1.3. OPEN LOOP CONTROL SYSTEM

The open loop control system is also known as control system without feedback or non feedback control systems. In open loop systems the control action is independent of the desired output. In this system the output is not compared with the reference input.

The component of the open loop systems are controller and controlled process. The controller may be amplifier, filter etc., depends upon the system. An input is applied to the controller and the output of the controller gives to the controlled process fand we get the output (desired).

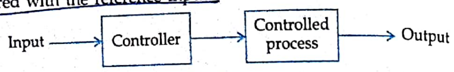


Fig. 1.5.

Examples :

1. Automatic washing machine is the example of the open loop systems. In the machine the operating time is set manually. After the completion of set time the machine will stops, with the result we may or may not get the desired (output) amount of cleanliness of washed cloths because there is no feedback is provided to the machine for desired output.

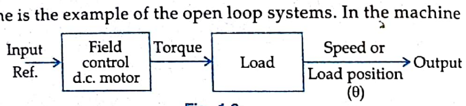


Fig. 1.6.

2. Immersion rod is another example of open loop system. The rod heats the water but how much heating is required is not sense by the rod because of no feedback to the rod.
3. A field control d.c. motor is the example of open loop system.
4. For automatic control of traffic the lamps of three different colours (red, yellow and green) are used. The time for each lamp is fixed. The operation of each lamp does not depends upon the density of the traffic but depends upon the fixed time. Thus, we can say that the control system which operates on the time basis is open loop system.

Advantages :
1. Open loop control systems are simple.
2. Open loop control systems are economical.
3. Less maintenance is required and not difficult.
4. Proper calibration is not a problem.

Disadvantages :
1. Open loop systems are inaccurate.
2. These are not reliable.
3. These are slow.
4. Optimization is not possible.

1.4. CLOSED LOOP CONTROL SYSTEM

Closed loop control systems are also known as feedback control systems. In closed loop control systems the control action is dependent on the desired output. If any system having one or more feedback paths forming a closed loop system.

In closed loop systems the output is compared with the reference input and error signal is produced. The error signal is fed to the controller to reduce the error and desired output is obtained.

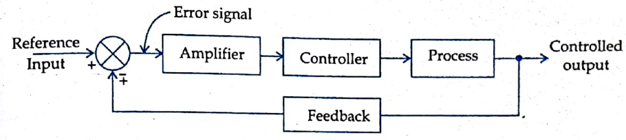


Fig. 1.7.

Example : In a room we need to regulate the temperature and humidity for comfortable living. Air-conditioners are provided with thermostat. By measuring the actual room temperature and compared it with desired temperature, an error signal is produced, the thermostat turns ON the compressor or OFF the compressor. The block diagram is shown in Fig. 1.8.

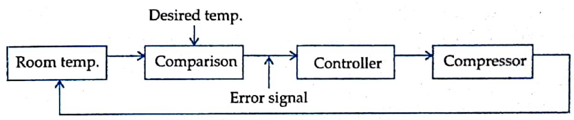


Fig. 1.8.

Advantages :
1. These systems are more reliable.
2. Closed loop systems are faster.
3. A number of variables can be handled simultaneously.
4. Optimization is possible.

Disadvantages :
1. Closed loop systems are expensive.
2. Maintenance difficult.
3. Complicated installation.

1.5. COMPARISON BETWEEN OPEN LOOP AND CLOSED LOOP

Table 1.1

S.No	Open Loop System	Closed Loop System
1.	These are not reliable.	These are reliable.
2.	It is easier to build.	It is difficult to build.
3.	If calibration is good, they perform accurately.	They are accurate because of feedback.
4.	Open loop systems are generally more stable.	These are less stable.
5.	Optimization is not possible.	Optimization is possible.

1.6. ELEMENTS OR COMPONENTS OF CLOSED LOOP SYSTEMS

The various components of closed loop system are shown in Fig. 1.9

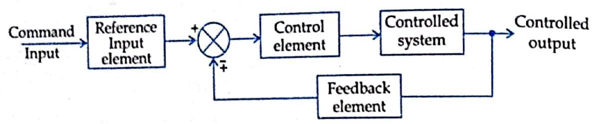


Fig. 1.9.

Command : The command is the externally produced input and independent of the feedback control system.

Reference input element : This produces the standard signals proportional to the command.

Error detector : The error detector receives the measured signal and compare it with reference input. The difference of two signals produces the error signal.

Control element : This regulates the output according to the signal obtained from error detector.

Controlled system : This represents what we are controlling by the feedback loop.

Feedback element : This element fed back the output to the error detector for comparison with the reference input.

TRANSFER FUNCTION

1.7. TRANSFER FUNCTION FOR SINGLE INPUT SINGLE OUTPUT SYSTEM

The transfer function is defined as the ratio of Laplace transform of the output to the Laplace transform of input with all initial conditions are zero.

Consider a linear system having input $r(t)$ and $c(t)$ is the output of the system, the input-output relation can be described by the following n^{th} order differential equation :

$$a_n\frac{d^nc(t)}{dt^n}+a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}}+.....+a_1\frac{dc(t)}{dt}+a_0c(t) = b_m\frac{d^mr(t)}{dt^m}+b_{m-1}\frac{d^{m-1}r(t)}{dt^{m-1}}+.....+b_1\frac{dr(t)}{dt}+b_0r(t) \qquad ...(1.1)$$

where 'a' and 'b' are constants.

Take the Laplace transform of equation (1.1)

$$(a_ns^n + a_{n-1}s^{n-1} + ... + a_1s + a_0)\, C(s) = (b_m s^m + b_{m-1}s^{m-1} + \; ..... + b_1s+b_0)R(s) \qquad ...(1.2)$$

We can define the transfer function as

$$G(s) = \frac{C(s)}{R(s)}$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{(b_ms^m+b_{m-1}s^{m-1}+...+b_1s+b_0)}{(a_ns^n+a_{n-1}s^{n-1}+...+a_1s+a_0)} \qquad ...(1.3)$$

In equation (1.3), if the order of the denominator polynomial is greater than the order of the numerator polynomial then the transfer function is said to be STRICTLY PROPER. If the order of both polynomials are same, then the transfer function is PROPER. The transfer function is said to be IMPROPER, if the order of numerator polynomial is greater than the order of denominator polynomial.

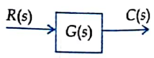


Fig. 1.10

Consider the block diagram of open loop control system Fig. 1.10 where $R(s)$ and $C(s)$ are the Laplace transform of input and output respectively, then the transfer function $G(s)$ can be expressed as

$$G(s) = \frac{C(s)}{R(s)} = \frac{£[c(t)]}{£[r(t)]} \qquad ...(1.4)$$

1.8. PROCEDURE

The following steps are involved to obtain the transfer function of the given system :

Step 1 : Write the differential equations for the given system.

Step 2 : Take the laplace transform of the equations obtained in step 1, with assumption; all initial conditions are zero.

Step 3 : Take the ratio of transformed output to input.

Step 4 : The ratio of transformed output to the input, obtained in step 3 is the required transfer function of the given system.

1.9. CHARACTERISTIC EQUATION OF A TRANSFER FUNCTION

The characteristic equation of a linear system can be obtained by equating the denominator polynomial of the transfer function to zero. Thus, the characteristic equation of the transfer function of equation (1.3) will be

$$a_ns^n + a_{n-1}s^{n-1} + ... + a_1s + a_0 = 0 \qquad ...(1.5)$$

1.10. POLES AND ZEROS OF A TRANSFER FUNCTION

Consider the equation (1.3), the numerator and denominator can be factored in m and n terms respectively, then the equation (1.3) can be expressed as

$$\frac{C(s)}{R(s)} = G(s) = \frac{K(s+z_1)(s+z_2)\left(as^2+bs+c\right)}{(s+p_1)(s+p_2)\left(As^2+Bs+C\right)} \qquad ...(1.6)$$

where $K = \frac{b_m}{a_n}$ is known as the gain factor, s is the complex frequency.

POLES : The poles of G(s) are those values of 's' which make G(s) tend to infinity. For example in equation (1.6) we have poles at $s = -p_1, s = -p_2$ and a pair of poles at

$$s = \frac{-B \pm \sqrt{B^2-4AC}}{2A} \qquad ...(1.7)$$

ZEROS : The zeros of G(s) are those values of 's' which make G(s) tend to zero. For example in eq. (1.6) we have zeros at $s_1 = -z_1, s_2 = -z_2$ and a pair of zeros at

$$s = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad ...(1.8)$$

If either poles or zeros coincide, then such type of poles or zeros are called multiple poles or multiple zeros, otherwise they are known as simple poles or simple zeros. Multiple poles are due to the repetitive factor in denominator and multiple zeros are due to the repetitive factor in numerator of a transfer function.

For example, consider the transfer function

$$G(s) = \frac{50(s+3)}{s(s+2)(s+4)^2} \quad \text{...(1.9)}$$

The above transfer function having the simple poles at $s = 0$, $s = -2$, multiple poles at $s = -4$ *i.e.* the pole of order 2 at $s = -4$ and simple zero at $s = -3$.

The above, mentioned poles and zeros are of finite values. If we consider the entire 's' plane including infinity then two cases arises.

1. If the no. of zeros are less than the no of poles *i.e.*, $Z < P$ then the value of the transfer function becomes zero for $s \to \infty$. Hence we can say that there are zeros at infinity and the order of such zeros is $P - Z$. For example in equation (1.9) there are four finite poles at $s = 0, -2, -4$ and -4; there is one finite zero at $s = -3$ but there are three zeros at infinity $P - Z = 4 - 1 = 3$. Therefore, the function has a total of four poles and four zeros in the entire s-plane including infinity.
2. If the no. of poles are less than the no. of zeros $P < Z$ then the value of the transfer function becomes infinity for $s \to \infty$. Hence we can say there are poles at infinity ($s \to \infty$) and the order of the poles will be $Z - P$.

Therefore, in addition to finite poles and zeros, if we consider poles and zeros at infinity, then for a rational function the total number of zeros will be equal to the total number of poles.

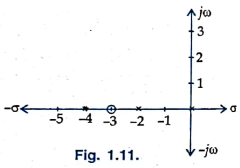


Fig. 1.11.

The pole is represented by 'X' and zero by 'O'. These symbols are used to locate the poles and zeros on s-plane. The pole-zero plot of equation (1.9) is shown in Fig. 1.11.

1.11. ADVANTAGES OF TRANSFER FUNCTION

1. Transfer function is a mathematical model of all system components and hence of the overall system and therefore it gives the gain of the system.
2. Since Laplace transform is used, it converts time domain equations to simple algebraic equations.
3. By the transfer function, poles, zeros and characteristic equation can be determine.
4. It helps in the study of stability of the system.
5. If transfer function is known, output response for any type of input can be determined easily.
6. The differential equation of the system can be obtained by replacing 's' with $\frac{d}{dt}$.
7. Transfer function does not depends on the input to the system.

1.12. DISADVANTAGES OF TRANSFER FUNCTIONS

1. Transfer function cannot be defined for non-linear system.
2. Transfer function is defined only for linear system.
3. From the transfer function, physical structure of a system cannot determined.
4. Initial conditions loose their importance.

1.13. IMPULSE RESPONSE

From equation (1.4) $\quad C(s) = R(s)\,G(s) \quad$ ...(1.10)

Suppose, a system is subjected to a unit impulse, then the output will be

$$C(s) = G(s) \quad \text{...(1.11)}$$

Because the Laplace transform of unit impulse function is unity.

The inverse Laplace of equation (1.11) will be

$$C(t) = g(t) \quad \text{...(1.12)}$$

where $g(t)$ is the unit impulse response of a system.

Therefore inverse Laplace of $G(s)$ is called impulse response or the transfer function of a system is the laplace transform of its impulse response.

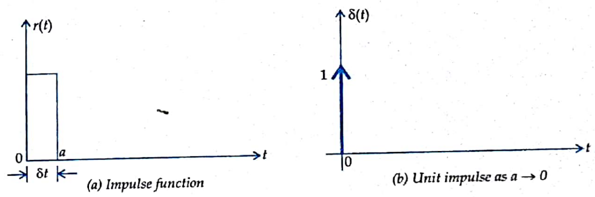


Fig. 1.12.

Practically, it is not possible to generate a true impulse. A pulse with less duration than the time constant of the system can be considered as an impulse and denoted by $\delta(t)$.

EXAMPLE 1.1. **Find the transfer function of the given network**

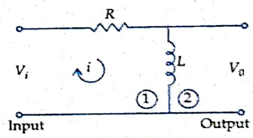


Fig. 1.13.

Solution : Step 1 : Apply KVL in mesh (1)

$$V_i = Ri + L\frac{di}{dt} \quad \text{...(1.13)}$$

Apply *KVL* in mesh (2)

$$V_0 = L\frac{di}{dt} \quad \text{...(1.14)}$$

Step 2 : Take Laplace transform of equations (1.13) and (1.14) with assumption that all initial conditions are zero.

$$V_i(s) = RI(s) + sLI(s) \quad \text{...(1.15)}$$

$$V_0(s) = sLI\,(s) \quad \text{...(1.16)}$$

Step 3 : Calculation of transfer function

$$\frac{V_0(s)}{V_i(s)} = \frac{sLI(s)}{(R+sL)\,I(s)}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{sL}{R+sL} \quad \text{...(1.17)}$$

Equation 1.17 is the required transfer function.

EXAMPLE 1.2. **Determine the transfer function of the electrical network shown in Fig. 1.14.**

Solution : Step 1 : Apply KVL in both meshes

$$E_i = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt \qquad ...(1.18)$$

$$E_0 = \frac{1}{C}\int idt \qquad ...(1.19)$$

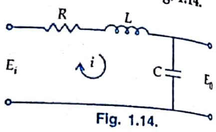


Fig. 1.14.

