

7 - Counting (continued)

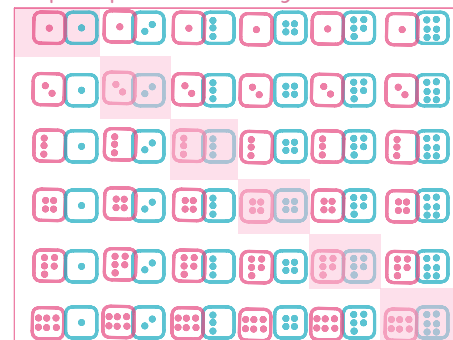
Example 1. Roll two dice. How many outcomes have the 2nd dice roll greater than the 1st dice roll?

Answer 1: To get the upper triangle of the sample space (right), subtracting the diagonal (6 outcomes) and divide by 2.

This gives $\frac{6^2 - 6}{2} = 15$ outcomes.

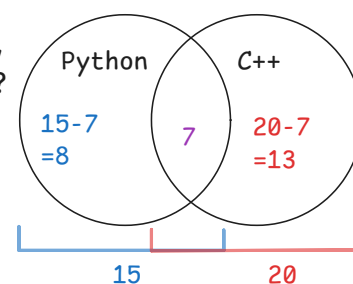
Answer 2: Sample two numbers without replacement from $\{1, 2, \dots, 6\}$ ("6 choose 2" = 15 ways), like $\{3, 1\}$, $\{4, 2\}$. Now note that any unordered set admits a natural order: increasing order, so $\{3, 1\} \rightarrow (1 < 3)$.

Sample space of rolling two dice:



Example 2. Of 30 students, 20 dislike Python, 15 dislike C++, and 7 dislike Python and C++. How many dislike Python or C++?

Answer: By PIE (below), the answer is $20 + 15 - 7$ (since $20 + 7$ counts the intersection of 7 students twice).



Principle of inclusion-exclusion. (PIE) $|A \cup B| = |A| + |B| - |A \cap B|$

Two events A and B are **mutually exclusive** or **disjoint** if $A \cap B = \emptyset$, in which case

$$|A \cap B| = 0 \quad \text{and} \quad |A \cup B| = |A| + |B|$$

Example 3. How many cards have black face or red number in a standard card deck?

Black face cards are $\spadesuit J, \spadesuit Q, \spadesuit K, \clubsuit J, \clubsuit Q, \clubsuit K$ totaling 6 cards

Red number cards are $A\heartsuit, 2\heartsuit, 3\heartsuit, \dots, 10\heartsuit, A\diamondsuit, 2\diamondsuit, 3\diamondsuit, \dots, 10\diamondsuit$ totaling 20 cards

Red & black cards are mutually exclusive. Overall, there are $20 + 6$ such cards.

Example 4. How many numbers from 1 to 100 are not divisible by 5?

Answer: "Count by complement" since the complement {numbers divisible by 5} is easier to count:

{Multiples of 5's in 1, ..., 100} \cup {Non-multiples of 5's in 1, ..., 100} = {1, 2, ..., 100}

Size 20 disjoint So size 100-20 Size 100

8 - Probability

Probability of an event is how often it occurs in the long run:

$$P(\text{Event}) = \lim_{n \rightarrow \infty} \frac{\text{Occurrences of the event in } n \text{ repeated experiments}}{n}$$

Law of Large Numbers. If each outcome is equally likely to occur, then

$$P(\text{Event}) = \frac{\# \text{ of outcomes in event}}{\# \text{ of outcomes in sample space}}$$

Example 5. Find the probability each occurs (in the long run):

(a) Get heads when flipping a fair coin.

$1/2$ since event = $\{H\}$ and sample space = $\{H, T\}$

(b) Roll two fair dice and getting 2nd roll $>$ 1st roll.

$15/36$ since this event has size 15 out of sample space of size $6 \times 6 = 36$.

(c) In Ex 2: pick a student (via SRS) who dislikes Python and C++.

(d) From a standard deck, draw a card that has black face or red number. $\frac{26}{52} = \frac{1}{2}$

(e) In a poker hand of 5 cards, find the probability of holding 2 aces and 3 jacks.

Answer: Sample space = {Having 5 cards from 52 cards} ("52 choose 5" outcomes)

Event = {Have 2 aces and 3 jacks}

= {Have 2 aces from 4} × {Have 3 J from 4} (6 × 4 = 24 outcomes)

Answer: $\frac{24}{\binom{52}{5}} \approx 0.9 \times 10^{-5}$

Exercises. Find the probability of the following events:

1. Getting a total of 7 or 11 when a pair of fair dice is tossed.
2. Getting a number that ends in 1 from 1, 2, 3, ..., 100 by using a random number generator.
3. Getting a number whose English words has the letter "x" from 1, 2, 3, ..., 100 by using a random number generator.
4. Getting (at least) two sixes in a row when rolling 3 fair dice?
(JP Morgan interview, easy)
5. Observing at least three heads in 10 flips of a fair coin.

Axioms of Probability. Given sample space S , given events E and F :

Axiom 1:

Axiom 2:

Axiom 3: If E and F are mutually exclusive, then

and

Principle of inclusion-exclusion: (Consequence)

Probability of complement: (Consequence) If E and E' are complements, then:

Exercise 1, revisited:

Exercise 5, revisited: