Unit - I

Solutions of Equations and Eigen Value Problems.

Iterative Method:

(1) Write the gn eqn $f(x) = 0$ into the form $x = \varphi(x)$

(2) Assume that $x = x_0$ be the root of the given eqn

(3) The first approximation to the root is gn by $x_1 = \varphi(x_0)$

|||ly $x_2 = \varphi(x_1)$

$x_3 = \varphi(x_2)$

$\vdots$

$x_n = \varphi(x_{n-1})$

x_n n^{th} iteration + the value of x_n is the root of the gn eqn.

$\Rightarrow x_n$ is the root of the gn eqn.

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(1) Find the root of the equation $\cos x = 3x - 1$, using iteration Method.

Soln

$f(x) = \cos x - 3x + 1$

$f(0) = \cos 0 - 3(0) + 1 = 2 \rightarrow +ve$

$f(1) = \cos 1 - 3(1) + 1 = 0 - 3(1) + 1 \rightarrow -ve$

$\therefore$ The root lies between 0 and 1

The eqn can be written as

$$\cos x - 3x + 1 = 0$$
$$-3x = -\cos x - 1$$
$$3x = \cos x + 1$$
$$x = \frac{1}{3}[1 + \cos x]$$

Let $\varphi(x) = \frac{1}{3}[1 + \cos x]$

$$\varphi'(x) = -\frac{1}{3}\sin x$$
$$|\varphi'(x)| = \frac{1}{3}\sin x$$

$$|\varphi'(0)| = 0 < 1$$
$$|\varphi'(1)| = \frac{1}{3}\sin 1 = 0.2804 < 1.$$

Let $x_0 = 0$

$x_1 = \varphi(x_0) = \frac{1}{3}(1 + \cos x_0) = \frac{1}{3}(1 + \cos 0)$

$x_1 = 0.6667$

$x_2 = \varphi(x_1) = \frac{1}{3}(1 + \cos x_1) = \frac{1}{3}(1 + \cos 0.6667)$

$x_2 = 0.5953$

$x_3 = \varphi(x_2) = \frac{1}{3}(1 + \cos x_2) = \frac{1}{3}(1 + \cos 0.5953)$

$x_3 = 0.6093$

$x_4 = \varphi(x_3) = \frac{1}{3}(1 + \cos x_3) = \frac{1}{3}(1 + \cos 0.6093)$

$x_4 = 0.6067$

$x_5 = \varphi(x_4) = \frac{1}{3}(1+\cos x_4) = \frac{1}{3}(1+\cos 0.6067)$

$x_5 = 0.6072$

$x_6 = \varphi(x_5) = \frac{1}{3}(1+\cos x_5) = \frac{1}{3}(1+\cos 0.6072)$

$x_6 = 0.6071$

$x_7 = \varphi(x_6) = \frac{1}{3}(1+\cos x_6) = \frac{1}{3}(1+\cos 0.6071)$

$x_7 = 0.6071$

$\therefore$ The required root is 0.6071.

(5) Solve the equation $x^2 - 2x - 3 = 0$ for the +ve root by iteration method.

Soln

$$x^2 - 2x - 3 = 0$$

$$f(x) = x^2 - 2x - 3$$

$f(0) = 0 - 2(0) - 3 = -3 \to -ve$

$f(1) = -4 \to -ve$

$f(2) = -3 \to -ve$

$f(3) = 0 \to +ve$

$\therefore$ The root lies between 2 and 3

$$x^2 = 2x + 3$$

$$x = \sqrt{2x+3}$$

$$\varphi(x) = \sqrt{2x+3} = (2x+3)^{1/2}$$

$$\varphi'(x) = \frac{1}{\cancel{2}}(2x+3)^{-1/2} \cdot \cancel{2}$$

$$|\varphi'(x)| = |(2x+3)^{-1/2}|$$

$$|\varphi'(2)| < 1 \quad \& \quad |\varphi'(3)| < 1$$

Take $x_0 = 2.5$

$$x_1 = \varphi(x_0) = \sqrt{2x_0+3} = \sqrt{2(2.5)+3} = 2.8284$$

$$x_2 = \varphi(x_1) = \sqrt{2x_1+3} = \sqrt{2(2.8284)+3} = 2.9422$$

$$x_3 = \varphi(x_2) = \sqrt{2x_2+3} = \sqrt{2(2.9422)+3} = 2.9807$$

$$x_4 = \varphi(x_3) = \sqrt{2x_3+3} = \sqrt{2(2.9807)+3} = 2.9936$$

$$x_5 = \varphi(x_4) = \sqrt{2x_4+3} = \sqrt{2(2.9936)+3} = 2.9979$$

$$x_6 = \varphi(x_5) = \sqrt{2x_5+3} = \sqrt{2(2.979)+3} = 2.9993$$

$$x_7 = \varphi(x_6) = \sqrt{2x_6+3} = \sqrt{2(2.9993)+3} = 2.9998$$

$$x_8 = \varphi(x_7) = \sqrt{2x_7+3} = \sqrt{2(2.9998)+3} = 2.9999$$

$$x_9 = \varphi(x_8) = \sqrt{2x_8+3} = \sqrt{2(2.9999)+3} = 2.9999$$

$\therefore$ The required root is 2.9999

(3) Solve by iteration Method $2x - \log_{10} x = 7$

Soln

$$2x - \log_{10} x - 7 = 0$$

$$f(x) = 2x - \log_{10} x - 7$$

$f(1) = -5 \rightarrow -ve$

$f(2) = -3.3010 \rightarrow -ve$

$f(3) = -1.4771 \rightarrow -ve$

$f(4) = 0.3979 \rightarrow +ve$

$\therefore$ The root lies between 3 and 4

$$2x = 7 + \log_{10} x$$

$$x = \frac{1}{2}\left[7 + \log_{10} x\right]$$

$$\therefore \phi(x) = \frac{1}{2}\left[7 + \log_{10} x\right]$$

$$\phi'(x) = \frac{1}{2}\left[\frac{1}{x}\log_{10} e\right]$$

$$|\phi'(x)| = \left|\frac{1}{2}\left[\frac{1}{x}\log_{10} e\right]\right| < 1 \text{ in } (3, 4)$$

Take $x_0 = 3.6$

$$x_1 = \phi(x_0) = \frac{1}{2}\left[\log_{10} x_0 + 7\right]$$

$$= \frac{1}{2}\left[\log_{10} 3.6 + 7\right]$$

$$= 3.7782$$

$$x_2 = \varphi(x_1) = \frac{1}{2}[\log_{10} x_1 + 7]$$
$$= \frac{1}{2}[\log_{10} 3.7782 + 7]$$
$$x_2 = 3.7886$$
$$x_3 = \varphi(x_2) = \frac{1}{2}[\log_{10} x_2 + 7]$$
$$= \frac{1}{2}[\log_{10} 3.7886 + 7]$$
$$x_3 = 3.7892$$
$$x_4 = \varphi(x_3) = \frac{1}{2}[\log_{10} x_3 + 7]$$
$$= \frac{1}{2}[\log_{10} 3.7892 + 7]$$
$$x_4 = 3.7893$$
$$x_5 = \varphi(x_4) = \frac{1}{2}[\log_{10} x_4 + 7]$$
$$= \frac{1}{2}[\log_{10} 3.7893 + 7]$$
$$x_5 = 3.7893$$

