Unit - III
Complex Functions

$y = f(x) \rightarrow$ Normal funcⁿ

If for each value of the complex variable z in a given region ~~...~~ x we have one or more values of w then w is said to be a complex funcⁿ of z.

$$w = f(z) \qquad (z = x+iy)$$
$$w = u+iv \quad \text{(where u \& v are funcⁿs of x \& y)}$$

If to each value of z there corresponds one & only one value of w, then w is said to be single value funcⁿ otherwise multivalue funcⁿ.

Eg: $w = \frac{1}{z}$ (single value funcⁿ)

$w = \sqrt{z}$ (multivalue funcⁿ)

Limit of a complex function:

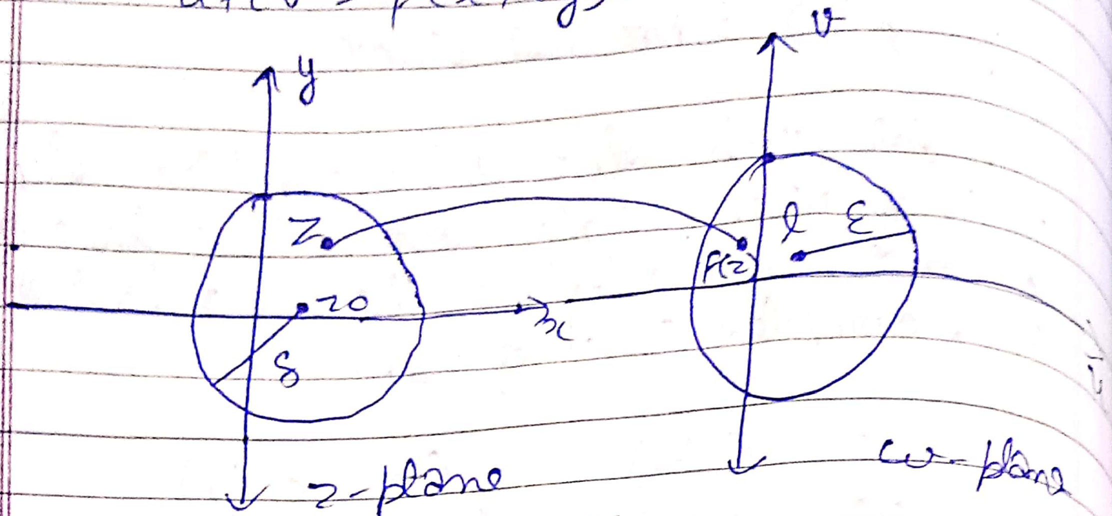
A funcⁿ $w = f(z)$ is said to tend to limit l as z approaches a point z_0 if for every real ε we can find a +ve real δ such that $|f(z) - l| < \varepsilon$ for $|z - z_0| < \delta$

$$\boxed{\lim_{z \to z_0} f(z) = l}$$

$|z-a| = r \rightarrow$ represents circle

$w = f(z)$

$u+iv = f(x+iy)$



Continuity of f(z):

A func^n f(z) is continuous at point z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$

Differentiability of f(z):

Let w be a func^n of variable z then the derivative of w w.r.t z is defined as

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \left[\frac{f(z+\delta z) - f(z)}{\delta z} \right]$$

provided limit exist

Analytic Function:

A func^n f(z) which is single value & have a unique derivative w.r.t z at all points of a region r, is called an analytic or regular func^n of z in that region.

Entire Function:

If a func^n is analytic every where in the entire plane

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then it is said to be an entire funcⁿ every where.

Eg: Polynomial is an entire funcⁿ.

→ Necessary & sufficient conditions for funcⁿ $f(z) = u + iv$ to be analytic in the region R:

1. $\frac{\delta u}{\delta x}, \frac{\delta u}{\delta y}, \frac{\delta v}{\delta x}, \frac{\delta v}{\delta y}$ are continuous funcⁿs of x & y in region R.

2. $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$, $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$ } for analytic funcⁿ. (Cauchy Reimann Eqⁿs or C.R Eqⁿs).

Q. Determine where the C.R Eqⁿs are satisfied for the funcⁿ $f(z) = 2x + ixy^2$

Ans. $u = 2x$ $\qquad$ $v = xy^2$

$\frac{\delta u}{\delta x} = 2$ $\qquad$ $\frac{\delta v}{\delta x} = y^2$

$\frac{\delta u}{\delta y} = 0$ $\qquad$ $\frac{\delta v}{\delta y} = 2xy$

$2 = 2xy \rightarrow ①$ C.R Eqⁿs
& $y^2 = 0 \rightarrow ②$

from eqⁿ ① & ② $y = 0$ & x has no value hence C.R eqⁿs are nowhere

$-\frac{px}{y^2} \times y$

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Q2. Determine p such that $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{px}{y}$ is analytic

Ans: $u = \frac{1}{2}\log(x^2+y^2) \qquad v = \tan^{-1}\frac{px}{y}$

$$\frac{\delta u}{\delta x} = \frac{x}{x^2+y^2} \qquad \frac{\delta v}{\delta x} = \frac{py}{y^2+p^2x^2}$$

$$\frac{\delta u}{\delta y} = \frac{y}{x^2+y^2} \qquad \frac{\delta v}{\delta y} = \frac{-px}{y^2+p^2x^2}$$

$$\Rightarrow \frac{y}{x^2+y^2} = -\frac{py}{y^2+p^2x^2} \rightarrow (1)$$

$$\frac{x}{x^2+y^2} = \frac{-px}{y^2+p^2x^2} \rightarrow (2)$$

C.R Eqns

So for $p=-1$ only both the above eqns are satisfied

Hence $p=-1$ is the solution

Harmonic Function: Any func^n $f(z)$ which has continuous partial derivatives of 1st & second order & satisfies the Laplace Eq^n.

$$\nabla^2 = \nabla.\nabla \rightarrow \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}$$

~~$\nabla^2 = 0$~~ $\boxed{\nabla^2 F = 0} \rightarrow$ Laplace eq^n

$$\Rightarrow \boxed{\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$$

$\Rightarrow$ u is said to be conjugate harmonic of v & v is said to be conjugate harmonic of u.

Orthogonal Function:

Any analytic func^n $f(z) = u + iv$ define two families of curve $u(x,y) = c_1$ & $v(x,y) = c_2$ which forms an orthogonal system.

$$\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = m_1 = -\frac{\delta u/\delta x}{\delta u/\delta y} \rightarrow (1)$$

Similarly $\frac{dy}{dx} = m_2 = -\frac{\delta v/\delta x}{\delta v/\delta y} \rightarrow (2)$

$$\Rightarrow m_1 \times m_2 = \frac{\delta u/\delta x}{\delta u/\delta y} \times \frac{\delta v/\delta x}{\delta v/\delta y}$$

$$= -\frac{\cancel{\delta v/\delta y}}{\cancel{\delta v}/\delta x} \cdot \frac{\cancel{\delta v}/\delta x}{\delta v/\cancel{\delta y}} \quad (\text{C.R eq}^{ns})$$

$\boxed{m_1 m_2 = -1} \rightarrow$ Condi^n for Orthogonal func^n

Milne Thomson Theorem:

$\boxed{\begin{matrix} x = z \\ y = 0 \end{matrix}} \rightarrow$ when we use Milne Thomson theorem as $z = \bar{z}$ (we use in this case)

$x + iy = x - iy$

$\Rightarrow y = 0$ Also $z = x + iy$

$\Rightarrow x = z$

Q. If $f(z) = u + iv$ is an analytic func^n of the complex variable z & $u - v = e^x(\cos y - \sin y)$ Find $f(z)$ in terms of z.

Ans $u - v = e^x(\cos y - \sin y)$

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$$\frac{\delta u}{\delta x} - \frac{\delta v}{\delta x} = e^{x}(\cos y - \sin y) \rightarrow (1)$$

$$\& \ \frac{\delta u}{\delta y} - \frac{\delta v}{\delta y} = -e^{x}(\sin y + \cos y)$$

$$\Rightarrow \cancel{+}\frac{\delta v}{\delta x} \cancel{+} \frac{\delta u}{\delta x} = \cancel{+}e^{x}(\sin y + \cos y) \rightarrow (2)$$

(using C.R eqn)

$\Rightarrow$ from eqn (1) & (2)

$$\frac{\delta u}{\delta x} = e^{x}\cos y$$

$$\therefore \frac{\delta v}{\delta x} = e^{x}\sin y$$

Also $f(z) = u + iv$

$$\frac{\delta f}{\delta x} = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$$

$$\frac{\delta f}{\delta x} = e^{x}\cos y + ie^{x}\sin y$$

Now use Milne Thomson theorem $(x=z)$ $\& \ y=0$

$$\Rightarrow \frac{\delta f}{\delta z} = e^{z}$$

$$\int \delta f = \int e^{z}\,\delta z$$

$$\boxed{f(z) = e^{z} + C}$$

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Q1 Determine the analytic funcn whose real part $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 2x + 1$. Also prove th the given funcn satisfy the laplace eqn.

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$$\nabla^2 u = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{\delta}{\delta x}(3x^2 - 3y^2 + 6x + 2)$$

$$= 6x + 6 = 6(x+1)$$

$$\frac{\delta^2 u}{\delta y^2} = \frac{\delta}{\delta y}(-6xy - 6y)$$

$$= -6x - 6$$
$$= -6(x+1)$$

$\boxed{\nabla^2 u = 0}$ $\Rightarrow$ satisfy laplace eq^n

$$\frac{\delta u}{\delta x} = 3x^2 - 3y^2 + 6x + 2, \quad \frac{\delta u}{\delta y} = -6xy - 6y$$

$$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}, \quad \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$$

$$\Rightarrow \frac{\delta v}{\delta x} = 6xy + 6$$

$$\Rightarrow \frac{\delta f}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}$$

$$= 3x^2 + 6x - 3y^2 + 2 + 6xy + 6$$
$$= 3x^2 - 3y^2 + 6x + 6xy + 8$$

Now applying Milne Thomson theorem,
$x = z, \ y = 0$

$$\Rightarrow \frac{\delta f}{\delta z} = 3z^2 + 6z + 8$$

$$\Rightarrow \int \delta f = \int (3z^2 + 6z + 8)\, \delta z$$

$$\boxed{f(z) = z^3 + 3z^2 + 8z + c}$$

* if imaginary part is to be found then put $z = x+iy$

Q2. Show that the funcⁿ $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic. Find its conjugate harmonic.

