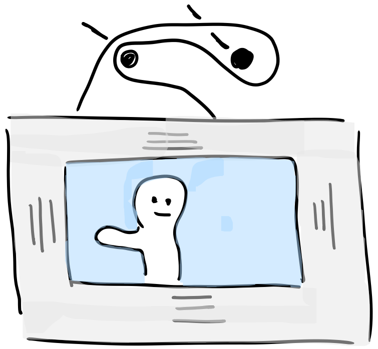
Hanging Pictures and Tiling Chessboards

MAN CHEUNG TSUI / MATH POSTDOC

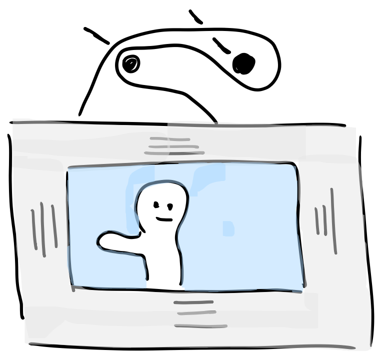
FSU / SOCIETY OF UNDERGRADUATE MATH STUDENTS

Hanging Pictures

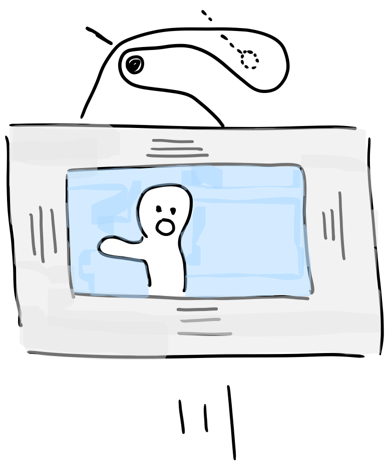
Hang a picture on two nails ...



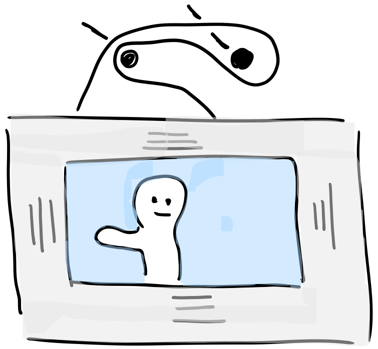
Hang a picture on two nails …



but remove any nail, the picture falls.

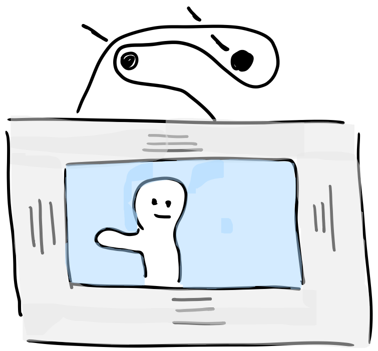


Hang a picture on two nails ...



but remove any nail, the picture falls.

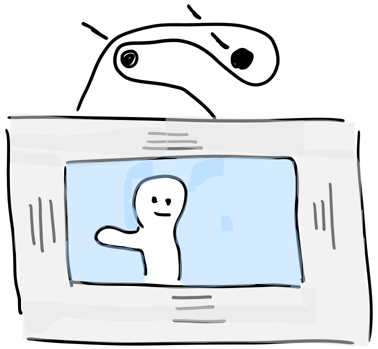
Hang a picture on two nails ...



but remove any nail, the picture falls.

Can you do this?

Hang a picture on two nails …



but remove any nail, the picture falls.

Can you do this?

Try three nails.

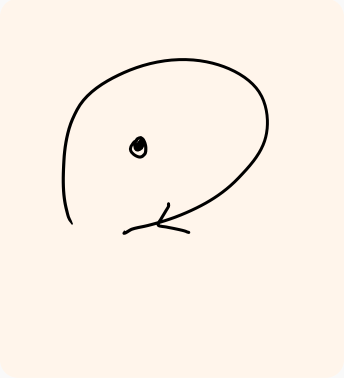
No nail.

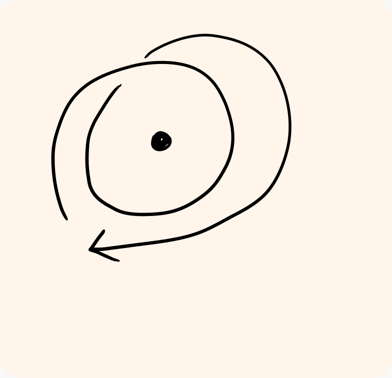
Every loop shrinks to a
constant loop (a point).

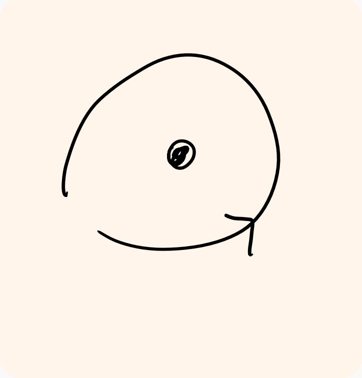
No nail.

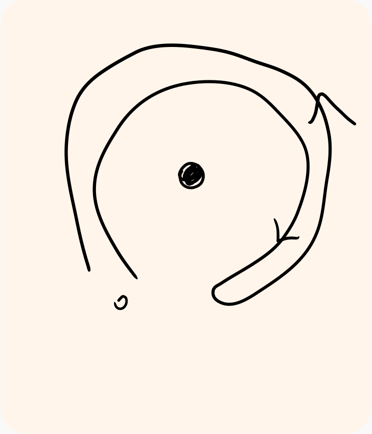
Every loop shrinks to a constant loop (a point).

One nail.





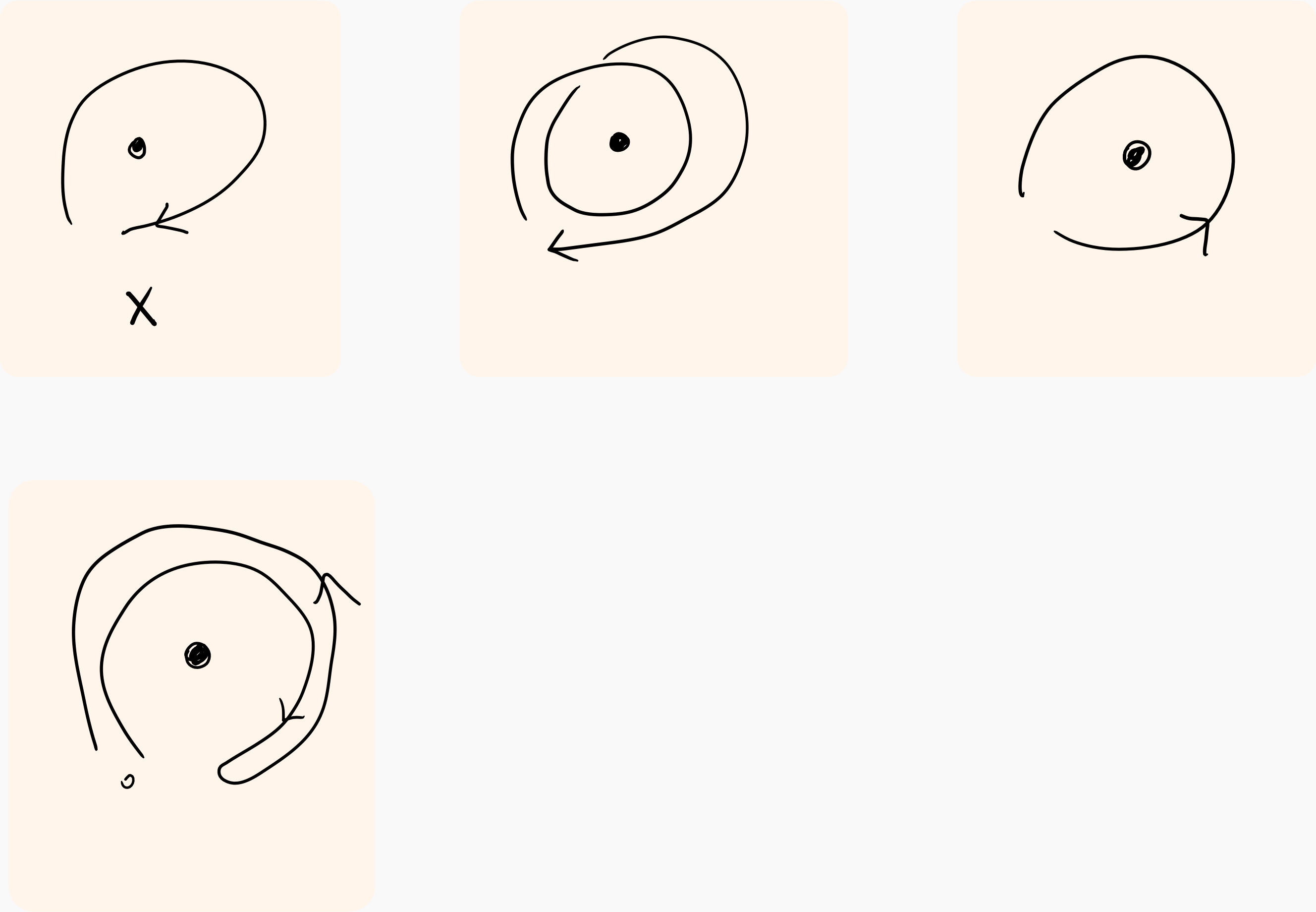




No nail.

Every loop shrinks to a constant loop (a point).

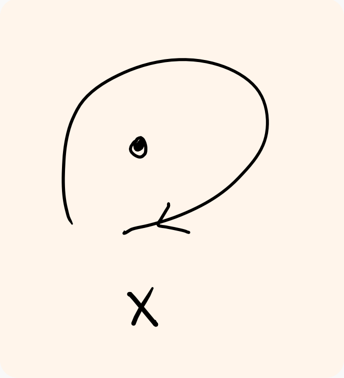
One nail.

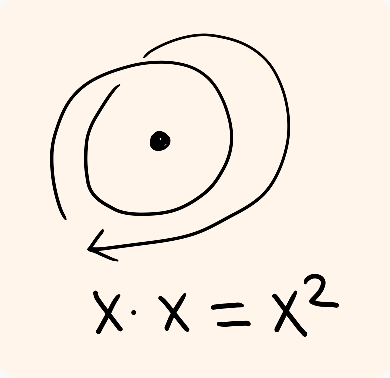


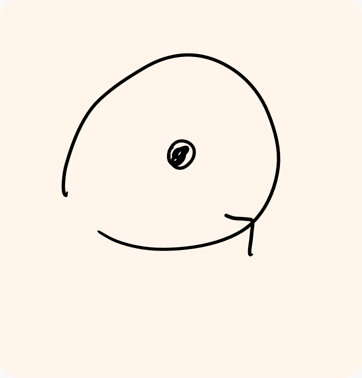
No nail.

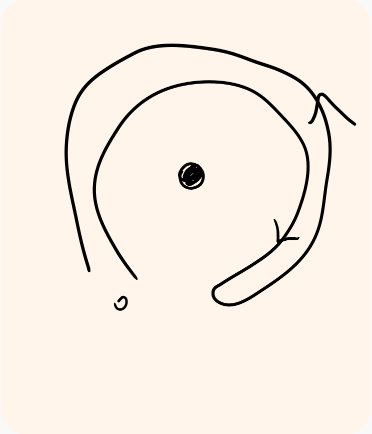
Every loop shrinks to a constant loop (a point).

One nail.





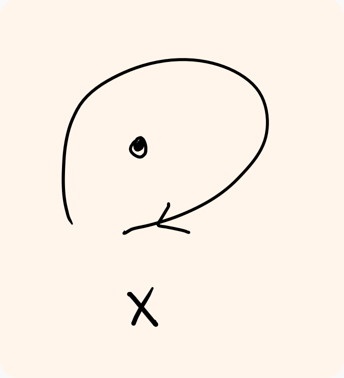


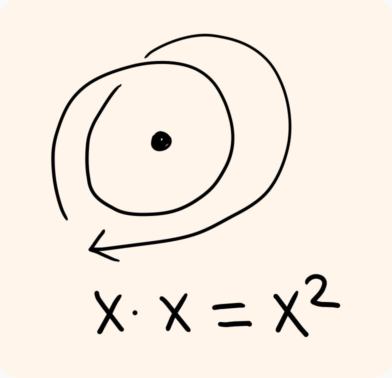


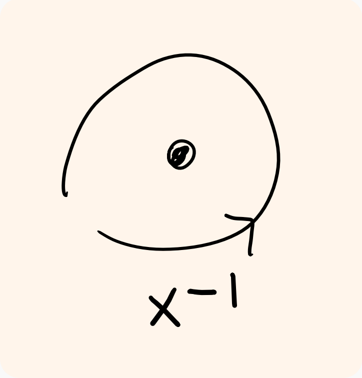
No nail.

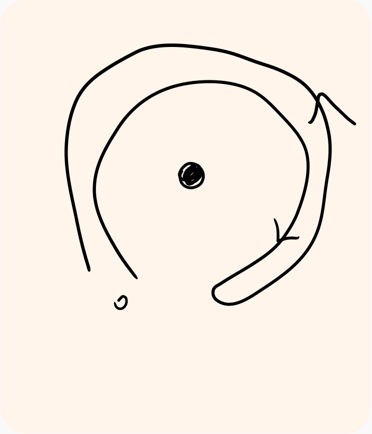
Every loop shrinks to a constant loop (a point).

One nail.





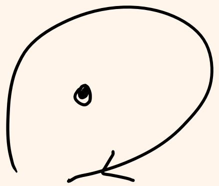




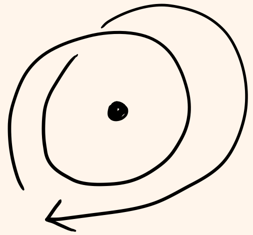
No nail.

Every loop shrinks to a constant loop (a point).

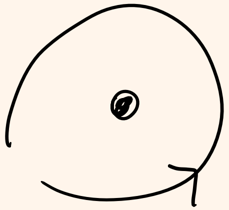
One nail.



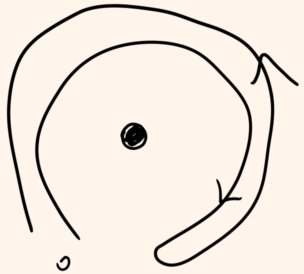
x



$x \cdot x = x^2$



x^{-1}

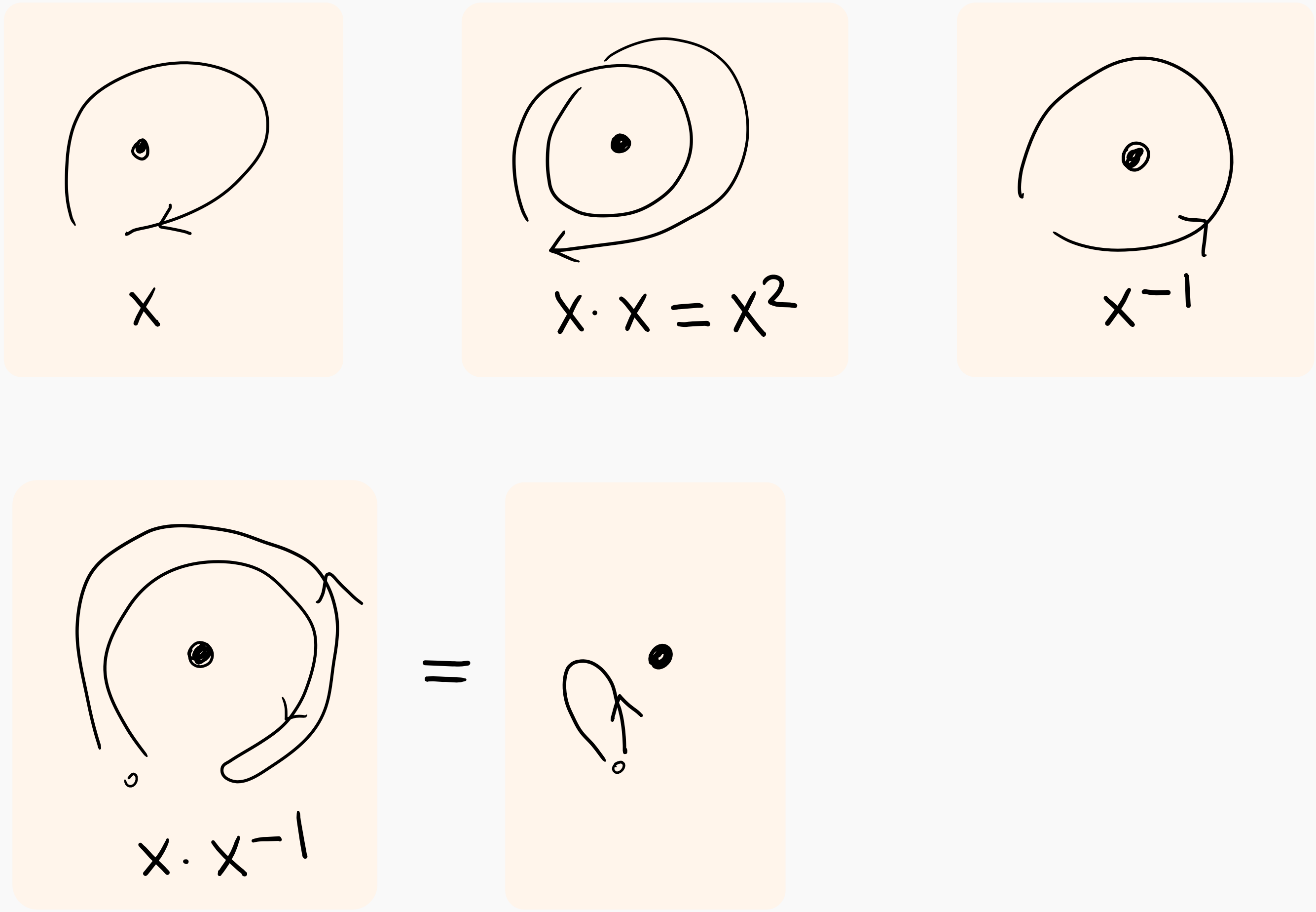


$x \cdot x^{-1}$

No nail.

Every loop shrinks to a constant loop (a point).

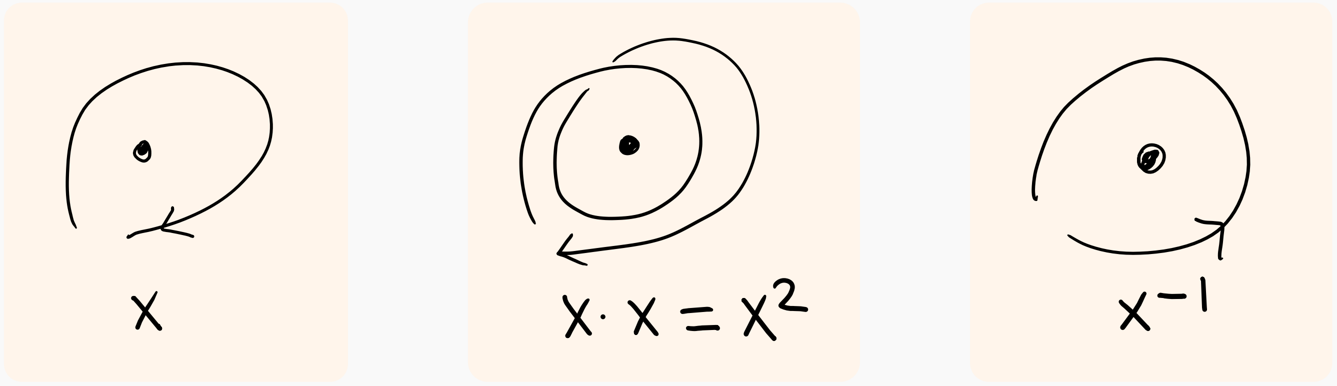
One nail.

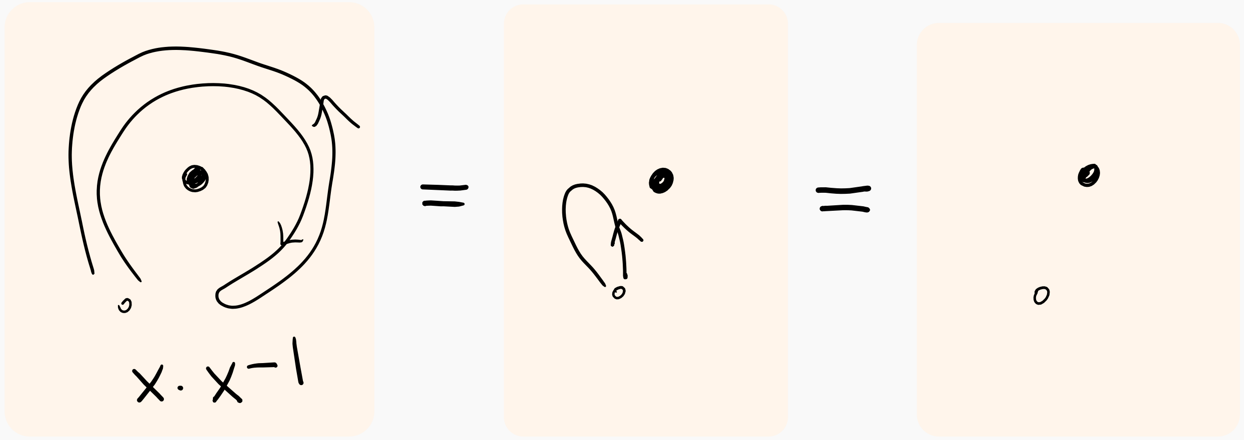


No nail.

Every loop shrinks to a constant loop (a point).

One nail.