$\therefore$ The required root is 3.7893

H.W 4) find the negative root of the eqn $x^3 - 2x + 5 = 0$

Gauss Jordan Method

① $2x - y + 6z = 22$
$x + 7y - 3z = -22$
$5x - 2y + 3z = 18$

Soln

$$[A, B] = \left[\begin{array}{ccc|c} 2 & -1 & 6 & 22 \\ 1 & 7 & -3 & -22 \\ 5 & -2 & 3 & 18 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 3 & 11 \\ 1 & 7 & -3 & -22 \\ 1 & -\frac{2}{5} & \frac{3}{5} & \frac{18}{5} \end{array}\right] \quad R_1 \rightarrow \frac{R_1}{2}, \quad R_3 \rightarrow \frac{R_3}{5}$$

$$= \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 3 & 11 \\ 0 & \frac{15}{2} & -6 & -33 \\ 0 & \frac{1}{10} & -\frac{12}{5} & -\frac{37}{5} \end{array}\right] \quad R_2 \rightarrow R_2 - R_1, \quad R_3 \rightarrow R_3 - R_1$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & -6 & -22 \\ 0 & 1 & -\frac{4}{5} & -\frac{22}{5} \\ 0 & 1 & -24 & -74 \end{array}\right] \quad R_1 \rightarrow -2R_1, \quad R_2 \rightarrow \frac{2}{15}R_2, \quad R_3 \rightarrow 10R_3$$

$$= \left[\begin{array}{cccc} -2 & 0 & -\frac{26}{5} & -\frac{88}{5} \\ 0 & 1 & -\frac{4}{5} & -\frac{22}{5} \\ 0 & 0 & -\frac{116}{5} & -\frac{348}{5} \end{array}\right] \quad R_1 \rightarrow R_1 - R_2, \quad R_3 \rightarrow R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} \frac{5}{13} & 0 & 1 & \frac{44}{13} \\ 0 & -\frac{5}{4} & 1 & \frac{11}{2} \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 \to R_1 \times \frac{-5}{26} \\ R_2 \to -\frac{5}{4} R_2 \\ R_3 \to -\frac{5}{116} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} \frac{5}{13} & 0 & 0 & \frac{5}{13} \\ 0 & -\frac{5}{4} & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_2 \to R_2 \times \frac{-4}{5} \\ R_1 \to R_1 \times \frac{13}{5} \end{array}$$

$x = 1, \quad y = -2, \quad z = 3.$

(2) Solve
$$x + 3y + 3z = 16$$
$$x + 4y + 3z = 18$$
$$x + 3y + 4z = 19$$

$$[A,B] = \left[\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1/3 & 1 & 1 & 16/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 \to \frac{R_1}{3}$$

$$= \left[\begin{array}{ccc|c} 1/3 & 0 & 1 & 10/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 \to R_1 - R_2$$

$$= \left[\begin{array}{ccc|c} 1/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 \to R_1 - R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 \to 3R_1$$

$x = 1, \quad y = 2, \quad z = 3$

(3) Solve

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

soln

$$[A, B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1/10 & 1/10 & 6/5 \\ 1 & 5 & 1/2 & 13/2 \\ 1 & 1 & 5 & 7 \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{10} \\ R_2 \to \frac{R_2}{2} \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & \frac{1}{10} & \frac{1}{10} & \frac{6}{5} \\ 0 & \frac{49}{10} & \frac{2}{5} & \frac{53}{10} \\ 0 & \frac{9}{10} & \frac{49}{10} & \frac{29}{5} \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{cccc} 10 & 1 & 1 & 12 \\ 0 & 1 & \frac{4}{49} & \frac{53}{49} \\ 0 & 1 & \frac{49}{9} & \frac{58}{9} \end{array}\right] \begin{array}{l} R_1 \to 10 R_1 \\ R_2 \to \frac{10}{49} R_2 \\ R_3 \to \frac{10}{9} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 10 & 0 & \frac{45}{49} & \frac{535}{49} \\ 0 & 1 & \frac{4}{49} & \frac{53}{49} \\ 0 & 0 & \frac{2365}{441} & \frac{2365}{441} \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} \frac{98}{9} & 0 & 0 & \frac{107}{9} \\ 0 & \frac{49}{4} & 1 & \frac{53}{4} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_2 \to \frac{49}{45} R_1 \\ R_2 \to \frac{49}{4} R_2 \\ R_3 \to \frac{441}{2365} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} \frac{98}{9} & 0 & 0 & \frac{98}{9} \\ 0 & \frac{49}{4} & 0 & \frac{49}{4} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_3 \\ R_1 \to R_1 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 \times \frac{9}{98} \\ R_2 \to R_2 \times \frac{4}{49} \end{array}$$

$x = 1 \quad y = 1 \quad z = 1$

Inverse of a Matrix

Gauss Jordan Method

(1) Find the inverse of $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$ using Gauss Jordan Method.

Soln

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$$

$$(A/I) = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{1} \\ R_2 \to \frac{R_2}{1} \\ R_3 \to \frac{R_3}{-2} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & -1/2 \end{array}\right] R_2 \to$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1/2 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & -1 & 0 & -\frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{R_1}{1} \\ R_2 \to \frac{R_2}{2} \\ R_3 \to \frac{R_3}{1} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} \frac{1}{6} & 0 & 1 & \frac{1}{4} & -\frac{1}{12} & 0 \\ 0 & -\frac{1}{3} & 1 & \frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \quad \begin{array}{l} R_1 \to R_1/6 \\ R_2 \to R_2/-3 \\ R_3 \to R_3/2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & -\frac{1}{4} \\ 0 & -\frac{1}{3} & 0 & \frac{5}{12} & \frac{1}{12} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 \times 6 \\ R_2 \to -3R_2 \end{array}$$

$$= [I / A]$$

$\therefore$ Inverse of A is $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

(2) find the inverse of the Matrix $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ using Gauss Jordan Method.

Soln

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ -1 & 4 & 10 \end{bmatrix}$$

$$[A/I] = \left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -15 & 6 & -5 & 0 & 1 & 0 \\ -1 & 4 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & -\frac{6}{15} & +\frac{1}{3} & 0 & -\frac{1}{15} & 0 \\ 1 & -4 & -10 & 0 & 0 & -1 \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{3} \\ R_2 \to \frac{R_2}{-15} \\ R_3 \to \frac{R_3}{-1} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{15} & \frac{[illegible]}{3} & -\frac{1}{3} & -\frac{1}{15} & 0 \\ 0 & -\frac{11}{3} & -\frac{31}{3} & -\frac{1}{3} & 0 & -1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} -3 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 10 & 5 & 1 & 0 \\ 0 & 1 & \frac{31}{11} & \frac{1}{11} & 0 & \frac{3}{11} \end{array}\right] \begin{array}{l} R_1 \to R_1(-3) \\ R_2 \to -15R_2 \\ R_3 \to \frac{-3}{11} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} -3 & 0 & -1 & -6 & -1 & 0 \\ 0 & 1 & 10 & 5 & 1 & 0 \\ 0 & 0 & -\frac{79}{11} & -\frac{54}{11} & -1 & \frac{3}{11} \end{array}\right] \begin{array}{l} R_3 \to R_3 - R_2 \\ R_1 \to R_1 - R_2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} +3 & 0 & 1 & 6 & 1 & 0 \\ 0 & \frac{1}{10} & 1 & \frac{1}{2} & \frac{1}{10} & 0 \\ 0 & 0 & 1 & \frac{1}{31} & 0 & \frac{3}{31} \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{-1} \\ R_2 \to \frac{R_2}{10} \\ R_3 \to \frac{11}{31} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & \frac{185}{31} & 1 & -\frac{3}{31} \\ 0 & \frac{1}{10} & 0 & \frac{29}{62} & \frac{1}{10} & -\frac{3}{31} \\ 0 & 0 & 1 & \frac{1}{31} & 0 & \frac{3}{31} \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{185}{93} & \frac{1}{3} & \frac{-1}{31} \\ 0 & 1 & 0 & \frac{290}{62} & 1 & \frac{-30}{31} \\ 0 & 0 & 1 & \frac{1}{31} & 0 & \frac{3}{31} \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{3} \\ R_2 \to R_2 \times 10 \end{array}$$