Ans2. $\nabla^2 u = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}$

$$\frac{\delta^2 u}{\delta x^2} = \frac{\delta}{\delta x}\left\{\frac{1}{\cancel{2}}\frac{\cancel{2}x}{x^2+y^2}\right\}$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$$

$$\frac{\delta^2 u}{\delta y^2} = \frac{\delta}{\delta y}\left\{\frac{1}{\cancel{2}}\frac{\cancel{2}y}{x^2+y^2}\right\}$$

$$\frac{\delta^2 u}{\delta y^2} = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

$$\nabla^2 u = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

Hence it satisfy laplace eqⁿ

$$\frac{\delta u}{\delta x} = \frac{x}{x^2+y^2}, \quad \frac{\delta u}{\delta y} = \frac{y}{x^2+y^2}, \quad \frac{\delta v}{\delta x} = \frac{-y}{x^2+y^2}$$

$$\frac{\delta f}{\delta x} = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$$

$(x=z \,\&\, y=0)$ $\frac{\delta f}{\delta z} = \frac{z}{z^2} = \frac{1}{z} \Rightarrow f(z) = \log z + c$

Also $(z = x+iy) \Rightarrow \boxed{f(z) = \log(x+iy) + c}$

nce ~~AA~~ $\log(a+ib) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\frac{b}{a}$ }

$\Rightarrow \boxed{f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}}$

Alternate $\frac{\delta v}{\delta y} = \frac{x}{x^2+y^2} \Rightarrow v = \tan^{-1}\left(\frac{y}{x}\right)$

Complex Integration:

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

Q1. Evaluate $\int_0^{2+i} \bar{z}^2\,dz$ along the line $y = \frac{x}{2}$

Ans1. As $z = x+iy \Rightarrow \bar{z} = x-iy$ As $y = \frac{x}{2}$

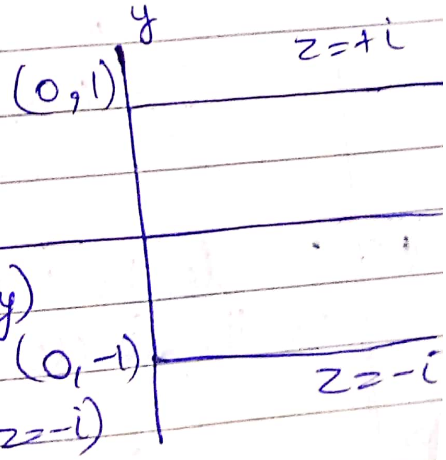
$\int_0^{2+i} \bar{z}^2\,dz = \int_0^{2+i} (x-iy)^2(dx+idy)$ $dy = \frac{dx}{2}$

$= \int_0^{2} \left(x - i\frac{x}{2}\right)^2\left(dx + i\frac{dx}{2}\right)$ $z = x+iy$, At $z=0$, $0 = x+0$

$= \left(1-\frac{i}{2}\right)^2\left(1+\frac{i}{2}\right)^2\int_0^2 x^2\,dx$. At $z = 2+i$

$= \frac{5}{3}(2-i)$ $2+i = x+iy$ $\Rightarrow x = 2$

Q2. Evaluate $\int_C |z|^2\,dz$ where c is straight line from $z=-i$ to $z=+i$



Ans2. $z = x+iy$ $|z| = \sqrt{x^2+y^2}$

$\Rightarrow \int_C |z|^2\,dz = \int_C (x^2+y^2)(dx+idy)$

$\Rightarrow x = 0$ $dx = 0$ (At $z=+i$ & $z=-i$)

$\sqrt{y} = t$ $\qquad \frac{1}{3}+\frac{1}{2}$ $\qquad t^2 dt$

$\frac{1}{2\sqrt{y}} dy = dt$ $\qquad 2\,\frac{t^3}{3}$

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$$\Rightarrow i\left[\frac{y^3}{3}\right]_{-1}^{1} = \frac{2}{3}i$$

Q3. Evaluate $\int_0^{1+i} (x^2+iy)\,dz$ along the path $y=x^2$

As3. $\int_0^1 (x^2+iy)(dx+idy)$ $\qquad \left(\text{As } \frac{dy}{2x} = dx\right)$

$$\Rightarrow \int_0^1 (y+iy)\left(\frac{dy}{2\sqrt{y}} + idy\right) \qquad (dy = 2x\,dx)$$

$$\Rightarrow (1+i)\int_0^1 \frac{y}{2\sqrt{y}}\,dy + y\,i\,dy$$

$$\Rightarrow (1+i)\left\{\frac{\cancel{2}}{\cancel{2}}\,\frac{y^{3/2}}{3} + \frac{iy^2}{2}\right\}$$

$$\Rightarrow (1+i)\int_0 y\left(\frac{1}{2\sqrt{y}} + i\right)dy$$

$$\Rightarrow -\frac{1}{6} + \frac{5}{6}i$$

Q4. Prove that $\int_C \frac{dz}{z-a} = 2\pi i$, where a is a circle $|z-a| = r$

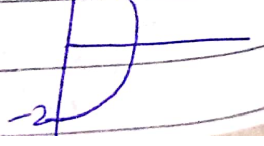
As4. $\Rightarrow z-a = re^{i\theta}$

$dz = ire^{i\theta}d\theta$

for $\{\theta = 0$ to $\theta = 2\pi\}$

$$= \int_0^{2\pi} \frac{i\cancel{r}\cancel{e^{i\theta}}\,d\theta}{\cancel{r}\cancel{e^{i\theta}}} = i(2\pi - 0) = 2\pi i$$

Q5. Evaluate the integral $\int_C \bar{z}\,dz$ along the right half of the circle $|z| = 2$



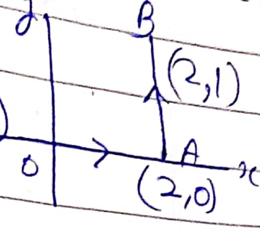
As5. $\bar{z} = x - iy = e^{i\theta}$

$z = e^{i\theta}$ & $r = 2$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} i e^{-i\theta} e^{i\theta} \, d\theta = i\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \pi i$$

Q. Evaluate $\int^{2+i} (\bar{z})^2 dz$ along the real axes to 2 and then vertically to $2+i$

Sol. $\int_{OA} (x-iy)^2 (dx + i dy) + \int_{AB} (x-iy)^2 (dx+idy)$



$$= \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 (i\,dy)$$

$$= \left[\frac{x^3}{3}\right]_0^2 + i\left\{4\left[y\right]_0^1 - i^2\left[\frac{y^3}{3}\right]_0^1 - 4i\left[\frac{y^2}{2}\right]_0^1\right\}$$

$$= \frac{8}{3} + 4i(1) + i\left(\frac{1}{3}\right) + 4^2\left(\frac{1}{2}\right)$$

$$= \frac{14}{3} + \frac{13}{3} i$$

Cauchy's Integral theorem:

If $f(z)$ is an analytic func^n & $f'(z)$ is continuous at each point within and on a closed curve C then, $\boxed{\int_C f(z)\, dz = 0}$

↓

Cauchy's Integral theorem

Cauchy's Integral Formula:

If $f(z)$ is analytic inside and on a closed curve C & a is any point within C then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

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14.3.19

Q1. Evaluate integral $\int_C \frac{z^2-z-1}{z-1}\,dz$ where C is the circle (i) $|z|=1$ (ii) $|z|=\frac{1}{2}$

Ans. (i) Here we will use formula (as point lies inside the circle).

$$= 2\pi i\, f(1)$$
$$= -2\pi i \quad (\text{as } f(1) = -1)$$

(ii) for $|z| = \frac{1}{2}$

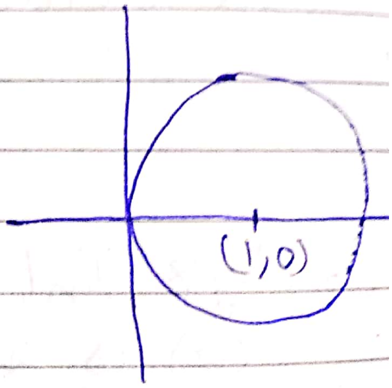
$$\int_C \frac{z^2-z-1}{z-1}\,dz = 0$$

as point 1 lie outside the circle $|z| = \frac{1}{2}$ & hence this funcⁿ is analytic everywhere inside the circle.

Q2. Evaluate integral $\int_C \frac{3z^2+z}{z^2-1}\,dz$ where C is the circle $|z-1|=1$.

Ans2.

$$\int_C \frac{3z^2+z}{(z+1)(z-1)}\,dz$$



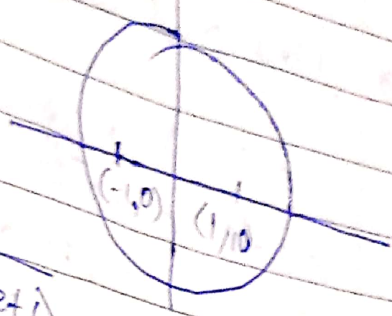
$$\int_C \frac{\left[\frac{3z^2+z}{(z+1)}\right]}{(z-1)}\,dz$$

$$= 2\pi i\, f(1)$$
$$= 4\pi i$$

Evaluate integral $\int \frac{3z^2+z}{z^2-1} dz$ where the circle is $|z| = 2$

$$\frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$$

$$= \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$$



$$\Rightarrow \int_C \frac{3z^2+z}{z^2-1} dz = \frac{1}{2}\int_C \frac{3z^2+z}{z-1} dz - \frac{1}{2}\int \frac{3z^2+z}{z+1} dz$$

$$= \frac{1}{2} 2\pi i \left[f(1) - f(-1) \right]$$

$$= \pi i \left[4 - 2 \right]$$

$$= 2\pi i$$

Q4. Evaluate F(2) & F(3) if $F(a) = \int_C \frac{2z^2-z-2}{z-a} dz$ where C is the circle $|z| = 2.5$.

Sol4. $F(2) = \int_C \frac{2z^2-z-2}{z-2} dz$

$$= 2\pi i f(2) \quad (\text{as } f(2) = 4)$$

$$= 8\pi i$$

$F(3) = \int_C \frac{2z^2-z-2}{z-3} dz$

$= 0$ (as point 3 write outside the circle hence this is analytic everywhere inside the circle by Cauchy's theorem)

Taylor's Series:

If $f(z)$ is analytic func^n inside the circle c with centre at $z=a$, then it can be expanded in the series

$$\boxed{f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots \infty}$$

which is convergent at every point inside c.

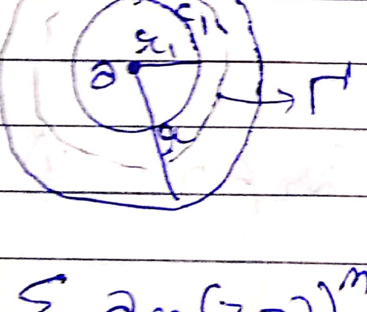
Laurent's Series:

If $f(z)$ is analytic in the ring shaped region R bounded by 2 concentric circles C & C_1 of radii r & r_1 with centre at a then for all z in R

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots \infty$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma} \frac{f(t)\,dt}{(t-a)^{n+1}}$

↳ (gamma being any curve encircling C_1)



⇒ $\Sigma a_n(z-a)^n$ → called the regular part of the expansion

⇒ $\Sigma a_n(z-a)^{-n}$ → called the principle part of the expansion.