Step 2 : Take Laplace transform of equations (1.18) and (1.19)

$$E_i(s) = RI(s) + sLI(s) + \frac{1}{Cs}I(s) = I(s)\left[R + sL + \frac{1}{Cs}\right]$$

$$E_i(s) = I(s)\left[\frac{RCs + s^2LC + 1}{Cs}\right] \qquad ...(1.20)$$

$$E_0(s) = \frac{1}{Cs}I(s) \qquad ...(1.21)$$

Step 3 : Determination of transfer function

$$\frac{E_0(s)}{E_i(s)} = \frac{I(s)}{Cs}\cdot\frac{Cs}{I(s)\left[s^2Lc + sRC + 1\right]}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{s^2LC + sRC + 1} \qquad \textbf{Ans.} \qquad ...(1.22)$$

EXAMPLE 1.3. **Obtain the transfer function $\frac{V_2(s)}{V_1(s)}$ for Fig. 1.15.**

Solution : Step 1 : KCL at node '*a*'

$$i = i_1 + i_2 \qquad ...(1.23)$$

$$i_1 = \frac{V_1 - V_2}{R_1}$$

$$i_2 = C\frac{d}{dt}(V_1 - V_2)$$

$$i = i_3 = \frac{V_2}{R_2}$$

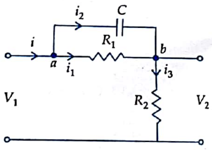


Fig. 1.15.

Put all these values in equation (1.23)

$$\frac{V_2}{R_2} = \frac{V_1 - V_2}{R_1} + C\frac{d}{dt}(V_1 - V_2) \qquad ...(1.24)$$

Step 2 : Take Laplace transform of equation (1.24)

$$\frac{V_2(s)}{R_2} = \frac{1}{R_1}V_1(s) - \frac{1}{R_1}V_2(s) + sC\,V_1(s) - sC\,V_2(s)$$

$$\frac{V_2(s)}{R_2} + \frac{1}{R_1}V_2(S) + sC\,V_2(s) = \frac{1}{R_1}V_1(s) + sC\,V_1(s)$$

$$V_2(s)\left[\frac{1}{R_1} + \frac{1}{R_2} + Cs\right] = V_1(s)\left[\frac{1}{R_1} + Cs\right]$$

Step 3 : Determination of transfer function

$$V_2(s)\left[\frac{R_1 + R_2 + sR_1R_2C}{R_1R_2}\right] = V_1(s)\left[\frac{1 + sCR_1}{R_1}\right]$$

$$\frac{V_2(s)}{V_1(s)} = \frac{R_2 + R_1R_2Cs}{R_1 + R_2 + R_1R_2Cs} \qquad \textbf{Ans.} \qquad ...(1.25)$$

EXAMPLE 1.4. **Find the transfer function of lag network shown in Fig. 1.16.**

Solution : Step 1 : Apply KVL in both meshes

$$e_i(t) = R_1 i(t) + R_2 i(t) + \frac{1}{C}\int i(t)\,dt \qquad ...(1.26)$$

$$e_0(t) = R_2 i(t) + \frac{1}{C}\int i(t)\,dt \qquad ...(1.27)$$

Step 2 : Laplace transform of equation (1.26) and (1.27)

$$E_i(s) = \left[R_1 + R_2 + \frac{1}{sC}\right]I(s)$$

$$E_0(s) = \left[R_2 + \frac{1}{sC}\right]I(s)$$

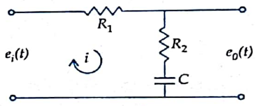


Fig. 1.16

Step 3 : Calculation of transfer function

$$E_0(s) = \frac{\left[R_2 + \frac{1}{sC}\right]I(s)}{\left[\frac{R_1sC + R_2sC + 1}{sC}\right]I(s)}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{1 + R_2sC}{1 + R_1sC + R_2sC} \qquad ...(1.28)$$

Equation (1.28) is the required transfer function.

EXAMPLE 1.5. **Determine the transfer function of Fig. 1.17.**

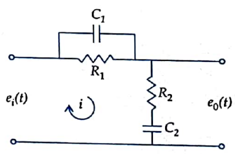


Fig. 1.17.

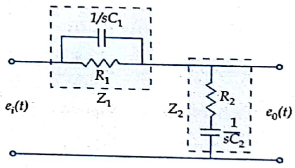


Fig. 1.18.

Solution : Step 1 : Calculation of Z_1 :

$$Z_1 = \frac{R_1 \cdot \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} = \frac{R_1}{R_1 C_1 s + 1} \quad \text{...(1.29)}$$

Step 2 : Calculation of Z_2 :

$$Z_2 = R_2 + \frac{1}{sC_2} = \frac{R_2 C_2 s + 1}{sC_2} \quad \text{...(1.30)}$$

Step 3 : Calculation of transfer function in terms of Z_1 and Z_2

$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \quad \text{...(1.31)}$$

Step 4 : Calculation of transfer function in terms of R_1, R_2, C_1 and C_2. Put the values of $Z_1(s)$ and $Z_2(s)$ from equations (1.29) & 1.30 in equation (1.31)

$$\frac{E_0(s)}{E_i(s)} = \frac{(1 + R_2 C_2 s)/sC_2}{\frac{R_1}{C_1 R_1 s + 1} + \frac{R_2 C_2 s + 1}{sC_2}}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{(1 + R_1 C_1 s)(1 + R_2 C_2 s)}{(1 + R_1 C_1 s)(1 + R_2 C_2 s) + R_1 C_2 s} \quad \text{...(1.32)}$$

The above equation is the required transfer function of the given circuit.

EXAMPLE 1.6. Determine the transfer function of given transformer coupled circuit (Fig. 1.19).

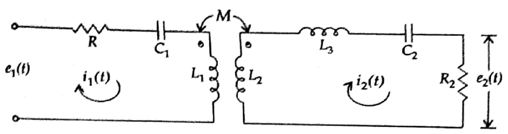


Fig. 1.19.

Solution : Apply KVL in both meshes

$$e_i(t) = Ri_1(t) + \frac{1}{C_1}\int i_1\, dt + L_1 \frac{di_1(t)}{dt} - M\frac{di_2(t)}{dt} \quad \text{...(1.33)}$$

$$e_2(t) = R_2 i_2(t) \quad \text{...(1.34)}$$

$$0 = R_2 i_2(t) + (L_3 + L_2)\frac{d}{dt} i_2(t) + \frac{1}{C_2}\int i_2(t)dt - M\frac{d}{dt} i_1(t) \quad \text{...(1.35)}$$

Take Laplace transform of equation (1.33), (1.34) and (1.35)

$$E_1(s) = I_1(s)\left[R + \frac{1}{C_1 s} + SL_1\right] - sMI_2(s) \quad \text{...(1.36)}$$

$$E_2(s) = R_2 I_2(s) \quad \text{...(1.37)}$$

$$0 = I_2(s)[R_2 + s(L_2 + L_3) + 1/sC_2] - s\,MI_1(s) \quad \text{...(1.38)}$$

Solving the equations (1.36), (1.37) and (1.38). The required transfer function.

$$G(s) = \frac{s^3 R_2 C_1 C_2 M}{\left[sR_2C_2 + s^2C_2(L_2 + L_3) + 1\right]\left[s^2L_1C_1 + sC_1R_1 + 1\right] - M^2 s^4 C_1 C_2} \quad \text{Ans.}$$

EXAMPLE 1.7. A system having input $x(t)$ and output $y(t)$ is represented by,

$$\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + 5x(t)$$

Find the transfer function of the system.

Solution : Given equation,

$$\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + 5x(t)$$

Taking Laplace transform

$$sY(s) + 4Y(s) = sX(s) + 5X(s)$$

$$Y(s)\,(s+4) = X(s)\,(s+5)$$

$$\frac{Y(s)}{X(s)} = \frac{s+5}{s+4}$$

$\therefore$ Transfer function of the system $G(s) = \dfrac{Y(s)}{X(s)} = \dfrac{s+5}{s+4}$

EXAMPLE 1.8. The transfer function of the system is given by,

$$G(s) = \frac{4s+1}{s^2 + 2s + 3}$$

Find the differential equation of the system having input $x(t)$ and output $y(t)$.

Solution : $$G(s) = \frac{Y(s)}{X(s)} = \frac{4s+1}{s^2+2s+3}$$

$$X(s)\,[4s + 1] = Y(s)[s^2 + 2s + 3]$$

$$4sX(s) + X(s) = s^2Y(s) + 2sY(s) + 3Y(s)$$

Taking inverse Laplace transform, we have

$$4\frac{dx(t)}{dt} + x(t) = \frac{dy^2(t)}{dt^2} + 2\frac{dy(t)}{dt} + 3y(t)$$

Required differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 3y(t) = 4\frac{dx(t)}{dt} + x(t)$$

1.25. BLOCK DIAGRAM REDUCTION

When a number of blocks are connected, the overall transfer function can be obtained by block diagram reduction technique. The following rules are associated with the block reduction technique.

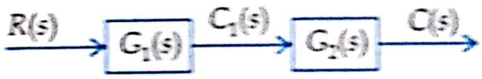


Fig. 1.90.

Rule No. 1. Blocks in Cascade

When two or more blocks are in cascade, the resultant block is a product of the individual block transfer function. Consider the two blocks are in cascade shown in Fig. 1.90.

From Fig. 1.90

$$\frac{C_1(s)}{R(s)} = G_1(s) \qquad \text{...(1.117}a\text{)}$$

$$\frac{C(s)}{C_1(s)} = G_2(s) \qquad \text{...(1.117}b\text{)}$$

From equation (1.117*a*) and (1.117*b*)

$$\frac{C_1(s)}{R(s)} \cdot \frac{C(s)}{C_1(s)} = \frac{C(s)}{R(s)}$$

$$= G_1(s) \cdot G_2(s)$$

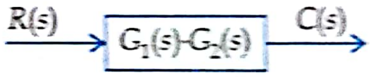


Fig. 1.91.

The equivalent diagram is shown in Fig. 1.91.

Rule No 2. Blocks in parallel

When two or more blocks are connected in parallel as shown in Fig. 1.92 (*a*), the resultant block is the sum of individual block transfer function.

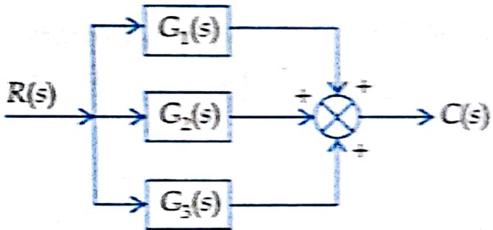


Fig. 1.92 (a).

From Fig. 1.92 (*b*)

$$C(s) = R(s)\,G_1(s) + R(s)\,G_2(s) + R(s)\,G_3(s)$$

$$= R(s)\,[G_1(s) + G_2(s) + G_3(s)]$$

$$\therefore \quad \frac{C(s)}{R(s)} = G_1(s) + G_2(s) + G_3(s)$$

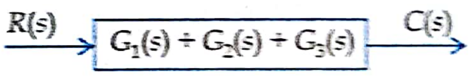


Fig. 1.92 (b).

The equivalent diag. is shown in Fig. 1.92 (*b*)

Rule No 3. Moving a take off point ahead of a block

If a take off point is moved ahead of a block, a block with same transfer function will introduce in the branch of a take off point, as shown in Fig. 1.92 (*c*).

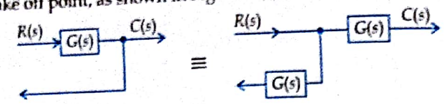


Fig. 1.92 (c).

Rule No. 4. Moving a take off point after the block

If a take off point is moved after the block, a block with the reciprocal of the transfer function is introduced in the branch of a take off point as shown in Fig. 1.92 (*d*).

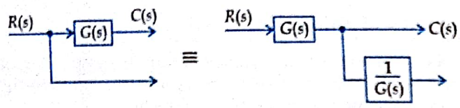
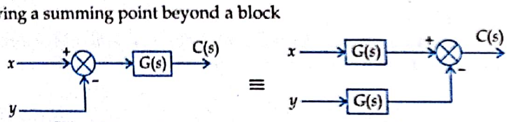


Fig. 1.92 (d).

Rule No. 5. Moving a summing point beyond a block



$C(s) = G(s)\cdot(x - y)$

$C(s) = G(s)x - G(s)y$
$= G(s)(x - y)$

Fig. 1.92 (e).

Rule No. 6. Moving a summing point ahead of a block

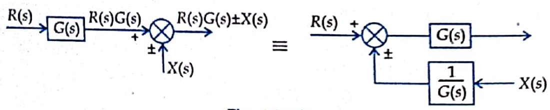


Fig. 1.92 (f)

Rule No. 7. Interchanging two summing points

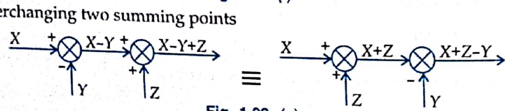


Fig. 1.92. (g)

Rule No. 8. Moving a take off point beyond a summing point

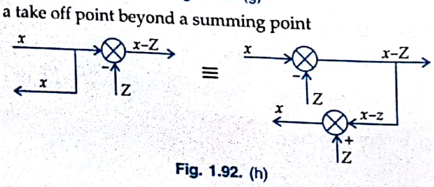


Fig. 1.92. (h)

Rule No. 9. Moving a take off point ahead of a summing point

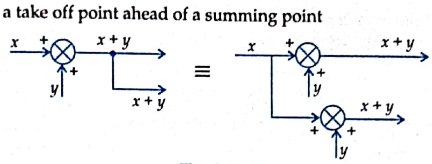


Fig. 1.92 (i)

Rule No. 10. Eliminating a forward loop

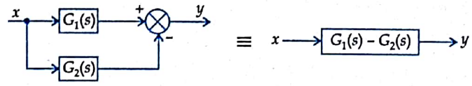


Fig. 1.92 (j)

EXAMPLE 1.32. **Derive the transfer function using block reduction technique.**

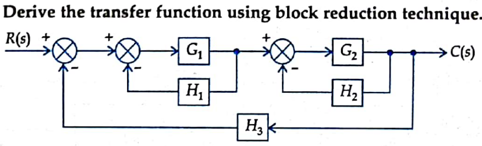


Fig. 1.93.

