A Robust Design for MISO Physical-Layer Multicasting over Line-of-Sight Channels

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***Abstract*—This paper studies a robust design problem for far-field line-of-sight (LOS) propagation channels where phase errors are present. Compared with the commonly used additive error model, the phase error model is more suitable for capturing the uncertainty in an LOS propagation channel, as the dominant source of uncertainty lies in the phase. We consider a multiple-input single-output (MISO) multicast scenario, in which our goal is to design a beamformer that minimizes the transmit power while satisfying probabilistic signal-to-noise ratio (SNR) constraints. In particular, the probabilistic constraints give rise to a new computational challenge, as they involve random trigonometric forms. In this work, we propose to first approximate the random trigonometric form by its second-order Taylor expansion and then tackle the resulting random quadratic form using a Bernstein-type inequality. It follows that an approximately optimal beamformer can be obtained using the standard semidefinite relaxation technique. Such a design approach is applicable to both independent and correlated phase errors. In the simulations, we first show that if a non-robust design (i.e., one that does not take phase errors into account) is used, then the whole system may collapse. We then show that our proposed method is less conservative than the existing robust design based on Gaussian approximation and thus requires a lower power budget.**

***Index terms*—** MISO, multicast, line-of-sight (LOS), phase error, robust, beamforming.

I. Introduction

In wireless communication systems, a channel is called line-of-sight (LOS) if a direct link between the transmitter and the receiver is always present. Information delivery over LOS channels is an important kind of data transmission in modern wireless systems, with numerous applications including satellite communications, indoor communications, and near-base-station communications. A typical far-field multiple-input single-output (MISO) LOS channel [1, Chapter 7.22] takes the form $\boldsymbol{h} = a \cdot \boldsymbol{e}$ with $\boldsymbol{e} = \exp\left(-i2\pi d/\lambda_c\right)\left[1, \exp(i2\pi\theta t), ..., \exp(i2\pi(n-1)\theta t)\right]^T$, where λ_c is the carrier wavelength, d is the distance between the transmitter and the receiver, θ is the normalized receiver antenna separation (normalized to the unit of the carrier wavelength), and a is the attenuation of the path. The magnitude of the LOS channel $\boldsymbol{h}$ is determined solely by the path attenuation a, which usually varies slowly and is easy to estimate in practice [2]–[4]. By contrast, the phase $\boldsymbol{e}$ depends on many factors such as distance, antenna position, and oscillator offsets.[1] In real-world systems, errors in those factors may originate from long distance transmission delay [5], [6] (e.g., for satellite channels), asynchronous carrier frequency (e.g., cheap oscillators in mobile terminals), or arrival delay at different antennas. Therefore, phase error has a more dominant effect in LOS channels. Moreover, the phase error at different antennas could be correlated due to different inducing factors. Although there is a vast literature on robust beamforming for LOS channels, to our best knowledge, not much has been done to incorporate phase error in the noise model. Two related works we are aware of are those by El-Keyi and Champagne [3] and Gharanjik et al [2]. In [3], the authors study collaborative uplink beamforming for LOS channels with phase errors and model them as $\boldsymbol{e} + \boldsymbol{\Delta}$ with $\boldsymbol{\Delta}$ being the uncertainty. This model is additive in nature (with respect to $\boldsymbol{e}$) and the channel magnitude is subject to change. Recently, the authors of [2] propose modeling the phase error by adding a Gaussian noise directly to the phases of the entries of $\boldsymbol{e}$. Such a multiplicative noise model is attractive from a modeling perspective. However, this model results in a substantially more difficult design problem.

In this paper, we model the phase error as a Gaussian vector with zero mean and a given covariance matrix, which represents the correlation of the phase error among different antennas. We are interested in the robust beamforming design for the MISO downlink multicast LOS channel under the same phase error model as in [2] (the applicability of our approach is actually not limited to MISO multicasting). Specifically, we consider the power minimization problem subject to probabilistic outage constraints. Such constraints involve random trigonometric forms, which have rarely been addressed in the beamforming literature. To obtain more tractable approximations of the probabilistic constraints, we propose to approximate the random trigonometric form by a second-order Taylor expansion and then apply the Bernstein-type inequality approach in [7], [8]. The resulting formulation can then be tackled by the semidefinite relaxation (SDR) technique, thereby leading to an approximately optimal beamformer. To demonstrate the effectiveness of our approach, we compare our method with that in [2] by simulations. The results show that the proposed design approach is less conservative and require less transmit power than the Gaussian approximation (GA) approach in [2].