$\therefore = [I \mid A]$

$\therefore$ inverse of A is $\begin{bmatrix} \frac{85}{93} & \frac{1}{3} & \frac{-1}{31} \\ \frac{290}{62} & 1 & \frac{-30}{31} \\ \frac{1}{31} & 0 & \frac{3}{31} \end{bmatrix}$

3) Using Gauss Jordan Method find the inverse of $\begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$

Soln

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$$

$$(A/I) = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & \frac{4}{3} & \frac{5}{3} & 0 & \frac{1}{3} & 0 \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{1} \\ R_2 \to \frac{R_2}{3} \\ \cancel{R} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & \frac{4}{3} & \frac{8}{3} & -1 & \frac{1}{3} & 0 \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right] R_2 \to R_2 - R_1$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{2} & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 7/6 & 0 & 0 & -\frac{1}{6} \end{array}\right] \begin{array}{l} R_2 \to \frac{3}{4} R_2 \\ R_3 \to \frac{R_3}{-6} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & \cancel{2} & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{5}{6} & 3/4 & -1/4 & \frac{-1}{6} \end{array}\right] \; R_3 \to R_3 - R_2$$

$$= \left[\begin{array}{ccc|ccc} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & -3/8 & \frac{1}{8} & 0 \\ 0 & 0 & 1 & -\frac{9}{10} & \frac{3}{10} & \frac{1}{5} \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{-1} \\ R_2 \to \frac{R_2}{2} \\ R_3 \to -\frac{6}{5} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -1/10 & -3/10 & -1/5 \\ 0 & 1/2 & 0 & 21/40 & -7/40 & -1/5 \\ 0 & 0 & 1 & -9/10 & 3/10 & 1/5 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/10 & 3/10 & 1/5 \\ 0 & 1 & 0 & 21/20 & -7/20 & -2/5 \\ 0 & 0 & 1 & -9/10 & 3/10 & 1/5 \end{array}\right]$$

$$= [I / A]$$

Inverse of A is $\begin{bmatrix} 1/10 & 3/10 & 1/5 \\ 21/20 & -7/20 & -2/5 \\ -9/10 & 3/10 & 1/5 \end{bmatrix}$

Gauss Jacobi Method

① Solve the following eqns by Gauss Jacobi Method

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

$x = \frac{17 - y + 2z}{20}$	$y = \frac{-18 + z - 3x}{20}$	$z = \frac{25 - 2x + 3y}{20}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 0.85$	$y_1 = -0.9$	$z_1 = 1.25$
$x_2 = 1.02$	$y_2 = -0.965$	$z_2 = 1.03$
$x_3 = 1.0013$	$y_3 = -1.0015$	$z_3 = 1.0033$
$x_4 = 1.0004$	$y_4 = -1.0001$	$z_4 = 0.9996$
$x_5 = 0.9999$	$y_5 = -1.0001$	$z_5 = 0.9999$
$x_6 = 1$	$y_6 = -1$	$z_6 = 1$
$x_7 = 1$	$y_7 = -1$	$z_7 = 1$

$\therefore \quad x = 1, \quad y = -1, \quad z = 1$

② Solve
$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

$x = \frac{32-4y+z}{28}$	$y = \frac{35-4x-2x}{17}$	$z = \frac{24-x-3y}{10}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 1.1429$	$y_1 = 2.0588$	$z_1 = 2.4$
$x_2 = 0.9345$	$y_2 = 1.3597$	$z_2 = 1.6681$
$x_3 = 1.0082$	$y_3 = 1.5564$	$z_3 = 1.898$
$x_4 = 0.9883$	$y_4 = 1.4935$	$z_4 = 1.8323$
$x_5 = 0.9949$	$y_5 = 1.514$	$z_5 = 1.8531$
$x_6 = 0.9931$	$y_6 = 1.5058$	$z_6 = 1.847$
$x_7 = 0.9937$	$y_7 = 1.5074$	$z_7 = 1.8490$
$x_8 = 0.9936$	$y_8 = 1.5069$	$z_8 = 1.8484$
$x_9 = 0.9936$	$y_9 = 1.5070$	$z_9 = 1.8486$
$x_{10} = 0.9936$	$y_{10} = 1.5070$	$z_{10} = 1.8485$
$x_{11} = 0.9936$	$y_{11} = 1.5070$	$z_{11} = 1.8485$

$\therefore$ The soln is

$x = 0.9936 \quad y = 1.5070 \quad z = 1.8485$

(3) Solve
$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

$x = \frac{85 - 6y + z}{27}$	$y = \frac{72 - 6x - 2z}{15}$	$z = \frac{110 - x - y}{54}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 3.148$	$y_1 = 4.8$	$z_1 = 2.037$
$x_2 = 2.157$	$y_2 = 3.269$	$z_2 = 1.890$
$x_3 = 2.492$	$y_3 = 3.685$	$z_3 = 1.937$
$x_4 = 2.401$	$y_4 = 3.545$	$z_4 = 1.923$
$x_5 = 2.432$	$y_5 = 3.583$	$z_5 = 1.927$
$x_6 = 2.423$	$y_6 = 3.570$	$z_6 = 1.926$
$x_7 = 2.426$	$y_7 = 3.574$	$z_7 = 1.926$
$x_8 = 2.425$	$y_8 = 3.573$	$z_8 = 1.926$

$x = 2.425 \quad y = 3.573 \quad z = 1.926.$

Gauss Seidal Iteration Method

① Solve
$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Soln:

$x = \frac{17 - y + 2z}{20}$	$y = \frac{-18 - 3x + z}{20}$	$z = \frac{25 - 2x + 3y}{20}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 0.82$	$y_1 = -1.0275$	$z_1 = 1.0109$
$x_2 = 1.0025$	$y_2 = -0.9998$	$z_2 = 0.9998$
$x_3 = 1.0000$	$y_3 = -1.000$	$z_3 = 1.0000$
$x_4 = 1.0000$	$y_4 = -1.0000$	$z_4 = 1.0000$

$x = 1 \quad y = -1 \quad z = 1$

② Solve
$$4x + 2y + z = 14$$
$$x + 5y - z = 10$$
$$x + y + 8z = 20$$

$x = \frac{85-6y+z}{27}$	$y = \frac{72-6x-2z}{15}$	$z = \frac{110-x-y}{54}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 3.148$	$y_1 = 3.541$	$z_1 = 1.913$
$x_2 = 2.432$	$y_2 = 3.572$	$z_2 = 1.926$
$x_3 = 2.426$	$y_3 = 3.573$	$z_3 = 1.926$
$x_4 = 2.426$	$y_4 = 3.573$	$z_4 = 1.926$

$\therefore$ $x = 2.426$
$y = 3.573$
$z = 1.926$.

