t)$, must satisfy

$$\boxed{h(t) * g(t) = \delta(t)}$$

$[\delta(t) \to h(t) \to g(t) \to \delta(t)]$

D. Stability of LTI Systems

An LTI system is said to stable if the impulse response is absolutely integrable, *i.e.*, if

$$\boxed{\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty}$$

More general, an LTI system is said to be stable if the impulse response approaches zero as $t \to \infty$.

E. Commutative Property

The output of an LTI system of impulse response $h(t)$ to input $x(t)$ is equal to the output of system of impulse response $x(t)$ to input $h(t)$, *i.e.*,

$$y(t) = \boxed{x(t) * h(t) = h(t) * x(t)}$$

F. Distributive Property

The output of an LTI system of impulse response $[h_1(t) + h_2(t)]$ to input $x(t)$ is equal to the sum of the output of system with impulse response $h_1(t)$ to input $x(t)$ and system with impulse response $h_2(t)$ to input $x(t)$, *i.e.*,

$$y(t) = \boxed{x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t).}$$

The distributive property is useful when two or more systems are connected in parallel.

Also, as a consequence of both commutative and distributive properties, we have

$$y(t) = [x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

which simply state that the response of an LTI system to the sum of two inputs must equal the sum of the responses to these signals individually.

G. Associative Property

The output of an LTI system of impulse response $[h_1(t) * h_2(t)]$ to input $x(t)$ is equal to the output of the system with impulse response $h_2(t)$ to input $[x(t) * h_1(t)]$, *i.e.*,

$$y(t) = \boxed{x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)}$$

***Note :** All instantaneous (memoryless) systems are causal, since the output responds only to the present value of input.*

The associative property is useful when two or more systems are connected in series or in cascade.

Also, as a consequence of this property, we can say that the overall system response doesnot depend upon the order of the systems in the cascade.

2.7. SYSTEM MODELING IN TERMS OF DIFFERENTIAL EQUATIONS

Mostly systems are dynamic systems characterized by variables that are function of time. They are described by dynamic equations for the purposes of mathematical analysis. The process of obtaining the desired mathematical description of the system is known as "modelling." The basic models of dynamic physical systems are differential equations obtained by application of the suitable laws of nature.

The dynamics of many physical systems results from the transfer, loss and storage of mass or energy. There are many physical laws such as conservation of mass, conservation of energy, conservation of momentum, conservation of charge, and Newton's laws of motion. These physical laws alone donot provide enough information to write the equations that describe the system. Specific arrangement or the way the system elements are interconnected as well as the empirically based descriptions for some or all of the system elements are needed to develop the equations of the physical systems. These equations are then combined to produce a composite mathematical model of the system which usually takes the form of differential equations with time as independent variable.

A causal continuous-time LTI systems are conveniently described by input-output relationship in the form of differential equation. Its dynamic equation would be linear constant coefficients differential equation as given by:

$$\boxed{\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt^k}} \quad \text{...(i)}$$

or $$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \ldots + b_1 \frac{dx}{dt} + b_0 x \quad \text{...(2)}$$

Where $x(t)$ is input, $y(t)$ is output and t is time.

When above equation (2) describe real physical systems, the coefficients, $a_n, a_{n-1}, \ldots, a_0, b_m, b_{m-1}, \ldots, b_0$, are all real and n, the order of the differential equation (output), is greater than or equal to m, the order of the forcing function (input). This condition $n \geq m$ reflects the fact that the system cannot respond to an input before the input has been applied. ?

More generally, to solve a differential equation, we must specify one or more auxiliary (initial, final, etc.) conditions, and once these are specified, we can then, obtain an expression for the output in terms of the input. In other words, we can say, a differential equation such as equation (2) describes a constraint between the input and the output of a system, but to characterize the system completely, we must also specify auxiliary conditions. Different auxiliary conditions lead to different relationships between the input and the output or different solutions of the differential equation.

We will solve the differential equations in chapter - 3 by classical method or first principle and in chapter - 5 by best suited mathematical technique *i.e.* Laplace transform for interpretation of results.

2.7.1. Electrical System

The resistor (R), inductor (L), and capacitor (C) are three basic elements of an electrical system. For the electrical system consisting of R, L and C are in series as shown in figure 2.34(a), the voltage equation can be written, using Kirchoff's voltage law (conservation of energy) as follows:

Scanned by CamScanner

Applied voltage = sum of the voltage drops in the circuit

$$v(t) = v_R(t) + v_L(t) + v_C(t)$$

$$v(t) = R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt$$

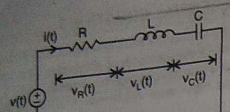


Fig. 2.34(a).

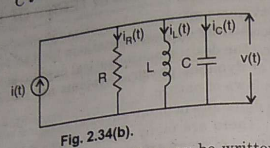


Fig. 2.34(b).

And for the electrical system of figure 2.34(*b*), the current equation can be written, using Kirchoff's current law (conservation of charge) as follows:

$$i(t) = i_R(t) + i_L(t) + i_C(t)$$

or

$$i(t) = \frac{v(t)}{R} + \frac{1}{L}\int v(t)dt + C\frac{dv(t)}{dt}$$

2.7.2. Mechanical System

The mass (*M*), spring (*K*) and damper (*D*) are the three basic elements of a mechanical translation system. For the mechanical system consisting of these three as shown in figure 2.35, the force equation can be written, using Newton's third law as follows:

Applied force = sum of the reaction forces

$$f(t) = M\frac{du(t)}{dt} + K\int u(t)dt + D\,u(t)$$

or

$$f(t) = M\frac{d^2x(t)}{dt^2} + Kx(t) + D\frac{dx(t)}{dt}$$

where $u(t)$ is the velocity and $x(t)$ is the displacement.

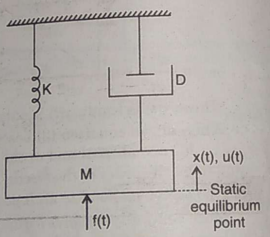


Fig. 2.35.

EXAMPLE 2.19 **Two ramp functions are given by**

$$f_1(t) = mt\,u(t)$$

$$f_2(t) = m'(t-a)\,u(t-a)$$

where m and m' are two slopes (+ve) and $m > m'$.

Draw the final wave from adding these two functions.

Solution : The addition of two functions $f_1(t)$ as shown in figure 2.36(*a*) and $f_2(t)$ as shown in figure 2.36(*b*) is as shown in figure 2.36(*c*).

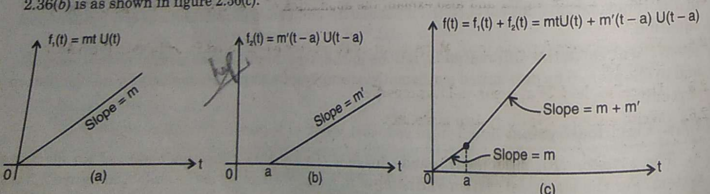


Fig. 2.36.

EXAMPLE 2.20 **A continuous-time signal $x(t)$ is shown in figure 2.37(*a*). Sketch and label each of the following signals.**

$x(t-2)$; $x(2t)$; $x(t/2)$ and $x(-t)$ (U.P.T.U. 2005)

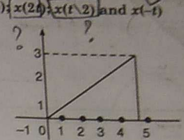


Fig. 2.37(a).

Solution :

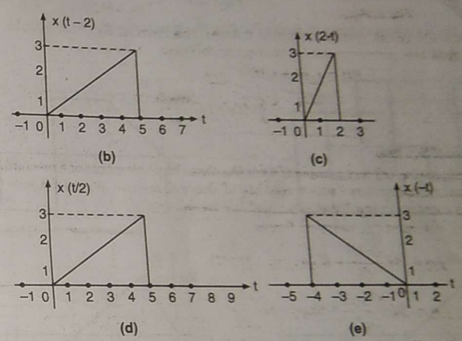


Fig. 2.37.

EXERCISES

2.1. Define a signal and also describe the different types of the signals.

2.2. Express the standard signals (or singularity functions) in mathematical and graphical forms and also write the relationship between them.

2.3. What is a gate signal and also explain the application of gate signal with the help of suitable example.

2.4. What is Direct formula (or K.M. formula). Illustrate the use of Direct formula with the help of an example.

2.5. Describe the LTI systems and its properties.

2.6. Write a short note on system modelling.

Scanned by CamScanner

PROBLEMS

2.1. Express $v(t)$, graphed in figure P.2.1, using the step signals.

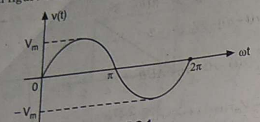


Fig. P.2.1.