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[1]The LOS channel with reflected paths can be similarly modeled as a sum of $\boldsymbol{h}$'s. Moreover, phase uncertainty also exists in non-LOS channels. The design for such channels is beyond the scope of this work.

II. System Model and Problem Formulation

We consider the physical-layer multicasting channel, where a base station with n antennas intends to transmit a common signal to m single-antenna receivers. Suppose that the channel is quasi-static and we consider the transmit design for each code block. As such, we shall omit the time index from the notations henceforth. For $j = 1, \ldots, m$, let $\boldsymbol{h}_j \in \mathbb{C}^n$ denote the estimated channel for user j and we focus on the problem of beamforming design that is robust against the phase error of the channel. Specifically, we model the phase error of the j-th channel by a Gaussian random vector $\boldsymbol{\theta}_j$. In general, the i-th entry of $\boldsymbol{\theta}_j$ satisfies $-(2k\pi+\pi) \le \theta_{ji} \le (2k\pi+\pi), k \in \mathbb{Z}$. Since the phase error is usually small compared to π, it can be assumed to follow a Gaussian distribution. In this paper, we model the phase error as $\boldsymbol{\theta}_j \sim \mathcal{N}(0, \sigma^2 \boldsymbol{C}_j)$, where $\boldsymbol{C}_j \succeq \boldsymbol{0}$, $\mathrm{Tr}(\boldsymbol{C}_j) = n$, and σ^2 is a scaling factor. Denoting $\boldsymbol{e}_j = (e^{i\theta_{j1}}, \ldots, e^{i\theta_{jn}})^T$, the corrupted channel for user j can be expressed as

$$\hat{\boldsymbol{h}}_j = \boldsymbol{h}_j \odot \boldsymbol{e}_j, \tag{1}$$

where $\odot$ is the entry-wise product.

Let $s \in \mathbb{C}$ be the common unit-power signal intended for all users. Before transmission, the signal will be precoded by a precoding vector $\boldsymbol{w} \in \mathbb{C}^n$ and the resulting signal for transmission is $\boldsymbol{w}s$. After passing through the channels, the signal obtained at the j-th receiver is

$$g_j = \hat{\boldsymbol{h}}_j^H \boldsymbol{w} s + \epsilon_j, \quad j = 1, ..., m, \tag{2}$$

where ϵ_j is the additive noise at user j and is assumed to be Gaussian distributed with mean zero and variance η_j^2. Also, the random variables/vectors $\epsilon_1, \ldots, \epsilon_m, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$ are assumed to be independent. As usual, the quality-of-service (QoS) at user j is measured by the SNR; i.e.,

$$\mathrm{SNR}_j(\boldsymbol{w}\boldsymbol{w}^H) = \boldsymbol{e}_j^H M_j(\boldsymbol{w}\boldsymbol{w}^H)\boldsymbol{e}_j / \eta_j^2, \tag{3}$$

where M_j is the operator on the space of Hermitian matrices given by

$$M_j(\boldsymbol{W}) = \boldsymbol{W} \odot (\boldsymbol{h}_j \boldsymbol{h}_j^H)^T. \tag{4}$$

We say that user j is in outage if $\mathrm{SNR}_j < \gamma$, where $\gamma > 0$ is a prescribed threshold. A natural formulation of the beamforming design problem is to minimize the power while maintaining a low outage probability:

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \|\boldsymbol{w}\|_2^2 \\ \text{s.t.} \quad & \Pr(\mathrm{SNR}_j(\boldsymbol{w}\boldsymbol{w}^H) > \gamma) \ge 1 - \rho_j, \; j = 1, \ldots, m. \end{aligned} \tag{5}$$