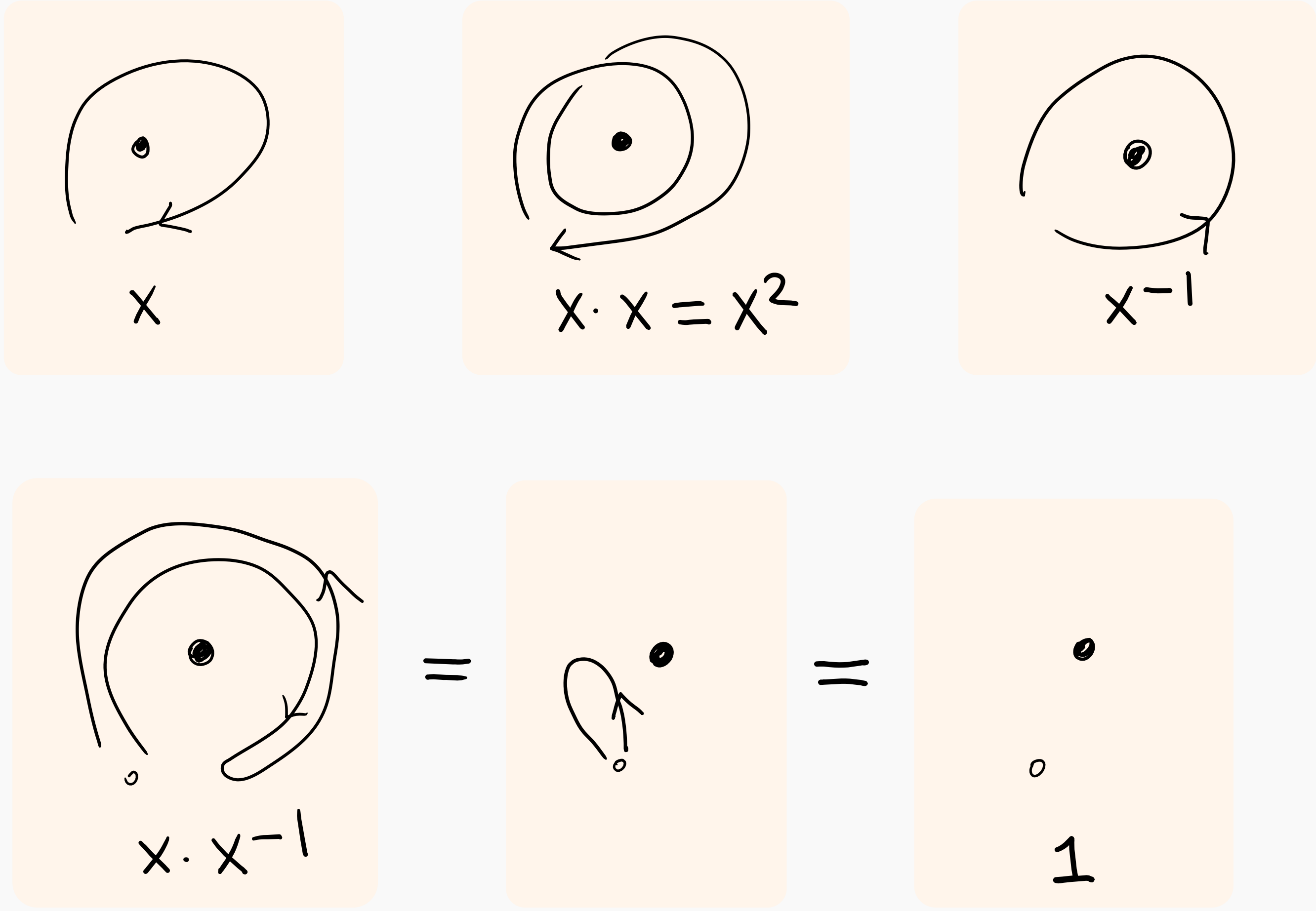




No nail.

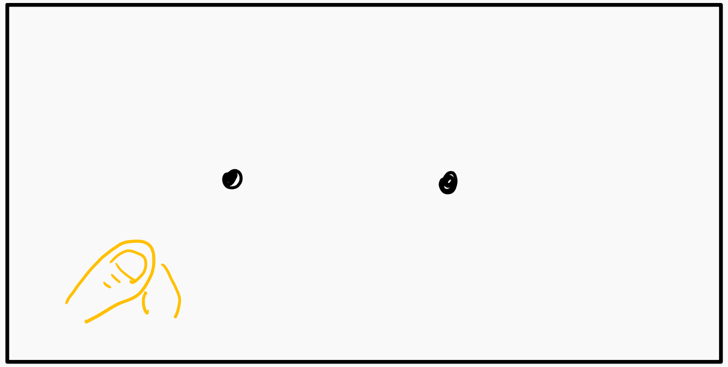
Every loop shrinks to a constant loop (a point).

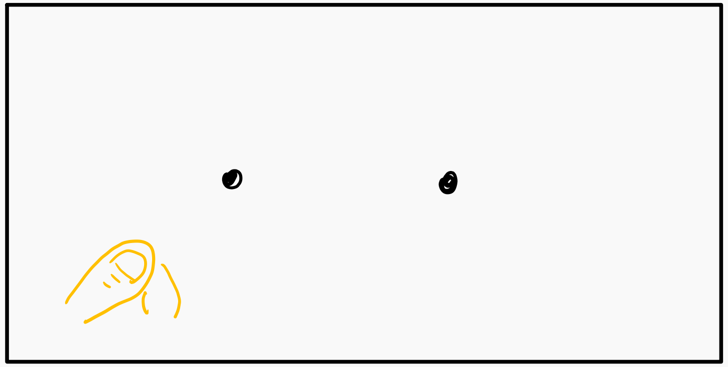
One nail.



(constant loop)

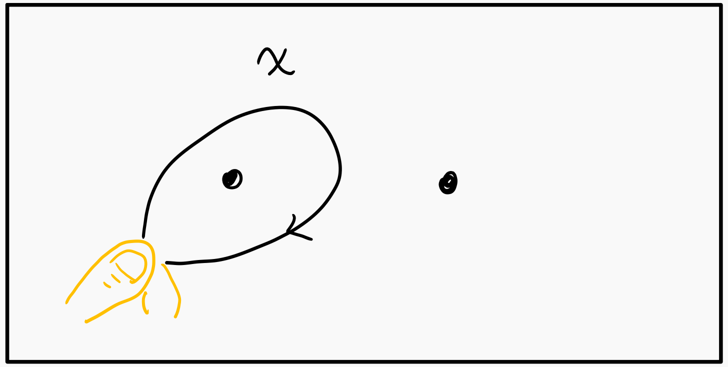
With two nails,

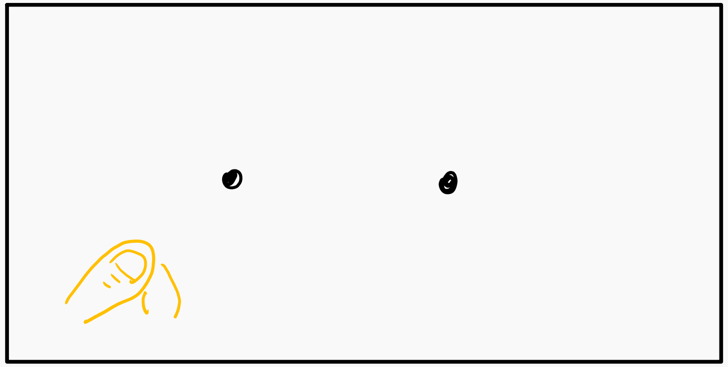




Finger position is “basepoint”

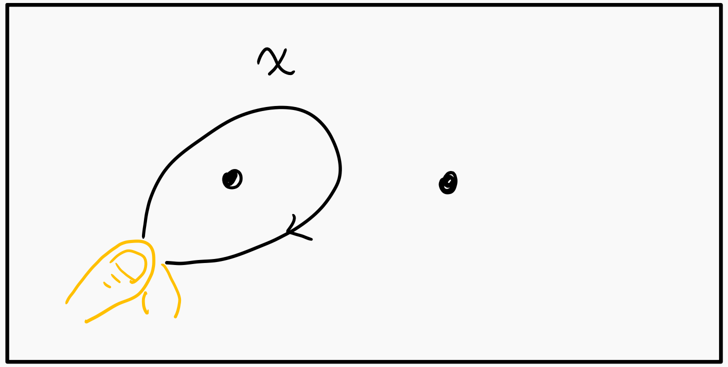
With two nails,

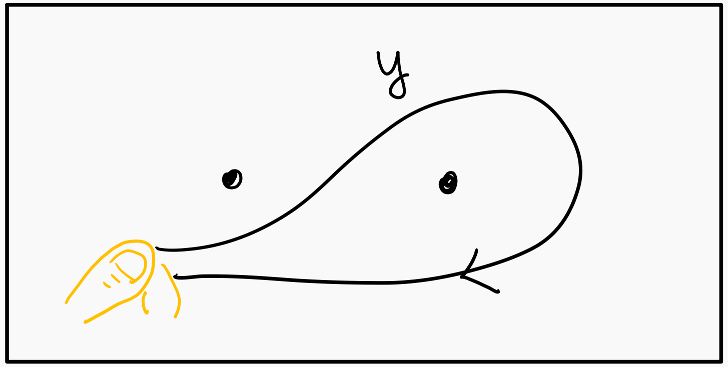




Finger position is "basepoint"

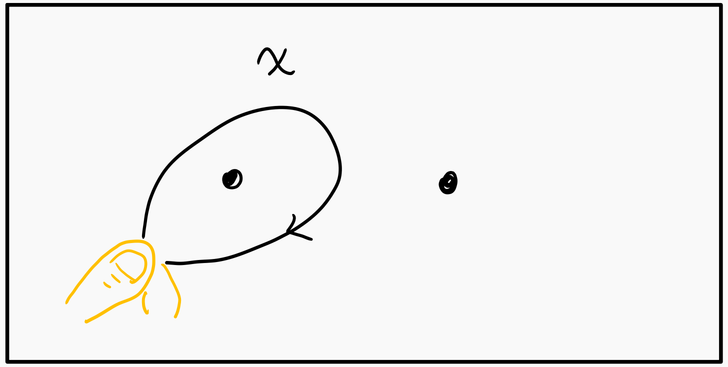
With two nails,

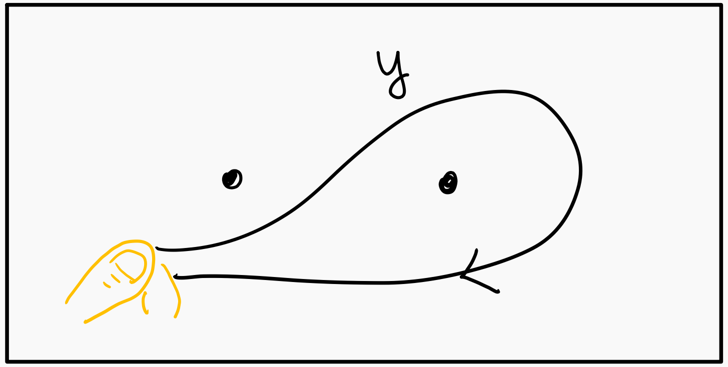


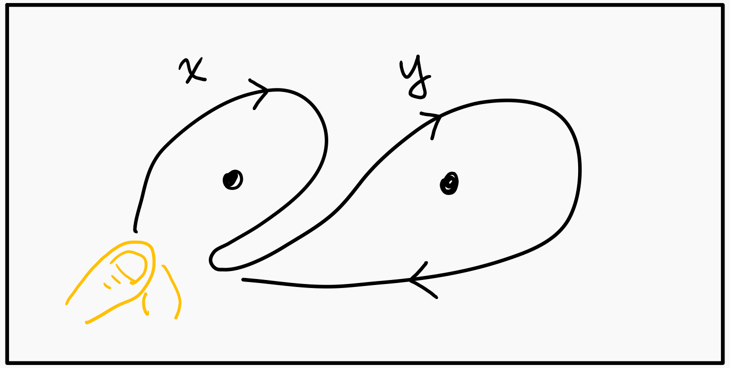


Finger position is “basepoint”

With two nails,

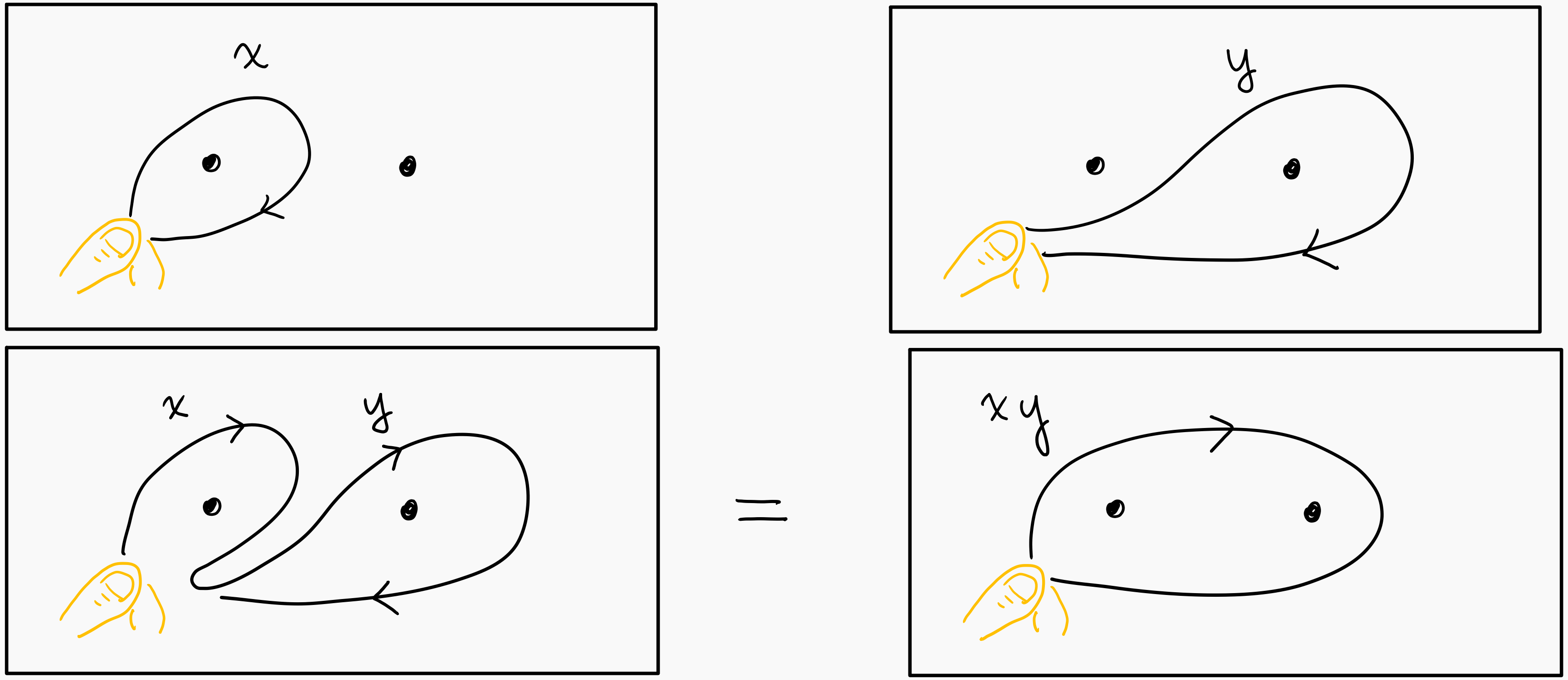






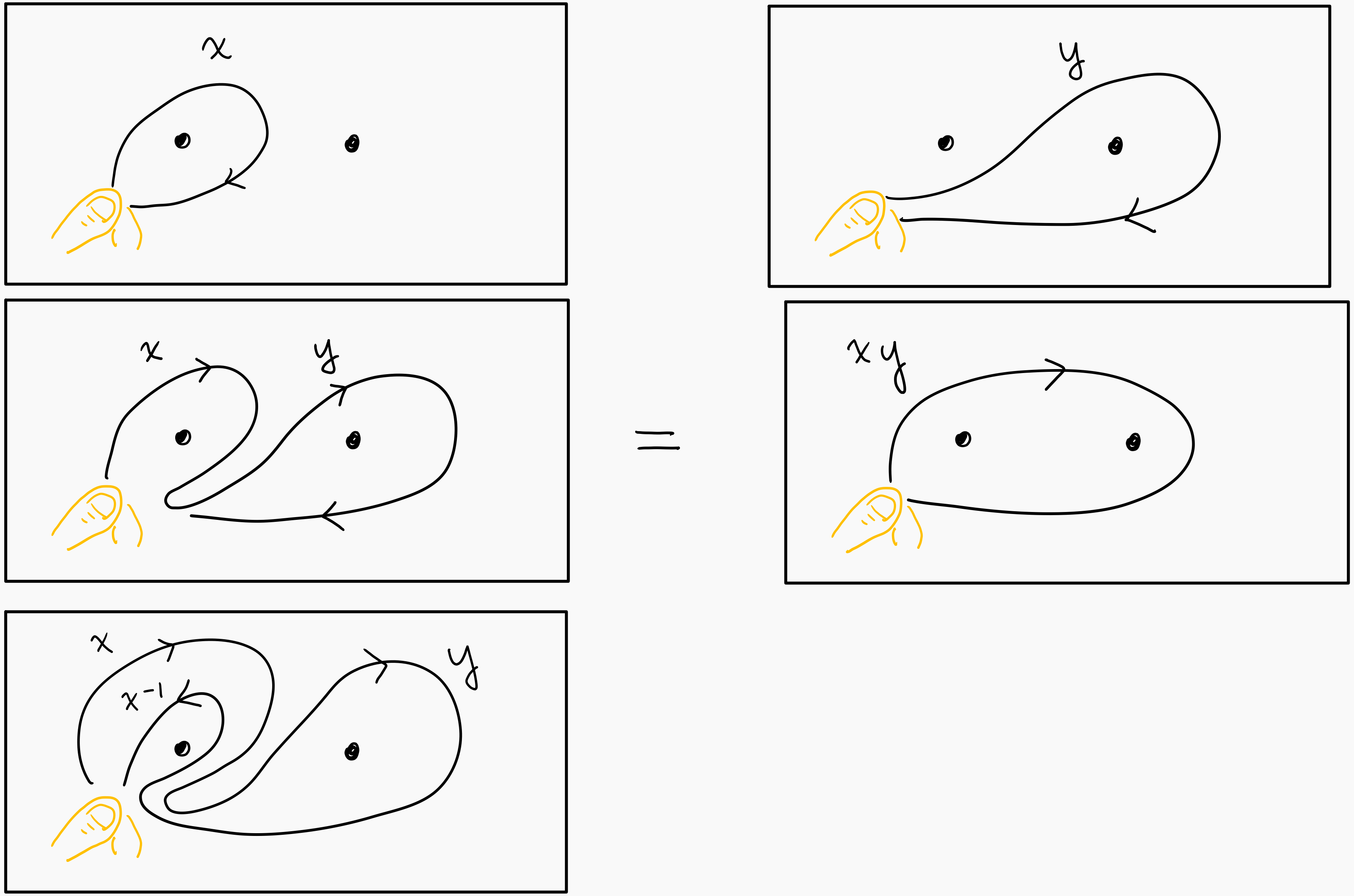
Finger position is “basepoint”

With two nails,

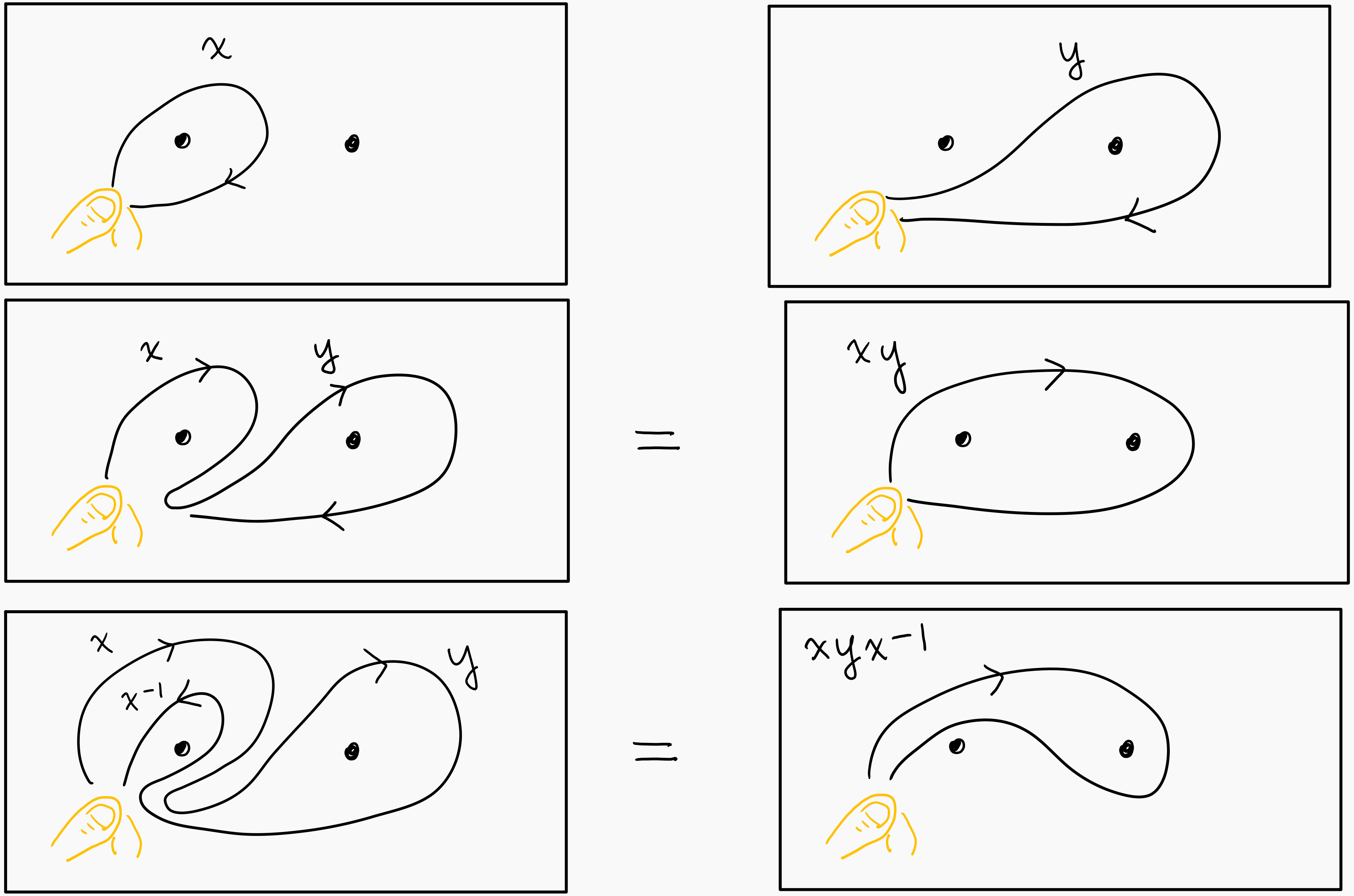


Finger position is “basepoint”

