Lesson 19 Notes on Logic

We wish to formalize natural language (English) to symbols so that we can analyze them better. But not all sentences can be analyzed:

Definition. A **proposition** or **statement** is a declarative sentence that is always true or always false.

Not propositions:

- "I can haz cheezeburger?" — Questions don't have intrinsic truth values.
- "Fly, you fools!" — Demand.
- "Pizza tastes good." — Opinion.
- "Pizza" — Not even a sentence.

Examples of propositions:

- "$3 + x = 5$ if $x = 2$" — A true proposition.
- "$3 + x = 5$ if $x = 0$" — A false proposition since $3 + 0 \neq 5$.
- "There are 7 days in a week." — True proposition.

We can combine propositions p and q by logical connectives "and", "or", and "not":

- $\neg p$ is the **negation** of p. It is also read as "not p" or "it is not true that p". It is true precisely when p is false.
- $p \wedge q$ is the **conjunction** of p and q. It is also read as "p and q" or "p but q". It is true precisely when both p and q are simultaneously true.
- $p \vee q$ is the **disjunction** of p and q. It is also read as "p or q". It is true precisely when at least one of p and q is true.

Example 1.
Let $p =$"It is hot" and $q =$"It is sunny". Then:

- $\neg p =$"It is not hot."
- $p \wedge q =$"It is hot and sunny."
- $\neg p \wedge \neg q =$"It is neither hot nor sunny."
- $\neg p \wedge q =$"It is sunny but not hot."

Example 2. (Olivia Rodrigo) "Good for you, you look happy and healthy, not me."
Here it is hard to translate "good for you". But we can translate the remaining sentence as:

"You look happy" $\wedge$ "You look healthy" $\wedge$ $\neg$ "I look happy" $\wedge$ $\neg$ "I look healthy"

We can also analyze whether propositions are true or false. A **truth assignment** is an assignment of true (T) or false (F) to p, q, r, s, ... A **truth table** lists all such assignments and the truth value of the proposition. Two propositions are **logically equivalent**, written $p \equiv q$, if their truth values are the same under every truth assignment.

For instance, here is the truth table for $\neg$, which flips true to false, and false to true:

p	$\neg p$
T	F
F	T

And here are the truth tables of the other logical connectives:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Theorem. (De Morgan's law)

- Negating an "and" statement negates each part of the statement and flips "and" to an "or", or in symbols

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

- Negating an "or" statement negates each part of the statement and flips "or" to an "or", or in symbols

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Proof of the first case.

We write out the truth table for $\neg(p \wedge q)$:

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

We write out the truth table for $\neg p \wedge \neg q$:

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Their final columns match, so $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

Example 4. What is another way to say "It is neither hot nor sunny"?

Answer. "It is neither hot nor sunny" is the shortened form of "It is not (either hot or sunny)", which by de Morgan's law simplifies to "It is not hot and it is not sunny" or "It is not hot and not sunny".

This is the first application of how mathematics (logic) explains how English works.

Example 5. Find the negation of "It is hot or sunny."

Answer. "It is not hot and it is not sunny." — De Morgan's law requires negating each part of the sentence, as well as interchanging "and" with "or".