Here, ρ_j's are the outage probabilities. Assuming $\eta_j^2 = 1, \forall j$ and letting $\boldsymbol{W} = \boldsymbol{w}\boldsymbol{w}^H$, we can equivalently write (5) as

$$\text{(P)} \quad \min_{\boldsymbol{W}} \quad \mathrm{Tr}(\boldsymbol{W})$$

$$\text{s.t.} \quad \Pr(\boldsymbol{e}_j^H M_j(\boldsymbol{W})\boldsymbol{e}_j > \gamma) \ge 1 - \rho_j, \; j = 1, \ldots, m, \tag{6}$$

$$\mathrm{Rank}(\boldsymbol{W}) = 1, \quad \boldsymbol{W} \succeq 0. \tag{7}$$

Recall that we have $\boldsymbol{e}_j = (e^{i\theta_{j1}}, \ldots, e^{i\theta_{jn}})^T$ and $\boldsymbol{\theta}_j \sim \mathcal{N}(0, \sigma^2 \boldsymbol{C}_j)$. Problem (P) is difficult to handle since we do not have a good analytic description of the probabilistic constraint (6). In [2], the authors work around this by approximating $\boldsymbol{e}_j^H M_j(\boldsymbol{W})\boldsymbol{e}_j$ by a Gaussian random variable with matching mean and variance. The aim of this paper is to derive a more accurate and efficient alternative to handle (6) by approximating the error vector $\boldsymbol{e}_j$.

III. Approximating the probabilistic constraints

In this section, we use a Taylor series approximation and the Bernstein-type inequality to tackle (6). Specifically, by considering the second-order Taylor expansion of the random variable $\boldsymbol{e}_j^H M_j(\boldsymbol{W})\boldsymbol{e}_j$, we transform the outage event into the large deviation of a quadratic function in Gaussian random variables. The probabilistic constraint can then be handled by the Bernstein-type inequality [7], [8]. Finally, by employing the SDR technique, we obtain a tractable approximation of the probabilistic constraint (6). Let us first introduce some notations. Denote by $\mathcal{S}^n$ and $\mathcal{K}^n$ the sets of $n \times n$ real symmetric and skew-symmetric matrices respectively. Given any arbitrary matrix $\boldsymbol{A}$ with its (i,j)-th element denoted by A_{ij}, let $L : \mathbb{R}^{n\times n} \to \mathbb{R}^{n \times n}$ be the linear map given by

$$(L(\boldsymbol{A}))_{kl} = \begin{cases} A_{kk} - \sum_j A_{kj} & \text{for } k = l, \\ A_{kl} & \text{for } k \ne l \end{cases}$$

and $f : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ be the linear map given by

$$(f(\boldsymbol{A}))_k = 2 \sum_j A_{kj}.$$

For $\boldsymbol{A} \in \mathcal{S}^n$, define $\lambda^-(\boldsymbol{A}) = \max\{\lambda_{\max}(-\boldsymbol{A}), 0\}$. To build our approximation, we need the following two lemmas, whose proofs are relegated to the appendix.

Lemma 1 *Let $\boldsymbol{\theta} \in \mathbb{R}^n$, $\boldsymbol{x} = [e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n}]^T$, and $\boldsymbol{M}$ be an $n \times n$ Hermitian matrix with symmetric real part $\boldsymbol{A} \in \mathcal{S}^n$ and skew-symmetric imaginary part $\boldsymbol{B} \in \mathcal{K}^n$. Then, the second-order Taylor approximation of $\boldsymbol{x}^H \boldsymbol{M} \boldsymbol{x}$ is given by*

$$\boldsymbol{x}^H \boldsymbol{M} \boldsymbol{x} \approx \sum_{k,l} M_{kl} + \boldsymbol{\theta}^T L(\boldsymbol{A})\boldsymbol{\theta} + f(\boldsymbol{B})^T \boldsymbol{\theta}. \tag{8}$$

Lemma 1 allows us to approximate the hard probabilistic constraint (6) by a more well-studied one. In particular, if $\boldsymbol{\theta} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{C})$ and σ is small, then we should have

$$\Pr\big(\boldsymbol{x}^H \boldsymbol{M} \boldsymbol{x} \ge \gamma\big) \tag{9}$$

$$\approx \Pr\left(\sum_{k,l} M_{kl} + \sigma^2 \boldsymbol{\xi}^T \boldsymbol{C}^{\frac{1}{2}} L(\boldsymbol{A}) \boldsymbol{C}^{\frac{1}{2}} \boldsymbol{\xi} + \sigma f(\boldsymbol{B})^T \boldsymbol{C}^{\frac{1}{2}} \boldsymbol{\xi} \ge \gamma \right),$$

where $\boldsymbol{\xi} \sim \mathcal{N}(0, \boldsymbol{I})$ is the standard n-dimensional Gaussian random vector. Given the above approximation, it suffices to approximate the right hand side of (9), which can be achieved using the following lemma:

