CS7290 Causal Modeling in Machine Learning: Homework 3

Submission guidelines

Use a Jupyter notebook and/or R Markdown file to combine code and text answers. Compile your solution to a static PDF document(s). Submit both the compiled PDF and source files. The TA's will recompile your solutions, and a failing grade will be assigned if the document fails to recompile due to bugs in the code. If you use Google Collab, send the link as well as downloaded PDF and source files.

Background

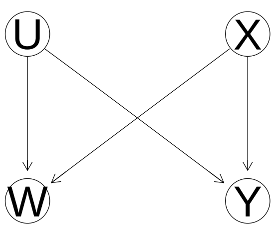
This assignment is going to cover several topics, including some that haven't been taught at the time this was assigned. We will cover those topics in subsequent classes.

- Recognizing valid adjustment sets
- Covariate adjustment with parent and back-door criterion
- Front-door criterion
- Propensity matching and inverse probability weighting
- Intro to structural causal models

Question 1: Valid adjustment sets

1.1

The following DAG represents a causal model of user behavior in an app.



U represents the user specific preferences. X represents the introduction of a feature designed to make users make certain in-app purchases, Y was whether or not the user made the purchase, W represents app usage after the feature is introduced.

1.1.a

You are interested in estimating the causal effect of X on Y. What is the valid adjustment set? Valid adjustment set is the set of variables that if you adjust, you will get the unbiased results. (3 points)

There is nothing to adjust as there's no backdoor path. Valid adjustment set is {}

1.1.b

What would happen if you adjusted for W? Be specific. (2 points)

If W is conditioned, then the path U, W, X becomes active and hence there's a backdoor path that affects Y.

1.1.c

Suppose you want to assess the effect of X on Y for users who have a high amount of app usage. Fill in the blanks on the right-hand-side for the adjustment formula of interest:

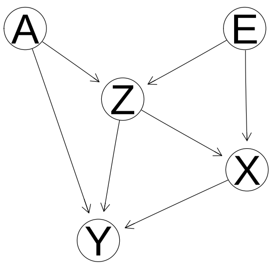
$$P(Y = y|do(X = x), W = high) = \sum_{?} P(Y = y|?)P(?|?) \tag{1}$$

(4 points)

$$P(Y = y|do(X = x), W = high) = \sum_{u \forall U} P(Y = y|X = x, W = high, U = u)P(U = u|W = High) \tag{2}$$

1.2

Consider the following DAG.



You are interest in estimating the causal effect of X on Y.

1.2.a

Is the set containing only Z a valid adjustment set? Why or why not? (2 points) No, If we adjust for z , then the triplet A, Z and E (path) becomes active. Hence, we need to condtion on A or E {A, Z} or {E, Z}

1.2.b

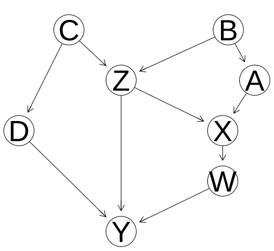
List all of the adjustment sets that blocks all the back doors(there are three) and write the adjustment formula for each adjustment set. (3 points)

{A, Z} {E, Z} {A, E, Z}

1.2.c

Suppose that E and A are both observable, but observing E costs \$10 per data point and observing A costs \$5 per data point. Which conditioning set do you go with? (1 point)

{A, Z} ## 1.3 Consider the following DAG:



1.3.a

List all of the sets of variables that satisfy the backdoor criterion to determine the causal effect of X on Y. (3 points)

{ A, Z } { B, Z } { A, B, Z } { Z, C } { A, Z, C } { B, Z, C } { A, B, Z, C } { Z, D } { A, Z, D } { B, Z, D } { A, B, Z, D } { Z, C, D } { A, Z, C, D } { B, Z, C, D } { A, B, Z, C, D }

1.3.b

List all of the minimal sets of variables that satisfy the backdoor criterion to determine the causal effect of X on Y (i.e., any set of variables such that, if you removed any one of the variables from the set, it would no longer meet the criterion). (3 points) { D, Z } { C, Z } { B, Z } { A, Z }

1.3.c

List all the minimal sets of variables that need to be measured in order to identify the effect of D on Y. (3 points)

{ W, Z } { X, Z } { A, Z } { B, Z } { C }

1.3.d

Now suppose we want to know the causal effect of intervening on 2 variables. List all the minimal sets of variables that need to be measured in order to identify the effect of set {D, W} on Y, i.e., $P(Y = y|do(D = d), do(W = w))$. (3 points)

{ Z } { C, X }

In [1]:
```
import pyro
import pyro.distributions as dist
from pyro.infer import Importance, EmpiricalMarginal
import matplotlib.pyplot as plt
import torch
import numpy as np
import pandas as pd
import random
```

In [2]:
```
# Giving the alias and cpt tables for the model
X_alias = ["promotion0", "promotion1"]
Y_alias = ["not-renewed", "renewed"]
Z_alias = ["unhappy", "happy"]


Z_prob = torch.tensor([0.5092,0.4908])
X_prob = torch.tensor([[0.247, 0.753], [0.763, 0.237]])
Y_prob = torch.tensor([[[0.068, 0.932], [0.267, 0.733]],
                      [[0.131, 0.869], [0.313, 0.687]]])
```

2.1.a

Build the model with Pyro using the values in the table. Use `pyro.condition` to calculate the causal effect by adjusting for happiness. (5 points)

In [3]:
```
def model():
    Z = pyro.sample("Z", dist.Categorical(probs=Z_prob))
    X = pyro.sample("X", dist.Categorical(probs=X_prob[Z]))
    Y = pyro.sample("Y", dist.Categorical(probs=Y_prob[X][Z]))
    return{'X': X, 'Y': Y, 'Z': Z}
```

You are interested in the average causal effect $P(Y=1|\text{do}(X=0)) - P(Y=1|\text{do}(X=1))$

In [4]:
```
def adjustmentFormula(x_cond, z_probs):
    Z_conditons=[{'Z':torch.tensor(0)}, {'Z': torch.tensor(1)}]
    effect = 0.0
    for idx in range(len(z_probs)):
        data = {}
        data.update(x_cond)
        data.update(Z_conditons[idx])

        conditioned_model = pyro.condition(model, data= data)
        T_samples = [conditioned_model()['Y'] for _ in range(1000)]
        T_unique, T_counts = np.unique(T_samples, return_counts=True)
        Y_prob = T_counts[1]/ 1000
        effect += (Y_prob * z_probs[idx])
    return effect
```

In [5]:
```
effect_Y1_X0 = adjustmentFormula({"X": torch.tensor(0)}, [0.5092, 0.4
908])
```

In [6]:
```
effect_Y1_X1 = adjustmentFormula({"X": torch.tensor(1)}, [0.5092, 0.4
908])
```

In [7]:
```
effect_Y1_X0 - effect_Y1_X1
```

Out[7]:
```
0.05323920000000004
```

2.1.b

Use `pyro.do` to calculate the causal effect by adjusting for happiness. (5 points)

In [8]:
```
def interventionUsingDo(x_cond):
    intervention_model = pyro.do(model, data = x_cond)
    Y_posterior_intervened = pyro.infer.Importance(intervention_model
, num_samples=10000).run()
    Y_marginal_intervened = EmpiricalMarginal(Y_posterior_intervened,
"Y")
    Y_samples_intervened = [Y_marginal_intervened().item() for _ in r
ange(10000)]
    Y_unique_intervened, Y_counts_intervened = np.unique(Y_samples_in
tervened, return_counts=True)
    return Y_counts_intervened[1]/ len(Y_samples_intervened)
```

In [9]:
```
effect_Y1_X0_do = interventionUsingDo({'X': torch.tensor(0)})
```

In [10]:
```
effect_Y1_X1_do = interventionUsingDo({'X': torch.tensor(1)})
```

In [11]:
```
effect_Y1_X0_do - effect_Y1_X1_do
```

Out[11]:
```
0.05959999999999986
```

Both the causal effect obtained in 2.1.a and 2.a.b are similar.

2.2

You are a data scientist investigating the effects of social media use on purchasing a product. You assume the dag shown below. User info here is unobserved. One of the team members argues that social media usage does not drive purchase based on Table 1. Only 15% social media user made the purchase, while 90.25% non social media users made the purchase. Moreover, within each group, no-adblock and adblock, social media users show a much lower rate of purchase than non social media users. However, another team member argues that social media usage increases purchases. When we look at each group, social media user and non social media user as show in Table 2 (Table 1 and Table 2 both represent the same dataset), advertisement increases purchases in both groups. Among social media users, purchases increases from 10% to 15% for people who have seen advertisement. Among non social media users, purchases increases from 90% to 95% for people who have seen advertisement. Which view is right?