2.2. Express the waveforms shown in figure *P.2.2* by the standard signals.

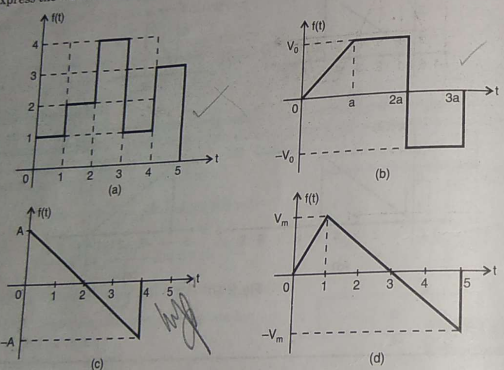


Fig. P.2.2.

2.3. Determine the current waveform through the inductor of 0.1 H, if the voltage waveform across it is shown in figure P.2.3.

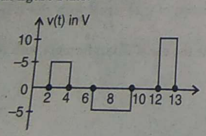


Fig. P. 2.3.

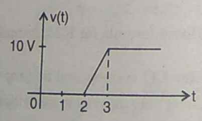


Fig. P. 2.4.

2.4. The voltage wave form across the capacitor of 0.2 μF is shown in figure P.2.4. Determine the current waveform through it.

ANSWERS

2.1. $v(t) = V_m \sin\omega t[U(t) - U(t-2\pi)]$

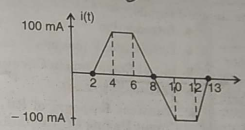
2.2. (*a*) $f(t) = U(t) + U(t-1) + 2U(t-2) - 3U(t-3) + 2U(t-4) - 3U(t-5)$

(*b*) $f(t) = \frac{V_0}{a}[r(t) - r(t-a)] - 2V_0\,U(t-2a) + V_0\,U(t-3a)$

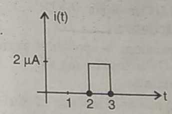
(*c*) $f(t) = AU(t) - \frac{A}{2}[r(t) - r(t-4)] + AU(t-4)$

(*d*) $f(t) = V_m\,r(t) - \frac{3}{2}V_m r(t-1) + \frac{V_m}{2} r(t-5) + V_m U(t-5)$

2.3.

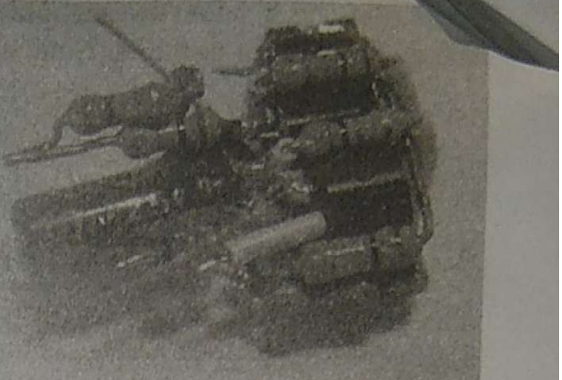


2.4.



4

Chapter



Laplace Transform

Inside This Chapter

4.1. INTRODUCTION

The Laplace transform is one of the mathematical tools used for the solution of linear ordinary integro-differential equations. (Mostly continuous-time systems are described by integro-differential equations). In comparison (as shown in figure 4.1.) with the classicial method of solving linear integro-differential equations, the Laplace transform method has the following two attractive features :

(*i*) The homogeneous equation and the particular integral of the solution are obtained in one operation.

(*ii*) The Laplace transform converts the integro-differential equation into an algebraic equation in s (Laplace operator). It is then possible to manipulate the algebraic equation by simple algebraic rules to obtain the expression in suitable forms. The final solution is obtained by taking the inverse Laplace transform.

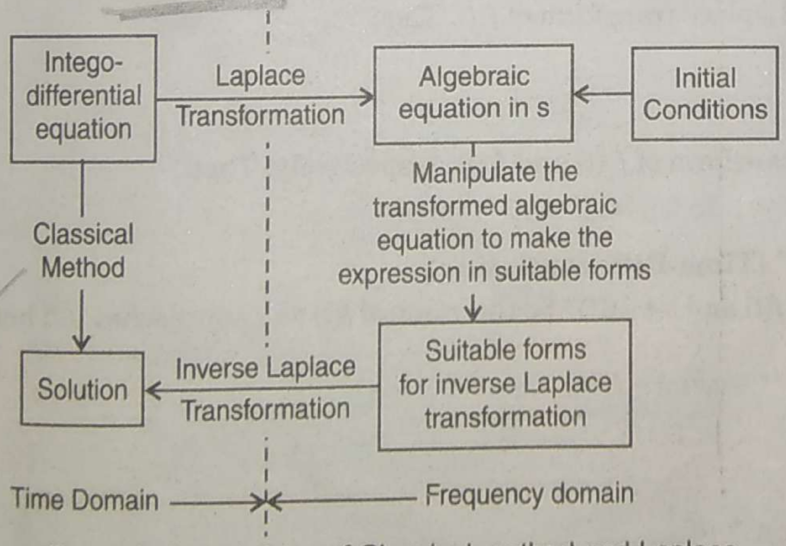


Fig. 4.1. Comparison of Classical method and Laplace transform method

Scanned by CamScanner

From the 2nd feature of Laplace transform over classical method, Laplace transformation is somewhat similar to logarithmic operation. To find the product or quotient of two number, we find

(i) logarithm of two numbers

(ii) add or subtract

(iii) take antilogarithm to get product or quotient

4.2. DEFINITION OF THE LAPLACE TRANSFORM

The Laplace transform method is a powerful technique for solving circuit problems. We define a Laplace transform as follows :

For the time function $f(t)$ which is zero for $t < 0$ and that satisfy the condition

$$\int_0^{\infty} |f(t)e^{-\sigma t}|\, dt < \infty$$

for some real and positive σ, the Laplace transform of $f(t)$ is defined as

$$\boxed{F(s) = \pounds[f(t)] = \int_0^{\infty} f(t)e^{-st}\, dt}$$

The variable s is referred to as the *Laplace operator*, which is complex variable, *i.e.*, $s = \sigma + j\omega$. And the functions $f(t)$ and $F(s)$ are known as *Laplace transform pair.*

4.3. INVERSE LAPLACE TRANSFORMATION

Given the Laplace transform $F(s)$, the operation of obtaining $f(t)$ is termed the *inverse Laplace transformation* and is denoted by

$$\boxed{f(t) = \pounds^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\, ds}$$

Though one could evaluate the inverse transform of a function F(s) by using above equation, normally the transform table is used to obtain the inverse transformation.

4.4. PROPERTIES OF LAPLACE TRANSFORM

1. Multiplication by a constant

Let k be a constant and $F(s)$ be the Laplace transform of $f(t)$. Then

$$\pounds[kf(t)] = kF(s)$$

2. Sum and Difference

Let $F_1(s)$ and $F_2(s)$ be the Laplace transform of $f_1(t)$ and $f_2(t)$, respectively. Then

$$\pounds[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

3. Differentiation with respect to "t" (Time-Differentiation)

Let $F(s)$ be the Laplace transform of $f(t)$ and let $f(0^+)$* be the value of $f(t)$ as t approaches 0. Then,

$$\pounds\left[\frac{df(t)}{dt}\right] = sF(s) - \lim_{t \to 0} f(t) = sF(s) - f(0^+)$$

Proof :

$$F(s) = \pounds[f(t)] = \int_0^{\infty} f(t) \cdot e^{-st}\, dt$$

* In some books $f(0^-)$ is written instead of $f(0^+)$.