Algorithm 1 Gaussian Randomized Rounding

Input: Optimal solution $\boldsymbol{W}^\star$ to (APR), number of trials I.
if $\mathrm{Rank}(\boldsymbol{W}^\star) > 1$ **then**
  for $i = 1, 2, \ldots, I$ **do**
    Generate $\tilde{\boldsymbol{w}}_i \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{W}^\star)$.
    Scale $\tilde{\boldsymbol{w}}_i$ so that $(A_j(\tilde{\boldsymbol{w}}_i\tilde{\boldsymbol{w}}_i^H), B_j(\tilde{\boldsymbol{w}}_i\tilde{\boldsymbol{w}}_i^H)) \in S_{\gamma,\rho}$ for all $j = 1, \ldots, m$.
  end for
  Set $\hat{\boldsymbol{w}} = \mathrm{argmin}_i \mathrm{Tr}(\tilde{\boldsymbol{w}}_i\tilde{\boldsymbol{w}}_i^H)$.
else
  Set $\hat{\boldsymbol{w}}$ such that $\hat{\boldsymbol{w}}\hat{\boldsymbol{w}}^H = \boldsymbol{W}^\star$.
end if
Output: A beamformer $\hat{\boldsymbol{w}}$ with $\hat{\boldsymbol{w}}\hat{\boldsymbol{w}}^H$ feasible to (AP).

Lemma 2 *Let $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{Q}, \boldsymbol{b}, y, z)$ be a solution to the system*

$$\begin{cases} \sum_{k,l} A_{kl} + \mathrm{Tr}(\boldsymbol{Q}) - 2z\sqrt{\log \rho^{-1}} + 2y \log \rho \geq \gamma, \\ \sqrt{\|\boldsymbol{Q}\|_F^2 + \frac{1}{2}\|\boldsymbol{b}\|_2^2} \leq z, \\ y\boldsymbol{I} + \boldsymbol{Q} \succeq \boldsymbol{0}, \\ \boldsymbol{Q} = \sigma^2 \boldsymbol{C}^{\frac{1}{2}} L(\boldsymbol{A}) \boldsymbol{C}^{\frac{1}{2}}, \\ \boldsymbol{b} = \sigma \boldsymbol{C}^{\frac{1}{2}} f(\boldsymbol{B}), \\ y \geq 0, \\ \boldsymbol{Q}, \boldsymbol{A} \in \mathcal{S}^n, \boldsymbol{B} \in \mathcal{K}^n. \end{cases} \tag{$S_{\gamma,\rho}$}$$

Then, $\boldsymbol{M} = \boldsymbol{A} + i\boldsymbol{B}$ satisfies

$$\Pr\left(\sum_{k,l} M_{kl} + \sigma^2 \boldsymbol{\xi}^T \boldsymbol{C}^{\frac{1}{2}} L(\boldsymbol{A}) \boldsymbol{C}^{\frac{1}{2}} \boldsymbol{\xi} + \sigma f(\boldsymbol{B})^T \boldsymbol{C}^{\frac{1}{2}} \boldsymbol{\xi} \geq \gamma\right) \geq 1 - \rho. \tag{10}$$

With a slight abuse of notation, we write $(\boldsymbol{A}, \boldsymbol{B}) \in S_{\gamma,\rho}$ if there exist $\boldsymbol{Q}$, $\boldsymbol{b}$, y, and z such that $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{Q}, \boldsymbol{b}, y, z)$ is feasible for system $(S_{\gamma,\rho})$. Also, let $A_j(\boldsymbol{W})$ and $B_j(\boldsymbol{W})$ be the real and imaginary parts of $M_j(\boldsymbol{W})$, respectively. Using Lemmas 1 and 2, we obtain the following approximation of Problem (P):

$$\begin{aligned} \text{(AP)} \quad & \min_{\boldsymbol{W} \succeq 0} \quad \mathrm{Tr}(\boldsymbol{W}) \\ & \text{s.t.} \quad (A_j(\boldsymbol{W}), B_j(\boldsymbol{W})) \in S_{\gamma,\rho}, \; j = 1, \ldots, m, \quad (11) \\ & \qquad \mathrm{Rank}(\boldsymbol{W}) = 1. \end{aligned}$$