Solution : Step 1: There are two internal closed loops, remove these loops by using the equation (1.115*f*).

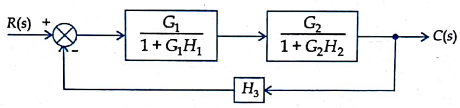


Fig. 1.93 (a)

Step 2: Two blocks are in cascade, use the rule no. 1.

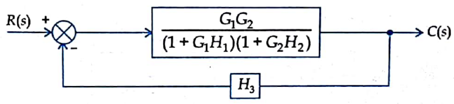
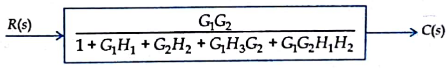


Fig. 1.93 (b)

Step 3: Figure 1.93 (*b*) is a closed loop, again use equation (1.115*f*)



$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1G_2}{1 + G_1H_1 + G_2H_2 + G_1G_2H_3 + G_1G_2H_1H_2} \quad \textbf{Ans.}$$

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EXAMPLE 1.33. Find the overall transfer function of the system shown in Fig. 1.84 (*e*).

Solution : Step 1 : Shift the pick off point beyond the block $1/sC_2$

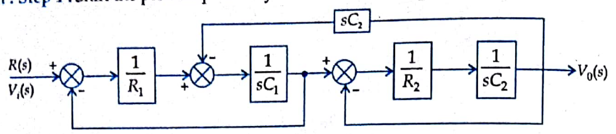


Fig. 1.94 (a)

Step 2 : Two blocks are in cascade

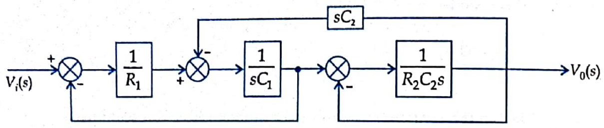


Fig. 1.94 (b)

Step 3 :

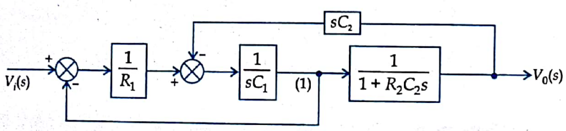


Fig. 1.94 (c)

Step 4 :

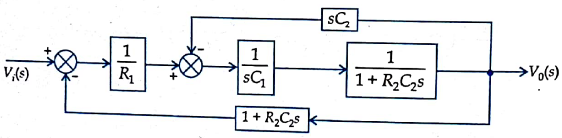


Fig. 1.94 (d)

Step 5 :

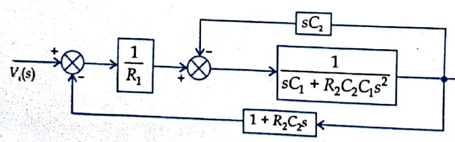


Fig. 1.94 (e)

Step 6 :

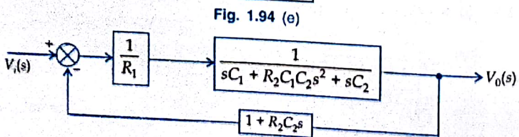


Fig. 1.94 (f)

Step 7 :

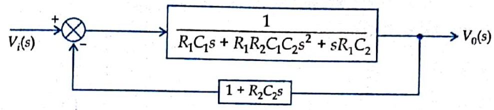
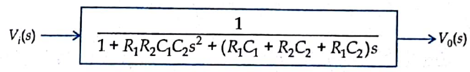


Fig. 1.94 (g)



$$\therefore \quad \frac{V_0(s)}{V_i(s)} = \frac{1}{s^2R_1R_2C_1C_2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1} \quad \textbf{Ans.}$$

EXAMPLE 1.34. Determine the ratio $C(s)/R(s)$ for the system shown in Fig. 1.95.

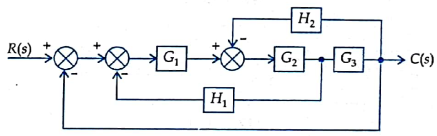


Fig. 1.95 (a).

Solution : Step 1 : Shift the takeoff point beyond the block G_3

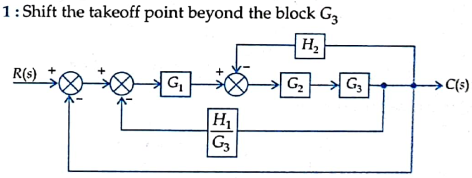


Fig. 1.95 (b)

Step 2 : G_2 and G_3 are in cascade

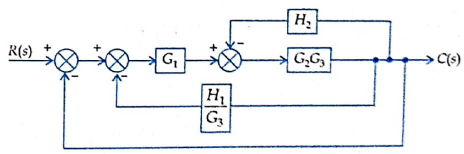


Fig. 1.95 (c)

Step 3 :

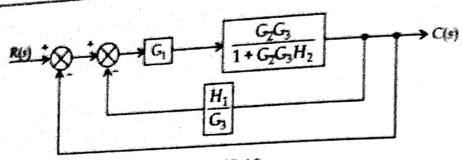


Fig. 1.95 (d)

Step 4 :

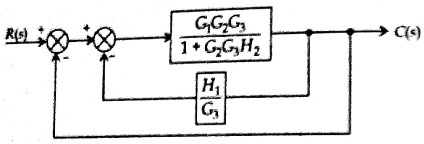


Fig. 1.95 (e)

Step 5 :

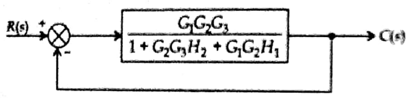
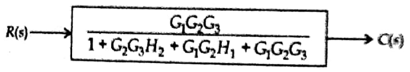


Fig. 1.95 (f)

Step 6 :



$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1+G_1G_2G_3} \quad \textbf{Ans.}$$

EXAMPLE 1.35. Determine the ratio C/R.

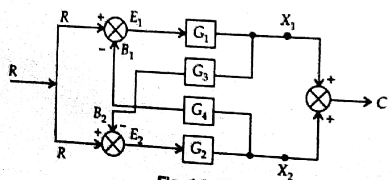


Fig. 1.96.

Solution :

$$E_2 = R - B_2$$
$$B_2 = X_1G_3 \quad \text{...(1.118}a\text{)}$$
$$X_2 = E_2G_2 \quad \text{...(1.118}b\text{)}$$
$$X_2 = G_2(R - B_2)$$
$$X_2 = G_2(R - X_1G_3)$$
$$B_1 = X_2G_4 \quad \text{...(1.118}c\text{)}$$
$$B_1 = G_4G_2\ (R - X_1G_3) \quad \text{...(1.118}d\text{)}$$

$$E_1 = R - B_1 = R - G_2G_4(R - X_1G_3) \quad \text{...(1.118}e\text{)}$$
$$X_1 = E_1G_1$$
$$X_1 = G_1\,[R - G_2G_4(R - X_1G_3)] \quad \text{...(1.118}f\text{)}$$
$$\therefore \quad X_1 = G_1R - G_1G_2G_4R + G_1G_2G_3G_4X_1 \quad \text{...(1.118}g\text{)}$$
$$= R(G_1 - G_1G_2G_4) + G_1G_2G_3G_4X_1$$
$$\therefore \quad X_1(1 - G_1G_2G_3G_4) = R\,(G_1 - G_1G_2G_4)$$
$$\frac{X_1}{R} = \frac{G_1 - G_1G_2G_4}{1 - G_1G_2G_3G_4} \quad \text{...(1.118}h\text{)}$$

From equation (1.118*h*) put the value of x_1 in equation (1.118*c*)

$$X_2 = G_2\left[R - \frac{G_1(1-G_2G_4)G_3R}{1-G_1G_2G_3G_4}\right] = \frac{G_2R(1-G_1G_3)}{1-G_1G_2G_3G_4} \quad \text{...(1.118}i\text{)}$$

$$\therefore \quad \frac{x_2}{R} = \frac{G_2(1-G_1G_3)}{1-G_1G_2G_3G_4}$$

$$\therefore \quad \frac{C}{R} = \frac{X_1}{R} + \frac{X_2}{R} = \frac{G_1+G_2-G_1G_2G_4-G_1G_2G_3}{1-G_1G_2G_3G_4} \quad \textbf{Ans.}$$

EXAMPLE 1.36. Determine the transfer function C_1/R_1, C_2/R_2, C_1/R_2 and C_2/R_1 from the block diagram shown in Fig. 1.97. *(R.M.L. Univ. 2002, Control System)*

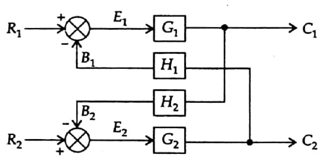


Fig. 1.97.

Solution :

$$E_2 = R_2 - B_2 \quad \text{...(1.119}a\text{)}$$
$$B_2 = C_1H_2 \quad \text{...(1.119}b\text{)}$$
$$C_2 = E_2G_2 \quad \text{...(1.119}c\text{)}$$

Put the value of E_2 from equation (1.119*a*) in equation (1.119*c*)

$$C_2 = G_2(R_2 - B_2) \quad \text{...(1.119}d\text{)}$$

From equation (1.119*b*) put the value of B_2 in equation (1.119*d*)

$$C_2 = G_2(R_2 - C_1H_2) \quad \text{...(1.119}e\text{)}$$
$$B_1 = C_2H_1 \quad \text{...(1.119}f\text{)}$$

Put the value of C_2 from equation (1.119*e*) in equation (1.119*f*)

$$B_1 = G_2H_1\,(R_2 - C_1H_2) \quad \text{...(1.119}g\text{)}$$
$$E_1 = R_1 - B_1$$
$$E_1 = R_1 - G_2H_1(R_2 - C_1H_2) \quad \text{...(1.119}h\text{)}$$

Also, $C_1 = E_1G_1$. ...(1.119*i*)

From equation (1.119h) put the value of E_1 in equation (1.119i)

$$C_1 = G_1\,[R_1 - G_2H_1\,(R_2 - C_1H_2)] \quad \text{...(1.119}j\text{)}$$

$$C_1(1 - G_1G_2H_1H_2) = G_1R_1 - G_1G_2H_1R_2 \quad \text{...(1.119}k\text{)}$$

When $R_1 = 0$ from equation (1.119k)

$$\frac{C_1}{R_2} = \frac{-G_1G_2H_1}{1-G_1G_2H_1H_2} \quad \textbf{Ans.} \quad \text{...(1.119}l\text{)}$$

From equation (1.119l) put the value of C_1 in equation (1.119e)

$$C_2 = G_2\left[R_2 + \frac{G_1G_2H_1H_2R_2}{1-G_1G_2H_1H_2}\right]$$

$$\therefore \quad \frac{C_2}{R_2} = \frac{G_2}{1-G_1G_2H_1H_2} \quad \textbf{Ans.} \quad \text{...(1.119}m\text{)}$$

When $R_2 = 0$, equation (1.119k) becomes

$$C_1 = G_1\,[R_1 + C_1\,G_2H_1H_2]$$

$$C_1 = G_1R_1 + C_1G_1G_2H_1H_2$$

$$\therefore \quad \frac{C_1}{R_1} = \frac{G_1}{1-G_1G_2H_1H_2} \quad \textbf{Ans.} \quad \text{...(1.119}n\text{)}$$

When $R_2 = 0$, equation (1.119e) becomes

$$C_2 = -G_2C_1H_2$$

From equation (1.119n) put the value of C_1

$$\therefore \quad C_2 = \frac{-G_1G_2R_1H_2}{1-G_1G_2H_1H_2}$$

$$\frac{C_2}{R_1} = \frac{-G_1G_2H_2}{1-G_1G_2H_1H_2} \quad \textbf{Ans.} \quad \text{...(1.119}n'\text{)}$$

EXAMPLE 1.37. Determine the ratio $C(s)/R(s)$.

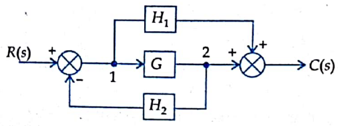
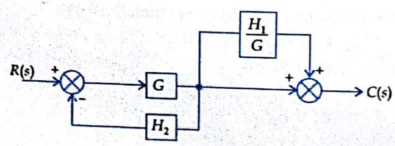
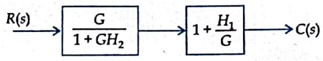


Fig. 1.98.

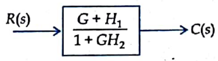
Solution : Step 1 : Shift the takeoff point beyond block G.



Step 2 :



Step 3 :



$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G+H_1}{1+GH_2} \quad \textbf{Ans.}$$

EXAMPLE 1.38. Find the ratio $C(s)/R(s)$ of the system shown in Fig. 1.99.

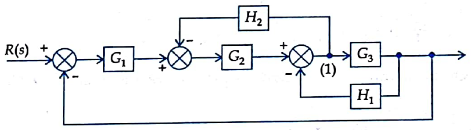


Fig. 1.99 (a).

Solution : Step 1: Shift the take off point beyond block G_3.

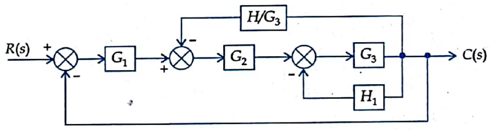


Fig. 1.99 (b).

Step 2 :

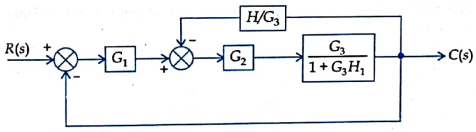


Fig. 1.99 (c)

Step 3 : Blocks are in cascade

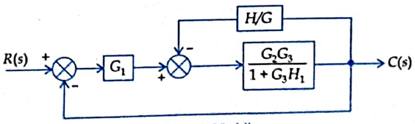


Fig. 1.99 (d)

Step 4 :

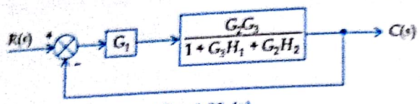


Fig. 1.99 (e)

Step 5 :

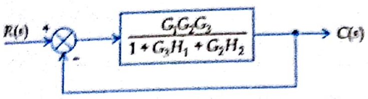


Fig. 1.99 (f)

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_3H_1+G_2H_2+G_1G_2G_3} \quad \text{Ans.}$$

EXAMPLE 1.39. Determine the transfer matrix MIMO system of example 1.34.

