

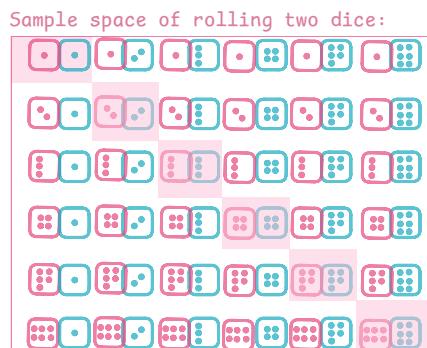
## 7 - Counting (continued)

**Example 1.** Roll two dice. How many outcomes have the 2nd dice roll greater than the 1st dice roll?

Answer 1: To get the upper triangle of the sample space (right), subtracting the diagonal (6 outcomes) and divide by 2.

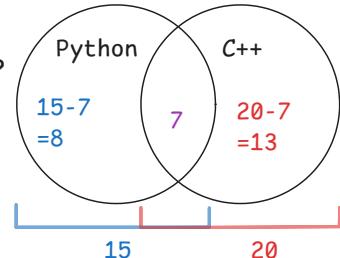
This gives  $\frac{6^2 - 6}{2} = 15$  outcomes.

Answer 2: Sample two numbers without replacement from  $\{1, 2, \dots, 6\}$  ("6 choose 2" = 15 ways), like  $\{3, 1\}$ ,  $\{4, 2\}$ . Now note that any unordered set admits a natural order: increasing order, so  $\{3, 1\} \rightarrow (1 < 3)$ .



**Example 2.** Of 30 students, 20 dislike Python, 15 dislike C++, and 7 dislike Python and C++. How many dislike Python or C++?

Answer: By PIE (below), the answer is  $20+15-7$  (since  $20+7$  counts the intersection of 7 students twice).



**Principle of inclusion-exclusion. (PIE)**  $|A \cup B| = |A| + |B| - |A \cap B|$

Two events A and B are **mutually exclusive** or **disjoint** if  $A \cap B = \emptyset$ , in which case  $|A \cap B| = 0$  and  $|A \cup B| = |A| + |B|$ .

**Example 3.** How many cards have black face or red number in a standard card deck?

Black face cards are ♠J, ♠Q, ♠K, ♣J, ♣Q, ♣K totaling 6 cards

Red number cards are A♥, 2♥, 3♥, . . . , 10♥, A♦, 2♦, 3♦, . . . , 10♦ totaling 20 cards

Red & black cards are mutually exclusive. Overall, there are 20+6 such cards.

**Example 4.** How many numbers from 1 to 100 are not divisible by 5?

Answer: "Count by complement" since the complement {numbers divisible by 5} is easier to count:

$$\{ \text{Multiples of 5's in } 1, \dots, 100 \} \cup \{ \text{Non-multiples of 5's in } 1, \dots, 100 \} = \{1, 2, \dots, 100\}$$

Size 20

↑  
disjoint

So size 100-20

Size 100

## 8 - Probability

**Probability** of an event is how often it occurs in the long run:

$$P(\text{Event}) = \lim_{n \rightarrow \infty} \frac{\text{Occurrences of the event in } n \text{ repeated experiments}}{n}$$

**Law of Large Numbers.** If each outcome is equally likely to occur, then

$$P(\text{Event}) = \frac{\# \text{ of outcomes in event}}{\# \text{ of outcomes in sample space}}$$

**Example 5.** Find the probability each occurs (in the long run):

(a) Get heads when flipping a fair coin.

1/2 since event = {H} and sample space = {H, T}

(b) Roll two fair dice and getting 2nd roll > 1st roll.

15/36 since this event has size 15 out of sample space of size 6x6 = 36.

(c) In Ex 2: pick a student (via SRS) who dislikes Python and C++.

(d) From a standard deck, draw a card that has black face or red number.  $\frac{26}{52} = \frac{1}{2}$

(e) In a poker hand of 5 cards, find the probability of holding 2 aces and 3 jacks.

Answer: Sample space = {Having 5 cards from 52 cards} ("52 choose 5" outcomes)

Event = {Have 2 aces and 3 jacks}

= {Have 2 aces from 4} x {Have 3 J from 4} ( $6 \times 4 = 24$  outcomes)

Answer:  $\frac{24}{\binom{5}{2}} \approx 0.9 \times 10^{-5}$

**Exercises.** Find the probability of the following events:

1. Getting a total of 7 or 11 when a pair of fair dice is tossed.

2. Getting a number that ends in 1 from 1, 2, 3, ..., 100 by using a random number generator.

3. Getting a number whose English words has the letter "x" from 1, 2, 3, ..., 100 by using a random number generator.

4. Getting (at least) two sixes in a row when rolling 3 fair dice?  
(JP Morgan interview, easy)

5. Observing at least three heads in 10 flips of a fair coin.

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