$x = \frac{14-2y-z}{4}$	$y = \frac{10-x+z}{5}$	$z = \frac{20-x-y}{8}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 3.5$	$y_1 = 1.3$	$z_1 = 1.9$
$x_2 = 2.375$	$y_2 = 1.905$	$z_2 = 1.965$
$x_3 = 2.056$	$y_3 = 1.982$	$z_3 = 1.995$
$x_4 = 2.010$	$y_4 = 1.997$	$z_4 = 1.999$
$x_5 = 2.002$	$y_5 = 1.999$	$z_5 = 2$
$x_6 = 2.001$	$y_6 = 2$	$z_6 = 2$
$x_7 = 2$	$y_7 = 2$	$z_7 = 2$
$x_8 = 2$	$y_8 = 2$	$z_8 = 2$

$\therefore \quad x = 2, \; y = 2, \; z = 2$

(3) Solve
$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

Eigen Values of a Matrix by Power Method

(1) Find the numerically largest eigen value of $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ and its corresponding eigen vector by power method, taking the initial eigen vector as $(1\ 0\ 0)^T$ (upto three decimal places.

Soln

(1) Given $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$$

$$AX_1 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ 1 \\ 2 \end{pmatrix} = 25\begin{pmatrix} 1 \\ 0.04 \\ 0.08 \end{pmatrix} = 25X_2$$

$$AX_2 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.04 \\ 0.08 \end{pmatrix} = \begin{pmatrix} 25.2 \\ 1.12 \\ 0.08 \end{pmatrix} = 25.2\begin{pmatrix} 1 \\ 0.0444 \\ 0.0667 \end{pmatrix}$$

$$= 25.2X_3$$

$$AX_3 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.0444 \\ 0.0667 \end{pmatrix} = \begin{pmatrix} 25.1778 \\ 1.1332 \\ 1.7337 \end{pmatrix} = 25.1778\begin{pmatrix} 1 \\ 0.0450 \\ 0.0688 \end{pmatrix}$$

$$= 25.1778X_4$$

$$A X_4 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0.0450 \\ 0.06888 \end{pmatrix} = \begin{pmatrix} 25.1826 \\ 1.135 \\ 1.7248 \end{pmatrix}$$

$$= 25.1826 \begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix} = 25.1826 X_5$$

$$A X_5 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix} = \begin{pmatrix} 25.1821 \\ 1.1353 \\ 1.7260 \end{pmatrix}$$

$$= 25.1821 \begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix} = 25.1821 X_6$$

Dominant eigen value $\lambda = 25.1821$

corresponding eigen vector is $\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix}$

(2) Determine by power method the largest eigen value and the corresponding eigen vector of the Matrix $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$

Soln

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 0.3333 \\ 1 \\ -0.333 \end{bmatrix} = 3X_2$$

$$AX_2 = \begin{bmatrix} 3.6666 \\ 1.6667 \\ 0.3337 \end{bmatrix} = 3.6666\begin{bmatrix} 1 \\ 0.4546 \\ 0.0910 \end{bmatrix} = 3.6666\, X_3$$

$$AX_3 = \begin{bmatrix} 2.2728 \\ 4.2732 \\ 1.7284 \end{bmatrix} = 4.2732\begin{bmatrix} 0.5319 \\ 1 \\ 0.4045 \end{bmatrix} = 4.2732\, X_4$$

$$AX_4 = \begin{bmatrix} 3.1274 \\ 5.2137 \\ 7.5131 \end{bmatrix} = 7.5131\begin{bmatrix} 0.4163 \\ 0.6939 \\ 1 \end{bmatrix} = 7.5731\, X_5$$

$$AX_5 = \begin{bmatrix} 1.498 \\ 6.6367 \\ 12.3593 \end{bmatrix} = 12.3593\begin{bmatrix} 0.1212 \\ 0.5370 \\ 1 \end{bmatrix} = 12.3596\, X_6$$

$$AX_6 = \begin{bmatrix} 0.7322 \\ 5.4376 \\ 12.0268 \end{bmatrix} = 12.0268\begin{pmatrix} 0.0609 \\ 0.4521 \\ 1 \end{pmatrix} = 12.0268\, X_7$$

$$AX_7 = \begin{pmatrix} 0.4172 \\ 5.0869 \\ 11.7473 \end{pmatrix} = 11.7475 \begin{pmatrix} 0.0355 \\ 0.4330 \\ 1 \end{pmatrix} = 11.7475\, X_8$$

$$AX_8 = \begin{pmatrix} 0.3345 \\ 4.9725 \\ 11.6965 \end{pmatrix} = 11.6965 \begin{pmatrix} 0.0286 \\ 0.4251 \\ 1 \end{pmatrix} = 11.6965\, X_9$$

$$AX_9 = \begin{pmatrix} 0.3039 \\ 4.936 \\ 11.6718 \end{pmatrix} = 11.6718 \begin{pmatrix} 0.0260 \\ 0.4229 \\ 1 \end{pmatrix} = 11.6718 \begin{pmatrix} 0.0260 \\ 0.4229 \\ 1 \end{pmatrix}$$

$$AX_{10} = \begin{pmatrix} 0.2967 \\ 4.9238 \\ 11.6656 \end{pmatrix} = 11.6656 \begin{pmatrix} 0.0253 \\ 0.4221 \\ 1 \end{pmatrix} = 11.6656\, X_{11}$$

$$AX_{11} = \begin{pmatrix} 0.2916 \\ 4.9201 \\ 11.6631 \end{pmatrix} = 11.6631 \begin{pmatrix} 0.025 \\ 0.4219 \\ 1 \end{pmatrix} = 11.6631\, X_{12}$$

$$AX_{12} = \begin{pmatrix} 0.2907 \\ 4.9188 \\ 11.6626 \end{pmatrix} = 11.6626 \begin{pmatrix} 0.0249 \\ 0.4218 \\ 1 \end{pmatrix} = 0.6626\, X_{13}$$

$$AX_{13} = \begin{pmatrix} 0.2903 \\ 4.9183 \\ 11.6623 \end{pmatrix} = 11.6623 \begin{pmatrix} 0.0249 \\ 0.4217 \\ 1 \end{pmatrix} = 11.6623\, X_{14}$$

$$AX_{14} = \begin{pmatrix} 0.29 \\ 4.9181 \\ 11.6619 \end{pmatrix} = 11.6619 \begin{pmatrix} 0.0249 \\ 0.4217 \\ 1 \end{pmatrix} = 11.6619\, X_{15}$$

$$AX_{15} = \begin{pmatrix} 0.29 \\ 4.9181 \\ 11.6619 \end{pmatrix} = 11.6619 \begin{pmatrix} 0.0249 \\ 0.4217 \\ 1 \end{pmatrix} = 11.6619\, X_{16}$$