(but we don't use these series

Expand $f(z) = \frac{1}{z(z^2-3z+2)}$ in the region $1 < |z| < 2$

$f(z) = \frac{1}{z(z-2)(z-1)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$

$\Rightarrow 1 = A(z-1)(z-2) + Bz(z-2) + Cz(z-1)$

$A + B + C = 0$

$-3A - 2B - C = 0$

$2A = 1$

$A = \frac{1}{2}$

$\Rightarrow B + C = -\frac{1}{2}$ & $2B + C = -\frac{3}{2}$

$B = -1$ & $C = \frac{1}{2}$

$\Rightarrow f(z) = \frac{1}{2z} - \frac{1}{(z-1)} + \frac{1}{2(z-2)}$

$= \frac{1}{2z} - (z-1)^{-1} + \frac{1}{2}(z-2)^{-1}$

$= \frac{1}{2z} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{4}\left(1 - \frac{z}{2}\right)^{-1}$

$= \frac{1}{2z} - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \infty\right]$

$-\frac{1}{4}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots \infty\right]$

$=$ (1 term more after simplification)

Q2 Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the region $|z| < 1$.

Ans 2. $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$= \frac{1}{z-2} - \frac{1}{z-1}$

$= (z-2)^{-1} - (z-1)^{-1}$

$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + 1(1-z)^{-1}$

$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\dots\infty\right] + \left[1+z+z^2+z^3+\dots\infty\right]$

$= \left[-\frac{1}{2}-\frac{z}{4}-\frac{z^2}{8}-\dots\infty\right] + \left[1+z+z^2+z^3+\dots\infty\right]$

$= \frac{1}{2}+\frac{3z}{4}+\frac{7}{8}z^2+\dots\infty$

(It is Taylor's series as only +ve terms of z are there)

Q3. Find Laurent series for the func^n $f(z) = \frac{7z-2}{z^3-z^2-2z}$ in the region $1<|z+1|<3$

Ans 3. $f(z) = \frac{7z-2}{z(z^2-z-2)} = \frac{A}{z} + \frac{B}{(z-2)} + \frac{C}{(z+1)}$

$C = -1$

$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$

$A+B+C=0$ $\quad -2A=-2$

$-A+B-2C=7$ $\quad A=1$

$C=-3$

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$$f(z) = \frac{1}{2} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$ Let $z+1 = u$, $1 < |u| < 3$

$$= (u-1)^{-1} + 2(u-3)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left[1-\frac{u}{3}\right]^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left[1+\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\ldots\infty\right]$$
$$-\frac{2}{3}\left[1+\frac{u}{3}+\frac{u^2}{9}+\ldots\infty\right] - \frac{3}{u}$$

Put $u = z+1$

$$= -\frac{3}{u} + \left[\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\frac{1}{u^4}+\ldots\infty\right] + \left[-\frac{2}{3}-\frac{2u}{9}-\frac{2u^2}{27}\ldots\infty\right]$$

$$= -\frac{2}{3} - \frac{2}{z+1} + \frac{1}{(z+1)^2} - \frac{2(z+1)}{9} - \frac{2(z+1)^2}{27} \ldots\ldots\infty$$

Zeroes of an Analytic Function:

→ Let $f(z)$ be an analytic function, if $f(z_0) = 0$ then z_0 is called the zero of $f(z)$ func^n.

→ If $f(z_0) = 0$ but $f'(z_0) \neq 0$ then z_0 is called simple zero or zero of first order.

→ If $f(z_0) = 0$ & $f'(z_0) = 0$ but $f''(z_0) \neq 0$ then z_0 is called zero of second order.

Eg: $f(z) = 1 - \cos z$

$$1 - \cos z = 0$$
$$\cos z = 1$$
$$\cos z = \cos 2n\pi$$
$$\Rightarrow \boxed{z = 2n\pi}$$

$$f'(z) = \sin z \ (\text{at } 2n\pi) = 0$$
$$f''(z) = \cos z \ (\text{at } 2n\pi) \neq 0$$
$\Rightarrow$ second order zero.

Singular point:
A point z_0 at which a func^n $f(z)$ is not analytic is known as singular point or singularity of the func^n.

(i) Isolated Singularity: If the func^n $f(z)$ is analytic at every point in the neighbourhood of a point z_0 except at z_0 itself, then z_0 is called an isolated singularity.
Eg: $f(z) = \frac{1}{z}$ is an isolated singularity at $z = 0$

(ii) Removable Singularity: If $f(z)$ is not defined at $z = z_0$ but limit of $f(z)$ at $z \to z_0$ exists, then $z = z_0$ is called removable singularity of $f(z)$. In this case the principle part of laurent series is zero.
Eg: $f(z) = \frac{\sin z}{z}$

Also $\frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right]$

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$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

(as no. -ve powers of z is there in this hence it is Removable singularity)

(iii) **Essential Singularity:**

In Laurent's expansion of $f(z)$ about $z = z_0$ if the principle part contains infinite no. of terms then $z = z_0$ is called an essential singularity of $f(z)$.

Eg: $f(z) = \sin\left(\frac{1}{z}\right)$

$$= \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} + \dots \infty$$

it is an essential singularity

(iv) **Pole:**

If all the terms in the principle part of the Laurent's series after the n^{th} term are missing, then the singularity at $z = z_0$ is called a pole of order n.

Note: A pole of order 1 is called a simple pole

Eg: $f(z) = \frac{z}{(z-2)^2} + \frac{2z^2}{(z+3)}$

$\Rightarrow z = 2$ is the pole of order 2

$z = -3$ " " simple pole

Q. Find the nature and location of singularities of the following func^n.

(i) $f(z) = \frac{z - \sin z}{z^2}$

$z = 0$ (singular point)

$$f(z) = \frac{\cancel{z} - \left(\cancel{z} - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)}{z^2}$$

$$= \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} + \cdots$$

$\Rightarrow z = 0$ is the removable singularity as the principle part is 0.

(ii) $f(z) = \frac{1 - e^{2z}}{z^3}$

$z = 0$ (singular point)

$$\frac{1 - \left[1 + 2z + \frac{4z^2}{2!} + \frac{8z^3}{3!} + \cdots\right)}{z^3}$$

$$= \underbrace{\frac{2}{z^2} + \frac{2}{z}}_{} + \frac{8}{3!} + \cdots$$

(principle part after 2nd term is missing)

$z = 0$ is the pole of order 2

(iii) $f(z) = (z+1) \sin \frac{1}{z-2}$

$z = 2$ (singular point)

Let $z - 2 = t$

$f(z) \Rightarrow (t+3) \sin \frac{1}{t}$

$$(t+3)\left[\frac{1}{t} - \frac{1}{3!\,t^3} + \frac{1}{5!\,t^5} \cdots \infty\right]$$

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$= 1 - \frac{1}{3!t^2} + \frac{1}{5!t^4} - \ldots$

Put $t = z-2$

$\Rightarrow$ it is an essential singularity as principle part is infinite

Residue of a complex function:

The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around a singularity is called the residue of $f(z)$ at that point.

So, in the Laurent's expansion about point 'a' i.e. $f(z) = a_0 + a_1(z-a) - a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + \ldots$

where $a_n = \frac{1}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^{n+1}}$

$$\boxed{a_{-1} = \operatorname*{Res}_{z=a} f(z) = \frac{1}{2\pi i}\int_C f(z)\,dz}$$

$\downarrow$ Residue

$\Rightarrow$ If $f(z)$ has a simple pole at $z=a$ then residue of $f(z)$ at $z=a$ is,

$$\boxed{\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a}[(z-a)f(z)]}$$

$\Rightarrow$ If $f(z)$ has a pole of order m then residue of $f(z)$ at $z=a$ is,

$$\boxed{\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right]_{z=a}}$$

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Cauchy's Residue Theorem:

If $f(z)$ is analytic in closed curve C except at a finite number of singular points within C, then

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of the residues at the singular point within C}$$

Q1. Find the pole of the func^n $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ & the residue at each pole, hence evaluate $\int_C f(z)\,dz$ where c is the circle $|z| = 2.5$

Ans. Poles $\Rightarrow$ $z = 1$ ~~of~~ order 2
$z = -2$ order 1 or simple pole

$$\underset{z=-2}{\text{Res.}} f(z) = \lim_{z\to -2}\left[\frac{\cancel{(z+2)}\, z^2}{(z-1)^2 \cancel{(z+2)}}\right] = \frac{4}{9}$$

$$\cancel{\underset{z=1}{\text{Res.}} f(z) = \lim_{z\to 1}\left[\frac{(z-1)^2 z^2}{(z-1)^2(z+2)}\right]}$$

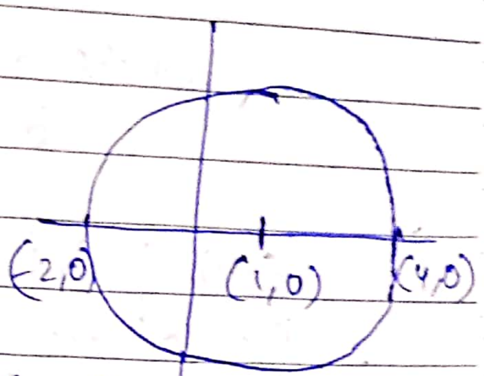
$$\underset{z=1}{\text{Res.}} f(z) = \left[\frac{d}{dz}\left(\frac{\cancel{(z-1)^2}\, z^2}{\cancel{(z-1)^2}(z+2)}\right)\right]_{z=1} = \frac{5}{9}$$

$$\int_C f(z)\,dz = \int_C \frac{z^2}{(z-1)^2(z+2)}\,dz \quad \text{(as both points lie inside the circle)}$$

$$= 2\pi i \times \left(\frac{4}{9} + \frac{5}{9}\right) = 2\pi i$$

Q. Evaluate $\int_C \frac{e^z \, dz}{(z+1)^2(z-2)}$ where c is the circle $|z-1| = 3$

Sol: $z = -1$ order 2
$z = 2$ simple pole
both poles lie inside the circle



$$\underset{z=2}{\text{Res}}\, f(z) = \lim_{z \to 2} \left[\frac{\cancel{(z-2)}\, e^z}{(z+1)^2 \cancel{(z-2)}} \right]$$

$$= \frac{e^2}{9}$$

$$\underset{z=-1}{\text{Res}}\, f(z) = \left[\frac{d}{dz} \cancel{(z+1)^2} \frac{e^z}{\cancel{(z+1)^2}(z-2)} \right]_{z=-1}$$

$$= \left[\frac{e^z(z-2) - e^z}{(z-2)^2} \right]_{z=-1}$$

$$= \frac{-4}{9e}$$

$$\Rightarrow \int_C f(z)\,dz = \int_C \frac{e^z}{(z+1)^2(z-2)}\,dz$$

$$= 2\pi i \times \left(\frac{e^2}{9} - \frac{4}{9e} \right)$$

Q3. Evaluate $\int_C \frac{z-3}{z^2+2z+5}\,dz$ where c is the
(i) circle $|z| = 1$.

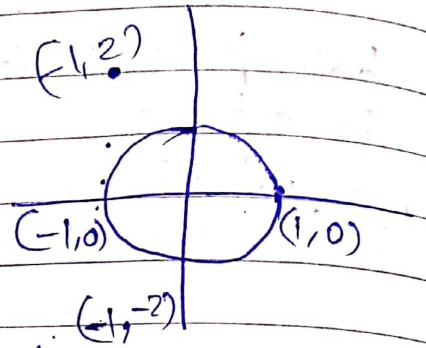
Ans3: (i) $f(z) = \int_C \frac{z-3}{(z-a)(z-b)}\,dz$ $\qquad z = \frac{-2 \pm \sqrt{4-20}}{2}$

$z = \frac{-2+4i}{2}$

where $a = -1+2i$ $(-1,2)$

$b = -1-2i$ $(-1,-2)$ $\qquad z = -1 \pm 2i$

Both points lie outside the circle hence the ~~ites~~ value is 0.

