Lesson 20 Notes on Implication

Although "and" and "or" are normally different, "*Children and seniors pay half price.*" and "*Children or seniors pay half price.*" sound the same: in fact, they are the same logically speaking via the notion of implication.

We write $p \Rightarrow q$ to mean "p implies q" or "if p then q", so we model

> "If you're born in the US, then you're a US citizen"

as $p \Rightarrow q$ where $p =$"*You are born in the US*" and $q =$"*You are a US citizen*".

The truth values for $p \Rightarrow q$ are given by:

p	q	$p \Rightarrow q$	**Sample sentence**
T	T	T	If I am a truth-teller, then I tell truths.
T	F	F	If I am a truth-teller, then I tell lies.
F	T	T	If I am a liar, then I tell truths.
F	F	T	If I am a liar, then I tell lies.

The reason we label "*If I am a liar, then I tell truths.*" a true statement is that we cannot trust liars: they are capable of both telling truths and lies.

We now use this truth table to solve problems where we know partial information about a situation:

Example 1. Suppose that "If Al scores $\geq 90\%$, then Al gets an A." is a true statement. What other information can we deduce from each case below?

(a) Al scores $\geq 90\%$.

(b) Al gets an A.

(c) Al does not get an A.

Answer:

(a) We easily deduce Al gets and A.

(b) We are given that $p \Rightarrow q$ is true, where p="If Al scores $\geq 90\%$" and q="Al gets an A". We are also given that q is true. Therefore we can eliminate rows 2 and 4 below (since q is false there) and eliminate row 2 below (since $p \Rightarrow q$ is false there).

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Therefore we are left with rows 1 and 3, but in these cases we only have p can be true or it can be false, so we get no information about p.

(c) Again we analyze the truth table, given now that $p \Rightarrow q$ is true, and that q is false. So now we can eliminate rows 1, 2, 4 from the truth table. This leaves us with row 3, i.e. p is false. In other words, we deduce that Al does not score $\geq 90\%$.

Another way we can understand (c) above is through the contrapositive of a statement.

Definition. The **contrapositive** of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$.

Example 2. The contrapositive of "*If I eat beans, then I fart.*" is "*If I did not fart, then I did not eat beans.*"

Intuitively, the contrapositive of a statement is logically equivalent to the statement itself: so a true statement will also have a true contrapositive, and a false statement will also have a false contrapositive. We can also prove this, that $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$, by showing the truth table of $\neg q \Rightarrow \neg p$ has the same final output column as that of $p \Rightarrow q$:

p	q	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Example 1 (c) revisited. Because we assume "Al does not get an A." is true, we easily deduce that Al does not score $\geq 90\%$ using the contrapositive form of the original statement:

Original: "If Al scores $\geq 90\%$, then Al gets an A."

Contrapositive: "If Al does not get an A, then Al does not score $\geq 90\%$."

We finally analyze why "*Children and seniors pay half price.*" and "*Children or seniors pay half price.*" are the same. Let:

- C = "You are a child."
- S = "You are a senior."
- H = "You pay half price."

Then "*Children and seniors pay half price.*" is modelled as

"(If you are a child, then you pay half price) and (If you are a senior, then you pay half price)"

and "*Children or seniors pay half price.*" is modelled as

"If you are a child or if you are a senior, then you pay half price"

In symbols, we should check using a truth table that

$$(C \Rightarrow H) \wedge (S \Rightarrow H) \equiv (C \vee S \Rightarrow H)$$