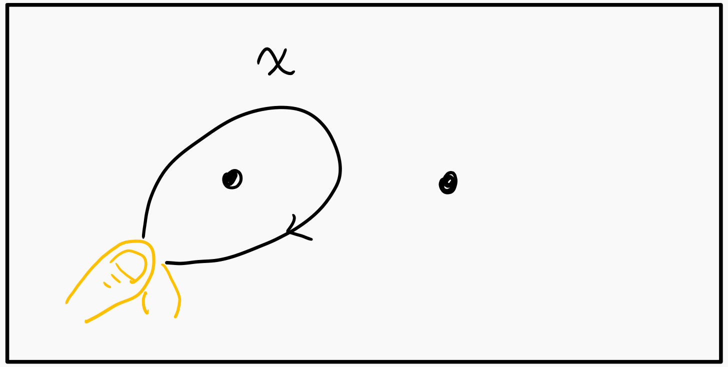
With two nails,

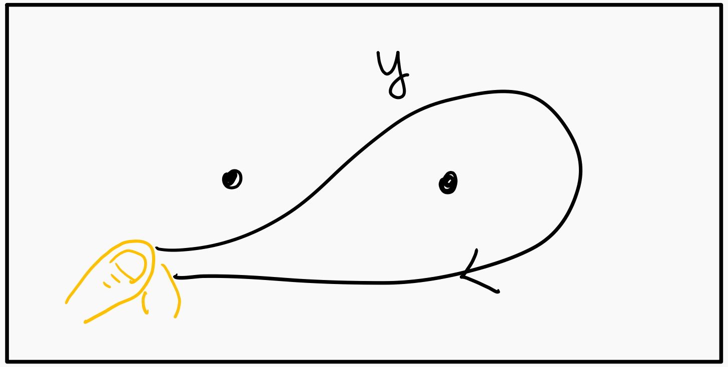


With two nails,



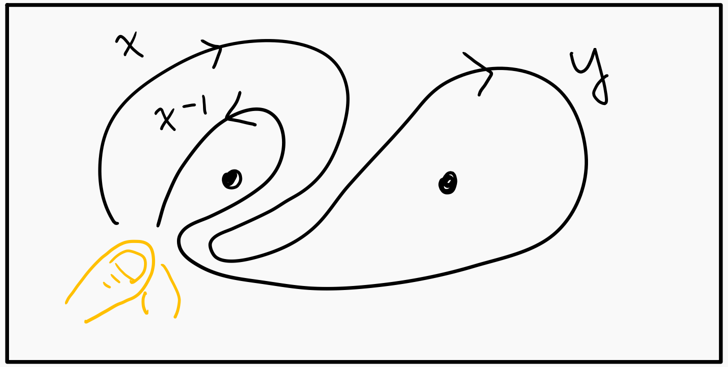
With two nails,



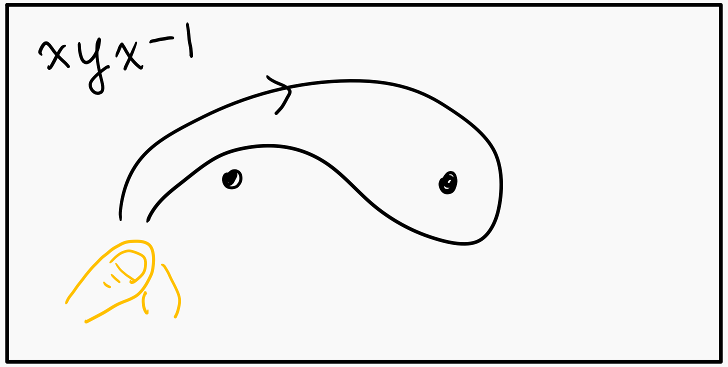


Such a loop = string of letters in alphabet x, y

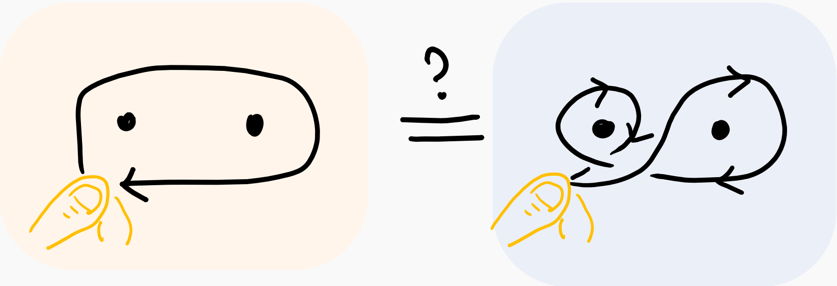
= word in generators x, y



=

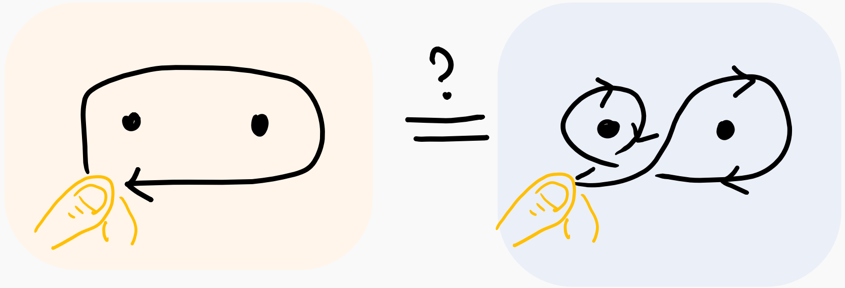


With two nails,



$$xy \stackrel{?}{=} yx$$

With two nails,

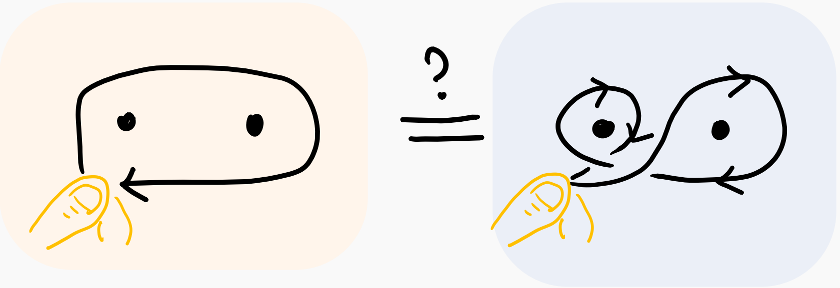


$$xy \stackrel{?}{=} yx$$

$$\Big\downarrow \cdot x^{-1}$$

Experiment!

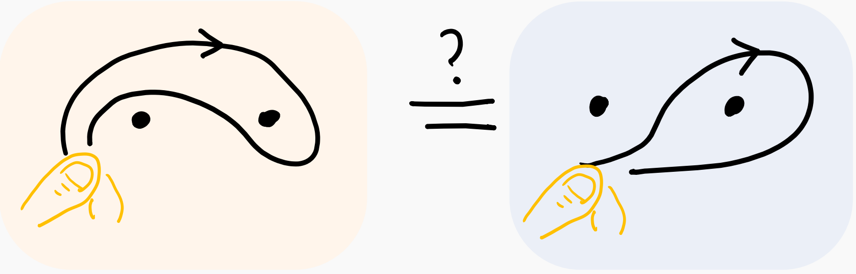
With two nails,



$$xy \stackrel{?}{=} yx$$

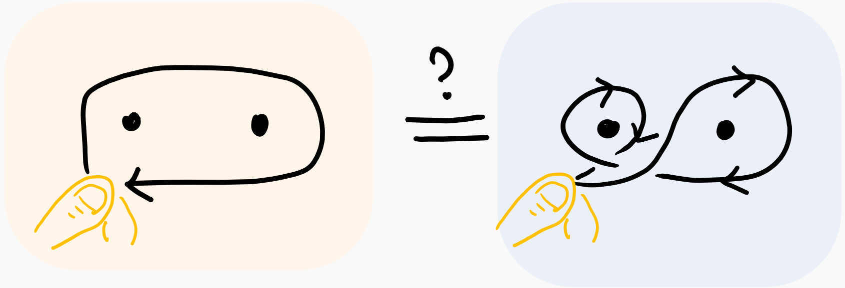
$$\Big\downarrow \cdot x^{-1}$$

Experiment!



$$xyx^{-1} \stackrel{?}{=} y$$

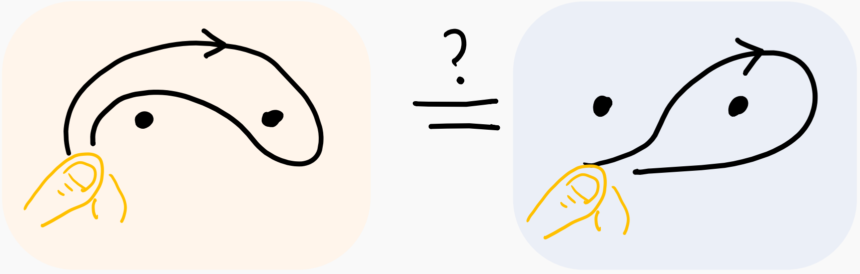
With two nails,



$$xy \stackrel{?}{=} yx$$

$$\Big\downarrow \cdot x^{-1}$$

Experiment!

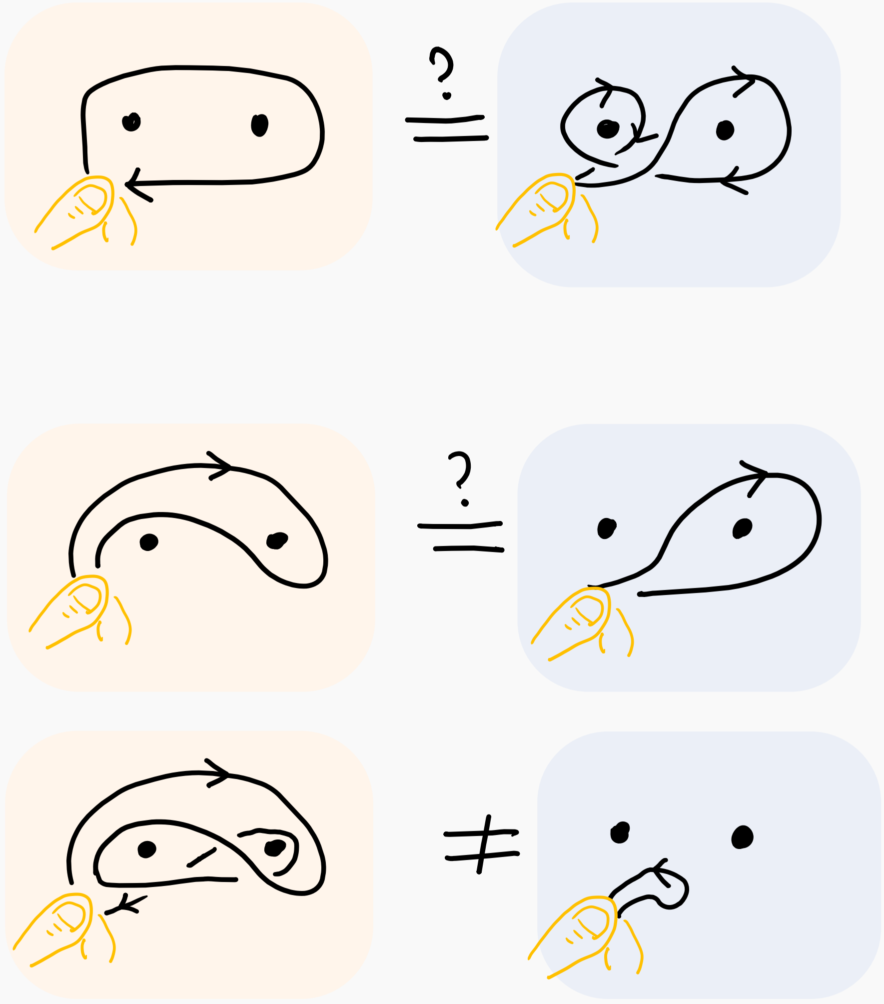


$$xyx^{-1} \stackrel{?}{=} y$$

$$\Big\downarrow \cdot y^{-1}$$

With two nails,

Experiment!



not fall falls

$$xy \overset{?}{=} yx$$

$$\downarrow \cdot x^{-1}$$

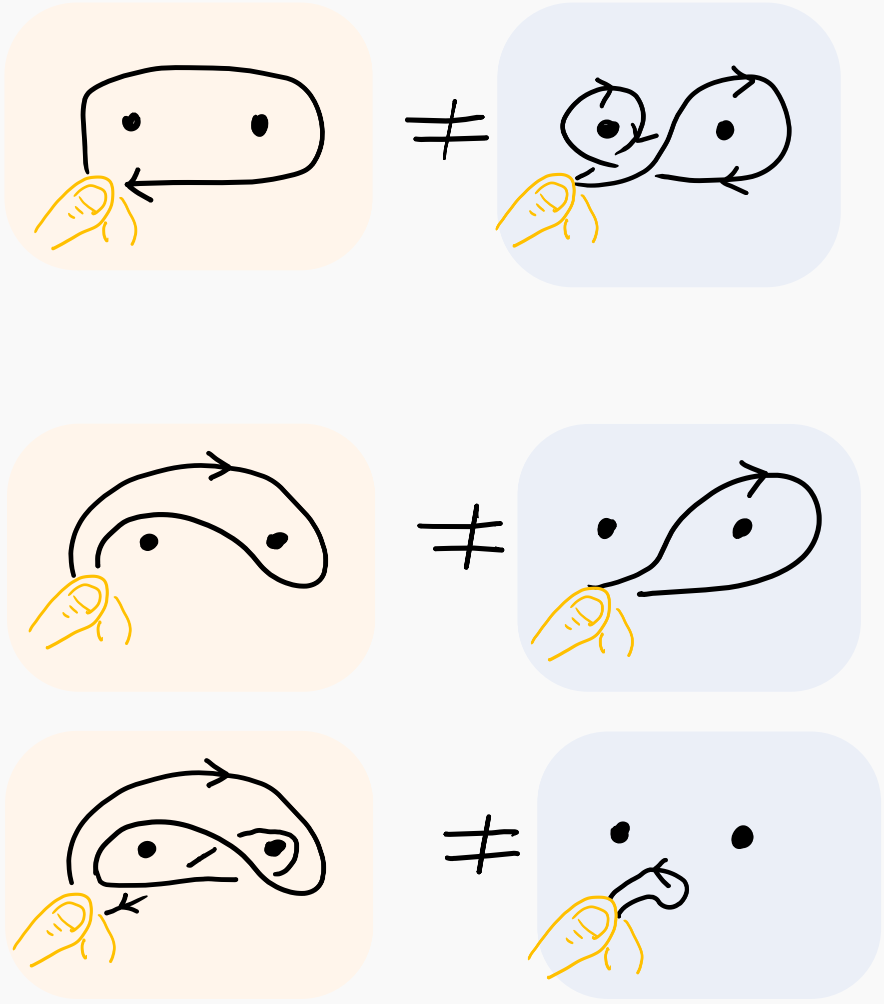
$$xyx^{-1} \overset{?}{=} y$$

$$\downarrow \cdot y^{-1}$$

$$xyx^{-1}y^{-1} \neq 1$$

With two nails,

Experiment!

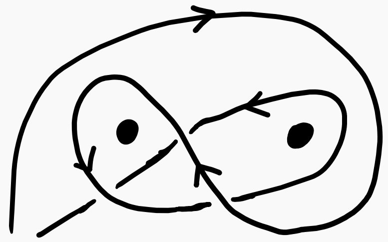


not fall falls

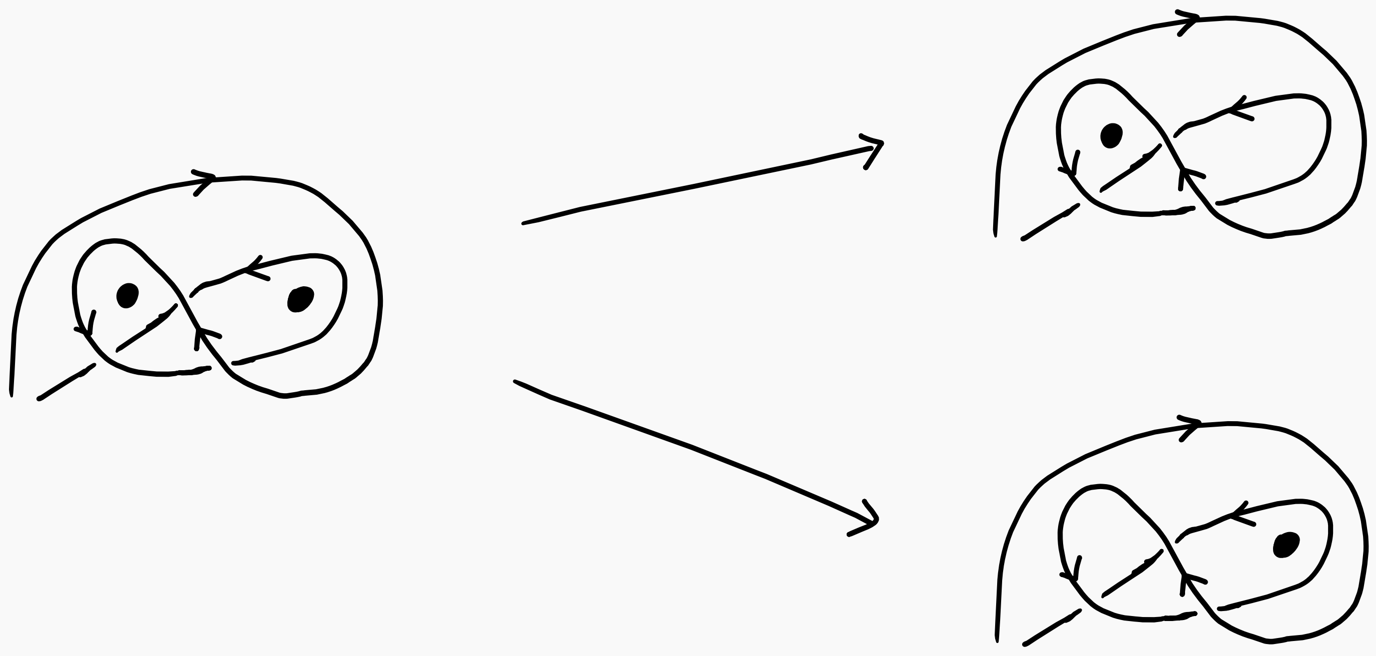
$$\begin{aligned} xy &\neq yx \\ & \Big\downarrow \cdot x^{-1} \\ xyx^{-1} &\neq y \\ & \Big\downarrow \cdot y^{-1} \\ xyx^{-1}y^{-1} &\neq 1 \end{aligned}$$

Picture-hanging solution for two nails.

On two nails



Picture-hanging solution for two nails.

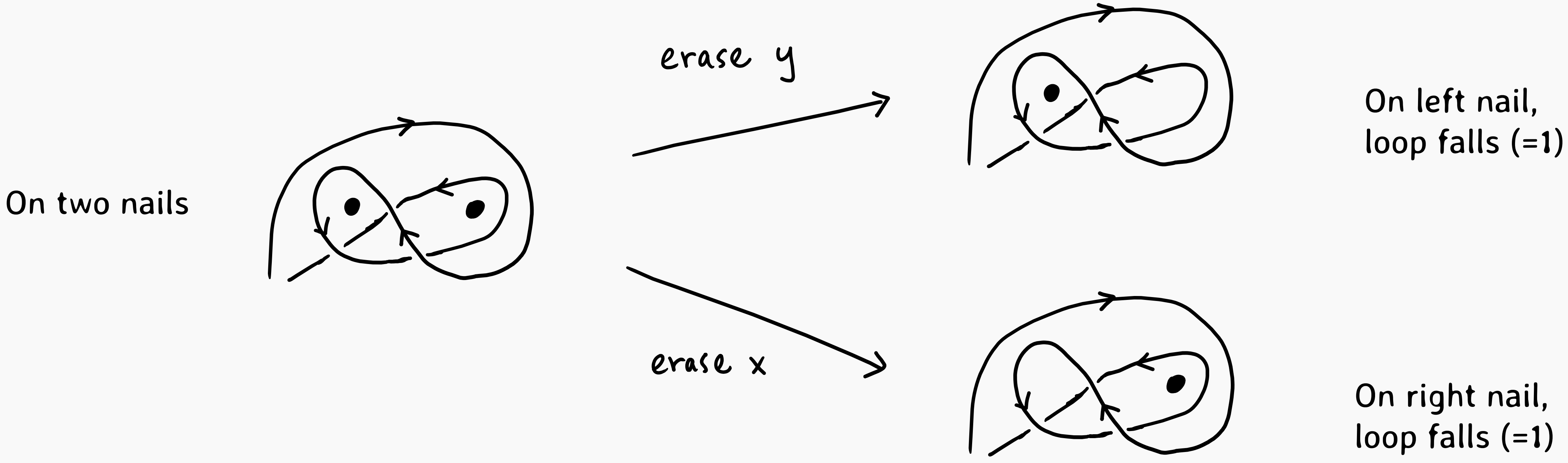


On two nails

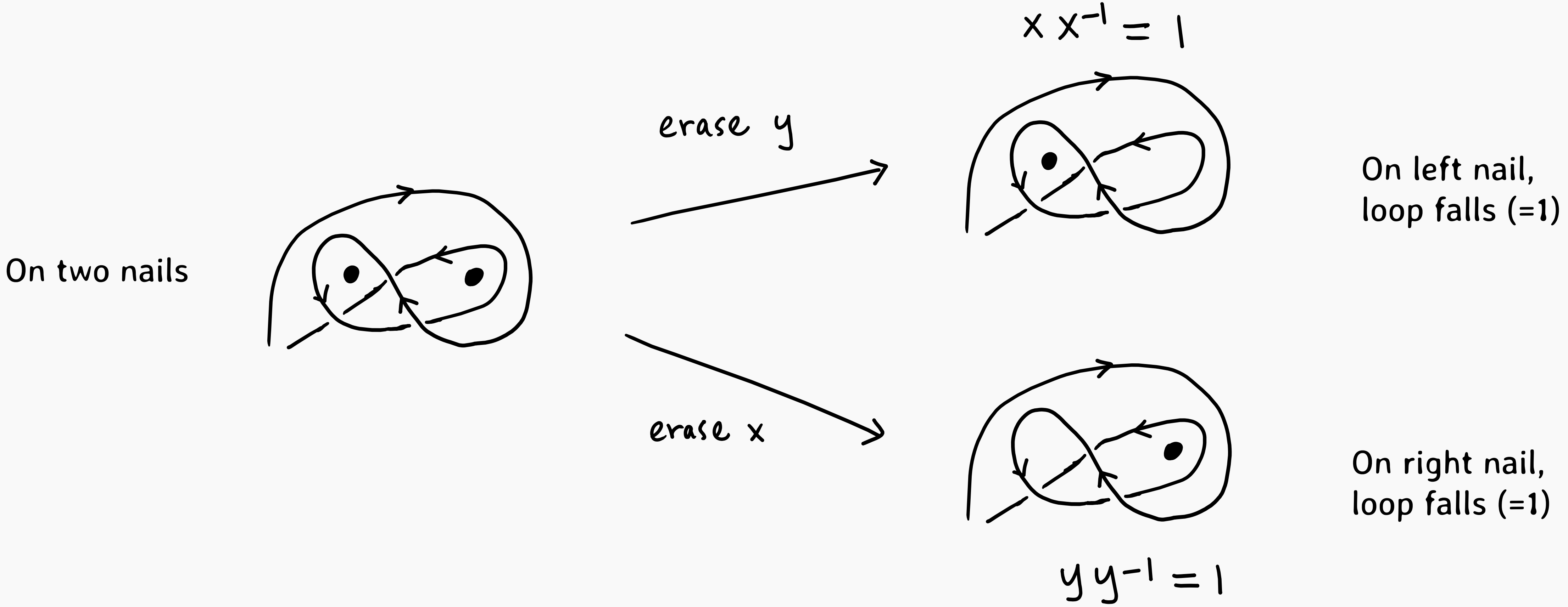
On left nail, loop falls (=1)

On right nail, loop falls (=1)

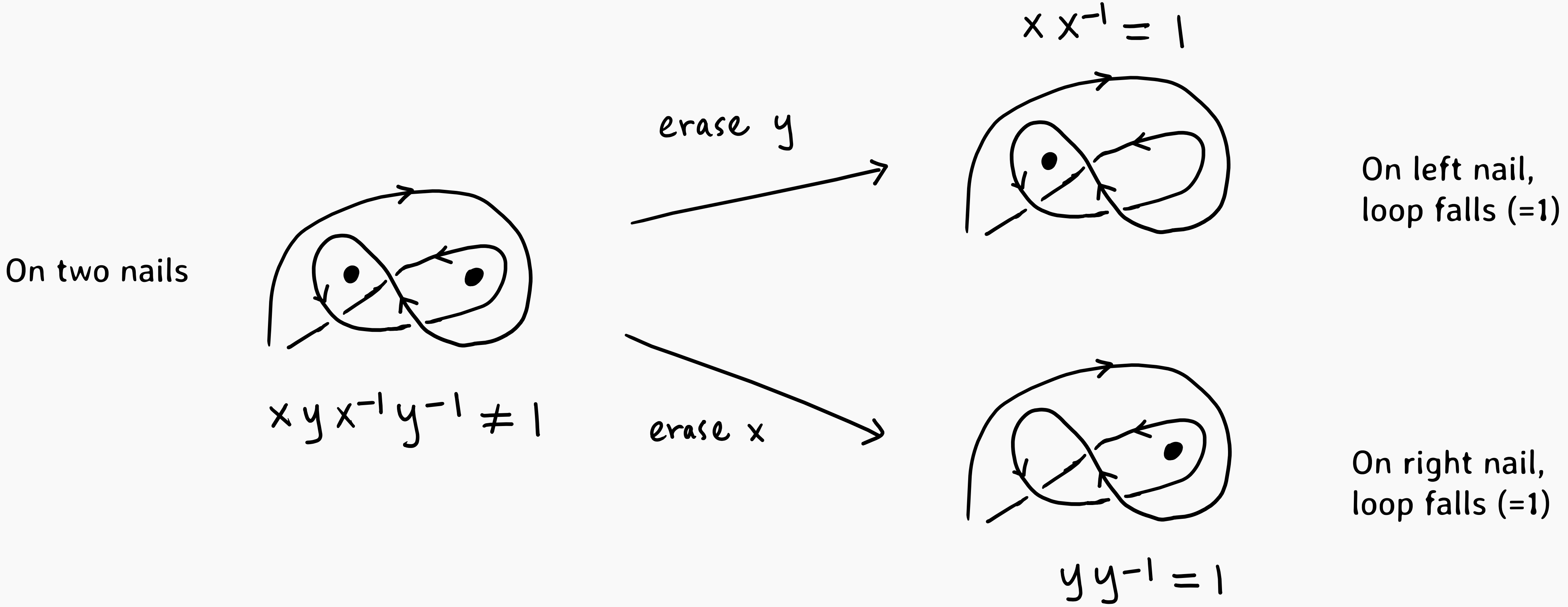
Picture-hanging solution for two nails.