In [12]:
```
# Giving the alias and cpt tables for the model
X_alias = ["no-social", "social"]
Y_alias = ["no-ad", "ad"]
Z_alias = ["no-purchase", "purchase"]


X2_prob = torch.tensor([0.5, 0.5])
Z2_prob = torch.tensor([[0.95, 0.05], [0.05, 0.95]])
Y2_prob = torch.tensor([[0.14, 0.86], [0.81, 0.19]])


def socialMediaModel():
    X_2 = pyro.sample("X2", dist.Categorical(probs=X2_prob))
    Z_2 = pyro.sample("Z2", dist.Categorical(probs=Z2_prob[X_2]))
    Y_2 = pyro.sample("Y2", dist.Categorical(probs=Y2_prob[Z_2]))
    return{'X2': X_2, 'Y2': Y_2, 'Z2': Z_2}
```

2.2.a

User info is unobserved. Use `pyro.condition` to calculate the causal effect of social media on product purchase using front-door adjustment (Section 3.4 in [Front Door Criterion (http://bayes.cs.ucla.edu/PRIMER/primer-ch3.pdf)](http://bayes.cs.ucla.edu/PRIMER/primer-ch3.pdf)).(5 points)

In [13]:
```
def frontDoorAdjustment(x_conds, x_prob, x_idx):
    Z_conditions = [{"Z2": torch.tensor(0)}, {"Z2": torch.tensor(1)}]
    cumProb = 0.0
    for idx in range(len(Z_conditions)):
        first_condition = {}
        first_condition.update(x_conds[x_idx])
        z_condition_model = pyro.condition(fn=socialMediaModel, data
= first_condition)
        Z_samples = [z_condition_model()['Z2'] for _ in range(1000)]
        Z_unique, Z_counts = np.unique(Z_samples, return_counts=True)
        Z_prob = Z_counts[idx]/ 1000

        # Finding the sum part for each z - iteration
        innerProb = 0.0
        for idx1 in range(len(x_conds)):
            second_condition = {}
            second_condition.update(Z_conditions[idx])
            second_condition.update(x_conds[idx1])
            y_condition_model = pyro.condition(socialMediaModel, data
=second_condition)
            Y_samples = [y_condition_model()['Y2'] for _ in range(100
0)]
            Y_unique, Y_counts = np.unique(Y_samples, return_counts=T
rue)
            Y_prob = Y_counts[1]/ 1000
            innerProb += (Y_prob * x_prob[idx1])
        cumProb += (Z_prob * innerProb)
    return cumProb
```

In [14]:
```
X_conds = [{"X2": torch.tensor(0)}, {"X2": torch.tensor(1)}]
X_prob_1 = [0.5, 0.5]
```

In [15]:
```
frontDoorAdjustment(X_conds, X_prob_1, 0) -  frontDoorAdjustment(X_co
nds, X_prob_1, 1)
```

Out[15]:
```
0.6186054999999999
```

2.2.b

Verify your result using do-calculus with `pyro.do` .$(P(Y=1|\text{do}(X=0)) - P(Y=1|\text{do}(X=1)))$ (5 points)

In [16]:
```
def interventionUsingDoSocial(x_cond):
    intervention_model = pyro.do(socialMediaModel, data = x_cond)
    Y_posterior_intervened = pyro.infer.Importance(intervention_model
, num_samples=10000).run()
    Y_marginal_intervened = EmpiricalMarginal(Y_posterior_intervened,
"Y2")
    Y_samples_intervened = [Y_marginal_intervened().item() for _ in r
ange(10000)]
    Y_unique_intervened, Y_counts_intervened = np.unique(Y_samples_in
tervened, return_counts=True)
    return Y_counts_intervened[1]/ len(Y_samples_intervened)
```

In [17]:
```
interventionUsingDoSocial({"X2": torch.tensor(0)}) - interventionUsin
gDoSocial({"X2": torch.tensor(1)})
```

