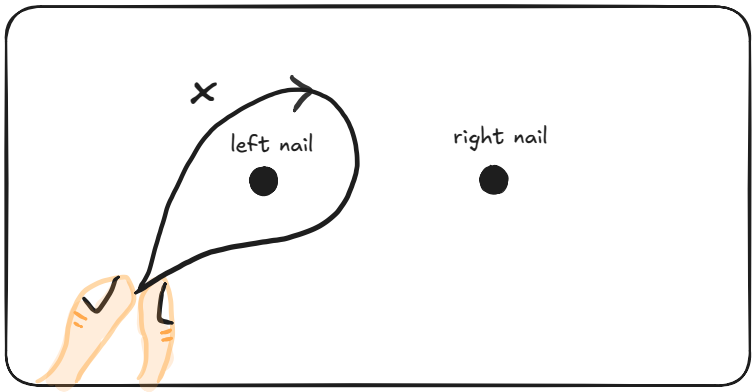
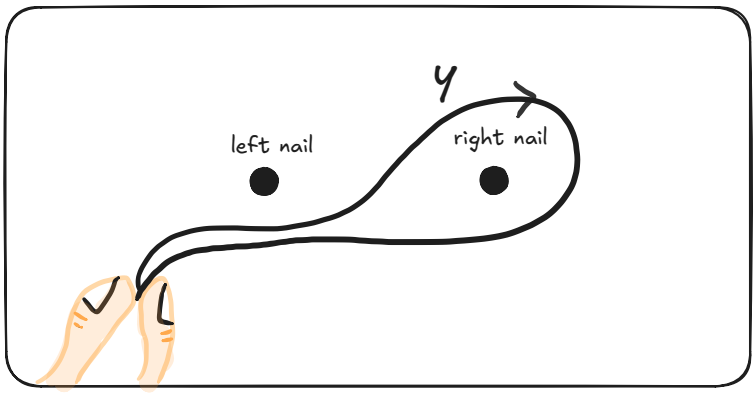
Lesson 16 — Hanging pictures

Problem 1

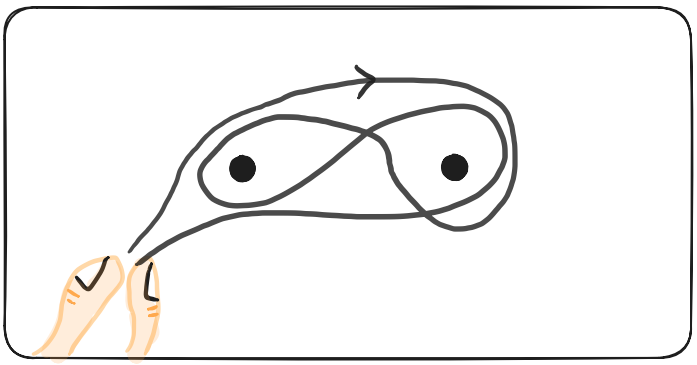
Suppose a clockwise loop around the left nail is represented by x and a clockwise loop around the right nail is represented by y:



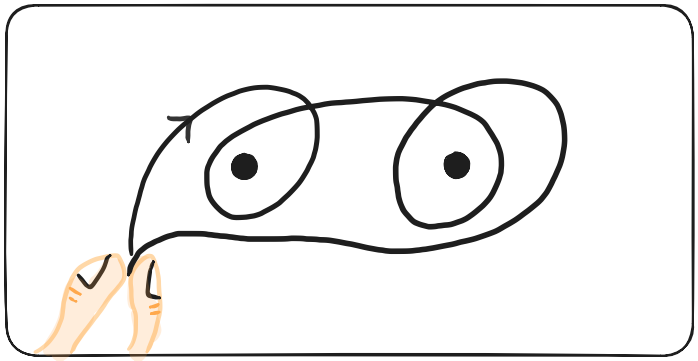


Problem: Find algebraic expressions for the following two loops in the pictures below.