Solution :

$$G_{11} = \left.\frac{C_1}{R_1}\right|_{R_2=0}, \qquad G_{21} = \left.\frac{C_2}{R_1}\right|_{R_2=0}$$

$$G_{12} = \left.\frac{C_1}{R_2}\right|_{R_1=0}, \qquad G_{22} = \left.\frac{C_2}{R_2}\right|_{R_1=0}$$

$$\therefore \quad G(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

$$= \begin{bmatrix} G_1 & -G_1G_2H_1 \\ -G_1G_2H_2 & G_2 \end{bmatrix} \frac{1}{1-G_1G_2H_1H_2} \quad \text{Ans.}$$

EXAMPLE 1.40. Find the transfer function of the system shown in Fig. 1.100.

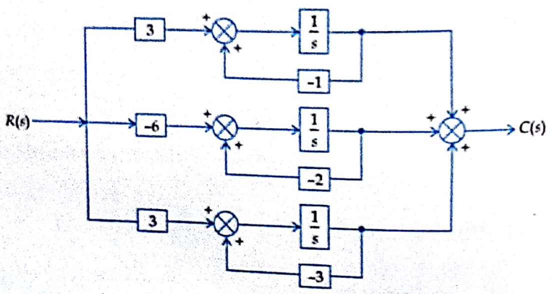


Fig. 1.100 (a)

Solution : Step 1 :

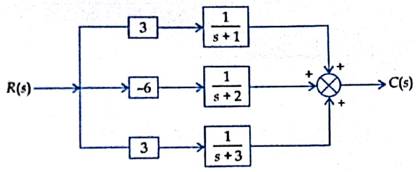


Fig. 1.100 (b)

Step 2 :

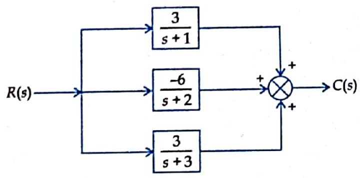


Fig. 1.100 (c)

Step 3 :

$$\frac{C(s)}{R(s)} = \frac{3}{s+1} + \frac{-6}{s+2} + \frac{3}{s+3}$$

$$= \frac{6}{s^3+6s^2+11s+6} \quad \text{Ans.}$$

EXAMPLE 1.41. Find the total transfer function $\frac{C}{R}$, using block reduction technique.

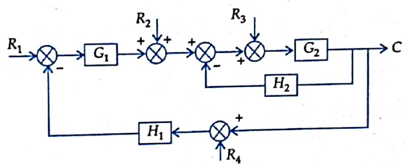


Fig. 1.100 (d)

Solution: Consider $R_2 = 0$, $R_3 = 0$, $R_4 = 0$

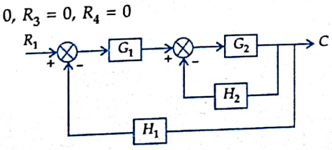


Fig. 1.100 (e)

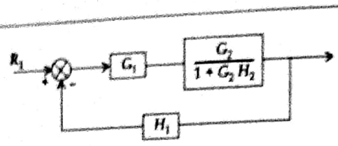


Fig. 1.100 (f)

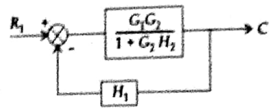


Fig. 1.100 (g)

$$\therefore \quad \frac{C}{R_1} = \frac{G_1G_2}{1+G_2H_2+G_1G_2H_1}$$

Now consider $R_1 = 0$, $R_3 = 0$, $R_4 = 0$

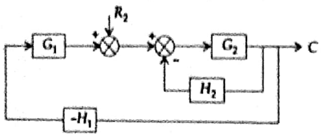


Fig. 1.100 (h)

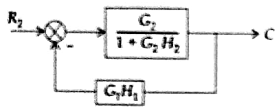


Fig. 1.100 (i)

$$\therefore \quad \frac{C}{R_2} = \frac{G_2}{1+G_2H_2+G_1G_2H_1}$$

Consider $R_1 = R_2 = R_4 = 0$

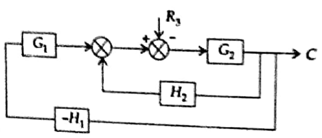


Fig. 1.100 (j)

$$\therefore \quad \frac{C}{R_3} = \frac{-G_2}{1+G_2H_2+G_1G_2H_1}$$

Consider $R_1 = R_2 = R_3 = 0$

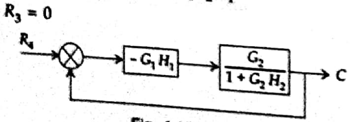


Fig. 1.100 (k)

$$\frac{C}{R_4} = \frac{-G_1G_2H_1}{1+G_2H_2+G_1G_2H_1}$$

Total transfer function

$$\frac{C}{R} = \frac{C}{R_1}+\frac{C}{R_2}+\frac{C}{R_3}+\frac{C}{R_4}$$

$$\frac{C}{R} = \frac{G_1G_2-G_1G_2H_1}{1+G_1H_2+G_1G_2H_1}$$

1.26. BLOCK DIAGRAM AND TRANSFER FUNCTION OF D.C. MOTOR

1.26.1. Armature Controlled dc Motor

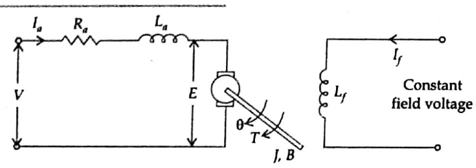


Fig. 1.101 (a)

Consider the armature controlled dc motor and assume that the demagnetizing effect of armature reaction is neglected, magnetic circuit is assumed linear and field voltage is constant *i.e.*, I_f = constant.

Let R_a = Armature resistance

L_a = Armature self inductance caused by armature flux

i_a = Armature current

i_f = Field current

E = Induced emf in armature

V = Applied voltage

T = Torque developed by the motor

θ = Angular displacement of the motor shaft

J = Equivalent moment of inertia of motor shaft and load referred to the motor

B = Equivalent coefficient of friction of motor and load referred to the motor

Apply KVL in armature circuit

$$V = R_a i_a + L\frac{di_a}{dt} + E \qquad \text{...(1.120)}$$

Since, field current I_f is constant, the flux ϕ will be constant

When armature is rotating, an emf is induced

$$E \propto \phi\,\omega$$

$$E = K_b\omega$$

or,

$$E = K_b\frac{d\theta}{dt} \qquad \text{...(1.121)}$$

where, ω = angular velocity

K_b = back e.m.f constant

Now, the torque T delivered by the motor will be the product of armature current and flux.

$$T \propto \phi\, I_a$$

$$T = K\, I_a \qquad ...(1.122)$$

where K = motor torque constant

The dynamic equation with moment of inertia and coefficient of friction will be

$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \qquad ...(1.123)$$

Take the Laplace transform of equations (1.120), (1.121), (1.122) and (1.123)

$$V(s) - E(s) = I_a(s)\,(R_a + sL_a)$$

$$E(s) = K_b\, s\theta(s)$$

$$T(s) = K\, I_a(s)$$

$$T(s) = (s^2J + sB)\,\theta(s)$$

$$T(s) = (SJ + B)\, S\theta(s)$$

The block diagram for each equation

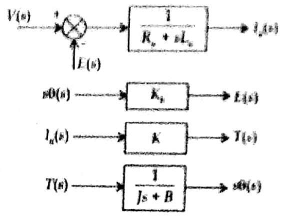


Fig. 1.101 (b)

Combine all four block diagrams

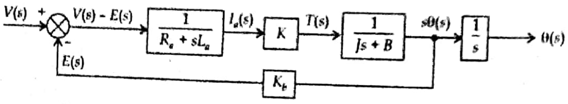


Fig. 1.102 (a). Block diagram of armature controlled dc motor

Now determine the transfer function by block reduction method.

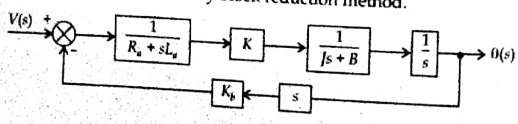


Fig. 1.102 (b)

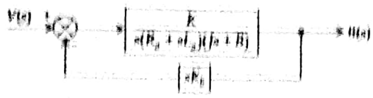


Fig. 1.102 (c)

$$\frac{\theta(s)}{V(s)} = \frac{K}{(R_a + sL_a)(Js + B)s + KK_b s} \qquad ...(1.124)$$

Equation (1.124) can be written as

$$\frac{\theta(s)}{V(s)} = \frac{K}{R_a\left(1 + s\frac{L_a}{R_a}\right)sB\left(1 + s\frac{J}{B}\right) + KK_b s}$$

Put $\frac{L_a}{R_a} = \tau_a$ time constant of armature circuit

$\frac{J}{B} = \tau_m$ = mechanical time constant

Equation (1.124) becomes

$$\frac{\theta(s)}{V(s)} = \frac{K}{sR_aB(1 + s\tau_a)(1 + s\tau_m) + KK_b s} \qquad ...(1.125)$$

From the block diagram (1.102) it is clear that it is a closed loop system. The effect of the back emf is represented by the feedback signal proportional to the speed of the motor.

1.26.2. Field Control dc Motor

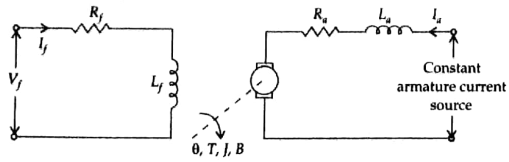


Fig. 1.103.

1. A constant current I_a is fed to the armature.
2. Flux is proportional to the field current.

$$\phi \propto I_f$$

$$\phi = K_f I_f \qquad ...(1.126)$$

3. Apply KVL in field circuit

$$V_f = R_f I_f + L_f \frac{d}{dt} I_f \qquad ...(1.127)$$

4. Torque developed by the motor is proportional to the flux and armature current.

$$T \propto \phi\, I_a$$

From equation (1.126) put the value of ϕ

$$T = K'\, K_f\, I_a I_f$$

$$T = KK_f\, I_f \qquad ...(1.128)$$

where $K = K'I_a$

Dynamic equation of torque in terms of J and B

$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \qquad ...(1.129)$$

Laplace transformation of equations (1.127), (1.128) and (1.129)

$$V_f(s) = R_f I_f(s) + sL_f I_f(s) = I_f(s)\,[R_f + sL_f]$$

$$I_f = \frac{V_f(s)}{R_f + sL_f} \qquad ...(1.130)$$

$$T(s) = KK_f I_f(s) \qquad ...(1.131)$$

$$T(s) = \theta(s)\,[s^2J + sB] \qquad ...(1.132)$$

Put the value of $I_f(s)$ from equation (1.130) in (1.131)

$$T(s) = K \cdot K_f \cdot \frac{V_f(s)}{R_f + sL_f} \qquad ...(1.133)$$

From equations (1.132) and (1.133)

$$\theta(s)\,[s^2J + sB] = K \cdot K_f \cdot \frac{V_f(s)}{R_f + sL_f}$$

$$\therefore \quad \frac{\theta(s)}{V_f(s)} = \frac{KK_f}{s(sJ + B)(R_f + sL_f)} \qquad ...(1.134)$$

Equation (1.134) can be written as

$$\frac{\theta(s)}{V_f(s)} = \frac{KK_f}{R_f Bs\left(1+s\frac{J}{B}\right)\left(1+s\frac{L_f}{R_f}\right)} = \frac{KK_f}{R_f Bs(1+s\tau_m)(1+s\tau_f)}$$

where $\tau_m = \frac{J}{B}$ = mechanical time constant

$\tau_f = \frac{L_f}{R_f}$ = time constant for field circuit

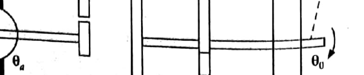


Fig. 1.104. Block diagram of field controlled dc motor.

1.26.3 Position Control System (Positional Servomechanism)

Fig. 1.105

The servomechanism as shown in Fig. 105 is used to control the position of mechanical load with reference position. The desired angular position θ_i (*i.e.* input) is set on the potentiometer P_1. The potentiometer P_2 is coupled with the output shaft. This pair of potentiometer forms an error-measuring device i.e. act as the error detector. This error detector produces an error voltage proportional to $\theta_i - \theta_o$. An amplifier with high input impedance amplifies this voltage. The amplified voltage is applied to the armature circuit of D.C. motor. The field of the D.C. motor is excited by a fixed voltage source. The armature of the motor is coupled with the mechanical load through gear which reduces the speed in ratio $n : 1$. If an error exists, the motor develops a torque to rotate the output load in such a way as to reduce the error to zero.

let 'e' is the error signal.

Armature voltage $\quad V_a = Ae \qquad ...(1.135)$

When armature is rotating an e.m.f. is induced

$$E \;\alpha\; \phi\,\omega$$

$$E = K_1\,\omega$$

$$E = K_1\frac{d\theta_a}{dt} = K_1\,\dot{\theta}_a \qquad ...(1.136)$$

where, K_1 is the back e.m.f. constant

$$V_a = R_a i_a + E$$

$$V_a = R_a\, i_a + K_1\,\dot{\theta}_a$$

$$\therefore \quad i_a = \frac{V_a - K_1\,\dot{\theta}_a}{R_a} = \frac{V_a - K_1 n\,\dot{\theta}_o}{R_a} \qquad \because \dot{\theta}_a = n\dot{\theta}_o \qquad ...(1.137)$$

where, i_a is the armature current.