Let $\quad f(t) = u \quad ; \text{ then } \left[\frac{df(t)}{dt}\right] \cdot dt = du$

and $\quad e^{-st}\, dt = dv \quad ; \text{ then } v = -\frac{1}{s}e^{-st}$

On integrating

$$F(s) = \int_0^{\infty} u\, dv = uv\Big|_0^{\infty} - \int_0^{\infty} v\, du$$

$$= f(t) \cdot \left(-\frac{1}{s}e^{-st}\right)\Bigg|_0^{\infty} - \int_0^{\infty}\left(-\frac{1}{s}e^{-st}\right)\left[\frac{df(t)}{dt}\right] dt \qquad (\therefore \text{ putting the values})$$

$$F(s) = \frac{1}{s} \cdot f(0^+) + \frac{1}{s}\int_0^{\infty} e^{-st}\left[\frac{df(t)}{dt}\right] dt = \frac{1}{s} \cdot f(0^+) + \frac{1}{s}\pounds\left[\frac{df(t)}{dt}\right]$$

Therefore,

$$\pounds\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$$

Thus the Laplace transform of the second derivative of $f(t)$ as

$$\pounds\left[\frac{d^2f(t)}{dt^2}\right] = \pounds\left[\frac{d}{dt}\left\{\frac{df(t)}{dt}\right\}\right] = s\,\pounds\left[\frac{df(t)}{dt}\right] - \frac{df(t)}{dt}\bigg|_{t=0}$$

$$= s[sF(s) - f(0^+)] - f^1(0^+) = s^2F(s) - sf(0^+) - f^1(0^+)$$

(where $f^1(0^+)$ is the value of the first derivative of $f(t)$ as t approaches 0)

In general,

$$\pounds\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s) - s^{n-1}f(0^+) - s^{n-2}f^1(0^+) - \ldots\ldots - sf^{n-2}(0^+) - f^{n-1}(0^+)$$

4. Integration by "t" (Time-Integration)

If $\quad \pounds[f(t)] = F(s)$

then, the Laplace transform of the first integral of $f(t)$ is given by

$$\pounds\left[\int_0^t f(t)\, dt\right] = \frac{F(s)}{s}$$

Proof :

$$\pounds\left[\int_0^t f(t)\, dt\right] = \int_0^{\infty}\left[\int_0^t f(t)\, dt\right]e^{-st}\, dt$$

Let $\quad u = \int_0^t f(t)\, dt \; ; \text{ then } du = f(t)\, dt$

and $\quad dv = e^{-st}\, dt \quad ; \text{ then } v = -\frac{1}{s}e^{-st}$

On integrating

$$\pounds\left[\int_0^t f(t)\, dt\right] = \int_0^{\infty} u\, dv = uv\Big|_0^{\infty} - \int_0^{\infty} v\, du = -\frac{1}{s}e^{-st} \cdot \int_0^t f(t)\, dt\Bigg|_0^{\infty} + \frac{1}{s}\int_0^{\infty} f(t)e^{-st}\, dt$$

$$= 0 - 0 + \frac{1}{s}\pounds[f(t)] = \frac{F(s)}{s}$$

In general,

$$\pounds\left[\int_0^{t_1}\int_0^{t_2} \ldots\ldots \int_0^{t_n} f(t)dt_1 \cdot dt_2 \ldots\ldots dt_n\right] = \frac{F(s)}{s^n}$$

Scanned by CamScanner

Note : The Laplace transform of the indefinite integral is given as

$$£\left[\int f(t)\,dt\right] = £\left[\int_0^t f(t)\,dt + f^{-1}(0^+)\right] = \frac{F(s)}{s} + \frac{f^{-1}(0^+)}{s}$$

(where $f^{-1}(0^+)$ is the value of the integral $f(t)$ as t approaches zero)

5. Differentiation with respect to "s" *(Frequency-Differentiation)*

The differentiation in the s-domain corresponds to the multiplication by "t" in the time domain, i.e.

$$£[t \cdot f(t)] = -\frac{dF(s)}{ds}$$

Proof :

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)\cdot\left(\frac{d}{ds}e^{-st}\right)dt$$

$$= \int_0^\infty f(t)\cdot e^{-st}\cdot(-t)\,dt = -\int_0^\infty t\cdot f(t)\cdot e^{-st}\,dt = -£[t\cdot f(t)]$$

6. Integration by "s" *(Frequency-Integration)*

The integration of $F(s)$ in s-domain corresponds to division by "t" in the time domain, *i.e.*

$$£\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\cdot ds$$

Proof :

$$\int_s^\infty F(s)\cdot ds = \int_s^\infty\left[\int_0^\infty f(t)\,e^{-st}\,dt\right]ds = \int_0^\infty f(t)\int_s^\infty e^{-st}ds\,dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt = \int_0^\infty f(t)\cdot\left(0 - \frac{e^{-st}}{-t}\right)dt$$

$$= \int_0^\infty f(t)\cdot\frac{e^{-st}}{t}\,dt = £\left[\frac{f(t)}{t}\right]$$

7. Shifting Theorem

[a] *Shifting in time (Time-shifting)*: The Laplace transform of a shifted or delayed function is given as

$$£[f(t-a)\cdot U(t-a)] = e^{-as}F(s)$$

Proof :

Let $t - a = y$, then, $dt = dy$

$$£[f(t-a)\cdot U(t-a)] = \int_{-a}^\infty f(y)\cdot U(y)\,e^{-s(y+a)}\,dy$$

$$= \int_0^\infty f(y)e^{-s(y+a)}\,dy \qquad \text{(since } U(y) = 0\,;\, y < 0)$$

$$= e^{-as}\int_0^\infty f(y)e^{-sy}\,dy = e^{-as}.F(s)$$

[b] *Shifting in frequency (Frequency shifting)*: The Laplace transform of e^{-at} times a function is equal to the Laplace transform of that function, with s is replaced by $(s + a)$.

Proof :

$$£[e^{-at}f(t)] = \int_0^\infty e^{-at}f(t)e^{-st}\,dt = \int_0^\infty f(t)\cdot e^{-(s+a)t}\,dt = F(s+a)$$

Note :

$$£[e^{at}f(t)] = F(s-a)$$

8. Initial Value Theorem

If the function $f(t)$ and its first derivative $\frac{df(t)}{dt}$ are both Laplace trans-formable, then

$$\boxed{f(0^+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty}[s\cdot F(s)]}$$

Proof : Using time-differentiation property,

$$s.F(s) - f(0^+) = \int_0^\infty \frac{df(t)}{dt}\,e^{-st}\,dt$$

or

$$s.F(s) = f(0^+) + \int_0^\infty \frac{df(t)}{dt}\,e^{-st}\,dt$$

$$\lim_{s\to\infty}[s.F(s)] = f(0^+) + \lim_{s\to\infty}\int_0^\infty \frac{df(t)}{dt}\,e^{-st}\,dt$$

$$\lim_{s\to\infty}[s.F(s)] = f(0^+) + \int_0^\infty \frac{df(t)}{dt}\left(\lim_{s\to\infty} e^{-st}\right)dt = f(0^+)$$

9. Final Value Theorem

The final value of a function $f(t)$ is given as

$$\boxed{f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0}[s\cdot F(s)]}$$

Proof : Using time-differentiation property, and we let $s \to 0$,

$$\lim_{s\to 0}[s.F(s) - f(0^+)] = \lim_{s\to 0}\int_0^\infty \frac{df(t)}{dt}\,e^{-st}\,dt = \int_0^\infty \frac{df(t)}{dt}\left(\lim_{s\to 0} e^{-st}\right)dt$$

$$= \int_0^\infty \frac{df(t)}{dt}\,dt = f(t)\big|_0^\infty = \lim_{t\to\infty}[f(t) - f(0)]$$

or

$$\lim_{s\to 0}[s.F(s)] = \lim_{t\to\infty} f(t) \qquad \text{(since } f(0^+) = f(0))$$

10. Theorem for Periodic Functions

The Laplace transform of a periodic function (wave) with period T is equal to $\frac{1}{1-e^{-Ts}}$ times the Laplace transform of the first cycle of that function (wave).

Proof :

Let $f_1(t)$, $f_2(t)$, $f_3(t)$,..... be the functions describing the first, second, third, ...cycles of a periodic function $f(t)$ whose time period is T. Then

$$f(t) = f_1(t) + f_2(t) + f_3(t) + ...$$

$$= f_1(t) + f_1(t-T)\,U(t-T) + f_1(t-2T)\,U(t-2T) + ...$$

Now, let $£[f_1(t)] = F_1(s)$

Therefore, by shifting theorem (property 7[a] of L.T.), we get

$$£[f(t)] = F_1(s) + e^{-Ts}.F_1(s) + e^{-2Ts}F_1(s) + ...$$

$$= F_1(s)\,[1 + e^{-Ts} + e^{-2Ts} + ...]$$

$$= \frac{1}{1-e^{-Ts}}\cdot F_1(s)$$

Note : Final value theorem doesnot apply when $f(t)$ is a periodic function.