Out[17]:
```
0.6065999999999999
```

Both the causal effect obtained in 2.2.a and 2.2.b are similar

3.1 Defining the propensity function

In [18]:
```
def propensity(x, z):
    return X_prob[z][x]
```

In [19]:
```
propensity(0, 1)
```

Out[19]:
```
tensor(0.7630)
```

3.2 Displaying the 10 samples generated for the model given in 2.1

In [20]:
```
xs = []
ys = []
zs = []
ps = []
trace_handler = pyro.poutine.trace(model)
for i in range(1000):
    trace = trace_handler.get_trace()
    x = trace.nodes['X']['value']
    y = trace.nodes['Y']['value']
    z = trace.nodes['Z']['value']
    log_prob = trace.log_prob_sum()
    p = np.exp(log_prob)
    xs.append(int(x))
    ys.append(int(y))
    zs.append(int(z))
    ps.append(p)
data = pd.DataFrame({"X": xs, "Y": ys, "Z": zs, "P": ps})
```

In [21]:
```
data.head(10)
```

Out[21]:

	X	Y	Z	P
0	1	1	0	tensor(0.3332)
1	1	1	0	tensor(0.3332)
2	0	1	1	tensor(0.2745)
3	1	0	0	tensor(0.0502)
4	1	1	0	tensor(0.3332)
5	0	0	1	tensor(0.1000)
6	1	1	0	tensor(0.3332)
7	0	1	1	tensor(0.2745)
8	1	1	1	tensor(0.0799)
9	0	1	1	tensor(0.2745)

3.3 Computing the inverse probability weighting using propensity score

In [22]:
```
# Using the data generated above, i re-weight the joint probability b
y propensity.

data["inverse_weighted_prob"] = data.apply(lambda x: x["P"]/ (propens
ity(x["X"], x["Z"])), axis=1)
```

In [23]:
```python
data.drop_duplicates(subset=['X', 'Y', 'Z'], inplace=True)
data.reset_index(drop=True)
```

Out[23]:

	X	Y	Z	P	inverse_weighted_prob
0	1	1	0	tensor(0.3332)	tensor(0.4425)
1	0	1	1	tensor(0.2745)	tensor(0.3598)
2	1	0	0	tensor(0.0502)	tensor(0.0667)
3	0	0	1	tensor(0.1000)	tensor(0.1310)
4	1	1	1	tensor(0.0799)	tensor(0.3372)
5	0	1	0	tensor(0.1172)	tensor(0.4746)
6	1	0	1	tensor(0.0364)	tensor(0.1536)
7	0	0	0	tensor(0.0086)	tensor(0.0346)

In [24]:
```python
combinations = data[['X', 'Y', 'Z']].apply(lambda x: x.to_dict(), axi
s=1).values
```

In [25]:
```python
inv_weighted_prob = list(data["inverse_weighted_prob"])
```

3.4 Generating samples from the inverse weighting probability distribution

In [26]:
```python
weighted_samples = pd.DataFrame.from_records(random.choices(combinati
ons, inv_weighted_prob,k=1000))
```

In [27]:
```python
weighted_samples.head()
```

Out[27]:

	X	Y	Z
0	1	1	1
1	0	1	0
2	0	1	1
3	0	1	0
4	1	0	1

3.5 Checking whether the causal estimate is same as 2.1

In [28]:
```python
def filterIt(df, var, value):
    return df[df[var]==value]
```

In [29]:
```python
def computeProb(df, var, value):
    return sum(df[var]==value)/(df.shape[0])
```

In [30]:
```
def causalEffect(df, xval):
    cumProb = 0.0
    for z in range(df["Z"].nunique()):
        ydf = filterIt(filterIt(df, "Z", z), "X", xval)
        yprob = computeProb(ydf, "Y", 1)
        zprob = computeProb(df, "Z", z)
        cumProb+= (yprob * zprob)
    return cumProb
```

In [31]:
```
causalEffect(weighted_samples, 0) - causalEffect(weighted_samples, 1)
```

Out[31]:
```
0.06200804901366552
```

The causal estimate is similar to what is obtained in 2.1

4.1 Defining the model and generating 10 samples

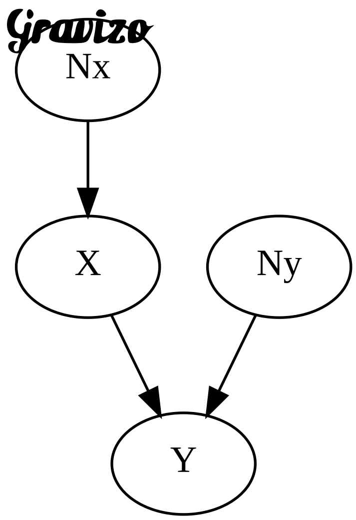
In [32]:
```
def scm1():
    X = pyro.sample('X', dist.Normal(0.0, 1.0))
    Y = X ** 2 + pyro.sample('Ny', dist.Normal(0.0, 1.0))
    Y = pyro.sample('Y', dist.Normal(Y, 0.001))
    return X, Y
```