Torque delivered by the motor will be the product of armature current and flux. Since flux is constant, then the torque will be

$$T_a = K_2\, i_a \qquad ...(1.138)$$

Now, combine the equations (1.135), (1.136), (1.137) and (1.138)

$$T_a = \frac{AK_2 e}{R_a} - \frac{K_1K_2\, n\,\dot{\theta}_o}{R_a} \qquad ...(1.139)$$

Take the Laplace transform of equation (1.139)

$$T_a(s) = \frac{A\,K_2}{R_a}E(s) - \frac{K_1K_2\,n}{R_a}\,s\,\theta_o(s) \qquad ...(1.140)$$

Put $\quad K_3 = \frac{A\,K_2}{R_a}$

and $\quad K_4 = \frac{K_1K_2\,n}{R_a}$, then

$$T_a(s) = K_3\,E(s) - K_4\, s.\theta_o(s) \qquad ...(1.141)$$

let J_a = Armature inertia

J_L = load inertia

B_a = Armature viscous friction

B_L = load viscous friction

Multiply the load inertia J_L and viscous friction B_L by n^2 when referred to the armature shaft. Therefore, the total inertia and total viscous friction at the motor shaft is given by

$$J' = J_a + n^2 J_L \quad ...(1.142)$$

$$B' = B_a + n^2 B_L \quad ...(1.143)$$

Now the equation of the mechanical motion is given by

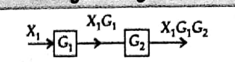
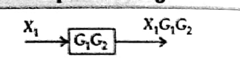
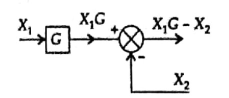
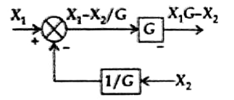
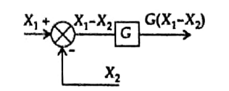
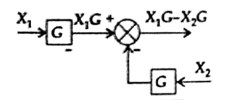
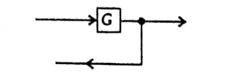
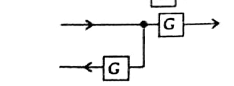
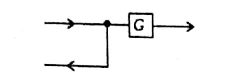
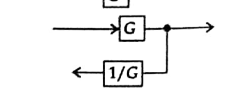
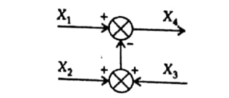
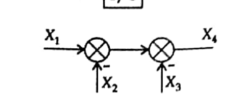
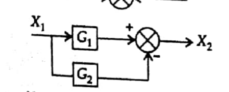
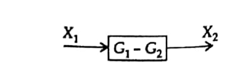
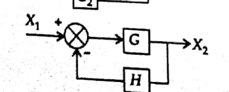
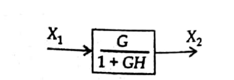
$$J' \ddot{\theta}_a + B' \dot{\theta}_a = T_a \quad ...(1.144)$$

or $$T_a = J' n \ddot{\theta}_o + B' n \dot{\theta}_o \quad ...(1.145)$$

Put $J = J'n$

and $B'n = B$

$$T_a = J \ddot{\theta}_o + B \dot{\theta}_o$$

Rule	Original Diagram	Equivalent Diagram
1. Blocks in cascade		
2. Moving a summing point ahead of a block		
3. Moving a summing point beyond a block		
4. Moving a take-off point ahead of a block		
5. Moving a take-off point beyond a block		
6. Rearrangement of summing point		
7. Eliminating a forward loop		
8. Eliminating feedback loop		

Take Laplace transform

$$T_a(s) = J s^2 \theta_o(s) + B s \theta_o(s)$$

$$T_a(s) = (J s^2 + B s)\theta_o(s) \quad ...(1.146)$$

From equation (1.141) and (1.146)

$$(J s^2 + B s)\theta_o(s) = K_3 E(s) - K_4\, s.\theta_o(s) \quad ...(1.147)$$

Also, $$E_s = K_5 [\theta_i(s) - \theta_o(s)] \quad ...(1.148)$$

Put this value in equation (1.147) we get

$$\theta_o(s)\,[J s^2 + B s + K_3 K_5 + s K_4] = K_3 K_5\, \theta_i(s)$$

or, $$\frac{\theta_o(s)}{\theta_i(s)} = \frac{K_3 K_5}{J s^2 + (B + K_4) s + K_3 K_5} \quad ...(1.149)$$

equation (1.149) is the closed loop transfer function.

1.27. SIGNAL FLOW GRAPH

The process of block diagram reduction technique is time consuming because at every stage modified block diagram is to be redrawn. A simple method was developed by S.J. Mason which is known as signal flow graph. This method is very simple and does not require any reduction technique. Signal flow graph is applicable to the linear systems.

A signal flow graph is a diagram which represents a set of simultaneous equations.

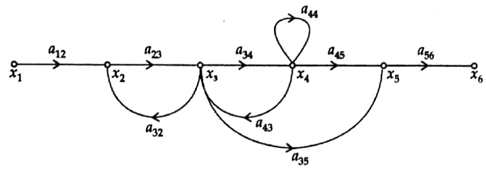


Fig. 1.106

Signal flow graph consists of nodes and these nodes are connected by a directed line called branches. Every branch of signal flow graph having an arrow, which reprsents the flow of signal.

The following terms are associated with the signal flow graph.

1. Input node or source node : An input node is a node which has only outgoing branches. For example x_1 is the input node.
2. Output node or sink node : An output node is a node that has only one or more incoming branches. For example, x_6 is the output node.
3. Mixed nodes : A node having incoming and outgoing branches is known as mixed nodes. For example x_2, x_3, x_4, and x_5 are the mixed nodes.
4. Transmittance : Transmittance also known as transfer function, which is normally written on the branch near the arrow. For example a_{12}, a_{23} etc.
5. Forward path : Forward path is a path which originates from the input node and terminates at the output node and along which no node is traversed more than once.

For example, in Fig 1.101 there are two forward paths.

1. x_1 to x_2 to x_3 to x_4 to x_5 to x_6
2. x_1 to x_2 to x_3 to x_5 to x_6

6. **Loop :** Loop is a path that originates and terminates on the same node and along which no other node is traversed more than once.

   For example, x_2 to x_3 to x_2

   x_3 to x_4 to x_3

7. **Self loop :** It is a path which originates and terminates on the same node. For example x_4 to x_4
8. **Path gain :** The product of the branch gains along the path is called path gain. For example the gain of the path x_1 to x_2 to x_3 to x_4 to x_5 to x_6 is $a_{12}\, a_{23}\, a_{34}\, a_{45}\, a_{56}$
9. **Loop gain :** The gain of the loop is known as loop gain. For example, the gain of the loop x_2 to x_3 to x_2 is $a_{23}\, a_{32}$.
10. **Non-touching loops :** Non-touching loops having no common nodes branch and paths. For example the loops x_2 to x_3 to x_2, and x_4 to x_4 are non-touching loops.

1.28. PROPERTIES OF SIGNAL FLOW GRAPH

1. Signal flow graph is applicable to linear time-invariant systems.
2. The signal flow is only along the direction of arrows.
3. The value of variable at each node is equal to the algebraic sum of all signals entering at that node.
4. The gain of signal flow graph is given by Mason's formula.
5. The signal gets multiplied by the branch gain when it travels along it.
6. The signal flow graph is not be the unique property of the system.

1.29. COMPARISON OF BLOCK DIAGRAM AND SIGNAL FLOW GRAPH METHOD

S.No.	Block Diagram	SFG
1.	Applicable to linear time invariant systems only.	Applicable to linear time invariant system.
2.	Each element is represented by block.	Each variable is represented by node.
3.	Summing point and take off points are separate.	Summing and take off points are absent.
4.	Self loop do not exist.	Self loop can be exist.
5.	It is time consuming method.	Require less time by using Mason gain formula.
6.	Block diagram is required at each and every step.	At each step it is not necessary to draw SFG.
7.	Transfer function of the element is shown inside the corresponding block.	Transfer function is shown along the branches connecting the nodes.
8.	Feedback path is present.	Feedback loops are used.

1.30. CONSTRUCTION OF SIGNAL FLOW GRAPH FROM EQUATIONS

Consider the following sets of equations

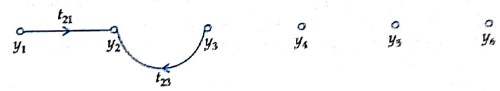
$$y_2 = t_{21} y_1 + t_{23} y_3 \quad \text{...(1)}$$
$$y_3 = t_{32} y_2 + t_{33} y_3 + t_{31} y_1 \quad \text{...(2)}$$
$$y_4 = t_{43} y_3 + t_{42} y_2 \quad \text{...(3)}$$
$$y_5 = t_{54} y_4 \quad \text{...(4)}$$
$$y_6 = t_{65} y_5 + t_{64}\, y_4 \quad \text{...(5)}$$

where y_1 is the input and y_6 is the output.

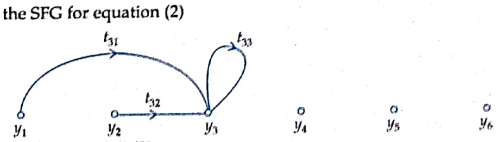
First of all draw the nodes. In the given example there are six nodes. From the first equation it is clear that the y_2 is the sum of two signals. Similarly, y_3 is the sum of three signals and so on. Insert the branches with proper transmittance to connect the nodes.

Step 1 : Draw the nodes

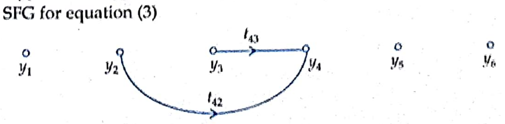
Step 2 : Draw the SFG for equation (1)



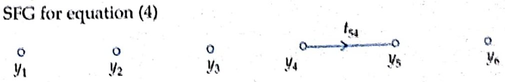
Step 3 : Draw the SFG for equation (2)



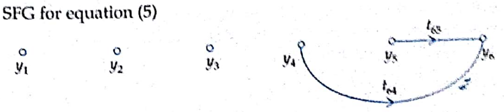
Step 4 : Draw SFG for equation (3)



Step 5 : Draw SFG for equation (4)



Step 6 : Draw SFG for equation (5)



Step 7 : Draw the complete signal flow graph with the help of above graphs.

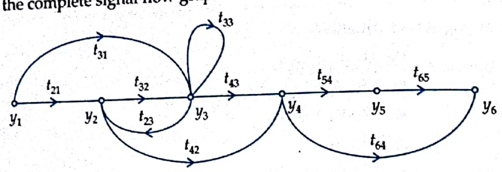


Fig. 1.107.

1.31. SIGNAL FLOW GRAPH FOR DIFFERENTIAL EQUATIONS

Consider the following defferential equation

$$y''' + 3y'' + 5y' + 2y = x \quad ...(1.150)$$

Step 1 : Solve the eqn 1.135 for the highest order

$$y''' = x - 3y'' - 5y' - 2y$$

Step 2 : Consider the left hand term (highest order derivative) as dependent variable and all other terms on right hand side as independent variables.

Construct the branches of signal flow graph as shown in Fig. 1.108(*a*).

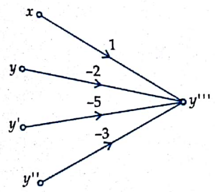


Fig. 1.108 (a)

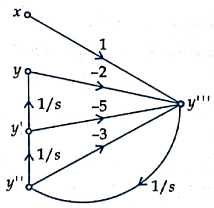
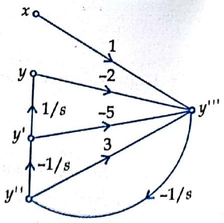


Fig. 1.108 (b)

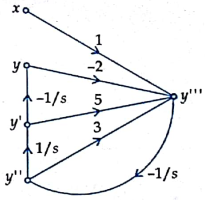
Step 3 : Connect the nodes of highest order derivative to the node whose order is lower than this and so on. The flow of the signal will be from higher order node to the lower order node and transmittance will be $1/s$ as shown in Fig. 1.108(*b*).

Step 4 : Reverse the sign of a branch connecting the p^{th} node to the q^{th} node of a signal flow graph without disturbing the transfer function.

Consider the Fig. 1.108(*b*), reverse the sign of the branch connecting y''' to y'', it is necessary to reverse the sign of all remaining branches entering as well as leaving the q^{th} node.



Similarly, reverse the sign of branch connecting y'' to y'.



By reversing the sign, we have already reverse the sign of branch connecting y' to y and therefore further reversal of sign is not required.

Step 5 : Redraw the signal flow graph (SFG).

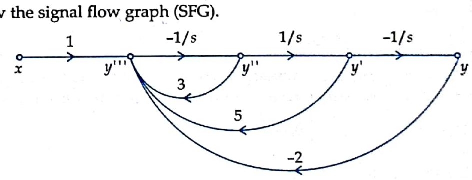


Fig. 1.108 (c)

1.32. CONSTRUCTION OF SIGNAL FLOW GRAPH FROM BLOCK DIAGRAM

Rules

1. All variables, summing points and take off points are represented by nodes.
2. If a summing point is placed before a take off point in the dirction of signal flow, in such case represent the summing point and takeoff point by a single node.
3. If a summing point is placed after a takeoff point in the direction of signal flow, in such case, represent the summing point and takeoff point by separate nodes connected by a branch having transmittance unity.

Consider the block diagram shown in Fig 1.109(*a*), the corresponding SFG is shown in Fig. 1.109(*b*).

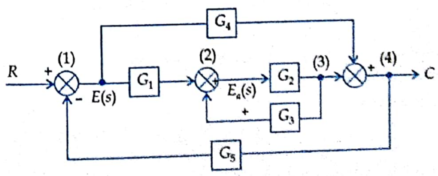


Fig. 1.109 (a)

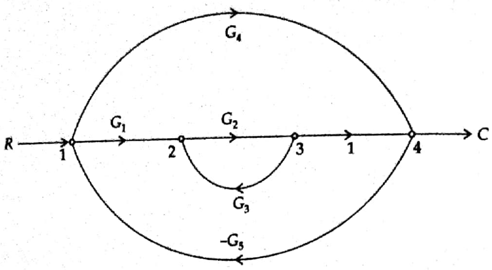


Fig. 1.109 (b)

1.33. MASON'S GAIN FORMULA

The overall transmittance or graph transmittance between the source node and sink node is given by Mason's gain formula. Mason's gain formula for signal flow graph is as follows :

$$T = \frac{\Sigma g_k \Delta_k}{\Delta} \quad ...(1.151)$$

where T = transfer function

Δ = 1 – [sum of all individual loop gain] + [sum of all possible gain products of two non-touching loops] – [sum of all possible gain products of three non-touching loops] + ...

g_k = gain of the k^{th} forward path

Δ_k = the part of Δ not touching the k^{th} forward path.