Scanned by CamScanner

11. Convolution Theorem

Given two functions $f_1(t)$ and $f_2(t)$, which are zero for $t < 0$. If $\pounds[f_1(t)] = F_1(s)$ and $\pounds[f_2(t)] = F_2(s)$, then $\boxed{\pounds^{-1}[F_1(s) \cdot F_2(s)] = f_1(t) * f_2(t)}$ is called the convolution of $f_1(t)$ and $f_2(t)$ and is equal to

$$\boxed{\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau} \quad \text{or} \quad \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau$$

12. Time-Scaling

If Laplace transform of $f(t)$ is $F(s)$, then

$$\boxed{\pounds[f(at)] = \frac{1}{a} \cdot F\left(\frac{s}{a}\right)}$$

Table 4.1. Laplace Transform Pairs

S. No.	$f(t)^*$	$F(s) = \int_0^\infty f(t) e^{-st}\, dt$
1.	1 or $U(t)$, K	$\frac{1}{s}$, $\frac{K}{s}$
2.	t (t^n)	$\frac{1}{s^2}$ $\left(\frac{n!}{s^{n+1}}\right)$
3.	$\delta(t)$	1
4.	$e^{\pm at}$	$\frac{1}{s \mp a}$
5.	$t\, e^{\pm at}$	$\frac{1}{(s \mp a)^2}$
6.	$\sin \omega t$	$\boxed{\frac{\omega}{s^2 + \omega^2}}$
7.	$\cos \omega t$	$\boxed{\frac{s}{s^2 + \omega^2}}$
8.	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
9.	$e^{-at} \cos \omega t$	$\frac{(s+a)}{(s+a)^2 + \omega^2}$
10.	$\sinh at$	$\boxed{\frac{a}{s^2 - a^2}}$
11.	$\cosh at$	$\boxed{\frac{s}{s^2 - a^2}}$
12.	$e^{\pm at} f(t)$	$F(s \mp a)$
13.	$f(t \pm t_0)$	$e^{\pm t_0 s} F(s)$

*All $f(t)$ should be thought of as being multiplied by $U(t)$, *i.e.*, $f(t) = 0$ for $t < 0$.

EXAMPLE 4.1 Find the Laplace transform of the following standard signals (functions)

(a) The unit step function $U(t)$.
(b) The delayed step function $KU(t-a)$.
(c) The ramp function $Kr(t)$ or $Kt\,U(t)$.
(d) The delayed unit ramp function $r(t-a)$.
(e) The unit impulse function $\delta(t)$.
(f) The unit doublet function $\delta'(t)$.

Solution :

(a)
$$f(t) = U(t)$$
$$F(s) = \int_0^\infty U(t) \cdot e^{-st}\, dt$$

By the definition of $U(t)$ given in chapter 2, we have

$$= \int_0^\infty e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$$

(b)
$$f(t) = KU(t-a)$$
$$F(s) = \int_0^\infty KU(t-a) e^{-st}\, dt$$

By the definition of $U(t-a)$ given in chapter 2, we have

$$= \int_a^\infty K e^{-st}\, dt = K \left.\frac{e^{-st}}{-s}\right|_a^\infty = K \frac{e^{-as}}{s}$$

(Alternatively we can find directly using property 7[*a*])

(c)
$$f(t) = Kr(t) = Kt\, U(t)$$
$$F(s) = \int_0^\infty KtU(t) e^{-st}\, dt = \int_0^\infty Kt e^{-st}\, dt = K\left[\left. t \cdot \frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-st}}{-s}\, dt\right]$$
$$= K[0-0] + \frac{K}{s}\int_0^\infty e^{-st}\, dt = -\left.\frac{K}{s^2} e^{-st}\right|_0^\infty = \frac{K}{s^2}$$

(d)
$$f(t) = r(t-a) = (t-a)\, U(t-a)$$
$$F(s) = \int_0^\infty (t-a) U(t-a) e^{-st}\, dt$$
$$= \int_a^\infty (t-a) e^{-st}\, dt = \left.(t-a)\frac{e^{-st}}{-s}\right|_a^\infty - \int_a^\infty 1 \cdot \frac{e^{-st}}{-s}\, dt$$
$$= 0 - 0 + \frac{1}{s}\int_a^\infty e^{-st}\, dt = \left.\frac{1}{s} \cdot \frac{e^{-st}}{-s}\right|_a^\infty = \frac{e^{-as}}{s^2}$$

(Alternatively we can find directly using property 7[*a*])

(e)
$$f(t) = \delta(t)$$
$$F(s) = \int_0^\infty \delta(t) e^{-st}\, dt$$

By the definition of $\delta(t)$ given in chapter 2, we have

$$F(s) = \left. e^{-st}\right|_{t=0} = 1 \qquad \text{(Since } \delta(t) = 1 \text{ only at } t = 0\text{)}$$

(f)
$$f(t) = \boxed{\delta'(t)} \qquad F(s) = s\pounds[\delta(t)] = s$$

$sF(s) - f(0^+)$?

EXAMPLE 4.2 Find the Laplace transform of the following functions :

(a) The exponential decay function $\boxed{Ke^{-at}}$
(b) The sinusoidal function $\sin \omega t$.
(c) The cosine function $\cos \omega t$.
(d) $e^{-at} \sin \omega t$
(e) $e^{-at} \cos \omega t$
(f) $e^{-at}\, t\, U(t)$
(g) $\sin h\, at$
(h) $\cos h\, at$

Solution : (a) $\quad f(t) = Ke^{-at}$

Scanned by CamScanner

$$F(s) = \int_0^\infty K e^{-at} e^{-st}\, dt = K\int_0^\infty e^{-(s+a)t}\, dt$$

$$= K \cdot \frac{e^{-(s+a)t}}{-(s+a)}\Big|_0^\infty = K\left[0 - \left\{-\frac{e^0}{s+a}\right\}\right] = \frac{K}{s+a}$$

(b) $$f(t) = \sin \omega t \qquad \left(\because \sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)$$

$$F(s) = \int_0^\infty \sin\omega t\, e^{-st}\, dt$$

$$= \int_0^\infty \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\cdot e^{-st}\, dt = \frac{1}{2j}\cdot\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right]$$

$$= \frac{1}{2j}\cdot\left[\frac{s+j\omega - s + j\omega}{(s+j\omega)(s-j\omega)}\right] = \frac{\omega}{s^2+\omega^2}$$

(c) As similar to above case

$$f(t) = \cos\omega t, \text{ then, } F(s) = \frac{s}{s^2+\omega^2}$$

Alternatively for (b) and (c) we know, that

$$£[e^{-at}] = \frac{1}{s+a}$$

put $a = j\omega$

$$£[e^{-j\omega t}] = \frac{1}{s+j\omega} \qquad [\because e^{-j\omega t} = \cos\omega t - j\sin\omega t]$$

$$£[\cos\omega t - j\sin\omega t] = \frac{1}{s+j\omega}\cdot\frac{s-j\omega}{s-j\omega}$$

[Multiplying Numerator and Denominator by $(s - j\omega)$]

$$= \frac{s-j\omega}{s^2+\omega^2}$$

Equating real and imaginaries on both sides, we have

$$£[\cos\omega t] = \frac{s}{s^2+\omega^2} \text{ and } £[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$$

(d) $$f(t) = e^{-at}\sin\omega t$$

$$F(s) = \frac{\omega}{(s+a)^2+\omega^2} \qquad \text{Using property 7[b]}$$

(e) $$f(t) = e^{-at}\cos\omega t$$

$$F(s) = \frac{s}{(s+a)^2+\omega^2} \qquad \text{Using property 7[b]}$$

(f) $$f(t) = e^{-at}\, t\, U(t)$$

$$F(s) = \frac{1}{(s+a)^2} \qquad \text{Using property 7[b]}$$

(g) We know that

$$\sinh at = \frac{1}{2}(e^{at} - e^{-at})$$

$$£[\sinh at] = \frac{1}{2}\left[\int_0^\infty e^{at}\cdot e^{-st}\, dt - \int_0^\infty e^{-at} e^{-st}\, dt\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2}$$

(h) As similar to above case

$$\cos h\, at = \frac{1}{2}(e^{at} + e^{-at})$$

$$£[\cos h\, at] = \frac{s}{s^2-a^2}$$

Note :

This is to point out that the following six functions are all different

(i) $f(t)$ (ii) $f(t)\, U(t)$

(iii) $f(t - t_0)$ (iv) $f(t-t_0)\, U(t)$

(v) $f(t)\, U(t - t_0)$ (vi) $f(t-t_0)\, U(t-t_0)$

If $f(t) = \sin\omega t$, then

(i) $f(t) = \sin\omega t$ (ii) $f(t)\cdot U(t) = \sin\omega t\, U(t)$

(iii) $f(t-t_0) = \sin\omega(t-t_0)$ (iv) $f(t-t_0)\, U(t) = \sin\omega(t-t_0)\cdot U(t)$

(v) $f(t)\, U(t-t_0) = \sin\omega t\cdot U(t-t_0)$ (vi) $f(t-t_0)\, U(t-t_0) = \sin\omega(t-t_0)\cdot U(t-t_0)$

The six functions given above are shown graphically in figure 4.2. from which it is obvious that they are different.