By dropping the non-convex rank-one constraint, we arrive at

$$\text{(APR)} \quad \min_{\boldsymbol{W} \succeq 0} \quad \mathrm{Tr}(\boldsymbol{W}) \qquad \text{s.t.} \quad \text{(11) satisfied.}$$

Problem (APR) is an instance of semidefinite programming (SDP) and hence can be solved efficiently by any off-the-shelf SDP solver [9]. The optimal solution $\boldsymbol{W}^\star$ to Problem (APR) can be of any rank because we drop the rank constraint. For ease of presentation, we consider the problem instances where Problems (P), (AP), and (APR) are feasible. Then, to extract a rank-one solution from $\boldsymbol{W}^\star$, we can use Gaussian rounding (see Algorithm 1) to obtain a vector beamformer $\hat{\boldsymbol{w}}$ such that $\hat{\boldsymbol{w}}\hat{\boldsymbol{w}}^H$ is approximately feasible to Problem (P). We remark that although Problem (APR) is convex and tractable, its feasible region is not necessarily included in that of (P) and hence theoretically the beamformer $\hat{\boldsymbol{w}}$ can fail to satisfy the original outage probability requirement. However, our numerical results in the next section show that the proposed approximation is actually *empirically safe* under our scenario settings: the outage probabilities given by $\hat{\boldsymbol{w}}$ are always smaller than ρ_j's.

IV. Numerical Simulations

In this section, we provide numerical simulations to demonstrate the superiority of the proposed robust design. The setup of the experiment is as follows: the number of transmit antennas is $n = 8$; the number of users served is $m = 16$; channels are generated by $\boldsymbol{h}_i \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I})$ independently; the noise power at each user is set to be 1; the phase error variance is $\sigma_j = 5/360, \forall j$; the SNR outage probability is set to be $\rho_j = 0.1, \forall j$. We consider the cases $\boldsymbol{\theta}_j \sim \mathcal{N}(0, \sigma^2\boldsymbol{I}), \forall j$ and $\boldsymbol{\theta}_j \sim \mathcal{N}(0, \sigma^2\boldsymbol{C}_j)$, where $\boldsymbol{C}_j$ are randomized covariance matrices with $\mathrm{Tr}(\boldsymbol{C}_j) = n$. In both cases, we averaged 100 channel realizations to get the plots and generated 10^4 phase errors to count the outage probability. If the principal eigenvalue is 10^4 times larger than the second large eigenvalue, we consider the SDR solution as rank-one.

In Figure 1, we show the minimum power required to satisfy the outage constraints as the SNR threshold varies from 0dB to 6dB when $\rho = 0.1$. Specifically, we compare the GA design in [2] with the proposed design (called TB in the legend). From the plots, we see that the optimal values of the SDRs serve as lower bounds of the respective beamforming schemes. In the legend, "SDR" refers to the values obtained by the SDRs prior to rounding and "BF" refers to the actual power required by the beamformers produced by the rounding. The figure shows that the proposed TB approach needs a lower power budget than the GA approach in [2]. It is worth noting that, for both independent errors (IE) and random correlated errors (CE), the minimum power required for transmit beamforming is almost the same. To investigate the conservatism of the different schemes, we plot the histogram of the SNR satisfaction probability of the obtained beamformers; see Figure 2. We see that for the non-robust approach, the channel errors are simply ignored and the problem reduces to the classical physical-layer multicasting design problem [10], [11]. The non-robust histogram reveals that if we do not take phase error into account, the outage probability requirement could be seriously violated and the system is totally unreliable. This demonstrates the importance of the robust designs. For the GA approach, the SNR satisfaction probability is always 1 under our setting. This means that the GA scheme sacrifices extra transmit power for unnecessary conservatism. This is also manifested in Figure 1. For the proposed TB approach, we see that the SNR satisfaction probability is less than 1 but exceeds the target threshold 0.9. Hence, the proposed TB approach is reliable but less conservative than the GA counterpart. Both figures demonstrate the superiority of the proposed TB design.

V. Conclusion

In this paper, we study the robust design for LOS channels with arbitrary correlated phase errors. We consider the MISO physical-layer multicasting scenario and aim at minimizing the transmit power subject to probabilistic SNR constraints.