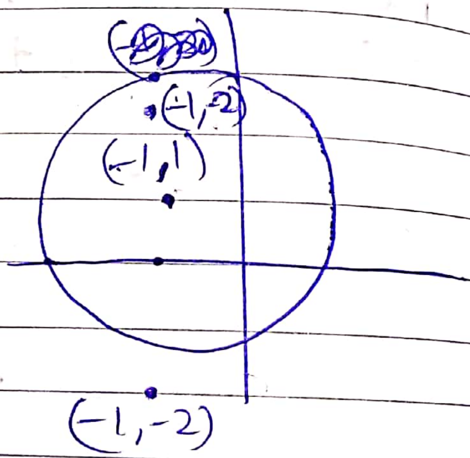


(ii) if c is the circle $|z+1-i| = 2$

$a = (-1,2)$

$b = (-1,-2)$

~~Rees~~ point a lie inside the ~~Bo~~ circle & point b lie outside it hence,



$$\underset{z=a}{\text{Res}}\, f(a) = \lim_{z \to a} \left[\frac{\cancel{(z-a)}(z-3)}{\cancel{(z-a)}(z-b)} \right]$$

$$= \frac{a-3}{a-b}$$

$$= \frac{-1+2i-3}{-\cancel{1}+2i+\cancel{1}+2i}$$

$$= \frac{-4+2i}{4i} = i + \frac{1}{2}$$

(iii) if c is the circle $|z+1+i| = 2$

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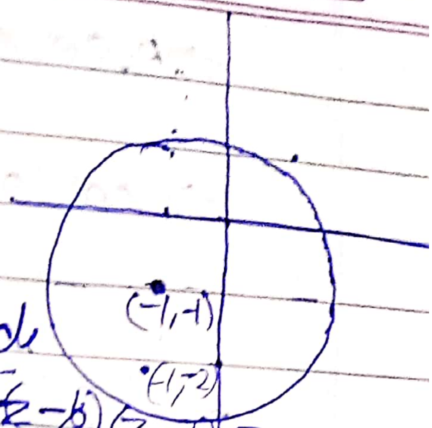
$a = (-1, 2)$
$b = (-1, -2)$



Now in this case point b lies inside the circle & a lies outside the circle hence, $\underset{z=b}{\text{Res}}\, f(b) = \lim_{z \to b} \left[\frac{(z-b)(z-3)}{(z-a)(z-b)} \right]$

$$= \frac{b-3}{b-a} = \frac{-1-2i-3}{-1-2i+1-2i} = \frac{1}{2} - i$$

Evaluation of Real definite integrals:

(a) Integration around the unit circle:

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

Put $e^{i\theta} = z$ & $e^{-i\theta} = \frac{1}{z}$

$\Rightarrow \cos\theta + i\sin\theta = z$

& $\cos\theta - i\sin\theta = \frac{1}{z}$

Also $e^{i\theta} = z$
$ie^{i\theta}d\theta = dz$
$d\theta = \frac{dz}{iz}$

$$\boxed{\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)}$$

& $\boxed{\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)}$

$\Rightarrow \int_C f(z)\, dz$ where c is unit circle

Q1. Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta$.

Ans:- Put $e^{i\theta} = z$

$2z^2 - 4z - z + 2$
$2z(z-2) - 1(z-2)$

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$\Rightarrow$ As $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ $\quad e^{i\theta} = z$
$\Rightarrow |z| = 1.$

$\Rightarrow \cos 3\theta = \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)$

$$\Rightarrow \int_0^{2\pi} f(z)\,dz = \int_C \frac{\frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)}{5 - \frac{4}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \frac{1}{2i} \int_C \frac{(z^6+1)\,dz}{z^3(5z - 2z^2 - 2)}$$

$$= \frac{-1}{2i} \int \frac{(z^6+1)\,dz}{z^3(2z^2 - 5z + 2)}$$

$$= \frac{-1}{2i} \int \frac{(z^6+1)\,dz}{z^3(2z-1)(z-2)}$$

$$= \frac{-1}{4i} \int \frac{(z^6+1)\,dz}{z^3\left(z - \frac{1}{2}\right)(z-2)}$$

$z = 0, \frac{1}{2}, 2$
(order 3, 1, 1)

$$\operatorname*{Res}_{z=1/2} f(z) = \frac{-1}{4i} \lim_{z \to 1/2} \left[\frac{\cancel{(z - 1/2)}\,(z^6+1)}{z^3 \cancel{(z - 1/2)} (z-2)} \right]$$

$\frac{65}{48}$

$$= \frac{65}{\cancel{64}} \times \frac{\cancel{8}}{1} \times \frac{-2}{3} = \frac{-65}{12} \times$$

$$= \frac{65}{48i}$$

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$$\operatorname*{Res}_{z=0} f(z) = \frac{-1}{4i\,2!}\left[\frac{d^2}{dz^2}\frac{\cancel{(z-0)^3}(z^6+1)}{\cancel{z^3}(z-\frac{1}{2})(z-2)}\right]_{z=0}$$

$$= \frac{1}{2} \times \frac{-1}{4i} \times \frac{21}{4} = \boxed{\frac{21i}{32}}$$

$$\operatorname*{Res}_{z=2} f(z) = -\frac{1}{4i}\lim_{z\to 2}\left[\frac{z^6+1}{z^3(z-\frac{1}{2})}\right] = \frac{-1}{4i} \times \frac{65\times 2}{48\times 3}$$

$$= \frac{65}{48} i .$$

2. Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ where $a>b$, hence show that $\int_0^{\pi} \frac{d\theta}{17-8\cos\theta} = \frac{\pi}{15}$

Ans 2. Put $e^{i\theta} = z$

$$\Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\& \; ie^{i\theta}\, d\theta = dz$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

$$\Rightarrow \int_0^{2\pi} f(z)\,dz = \int_0 \frac{1}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \frac{2}{i}\int \frac{1}{2za+bz^2+b}\,dz$$

$$= \frac{2}{i}\int \frac{1}{2az+b(z^2+1)}\,dz$$

$$= \frac{2}{i}\int \frac{1}{2az+b(z+i)(z-i)}\,dz$$

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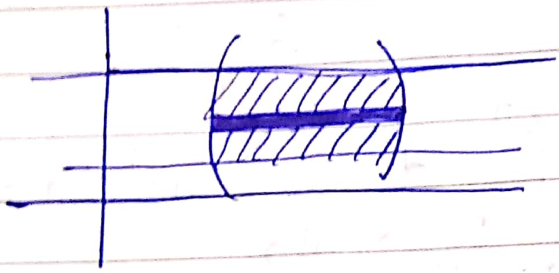
26.3.19

Unit – IV
Double Integral

• **Double** [Volume]

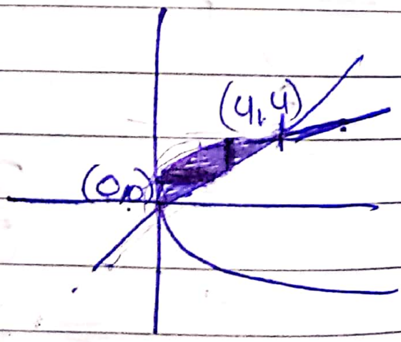
$$\int_{y=a}^{b}\int_{x=\phi_1(y)}^{\phi_2(y)} f(x,y)\,dx\,dy$$



[Area] $$\int_{y=a}^{b}\int_{x=\phi_1}^{\phi_2} dx\,dy$$

Q1. Evaluate double integral $\iint_R (x^2+y^2)\,dx\,dy$ where R is bounded by $y=x$ & $y^2=4x$

Ans. $$\int_{y=0}^{y=4}\int_{x=\frac{y^2}{4}}^{y} (x^2+y^2)\,dx\,dy$$



$$\int_0^4 \left(\frac{x^3}{3}+y^2x\right)_{x=\frac{y^2}{4}}^{y} dy$$

$$\int_0^4 \left[\left[\frac{y^3}{3}+y^3\right]-\left[\frac{y^6}{252}+\frac{y^4}{4}\right]\right]dy$$

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$$= \int_0^4 \left(\frac{4}{3}y^3 - \frac{y^6}{252} + \frac{y^4}{4}\right)dy$$

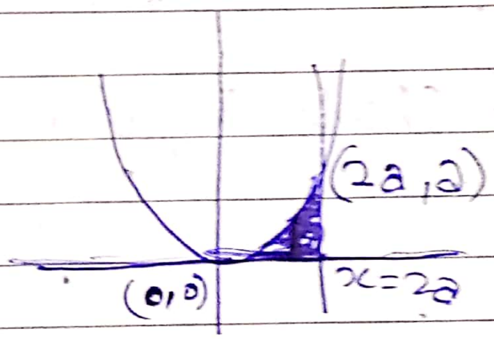
$$= \left[\frac{y^4}{3} - \frac{y^7}{1764} + \frac{y^5}{20}\right]_0^4$$

$$= \frac{256}{3} - \frac{4096}{441} + \frac{256}{5}$$

$$= \frac{2048}{15} - \frac{4096}{441}$$

2. Evaluate $\iint_R xy\, dy\, dx$, wher R is bounded by x-axis, $x = 2a$, $x^2 = 4ay$

$$\int_0^{2a}\int_{y=0}^{x^2/4a} xy\, dy\, dx$$



$y=0$ $\frac{x^2}{4a}$

$$\int_0^{2a} \left[\frac{xy^2}{2}\right]_0^{x^2/4a} dx$$

$$\int_0^{2a} \frac{x^5}{32a^2} dx$$

$$= \left[\frac{x^6}{192a^2}\right]_0^{2a}$$

$$= \frac{64a^{\cancel{6}4}}{192\cancel{a^2}}$$

$$= a^4$$

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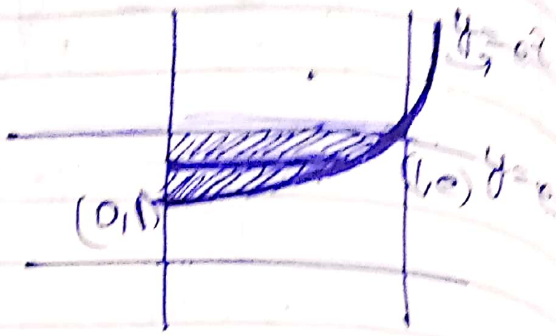
Q3. Change the order of integration & hence evaluate it $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$

Ans 3. $\int_{y=1}^{e} \int_{x=0}^{\log y} \frac{dx\,dy}{\log y}$



$\int_{y=1}^{e} \left[\frac{x}{\log y} \right]_0^{\log y} dy$

$y = e^x$
$y = e$

$\int_{y=1}^{e} (1-0)\,dy$

$\Rightarrow [y]_1^e = (e-1)$

Q4. Change the order of integration.
$\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dy\,dx$

$y = \sqrt{2ax}$
$y^2 = 2ax$

Ans 4. $y = \sqrt{2ax-x^2}$
$y^2 = 2ax - x^2 + a^2 - a^2$
$y^2 = -(x-a)^2 + a^2$

$(x-a)^2 + y^2 = a^2$
which is the eq^n of a circle whose centre $= (a,0)$, radius $= a$.