Picture-hanging solution for two nails.



Picture-hanging solution for two nails.



For a surface X and a point p on X

$$\pi_1(X, p) = \{ \text{loops on } X \text{ starting at } p \}$$

Here, two loops are “the same” if one loop deforms to the other loop.

For a surface X and a point p on X

$$\pi_1(X, p) = \{ \text{loops on } X \text{ starting at } p \}$$

Fundamental group of X with basepoint p

Here, two loops are “the same” if one loop deforms to the other loop.

For a surface X and a point p on X

$$\pi_1(X, p) = \{ \text{loops on } X \text{ starting at } p \}$$

Fundamental group of X with basepoint p

Here, two loops are “the same” if one loop deforms to the other loop.

$$\pi_1(\text{plane}, p) = \{1\}$$

For a surface X and a point p on X

$$\pi_1(X, p) = \{ \text{loops on } X \text{ starting at } p \}$$

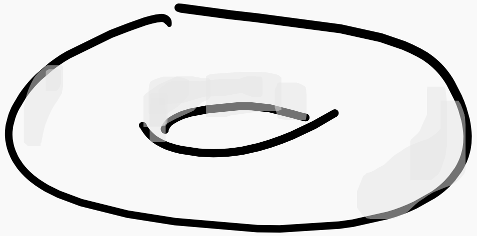
Fundamental group of X with basepoint p

Here, two loops are "the same" if one loop deforms to the other loop.

$$\pi_1(\text{plane}, p) = \{1\}$$

$$\pi_1(\text{plane missing one point}, p) = \langle x \rangle \leftarrow \text{words generated by } x$$

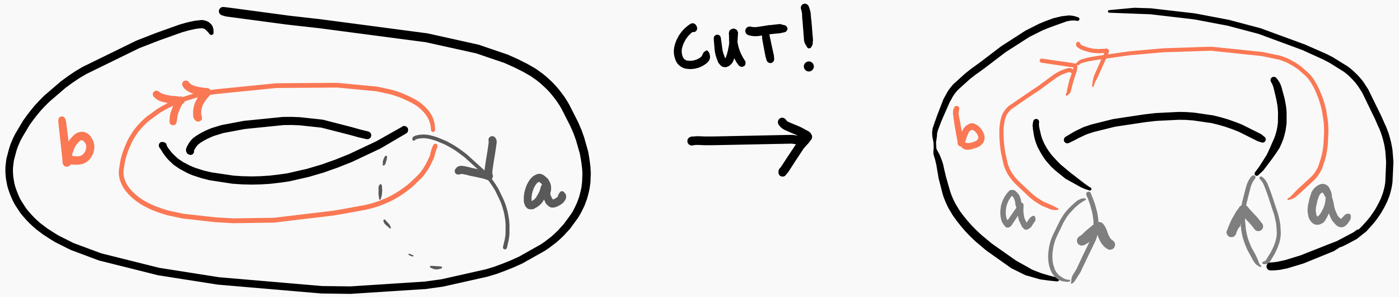
$$\pi_1(\text{plane missing two points}, p) = \langle x, y \rangle \leftarrow \text{words generated by } x, y$$

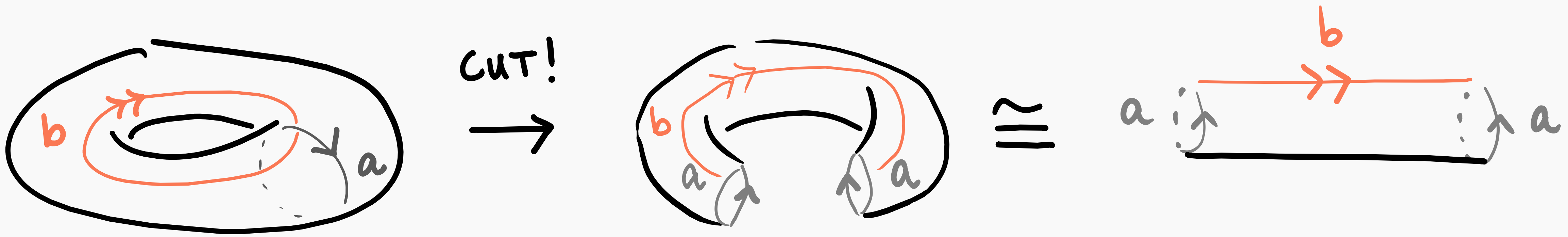
$\pi_1($  $, p) = ?$

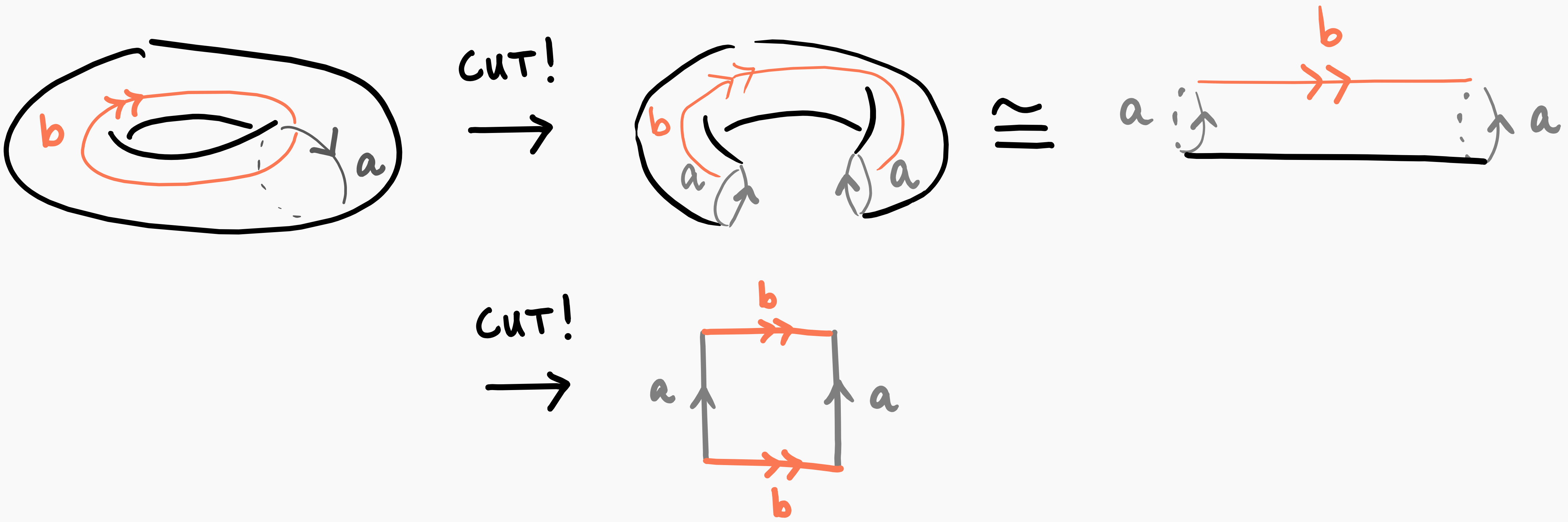

a

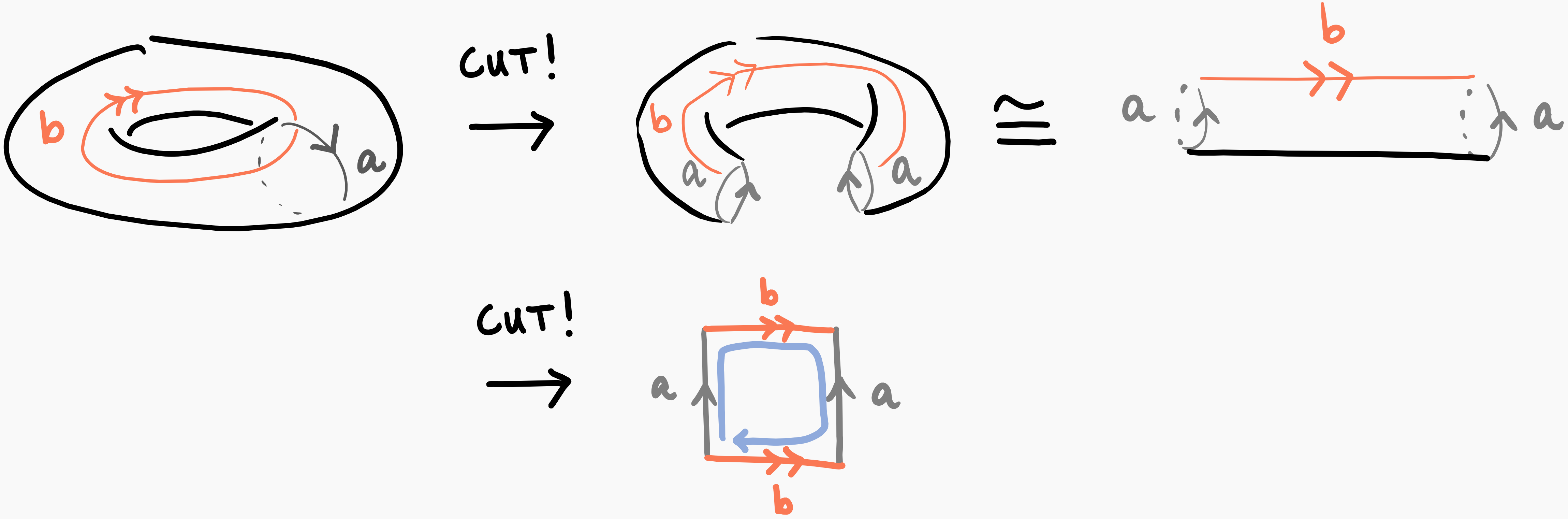
b
a

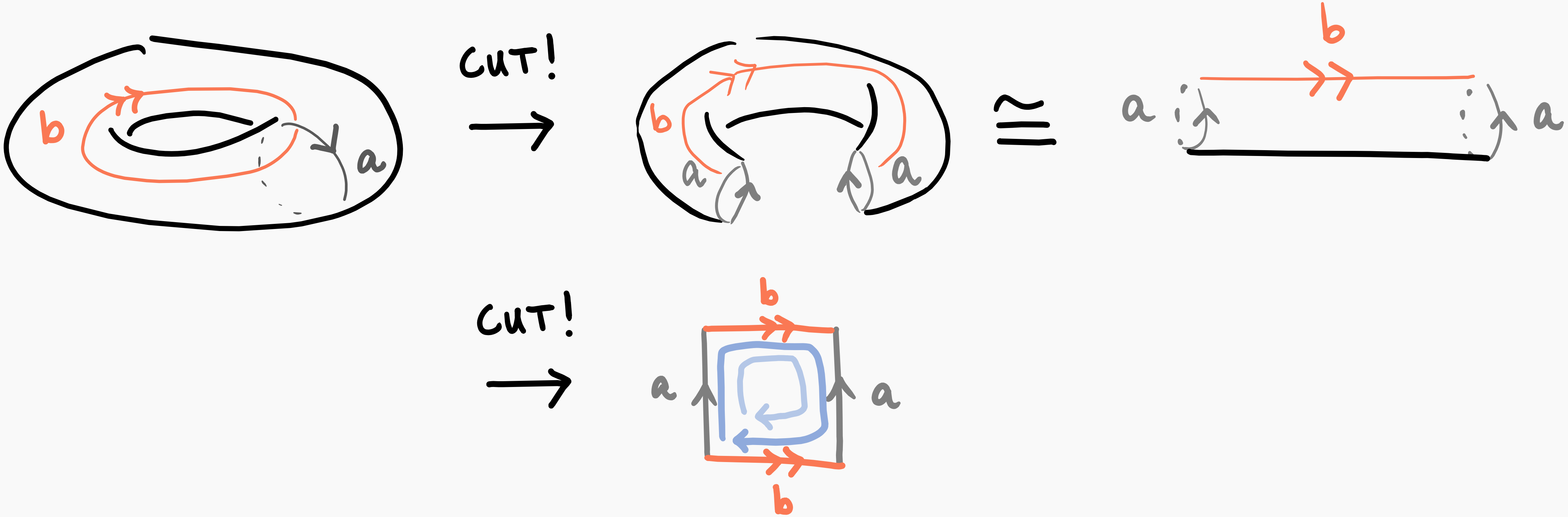
CUT! $\longrightarrow$

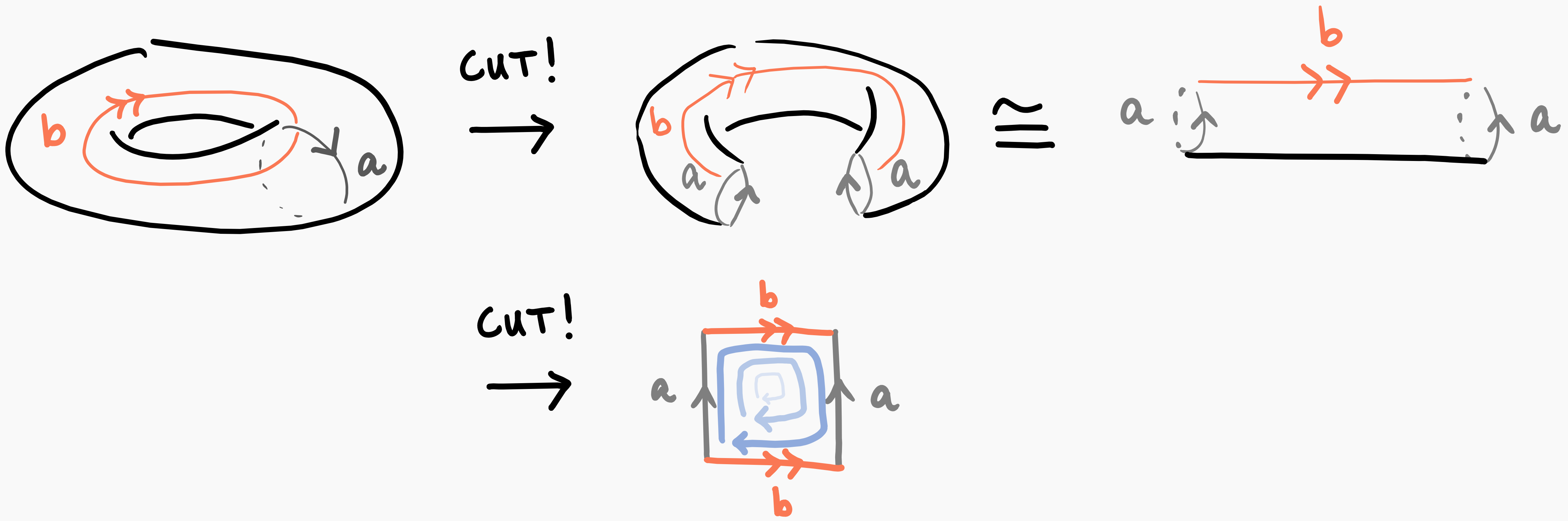

b
a
CUT!
b
a
a

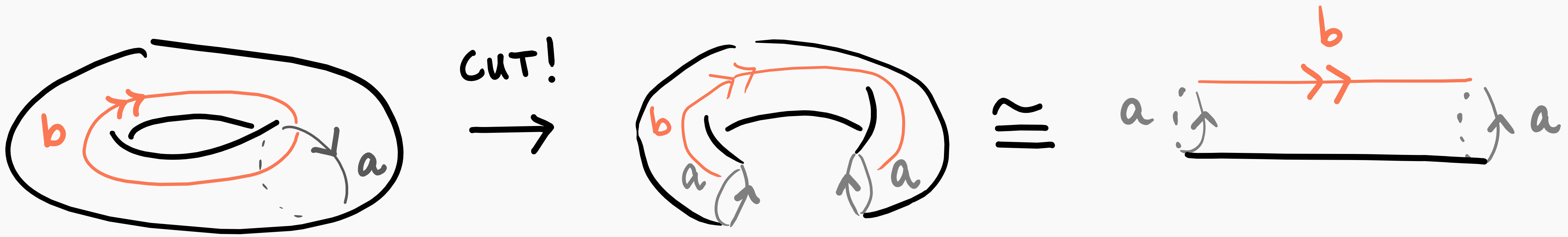

b
a
CUT!
b
a
a
≅
b
a
a

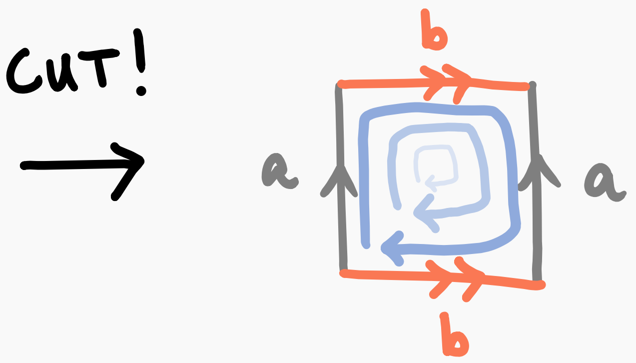

b
a
CUT!
b
a
a
≅
b
a
a
CUT!
b
a
a
b


b
a
CUT!
b
a
a
≅
b
a
a
CUT!
b
a
a
b

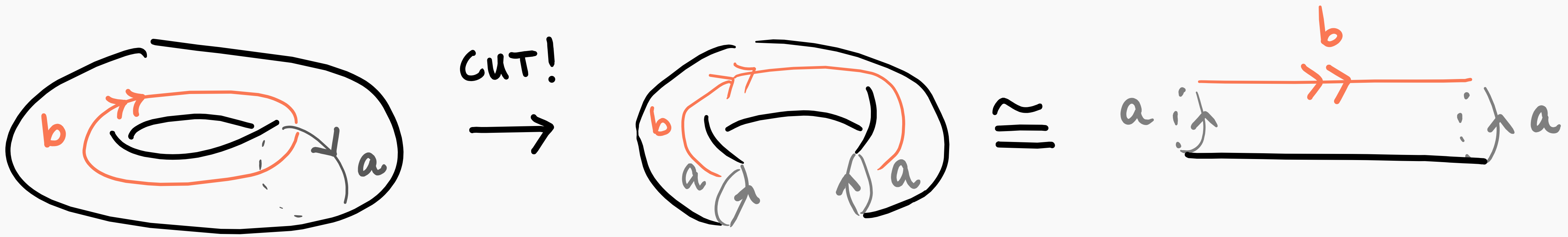

b
a
CUT!
b
a
a
b
a
a
CUT!
b
a
a
b

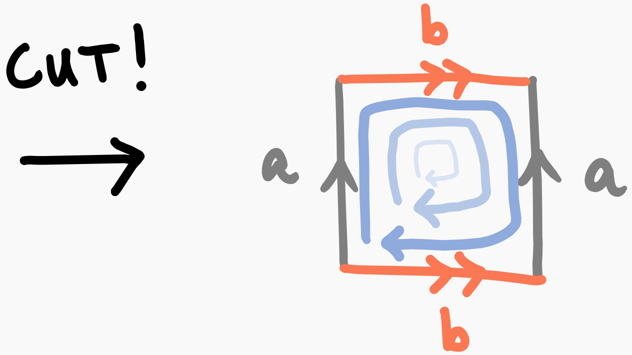

b
a
CUT!
b
a
a
b
a
a
CUT!
b
a
a
b





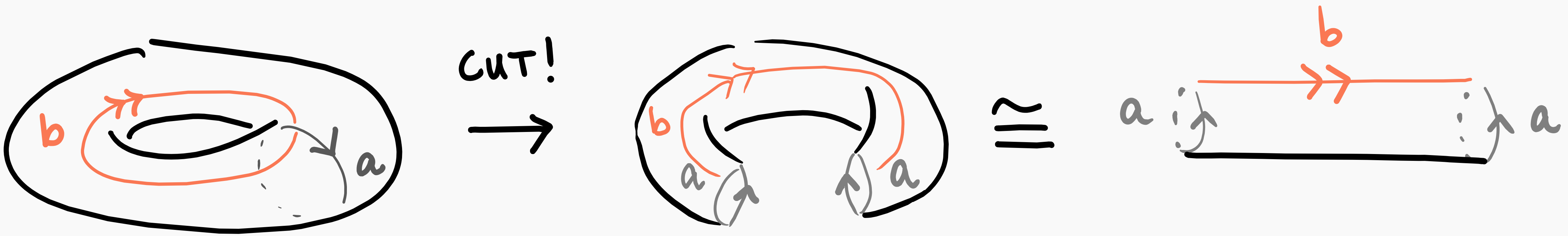
$$aba^{-1}b^{-1} = 1$$

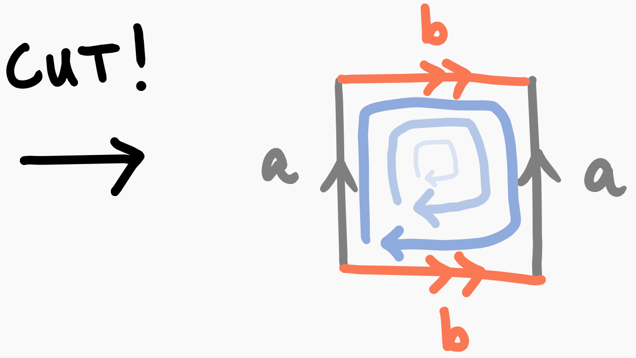




$$aba^{-1}b^{-1} = 1$$

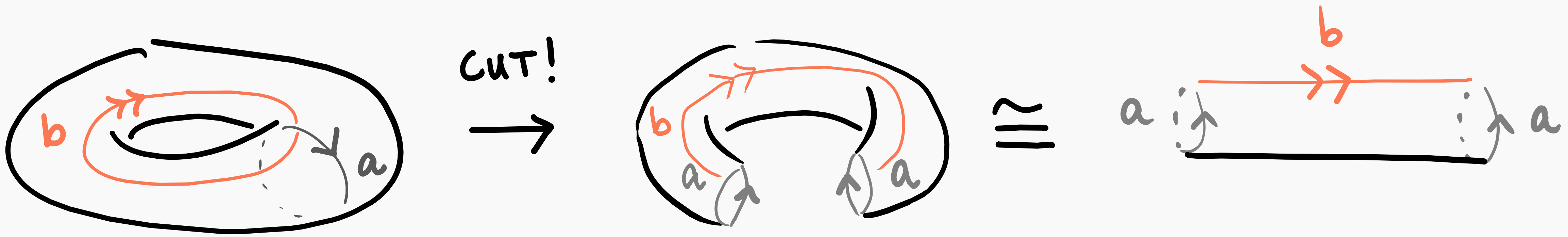
$\pi_1($  $, p) = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$

