Lesson 18 - Expected value

Question. When is a game of pure chance "fair"?

Example 1. (Coin flip)
Flip a fair coin. If heads, you gain $1. If tails, you lose $2. Should you play this game?

Answer:
No, 50% of the time you win $1 and 50% of the time you lose $2:

$$\left(\frac{1}{2}\right) \times \$1 + \left(\frac{1}{2}\right) \times (-\$2) = -\$0.50.$$

Each play, you expect to lose 50 cents on average so we should not play this game.

The **expected value** of a game is this weighted average: the sum of the probability of each outcome times the monetary gain/loss of that outcome.

Example 2. (Rolling two dice)
Two fair die are rolled. If both dice add to 7 or higher, you lose $1. Otherwise, you gain $1.
(a) What is the expected value of this game for you?
(b) Should you play this game?

Answer:
(a) We visualize how much money you get from each dice rolling outcome:

Die 1 \ Die 2	1	2	3	4	5	6
1	+1	+1	+1	+1	+1	-1
2	+1	+1	+1	+1	-1	-1
3	+1	+1	+1	-1	-1	-1
4	+1	+1	-1	-1	-1	-1
5	+1	-1	-1	-1	-1	-1
6	-1	-1	-1	-1	-1	-1

There are 15 out of the 36 outcomes which result in +$1.
There are 21 out of the 36 outcomes which result in -$1.

So the overall expected value of this game is

$$\left(\frac{15}{36}\right) \times \$1 + \left(\frac{21}{36}\right) \times (-\$1) = -\frac{6}{36} \approx -\$0.17.$$

So you expect an average loss of 17 cents with each play of this game.

(b) No we should not play this game.

Games in the long run

What happens in the long run?

Theorems:

- (Law of large numbers) If you play a game of pure chance enough times (many, many, many times), then your average amount of gain/loss is roughly the expected value of that game.
- (Gambler's ruin) A gambler with finite wealth who keeps playing the same game of pure chance indefinitely will eventually zero out (i.e. go bankrupt), *even if* the expected value of the game is positive.

How to counter these theorems?
Play very few times and hope you get lucky.

How does a casino counter your strategy?
Have jackpots to incentivize you to play for a long time.

Second counter the law of large numbers:
If you lose a game of pure chance, double down on your losses in your next bet to recoup all losses and also gain a profit. This is the **Martingale betting strategy**.

Does martingale work?

- No: you have finite $, you will much more likely zero out against the much more resourceful house (i.e. casino).
- Yes: if you are the richest person on Earth: https://www.reddit.com/r/poker/comments/1ijdu4m/is_this_one_of_the_greatest_poker_players_of_all/
- Casinos and professional gambling tournaments also prevent martingales from becoming an effective strategy by imposing **table limits**: min and max bets. The house can also stop the game at any time.
- To use martingale effectively, play a game of pure chance only against the poor and downtrodden with little $ left and where you can dictate when to stop playing to ensure they zero out first. (Don't do this.)