Chapter 3: Random Vectors and Matrix Algebra

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Definitions

- $\boldsymbol{y} = (y_1, \ldots, y_n)^T$: $n \times 1$
- $E(y_i) = \mu_i$
- $Var(y_i) = \sigma_{ii} = \sigma_i^2$
- $Cov(y_i, y_j) = \sigma_{ij}$

- $E(\boldsymbol{y}) = \big(E(y_1), \ldots, E(y_n)\big)^T = (\mu_1, \ldots, \mu_n)^T$: $n \times 1$

Definitions

$$
Var(\boldsymbol{y}) = \begin{pmatrix} Var(y_1) & Cov(y_1, y_2) & \cdots & Cov(y_1, y_n) \\ Cov(y_2, y_1) & Var(y_2) & \cdots & Cov(y_2, y_n) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(y_n, y_1) & Cov(y_n, y_2) & \cdots & Var(y_n) \end{pmatrix}
$$

$$
\begin{aligned}
Var(\boldsymbol{y}) &= (\sigma_{ij})_{n\times n} \\
&= E[\left(\boldsymbol{y} - E(\boldsymbol{y})\right)\left(\boldsymbol{y} - E(\boldsymbol{y})\right)^T] \\
&= E[(y_i - \mu_i)(y_j - \mu_j)]_{n\times n}
\end{aligned}
$$

y_i independent of $y_j \Rightarrow Cov(y_i, y_j) = 0$

Basic Properties

$\boldsymbol{A} = (a_{ij})_{m\times n}$, $\boldsymbol{b} = (b_1, \ldots, b_m)^T$, $\boldsymbol{c} = (c_1, \ldots, c_n)^T$

i $E(\boldsymbol{A}_{m\times n}\boldsymbol{y}_{n\times 1} + \boldsymbol{b}) = \boldsymbol{A}E(\boldsymbol{y}) + \boldsymbol{b}$

ii $Var(\boldsymbol{y}_{n\times 1} + \boldsymbol{c}) = Var(\boldsymbol{y})$

iii $Var(\boldsymbol{A}_{m\times n}\boldsymbol{y}_{n\times 1}) = \left(\boldsymbol{A}Var(\boldsymbol{y})\boldsymbol{A}^T\right)_{?\times ?}$

iv $Var(\boldsymbol{A}_{m\times n}\boldsymbol{y}_{n\times 1} + \boldsymbol{b}) =?$

Derivatives

i $f = f(\boldsymbol{y}) = f(y_1, \ldots, y_n)$
$\frac{d}{d\boldsymbol{y}} = \left(\frac{d}{dy_1} f, \ldots, \frac{d}{dy_n} f\right)^T$

ii $f = \boldsymbol{c}^T\boldsymbol{y} = \sum c_i y_i$ then $\frac{d}{d\boldsymbol{y}} f = \boldsymbol{c}$
$\boldsymbol{f} = \boldsymbol{A}\boldsymbol{y}$ then $\frac{d\boldsymbol{f}}{d\boldsymbol{y}} = \boldsymbol{A}^T$

iii $f = \boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y}$, where $\boldsymbol{A} = (a_{ij})_{m\times m}$ and $a_{ij} = a_{ji}$
$\frac{df}{d\boldsymbol{y}} = 2\boldsymbol{A}\boldsymbol{y}$
Verify (ii) and (iii).

Some Useful Results

i **Trace**: $tr(\boldsymbol{A}_{m\times m}) = \sum a_{ii}$
$tr(\boldsymbol{B}_{m\times n}\boldsymbol{C}_{n\times m}) = tr(\boldsymbol{C}_{n\times m}\boldsymbol{B}_{m\times n})$

ii **Rank of a Matrix**: $rank(\boldsymbol{A}) = \#$ of linearly independent columns/rows

iii Vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ are **linearly independent** if and only if
$c_1\boldsymbol{y}_1 + \cdots + c_m\boldsymbol{y}_m = \boldsymbol{0} \Rightarrow c_1 = \cdots = c_m = 0$

iv **Orthogonal Vectors and Matrices**

- Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal if $\boldsymbol{x}^T\boldsymbol{y} = 0$
- $\boldsymbol{A}_{m\times m}$ is an orthogonal matrix if
  $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$ and $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$