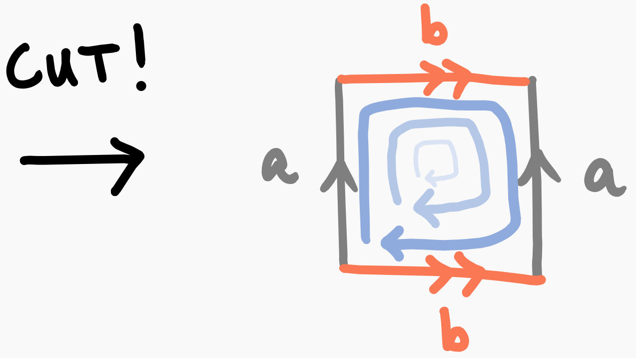




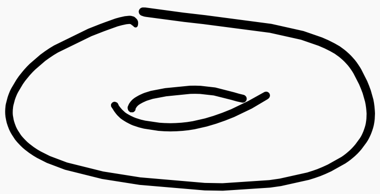
$$aba^{-1}b^{-1}=1$$

$\pi_1\big($  $, p\big) = \langle \underbrace{a, b}_{\text{generators}} \mid \underbrace{aba^{-1}b^{-1}=1}_{\text{relations (simplifying rules)}} \rangle$

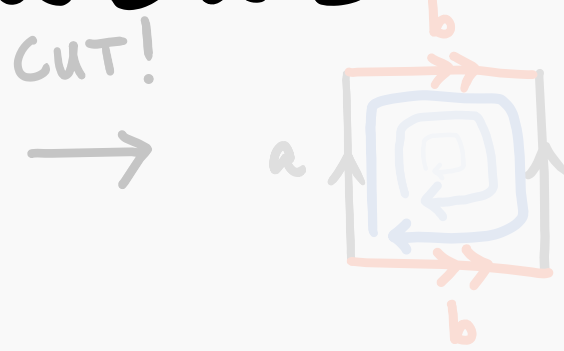




$$aba^{-1}b^{-1} = 1$$

$\pi_1($  $, p) = \langle \underbrace{a, b}_{\text{generators}} \mid \underbrace{ab = ba}_{\text{relations (simplifying rules)}} \rangle$

$$abaab = aabab = aaabb = a^3 b^2$$

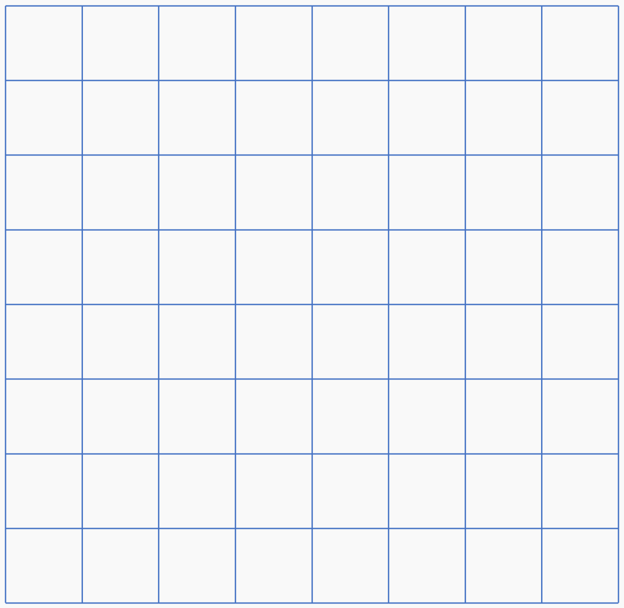


$$aba^{-1}b^{-1} = 1$$

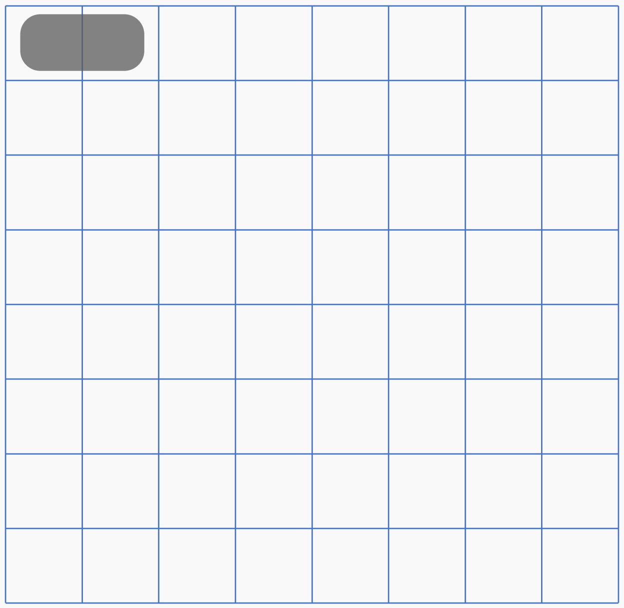
$$\pi_1\Big($$  $$, p\Big) = \langle \underbrace{a, b}_{\text{generators}} \mid \underbrace{ab = ba}_{\text{relations (simplifying rules)}} \rangle$$

Tiling Chessboards

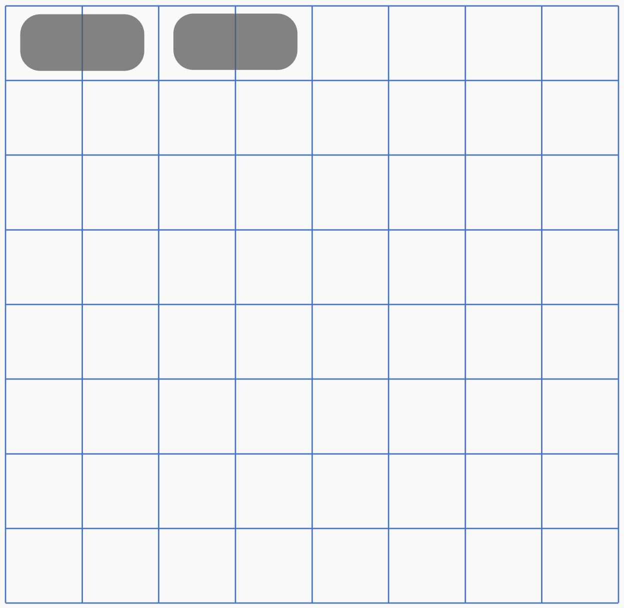
Tile a chessboard with dominoes



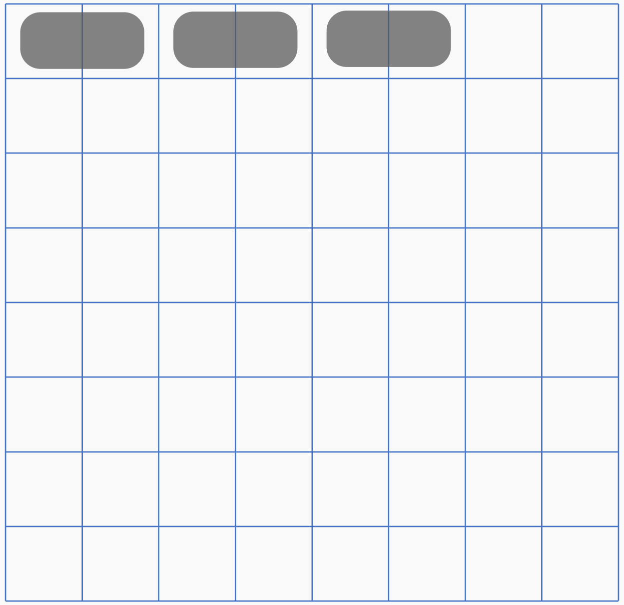
Tile a chessboard with dominoes



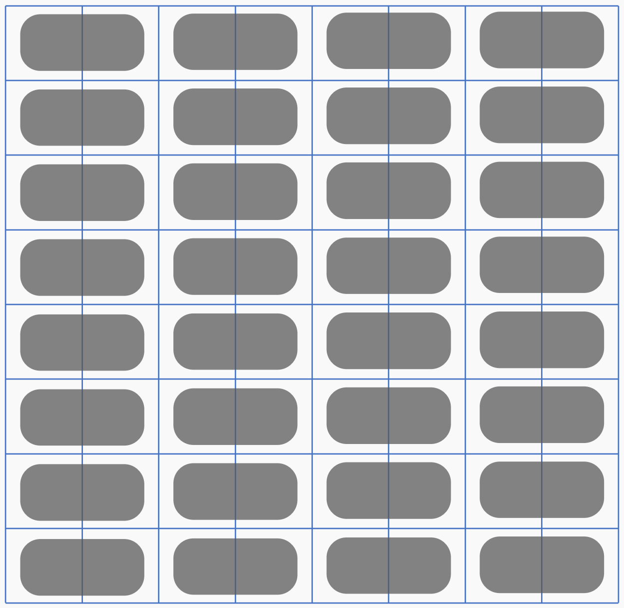
Tile a chessboard with dominoes



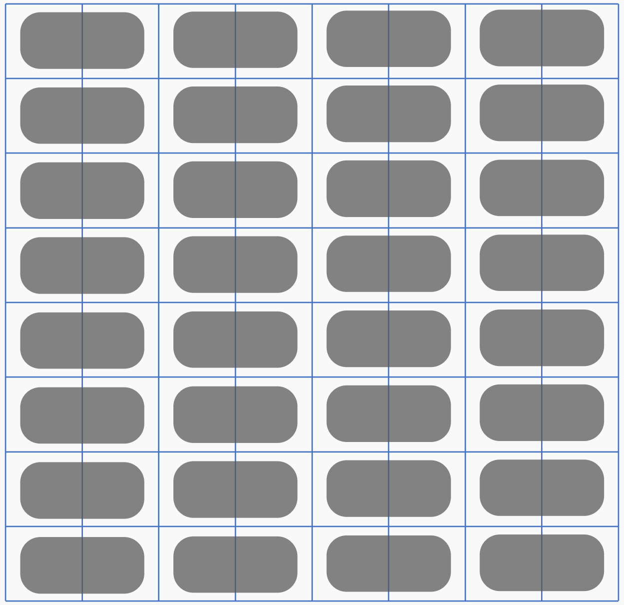
Tile a chessboard with dominoes



Tile a chessboard with dominoes

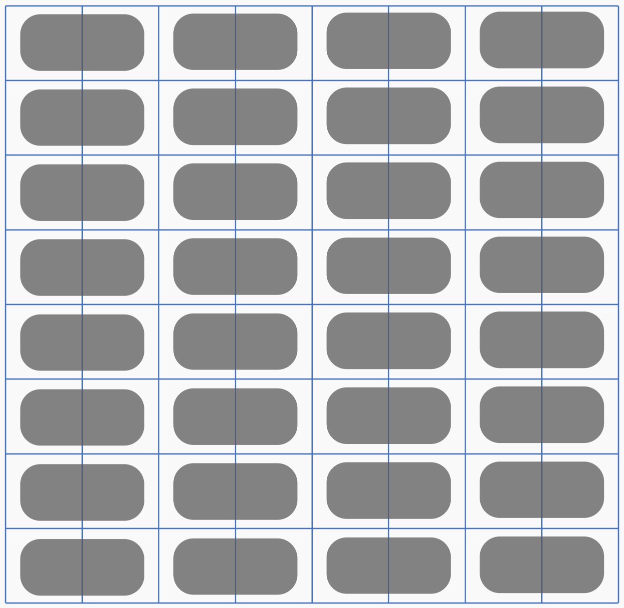


Tile a chessboard with dominoes



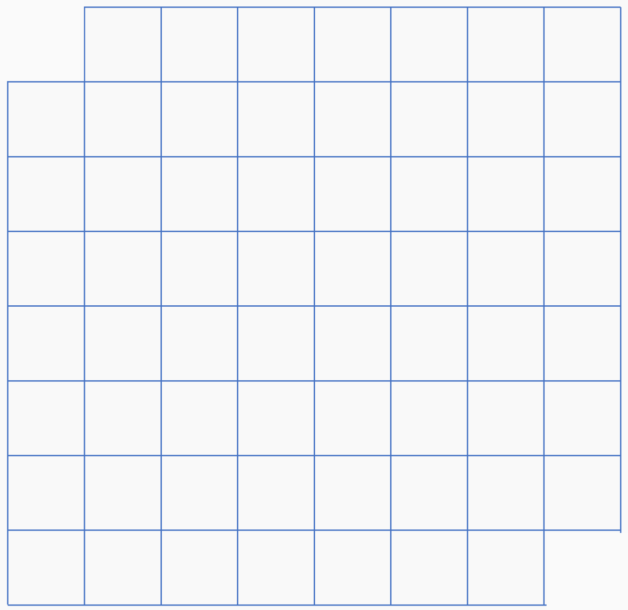
Yes!

Tile a chessboard with dominoes

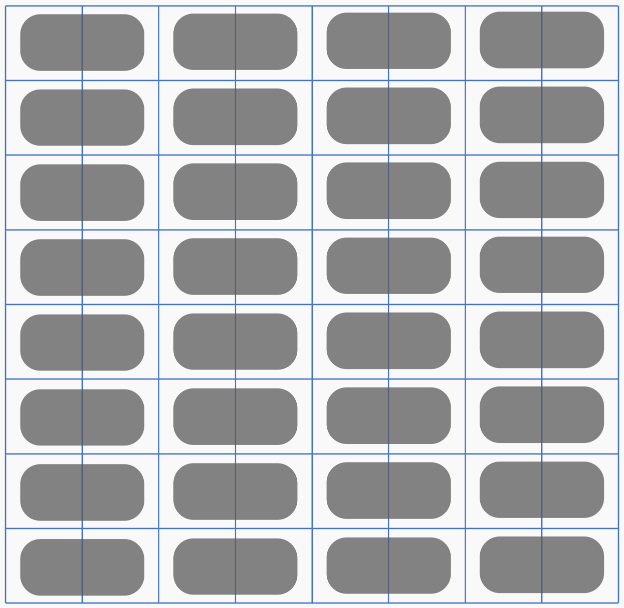


Yes!

... with the two corners removed

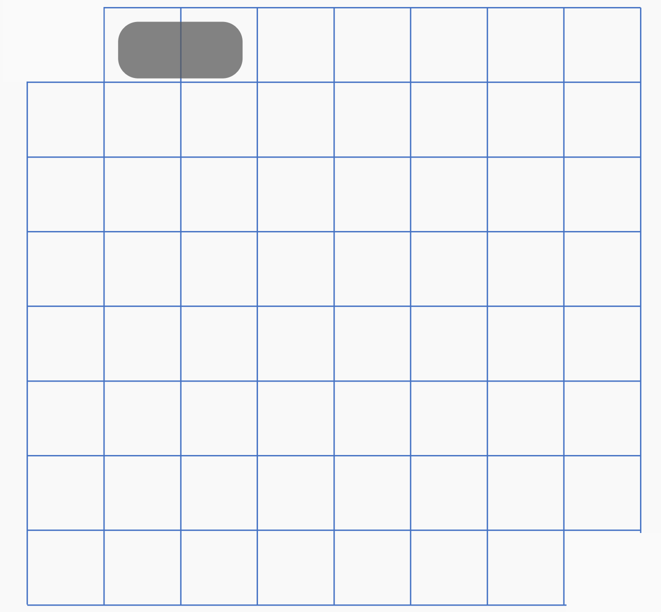


Tile a chessboard with dominoes

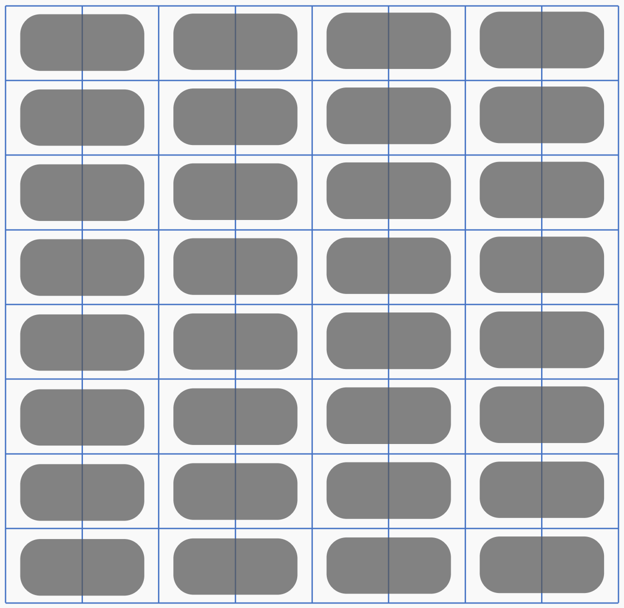


Yes!

... with the two corners removed

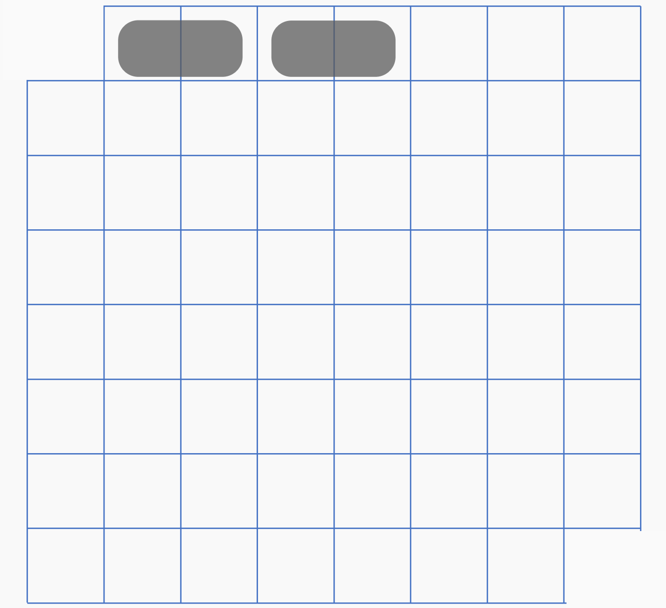


Tile a chessboard with dominoes

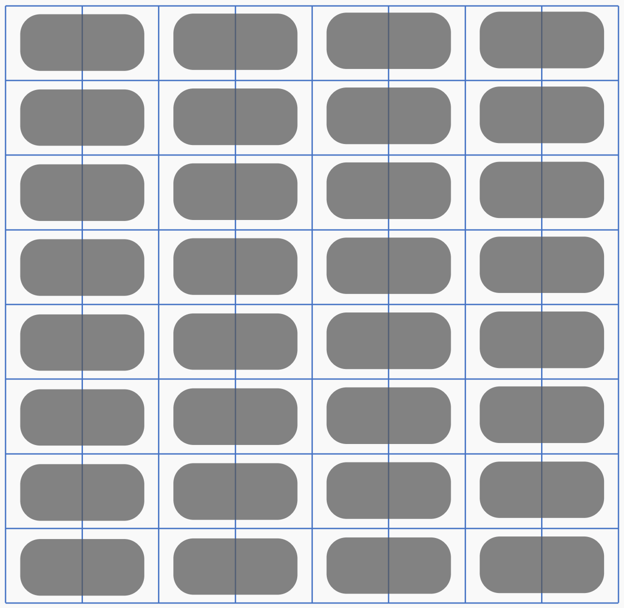


Yes!

... with the two corners removed

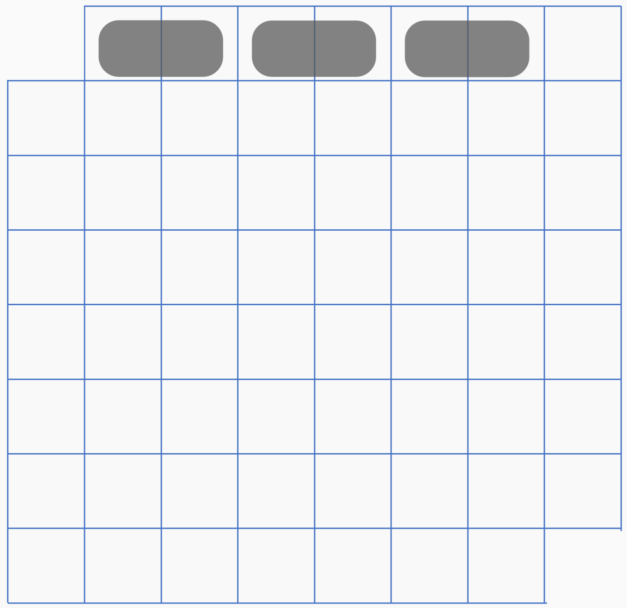


Tile a chessboard with dominoes

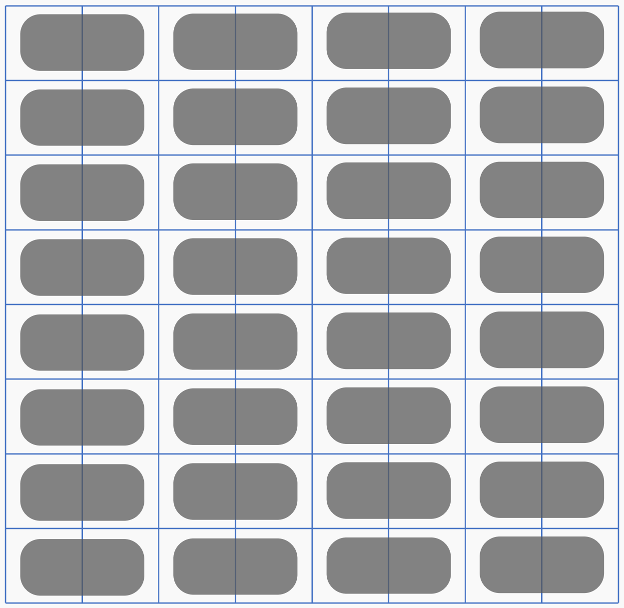


Yes!

... with the two corners removed

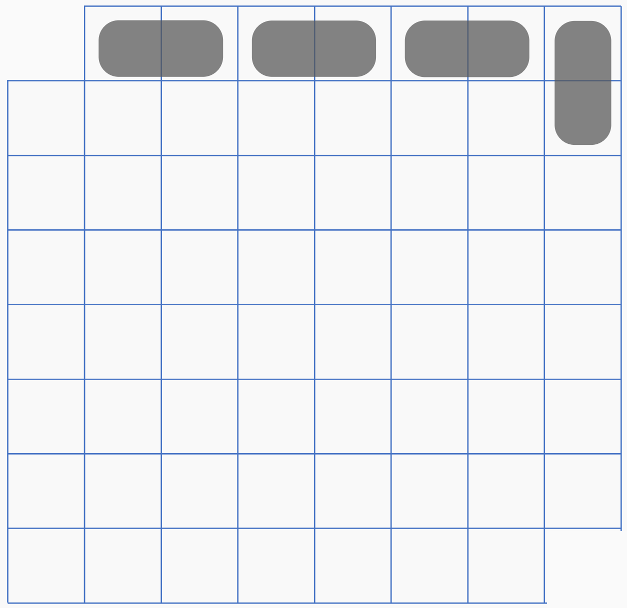


Tile a chessboard with dominoes

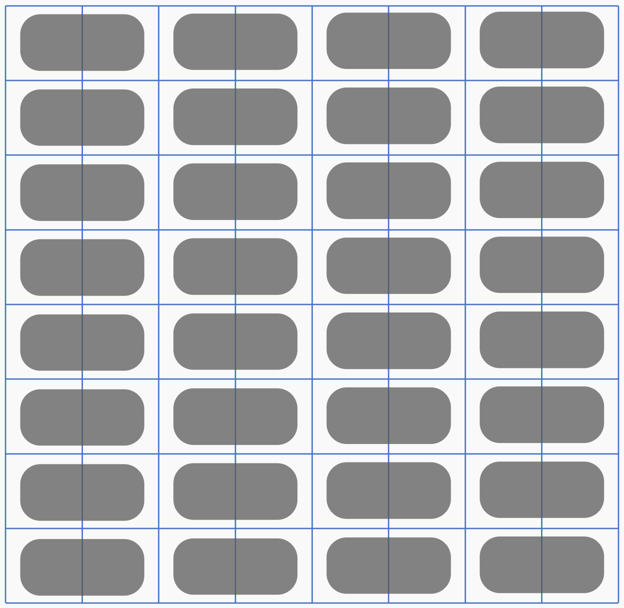


Yes!