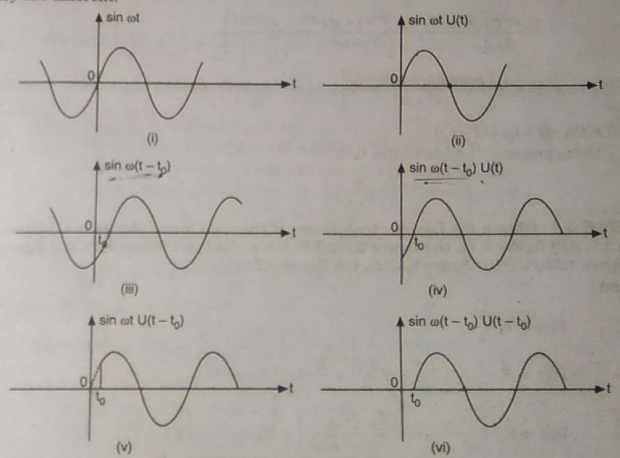


Fig. 4.2. Six different sinusoidal functions

EXAMPLE 4.3 **Find the Laplace transforms of the above six functions.**

Solution : (i) $$£[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$$

(ii) $$£[\sin\omega t\cdot U(t)] = £[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$$

Since the Laplace transform is for $0 < t < \infty$, which is the same for functions (i) and (ii).

Scanned by CamScanner

(iii) $$\pounds[\sin\omega(t-t_0)] = \pounds[\sin\omega t\cos\omega t_0 - \cos\omega t\sin\omega t_0]$$
$$= \frac{\omega}{s^2+\omega^2}\cos\omega t_0 - \frac{s}{s^2+\omega^2}\cdot\sin\omega t_0 = \frac{\omega\cos\omega t_0 - s\sin\omega t_0}{s^2+\omega^2}$$

(iv) $$\pounds[\sin\omega(t-t_0)\,U(t)] = \pounds[\sin\omega(t-t_0)] = \frac{\omega\cos\omega t_0 - s\sin\omega t_0}{s^2+\omega^2}$$

Since the L.T. is for $0 < t < \infty$, which is the same for functions (iii) and (iv).

$$\pounds[\sin\omega t\cdot U(t-t_0)] = \int_0^\infty \sin\omega t\cdot U(t-t_0)\cdot e^{-st}\,dt = \int_{t_0}^\infty \sin\omega t\cdot e^{-st}\,dt$$

$$= \frac{1}{2j}\int_{t_0}^\infty\left[e^{(-s+j\omega)t} - e^{-(s+j\omega)t}\right]dt = \frac{1}{2j}\left[0 - \frac{e^{(-s+j\omega)t_0}}{-s+j\omega} - 0 + \frac{e^{-(s+j\omega)t_0}}{-(s+j\omega)}\right]$$

$$= \frac{1}{2j}\left[\frac{e^{(-s+j\omega)t_0}}{s-j\omega} - \frac{e^{-(s+j\omega)t_0}}{(s+j\omega)}\right] = \frac{1}{2j}e^{-t_0 s}\left[\frac{e^{j\omega t_0}}{s-j\omega} - \frac{e^{-j\omega t_0}}{s+j\omega}\right]$$

$$= \frac{e^{-t_0 s}}{2j}\left[\frac{e^{j\omega t_0}(s+j\omega) - e^{-j\omega t_0}(s-j\omega)}{(s-j\omega)(s+j\omega)}\right]$$

$$= \frac{e^{-t_0 s}}{2j}\left[\frac{j\omega(e^{j\omega t_0}+e^{-j\omega t_0}) + s(e^{j\omega t_0}-e^{-j\omega t_0})}{s^2+w^2}\right]$$

$$= e^{-t_0 s}\left[\frac{\omega\cos\omega t_0 + s\sin\omega t_0}{s^2+\omega^2}\right]$$

(vi) $\pounds[\sin\omega(t-t_0)\cdot U(t-t_0)]$

Using property 7[a], shifting in time

$$= e^{-t_0 s}\cdot\left(\frac{\omega}{s^2+\omega^2}\right)$$

EXAMPLE 4.4 **Obtain the Laplace transforms of the waveforms shown in (*i*) figure 2.4, (*ii*) figure 2.7, (*iii*) figure 2.17, (*iv*) figure 2.19, (I.P. Univ., 2001) (*v*) figure 2.20, (*vi*) figure 2.22(*b*), (*vii*) figure 2.22(*c*), (*viii*) figure 2.22(*d*), (*ix*) figure 2.22(*e*).**

Solution :

(i) $$F(s) = K\frac{e^{t_1 s}}{s}$$

(ii) $$F(s) = K\cdot\frac{1}{s^2} - K\frac{e^{-s}}{s^2} - K\frac{e^{-s}}{s} = \frac{K}{s^2}(1-e^{-s}-se^{-s})$$

(iii) $$V(s) = V_m\cdot\left[\frac{\omega}{s^2+\omega^2} + e^{-\frac{T}{2}s}\cdot\frac{\omega}{s^2+\omega^2}\right] = V_m\cdot\frac{\omega}{s^2+\omega^2}\left(1+e^{-\frac{T}{2}s}\right)$$

(iv) $$F(s) = \frac{1}{s}\left[1 - 2e^{-as} + 2e^{-2as} - 2e^{-3as} + \ldots\right]$$

$$= \frac{1}{s}\left[1 - 2e^{-as}(1 - e^{-as} + e^{-2as} - \ldots)\right] = \frac{1}{s}\left[1 - \frac{2e^{-as}}{1+e^{-as}}\right] = \frac{(1-e^{-as})}{s(1+e^{-as})}$$

$$= \frac{e^{as/2} - e^{-as/2}}{s(e^{as/2}+e^{-as/2})}$$

Alternatively ways : (Using theorem for periodic functions)

$$F(s) = \frac{1}{1-e^{-2as}}\cdot F_1(s) = \frac{1}{1-e^{-2as}}\cdot\left[\frac{1}{s} - \frac{2e^{-as}}{s} + \frac{e^{-2as}}{s}\right]$$

(Since $f_1(t) = U(t) - 2U(t-a) + U(t-2a)$)

$$= \frac{1}{1-e^{-2as}}\cdot\frac{1}{s}\cdot(1-e^{-as})^2 = \frac{1-e^{-as}}{s(1+e^{-as})} = \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

(v) $$F(s) = \frac{1}{s} - \frac{3e^{-2s}}{s} + 4\frac{e^{-3s}}{s} - 2\frac{e^{-5s}}{s}$$

$$= \frac{1}{s}\left(1 - 3e^{-2s} + 4e^{-3s} - 2e^{-5s}\right)$$

(vi) $$F(s) = K\left[\frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}\right] - \frac{K}{s}e^{-2s} = \frac{Ke^{-s}}{s^2}\left(1-e^{-s}\right) - \frac{K}{s}e^{-2s}$$

(vii) $$F(s) = \frac{K}{a}\cdot\frac{1}{s^2}(1-e^{-as})$$

(viii) $$F(s) = \frac{K}{a}\cdot\frac{e^{-as}}{s^2}(1-e^{-as})$$

(ix) $$F(s) = K\cdot\left[\frac{1}{s^2} - 2\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}\right] = \frac{K}{s^2}\left[1 - 2e^{-s} + e^{-2s}\right] = \frac{K(1-e^{-s})^2}{s^2}$$

EXAMPLE 4.5 **Obtain the Laplace Transforms of the periodic waveforms as shown in (*i*) figure 2.23 (*ii*) figure 2.27.**

Solution : (Using theorem for periodic functions)

(i)
$$i'(t) = (2t-2)[U(t) - U(t-2)] + (-2t+6)[U(t-2) - U(t-4)]$$
$$= 2t\,U(t) - 2U(t) - (2t-2+2t-6)\,U(t-2) + (2t-6)\,U(t-4)$$
$$= 2tU(t) - 2U(t) - 4(t-2)\,U(t-2) + 2(t-4)\,U(t-4) + 2U(t-4)$$

$$I'(s) = \pounds[i'(t)] = \frac{2}{s^2} - \frac{2}{s} - 4\frac{e^{-2s}}{s^2} + 2\frac{e^{-4s}}{s^2} + \frac{2e^{-4s}}{s}$$

$$= \frac{2}{s^2}\left[1 - 2e^{-2s} + e^{-4s}\right] - \frac{2}{s}(1-e^{-4s}) = \frac{2}{s^2}(1-e^{-2s})^2 - \frac{2}{s}(1-e^{-4s})$$

Since Time period, $T = 4$, Therefore,

$$I(s) = \frac{1}{1-e^{-4s}}\cdot I'(s) = \frac{2}{s^2}\left(\frac{1-e^{-2s}}{1+e^{-2s}}\right) - \frac{2}{s} = \frac{2}{s}\left[\left(\frac{1}{s}\right)\tanh s - 1\right]$$