$\therefore$ The dominant eigen value is 11.6619

The corresponding eigen vector is

$$\begin{pmatrix} 0.0249 \\ 0.4217 \\ 1 \end{pmatrix}.$$

(3) Find the dominant eigen value and the corresponding eigen vector of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Soln

$$A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$AX_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot X_2$$

$$AX_2 = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix} = 7 \cdot X_3$$

$$AX_3 = \begin{pmatrix} 3.5714 \\ 1.8572 \\ 0 \end{pmatrix} = 3.5714\begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix} = 3.5714\, X_4$$

$$AX_4 = \begin{pmatrix} 4.12 \\ 2.04 \\ 0 \end{pmatrix} = 4.12\begin{pmatrix} 1 \\ 0.4951 \\ 0 \end{pmatrix} = 4.12\, X_5$$

$$AX_5 = \begin{pmatrix} 3.9706 \\ 1.9902 \\ 0 \end{pmatrix} = 3.9706 \begin{pmatrix} 1 \\ 0.5012 \\ 0 \end{pmatrix} = 3.9706 X_6$$

$$AX_6 = \begin{pmatrix} 4.0072 \\ 2.0024 \\ 0 \end{pmatrix} = 4.0072 \begin{pmatrix} 1 \\ 0.4997 \\ 0 \end{pmatrix} = 4.0072 X_7$$

$$AX_7 = \begin{pmatrix} 3.9982 \\ 1.9994 \\ 0 \end{pmatrix} = 3.9982 \begin{pmatrix} 1 \\ 0.5000 \\ 0 \end{pmatrix} = 3.9982 X_8$$

$$AX_8 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = 4 X_9$$

$$AX_9 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$$

Dominant eigen value is $\lambda = 4$

Corresponding eigen vector is $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$

Eigen Value of a Matrix by Jacobi Method for Symmetric Matrix

Let $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2a_{ij}}{a_{ii} - a_{jj}}\right)$$

$$D = P^T A P$$

(1) Apply Jacobi process to evaluate the eigen values and eigen vectors of the Matrix $\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$

Soln

$$A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$$

The largest non diagonal element is $a_{13} = a_{31} = 1$

$a_{11} = 5$, $a_{33} = 5$

$$\tan 2\theta = \left[\frac{2 a_{13}}{a_{11} - a_{33}}\right] = \frac{2}{5-5}$$

$$\tan 2\theta = \infty$$

$$2\theta = \tan^{-1} \infty$$

$$2\theta = \pi/2$$

$$\boxed{\theta = \pi/4}$$

$$P = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos \pi/2 & 0 & -\sin \pi/4 \\ 0 & 1 & 0 \\ \sin \pi/4 & 0 & \cos \pi/4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

Ist transformation

$$D = P^T A P$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The eigen values are 6, −2, 4

corresponding eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(2) Find all the eigen values and eigen vectors of the Matrix

$$\begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$ using Jacobi Method.

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Here the largest non diagonal element is $a_{13} = a_{31} = 2$.

$a_{11} = 1, a_{33} = 1$

$$S_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{4}{0}$$

$$\tan 2\theta = \infty$$

$$2\theta = \pi/2$$

$$\boxed{\theta = \pi/4}$$

$$S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$B_1 = S_1^{-1} A S_1$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

II Transformation

$$a_{12} = a_{21} = 2$$

$$a_{11} = 3 \quad a_{22} = 3$$

$$S_2 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2\times 2}{3-3} = \infty$$

$$2\theta = \tan^{-1}\infty$$

$$2\theta = \pi/2$$

$$\theta = \pi/4$$

$$S_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_2 = S_1^{-1} B_1 S_2$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\therefore$ A is reduced to the diagonal Matrix B_2.

Hence the eigen values of A is 5, 1, -1

$$S = S_1 S_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\therefore$ eigen vectors are $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Numerical Methods

Unit - 2

Interpolation and Approximation

Lagrange's interpolation formula (unequal intervals)

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

---

① Using Lagrange's formula, find the polynomial to the given data

x	0	1	3
y	5	6	50

Soln

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot y_2$$

Here $x_0 = 0 \quad x_1 = 1 \quad x_2 = 3$

$y_0 = 5 \quad y_1 = 6 \quad y_2 = 50$

$$y = f(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)}(5) + \frac{(x-0)(x-3)}{(1-0)(1-3)} \cdot (6)$$

$$+ \frac{(x-0)(x-1)}{} \cdot (50)$$

$$= \frac{(x-1)(x-3)}{3}(5) + \frac{x(x-3)}{-2}(6) + \frac{x(x-1)}{6}(50)$$

$$= \frac{5}{3}[x^2-4x+3] - 3[x^2-3x] + \frac{50}{6}[x^2-x]$$

$$= x^2\left[\frac{5}{3} - 3 + \frac{50}{6}\right] + x\left[-\frac{20}{3} + 9 - \frac{50}{6}\right] + \left[\frac{15}{3}\right]$$

$$= 7x^2 + (-6)x + 5$$

$$\boxed{y = f(x) = 7x^2 - 6x + 5}$$

(2) Using Lagrange's interpolation find $y(2)$ from the following data

x	0	1	3	4	5
y	0	1	81	256	625

Soln

$x_0 = 0 \quad x_1 = 1 \quad x_2 = 3 \quad x_3 = 4. \quad x_4 = 5$

$y_0 = 0 \quad y_1 = 1 \quad y_2 = 81 \quad y_3 = 256 \quad y_4 = 625$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \cdot y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \cdot y_3$$

Put $x = 2$

$$y(2) = \frac{(2-1)(2-3)(2-4)(2-5)}{(0-1)(0-3)(0-4)(0-5)}(0) + \frac{(2-0)(2-3)(2-4)(2-5)}{(1-0)(1-3)(1-4)(1-5)}(1) + \frac{(2-0)(2-1)(2-4)(2-5)}{(3-0)(3-1)(3-4)(3-5)}(81) + \frac{(2-0)(2-1)(2-3)(2-5)}{(4-0)(4-1)(4-3)(4-5)}(256) + \frac{(2-0)(2-1)(2-3)(2-4)}{(5-0)(5-1)(5-3)(5-4)}(625)$$

$$= \frac{(2)(-1)(-2)(-3)}{(1)(-2)(-3)(-4)} + \frac{(2)(1)(-2)(-3)}{(3)(2)(-1)(-2)}(81) + \frac{(2)(1)(-1)(-3)}{(4)(3)(1)(-1)}(256) + \frac{(2)(1)(-1)(-2)}{(5)(4)(2)(1)}(625)$$

$$= \frac{12}{24} + \frac{12}{12}(81) - \frac{6}{12}(256) + \frac{4}{40}(625)$$

$$= \frac{1}{2} + 81 - 128 + 62.5$$

$$= 0.5 + 81 - 128 + 62.5 = 16.$$

3) Use Lagrange's Method to find $\log_{10} 656$, given that $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$ and $\log_{10} 661 = 2.8202$.

soln

x	654	658	659	661
$y = \log_{10} x$	2.8156	2.8182	2.8189	2.8202

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

Put $x = 656$

$$y = f(656) = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \cdot (2.8156) + \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} (2.8182) + \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} (2.8189) + \frac{(656-654)(656-659)(656-659)}{(661-654)(661-659)(661-659)} \cdot (2.8202)$$

$$= \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)} (2.8156) + \frac{2(-3)(-5)}{4(-1)(-3)} (2.8182) + \frac{(2)(-2)(-5)}{(5)(1)(-2)} (2.8189) + \frac{(2)(-2)(-3)}{(7)(3)(2)} (2.8202)$$

$$= 0.6033 + 7.0455 - 5.6378 + 0.8058$$

$$= 2.8168.$$

4) Use Lagrange's formula to find the value of y at $x = 6$ from the following data

x :	3	7	9	10

Soln

$x_0 = 3 \quad x_1 = 7 \quad x_2 = 9 \quad x_3 = 10$

$y_0 = 168 \quad y_1 = 120 \quad y_2 = 72 \quad y_3 = 63.$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

Put $x = 6$

$$y = f(6) = \frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)}(168) + \frac{(6-3)(6-9)(6-10)}{(7-3)(7-9)(7-10)}(120) + \frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)}(72) + \frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)}(63)$$

$$= \frac{(-1)(-3)(-4)}{(-4)(-6)(-7)}(168) + \frac{(3)(-3)(-4)}{4(-2)(-3)}(120) + \frac{(3)(-1)(-4)}{(6)(2)(-1)}(72) + \frac{(3)(-1)(-3)}{(7)(3)(1)}(63)$$

$= 12 + 180 - 72 + 27$

$= 147$

5) Given the values

x	14	17	31	35
$f(x)$	68.7	64.0	44.0	39.1

find $f(27)$ by using Lagrange's interpolation formula.