In [33]:
```
trace_scm = pyro.poutine.trace(scm1)
for _ in range(10):
    trace = trace_scm.get_trace()
    x = trace.nodes['X']['value']
    y = trace.nodes['Y']['value']
    print(f' The value of x is {x} and the value of y is {y}')
```

```
 The value of x is -0.5521014332771301 and the value of y is 2.854936
5997314453
 The value of x is -0.08818693459033966 and the value of y is 0.83122
49183654785
 The value of x is 0.5461435317993164 and the value of y is 0.1130787
804722786
 The value of x is -0.18929176032543182 and the value of y is -0.9127
78913974762
 The value of x is 0.41813671588897705 and the value of y is -1.56906
48555755615
 The value of x is -0.2518658936023712 and the value of y is -1.13973
67715835571
 The value of x is 0.09925052523612976 and the value of y is 0.871267
3783302307
 The value of x is 1.3132648468017578 and the value of y is 0.6808860
898017883
 The value of x is 1.1282209157943726 and the value of y is 0.7220373
749732971
 The value of x is -1.107576847076416 and the value of y is 1.1398751
735687256
```

4.2

4.2.a The DAG can be represented as



4.2.b

1. The mean of the distribution P_Y^M is 0
2. The variance of the distribution P_Y^M is 17
3. Y = 4 * N(0,1) + N(0,1) --> The variance gets squared when there's a multiplicative factor. --> 16 + 1 --> 17

4.2.c

1. The mean of the distribution $P_Y^{M:do(X=2)}$ is 8
2. The variance of the distribution $P_Y^{M:do(X=2)}$ is 1
3. Y = 4 * 2 (setting the value of x =2) + 0 --> Mean.

4.2.d

1. The distribution $P_Y^{M:X=2}$ doesn't differ from $P_Y^{M:do(X=2)}$ distribution because there isn't any other source that affects X. So, conditioning it or intervening on it doesn't make any difference.

4.2.e is answered in the end , as the image is attached separately .

4.2.f

1. The mean of the distribution $P_X^{M:do(Y=2)}$ is 0
2. The variance of the distribution $P_X^{M:do(Y=2)}$ is 1
3. Since the sources of Y are removed. The values of X doesn't affect Y in any way. Hence the mean and variance of X remains same

4.2.g

In [34]:
```
def scm2():
    X = pyro.sample('X', dist.Normal(0.0, 1.0))
    Y = 4 * X + pyro.sample('Ny', dist.Normal(0.0, 1.0))
    Y = pyro.sample('Y', dist.Normal(Y, 0.001))
    return X, Y

trace_scm = pyro.poutine.trace(scm2)
for _ in range(10):
    trace = trace_scm.get_trace()
    x = trace.nodes['X']['value']
    y = trace.nodes['Y']['value']
    print(f' The value of x is {x} and the value of y is {y}')
```

```
 The value of x is 1.3833633661270142 and the value of y is 5.7381381
98852539
 The value of x is 0.07725183665752411 and the value of y is 0.390086
7700576782
 The value of x is -0.7231635451316833 and the value of y is -3.64859
79557037354
 The value of x is -1.0205142498016357 and the value of y is -3.05001
3542175293
 The value of x is 0.13883854448795319 and the value of y is 0.666816
9498443604
 The value of x is 0.6039068698883057 and the value of y is 1.5654387
474060059
 The value of x is 1.5974762439727783 and the value of y is 7.4561481
47583008
 The value of x is -0.814222514629364 and the value of y is -2.912876
844406128
 The value of x is 0.9170505404472351 and the value of y is 2.8552155
49468994
 The value of x is 0.5865408778190613 and the value of y is 2.7083709
239959717
```