(ii)
$$v'(t) = 1.G_{0,1}(t) + (-2)\,G_{1,2}(t)$$
$$= [U(t) - U(t-1)] - 2[U(t-1) - U(t-2)]$$
$$= U(t) - 3U(t-1) + 2U(t-2)$$

$$V'(s) = \frac{1}{s}(1 - 3e^{-s} + 2e^{-2s}) = \frac{1}{s}(1-e^{-s})(1-2e^{-s})$$

Since Time period, $T = 2$, therefore

$$V(s) = \frac{1}{1-e^{-2s}}\cdot E'(s) = \frac{1}{1-e^{-2s}}\cdot\frac{1}{s}\cdot(1-e^{-s})(1-2e^{-s}) = \frac{1}{s}\cdot\left(\frac{1-2e^{-s}}{1+e^{-s}}\right)$$

Alternative ways :

$$E(t) = U(t) - 3U(t-1) + 3U(t-2) - 3U(t-3) + \ldots$$

Scanned by CamScanner

$$E(s) = \pounds[E(t)] = \frac{1}{s}\left[1-3e^{-s}+3e^{-2s}-3e^{-3s}+\ldots\right]$$

$$= \frac{1}{s}\left[1-\frac{3e^{-s}}{1+e^{-s}}\right] = \frac{1}{s}\cdot\frac{1-2e^{-s}}{1+e^{-s}}$$

EXAMPLE 4.6 **Obtain the L.T. of the waveform as shown in figure 2.28.**

Solution : $f(t) = e^{-t/2}[U(t) - U(t-1) + U(t-2) - U(t-3) + U(t-4) \ldots\ldots]$

$$F(s) = \pounds[f(t)] = \frac{1}{s+\frac{1}{2}} - \frac{e^{-\left(s+\frac{1}{2}\right)}}{s+\frac{1}{2}} + \frac{e^{-2\left(s+\frac{1}{2}\right)}}{s+\frac{1}{2}} - \frac{e^{-3\left(s+\frac{1}{2}\right)}}{s+\frac{1}{2}} + \ldots\ldots$$

$$= \frac{1}{s+\frac{1}{2}}\cdot\left[1-e^{-\left(s+\frac{1}{2}\right)}+e^{-2\left(s+\frac{1}{2}\right)}-e^{-3\left(s+\frac{1}{2}\right)}+\ldots\ldots\right]$$

$$= \frac{1}{s+\frac{1}{2}}\cdot\left[\frac{1}{1+e^{-\left(s+\frac{1}{2}\right)}}\right] = \left(\frac{1}{s+0.5}\right)\left(\frac{1}{1+e^{-(s+0.5)}}\right)$$

EXAMPLE 4.7 **Obtain the Laplace transform of the waveform shown in figure 2.30(c).**

Solution : $i(t) = 1.5(1-e^{-4t})\,U(t) - 1.5\,[1-e^{-4(t-0.1)}]\,U(t-0.1)$

$$I(s) = 1.5\left[\frac{1}{s}-\frac{1}{s+4}\right] - 1.5\left[\frac{e^{-0.1s}}{s}-\frac{e^{-0.1s}}{s+4}\right]$$

$$= 1.5\,(1-e^{-0.1s})\left(\frac{1}{s}-\frac{1}{s+4}\right) = \frac{6(1-e^{-0.1s})}{s(s+4)}$$

EXAMPLE 4.8 **Obtain the Laplace transform of the waveform shown in figure 2.31.**

Solution : $v(t) = K[t^2\,U(t) - 4\,(t-1)\,U(t-1) - (t-2)^2\,U(t-2)]$

$$V(s) = K\left[\frac{2}{s^3}-\frac{4e^{-s}}{s^2}-\frac{2e^{-2s}}{s^3}\right] = K\cdot\frac{2}{s^3}\,[1-2\,se^{-s}-e^{-2s}]$$

EXAMPLE 4.9 **Obtain the Laplace transform of the periodic, rectified half-sine wave as shown in figure 4.3.**

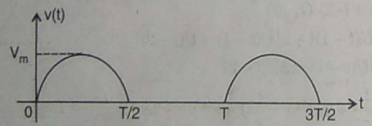


Fig. 4.3.

Solution : In example 4.4(*iii*), the Laplace transform of the single half-sine wave is

$$V_1(s) = \frac{V_m\omega}{s^2+\omega^2}\left(1+e^{\frac{T}{2}s}\right)$$

Using theorem for periodic functions.

Then $\quad V(s) = \pounds[v(t)] = \left(\frac{1}{1-e^{-Ts}}\right)V_1(s)$

$$= \left(\frac{1}{1-e^{-Ts}}\right)\cdot V_m\cdot\frac{\omega}{s^2+\omega^2}\left(1+e^{-\frac{T}{2}s}\right) = \frac{1}{\left(1-e^{-\frac{T}{2}s}\right)}\cdot V_m\cdot\frac{\omega}{(s^2+\omega^2)}$$

EXAMPLE 4.10 **Determine the Laplace transform of the periodic, rectified full-wave sine wave shown in figure 4.4.**

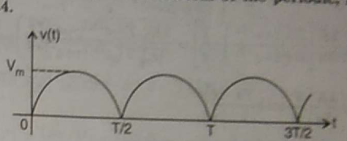


Fig. 4.4.

Solution : From example 4.9, the Laplace transform of the first cycle of the wave is given by

$$V_1(s) = \frac{V_m\omega}{s^2+\omega^2}\left(1+e^{-\frac{Ts}{2}}\right)$$

Therefore, the Laplace transform of the periodic wave form $v(t)$ of period $\frac{T}{2}$ is given by

$$V(s) = \left(\frac{1}{1-e^{-\frac{T}{2}s}}\right)\cdot\frac{V_m\omega}{s^2+\omega^2}\cdot\left(1+e^{-\frac{T}{2}s}\right)$$

$$= \frac{V_m\cdot\omega}{s^2+\omega^2}\cdot\left[\frac{1+e^{-\frac{T}{2}s}}{1-e^{-\frac{T}{2}s}}\right] = \frac{V_m\omega}{s^2+\omega^2}\cdot\left(\frac{e^{Ts/4}+e^{-Ts/4}}{e^{Ts/4}-e^{-Ts/4}}\right) = \frac{V_m\omega}{s^2+\omega^2}\cdot \mathrm{Coth}\,\frac{Ts}{4}$$

EXAMPLE 4.11 **Determine the Laplace transform of the waveform as shown in figure 4.5.**

Solution : Using the gate function, we can write the first cycle of the function as,

$$f_1(t) = \left\{-\frac{2K}{T}\cdot t+K\right\}G_{0,\,T}(t)$$

$$= -\frac{2K}{T}\left(t-\frac{T}{2}\right)[U(t)-U(t-T)]$$

$$= -\frac{2K}{T}\left(t-\frac{T}{2}\right)U(t)+\frac{2K}{T}\left(t-\frac{T}{2}\right)U(t-T)$$

$$= -\frac{2K}{T}\left(t-\frac{T}{2}\right)U(t)+\frac{2K}{T}\left(t-T+\frac{T}{2}\right)U(t-T)$$

$$= -\frac{2K}{T}\left[tU(t)-\frac{T}{2}U(t)-(t-T)U(t-T)-\frac{T}{2}U(t-T)\right]$$

$$= -\frac{2K}{T}\,[tU(t)-(t-T)\,U(t-T)-\frac{T}{2}\,\{U(t)+U(t-T)\}]$$

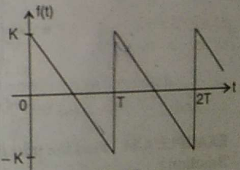


Fig. 4.5.