Consider the signal flow graph shown in Fig. 1.109 (*b*). There are two forward paths (*i*) 1-2-3-4 (*ii*) 1-4 therefore gain of the two paths will be

$$g_1 = G_1G_2$$
$$g_2 = G_4$$

There are three individual loops

$$L_1 = G_2G_3 \quad (2\text{-}3\text{-}2)$$
$$L_2 = -G_1G_2G_5 \quad (1\text{-}2\text{-}3\text{-}4\text{-}1)$$
$$L_3 = -G_4G_5 \quad (1\text{-}4\text{-}1)$$

Since all three loops touching the forward path g_1, therefore $\Delta_1 = 1 - 0 = 1$ The first loop L_1 do not touch the forward path g_2, therefore $\Delta_2 = 1 - G_2G_3$

There are two non touching loops L_1 and L_3

$\because$
$$L_1L_3 = -G_2G_3G_4G_5$$

Apply Mason's gain formula

$$T = \frac{C}{R} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta}$$

$$T = \frac{G_1G_2 + G_4(1 - G_2G_3)}{1 - (G_2G_3 - G_1G_2G_5 - G_4G_5) - (-G_2G_3G_4G_5)}$$

$$\frac{C}{R} = \frac{G_1G_2 + G_4 - G_2G_3G_4}{1 - G_2G_3 + G_1G_2G_5 + G_4G_5 - G_2G_3G_4G_5}$$

Note : where $\Delta = 1 - (L_1 + L_2 + L_3) + (L_1L_3)$

EXAMPLE 1.42. Draw the signal flow graph for the following set of equations.

$$x_2 = x_1 + ax_5$$
$$x_3 = bx_2 + cx_4$$
$$x_4 = dx_2 + ex_3$$
$$x_5 = fx_4 + gx_3$$
$$x_6 = x_5$$

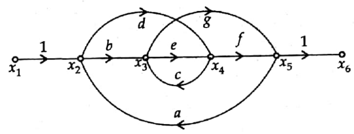


Fig. 1.110.

Solution : The required signal flow graph is shown in Fig. 1.110.

EXAMPLE 1.43. For the system represented by the given equations find the transfer function x_5/x_1 by the help of signal flow graph technique.

$$x_2 = a_{12}x_1 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5$$
$$x_3 = a_{23}x_2$$
$$x_4 = a_{34}x_3 + a_{44}x_4$$
$$x_5 = a_{35}x_3 + a_{45}x_4$$

where x_1 is the input variable and x_5 is the output variable. *(R.M.L. University Faizabad, 2002)*

Solution : There are two forward paths: (1) x_1 to x_2 to x_3 to x_4 to x_5

(2) x_1 to x_2 to x_3 to x_5

The gain of the paths $g_1 = a_{12}\, a_{23}\, a_{34}\, a_{45}$

$g_2 = a_{12}\, a_{23}\, a_{35}$

Gain of individual loops

$$L_1 = a_{23}\, a_{32}$$
$$L_2 = a_{23}\, a_{34}\, a_{42}$$
$$L_3 = a_{23}\, a_{34}\, a_{45}\, a_{52}$$
$$L_4 = a_{23}\, a_{35}\, a_{52}$$
$$L_5 = a_{44}$$

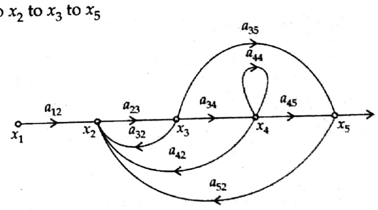


Fig. 1.111.

Gain of two nontouching loops

$$L_1L_5 = a_{23}\, a_{32}\, a_{44}$$
$$L_4\, L_5 = a_{23}\, a_{35}\, a_{52}\, a_{44}$$

Since, all the loops touches the forward path (1) $\therefore \Delta_1 = 1 - 0 = 1$

loop L_5 do not touch the second forward path $\therefore \Delta_2 = 1 - a_{44}$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + (L_1L_5 + L_4L_5)$$

$$\therefore \quad \frac{x_5}{x_1} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta}$$

$$\frac{x_5}{x_1} = \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{23}a_{35}(1 - a_{44})}{1 - (a_{23}a_{32} + a_{23}a_{34}a_{42} + a_{23}a_{34}a_{45}a_{52} + a_{23}a_{35}a_{52} + a_{44}) + (a_{23}a_{32}a_{44} + a_{23}a_{35}a_{52}a_{44})}$$

EXAMPLE 1.44. For the given signal flow grap find the ratio C/R.

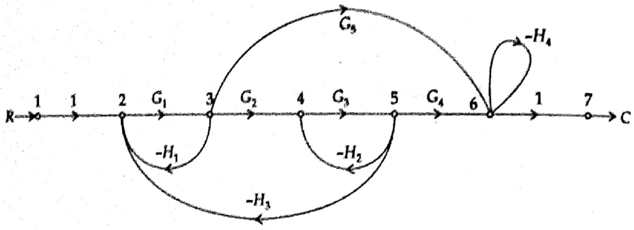


Fig. 1.112.

Solution: The gain of the forward paths $g_1 = G_1G_2G_3G_4$

$g_2 = G_1G_5$

Individual loops

$L_1 = -G_1H_1$

$L_2 = -G_3H_2$

$L_3 = -G_1G_2G_3H_3$

$L_4 = -H_4$

Two non-touching loop

$L_1L_2 = G_1H_1G_3H_2$

$L_1L_4 = G_1H_1H_4$

$L_2L_4 = G_3H_2H_4$

$L_3L_4 = G_1G_2G_3H_3H_4$

Three non-touching loops

$L_1L_2L_4 = -G_1H_1G_3H_2H_4$

$\Delta_1 = 1 - 0$

$\Delta_2 = 1 + G_3H_2$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_2 + L_1L_4 + L_2L_4 + L_3L_4) - (L_1L_2L_4)$$

$$\therefore \quad \frac{C}{R} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta}$$

$$\frac{C}{R} = \frac{G_1G_2G_3G_4 + G_1G_5(1 + G_3H_2)}{1 + G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + H_4 + G_1G_3H_1H_2 + G_1H_1H_4 + G_3H_2H_4 + G_1G_2G_3H_3H_4 + G_1G_3H_1H_2H_4}$$

EXAMPLE 1.45. Obtain signal flow graph representation for a system whose block diagram is given below and using Mason's gain formula determine the ratio $\frac{C}{R}$.

(R.M.L. Univ. Faizabad 2001; Linear System Theory)

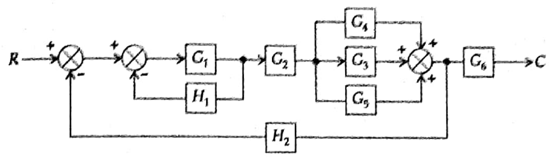


Fig. 1.113 (a)

Solution :

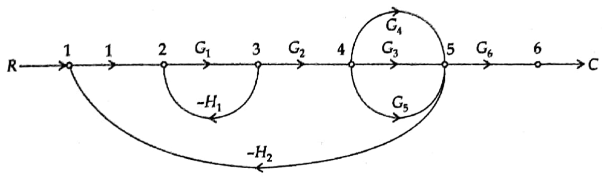


Fig. 1.113 (b)

The signal flow graph is shown in Fig. 1.108(*a*). Consider the SFG

$g_1 = G_1G_2G_3G_6 \qquad \Delta_1 = 1 - 0 = 1$

$g_2 = G_1G_2G_4G_6 \qquad \Delta_2 = 1 - 0 = 1$

$g_3 = G_1G_2G_5G_6 \qquad \Delta_3 = 1 - 0 = 1$

Individual loops

$L_1 = -G_1H_1$

$L_2 = -G_1G_2G_3H_2$

$L_3 = -G_1G_2G_4H_2$

$L_4 = -G_1G_2G_5H_2$

Two non-touching loops = None

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4)$$

$$\therefore \quad \frac{C}{R} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_6 + G_1G_2G_4G_6 + G_1G_2G_5G_6}{1 + G_1H_1 + G_1G_2G_3H_2 + G_1G_2G_4H_2 + G_1G_2G_5H_2}$$

EXAMPLE 1.46. Obtain the Transfer function for $\frac{C(s)}{R(s)}$ given block diagram using block diagram reduction technique or signal flow graph (Mason's gain formula).

(R.M.L. Univ. Faizabad 2001; Control System)

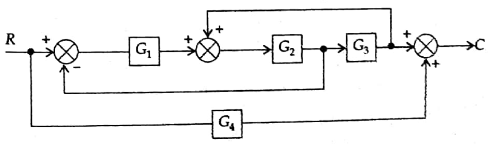


Fig. 1.114 (a)

Scanned by CamScanner

Solution : By signal flow graph SFG is shown in Fig. 1.114(*a*).

Gain of forward paths : $g_1 = G_1G_2G_3$

$g_2 = G_4$

$\Delta_1 = 1$

$\Delta_2 = 1 + G_1G_2 - G_2G_3$

Individual loops : $L_1 = -G_1G_2$

$L_2 = G_2G_3$

Two non-touching loops : None

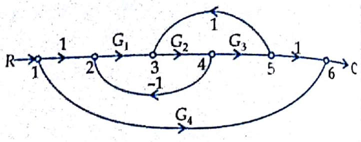


Fig. 1.114 (b)

$$\therefore \quad \frac{C}{R} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta}$$

$$\frac{C}{R} = \frac{G_1G_2G_3 + G_4(1 + G_1G_2 - G_2G_3)}{1 + G_1G_2 - G_2G_3}$$

EXAMPLE 1.47. Draw the signal flow graph and determine $\frac{C}{R}$ for the block diagram shown in Fig. 1.115. *(KNIT, Sultanpur, 1999-2000)*

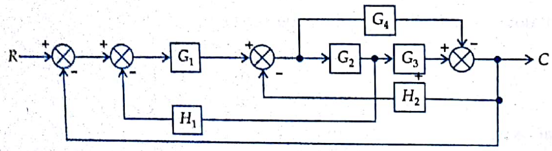


Fig. 1.115 (a)

Solution : The SFG is shown in Fig. 1.115(*a*)

The gain of the forward paths

$g_1 = G_1G_2G_3$ $\quad\quad \Delta_1 = 1$

$g_2 = -G_1G_4$ $\quad\quad \Delta_2 = 1$

Individual loops $\quad L_1 = -G_1G_2H_1$

$L_2 = -G_2G_3H_2$

$L_3 = -G_1G_2G_3$

$L_4 = G_1G_4$

$L_5 = G_4H_2$

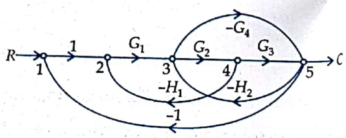


Fig. 1.115 (b)

$$\frac{C}{R} = \frac{g_1\Delta_1 + g_2\Delta_2}{\Delta}$$

$$\frac{C}{R} = \frac{G_1G_2G_3 - G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3 - G_1G_4 - G_4H_2}$$

EXAMPLE 1.48. Obtain the transfer function *C/R* of the block diagram shown in Fig. 1.116. *(KNIT, Sultanpur)*

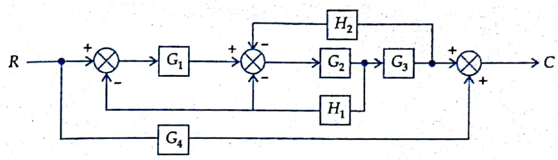


Fig. 1.116 (a)

Solution :

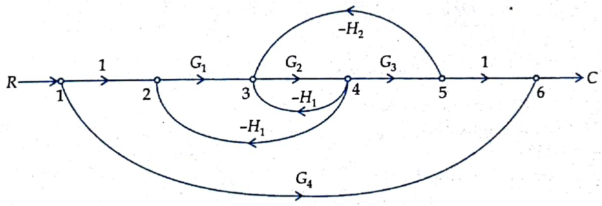


Fig. 1.116 (b)

The gain of the forward paths

$g_1 = G_1G_2G_3$

$g_2 = G_4$

Individual loops $\quad L_1 = -G_2H_1$

$L_2 = -G_1G_2H_1$

$L_3 = -G_2G_3H_2$

$\Delta_1 = 1$

$\Delta_2 = 1 - (-G_2H_1 - G_1G_2H_1 - G_2G_3H_2)$

$\Delta_2 = 1 + G_2H_1 + G_1G_2H_1 + G_2G_3H_2$

$\Delta = 1 - (L_1 + L_2 + L_3)$

$\Delta = 1 + G_2H_1 + G_1G_2H_1 + G_2G_3H_2$

$$\frac{C}{R} = \frac{G_1G_2G_3 + G_4(1 + G_2H_1 + G_1G_2H_1 + G_2G_3H_2)}{1 + G_2H_1 + G_1G_2H_1 + G_2H_2G_3}$$

EXAMPLE 1.49. Draw the signal flow graph for the network shown in Fig. 1.117, take V_3 as output node.