4.2.h

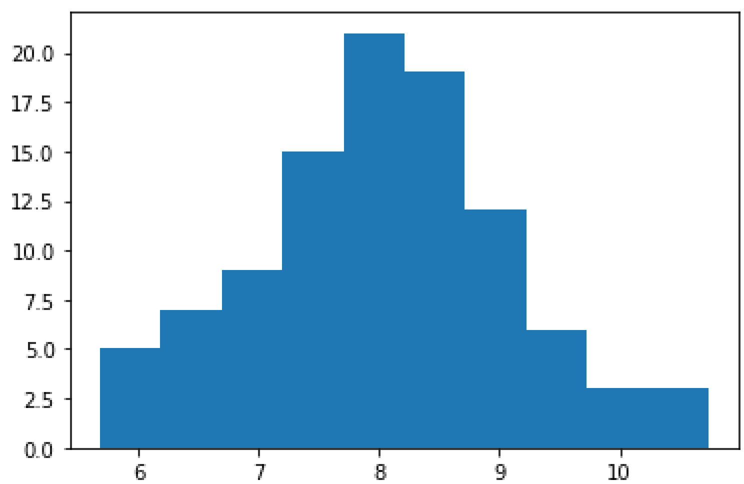
In [35]:
```
conditioned_model = pyro.poutine.do(scm2, data={"X": torch.tensor(2.0
)})
```

In [36]:
```
traced_model = pyro.poutine.trace(conditioned_model)
```

In [37]:
```
y_vals = []
for _ in range(100):
    trace = traced_model.get_trace()
    y = trace.nodes['Y']['value']
    y_vals.append(y)
```

In [38]:
```
plt.hist(y_vals)
```

Out[38]:
```
(array([ 5.,  7.,  9., 15., 21., 19., 12.,  6.,  3.,  3.]),
 array([ 5.6756163,  6.182397 ,  6.6891775,  7.1959577,  7.7027383,
         8.209518 ,  8.716299 ,  9.22308  ,  9.72986  , 10.236641 ,
        10.743422 ], dtype=float32),
 <a list of 10 Patch objects>)
```



In [39]:
```
np.mean(y_vals) #approximately around 8
```

Out[39]:
```
8.0432205
```

The mean is centered around 8, which is what we got for $P_Y^{M:do(X=2)}$ in 4.2.c and we stated in 4.2.d that the $P_Y^{M:(X=2)}$ distribution doesn't change much to $P_Y^{M:do(X=2)}$ distribution, which is what we see here

4.2.i

In [40]:
```
conditioned_model_y = pyro.poutine.condition(scm2, data={"Y": torch.t
ensor(2.0)})
```

In [50]:
```python
nuts_kernel = pyro.infer.NUTS(conditioned_model_y)
mcmc = pyro.infer.MCMC(nuts_kernel,
            num_samples=10)
mcmc.run()
```

```
Sample: 100%|██████████| 20/20 [00:05,  3.63it/s, step size=2.03e-04,
acc. prob=0.942]
```

In [51]:
```python
mcmc.get_samples()["X"]
```

Out[51]:
```
tensor([0.2253, 0.2256, 0.1948, 0.1952, 0.1953, 0.1953, 0.1951, 0.194
7, 0.2294,
        0.1509])
```

$$y = 4x$$

$$\begin{aligned}
Cov(x,y) &= E(xy) - E(x)E(y) \\
&= E(4x^2) - E(x)E(4x) \\
&= E(4x^2) - 4E(x^2) \\
&= 4[Var(x)] \\
&= 4 \times 1 \\
&= 4.
\end{aligned}$$

$$\rho = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{4}{1 \times \sqrt{17}} = 4/\sqrt{17}.$$

Bivariate Normal Conditional distribution takes the following form

$$X|Y = N\left(\mu_{\cancel{x}} + \frac{\sigma_x}{\sigma_y}(y-\mu_y)\rho,\ (1-\rho^2)\sigma_x^2\right)$$

$\mu_x = 0$, $\sigma_x = 1$

$\mu_y = 0$ $\sigma_y = \sqrt{17}$ → From 4.2.b

Setting $Y=2$

$$\mu_{x|y} = 0 + \frac{1}{\sqrt{17}} * \frac{4}{\sqrt{17}} * (2-0)$$

$$\mu_{x|y} = \frac{8}{17}.$$

$$\sigma^2_{x|y} = \left(1 - \left(\frac{4}{\sqrt{17}}\right)^2\right) * 1$$

$$= \left(1 - \frac{16}{17}\right)$$

$$= 1/17.$$