Soln

$x_0 = 14 \quad x_1 = 17 \quad x_2 = 31 \quad x_3 = 35$

$y_0 = 68.7 \quad y_1 = 64 \quad y_2 = 44 \quad y_3 = 39.1$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

Put $x = 27$

$$y = f(27) = \frac{(27-17)(27-31)(27-35)}{(14-17)(14-31)(14-35)} \cdot (68.7)$$
$$+ \frac{(27-14)(27-31)(27-35)}{(17-14)(17-31)(17-35)} (64.0)$$
$$+ \frac{(27-14)(27-17)(27-35)}{(31-14)(31-17)(31-35)} (44.0)$$
$$+ \frac{(27-14)(27-17)(27-31)}{(35-14)(35-17)(35-31)} (39.1)$$

$$= \frac{(10)(-4)(-8)}{(-3)(-17)(-21)} (68.7) + \frac{13(-4)(-8)}{(3)(-14)(-8)} (6$$
$$+ \frac{(13)(10)(-8)}{(17)(14)(-4)} (44) + \frac{(13)(10)(-4)}{(21)(18)(4)} (3$$

$$= -20.52 + 35.22 + 48.07 - 13.45$$

6) Find the Missing term in the following table using Lagranges interpolation

x	0	1	2	3	4
y	1	3	9	–	81

Soln

$x_0 = 0 \quad x_1 = 1 \quad x_2 = 3 \quad x_3 = 4$

$y_0 = 1 \quad y_1 = 3 \quad y_2 = 9 \quad y_3 = 81$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3.$$

Put $x = 3$

$$y = f(3) = \frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)}(1) + \frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)}(3) + \frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)}(9) + \frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)}(81)$$

$$= \frac{-2}{-8}\{-3\} + \frac{27}{2} + \frac{81}{4}$$

$$= 31.$$

7) Using Lagranges formula prove

$y_1 = y_3 - 0.3\,(y_5 - y_{-3}) + 0.2\,(y_{-3} + y_{-5})$

Soln y_{-5}, y_{-3}, y_3, y_5 Occur in the answers.

So we can have the table

x	-5	-3	3	5
y	y_{-5}	y_{-3}	y_3	y_5

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_{-5} + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_{-3} + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_3 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_5$$

put $x = 1$

$$y_1 = f(1) = \frac{(1+3)(1-3)(1-5)}{(-5+3)(-5-3)(-5-5)} \cdot y_{-5} + \frac{(1+5)(1-3)(1-5)}{(-3+5)(-3-3)(-3-5)} y_{-3} + \frac{(1+5)(1+3)(1-5)}{(3+5)(3+3)(3-5)} y_3 + \frac{(3+5)(3+3)(3-3)}{(5+5)(5+3)(5-3)} \cdot y_5$$

$$= \frac{(4)(-2)(-4)}{(-2)(-8)(-10)} \cdot y_{-5} + \frac{(6)(-2)(-4)}{(2)(-6)(-8)} \cdot y_{-3} + \frac{(6)(4)(-4)}{(8)(6)(-2)} \cdot y_3 + \frac{(6)(4)(-2)}{(10)(8)(2)} y_5$$

$= -0.24 \quad +0.54 \quad +4 \quad -0.34$

$$+\frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}.(38)$$

$$+\frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}(42)$$

$$= 37.23.$$

(2) Find the value of θ given $f(\theta) = 0.3887$ where $f(\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$ using the table

θ	21°	23°	25°
$f(\theta)$	0.3706	0.4068	0.4433.

Soln

Let $\theta = x$

$f(\theta) = f(x) = y$

x	21°	23°	25°
y	0.3706	0.4068	0.4433

$$x = f(y) = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}.x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}.x_1$$
$$+\frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}.x_2$$

Put $y = 0.3887$

$$x = f(0.3887) = \frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)}(21^\circ)$$
$$+\frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)}(23^\circ)$$
$$+\frac{(0.3887-0.3706)(0.3887-0.4068)}{\ldots}$$

Newton's divided difference formula : (unequal)

$$y = f(x) = y_0 + (x-x_0)\,\Delta f(x_0) + (x-x_0)(x-x_1)\,\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\,\Delta^3 f(x_0) + \cdots$$

① Using Newton's divided difference formula find $f(x)$ and $f(6)$ from the following data.

x :	1 x_0	2 x_1	7 x_2	8 x_3
$f(x)$:	1	5	5	4

Soln

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		$\frac{5-1}{2-1} = 4$		
2	5		$\frac{0-4}{7-1} = -\frac{4}{6}$	
		$\frac{5-5}{7-2} = 0$		$\frac{-\frac{1}{6}+\frac{4}{6}}{8-1} = \frac{1/2}{7} = \frac{1}{14}$
7	5		$\frac{-1-0}{8-2} = \frac{-1}{6}$	
		$\frac{4-5}{8-7} = -1$		
8	4			

$$y = f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + \cdots$$

$$= 1 + (x-1)(4) + (x-1)(x-2)\left(-\frac{4}{6}\right)$$

$$= x^3\left[\frac{1}{14}\right] + x^2\left[-\frac{4}{6}\right] - \frac{3}{14} - \frac{7}{14}]$$

$$+ x\left[4 + \frac{12}{6} + \frac{2}{14} + \frac{21}{14}\right] + \left[1 - 4 - \frac{8}{6} - \frac{14}{14}\right]$$

$$f(x) = \frac{1}{14}x^3 - \frac{29}{21}x^2 + \frac{107}{14}x - \frac{16}{63}$$

Put $x = 6$

$$y = f(6) = \frac{1}{14}(6)^3 - \frac{29}{21}(6)^2 + \frac{107}{14}(6) - \frac{16}{36}$$

$$\cancel{= 54 - 114 + 40.4 - 7.833}$$

$$= 15.428 - 49.714 + 45.857 - 0.444$$

$$= 11.127$$

2) Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

Soln

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	1245				
		$\frac{33-1245}{-1+4} = -404$			
-1	33		$\frac{-28+404}{0+4} = 94$		
		$\frac{5-33}{0+1} = -28$		$\frac{10-94}{2+4} = -14$	
0	5		$\frac{2+28}{2+1} = 10$		$\frac{13+14}{5+4} = 3$
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5+1} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1335-9}{5-2} = 442$			
5	1335				

$$y = f(x) = f(x_0) + (x-x_0)\Delta f(x) + (x-x_0)(x-x_1)\Delta^2 f(x) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta^4 f(x)$$