... with the two corners removed

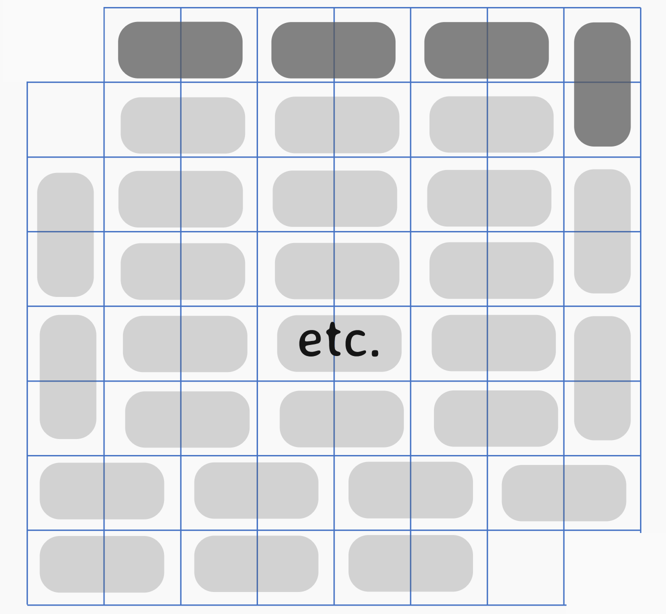


Tile a chessboard with dominoes



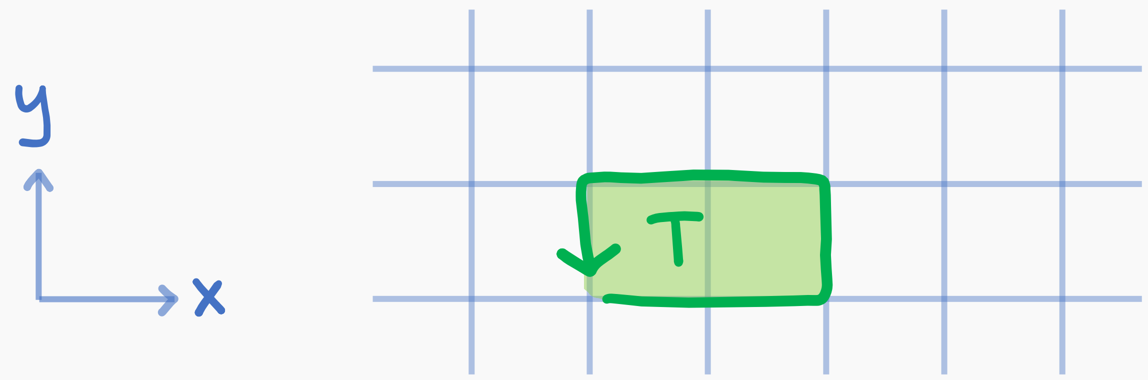
Yes!

... with the two corners removed

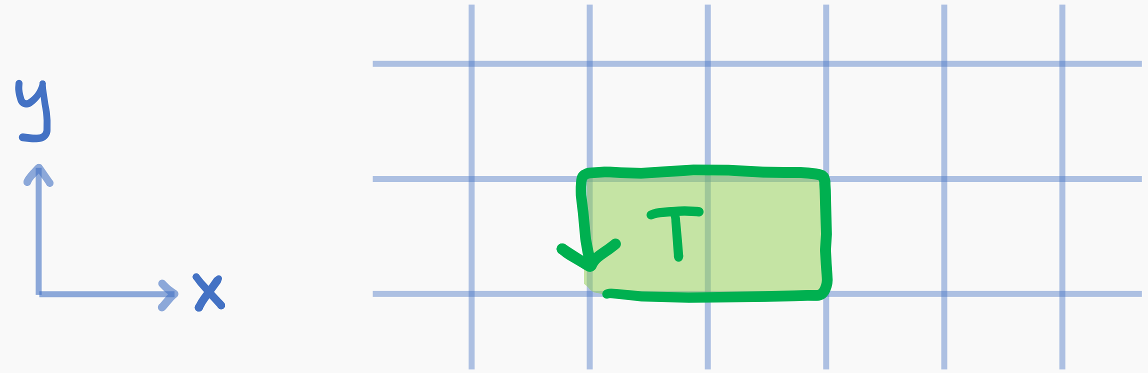


No! (But why?)

Assign word to region

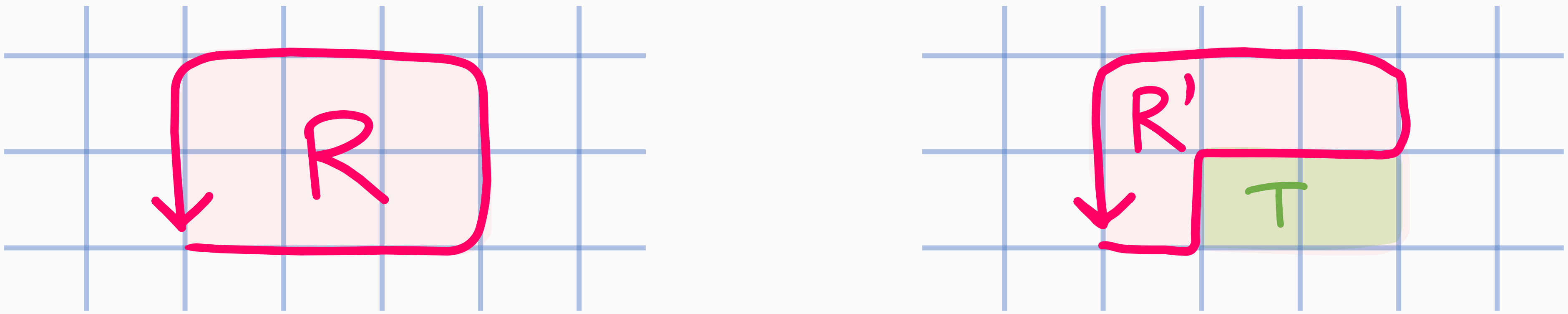


Assign word to region

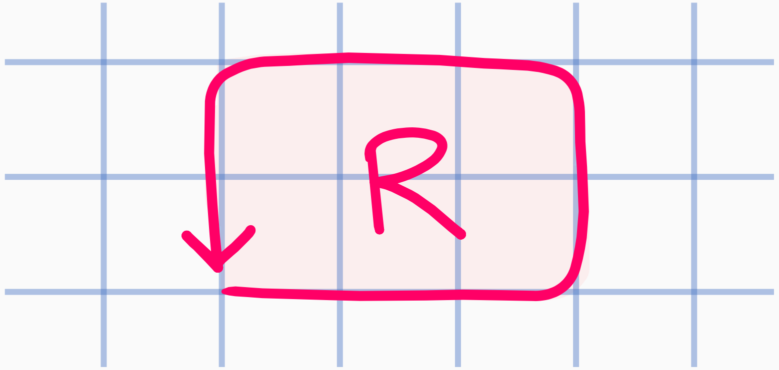


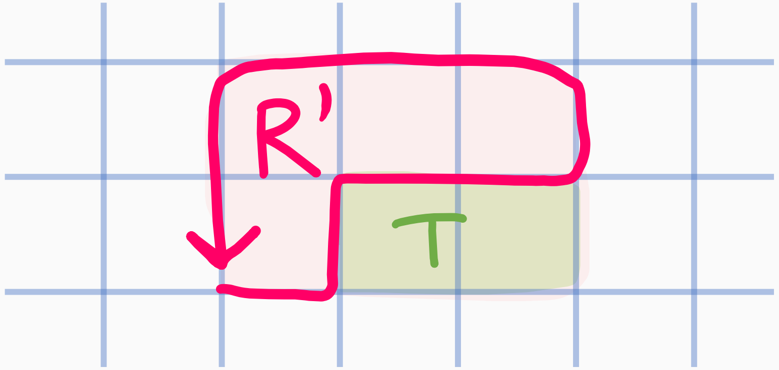
$$W(T) = x\,x\,y\,x^{-1}x^{-1}y^{-1} = x^2\,y\,x^{-2}\,y^{-1}$$

Shrink region R by tile shape T $\Rightarrow$ simplify W(R) by the rule W(T) = 1.



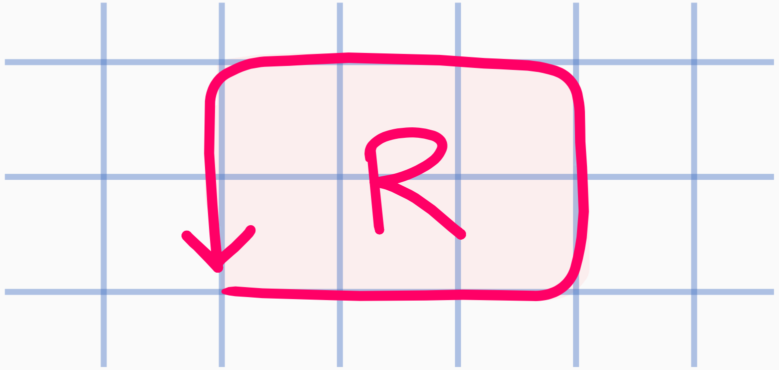
Shrink region R by tile shape T $\Rightarrow$ simplify W(R) by the rule W(T) = 1.

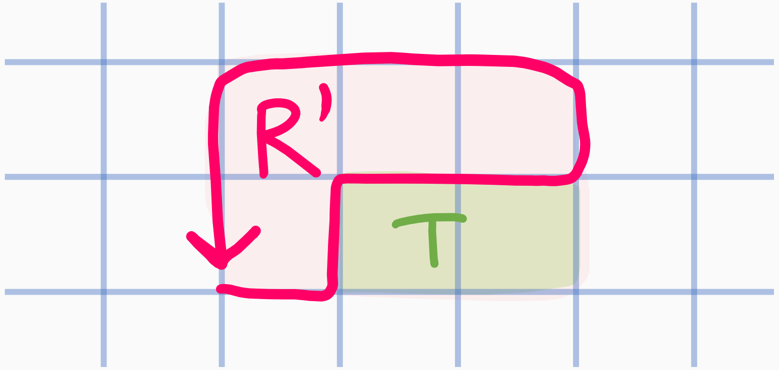




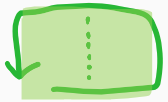
$$W(R) = xxxyyx^{-1}x^{-1}x^{-1}y^{-1}y^{-1} =$$

Shrink region R by tile shape T $\Rightarrow$ simplify W(R) by the rule W(T) = 1.



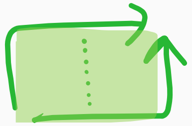


$$W(R) = xxxyyx^{-1}x^{-1}x^{-1}y^{-1}y^{-1} =$$

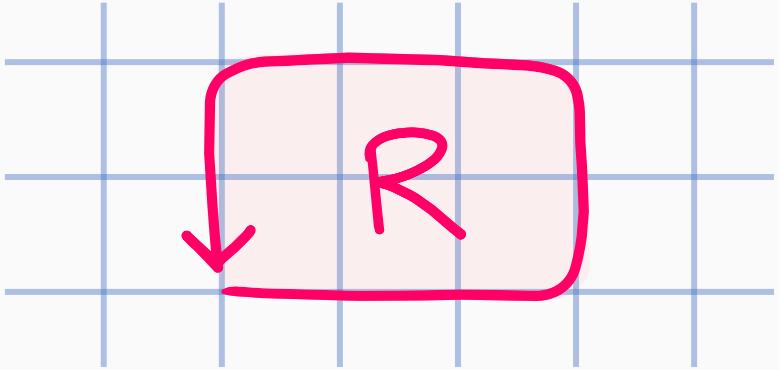


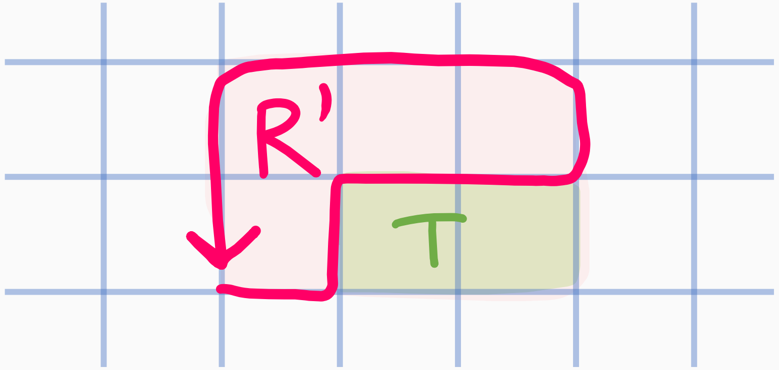
$$xxyx^{-1}x^{-1}y^{-1} = 1$$

or $$xxy = yxx$$

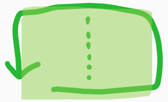


Shrink region R by tile shape T $\Rightarrow$ simplify W(R) by the rule W(T) = 1.



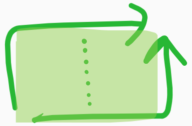


$$W(R) = x\underline{xxy}yx^{-1}x^{-1}x^{-1}y^{-1}y^{-1} =$$

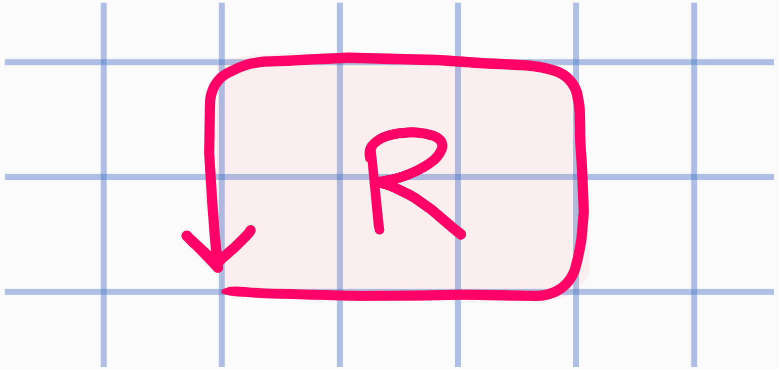


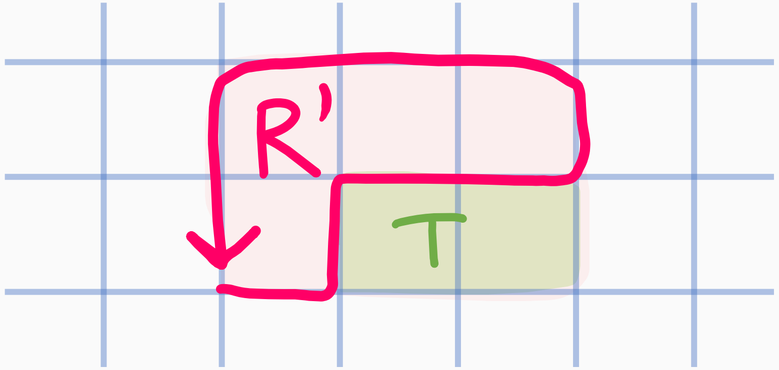
$$xxyx^{-1}x^{-1}y^{-1} = 1$$

or $xxy = yxx$

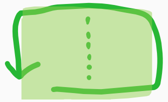


Shrink region R by tile shape T $\Rightarrow$ simplify W(R) by the rule W(T) = 1.



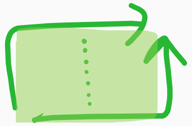


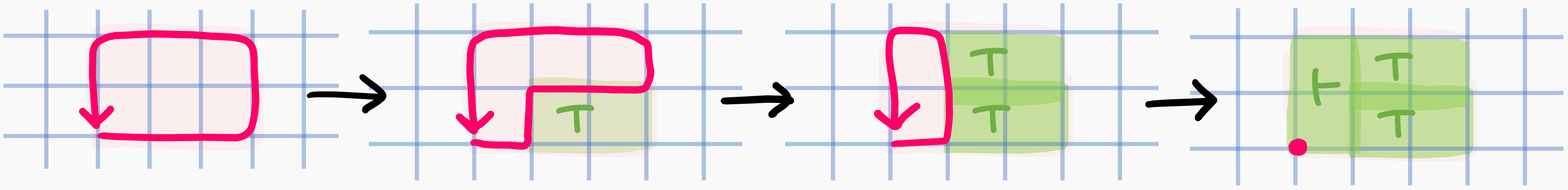
$$W(R) = x\underline{xxy}yx^{-1}x^{-1}x^{-1}y^{-1}y^{-1} = x\underline{yxx}yx^{-1}x^{-1}y^{-1}y^{-1} = W(R')$$



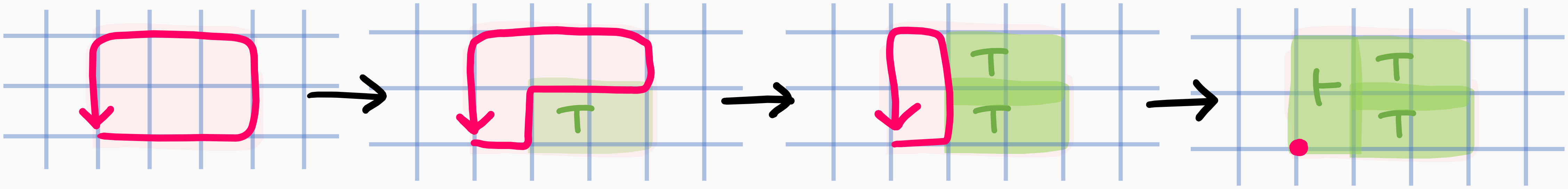
$$xxyx^{-1}x^{-1}y^{-1} = 1$$

or $xxy = yxx$



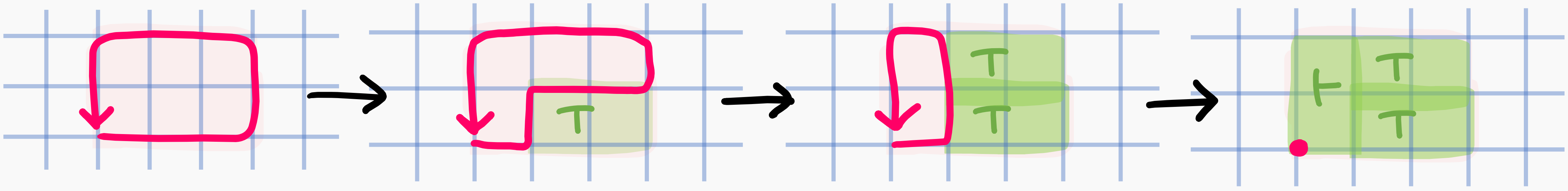


If R **can be tiled by tile shapes** $T_1, T_2, \ldots, T_r$**,**
Then $W(R)$ **simplifies to 1 by rules** $W(T_1) = 1, \ldots, W(T_r) = 1$**,**



If R **can be tiled by tile shapes** $T_1, T_2, \ldots, T_r$ **,**
Then $W(R)$ **simplifies to 1 by rules** $W(T_1) = 1, \ldots, W(T_r) = 1$**,**

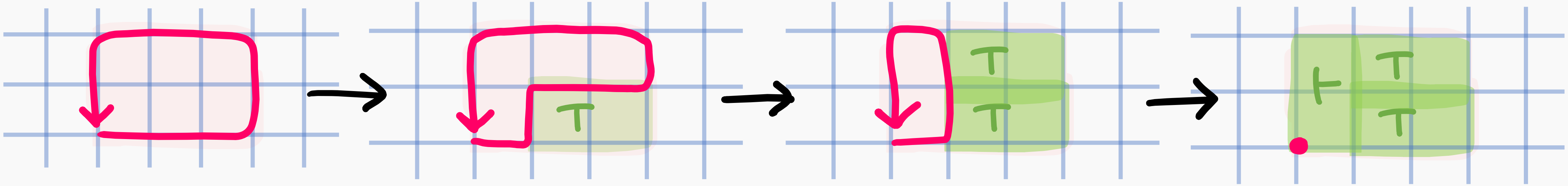
i.e. $W(R) = 1$ in $\langle x, y \mid W(T_1) = 1, \ldots, W(T_r) = 1\rangle$.



If R can be tiled by tile shapes $T_1, T_2, \ldots, T_r$,
Then $W(R)$ simplifies to 1 by rules $W(T_1) = 1, \ldots, W(T_r) = 1$,

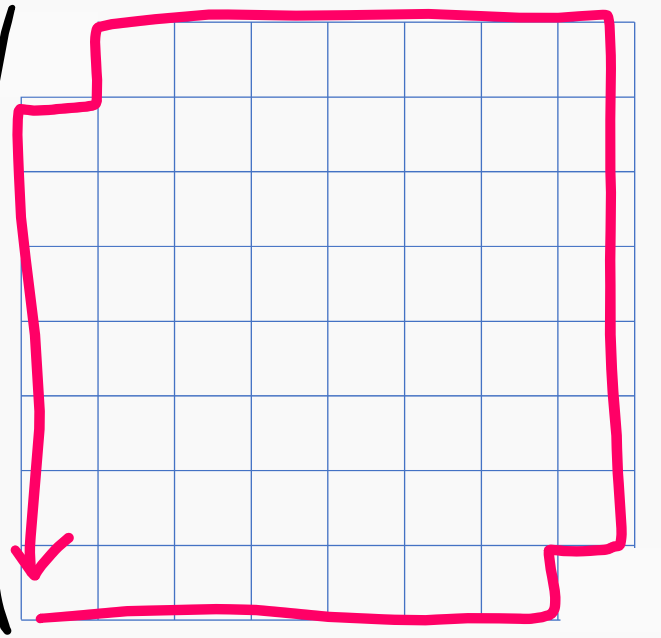
i.e. $W(R) = 1$ in $\langle x, y \mid W(T_1) = 1, \ldots, W(T_r) = 1 \rangle$.

