Oscillations, Wave and Optics

(SPRING 2025)

ASSIGNMENT-4

Topics: MATHS | Fourier Returns

Total Marks: 50

Date: 1st Apr, 2025

Due: 30th Apr, 2025 (EoD) No extension!

Problem-1: [30]

- Make a hand-drawn plot of the given functions.
- Find fourier series of the given functions.
- Plot the truncated fouier series (upto certain terms) using graphing calculators or programming.

(1) Sawtooth wave,

$$f(x) = \begin{cases} x, & \text{for } 0 \leq x < \pi \\ x - 2\pi, & \text{for } \pi \leq x < 2\pi \end{cases}$$

(2) Reverse sawtooth wave,

$$f(x) = \begin{cases} -\frac{1}{2}(\pi + x), & \text{for } -\pi \leq x < 0 \\ \frac{1}{2}(\pi - x), & \text{for } 0 \leq x < \pi \end{cases}$$

(3) Triangular wave,

$$f(x) = \begin{cases} -x, & \text{for } -\pi \leq x < 0 \\ x, & \text{for } 0 \leq x < \pi \end{cases}$$

(4)

$$f(x) = \begin{cases} 4x(1+x), & \text{for } -1 \leq x < 0 \\ 4x(1-x), & \text{for } 0 \leq x < 1 \end{cases}$$

(5) Full-wave rectifier,

$$f(x) = \begin{cases} \sin(\omega t), & \text{for } 0 \leq x < \pi/\omega \\ \sin(\omega t), & \text{for } -\pi/\omega \leq x < 0 \end{cases}$$

(6) Rectangular pulse ($n < \pi$)

$$\delta_n(x) = \begin{cases} n, & \text{for } |x| < \frac{1}{2n} \\ 0, & \text{for } \pi > |x| > \frac{1}{2n} \end{cases}$$

Problem-2: (Arfken 19.2.17) [10]

(1) Show that the Dirac delta function $\delta(x-a)$, expanded in a Fourier sine series in the half-interval $(0, L)$ $(0 < a < L)$ is given by,

$$\delta(x-a) = \frac{1}{2}\sum_{n=1}^{\infty} sin(\frac{n\pi a}{L}) sin(\frac{n\pi x}{L})$$

Note that this series actually describes $-\delta(x+a) + \delta(x-a)$ in the interval $(-L, L)$.
(2) By integrating both sides of the preceding equation from 0 to x, show that the cosine expansion of the square wave

$$f(x) = \begin{cases} 0, & \text{for } 0 \leq x < a \\ 1, & \text{for } a \leq x < L, \end{cases}$$

is,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right)\cos\left(\frac{n\pi x}{L}\right),$$

for $0 \leq x < L$.
(3) Show that the term $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(\frac{n\pi a}{l})$ is the average of f(x) on (0, L) **(3+4+3)**

Problem-3: (Arfken 19.2.20) [10]

(1) A string, clamped at x = 0 and at x = L, is vibrating freely. Its motion is described by the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2\frac{\partial^2 u(x,t)}{\partial x^2}$$

Assume a Fourier expansion of the form,

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

and determine the coefficients $b_n(t)$. The initial conditions are

$$u(x,0) = f(x) \; and \; \frac{\partial}{\partial t}u(x,0) = g(x)$$

Note (but don't care about this now), This is only half the conventional Fourier orthogonality integral interval. However, as long as only the sines are included here, the Sturm-Liouville boundary conditions are still satisfied and the functions are orthogonal.
(2) We assume now that the presence of a resisting medium will damp the vibrations according to the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2\frac{\partial^2 u(x,t)}{\partial x^2} - k\frac{\partial u(x,t)}{\partial t}$$

Introduce a Fourier expansion similar to above form. Again determine the coefficients $b_n(t)$. Take the initial and boundary conditions to be the same as above. Assume the damping to be small $(k^2 < \frac{4n\pi v}{L})$.
(3) Repeat, but assume the damping to be large $(k^2 > \frac{4n\pi v}{L})$. **4+3+3**