$$= 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)(x-0)(x-2)(3) + (x+4)(x+1)(x-0)(-14)$$

$$= 1245 - 404x - 1616 + (x^2+5x+4)94 + (x^2+5x+4)(3x^2-6x) + (x^2+5x+4)(-14x)$$

$$= 1245 - 404x - 1616 + 94x^2 + 470x + 376 + 3x^4 + 15x^3 + 12x^2 - 14x^3 - 70x^2 - 56x - 6x^3 - 30x^2 - 24x$$

$$= x^4[3] + x^3[-14+15-6] + x^2[94-70+12-30] + [1245-1616+376] + x[-404+470-56-24]$$

$$= 3x^4 + 5x^3 + 6x^2 - 14x + 5$$

③ Find the cubic polynomial from the following table using Newton's divided difference formula and hence find $f(4)$

x	0 x_0	1 x_1	2 x_2	5 x_3
y	2	3	12	147

Soln

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2			
		$\frac{3-2}{1-0}=1$		
1	3		$\frac{9-1}{2-0}=4$	
		$\frac{12-3}{2-1}=9$		$\frac{9-4}{5-0}=1$
2	12		$\frac{45-9}{5-1}=9$	
		$\frac{147-12}{5-2}=45$		
5	147			

$$y=f(x) = y_0 + (x-x_0)\Delta f(x) + (x-x_0)(x-x_1)\Delta^2 f(x) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x).$$

$$= 2 + (x-0)(1) + (x-0)(x-1)(4) + (x-0)(x-1)(x-2)(1)$$

$$= 2 + x + 4x^2 - 4x + (x^2-x)(x-2)$$

$$= 2 + x + 4x^2 - 4x + x^3 - x^2 - 2x^2 + 2x$$

$$= x^3 + x^2 - x + 2$$

Put $x=4$

$$y = f(4) = 4^3 + 4^2 - 4 + 2$$

$$= 78.$$

Cubic Spline Interpolation formula.

$$g(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} - (x_{i-1} - x)^3 M_i\right]$$
$$+ \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6} M_{i-1}\right]$$
$$- \frac{1}{h}(x_{i-1} - x)\left[y_i - \frac{h^2}{6} M_i\right]$$

where, $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$

with $M_0 = M_n = 0$

(1) Obtain cubic spline polynomial which best fits with the following data, given that $y_0'' = y_3'' = 0$

x	-1 x_0	0 x_1	1 x_2	x_3 2
y	-1 y_0	1 y_1	3 y_2	35 y_3

Soln

Given $M_0 = M_3 = 0$, $h = 1$

WKT $M_{i-1} + 4M_i + M_{i+1} = 6\left[y_{i-1} - 2y_i + y_{i+1}\right]$

Put $i = 1$

$$M_0 + 4M_1 + M_2 = 6\left[y_0 - 2y_1 + y_2\right]$$
$$4M_1 + M_2 = 6\left[-1 - 2 + 3\right]$$
$$4M_1 + M_2 = 0 \quad \text{------} \quad (1)$$

Put $i = 2$

$$M_1 + 4M_2 + M_3 = 6\left[y_1 - 2y_2 + y_3\right]$$
$$M_1 + 4M_2 = 6\left[1 - 6 + 35\right]$$

Solve ① & ②

$M_1 = -12 \quad M_2 = 48$

The cubic spline polynomial is

$$S(x) = \frac{1}{6}\left[(x_i - x)^3 M_{i-1} - (x_{i-1} - x)^3 M_i \right]$$
$$+ (x_i - x)\left[y_{i-1} - \frac{1}{6} M_{i-1} \right]$$
$$- (x_{i-1} - x)\left[y_i - \frac{1}{6} M_i \right]$$

Case(i) $-1 < x < 0$

Put $i = 1$

$$S(x) = \frac{1}{6}\left[(x_1 - x)^3 M_0 - (x_0 - x)^3 M_1 \right]$$
$$+ (x_1 - x)\left[y_0 - \frac{1}{6} M_0 \right] - (x_0 - x)\left[y_1 - \frac{1}{6} M_1 \right]$$

$$= \frac{1}{6}\left[-(-1-x)^3 (-12) \right] + (0 - x)(-1)$$
$$- (-1-x)\left[1 + \frac{12}{6} \right]$$

$$= \frac{1}{6}\left[-12 (1+x)^3 \right] + x + (1+x)(3)$$

$$= -2\left[1 + x^3 + 3x + 3x^2 \right] + x + 3 + 3x$$

$$= -2 - 2x^3 - 6x - 6x^2 + x + 3 + 3x$$

$$\boxed{S(x) = -2x^3 - 6x^2 - 2x + 1, \quad -1 < x < 0}$$

Case (ii) $0 < x < 1$

Put $i = 2$

$$S(x) = \frac{1}{6}\left[(x_2 - x_1)^3 M_1 - (x_1 - x)^3 M_2 \right.$$
$$+ (x_2 - x) \left[y_1 - \frac{1}{6} M_1 \right]$$
$$- (x_1 - x) \left[y_2 - \frac{1}{6} M_2 \right]$$

$$= \frac{1}{6}\left[(1-x)^3 (-12) - (0-x)^3 (48) \right]$$
$$+ (1-x)\left[1 - \frac{1}{6}(-12)\right] - (0-x)\left[3 - \frac{1}{6} \times 48\right]$$

$$= \frac{1}{6}\left[-12(1-x)^3 + 48x^3 \right] + 3(1-x) - 5x$$

$$= \frac{1}{6}\left[-12(1 - x^3 - 3x + 3x^2) + 48x^3 \right.$$
$$+ 3 - 3x - 5x$$

$$= \frac{1}{6}\left[-12 + 12x^3 + 36x - 36x^2 + 48x^3 \right]$$
$$+ 3 - 3x - 5x$$

$$= x^3[2 + 8] + x^2[-6] + x[6 - 8]$$
$$-2 + 3$$

$$\boxed{S(x) = 10x^3 - 6x^2 - 2x + 1, \quad 0 < x < 1}$$

Case (iii) $1 < x < 2$

Put $i = 3$

$$S(x) = \frac{1}{6}\left[(x_3 - x)^3 M_2 - (x_2 - x)^3 M_3 \right]$$
$$+ (x_3 - x)\left[y_2 - \frac{1}{6} M_2 \right] - (x_2 - x)$$
$$\left[y_3 - \frac{1}{6} M_3 \right]$$

$$= \frac{1}{6}\left[(2-x)^3 48 \right] + (2-x)\left[3 - \frac{1}{6} \times 48 \right]$$

$= 8(2-x)^3 + (2-x)(-5) - 35(1-x)$

$= 8[8 - x^3 - 12x + 6x^2] - 10 + 5x - 35 + 35x$

$= 64 - 8x^3 - 96x + 48x^2 - 10 + 5x - 35 + 35x$

$$\boxed{S(x) = -8x^3 + 48x^2 - 56x + 19, \quad 1 < x < 2}$$

The cubic Spline polynomial is

$$S(x) = \begin{cases} -2x^3 - 6x^2 - 2x + 1, & -1 < x < 0 \\ 10x^3 - 6x^2 - 2x + 1, & 0 < x < 1 \\ -8x^3 + 48x^2 - 56x + 19, & 1 < x < 2 \end{cases}$$

(2) From the following table

x	1 x_0	2 x_1	3 x_2
y	-8 y_0	-1 y_1	18 y_2

Compute $y(1\cdot5)$ and $y'(1)$ using cubic Spline.

