Oscillations, Wave and Optics

(SPRING 2025)

ASSIGNMENT-2

Topics: Coupled Oscillations and Transverse Standing Waves — Total : 40

Date: 17 th Feb, 2025 — **Due: 2nd Mar, 2025** (EoD)

Part:A | Coupled Oscillations **[25]**

(1) A linear triatomic molecule (e.g., carbon dioxide) consists of a central atom of mass M flanked by two identical atoms of mass m. The atomic bonds are represented as springs of spring constant k. Find the molecule's normal frequencies and modes of linear oscillation. **5**

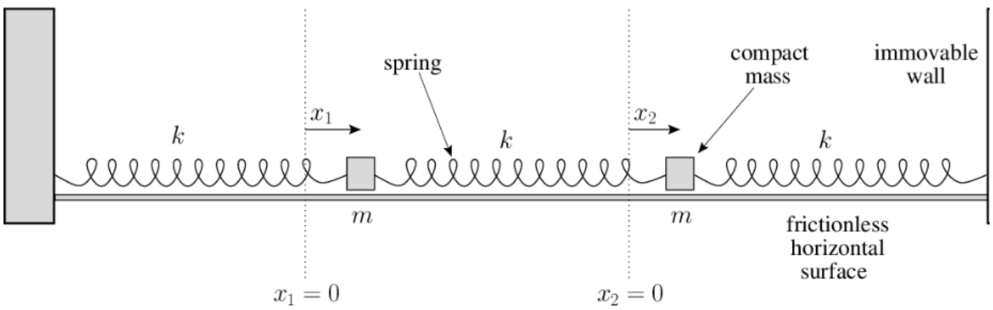

Figure 1

(2) Consider the mass-spring system as shown in the figure-1

a. Show that, when written in terms of the physical coordinates, the total energy of the system takes the form,

$$E = m[\frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \omega_0^2(x_1^2 - x_1x_2 + x_2^2)]$$

b. Furthermore, show that the total energy takes the form

$$E = m[(\dot{\eta}_1^2 + \dot{\eta}_2^2) + \omega_0^2(\eta_1^2 + 3\eta_2^2)]$$

when expressed in terms of the normal coordinates.

c. Hence, deduce that,

(i) $E = m(\mathcal{E}_1 + \mathcal{E}_2)$
(ii) $\mathcal{E}_1 = \dot{\eta}_1^2 + \omega_0^2\eta_1^2$
(iii) $\mathcal{E}_2 = \dot{\eta}_2^2 + 3\omega_0^2\eta_2^2$
(iv) $\frac{d\mathcal{E}_1}{dt} = 0$
(v) $\frac{d\mathcal{E}_2}{dt} = 0$

Here, $\mathcal{E}_1$ and $\mathcal{E}_2$ are the separately conserved energies per unit masses of the first and second normal modes, respectively. **5+5+(4 + 2 + 2 + 1 + 1)**

Part:B | Transverse Standing Waves **[15]**

(1) Figure-2 shows the left and right extremities of a linear LC network consisting of N identical inductors of inductance L, and $N+1$ identical capacitors of capacitance C. Let the instantaneous current flowing through the ith inductor be $I_i(t)$, for $i = 1, N$. Demonstrate from Kirchhoff's circuital laws that the currents evolve in time according to the coupled equations

$$\ddot{I}_i = \omega_0^2(I_{i-1} - 2I_i + I_{i+1})$$

for $i = 1, N$, where $\omega_0 = 1/\sqrt{LC}$, and $I_0 = I_{N+1} = 0$. Find the normal frequencies of the system. **2+3**

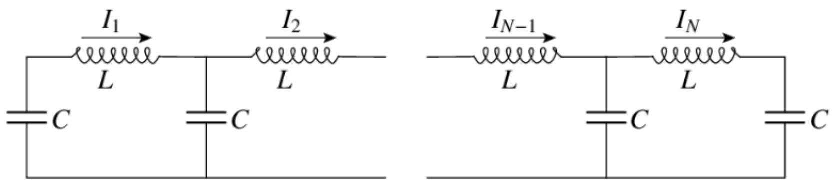

Figure 2

(2) The linear LC circuit considered in above question can be thought of as a discrete model of a uniform lossless transmission line (e.g., a co-axial cable). In this interpretation, $I_i(t)$ represents $I(x_i, t)$, where $x_i = i\delta x$. Moreover, $C = \mathcal{C}\delta x$, and $L = \mathcal{L}\delta x$, where $\mathcal{C}$ and $\mathcal{L}$ are the capacitance per unit length and the inductance per unit length of the line, respectively.

a. Show that, in the limit $\delta x \to 0$, the evolution equation for the coupled currents given in the above problem reduces to the wave equation,

$$\frac{\partial^2 I}{\partial t^2} = v^2 \frac{\partial^2 I}{\partial x^2}$$

b. If $V_i(t)$ is the potential difference (measured from the top to the bottom) across the $i+1$th capacitor (from the left) in the circuit shown in above problem, and $V(x,t)$ is the corresponding voltage in the transmission line, show that the discrete circuit equations relating the $I_i(t)$ and $V_i(t)$ reduce to

$$\frac{\partial V}{\partial t} = -\frac{1}{\mathcal{C}}\frac{\partial I}{\partial x}$$
$$\frac{\partial I}{\partial t} = -\frac{1}{\mathcal{L}}\frac{\partial V}{\partial x}$$

in the transmission-line limit.

c. Demonstrate that the voltage in a transmission line satisfies the wave equation

$$\frac{\partial^2 V}{\partial t^2} = v^2 \frac{\partial^2 V}{\partial x^2}.$$

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