We employ Taylor series, a Bernstein-type inequality, and the SDR technique to approximate the probabilistic SNR constraints by SDPs, from which an approximate beamformer can be obtained. According to our simulation results, the robust design is important when phase error is present. Furthermore, the proposed method is less conservative and requires lower transmit power when compared with the existing design approach based on Gaussian approximations. In real applications, except for satellite communications and some near-base-station transmissions, most of the communication channels are non-LOS with uncertainty in both phase and amplitude. Hence, our future direction is to extend the proposed method to study more general non-LOS channels.

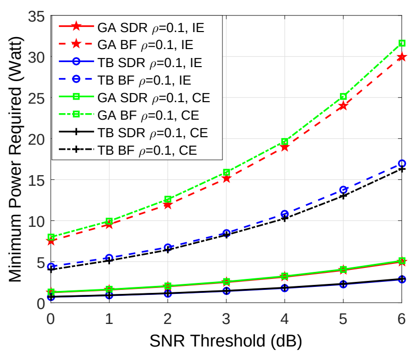

Fig. 1. Minimum power required versus the SNR threshold when the SNR satisfaction probability is equal to $\rho = 0.1$.

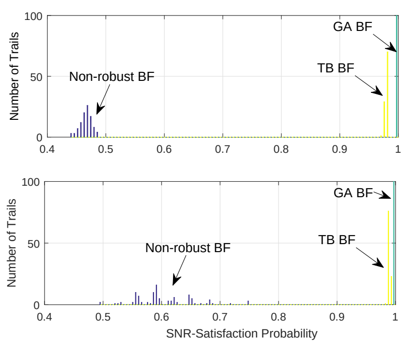

Fig. 2. Histogram of the SNR satisfaction probability. The upper figure is for independent phase errors and the lower figure is for phase errors with a given covariance matrix, which is randomly generated by MATLAB.

VI. Appendix

A. Proof of Lemma 1

Letting $\theta_{kl} = \theta_k - \theta_l$ and $M_{kl} = r_{kl}e^{i\phi_{kl}}$, we have

$$\begin{aligned}\boldsymbol{x}^H\boldsymbol{M}\boldsymbol{x} &= \sum_{k,l} M_{lk}e^{i(\theta_k-\theta_l)} \\ &= \sum_k M_{kk} + \sum_{k>l} M_{lk}e^{i\theta_{kl}} + \sum_{k>l} M_{kl}e^{i\theta_{lk}} \\ &= \sum_k M_{kk} + 2\sum_{k>l} r_{lk}\cos(\phi_{lk}+\theta_{kl}).\end{aligned}$$

To handle the cosine terms, we use Taylor approximation:

$$\begin{aligned}\cos(\phi_{lk}+\theta_{kl}) &= \cos\phi_{lk}\cos\theta_{kl} - \sin\phi_{lk}\sin\theta_{kl} \\ &\approx \cos\phi_{lk} - \frac{\theta_{kl}^2}{2}\cos\phi_{lk} - \theta_{kl}\sin\phi_{lk}.\end{aligned}$$

Thus,

$$\begin{aligned}\boldsymbol{x}^H\boldsymbol{M}\boldsymbol{x} \approx &= \sum_k M_{kk} + \sum_{k\neq l} r_{kl}\cos\phi_{kl} \\ &\quad - \sum_{k>l} A_{kl}\theta_{kl}^2 + 2\sum_{k>l} B_{kl}\theta_{kl} \quad (\because \phi_{kl} = -\phi_{lk}) \\ &= \sum_{k,l} M_{kl} - \sum_{k>l} A_{kl}\theta_{kl}^2 + 2\sum_{k>l} B_{kl}\theta_{kl}.\end{aligned}$$