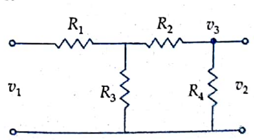


Fig. 1.117 (a)

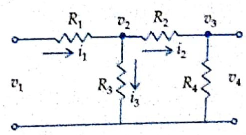


Fig. 1.117 (b)

Solution : From 1.112(*a*) : $i_1 = \frac{V_1}{R_1} - \frac{V_2}{R_1}$

$$V_2 = i_3 R_3 = R_3 (i_1 - i_2) = R_3 i_1 - R_3 i_2$$

$$i_2 = \frac{V_2}{R_2} - \frac{V_3}{R_2}$$

$$V_3 = R_4 i_2$$

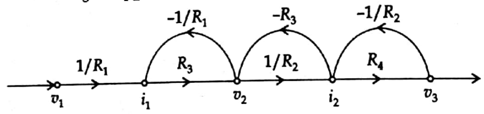


Fig. 1.117 (c) Signal flow graph.

EXAMPLE 1.50. Obtain the transfer function C/R from the signal flow graph shown in Fig. 1.118.

Solution : The gain of the forward paths

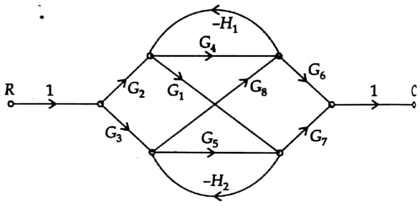


Fig. 1.118.

$$g_1 = G_2G_4G_6$$
$$g_2 = G_3G_5G_7$$
$$g_3 = G_2G_1G_7$$
$$g_4 = G_3G_8G_6$$
$$g_5 = -G_2G_1H_2G_8G_6$$
$$g_6 = -G_3G_8H_1G_1G_7$$

Individual loops : $L_1 = -G_4H_1$
$$L_2 = -G_5H_2$$
$$L_3 = H_1G_1H_2G_8$$

Two non-touching loops :

$$L_1L_2 = G_4H_1G_5H_2$$
$$\Delta_1 = 1 + G_5H_2$$
$$\Delta_2 = 1 + G_4H_1$$
$$\Delta_3 = 1$$
$$\Delta_4 = 1$$
$$\Delta_5 = 1$$
$$\Delta_6 = 1$$
$$\Delta = 1 + G_4 H_1 + G_5 H_2 - H_1G_1H_2G_8 + G_4H_1G_5H_2$$

$$\therefore \quad \frac{C}{R} = \frac{g_1\Delta_1 + g_2\Delta_2 + g_3\Delta_3 + g_4\Delta_4 + g_5\Delta_5 + g_6\Delta_6}{\Delta}$$

$$\frac{C}{R} = \frac{G_2G_4G_6(1+G_5H_2) + G_3G_5G_7(1+G_4H_1) + G_1G_2G_7 + G_3G_8G_6 - G_2G_1H_2G_8G_6 - G_3G_8H_1G_1G_7}{1+G_4H_1 + G_5H_2 - H_1G_1H_2G_8 + G_4H_1G_5H_2}$$

Ans.

EXAMPLE 1.51. Draw the block diagram, and signal flow graph and find out the transfer function of the circuit shown in Fig. 1.119(*a*).

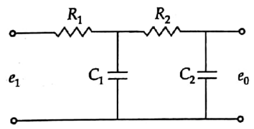


Fig. 1.119 (a)

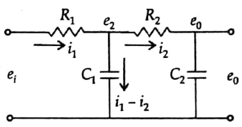


Fig. 1.119 (b)

Solution : From Fig. 1.119(*b*): $i_1 = \frac{e_i - e_2}{R_1}$

$$e_2 = \frac{1}{C_1}\int (i_1 - i_2)dt$$

$$i_2 = \frac{1}{R_2}(e_2 - e_0)$$

$$e_0 = \frac{1}{C_2}\int i_2\, dt$$

Take Laplace transform of above equations

$$I_1(s) = \frac{1}{R_1}[E_i(s) - E_0(s)]$$

$$E_2(s) = \frac{1}{sC_1}[I_1(s) - I_2(s)]$$

$$I_2(s) = \frac{1}{R_2}[E_2(s) - E_0(s)]$$

$$E_0(s) = \frac{1}{sC_2}[I_2(s)]$$

On the basis of above equation draw the block diagram

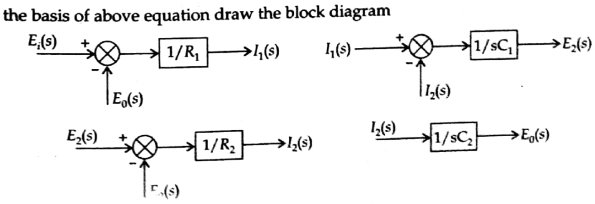
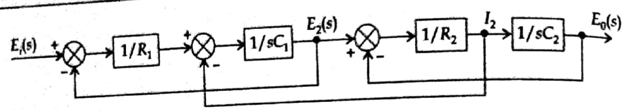


Fig. 1.119 (c). Block diagram



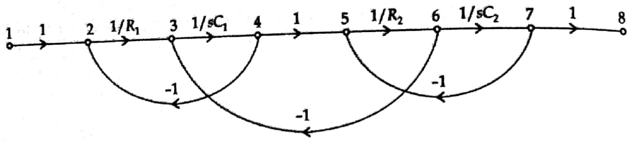


Fig. 1.119 (d). Signal flow graph

$$g_1 = \frac{1}{R_1R_2C_1C_2s^2} \qquad \Delta_1 = 1$$

$$L_1 = -\frac{1}{R_1C_1S} \qquad L_2 = -\frac{1}{R_2C_1s} \qquad L_3 = -\frac{1}{R_2C_2s}$$

Two non-touching loops

$$L_1L_3 = \frac{1}{s^2R_1R_2C_1C_2}$$

$$\frac{C}{R} = \frac{1/R_1R_2C_1C_2s^2}{1+\frac{1}{R_1C_1s}+\frac{1}{R_2C_1s}+\frac{1}{R_2C_2s}+\frac{1}{s^2R_1R_2C_1C_2}}$$

$$\frac{C}{R} = \frac{1}{1+R_1C_1s+R_1C_2s+R_2C_2s+s^2R_1R_2C_1C_2}$$ **Ans.**

1.34. BLOCK DIAGRAM FROM SIGNAL FLOW GRAPH

Step 1 : For given signal flow graph, write the system equations.
Step 2 : At each node consider the incoming branches only.
Step 3 : Add all incoming signals algebraically at a node.
Step 4 : For + or − sign in system equations use a summing point
Step 5 : For the gain of each branch of signal flow graph draw the block having the same transfer function as the gain of the branch.

Consider the following examples :

EXAMPLE 1.52. Draw the block diagram from the given signal flow graph.

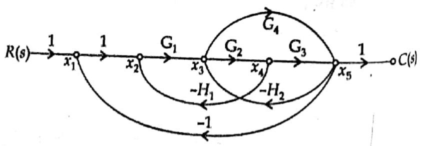


Fig. 1.120 (a)

Solution : The system equations are

At node x_1, the incoming branches are from $R(s)$ and x_5

$$x_1 = 1\ R(s) - 1 \cdot x_5$$

At node x_2, there are two incoming branches

$$x_2 = 1 \cdot x_1 - H_1x_4$$

At node x_3, there are two incoming branches

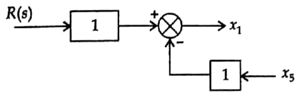
$$x_3 = G_1\,x_2 - H_2\,x_5$$

Similarly at node x_4 and x_5 the system equations are

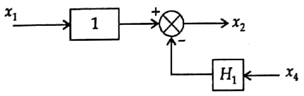
$$x_4 = G_2\,x_3$$
$$x_5 = G_3\,x_4 + G_4\,x_3$$

Draw the block diagram for each system equation

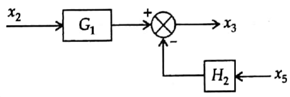
For $x_1 = R(s) - x_5$



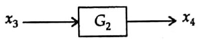
Block diagram for $x_2 = x_1 - H_1\,x_4$



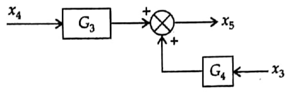
Block diagram for $x_3 = G_1x_2 - H_2\,x_5$



Block diagram for $x_4 = G_2\,x_3$



Block diagram for $x_5 = G_3\,x_4 + G_4\,x_3$



Combining all above block diagrams

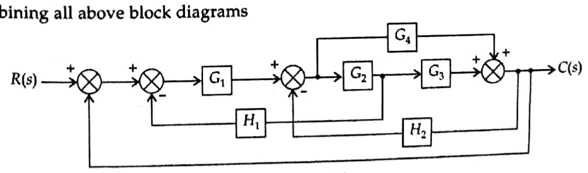


Fig. 1.120 (b)

The steady state error of the system is obtained by applying final value theorem.

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s.E(s) \quad ...(3.3)$$

$$e_{ss} = \lim_{s\to 0} s\cdot\frac{R(s)}{1+G(s)H(s)} \quad ...(3.4)$$

For unity feedback system $H(s) = 1$

$$e_{ss} = \lim_{s\to 0} s\cdot\frac{R(s)}{1+G(s)} \quad ...(3.5)$$

From the equation (3.4) or equation (3.5) it is clear that the steady state error depends on the input and open loop transfer function.

3.3. STATIC ERROR COEFFICIENTS

(a) Static-Position Error Constant (or Coefficient) K_p

The steady state error is given by equation (3.4)

$$e_{ss} = \lim_{s\to 0} s\cdot\frac{R(s)}{1+G(s)H(s)}$$

For unit step input $R(s) = \frac{1}{s}$, the steady state error is given by

$$e_{ss} = \lim_{s\to 0} s\cdot\frac{1}{s}\frac{1}{1+G(s)H(s)} = \frac{1}{1+\lim_{s\to 0} G(s)H(s)} = \frac{1}{1+K_p} \quad ...(3.6)$$

K_p = static position error constant = $\lim_{s\to 0} G(s)H(s)$

(b) Static Velocity Error Constant (or Coefficient) Kv

$$e_{ss} = \lim_{s\to 0} s.R(s)\cdot\frac{1}{1+G(s)H(s)}$$

Steady state error with a unit ramp input is given by $[R(s) = 1/s^2]$

$$e_{ss} = \lim_{s\to 0} s\cdot\frac{1}{s^2}\frac{1}{1+G(s)H(s)}$$

$$e_{ss} = \lim_{s\to 0}\frac{1}{s+sG(s)H(s)} = \lim_{s\to 0}\frac{1}{sG(s)H(s)} = \frac{1}{K_v} \quad ...(3.7)$$

Where $K_v = \lim_{s\to 0} sG(s)H(s)$ static velocity error coefficient.

(c) Static Acceleration Error Constant K_a

The steady-state error of the system with unit parabolic input is given by

$$R(s) = \frac{1}{s^3}$$

$$e_{ss} = \lim_{s\to 0} s\cdot\frac{1}{s^3}\cdot\frac{1}{1+G(s)H(s)} = \lim_{s\to 0}\frac{1}{s^2+s^2G(s)H(s)}$$

$$= \lim_{s\to 0}\frac{1}{s^2G(s)H(s)} = \frac{1}{K_a} \quad ...(3.8)$$

Where $K_a = \lim_{s\to 0} s^2\, G(s)\, H(s)$ = static acceleration constant.

3.4. STEADY-STATE ERROR FOR DIFFERENT TYPE OF SYSTEMS

1. (a) Type Zero System with Unit Step Input

$$R(s) = \frac{1}{s}$$

From equation (3.1)

$$G(s)\,H(s) = \frac{K(1+sT_1)(1+sT_2)...}{(1+sT_a)(1+sT_b)...} \quad ...(3.9)$$

$$K_P = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0}\frac{K(1+sT_1)(1+sT_2)...}{(1+sT_a)(1+sT_b)...} = K$$

From equation (3.6)

$$e_{ss} = \frac{1}{1+K_P} = \frac{1}{1+K} \qquad \boxed{e_{ss} = \frac{1}{1+K}}$$

Hence, for type zero system the static position error constant K_P is finite.

(b) Type '0' System with Unit Ramp Input

$$K_v = \lim_{s\to 0} s\cdot G(s)H(s) = \lim_{s\to 0} s\cdot\frac{K(1+sT_1)(1+sT_2)...}{(1+sT_a)(1+sT_b)...} = 0$$

$$\therefore \quad e_{ss} = \frac{1}{K_v} = \infty \qquad \boxed{e_{ss} = \infty}$$

(c) Type '0' System with Unit Parabolic Input

$$K_a = \lim_{s\to 0} s^2G(s)H(s) = \lim_{s\to 0} s^2\frac{K(1+sT_1)(1+sT_2)...}{(1+sTa)(1+sT_b)...}$$

$$K_a = 0$$

$$\therefore \quad e_{ss} = \frac{1}{K_a} = \infty \qquad \boxed{e_{ss} = \infty}$$

For type '0' system, the steady state error is infinite for ramp and parabolic inputs. Hence, the ramp and parabolic inputs are not acceptable.

2. (a) Type '1' System with Unit Step Input ($m = 1$)

$$G(s)\,H(s) = \frac{K(1+sT_1)(1+sT_2)...}{s(1+sTa)(1+sT_b)...}$$

$$K_P = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0}\frac{K(1+sT_1)(1+sT_2)...}{s(1+sT_a)(1+sT_b)...} = \infty$$

$$e_{ss} = \frac{1}{1+K_P} = 0 \qquad \boxed{e_{ss} = 0}$$

(b) Type '1' System with Unit Ramp Input

$$K_v = \lim_{s\to 0} s\cdot G(s)H(s) = \lim_{s\to 0} s\cdot\frac{K(1+sT_1)(1+sT_2)...}{s(1+sT_a)(1+sT_b)...}$$

$$K_v = K$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K} \qquad \boxed{e_{ss} = \frac{1}{K}}$$

(c) Type '1' System with Unit Parabolic Input

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} s^2 \frac{K(1+sT_1)(1+sT_2)...}{s(1+sT_a)(1+sT_b)...} = 0$$

$$\therefore \quad e_{ss} = \frac{1}{K_a} = \infty \qquad \boxed{e_{ss} = \infty}$$

Hence, from above relations for type '1' system, it is clear that for type '1' system step input and ramp inputs are acceptable and parabolic input is not acceptable.