$\Rightarrow x = a \pm \sqrt{a^2 - y^2}$

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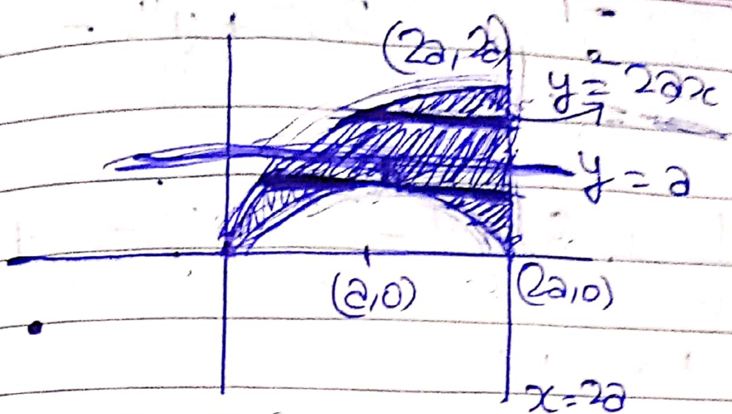
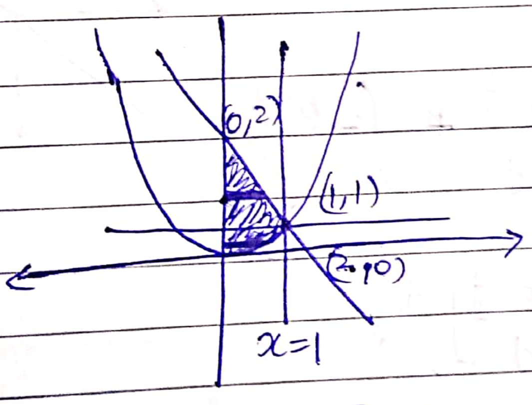


$$= \int_{y=0}^{a}\int_{x=y^2/2a}^{a-\sqrt{a^2-y^2}} f(x,y)\,dx\,dy + \int_{y=0}^{a}\int_{x=a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx\,dy +$$

$$\int_{y=a}^{2a}\int_{x=y^2/2a}^{2a} f(x,y)\,dx\,dy$$

5. Change the order of integration & hence evaluate it. $\int_{0}^{1}\int_{x^2}^{2-x} xy\,dy\,dx$

5.



$$= \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_{y=0}^{1}\left[\frac{x^2 y}{2}\right]_0^{\sqrt{y}} dy + \int_{y=1}^{2}\left[\frac{x^2 y}{2}\right]_0^{2-y} dy$$

$$= \int_{y=0}^{1}\frac{y^2}{2}\,dy + \int_{y=1}^{2}\frac{(2-y)^2 y}{2}\,dy$$

$-e^{-y}\,dy = dt$ $\quad e^{-y} = t$ $\quad -y = -\log t$ $\quad y = x$ $\quad x = 0$ $\quad x = \infty$ $\quad \frac{1}{y e^{y}}$

$$= \left[\frac{y^3}{6}\right]_0^1 + \left[2y^2 + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2$$

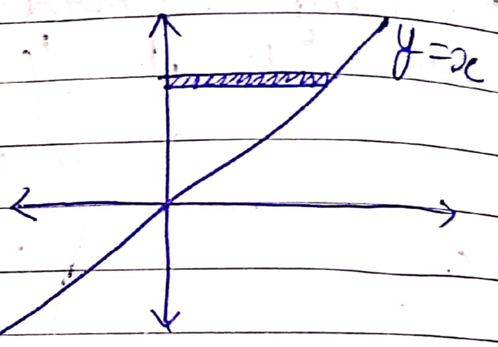
$$= \frac{1}{6} + 8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}$$

$$= 10 - \frac{113}{12}$$

$$= \frac{7}{12}$$

Q6. Change the order of integration & hence evaluate it. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

Ans 6.
$$= \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\,dy$$



$$\cancel{= \int_{y=0}^{\infty} \left[-\frac{e^{-y}}{y} + \int -\frac{e^{-y}}{y^2}\right]}$$

$$= \int_{y=0}^{\infty} \left[\frac{x e^{-y}}{y}\right]_0^y dy$$

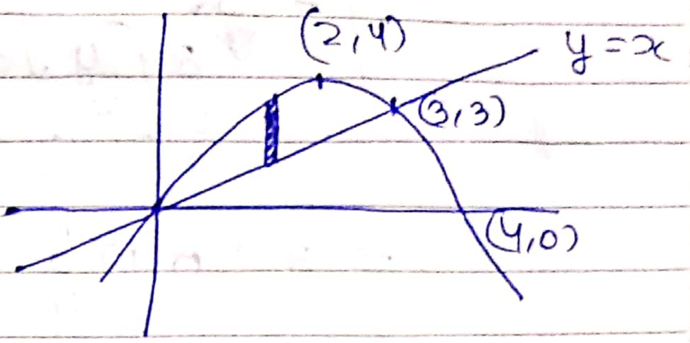
$$= \int_{y=0}^{\infty} e^{-y}\,dy$$

$$= \left[-e^{-y}\right]_0^\infty$$

$$\cancel{= \left[-\frac{1}{e^y}\right]_0^\infty} = 0 + 1$$

$$= 1$$

Q7. Find the area lying b/w the parabola $y = 4x - x^2$ & line $y = x$



Ans7. $y = 4x - x^2 - 4 + 4$

$y - 4 = -(x-2)^2$

$(x-2)^2 = -(y-4)$

$x^2 = -y$

$$= \int_{x=0}^{3} \int_{y=x}^{4x-x^2} dy\, dx$$

$$= \int_{x=0}^{3} [y]_{x}^{4x-x^2} dx$$

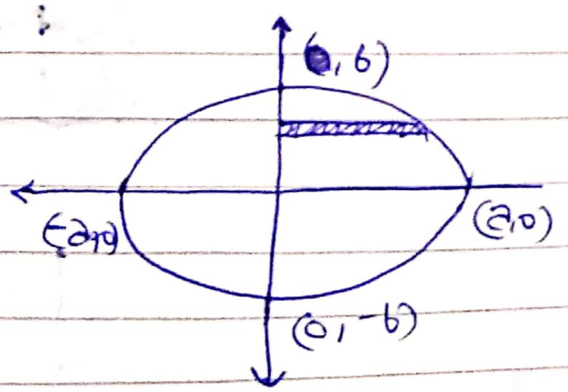
$$= \int_{x=0}^{3} [4x - x^2 - x]\, dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \left[\frac{27}{2} - 9\right]$$

$$= \frac{9}{2}$$

Q8. Find the area of the ellipse using double integration. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a > b$



Ans8. $$= \int_{y=0}^{b} \int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} dx\, dy$$

$$= \int_{y=0}^{b} \left(\frac{a}{b}\sqrt{b^2-y^2}\right) dy$$

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$$\left[\cancel{-\frac{2}{3b}\,a\,\frac{(b^2-y^2)^{3/2}}{2y}}\right]_0^b = \frac{a}{b}\int_{y=0}^{b}\sqrt{b^2-y^2}\,dy$$

$$\frac{a}{b}\left[\frac{y}{2}\sqrt{b^2-y^2}+\frac{b^2}{2}\sin^{-1}\frac{y}{b}\right]_0^b$$

$$\Rightarrow \frac{a}{b}\left[0+\frac{\pi b^2}{4}-0-0\right]=\frac{\pi ab}{4}$$

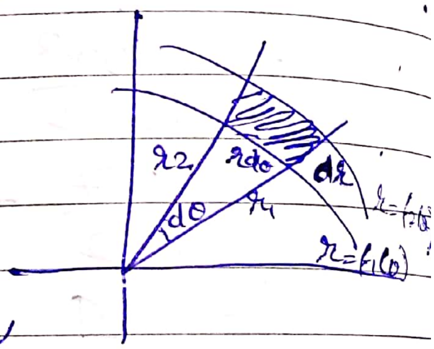
Area of ellipse $= 4\times\frac{\pi ab}{4} = \pi ab$

27.3.19

Double Integral in Polar Coordinates:

Volume integral: $\displaystyle\int_{\theta_1}^{\theta_2}\int_{r=r_1}^{r_2} f(r,\theta)\,dr\,d\theta$

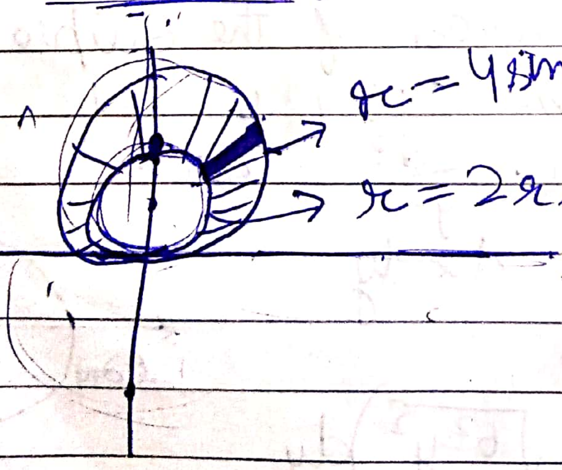
Area integral: $\displaystyle\int_{\theta=\theta_1}^{\theta_2}\int_{r=r_1}^{r_2} r\,dr\,d\theta$



Q1. Calculate double integral $\iint_R r^3\,dr\,d\theta$ ~~over the area~~

where R is the area b/w 2 circles $r=2\sin\theta$, $r=4\sin\theta$

Ans.



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$r = 2\sin\theta$ multiply by $r \Rightarrow r^2 = 2r\sin\theta$
$= 2y$

$x^2+y^2 = 2y$

$x^2+y^2-2y+1 = 1$

$x^2+(y-1)^2 = 1$

$C = (0,1)$ & $R = 1$

& for $r = 4\sin\theta$

$C = (0, \frac{4}{2}) = (0,2)$ & $R = \frac{4}{2} = 2$

$$= \int_0^{\pi} \int_{r=2\sin\theta}^{4\sin\theta} r^3 \, dr \, d\theta$$

$$= \int_0^{\pi} \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \int_0^{\pi} \left[\frac{256\sin^4\theta - 4\sin^4\theta}{4}\right] d\theta$$

$$= \int_0^{\pi} [64\sin^4\theta - 4\sin^4\theta] \, d\theta$$

$$= 60 \int_0^{\pi} \sin^4\theta \, d\theta$$

$$= 60\left[\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x\right]_0^{\pi} = \frac{45\pi}{2}$$

Q2. Evaluate double integral $r^2\sin\theta$ $\iint_R r^2 \sin\theta \, dr \, d\theta$ where R is the area of the cardioid $r = a(1+\cos\theta)$ above the initial line.