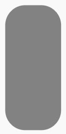
CONTRAPOSITIVE



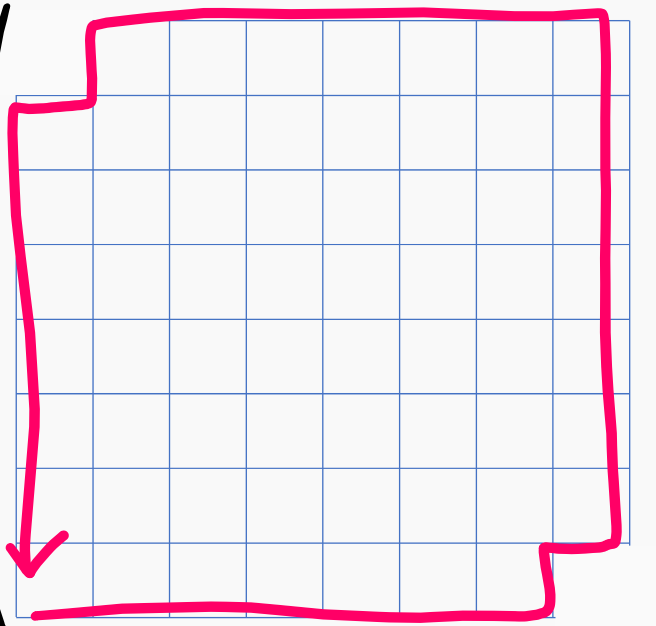
Theorem. (Conway-Lagarias, 1990)
If $W(R) \neq 1$ in $\langle x, y \mid W(T_1) = 1, \ldots, W(T_r) = 1 \rangle$,
Then R cannot be tiled by tile shapes $T_1, T_2, \ldots, T_r$.

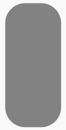
For the chessboard missing corners

W() not simplify to 1

using W() and W()

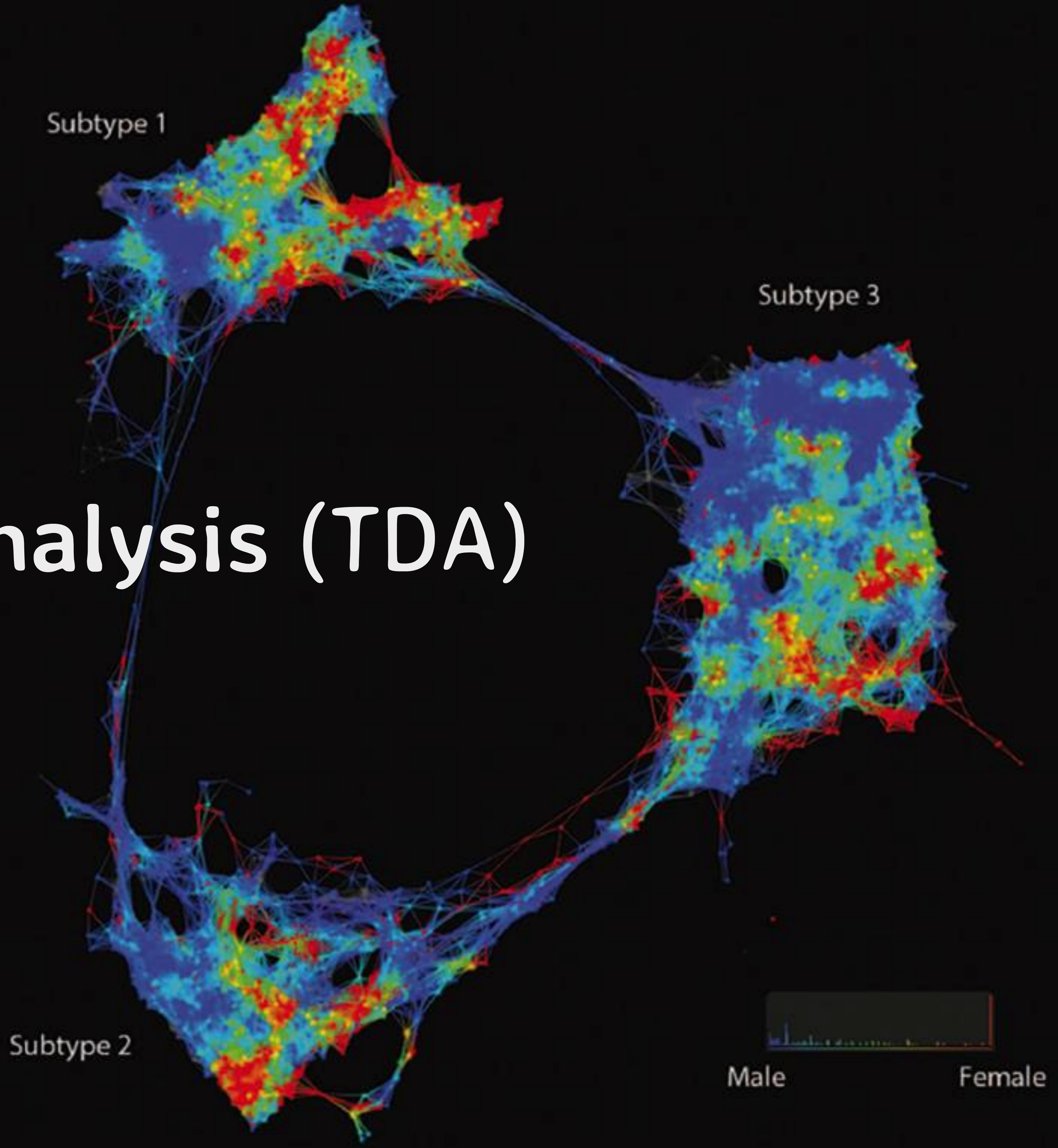
For the chessboard missing corners

W () not simplify to 1

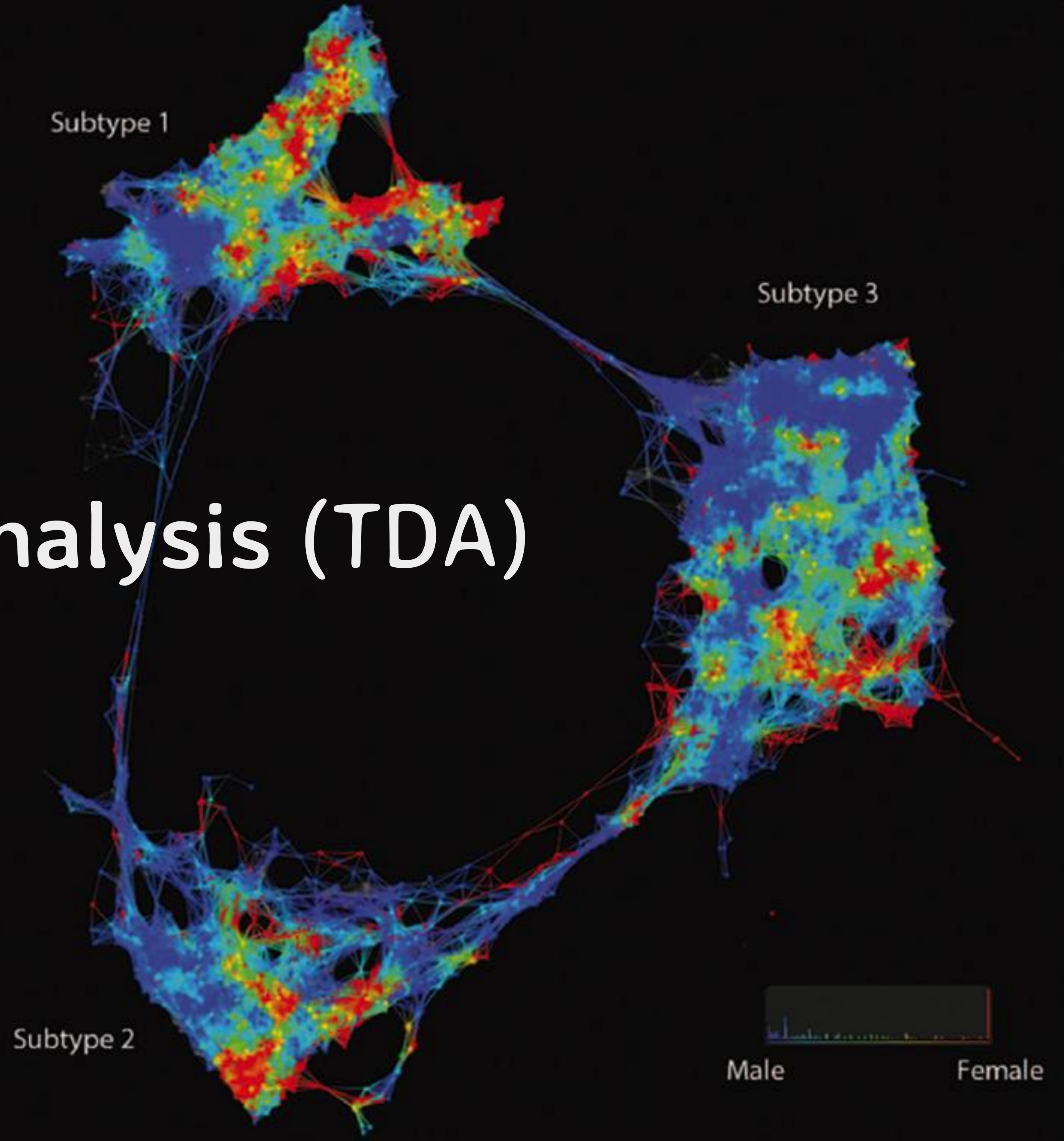
using W() and W()

Exercise.

Topological Data Analysis (TDA)



Three kinds of Type 2 Diabetes.
Discovered 2015.



Topological Data Analysis (TDA)

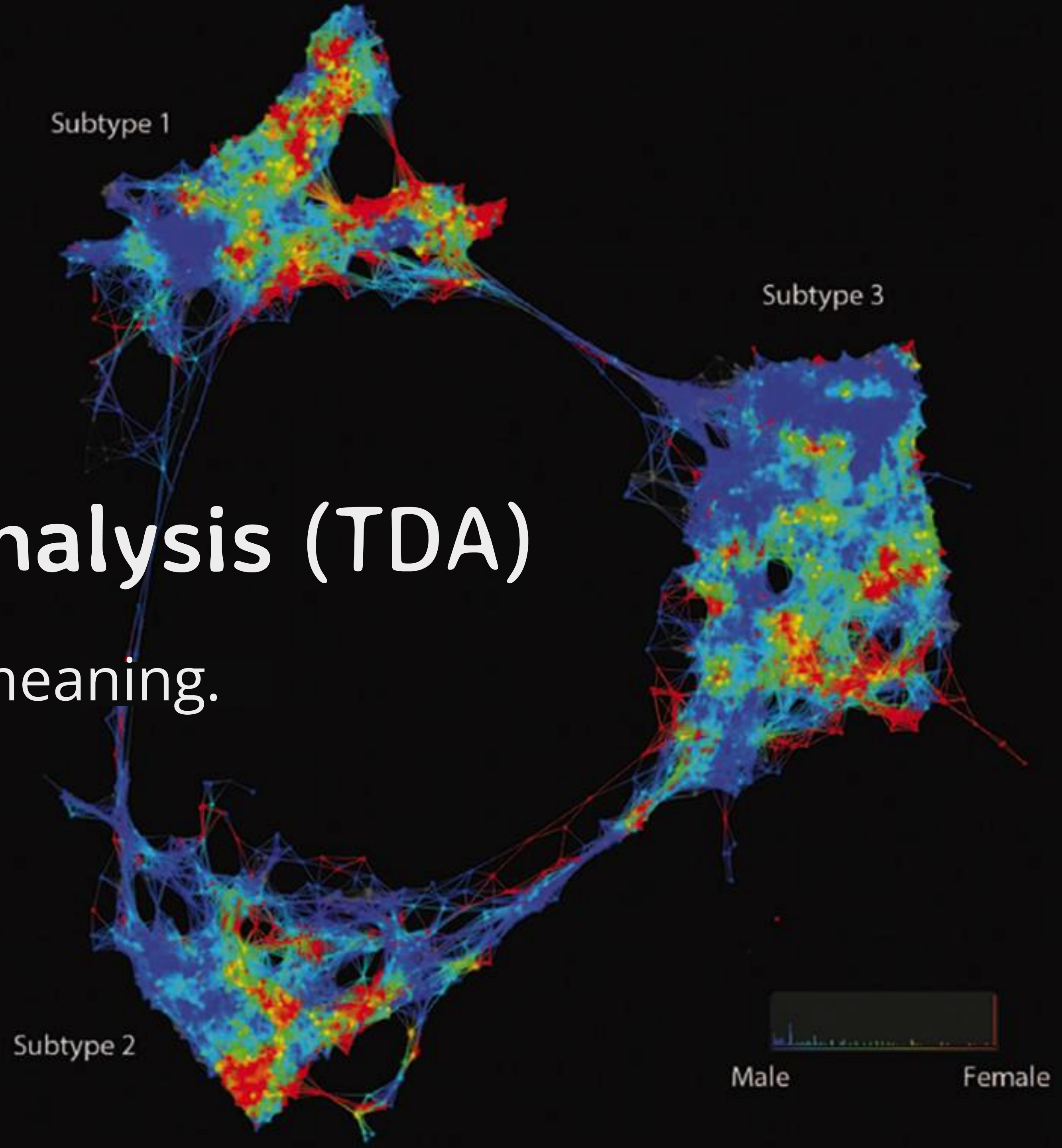
Data has shape.

Three kinds of Type 2 Diabetes.
Discovered 2015.

Topological Data Analysis (TDA)

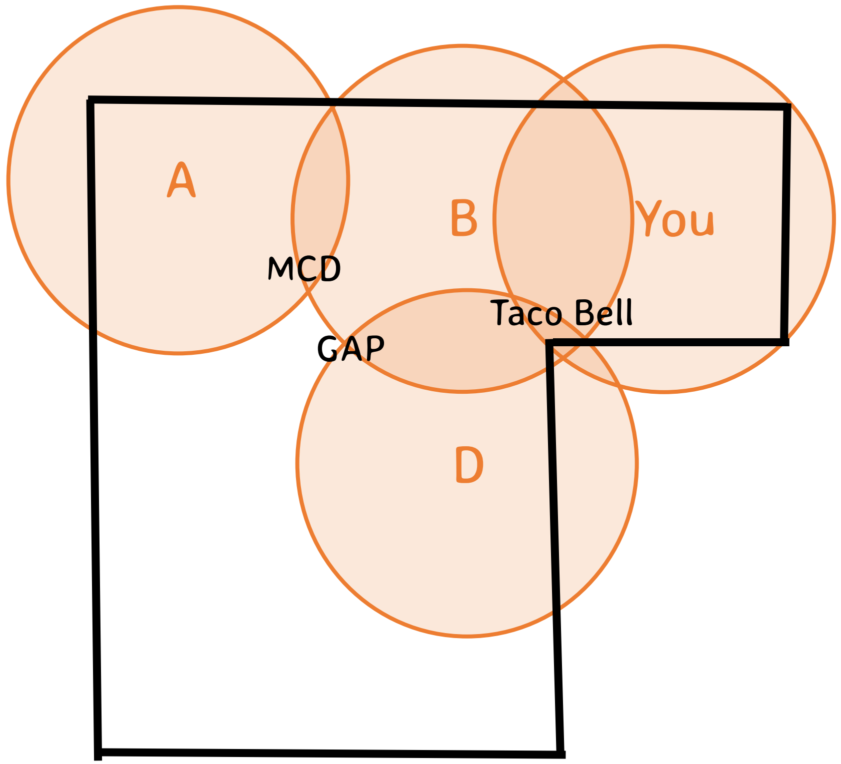
Data has shape. Shape has meaning.

Three kinds of Type 2 Diabetes.
Discovered 2015.



Coverage Problem

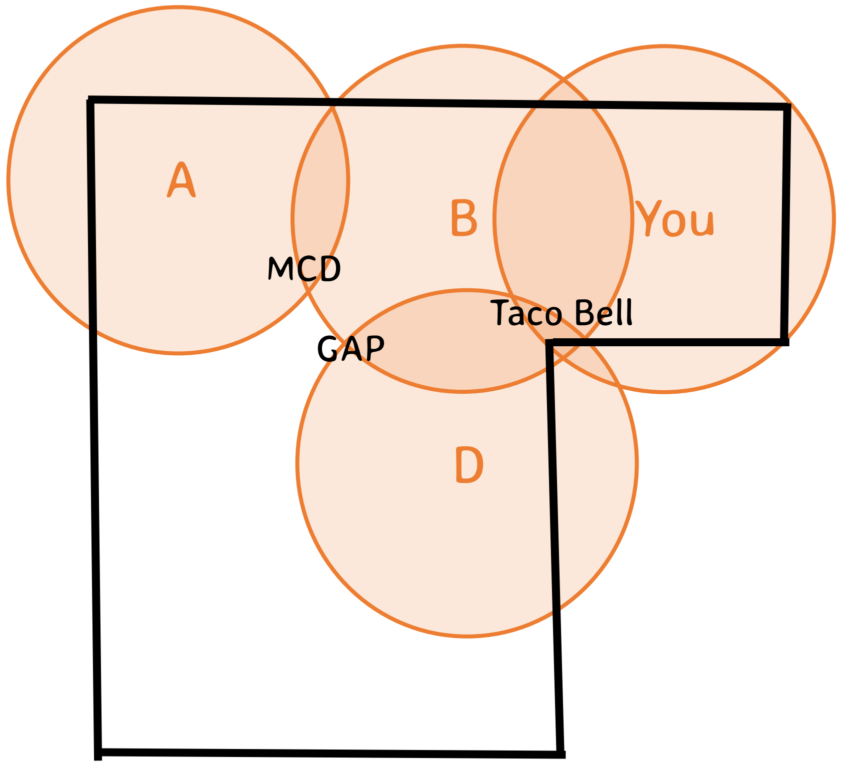
PERSON	SEES
You	Taco Bell
Aubrey	McDonald's
Becky	Taco Bell, McDonald's, GAP
Carlos	McDonald's, Gamestop, GAP
David	Taco Bell, Gamestop, GAP, Apple
Ellen	Gamestop, Foot Locker
Fabio	Apple, Foot Locker



Mall floorplan

Coverage Problem

PERSON	SEES
You	Taco Bell
Aubrey	McDonald's
Becky	Taco Bell, McDonald's, GAP
Carlos	McDonald's, Gamestop, GAP
David	Taco Bell, Gamestop, GAP, Apple
Ellen	Gamestop, Foot Locker
Fabio	Apple, Foot Locker



Mall floorplan

Assume

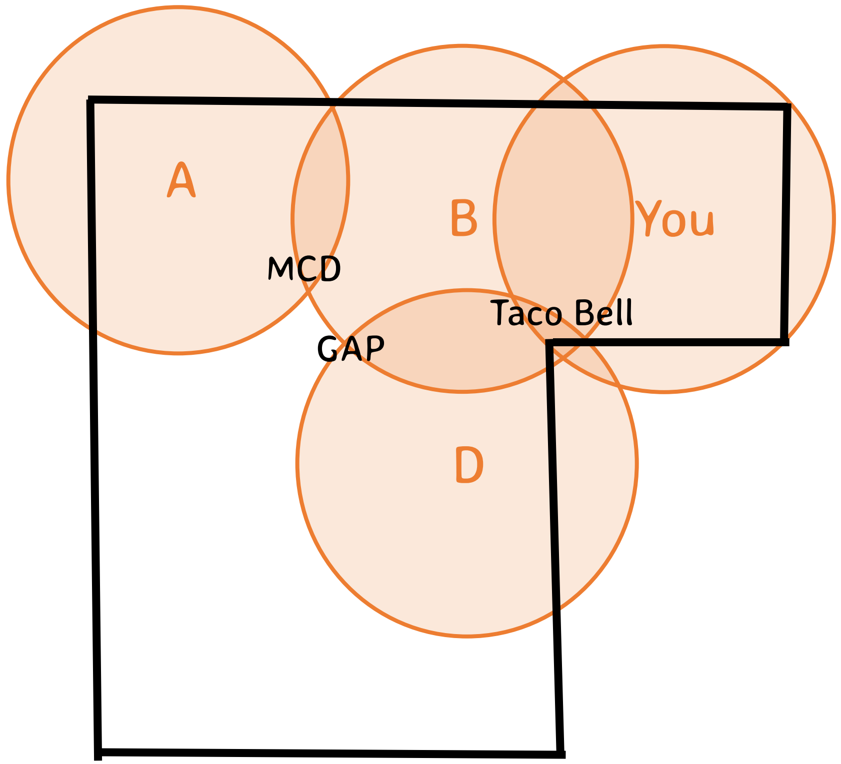
Everyone has the same sight radius

Phone calls reveal all common things any number of people see

Group sees entire mall periphery

Coverage Problem

PERSON	SEES
You	Taco Bell
Aubrey	McDonald's
Becky	Taco Bell, McDonald's, GAP
Carlos	McDonald's, Gamestop, GAP
David	Taco Bell, Gamestop, GAP, Apple
Ellen	Gamestop, Foot Locker
Fabio	Apple, Foot Locker



Mall floorplan

Assume

Everyone has the same sight radius

Phone calls reveal all common things any number of people see

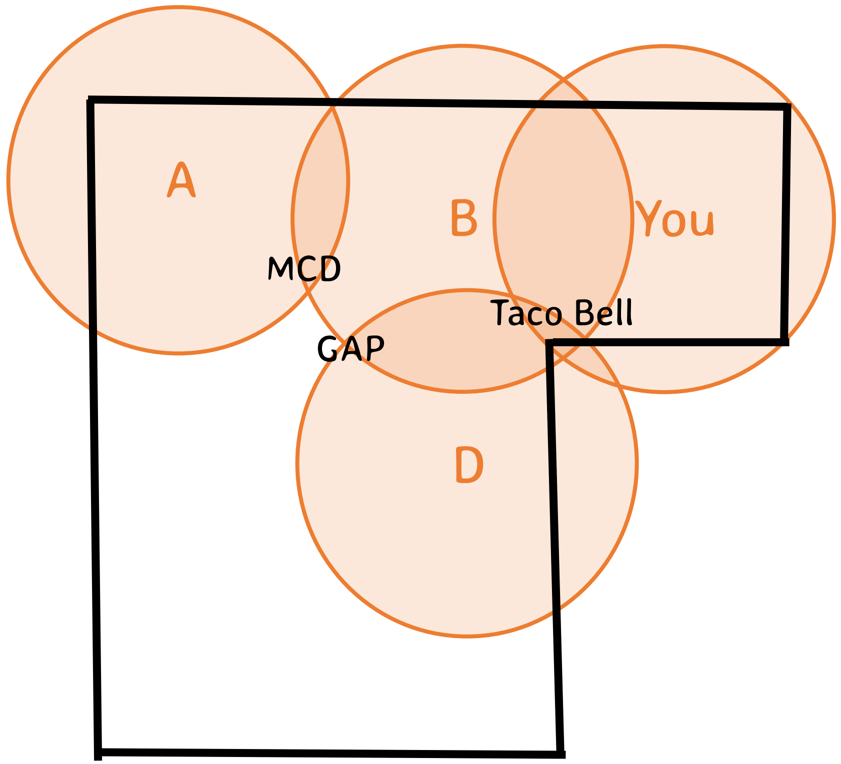
Group sees entire mall periphery

Question

Can the group see the entire mall premise?

