Oscillations, Wave and Optics

(SPRING 2025)

ASSIGNMENT-3

Topics: Longitudinal Standing Waves, Traveling Waves — Total : 40

Date: 3rd Mar, 2025 — **Due: 16th Mar, 2025** (EoD)

Part:A | Longitudinal Standing Waves [15]

(1) A simple model of an ionic crystal consists of a linear array of a great many equally-spaced atoms of alternating masses M and m, where $m < M$. The masses are connected by identical chemical bonds that are modeled as springs of spring constant K.

i. Show that the frequencies of the system's longitudinal modes of vibration either lie in the band 0 to $(2\,K/M)^{1/2}$ or in the band $(2\,K/m)^{1/2}$ to $[2\,K\,(1/M + 1/m)]^{1/2}$.

ii. Show that, in the long-wavelength limit, modes whose frequencies lie in the lower band are such that neighboring atomics move in the same direction, whereas modes whose frequencies lie in the upper band are such that neighboring atoms move in opposite directions. The lower band is known as the acoustic branch, whereas the upper band is known as the optical branch. **(10 + 5)**

Part:B | Traveling Waves [25]

(1) Demonstrate that for a transverse traveling wave propagating on a stretched string,

$$\langle \mathcal{I} \rangle = v\,\langle \mathcal{E} \rangle$$

where $\langle \mathcal{I} \rangle$ is the mean energy flux along the string due to the wave, $\langle \mathcal{E} \rangle$ is the mean wave energy per unit length, and v is the phase velocity of the wave. **3**

(2) A lossy transmission line has a resistance per unit length $\mathcal{R}$, in addition to an inductance per unit length $\mathcal{L}$, and a capacitance per unit length $\mathcal{C}$. The resistance can be considered to be in series with the inductance.

i. Demonstrate that the Telegrapher's equations generalize to,

$$\frac{\partial V}{\partial t} = -\frac{1}{\mathcal{C}}\frac{\partial I}{\partial x}$$
$$\frac{\partial I}{\partial t} = -\frac{\mathcal{R}}{\mathcal{L}}I - \frac{1}{\mathcal{L}}\frac{\partial V}{\partial x}$$

ii. Derive an energy conservation equation of the form

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial \mathcal{I}}{\partial x} = -\mathcal{R}I^2$$

where $\mathcal{E}$ is the energy per unit length along the line, and $\mathcal{I}$ the energy flux. Give expressions for $\mathcal{E}$ and $\mathcal{I}$. What does the right-hand side of the previous equation represent?

iii. Show that the current obeys the wave-diffusion equation

$$\frac{\partial^2 I}{\partial t^2} + \frac{\mathcal{R}}{\mathcal{L}}\frac{\partial I}{\partial t} = \frac{1}{\mathcal{L}\mathcal{C}}\frac{\partial^2 I}{\partial x^2}$$

iv. Consider the low-resistance, high-frequency, limit $\omega \gg \mathcal{R}/\mathcal{L}$. Demonstrate that a signal propagating down the line varies as

$$I(x,t) \simeq I_0 \cos[k(vt - x)]\mathrm{e}^{-x/\delta}$$
$$V(x,t) \simeq Z I_0 \cos[k(vt - x) - 1/(k\delta)]\mathrm{e}^{-x/\delta}$$

where $k = \omega/v$, $v = 1/\sqrt{\mathcal{L}\mathcal{C}}$, $\delta = 2\,Z/\mathcal{R}$, and $Z = \sqrt{\mathcal{L}/\mathcal{C}}$. Show that $k\,\delta \gg 1$; that is, the decay length of the signal is much longer than its wavelength. Estimate the maximum useful length of a low-resistance, high-frequency, lossy transmission line. **5+3+3+8**

(3) At normal incidence, the mean radiant power from the Sun illuminating one square meter of the Earth's surface is 1.35 kW. Show that the peak amplitude of the electric component of solar electromagnetic radiation at the Earth's surface is $1010\mathrm{V m}^{-1}$. Demonstrate that the corresponding peak amplitude of the magnetic component is $2.7\mathrm{A m}^{-1}$. **3**