(a)



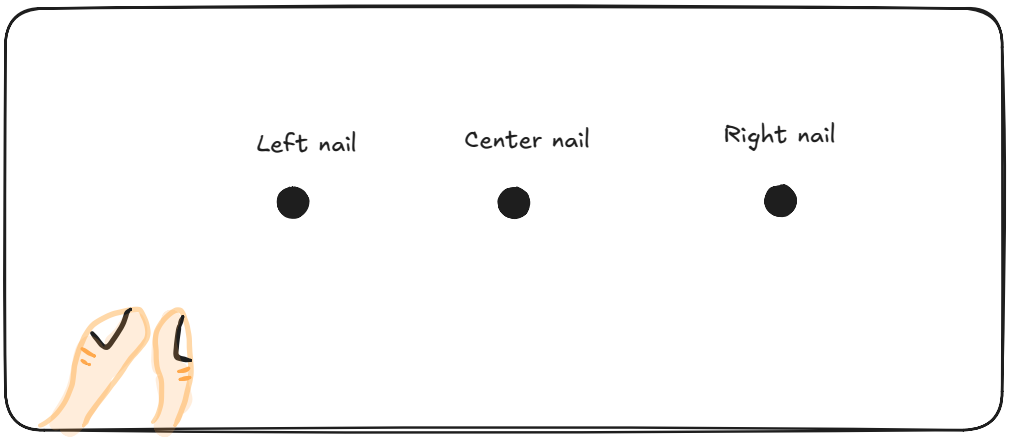
(b)



Problem 2

Suppose three wall nails are next to each other on the wall, and

- a clockwise loop around the left nail is represented by x,
- a clockwise loop around the center nail is represented by y,
- a clockwise loop around the right nail is represented by z.



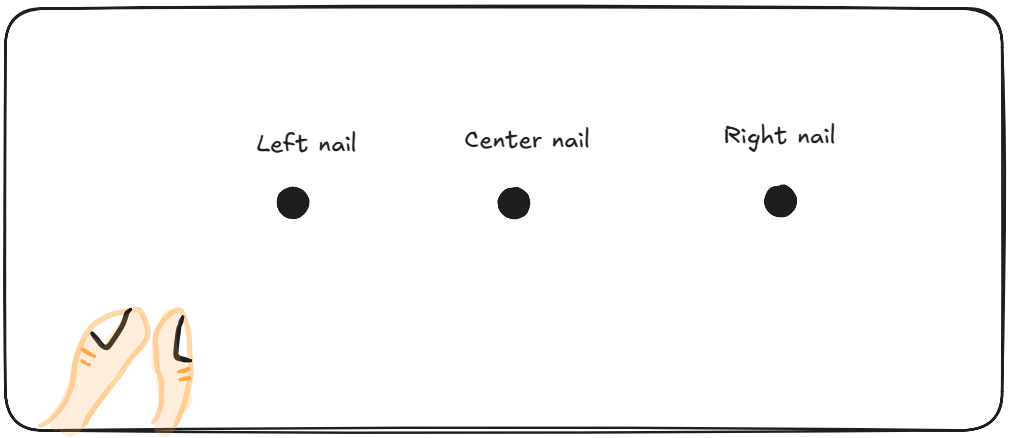
(a) Find a way to hang a picture on the three nails so that removing any one nail makes the picture not fall, but removing any two nails makes the picture fall.
(b) Draw the picture of the resulting loop, as well as write down the algebraic expression for resulting loop in terms of x, y, z.

(You can check your answer by physical demonstration with a piece of string.)

Problem 3

Suppose three wall nails are next to each other on the wall, and

- a clockwise loop around the left nail is represented by x,
- a clockwise loop around the center nail is represented by y,
- a clockwise loop around the right nail is represented by z.



Consider the algebraic expression $xyx^{-1}y^{-1}zyxy^{-1}x^{-1}z^{-1}$.

(a) Draw the picture of the loop that represents the expression $xyx^{-1}y^{-1}zyxy^{-1}x^{-1}z^{-1}$.
(b) If we remove the left nail, is the loop in (a) still attached to the center nail and right nail?
(c) If we remove the center nail, is the loop in (a) still attached to the left nail and right nail?
(d) If we remove the right nail, is the loop in (a) still attached to the left nail and center nail?

Remark: If we define the commutator $[x, y] = xyx^{-1}y^{-1}$, then the expression $xyx^{-1}y^{-1}zyxy^{-1}x^{-1}z^{-1}$ can be written simply as $[[x, y], z]$. The commutator has the nice property of satisfying the conditions $[x, 1] = 1$ and $[1, y] = 1$.