$1+\cos\theta = t$
$\sin\theta\, d\theta = dt$

$t^3\, dt$
$\frac{t^4}{4}$

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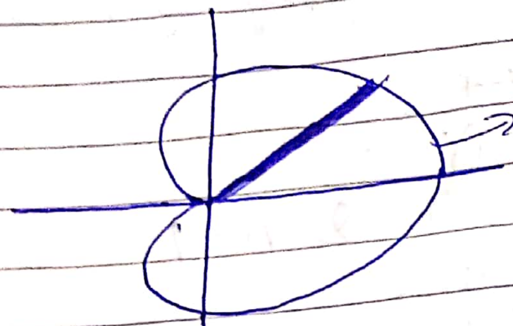
Ans 2.



$$\Rightarrow \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r^2 \sin\theta \, dr \, d\theta$$

$$\Rightarrow \int_{\theta=0}^{\pi} \left[\frac{r^3}{3} \sin\theta\right]_0^{a(1+\cos\theta)} d\theta$$

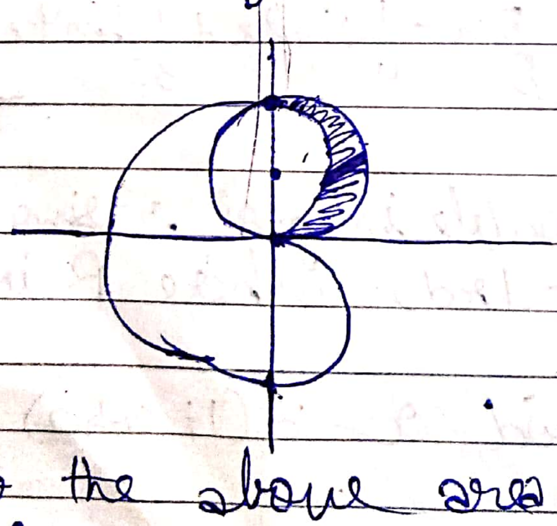
$$\Rightarrow \int_{\theta=0}^{\pi} \frac{a^3(1+\cos\theta)^3}{3} \sin\theta \, d\theta$$

$$= \frac{a^3}{3}\left[\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi}$$

$$= \frac{4a^3}{3}$$

Q3. Find the area lying inside the circle $r = a\sin\theta$ & outside the cardiod $r = a(1-\cos\theta)$ by double integration.

Ans 3.



$e^{x + \log y}$
$e^{x}\, e^{\log y}$

Acc to the above area we have the integral,

$\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta$

$2\cos\theta \sin\theta = \cos 2\theta - 1$

$$= \int_{0}^{\pi/2} \int_{r=a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{a^2\sin^2\theta}{2} - \frac{a^2(1-\cos\theta)^2}{2}\right] d\theta$$

$$= \frac{a^2}{2}\left[2\sin\theta - \frac{\sin 2\theta}{2} - \theta\right]_0^{\pi/2} = a^2 - \frac{a^2\pi}{4}$$

$$= a^2\left[1 - \frac{\pi}{4}\right]$$

Tripple Integration:

$$\int_{x=a}^{b} \int_{y=\psi_1(x)}^{\psi_2(x)} \int_{z=\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

$\iiint dx\,dy\,dz \rightarrow$ representing volume

Q1. $$\int_{0}^{\log 2} \int_{0}^{x} \int_{0}^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$$

$$= \int_{0}^{\log 2} \int_{0}^{x} \int_{0}^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$$

$$= \int \int e^{x+y}\left[ye^{x} - 1\right] dy\,dx$$

$$= \int_{0}^{\log 2} \int_{0}^{x} \left(y e^{2x+y} - e^{x+y}\right) dy\,dx$$

$$= \int_0^{\log 2} \left[e^{2x+y} + y e^{2x+y} - e^{x+y} \right]_0^x dx$$

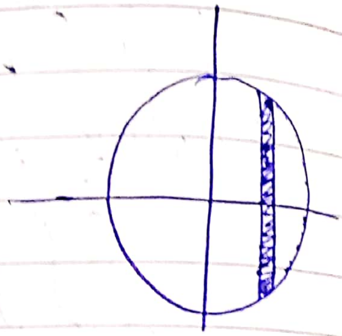
$$= \int_0^{\log 2} \left[e^{3x} + x e^{3x} - e^{2x} - e^{2x} + e^{x} \right] dx$$

$$= \int_0^{\log 2} \left[(1+x) e^{3x} - 2e^{2x} + e^{x} \right] dx$$

$$= \left[e^{3x} + \right.$$

Q2. Find the volume bounded by $x^2 + y^2 = 4$, $y + z = 4$ & $z = 0$

Ans2. $$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{4-y} dz\, dy\, dx.$$



$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-y \right] dy\, dx$$

$$= 2\int_{x=-2}^{2} \left[4y - \frac{y^2}{2} \right]_0^{\sqrt{4-x^2}} dx$$

$$= 2\int_{-2}^{2} \left[8\sqrt{4-x^2} - (4-x^2) \right] dx$$

$$= 32\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} - 16x + \frac{4x^3}{3} \right]_0^2$$

$$= 32\left[\pi - 32 + \frac{32}{3} - 0 \right] = 32\left[\pi - \frac{64}{3} \right]$$

$\sqrt{a^2-(x^2+y^2)}$ Date: / /

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Q3. Find the volume of the sphere $x^2+y^2+z^2=a^2$ using tripple & double integral.

A3. $= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx$

$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[2\sqrt{a^2-x^2-y^2}\right] dy\, dx$

$= 4 \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$

$= 4 \int_{-a}^{a} \left[\frac{x+y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2}{2}\sin^{-1}\frac{x+y}{a}\right]_0^{\sqrt{a^2-x^2}} dx$

$= 8 \int_{0}^{a} \Big[$

Change of Variable using Jacobian:

$$\iint f(x,y)\,dx\,dy \qquad x = f(u) \quad y = f(v)$$

$$\iint f(u,v)\,|J|\,du\,dv$$

$$J = \frac{d(x,y)}{d(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Q1. Evaluate $\iint_R \cos\left(\frac{x-y}{x+y}\right) dx\,dy$ where R is bounded by $x=0$, $y=0$ & $x+y=1$

A1. Put $x-y=u$ $\qquad x = \frac{u+v}{2}$

$x+y=v$ $\qquad y = \frac{v-u}{2}$

$$\Rightarrow \iint_{R'} \cos\left(\frac{u}{v}\right) |J|\,du\,dv$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{2}\iint_{R'} \cos\left(\frac{u}{v}\right) du\,dv$$

when $x=0 \Rightarrow \frac{1}{2}(u+v)=0$

$\Rightarrow u+v=0 \rightarrow (1)$

when $y=0 \Rightarrow \frac{1}{2}(v-u)=0$

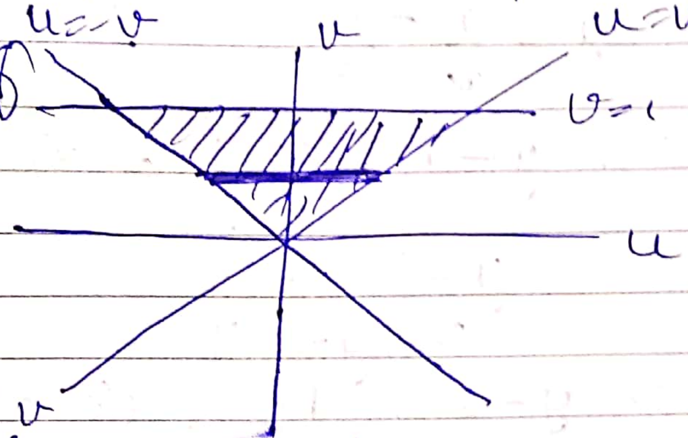
$\Rightarrow v-u=0 \rightarrow (2)$

when $x+y=1 \Rightarrow \frac{1}{2}(u+v)+\frac{1}{2}(v-u)=1$

$\Rightarrow v=1 \rightarrow (3)$

$\Rightarrow R'$ is the region bounded by $u+v=0$, $v-u=0$ & $v=1$

$\Rightarrow \iint$



$$\Rightarrow \frac{1}{2}\int_{v=0}^{1}\int_{u=-v}^{v} \cos\left(\frac{u}{v}\right) du\,dv$$

$$= \frac{2}{2}\int_{0}^{1}\left[v \sin\frac{u}{v}\right]_{0}^{v} dv$$

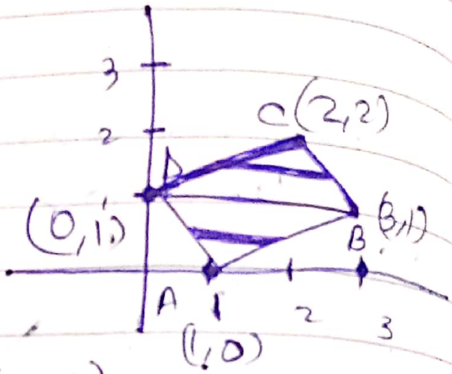
$$= \cancel{\int_{0}^{1}\left[\frac{\sin 1}{v} - 0\right] dv} = \int_{0}^{1} v\sin 1\, dv$$

$$= \cancel{\left[\sin 1 \log v\right]_{0}^{1}} = \left[\frac{v^2}{2}\sin 1\right]_{0}^{1}$$

$= \frac{1}{2}\sin 1 - 0$

$= \frac{1}{2}\sin 1$

Q2- Evaluate $\iint_R (x+y)^2\,dx\,dy$ where R is the parallelogram in x-y plane with vertices (1,0), (0,1), (3,1), (2,2) using the transformation $u = x+y$ & $v = x-2y$.



Ans2· Line eq^{n}s

$AB \Rightarrow y - 0 = \frac{1-0}{3-1}(x-1)$

$2y = \frac{x}{} - \frac{1}{}$

$BC \Rightarrow y - 1 = \frac{2-1}{2-3}(x-3)$

$y - 1 = -(x-3)$

$x + y = 4$

$CD \Rightarrow y - 2 = \frac{1-2}{0-2}(x-2)$

$2y - 4 = x - 2 \Rightarrow x - 2y + 2 = 0$

$DA \Rightarrow y - 0 = \frac{1-0}{0-1}(x-1)$

$x + y = 1$

$\text{area} = \int_{y=0}^{1}\int_{x=1-y}^{2y+1}(x+y^2)\,dx\,dy + \int_{y=1}^{2}\int_{x=2y-2}^{4-y}(x+y^2)\,dx\,dy$

$= \int_{y=0}^{1} 3y^3 + \frac{3}{2}y^2 + 3y = \left[\frac{3y^4}{4} + \frac{y^3}{2} + \frac{3y^2}{2}\right]_0^1 = \frac{11}{4}$

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Del operator:

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$\nabla f \rightarrow$ Gradient f
$\nabla . \vec{F} \rightarrow$ divergence $\vec{F}$
$\vec{\nabla} \times F \rightarrow$ Curl $\vec{F}$

→ Gradient f is a ~~vector~~ ~~scalar~~ vector normal to the surface f & has a magnitude equal to the rate of change of f along this normal. Gradient f gives the maximum rate of change of f.

