Prove: Sum of 2 rational numbers is rational.

Rational $(x) \to \exists ij(\text{int})\ x = \frac{i}{j}$ $\gcd(ij) = 1$

$\forall x, y$ Rational $(x) \wedge$ Rational $(y) \to$ Rational $(x+y)$

$$\left.\begin{array}{l} \text{Rational}(x) \to x = \frac{i_x}{j_x} \\ \text{Rational}(y) \to y = \frac{i_y}{j_y} \end{array}\right\} x+y = \frac{i_x}{j_x} + \frac{i_y}{j_y} = \frac{i_x j_y + i_y j_x}{j_x j_y}$$

$x+y = \frac{i_s}{j_s} \longrightarrow$ Rational $(x+y)$ QED

Indirect Method $p \to q \leftrightarrow \neg q \to \neg p$

ex: Prove if $3n+2$ is odd then n is odd.

Odd $(3n+2) \to \boxed{3n+2 = 2k+1}$ $n = \frac{2k-1}{3}$

Even $(n) \to$ Even $(3n+2)$

$\exists k\ n = 2k \to 3n = 6 \to 3n+2 = 6k+2$
$\quad -2 \quad = 2(3k+1)$

Pick $k' = 3k+1$ $\quad 3n+2 = 2k' \to$ Even$(3n+2)$

even $(n) \to$ even$(3n+2)$
odd $(3n+2) \to$ odd(n)

ex: If n^2 is odd, then n is odd

$odd(n^2) \to \exists k: n^2 = 2k+1 \to n = \sqrt{2k+1}$

Indirect: $even(n) \to even(n^2)$

$even(n) \to \exists k \; n = 2k \to n^2 = 4k^2 = 2(\underbrace{2k^2}_{k'})$ Contrapositive

$n^2 = 2k' \to even(n^2)$

$even(n) \to even(n^2)$

$odd(n^2) \to odd(n)$

$p \to q \leftrightarrow \neg q \to \neg p$

$\sqrt{2}$ is irrational $\nexists i\,j$ such that $\sqrt{2} = \frac{i}{j}$

By contradiction $\boxed{\sqrt{2} = \frac{i}{j}}$ $gcd(i,j) = 1$

$2 = \frac{i^2}{j^2} \to i^2 = 2j^2 \to even(i) \to even(j^2)$

$even(i) \to \exists k \; i = 2k$

$even(i) = \exists k \; i = 2k \Rightarrow i^2 = 4k^2$

$2 = \frac{i^2}{j^2} \to i^2 = 2j^2 \to 2j^2 = \overset{2}{4}k^2$

$j^2 = 2k^2$

$even(j^2) \to even(j)$

$even(i) \land even(j) \to gcd(i,j) = 2 \neq 1$

---

ex: Prove $|xy| = |x| \cdot |y|$ Proof by cases

$x \geq 0 \lor x < 0$

$y \geq 0 \lor y < 0$

$(x \geq 0 \land y \geq 0) \lor (x < 0 \land y \geq 0) \lor (x \geq 0 \land y < 0) \lor (x < 0 \land y < 0)$

$\to |xy| = |x| \cdot |y|$

Cases: $(x < 0 \land y \geq 0)$ $-xy =$

$|x| = -x$

$|y| = y$

Counterexamples:

$\forall x \; P(x)$

ex: Every pos int is the sum of the squares of 3 distinct integers.

Counter example: Pick 1

* Prove $\exists x \; P(x)$

  * Actually finding a C for which $P(x)$ (constructive proof).

  * Non-constructive Proofs

Goldbach: $\forall n \geq 2 \; \exists$ primes p, q : $n = \frac{p+q}{2}$