3. (a) Type '2' System with Unit Step Input

$$G(s)\ H(s) = \frac{K(1+sT_1)(1+sT_2)...}{s^2(1+sT_a)(1+sT_b)...}$$

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{K(1+sT_1)(1+sT_2)...}{s^2(1+sT_a)(1+sT_b)...} = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = 0 \qquad \boxed{e_{ss} = 0}$$

(b) Type '2' System with Unit Ramp Input

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} s \cdot \frac{K(1+sT_1)(1+sT_2)...}{s^2(1+sT_a)(1+sT_b)...} = \infty$$

$$\therefore \quad e_{ss} = \frac{1}{K_v} = 0 \qquad \boxed{e_{ss} = 0}$$

(c) Type '2' System with Unit Parabolic Input

$$K_a = \lim_{s\to 0} s^2 G(s)H(s)$$

$$= \lim_{s\to 0} s^2 \cdot \frac{K(1+sT_1)(1+sT_2)...}{s^2(1+sT_a)(1+sT_b)...} = K$$

$$\therefore \quad e_{ss} = \frac{1}{K_a} = \frac{1}{K} \qquad e_{ss} = \frac{1}{K}$$

Hence, for type '2' system all three inputs (step, ramp and parabolic) are acceptable. From Table 3.1, the diagonal elements are the finite values of steady state error.

Table 3.1.

	Type '0' System	Type '1' System	Type '2' System
Unit step input	$\frac{1}{1+K}$	0	0
Unit ramp input	∞	$\frac{1}{K}$	0
Unit parabolic input	∞	∞	$\frac{1}{K}$

EXAMPLE 3.1. The open loop transfer function of unity feedback system is given by

$$G(s) = \frac{50}{(1+0.1s)(s+10)}$$

Determine the static error coefficients K_p, K_v and K_a.

Solution :

$$K_p = \lim_{s\to 0} G(s)H(s)$$

$$= \lim_{s\to 0} \frac{50}{(1+0.1s)(s+10)} = 5$$

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} s \cdot \frac{50}{(1+0.1s)(s+10)} = 0$$

$$K_a = \lim_{s\to 0} s^2 \frac{50}{(1+0.1s)(s+10)} = 0$$

EXAMPLE 3.2. The forward path transfer function of a unity feedback control system is given by

$$G(s) = \frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)}$$

Determine the step, ramp and parabolic error coefficients. Also determine the type of the system.

Solution :

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)}$$

$$K_p = \infty \qquad \boxed{K_p = \infty}$$

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} s \cdot \frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)}$$

$$K_v = \infty \qquad \boxed{K_v = \infty}$$

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} s^2 \cdot \frac{5(s^2+2s+100)}{s^2(s+5)(s^2+3s+10)}$$

$$= 10 \qquad \boxed{K_a = 10}$$

In denominator the value of $m = 2$. Hence, the given system is type '2' system.

EXAMPLE 3.3. The block diagram of an electronic pacemaker is given in Fig. 3.2. Determine the steady state error for unit ramp input when K = 400 Also, determine the value of K for which the steady state error to a unit ramp will be 0.02.

Solution : Given that $K = 400$

$$R(s) = \frac{1}{s^2}$$

$$H(s) = 1$$

$$\therefore \quad G(s)\ H(s) = \frac{K}{s(s+20)}$$

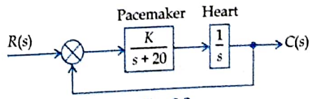


Fig. 3.2.

Steady state error is given by $e_{ss} = \lim_{s\to 0} s \cdot \frac{R(s)}{1+G(s)H(s)}$

$$e_{ss} = \lim_{s\to 0} s\cdot\frac{1}{s^2}\cdot\frac{1}{1+\frac{K}{s(s+20)}} = \lim_{s\to 0}\frac{s+20}{s(s+20)+400} = 0.05 \quad \text{Ans.}$$

Now $e_{ss} = 0.02$ (given)

$$e_{ss} = \lim_{s\to 0} s\cdot\frac{1}{s^2}\cdot\frac{1}{1+\frac{K}{s(s+20)}}$$

$$0.02 = \lim_{s\to 0}\frac{s+20}{s(s+20)+K}$$

$$0.02 = \frac{20}{K} \qquad \therefore \qquad \boxed{K = 1000} \quad \textbf{Ans.}$$

EXAMPLE 3.4. **For a unity feedback control system the forward path transfer function is given by**

$$G(s) = \frac{20}{s(s+2)(s^2+2s+20)}$$

Determine the steady state error of the system. When the inputs are : (*i*) 5, (*ii*) 5*t*, (*iii*) $\frac{3t^2}{2}$.

Solution:

(*i*) $r(t) = 5 \quad \therefore \quad R(s) = \frac{5}{s}$

$$e_{ss} = \lim_{s\to 0} s\cdot E(s) = \lim_{s\to 0} s\cdot\frac{R(s)}{1+G(s)H(s)}$$

$$e_{ss} = \lim_{s\to 0} s\cdot\frac{5}{s}\cdot\frac{1}{1+\frac{20}{s(s+2)(s^2+2s+20)}} = \lim_{s\to 0}\frac{5s(s+2)(s^2+2s+20)}{s(s+2)(s^2+2s+20)+20}$$

$\therefore \quad e_{ss} = 0$

(*ii*) $R(s) = \frac{5}{s^2}$

$$e_{ss} = \lim_{s\to 0} s\cdot\frac{5}{s^2}\cdot\frac{1}{1+\frac{20}{s(s+2)(s^2+2s+20)}} = \lim_{s\to 0} s\cdot\frac{5}{s^2}\cdot\frac{s(s+2)(s^2+2s+20)}{s(s+2)(s^2+2s+20)+20}$$

$$e_{ss} = 10$$

(*iii*) $R(s) = \frac{3}{s^3}$

$$e_{ss} = \lim_{s\to 0} s\cdot\frac{3}{s^3}\cdot\frac{s(s+2)(s^2+2s+20)}{s(s+2)(s^2+2s+20)+20}$$

$$e_{ss} = \infty$$

EXAMPLE 3.5. **The open loop transfer function of a unity feedback system is given by**

$$G(s) = \frac{108}{s^2(s+4)(s^2+3s+12)}$$

Find the static error coefficients and steady state error of the system when subjected to an input given by

$$r(t) = 2 + 5t + 2t^2$$

Solution :

$$K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0}\frac{108}{s^2(s+4)(s^2+3s+12)} = \infty$$

$$K_v = \lim_{s\to 0} s\cdot G(s) = \lim_{s\to 0} s\cdot\frac{108}{s^2(s+4)(s^2+3s+12)} = \infty$$

$$K_a = \lim_{s\to 0} s^2G(s) = \lim_{s\to 0} s^2\cdot\frac{108}{s^2(s+4)(s^2+3s+12)} = \frac{108}{48}$$

$$r(t) = 2 + 5t + 2t^2$$

$$R(s) = \frac{2}{s}+\frac{5}{s^2}+\frac{4}{s^3}$$

$$\therefore \quad e_{ss} = \frac{R_1}{1+K_p}+\frac{R_2}{K_v}+\frac{R_3}{K_a}$$

$$= \frac{2}{1+\infty}+\frac{5}{\infty}+\frac{4\times 48}{108} = 1.77 \qquad \boxed{e_{ss} = 1.77} \quad \textbf{Ans.}$$

EXAMPLE 3.6. **Determine the type of the following unity feedback systems for which the forward path transfer function are given**

(*a*) $G(s) = \frac{K}{(s+10)(s+5)}$ (*b*) $G(s) = \frac{20}{s(1+0.1s)(1+0.05s)}$

(*c*) $G(s) = \frac{20(s+2)}{s^3(s+2)(s^2+5s+5)}$ (*d*) $G(s) = \frac{K}{s(s+1)(s+2)}$

Solution : (*a*) Since in denominator the power of s is zero *i.e.*, $m = 0$, hence it is type zero system.

(*b*) type one system

(*c*) $m = 3$, type '3' system

(*d*) $m = 1$, type '1' system

EXAMPLE 3.7. **A servomechanism is designed to keep a radar antenna pointed at a flying aeroplane. If the aeroplane is flying with a velocity of 600 km/hr, at a range of 2 km and the maximum tracking error is to be within 0.1°, determine the required velocity error coefficient.** (*GATE, 1994*)

Solution : The block diagram of servomechanism is shown in Fig. 3.3.

Linear velocity $= 600$ km/hr

$r = 2$ km

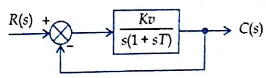


Fig. 3.3.

Angular velocity $\omega = \frac{600}{2} = 300$ rad/hr.

$$= \frac{300}{3600}\text{ rad/sec} = 0.083\text{ rad/sec.}$$

$$R(s) = \frac{0.083}{s^2}$$

$$e_{ss} = \lim_{s\to 0} s\cdot R(s)\cdot\frac{1}{1+G(s)H(s)} = \lim_{s\to 0} s\cdot\frac{0.083}{s^2}\cdot\frac{1}{1+\frac{K_v}{s(1+sT)}}$$

$$e_{ss} = \lim_{s\to 0}\frac{0.083}{s}\cdot\frac{s(1+sT)}{s(1+sT)+K_v} = \frac{0.083}{K_v} = 0.1^\circ\text{ given}$$

$$0.1 = \frac{0.083}{K_v}$$

$\therefore$ $K_v = 0.83$ per degree

$\therefore$ Required velocity error coefficient $K_v = 0.83$ per degree.

EXAMPLE 3.8. **For the system shown in Fig. 3.4. Determine K_p and e_{ss} for unit step input.**

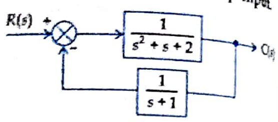


Fig. 3.4.

Solution :

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$= \lim_{s \to 0} \frac{1}{(s^2+s+2)(s+1)} = \frac{1}{2}$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\frac{1}{2}} = \frac{1}{3/2} = \frac{2}{3}$$

$\boxed{K_p = 1/2}$ $\boxed{e_{ss} = 2/3}$ **Ans.**

EXAMPLE 3.9. **For a given system shown in Fig. 3.5, determine the actuating signal $E_a(s)$. Also determine the position error constant for unit step input.**

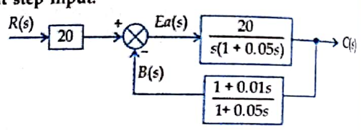


Fig. 3.5.

Solution :

$$E_a(s) = 20\,R(s) - B(s)$$

$$C(s) = \frac{20Ea(s)}{s(1+0.05s)}$$

$$B(s) = C(s) \cdot \frac{1+0.01s}{1+0.05s}$$

$$B(s) = \frac{20(1+0.01s)E_a(s)}{s(1+0.05s)^2}$$

$\therefore$
$$E_a(s) = 20R(s) - \frac{20(1+0.01s)E_a(s)}{s(1+0.05s)^2}$$

$$E_a(s)\left[1+\frac{20(1+0.01s)}{s(1+0.05s)^2}\right] = 20\,R(s)$$

$\therefore$
$$E_a(s) = \frac{20R(s)}{1+\dfrac{20(1+0.01s)}{s(1+0.05s)^2}}$$

$$E_a(s) = \frac{20s(1+0.05s)^2 \cdot R(s)}{s(1+0.05s)^2 + 20(1+0.01s)} \quad \textbf{Ans.}$$

For step input $R(s) = \dfrac{1}{s}$

$$E_a(s) = \frac{20(1+0.05s)^2}{s(1+0.05s)^2 + 20(1+0.01s)}$$

$\therefore$
$$e_{ss} = \lim_{s \to 0} s \cdot E(s) = \lim_{s \to 0} s \cdot \frac{20(1+0.05s)^2}{s(1+0.05s)^2 + 20(1+0.01s)} = 0$$

But $e_{ss} = \dfrac{1}{1+K_p}$ or $0 = \dfrac{1}{1+K_p}$

$\therefore$ $\boxed{K_p = \infty}$ **Ans.**

EXAMPLE 3.10. **Consider a unity feedback control system with the closed loop transfer function**

$$\frac{C(s)}{R(s)} = \frac{Ks+b}{s^2+as+b}$$

Determine the open loop transfer function. Show that the steady state error in the unit ramp input response is given by

$$e_{ss} = \frac{a-k}{b}$$

Solution : For closed loop control system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{Ks+b}{s^2+as+b} = \frac{G(s)}{1+G(s)}$$

$\therefore$
$$G(s) = \frac{Ks+b}{s^2+s(a-K)}$$

$\therefore$ Open loop transfer function $G(s) = \dfrac{Ks+b}{s^2+s(a-k)}$ **Ans.**

Steady state error $e_{ss} = \lim_{s \to 0} s \cdot R(s) \cdot \dfrac{1}{1+G(s)H(s)}$

$$e_{ss} = \lim_{s \to 0} s \cdot \frac{1}{s^2} \cdot \frac{1}{1+\dfrac{Ks+b}{s^2+s(a-K)} \cdot 1} = \lim_{s \to 0} \frac{1}{s} \cdot \frac{s^2+s(a-K)}{s^2+s(a-K)+Ks+b}$$

$$= \lim_{s \to 0} \frac{1}{s} \frac{s[s+a-K]}{s^2+s(a-K)+Ks+b} = \frac{a-K}{b}$$

$$e_{ss} = \frac{a-K}{b} \quad \textbf{Proved.}$$

3.5. DYNAMIC ERROR COEFFICIENTS

For the steady-state error, the static error coefficients gives the limited information.

The error function is given by

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} \quad \text{...(3.10)}$$

For unity feedback system

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} \quad \text{...(3.11)}$$

The equation (3.11) can be expressed in polynomial form (ascending power of 's')

$$\frac{E(s)}{R(s)} = \frac{1}{K_1} + \frac{1}{K_2}s + \frac{1}{K_3}s^2 + \ldots.. \quad \text{...(3.12)}$$