→ Divergence of the fluid velocity f represents the rate of fluid flow through unit volume. It is known as the fluid flux.

→ curl of any vector point func^n f Gives the ~~magn~~ measure of the angular velocity at any point of the vector field.

⇒ if $\nabla . \vec{F} = 0 \rightarrow$ Solenoidal or incompressible

⇒ Similarly if $\vec{\nabla} \times \vec{F} = 0 \rightarrow$ Irrotational or conservative

⇒ if F is irrotational, there exist a scalar point func^n ϕ such that

$$\vec{F} = \nabla \phi \quad \text{(Gradient)}$$

ϕ is called the scalar potential of F.

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Q1. Show that $\bar{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3z^2x\hat{k}$ is conservative vector field. Find its scalar potential.

Ans 1. $\bar{\nabla} \times \bar{F} = 0$ (for conservative)

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2 & 3z^2x \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial(3z^2x)}{\partial y} - \frac{\partial(x^2)}{\partial z}\right] + \hat{j}\left[\frac{\partial(2xy+z^3)}{\partial z} - \frac{\partial(3z^2x)}{\partial x}\right] + \hat{k}\left[\frac{\partial x^2}{\partial x} - \frac{\partial(2xy+z^3)}{\partial y}\right]$$

$= 0$

Also $\bar{F} = \nabla\phi$

$$\Rightarrow (2xy+z^3)\hat{i} + x^2\hat{j} + 3z^2x\hat{k} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

On comparing the above eq^n we get,

$$\frac{\partial\phi}{\partial x} = 2xy + z^3 \rightarrow (1)$$

$$\frac{\partial\phi}{\partial y} = x^2 \rightarrow (2)$$

$$\frac{\partial\phi}{\partial z} = 3z^2x \rightarrow (3)$$

$\Rightarrow$ Integrate eq^n (1) w.r.t x.

$$\phi = x^2y + z^3x + f(y, z) \rightarrow (4)$$

diff. above eq^n partially w.r.t y

$$\frac{\partial\phi}{\partial y} = x^2 + \frac{\partial f}{\partial y} \rightarrow (5)$$

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On comparing eq^n (5) & (2) we get,

$$\frac{\partial f}{\partial y} = 0$$

$$\Rightarrow f = \psi(z)$$

Again diff. eq^n (4) w.r.t z

$$\frac{\partial \phi}{\partial z} = 3z^2 x + \frac{d\psi}{dz} \rightarrow (6)$$

On comparing eq^n (3) & (6) we get

$$\frac{d\psi}{dz} = 0$$

$$\Rightarrow \psi = C$$

Hence $\boxed{\phi = x^2 y + z^3 x + \psi(z)}$ (where $\psi(z) = C$)

Q2. Find the values of λ & μ so that the surface $\lambda x^2 - \mu yz = (\lambda+2)x$ & $4x^2y + z^3 = 4$ intersect orthogonally at point $(1,-1,2)$

Ans2. $f \Rightarrow \lambda x^2 - \mu yz = (\lambda + z)x$

$g \Rightarrow 4x^2y + z^3 = 4$

$\nabla f \cdot \nabla g = 0$

$\nabla f \Rightarrow (2\lambda x - \lambda + z)\hat{i} + (-\mu z)\hat{j} + (-\mu y$

$\nabla g \Rightarrow 8xy\,\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

$\nabla f \cdot \nabla g = 0 \Rightarrow -2\lambda + \mu - 3 = 0 \rightarrow (1)$

Satisfy point $(1,-1,2)$ in func^n f

$\lambda + 2\mu - \lambda - 2 = 0$

$\Rightarrow \boxed{\mu = 1}$

Put value of μ in eq^n (I)

$\Rightarrow -2\lambda + 1 - 3 = 0$

$-2\lambda - 2 = 0$

$\boxed{\lambda = -1}$

Q3. Prove that the divergence of curl $\bar{F}$ is equal to 0.

Ans 3. div. (Curl $\bar{F}$) $= 0$

Let $\bar{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\text{Curl } \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\Rightarrow \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \hat{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\text{div. (Curl } F) = \cancel{\frac{\partial^2 f_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 f_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 f_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 f_1}{\partial y \partial z}} + \cancel{\frac{\partial^2 f_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 f_1}{\partial z \partial y}} = 0$$

$-4+1$

Similarly curl $(\nabla\phi) = 0$

Directional Derivatives:

Directional derivative of f in the dir^n^ is equal to,

$$\boxed{\nabla f \cdot \frac{\vec{a}}{|\vec{a}|}} \quad \text{or} \quad \boxed{\nabla f \cdot \hat{a}}$$

Q4. ~~[illegible]~~ What is the directional derivative of $\phi = xy^2 + yz^3$ at $(2,-1,1)$ in the direction of the normal to the surface $F = x\log z - y^2 - 4$ at $(-1,2,1)$

Ans 4. $\vec{a} = \nabla f$ at $(-1,2,1)$

$= \log z\,\hat{i} - 2y\hat{j} + \frac{x}{z}\hat{k}$ at $(-1,2,1)$

$\vec{a} = -4\hat{j} - \hat{k}$

$|\vec{a}| = \sqrt{(-4)^2 + (-1)^2} = \sqrt{17}$.

dir^n^ derivative of $\phi = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$ at $(2,-1,1)$

$= (y^2\hat{i} + (2xy+z^3)\hat{j} + 3yz^2\hat{k}) \frac{(-4\hat{j}-\hat{k})}{\sqrt{17}}$

$= \frac{[0 - 4(2xy+z^3) - 3yz^2]}{\sqrt{17}}$ at $(2,-1,1)$

$= \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$

$$\boxed{\therefore \text{Ans} = \frac{15}{\sqrt{17}}}$$

Line Integral :

$\int_C \vec{F} \cdot d\vec{r}$

where $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

→ If $\vec{F}$ represents the velocity of fluid then the line integral $\int_C \vec{F} \cdot d\vec{r}$ is called circulation of $\vec{F}$ around the curve C.

→ if $\vec{F}$ represents the force acting on a particle moving along ~~[illegible]~~ an arc AB (from A to B), then the total work done is equal to $\int_A^B \vec{F} \cdot d\vec{r}$

→ if $\vec{F}$ is the gradient of some scalar point funcⁿ ϕ then the line integral of $\vec{F}$ is independent of the path.

$$\boxed{\int_A^B \bar{F} \cdot d\bar{r} = \phi(B) - \phi(A)}$$

Q5 Evaluate line integral $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$

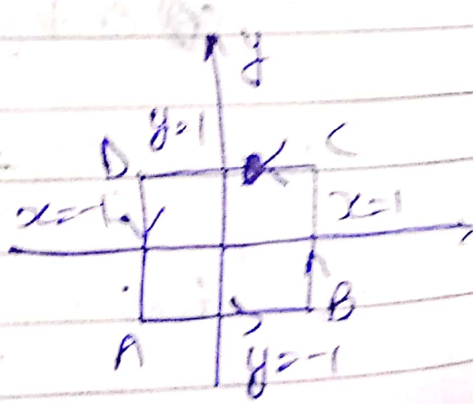
where C is the square formed by the lines $x = \pm 1$ & $y = \pm 1$.

Ans For line AB

$y = -1$

$dy = 0$

Put this in the integral



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$$\Rightarrow \int_{-1}^{1}(x^2-x)\,dx + \int_{-1}^{1}(1+y^2)\,dy + \int_{1}^{-1}(x^2+x)\,dx + \int_{1}^{-1}(1+y^2)\,dy$$

$$\Rightarrow \int_{-1}^{1}(x^2-x)\,dx + \int_{-1}^{1}\cancel{(1+y^2)}\,dy + \int_{1}^{-1}(x^2+x)\,dx - \int_{-1}^{1}\cancel{(1+y^2)}\,dy$$

$$\Rightarrow \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{1} + \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{1}^{-1}$$

$$\Rightarrow \left[\frac{1}{3}-\frac{1}{2}+\frac{1}{3}+\frac{1}{2}\right] + \left[\frac{-1}{3}+\frac{1}{2}-\frac{1}{3}-\frac{1}{2}\right]$$

$$= \frac{2}{3}-\frac{2}{3}=0$$

5. Find the work done in moving a particle in force field $F = 3x^2\hat{i}+(2xz-y)\hat{j}+z\hat{k}$ along the line A $(0,0,0)$ to B$(2,1,3)$

Sol. $W = \int_A^B 3x^2\,dx + (2xz-y)\,dy + z\,dz$

Eq^n of line

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$\Rightarrow W = \int_{t=0}^{1} (24t^2 + 12t^2 - t + 9t)\,dt$

because when $x = 2t$
$y = t$
$z = 3t$

$\Rightarrow$ when $t = 0$ then $(x, y, z) = (0,0,0)$
$\Rightarrow$ when $t = 1$ then $(x, y, z) = (2,1,3)$
hence limit of $t = 0$ to 1.

$$W = \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_0^1$$

$$W = [12 + 4 - 0]$$

$$W = 16$$

4.4.19 **Surface Integral:**

$$\iint_S \vec{F}.\hat{n}.dS$$

where $\hat{n} \rightarrow$ unit vector normal to the outw plane.

- X-Y plane:

$$dS = \frac{dx\,dy}{|\hat{n}.\hat{k}|}$$

- Y-Z plane:

$$dS = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

z-x Plane:

$$dS = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

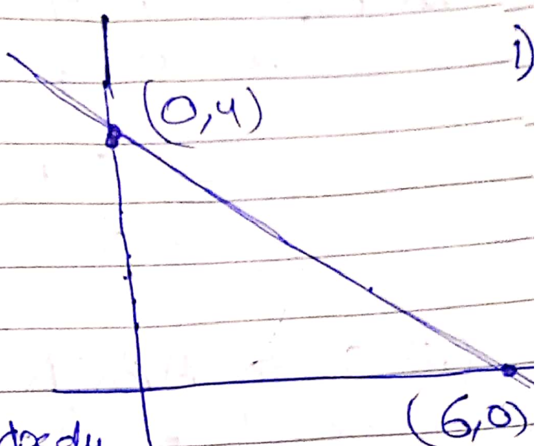
Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$ where $F = 6z\hat{i} - 4\hat{j} + y\hat{k}$ & S is the portion of $2x+3y+6z=12$ in x-y plane.