Since $\boldsymbol{A} \in \mathcal{S}^n$ and $\boldsymbol{B} \in \mathcal{K}^n$, we have $\sum_{k>l} A_{kl}\theta_{kl}^2 = \frac{1}{2}\sum_{k,l} A_{kl}\theta_{kl}^2$ and $2\sum_{k>l} B_{kl}\theta_{kl} = \sum_{k,l} B_{kl}\theta_{kl}$. Therefore, we have

$$\begin{aligned}&- \sum_{k>l} A_{kl}\theta_{kl}^2 + 2\sum_{k>l} B_{kl}\theta_{kl} \\ &= -\sum_k\left(\theta_k^2\sum_l A_{kl}\right) + \sum_{k,l} A_{kl}\theta_k\theta_l + 2\sum_k\left(\theta_k\sum_l B_{kl}\right) \\ &= \boldsymbol{\theta}^T L(\boldsymbol{A})\boldsymbol{\theta} + f(\boldsymbol{B})^T\boldsymbol{\theta},\end{aligned}$$

which implies that $\boldsymbol{x}^H\boldsymbol{M}\boldsymbol{x} \approx \sum_{k,l} M_{kl} + \boldsymbol{\theta}^T L(\boldsymbol{A})\boldsymbol{\theta} + f(\boldsymbol{B})^T\boldsymbol{\theta}$. This completes the proof.

B. Proof of Lemma 2

By [8, Fact 1], we have

$$\begin{aligned}&\Pr\Big(\sigma^2\boldsymbol{\xi}^T\boldsymbol{L}\boldsymbol{\xi} + \sigma\boldsymbol{f}^T\boldsymbol{\xi} \geq \sigma^2\mathrm{Tr}(\boldsymbol{L}) - 2\sigma^2\lambda^-(\boldsymbol{L})\tau \\ &-2\sqrt{\sigma^4\|\boldsymbol{L}\|_F^2 + \frac{1}{2}\sigma^2\|\boldsymbol{f}\|_2^2}\sqrt{\tau}\Big) \geq 1 - e^{-\tau},\end{aligned} \tag{12}$$

where $\boldsymbol{L} = \boldsymbol{C}^{\frac{1}{2}}L(\boldsymbol{A})\boldsymbol{C}^{\frac{1}{2}}$ and $\boldsymbol{f} = \boldsymbol{C}^{\frac{1}{2}}f(\boldsymbol{B})$. Define $\zeta : (0,\infty) \to (-\infty, \sigma^2\mathrm{Tr}(\boldsymbol{L})]$ to be the function

$$\zeta(\tau) = \sigma^2\mathrm{Tr}(\boldsymbol{L}) - 2\sigma^2\lambda^-(\boldsymbol{L})\tau - 2\sqrt{\sigma^4\|\boldsymbol{L}\|_F^2 + \frac{1}{2}\sigma^2\|\boldsymbol{f}\|_2^2}\sqrt{\tau}.$$

Since ζ is strictly decreasing, ζ^{-1} is well defined on $(-\infty, \sigma^2\mathrm{Tr}(\boldsymbol{L})]$. If $\gamma \leq \sum_{k,l} A_{kl} + \sigma^2\mathrm{Tr}(\boldsymbol{L})$, then

$$\bar{\gamma} := \gamma - \sum_{k,l} A_{kl} \leq \sigma^2\mathrm{Tr}(\boldsymbol{L}).$$

Thus (12) can be rewritten as

$$\Pr\left(\sum_{k,l} M_{kl} + \boldsymbol{\theta}^T\boldsymbol{L}\boldsymbol{\theta} + \boldsymbol{f}^T\boldsymbol{\theta} \geq \gamma\right) \geq 1 - e^{-\zeta^{-1}(\bar{\gamma})}. \tag{13}$$

This leads to the following sufficient condition for (10):

$$1 - e^{-\zeta^{-1}(\bar{\gamma})} \geq 1 - \rho \Leftrightarrow \zeta^{-1}(\bar{\gamma}) \geq -\log\rho \Leftrightarrow \bar{\gamma} \leq \zeta(-\log\rho).$$

Written out explicitly, the above becomes

$$\begin{aligned}&\sum_{k,l} A_{kl} + \sigma^2\mathrm{Tr}(\boldsymbol{L}) + 2\sigma^2\lambda^-(\boldsymbol{L})\log\rho \\ &- 2\sqrt{-\log\rho}\sqrt{\sigma^4\|\boldsymbol{L}\|_F^2 + \frac{\sigma^2}{2}\|\boldsymbol{f}\|_2^2} \geq \gamma.\end{aligned} \tag{14}$$

This constraint can equivalently be expressed as the system $(S_{\gamma,\rho})$. Note that from this constraint we always have $\gamma \leq \sum_{k,l} A_{kl} + \sigma^2\mathrm{Tr}(\boldsymbol{L})$. This completes the proof.

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