Scanned by CamScanner

$$\pounds[f_1(t)] = -\frac{2K}{T}\left[\frac{1}{s^2}-\frac{e^{-Ts}}{s^2}-\frac{T}{2}\left\{\frac{1}{s}+\frac{e^{-Ts}}{s}\right\}\right]$$

or

$$F_1(s) = \frac{2K}{Ts}\left\{\frac{T}{2}(1+e^{-Ts})-\frac{1}{s}(1-e^{-Ts})\right\}$$

Therefore, the Laplace transform of the periodic waveform $f(t)$ of period T is given by

$$F(s) = \frac{1}{1-e^{-Ts}}\cdot F_1(s) = \frac{1}{1-e^{-Ts}}\cdot\frac{2K}{Ts}\left\{\frac{T}{2}(1+e^{-Ts})-\frac{1}{s}(1-e^{-Ts})\right\}$$

$$= \frac{2K}{Ts}\left[\frac{T}{2}\left(\frac{1+e^{-Ts}}{1-e^{-Ts}}\right)-\frac{1}{s}\right] = \frac{2K}{Ts}\left[\frac{T}{2}\left(\frac{e^{Ts/2}+e^{-Ts/2}}{e^{Ts/2}-e^{-Ts/2}}\right)-\frac{1}{s}\right]$$

$$= \frac{2K}{Ts}\left[\frac{T}{2}\coth\frac{Ts}{2}-\frac{1}{s}\right]$$

EXAMPLE 4.12 **Determine the initial value $f(0^+)$, if**

$$F(s) = \frac{2(s+1)}{s^2+2s+5}.$$

Solution : $f(0^+) = \lim_{s\to\infty}[s.F(s)]$

$$= \lim_{s\to\infty}\left[\frac{2s(s+1)}{s^2+2s+s}\right] = \lim_{s\to\infty}\left[\frac{2+\frac{2}{s}}{1+\frac{2}{s}+\frac{5}{s^2}}\right] = 2$$

EXAMPLE 4.13 **For the current $i(t) = 5\,U(t) - 3e^{-2t}$, find $I(s)$ and hence to determine the value of $i(0^+)$ and $i(\infty)$.**

Solution :

$$i(t) = 5U(t) - 3e^{-2t}$$

$$I(s) = \frac{5}{s}-\frac{3}{s+2} = \frac{2s+10}{s(s+2)}$$

Therefore, $i(0^+) = \lim_{s\to\infty}[s.I(s)] = \lim_{s\to\infty}\left[\frac{2s+10}{s+2}\right]$

$$= \lim_{s\to\infty}\left[\frac{2+\frac{10}{s}}{1+\frac{2}{s}}\right] = 2$$

And $i(\infty) = \lim_{s\to 0}[s.I(s)] = \lim_{s\to 0}\left[\frac{2s+10}{s+2}\right] = 5$

EXAMPLE 4.14 **Find the initial value of the function, $f(t) = 9-2\,e^{-5t}$.**

Solution :

$$f(0^+) = \lim_{t\to 0}f(t) = \lim_{t\to 0}(9-2\,e^{-5t}) = 9-2\,e^0 = 7$$

EXAMPLE 4.15 **Given the function $F(s) = \frac{5s+3}{s(s+1)}$. Find the initial value $f(0^+)$, final value $f(\infty)$, and the corresponding time function $f(t)$.**

(I.P. Univ., 2001)

Solution : Initial value, $f(0^+) = \lim_{s\to\infty}[s.F(s)] = \lim_{s\to\infty}\left[\frac{5s+3}{s+1}\right]$

$$= \lim_{s\to\infty}\left[\frac{5+\frac{3}{s}}{1+\frac{1}{s}}\right] = 5$$

Final value, $f(\infty) = \lim_{s\to 0}[s.F(s)] = \lim_{s\to 0}\left[\frac{5s+3}{s+1}\right] = 3$

And, $F(s) = \frac{5s+3}{s(s+1)} = \frac{3}{s}+\frac{2}{s+1}$ (Using partial fraction expansion)

Therefore, $f(t) = \pounds^{-1}[F(s)] = (3+2\,e^{-t})\,U(t)$

Alternative ways :

$$f(0^+) = \lim_{t\to 0}f(t) = \lim_{t\to 0}(3+2\,e^{-t}) = 3+2 = 5$$

$$f(\infty) = \lim_{t\to\infty}f(t) = \lim_{t\to\infty}(3+2\,e^{-t}) = 3+0 = 3$$

EXAMPLE 4.16 **Without finding the inverse Laplace transform of $F(s)$, determine $f(0^+)$ and $f(\infty)$ for each of the following functions :**

(i) $\frac{4e^{-2s}(s+50)}{s}$ *(ii)* $\frac{s^2+6}{s^2+7}$ *(U.P.T.U., 2001)*

Solution : Since we know that

$$f(0^+) = \lim_{s\to\infty}s.F(s) \quad\text{and}\quad f(\infty) = \lim_{s\to 0}s.F(s)$$

(i) $f(0^+) = \lim_{s\to\infty}s\cdot\frac{4e^{-2s}(s+50)}{s} = \lim_{s\to\infty}4e^{-2s}(s+50) = \mathbf{0}$

$$f(\infty) = \lim_{s\to 0}4e^{-2s}(s+50) = 4.(50) = \mathbf{200}$$

(ii) $f(0^+) = \lim_{s\to\infty}s\cdot\left(\frac{s^2+6}{s^2+7}\right) = \lim_{s\to\infty}\frac{1+\frac{6}{s^2}}{\frac{1}{s}+\frac{7}{s^3}} = \infty$

$$f(\infty) = \lim_{s\to 0}\frac{s(s^2+6)}{(s^2+7)} = \mathbf{0}$$

EXAMPLE 4.17 **Find the Laplace transform of the time function shown in figure 4.6.**

(U.P.T.U., 2001)

Solution : $f(t) = \frac{5}{2}(t-2)\,G_{2,4}(t)$

$$= \frac{5}{2}(t-2)\,[U(t-2)-U(t-4)]$$

$$= \frac{5}{2}(t-2)\,U(t-2)-\frac{5}{2}(t-2)\,U(t-4)$$

$$= \frac{5}{2}r(t-2)-\frac{5}{2}(t-4+2)\,U(t-4)$$

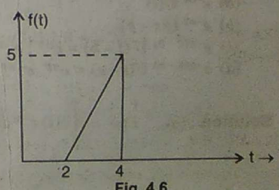


Fig. 4.6.

Scanned by CamScanner

$$f(t) = \frac{5}{2}r(t-2) - \frac{5}{2}r(t-4) - 5U(t-4)$$

So, $F(s) = £\,[f(t)]$

$$= \frac{5}{2}\frac{e^{-2s}}{s^2} - \frac{5}{2}\frac{e^{-4s}}{s^2} - \frac{5e^{-4s}}{s}$$

$$= \frac{5e^{-2s}}{2s^2}\left[1 - e^{-2s} - 2s\,e^{-2s}\right]$$

EXAMPLE 4.18 **Without finding inverse Laplace Transform of $F(s)$, determine $f(0^+)$) and $f(\infty)$ for the following function :**

$$F(s) = \frac{5s^3 - 1600}{s(s^3 + 18s^2 + 90s + 800)}$$ (U.P.T.U., 2002)

Solution : $f(0^+) = \lim_{s\to\infty} s.F(s)$

$$= \lim_{s\to\infty} \cdot \frac{5s^3 - 1600}{s^3 + 18s^2 + 90s + 800} = \lim_{s\to\infty} \cdot \frac{5 - \frac{1600}{s^3}}{1 + \frac{18}{s} + \frac{90}{s^2} + \frac{800}{s^3}} = 5$$

$$f(\infty) = \lim_{s\to 0} s \cdot F(s) = -\frac{1600}{800} = -2$$

EXAMPLE 4.19 **Determine Laplace Transform of the following wave shown in figure 4.7.** (U.P.T.U., 2002)

Solution : From the wave form shown in figure 4.7,

$$f(t) = 4G_{0,3}(t) + (-2t + 10)\,G_{3,5}(t)$$
$$= 4\,[U(t) - U(t-3)] - 2\,(t-5)\,[U(t-3) - U(t-5)]$$
$$= 4\,U(t) - 4U(t-3) - 2(t - 5 + 2 - 2)\,U(t-3) + 2(t-5)\,U(t-5)$$
$$= 4\,U(t) - 2(t-3)\,U(t-3) + 2(t-5)\,U(t-5)$$

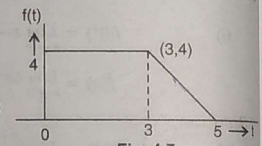


Fig. 4.7.

Therefore, Laplace transform of the waveform is

$$F(s) = £\,[f(t)]$$
$$= \frac{4}{s} - \frac{2e^{-3s}}{s^2} + \frac{2\,e^{-5s}}{s^2}$$

EXAMPLE 4.20 **Find the Laplace transforms of the following functions :**

(a) $e^{-at}\,U(t)$
(b) $e^{-at}\,U(t-b)$
(c) $e^{-a(t-b)}\,U(t-b) = e^{ab}\,.\,e^{-at}\,U(t-b)$
(d) $e^{-a(t-b)}\,U(t-c) = e^{ab}\,.e^{-at}\,U(t-c)$.