Sol. For x-y plane

$z=0$

$\Rightarrow 2x+3y=12$

or $\frac{x}{6}+\frac{y}{4}=1$



$\hat{n} = \hat{k}$ so, $dS = dx\,dy$

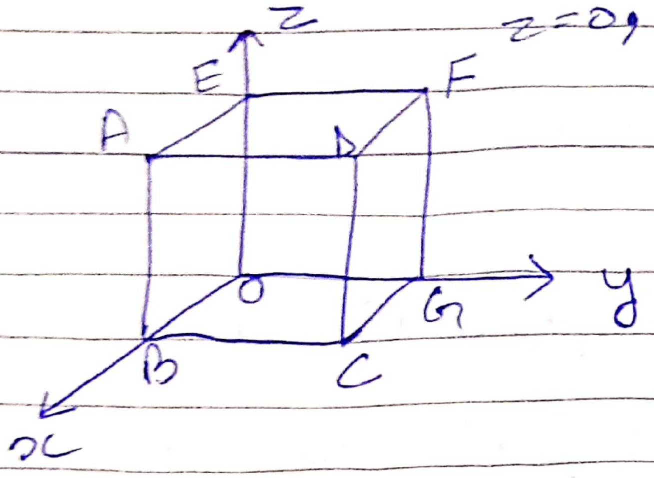
$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dS = \int_{y=0}^{4}\int_{x=0}^{\frac{12-3y}{2}} y\,dx\,dy$$

$$= \int_0^4 y\left[\frac{12-3y}{2}\right]dy$$

$$= \int_0^4 \left(6y - \frac{3}{2}y^2\right)dy$$

$$= \left[\frac{6y^2}{2} - \frac{y^3}{2}\right]_0^4 = 16$$

Q2. Evaluate $\iint_S F\cdot \hat{n}\,dS$ where S is the surface of cube bounded by $x=0,\ x=1$
$F = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ $\quad y=0,\ y=1$
$z=0,\ z=1$



Ans (i) $S_1 \rightarrow OEFG$

$x=0,\quad \hat{n}=-i,\quad dS = dy\,dz$

$$= -\iint_{S_1} 4xz\,dy\,dz = 0 \quad (\text{because } x=0)$$

(ii) $S_2 \rightarrow ABCD$

$x=1,\quad \hat{n}=\hat{i},\quad dS = dy\,dz$

$$\Rightarrow \iint_{S_2} 4z\,dy\,dz = \int_0^1\int_0^1 4z\,dy\,dz = 2$$

(iii) $S_3 \rightarrow OBAE$

$y=0,\quad \hat{n}=-\hat{j},\quad ds = dx\,dz$

$$\Rightarrow \iint_{S_3} y^2\,dx\,dz = 0 \quad (\text{because } y=0)$$

(iv) $S_4 \rightarrow DFGC$

$y=1,\quad \hat{n}=\hat{j},\quad dS = dx\,dz$

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$$= -\iint_{S_4} y^2 \, dx\,dz = -\int_0^1\int_0^1 dx\,dz = -1$$

v) $S_5 \rightarrow OBCG$

$z=0$, $\hat{n} = -\hat{k}$, $dS = dx\,dy$

$$\Rightarrow -\iint_{S_5} yz \, dx\,dy = 0 \text{ (because } z=0\text{)}$$

vi) $S_6 \rightarrow ADFE$

$z=1$, $\hat{n} = \hat{k}$, $dS = dx\,dy$

$$\Rightarrow \iint_{S_6} yz\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy \quad (z=1)$$

$$\Rightarrow \frac{1}{2}$$

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dS = 2-1+\frac{1}{2} = \frac{3}{2}$$

Volume Integral: $\iiint_V \vec{F}\cdot dV$

If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ then evaluate $\iiint_V \operatorname{div}\vec{F}\,dV$ where V is bounded by the plane $x=0, y=0, z=0$ & $2x+2y+z=4$

$$\Rightarrow \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0} (4x - 2x)\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} (2x)\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} \Big[2xz\Big]_0^{4-2x-2y} dy\,dx$$

$$= \int_0^2\int_0^{2-x} 2x(4-2x-2y)\,dy\,dx$$

$$= \int_0^2 \Big[8xy - 4x^2y - 2xy^2\Big]_0^{2-x} dx$$

$$= \int_0^2 2x^3 - 8x^2 + 8x\,dx$$

$$= \left[\frac{x^4}{2} - \frac{8}{3}x^3 + 4x^2\right]_0^2 = \frac{8}{3}$$

Green's Theorem:

If $\phi(x,y)$, $\psi(x,y)$, ψ_x & ϕ_y be continuous in the region E of the x-y plane bounded by a closed curve C, then

$$\int_C \phi\,dx + \psi\,dy = \iint_E \left(\frac{d\psi}{dx} - \frac{d\phi}{dy}\right) dx\,dy$$

Stoke's Theorem:

If S is an open surface bounded by a closed curve C & $\vec{F}$ be any continuous differential vector point function then,

$$\int_C \vec{F}\cdot\vec{dr} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS$$

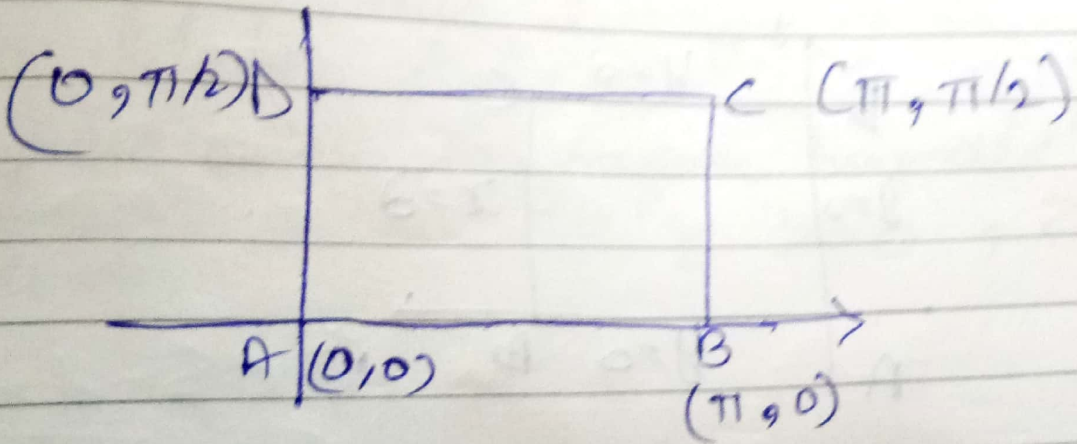
where $\hat{n}$ = unit outward normal vector.

Gauss Divergence Theorem:

If $\vec{F}$ is any continuous, differentiable vector point function then,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$$

Q. Evaluate by Green's Theorem $\int_C e^{-x}(\sin y\,dx + \cos y\,dy)$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(0,\pi/2)$, $(\pi,\pi/2)$



$\phi = e^{-x}\sin y$ $\qquad \psi = e^{-x}\cos y$

$$= \int_{y=0}^{\pi/2}\int_{x=0}^{\pi} -2e^{-x}\cos y\,dx\,dy$$

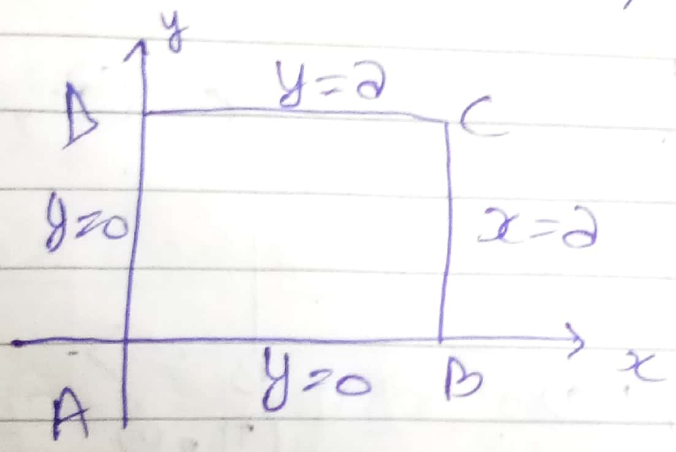
Q2. If S is any closed surface then prove that $\iint_S \text{curl}\, \vec{F}\cdot\hat{n}\, dS = 0$

Ans2. $\iint_S \text{curl}\, \vec{F}\cdot\hat{n}\, dS = \iiint_V \text{div}\,(\text{curl}\,\vec{F})\, dV = 0$

(using gauss divergence theorem)

Q3. Verify Stoke's theorem for the function $\bar{F} = x^2\hat{i} + xy\hat{j}$ integrated around the square in the plane $z=0$ & bounded by the lines $x=0$, $x=a$, $y=0$, $y=a$.

Ans 3.



Stoke's theorem :

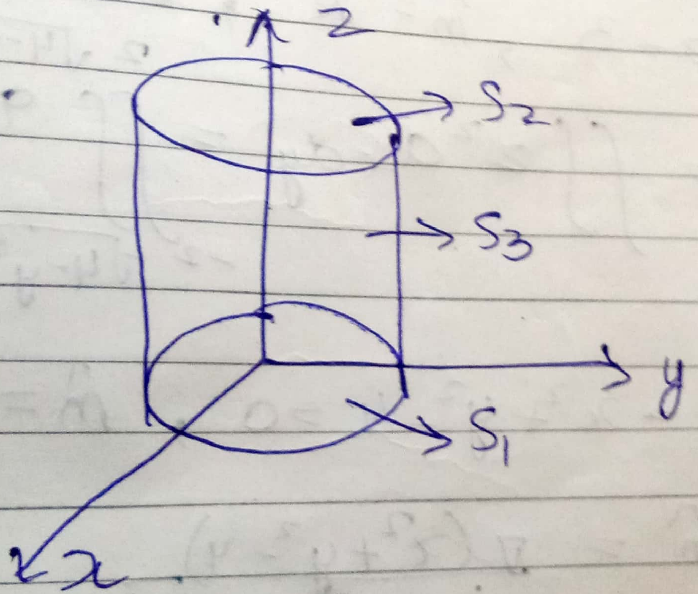
$$\int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\, \bar{F}\cdot\hat{n}\, dS$$

$$L.H.S = \int_{AB} + \int_{BC} + \int_{CA} + \int_{AD}$$

$$= \int_{y=0}^{2} \int_{x=0}^{2} (\text{curl}\,\vec{F}) \cdot \hat{k}\, dx\, dy$$

Q4. Verify Gauss divergence theorem for the vector field $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the ~~sphere~~ cylinder $x^2+y^2=4$, $z=0$, $z=3$.

Sol.



Gauss divergence theorem

$$\iint \vec{F} \cdot \hat{n}\, dS = \iiint_V \text{div.}\, \vec{F}\, dV$$

R.H.S.:

$$\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)\,dz\,dy\,dx$$

L.H.S.:

S_1 : $z=0$, $\hat{m}=-\hat{k}$, $dS=dx\,dy$

$$\iint_{S_1} = -\iint z^2\,dx\,dy = 0 \quad (\text{as } z=0)$$

S_2 : $z=3$, $\hat{m}=\hat{k}$, $dS=dx\,dy$

$$\iint_{S_2} = \iint z^2\,dx\,dy = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 9\,dx\,dy$$

S_3 : $f=x^2+y^2-4=0$, $\hat{m}=\dfrac{\nabla f}{|\nabla f|}$

$$\hat{m} = \frac{\nabla(x^2+y^2-4)}{|\nabla(x^2+y^2-4)|}$$

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another way of S_3:

$x = 2\cos\theta$

$y = 2\sin\theta$

dot product of f & $\hat{n}$

$$\iint_{S_3} (2x^2 - y^3)\, dS_3$$

$$\Rightarrow \int_{z=0}^{3} \int_{\theta=0}^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\, 2\, d\theta\, dz$$