Some Useful Results

- **Eigenvalues & Eigenvectors of a square matrix**
  A non-zero vector $\boldsymbol{x}_{m\times 1}$ is called an eigenvector of $\boldsymbol{A}_{m\times m}$ if there exists a real number λ such that $\boldsymbol{Ax} = \lambda \boldsymbol{x}$, and we call λ the eigenvalue (with respect to $\boldsymbol{x}$)
  Note: If $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$, so is $c\boldsymbol{x}$ for any real number $c \neq 0$

vi **Decomposition of a symmetric matrix**
$\boldsymbol{A}_{m\times m}$ is symmetric if $\boldsymbol{A}^T = \boldsymbol{A}$
If $\boldsymbol{A}_{m\times m}$ is symmetric and real, then all its eigenvalues $\lambda_1, \ldots, \lambda_m$ are real, and there exits an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{P\Lambda P}^T$, where $\boldsymbol{\Lambda}$ is diagonal with eigenvalues of $\boldsymbol{A}$ on the diagonal

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_m \end{pmatrix}$$

$\boldsymbol{P} = \left[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\right]$, where $\boldsymbol{x}_i$ is the eigenvalue corresponding to λ_i; $i = 1, \ldots, m$.

Some Useful Results

Note: For a symmetric matrix $\boldsymbol{A}$, if $\boldsymbol{x}_1$ corresponds to λ_1, $\boldsymbol{x}_2$ corresponds to λ_2, and $\lambda_1 \neq \lambda_2$, then $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are orthogonal

vii **Idempotent Matrix**
$\boldsymbol{A}_{m\times m}$ is idempotent if $\boldsymbol{A}^2 = \boldsymbol{A}$
Note: If $\boldsymbol{A}$ is idempotent, then all its eigenvalues are either 1 or 0

Multivariate Normal Distributions

Why normal? Central limit theorem + nice properties.

The random vector $\boldsymbol{y} = (y_1, \ldots, y_n)^T$ follows a multivariate normal distribution with pdf

$$f(\boldsymbol{y}) = \left(\frac{1}{2\pi}\right)^{n/2} |\Sigma|^{-1/2} \exp\{-\frac{1}{2}(\boldsymbol{y} - \boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{y} - \boldsymbol{\mu})\},$$

where $\boldsymbol{\mu} = E(\boldsymbol{y})_{n\times 1}$ and $\Sigma = Var(\boldsymbol{y}) = (\sigma_{ij})_{n\times n}$.

Multivariate Normal Distributions

i We also write $\boldsymbol{y} \sim MVN(\boldsymbol{\mu}, \Sigma)$.

ii Marginal normality: $y_i \sim N(\mu_i, \sigma_{ii})$.

iii $y_1, \ldots, y_n$ are independent if and only if Σ is diagonal.

iv If $\boldsymbol{y} \sim MVN(\boldsymbol{\mu}, \Sigma)$ and $\boldsymbol{z}_{m\times 1} = \boldsymbol{A}_{m\times n}\boldsymbol{y}_{n\times 1}$, then $\boldsymbol{z} \sim MVN(\boldsymbol{A}\boldsymbol{\mu}, \boldsymbol{A}\Sigma\boldsymbol{A}^T)$.

v If $\boldsymbol{y} \sim MVN(\boldsymbol{0}, \sigma^2\boldsymbol{I}_{n\times n})$, then $\frac{1}{\sigma^2}\boldsymbol{y}^T\boldsymbol{y} \sim \chi^2_{(n)}$.

vi If $\boldsymbol{y} \sim MVN(\boldsymbol{\mu}, \Sigma)$, then $\boldsymbol{A}\boldsymbol{y}$ and $\boldsymbol{B}\boldsymbol{y}$ are independent if and only if $\boldsymbol{A}\Sigma\boldsymbol{B}^T = \boldsymbol{0}$.