Solution : (a) $F(s) = \int_0^\infty f(t)\,e^{-st}dt = \int_0^\infty e^{-at}.U(t)\,e^{-st}dt = \frac{1}{s+a}$

(Alternatively we can find directly using properly 7 [b])

(b) $$F(s) = \int_0^\infty e^{-at}\,U(t-b)\,e^{-st}\,dt = \int_b^\infty e^{-(s+a)t}\,dt$$

$$= \left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_b^\infty = \frac{e^{-(s+a)b}}{(s+a)}$$

(Alternatively we can find directly using property 7[a] and property 7[b])

(c) $$F(s) = \int_0^\infty e^{-a(t-b)}\,U(t-b)\,e^{-st}dt$$

$$= e^{ab}\int_b^\infty e^{-(s+a)t}dt = e^{ab}.\left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_b^\infty = e^{ab}.\frac{e^{-(s+a)b}}{s+a} = \frac{e^{-bs}}{s+a}$$

(Alternatively we can find directly using property 7[a])

(d) $$F(s) = \int_0^\infty e^{-a(t-b)}\,U(t-c)\,e^{-st}dt = e^{ab}\int_c^\infty e^{-at}.\,e^{-st}dt$$

$$= e^{ab}.\left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_c^\infty = e^{ab}.\frac{e^{-(s+a)c}}{s+a}$$

EXAMPLE 4.21 **Find the Laplace transform of the waveform shown in figure 4.8(a).** (U.P.T.U., 2002 C.O.)

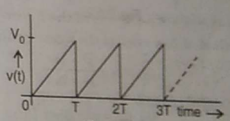


Fig. 4.8(a).

Solution : Let $v_1(t)$ be the first cycle of the waveform of the figure 4.8(a), as shown in figure 4.8(b).

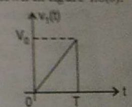


Fig. 4.8(b).

So, $$v_1(t) = \frac{V_0}{T}t\,G_{0,T}(t)$$
$$= \frac{V_0 t}{T}[U(t) - U(t-T)]$$
$$= \frac{V_0 t}{T}U(t) - \frac{V_0}{T}t\,U(t-T)$$
$$= \frac{V_0 t}{T}U(t) - \frac{V_0}{T}(t-T)\,U(t-T) - V_0\,U(t-T)$$

or $$V_1(s) = \frac{V_0}{T}\left[\frac{1}{s^2} - \frac{e^{-Ts}}{s^2}\right] - V_0\frac{e^{-Ts}}{s}$$

Therefore, the Laplace transform of the periodic waveform $U(t)$ of period T is given by

$$V(s) = \frac{1}{1-e^{-Ts}}\,.\,V_1(s) = \frac{V_0}{Ts^2} - \frac{V_0\,e^{-Ts}}{s\,(1-e^{-Ts})}$$

Scanned by CamScanner

EXERCISES

4.1. Define the Laplace transform and inverse Laplace transform.
4.2. State and prove all the properties of Laplace transform.

PROBLEMS

4.1. Find the Laplace transforms of the following functions (signals).

(i) $f(t) = K\,U(t)$
(ii) $f(t) = K\,r(t-a)$
(iii) $f(t) = \delta(t-1)$
(iv) $f(t) = e^{-at}\sin\omega t$
(v) $f(t) = e^{-at}\,t^2$
(vi) $f(t) = e^{-at}\cosh bt$
(vii) $f(t) = \sinh\omega t$
(viii) $f(t) = 10\,U(t) - \delta(t) - 5\,\delta(t-4)$
(ix) $f(t) = e^{-3t}(\cos 2t + \frac{5}{2}\sin 2t)$

4.2. Find the Laplace transform of the function shown in figure P.4.2.

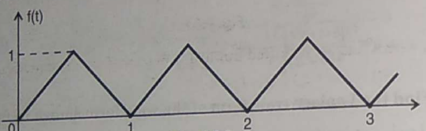


Fig. P.4.2.

4.3. Find the initial and final values, if they exist, of the signals with Laplace transforms given below:

(a) $F(s) = \dfrac{s+10}{s^2+3s+2}$
(b) $F(s) = \dfrac{s^2+5s+7}{s^2+3s+2}$

4.4. Find the initial values of the signals given below :

(a) $f(t) = e^{-at}\cos\omega t\,U(t)$
(b) $f(t) = e^{-at}\sin\omega t\,U(t)$
(c) $f(t) = [e^{-2t} + e^{-t}\cos 3t]\,U(t)$
(d) $f(t) = \dfrac{1+2t+5t^3}{1+6t}$

4.5. Find the final values of the signals with Laplace transforms given below :

(a) $F(s) = \dfrac{5}{s(s^2+s+2)}$
(b) $F(s) = \dfrac{7}{s(s+3)^2}$
(c) $F(s) = \dfrac{2}{3} - \dfrac{1}{s+3}$
(d) $f(t) = \dfrac{1+2t+5t^3}{1+6t}$

4.6. Find the Laplace transform of the waveform shown is figure P.4.6.

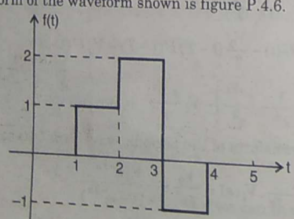


Fig. P.4.6.

4.7. Find the Laplace transform of the waveform shown in figure P.2.2(b).

4.8. A rectangular pulse $x(t)$ as shown in figure P.4.8(a), find its Laplace transform $X(s)$. Also show that the inverse Laplace transform of $X^2(s)$ gives the triangular waveform $y(t)$ as shown in figure P.4.8(b).

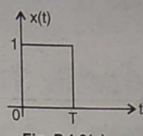


Fig. P.4.8(a).

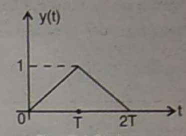


Fig. P.4.8(b).

4.9. Determine the signal $x(t)$, whose first derivative is as shown in figure P.4.9.

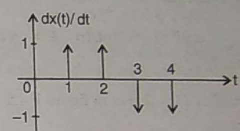


Fig. P.4.9.

4.10. A ramp function which has been shifted upward is described by the equation.

$$f(t) = k(t + t_o)\,U(t)\ ;\ t_o > 0$$

(a) Sketch $f(t)$, (b) Determine $F(s)$.

4.11. Determine the Laplace transform for the staircase waveform as shown in figure P. 4.11.

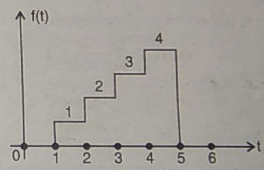


Fig. P.4.11.

ANSWERS

4.1. (i) $\dfrac{K}{s}$ (ii) $\dfrac{Ke^{-as}}{s^2}$ (iii) e^{-s}

(iv) $\dfrac{\omega}{(s+a)^2+\omega^2}$ (v) $\dfrac{2}{(s+a)^3}$ (vi) $\dfrac{(s+a)}{(s+a)^2-b^2}$

(vii) $\dfrac{\omega}{s^2-\omega^2}$ (viii) $\dfrac{10}{s} - 1 - 5e^{-4s}$ (ix) $\dfrac{s+8}{s^2+6s+13}$

Scanned by CamScanner

4.2. $F(s) = \dfrac{2(1-e^{-0.5s})}{s^2(1+e^{-0.5s})} = \dfrac{2}{s^2}\tan h\, 0.25s$

4.3. (*a*) 1,0. (*b*) ∞, 0

4.4. (*a*) 1 (*b*) 0 (*c*) 2 (*d*) 1

4.5. (*a*) $\dfrac{5}{2}$ (*b*) $\dfrac{7}{9}$ (*c*) 0

4.6. $F(s) = \dfrac{1}{s}(e^{-s} + e^{-2s} - 3e^{-3s} + e^{-4s})$

4.7. $F(s) = \dfrac{V_0}{as^2}(1-e^{-as}) - 2V_0\dfrac{e^{-2as}}{s} + \dfrac{V_0 e^{-3as}}{s}$

$$= \frac{V_0}{as^2}(1-e^{-as}) - \frac{V_0 e^{-2as}}{s}(2-e^{-as})$$

4.8. $X(s) = \dfrac{1}{s}(1-e^{-Ts})$

4.9.

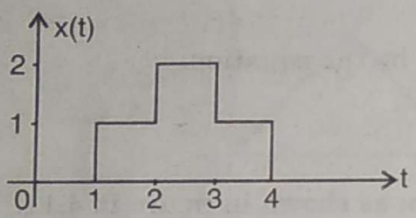


Fig. A. 4.9.

4.10. (*a*)

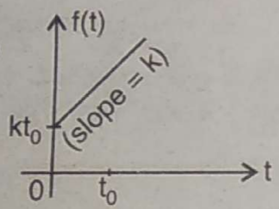


Fig. A. 4.10.

(*b*) $F(s) = K\left(\dfrac{1}{s^2} + \dfrac{t_o}{s}\right)$

4.11. $F(s) = \dfrac{1}{s}[e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} - 4e^{-5s}]$

Scanned by CamScanner