Coverage Problem

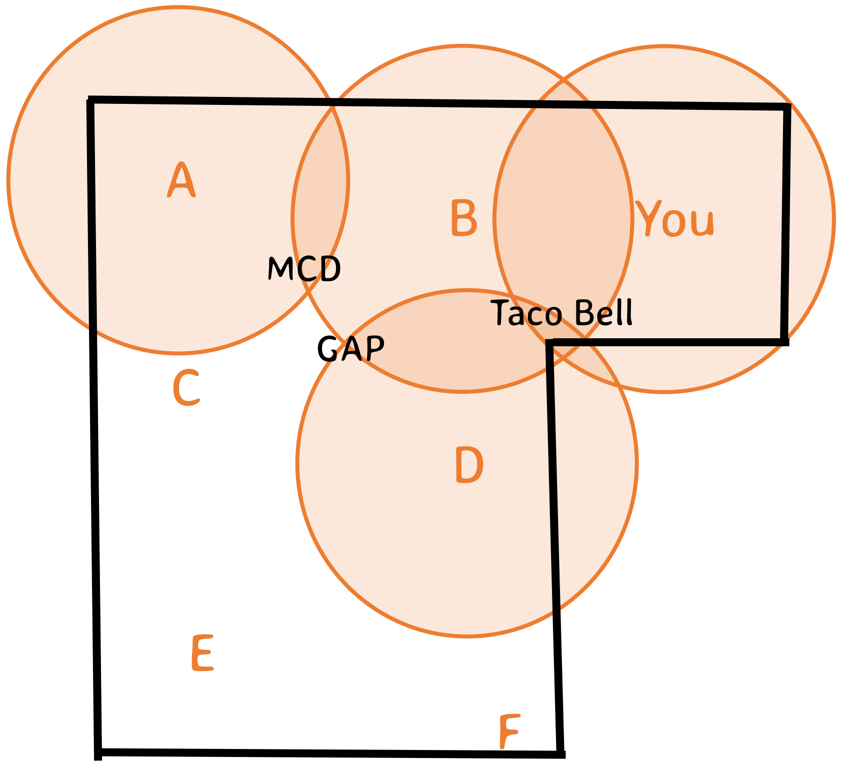
PERSON	SEES
You	Taco Bell
Aubrey	McDonald's
Becky	Taco Bell, McDonald's, GAP
Carlos	McDonald's, Gamestop, GAP
David	Taco Bell, Gamestop, GAP, Apple
Ellen	Gamestop, Foot Locker
Fabio	Apple, Foot Locker



Build
Simplicial Complex

Coverage Problem

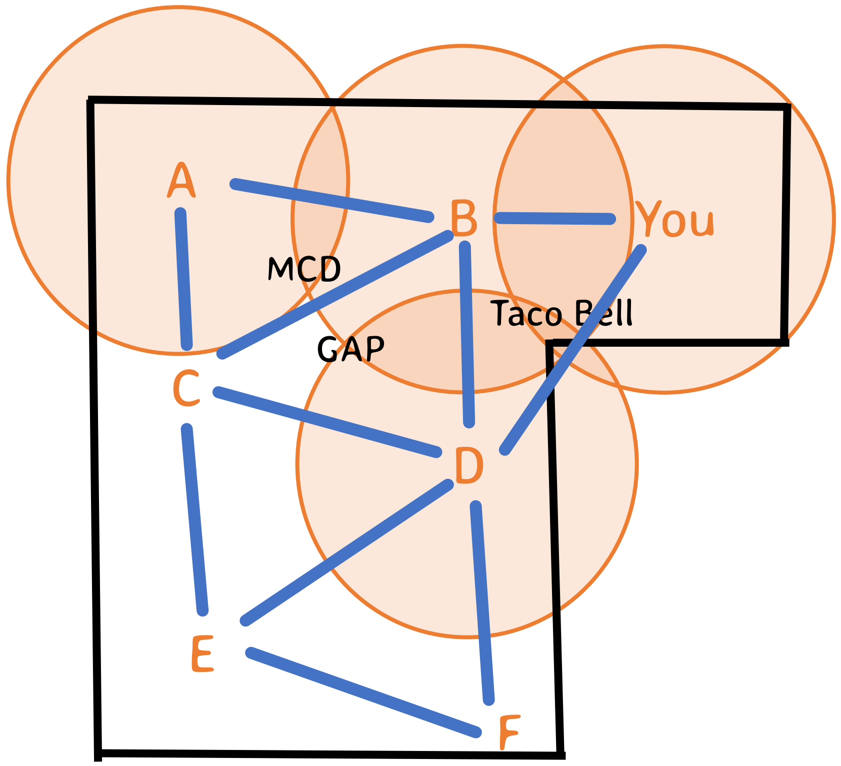
PERSON	SEES
You	Taco Bell
Aubrey	McDonald’s
Becky	Taco Bell, McDonald’s, GAP
Carlos	McDonald’s, Gamestop, GAP
David	Taco Bell, Gamestop, GAP, Apple
Ellen	Gamestop, Foot Locker
Fabio	Apple, Foot Locker



Build
Simplicial Complex

Coverage Problem

PERSON	SEES
You	Taco Bell
Aubrey	McDonald's
Becky	Taco Bell, McDonald's, GAP
Carlos	McDonald's, Gamestop, GAP
David	Taco Bell, Gamestop, GAP, Apple
Ellen	Gamestop, Foot Locker
Fabio	Apple, Foot Locker



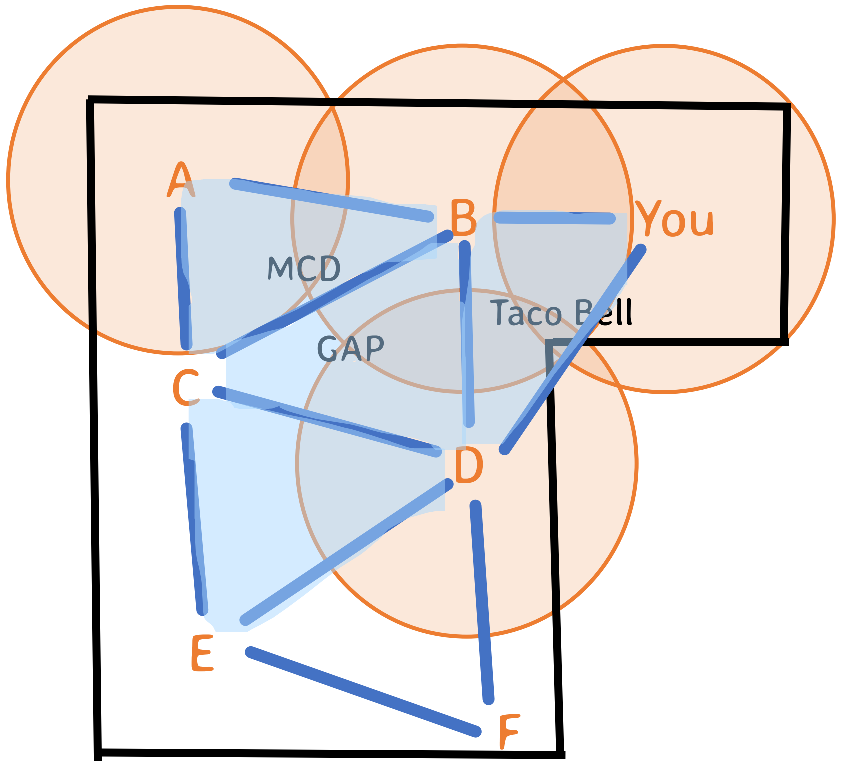
Build

Simplicial Complex

Edge: 2 people see same store

Coverage Problem

PERSON	SEES
You	Taco Bell
Aubrey	McDonald’s
Becky	Taco Bell, McDonald’s, GAP
Carlos	McDonald’s, Gamestop, GAP
David	Taco Bell, Gamestop, GAP, Apple
Ellen	Gamestop, Foot Locker
Fabio	Apple, Foot Locker



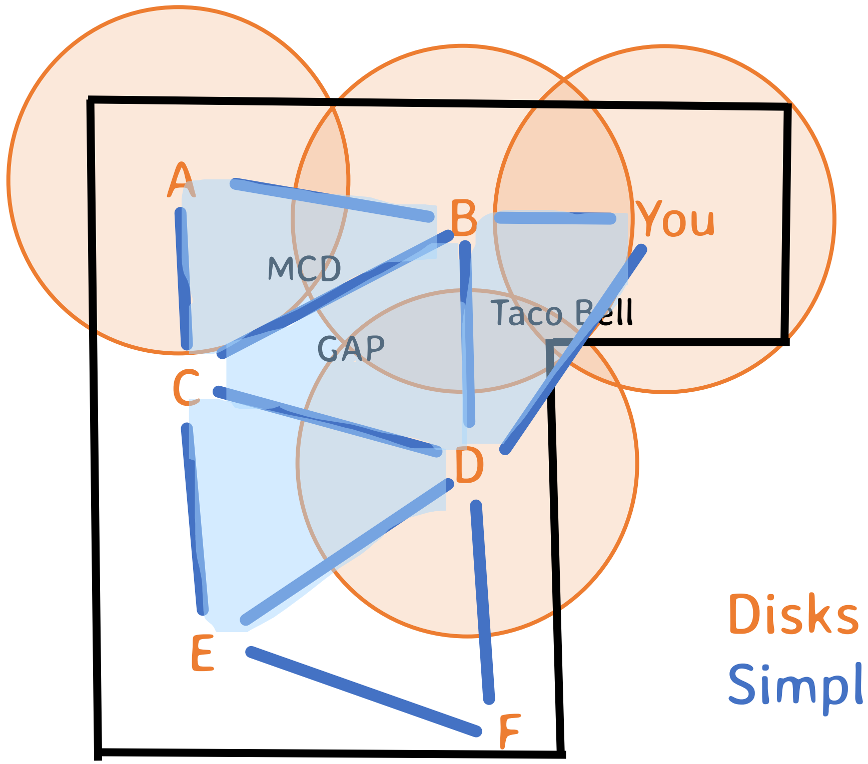
Build
Simplicial Complex

Edge: 2 people see same store

Face: 3 people see same store

Coverage Problem

PERSON	SEES
You	Taco Bell
Aubrey	McDonald's
Becky	Taco Bell, McDonald's, GAP
Carlos	McDonald's, Gamestop, GAP
David	Taco Bell, Gamestop, GAP, Apple
Ellen	Gamestop, Foot Locker
Fabio	Apple, Foot Locker



Build
Simplicial Complex

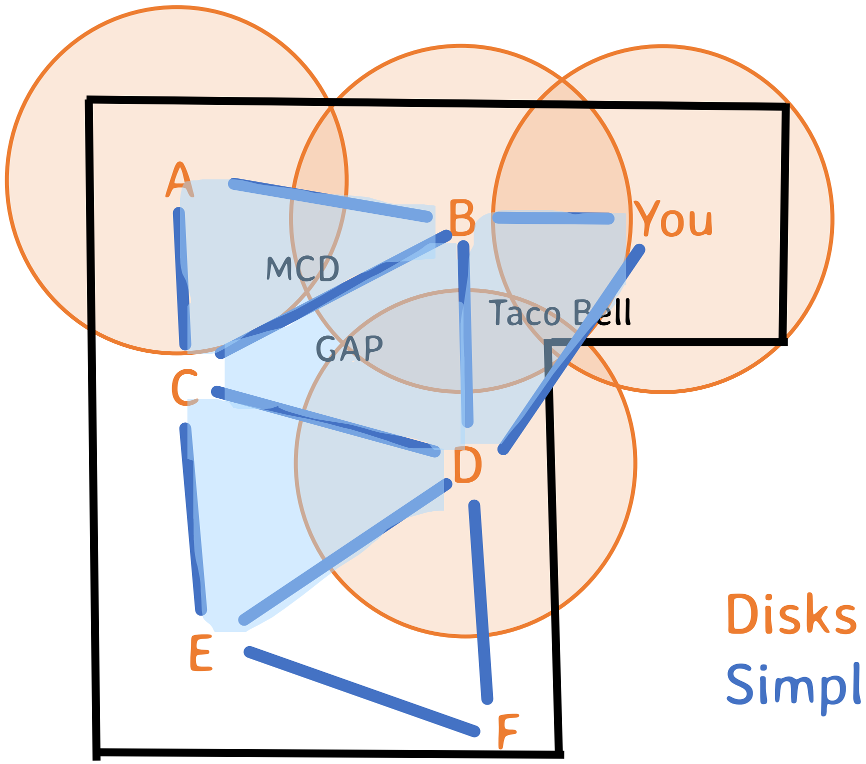
Edge: 2 people see same store

Face: 3 people see same store

Disks cover region ⇒
Simplicial complex has no "holes".

Coverage Problem

PERSON	SEES
You	Taco Bell
Aubrey	McDonald's
Becky	Taco Bell, McDonald's, GAP
Carlos	McDonald's, Gamestop, GAP
David	Taco Bell, Gamestop, GAP, Apple
Ellen	Gamestop, Foot Locker
Fabio	Apple, Foot Locker



Build
Simplicial Complex

Edge: 2 people see same store

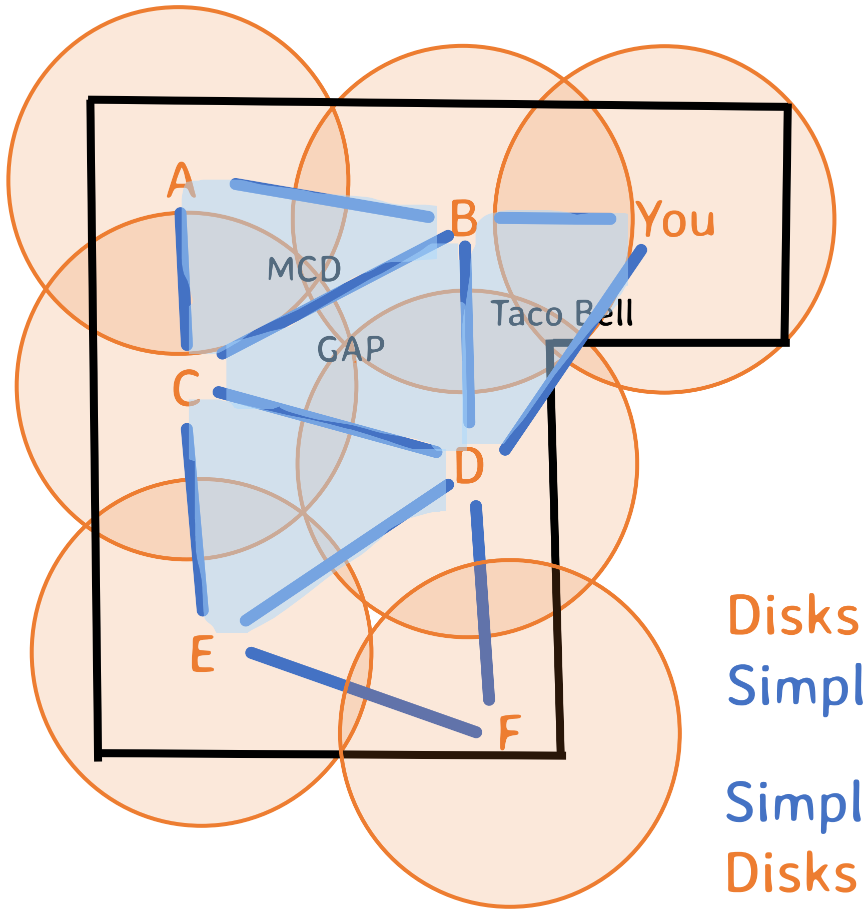
Face: 3 people see same store

Disks cover region ⇒
Simplicial complex has no "holes".

Simplicial complex has "holes" ⇒
Disks do not cover region.

Coverage Problem

PERSON	SEES
You	Taco Bell
Aubrey	McDonald's
Becky	Taco Bell, McDonald's, GAP
Carlos	McDonald's, Gamestop, GAP
David	Taco Bell, Gamestop, GAP, Apple
Ellen	Gamestop, Foot Locker
Fabio	Apple, Foot Locker



Build
Simplicial Complex

Edge: 2 people see same store

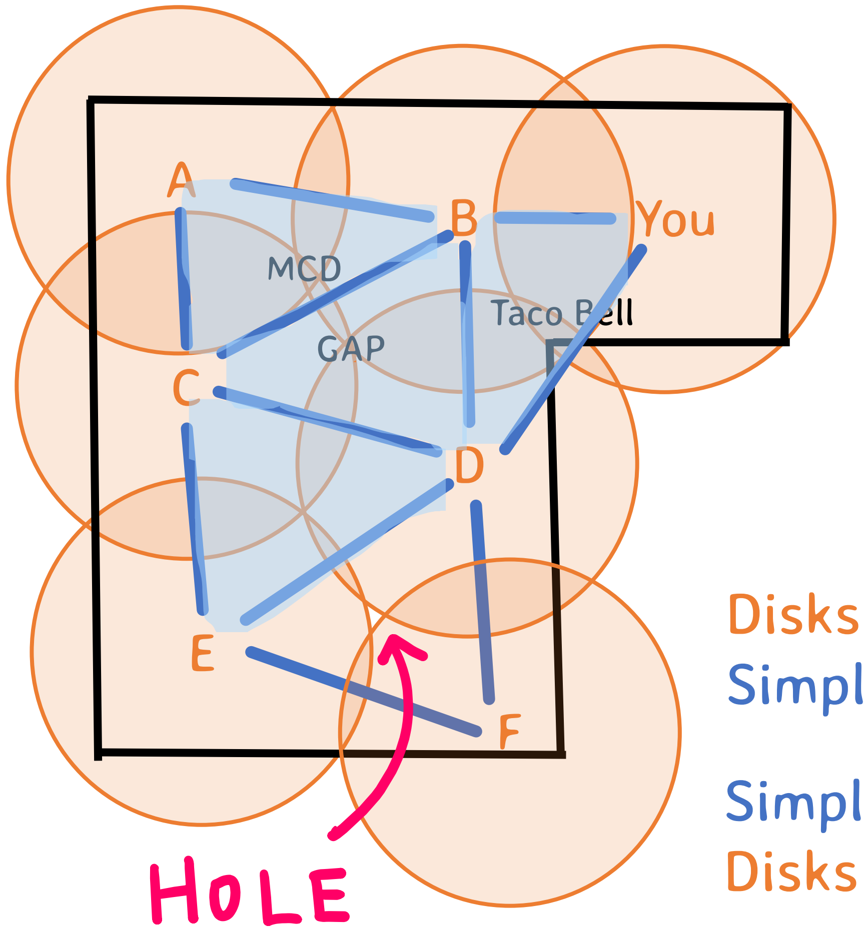
Face: 3 people see same store

Disks cover region ⇒
Simplicial complex has no "holes".

Simplicial complex has "holes" ⇒
Disks <u>do not</u> cover region.

Coverage Problem

PERSON	SEES
You	Taco Bell
Aubrey	McDonald's
Becky	Taco Bell, McDonald's, GAP
Carlos	McDonald's, Gamestop, GAP
David	Taco Bell, Gamestop, GAP, Apple
Ellen	Gamestop, Foot Locker
Fabio	Apple, Foot Locker



Build
Simplicial Complex

Edge: 2 people see same store

Face: 3 people see same store

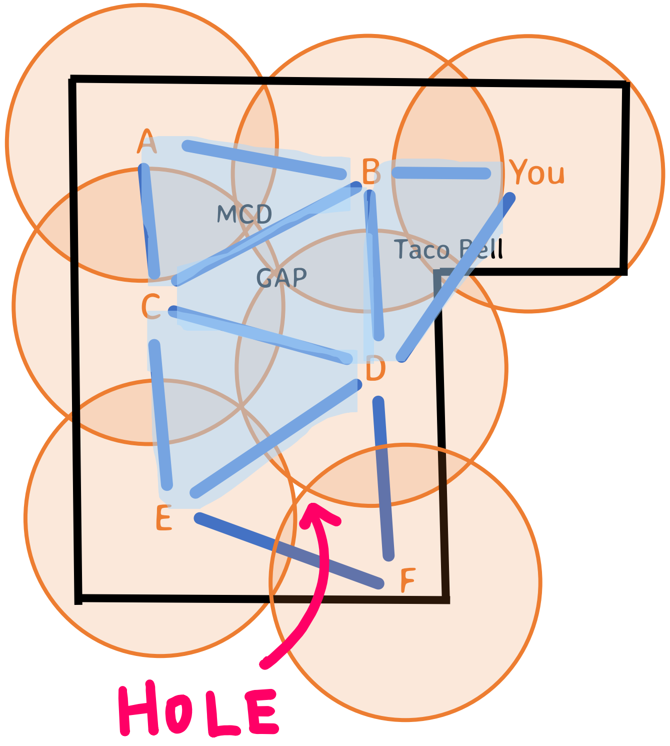
Disks cover region ⇒
Simplicial complex has no "holes".

Simplicial complex has "holes" ⇒
Disks <u>do not</u> cover region.

Coverage Problem

Sensor network (drones, etc.) used for:

- surveillance (forest fire),
- ensure wifi coverage

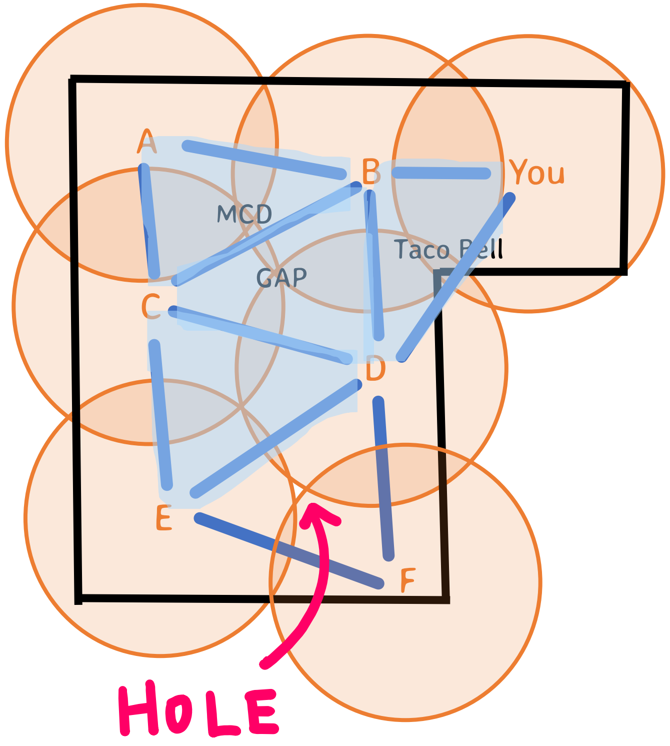


Coverage Problem

Sensor network (drones, etc.) used for:

- surveillance (forest fire),
- ensure wifi coverage

Computers use homology groups to detect "holes" in coverage.



RESOURCES ON TOPOLOGY

- Tadashi Tokieda's lectures on topology on YouTube. (Prerequisite: Calculus 3)
  < https://www.youtube.com/playlist?list=PLTBqohhFNBE_09L0i-lf3fYXF5woAbrzJ >

  Accompanying notes: "Topology in Four Days" in An Introduction to the Geometry and Topology of Fluid Flows.

RESOURCES ON TOPOLOGY

- Tadashi Tokieda's lectures on topology on YouTube. (Prerequisite: Calculus 3)
  < https://www.youtube.com/playlist?list=PLTBqohhFNBE_09L0i-lf3fYXF5woAbrzJ >

  Accompanying notes: "Topology in Four Days" in An Introduction to the Geometry and Topology of Fluid Flows.

RESOURCES ON TDA

- Learn more: talk to Thomas Needham or Washington Mio in FSU math department.
- Gunnar Carlsson, The Shape of Big Data
  < https://www.youtube.com/watch?v=L9iiJa1nZZk >
- Diabetes subtypes: < https://towardsdatascience.com/identification-of-type-2-diabetes-subgroups-through-topological-data-analysis-of-patient-similarity-91838f2ccf74 >
- An example of a topology-based algorithm called Mapper (2007)
  < https://www.youtube.com/watch?v=DD0_zPlEsqY >
- de Silva, Ghrist, Homological Sensor Networks
  < https://www.ams.org/notices/200701/fea-ghrist.pdf >