Soln

Take $M_0 = M_2 = 0$, $h = 1$

W.k.T $M_{i-1} + 4M_i + M_{i+1} = 6[y_{i-1} - 2y_i + y_{i+1}]$

Put $i = 1$

$M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$

$4M_1 = 6[-8 + 2 + 18]$

$4M_1 = 72$

$\boxed{M_1 = 18}$

The cubic spline polynomial is

$$S(x) = \frac{1}{6}\left[(x_i - x)^3 M_{i-1} - (x_{i-1} - x)^3 M_i\right]$$
$$+ (x_i - x)\left[y_{i-1} - \frac{1}{6}M_{i-1}\right]$$
$$- (x_{i-1} - x)\left[y_i - \frac{1}{6}M_i\right]$$

Case (i) $1 < x < 2$

Put $i = 1$

$$S(x) = \frac{1}{6}\left[(x_1 - x)^3 M_0 - (x_0 - x)^3 M_1\right]$$
$$+ (x_1 - x)\left[y_0 - \frac{1}{6}M_0\right]$$
$$- (x_0 - x)\left[y_1 - \frac{1}{6}M_1\right]$$
$$= \frac{1}{6}\left[(2-x)^3(0) - (1-x)^3(18)\right]$$
$$+ (2-x)\left[-8 - \frac{1}{6}(0)\right]$$
$$- (1-x)\left[-1 - \frac{1}{6}(18)\right]$$
$$= \frac{1}{6}\left[-(1-x)^3(18)\right] + (2-x)(-8)$$
$$- (1-x)\left[-1-3\right]$$
$$= -18(1-x)^3 - 8(2-x) + 4(1-x)$$
$$= -18(1-x)^3 - 16 + 8x + 4 - 4x$$
$$\boxed{S(x) = -18(1-x)^3 + 4x - 12, \quad 1 < x < 2}$$

Put $x = 1.5$

$$y(1.5) = S(1.5) = -18(1-1.5)^3 + 4(1.5) - 12$$
$$= -5.625$$

$y'(1) = 9(0) + 4 = 4$

$\boxed{y'(1) = 4}$ $\boxed{y(1.5) = -5.625}$

(3) Find the cubic spline interpolation

x :	1	2	3	4	5
f :	1 y_0	0 y_1	1 y_2	0 y_3	1 y_4

Soln

Take $M_0 = M_4 = 0$, $h = 1$

WKT
$$M_{i-1} + 4M_i + M_{i+1} = 6[y_{i-1} - 2y_i + y_{i+1}]$$

Put $i=1$
$$M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$$
$$4M_1 + M_2 = 6[1 - 0 + 1] = 12$$
$$4M_1 + M_2 = 12 \text{ ——— } (1)$$

Put $i = 2$
$$M_1 + 4M_2 + M_3 = 6[y_1 - 2y_2 + y_3]$$
$$= 6[0 - 2 + 0]$$
$$M_1 + 4M_2 + M_3 = -12 \text{ ——— } (2)$$

Put $i = 3$
$$M_2 + 4M_3 + M_4 = 6[y_2 - 2y_3 + y_4]$$
$$M_2 + 4M_3 = 6[1 - 0 + 1]$$
$$M_2 + 4M_3 = 12 \text{ ——— } (3)$$

from (1) & (2)

$4 \times (1) \Rightarrow 16M_1 + 4M_2 = 48$

from ② & ③

$$② \Rightarrow M_1 + 4M_2 + M_3 = -12$$

$$4 \times ③ \Rightarrow 4M_2 + 16M_3 = 48$$

$$M_1 - 15M_3 = -60 \quad \text{——} ⑤$$

Solve ④ & ⑤

$$\boxed{M_3 = \frac{30}{7}}$$

$$⑤ \Rightarrow M_1 = -60 + 15M_3$$

$$M_1 = -60 + \frac{450}{7}$$

$$\boxed{M_1 = \frac{30}{7}}$$

$$① \Rightarrow 4M_1 + M_2 = 12$$

$$M_2 = 12 - 4M_1$$

$$= 12 - 4\left(\frac{30}{7}\right)$$

$$\boxed{M_2 = \frac{-36}{7}}$$

The cubic Spline polynomial is

$$S(x) = \frac{1}{6}\left[(x_i - x)^3 M_{i-1} - (x_{i-1} - x)^3 M_i\right] + (x_i - x)\left[y_{i-1} - \frac{1}{6}M_{i-1}\right] - (x_{i-1} - x)\left[y_i - \frac{1}{6}M_i\right]$$

Case (i) $-1 < x < 2$

Put $i = 1$

$$S(x) = \frac{1}{6}\left[(x_1 - x)^3 M_0 - (x_0 - x)^3 M_1\right] + (x_1 - x)\left[y_0 - \frac{1}{6}M_0\right] - (x_0 - x)\left[y_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}\left[(2 - x)^3 (0) - (1 - x)^3 \left(\frac{30}{7}\right)\right] + (2 - x)\left[1 - \frac{1}{6}(0)\right]$$

$(a-b)$
$= a^3 - 3$

$$= \frac{1}{6}\left[-(1-x)^3\left(\frac{30}{7}\right)\right] + (2-x)[1] - (1-x)\left[-\frac{1}{6}\,\frac{30}{7}\right]$$

$$= \frac{1}{6}\left[-\frac{30}{7}(1-x)^3 + (2-x) + \frac{5}{7}(1-x)\right]$$

$$= \frac{1}{6}\left[-\frac{30}{7}\left[1^3 - x^3 - 3x + 3x^2\right] + 2 - x + \frac{5}{7} - \frac{5}{7}x\right]$$

$$= \cancel{-\frac{5}{7}} + \frac{5}{7}x^3 + \frac{15}{7}x + 15x^2 + 2 - x + \cancel{\frac{5}{7}} - \frac{5}{7}x$$

$$= \frac{5}{7}x^3 + 15x^2 + x\left(\frac{15}{7} - 1 - \frac{5}{7}\right)$$

$$S(x) = \frac{5}{7}x^3 + 15x^2 + \frac{3}{7}x + 2, \quad 1 \le x \le 2$$

Case (ii) $2 < x < 3$.

Put $i = 2$.

$$S(x) = \frac{1}{6}\left[(x_2 - x)^3 M_1 - (x_1 - x)^3 M_2\right] + (x_2 - x)\left[y_1 - \frac{1}{6}M_1\right] - (x_1 - x)\left[y_2 - \frac{1}{6}M_2\right]$$

$$= \frac{1}{6}\left[(3-x)^3\frac{30}{7} - (2-x)\left(\frac{-36}{7}\right)\right] + (3-x)\left[0 - \frac{1}{6}\left(\frac{30}{7}\right)\right] - (2-x)\left[1 - \frac{1}{6}\left(-\frac{36}{7}\right)\right]$$

$$= \frac{1}{6}\left[\frac{30}{7}(3-x)^3 + \frac{36}{7}(2-x)\right] + (3-x)\left(-\frac{5}{7}\right)$$

$$- (2-x)\left[1+\frac{5}{7}\right]$$

$$= \frac{5}{7}\left[27-27x+9x^2-x^3\right] + \frac{6}{7}\left[4+x^2-4x\right]$$

$$-\frac{15}{7} + \frac{5x}{7} - \frac{26}{7} + \frac{13x}{7}$$

$$= x^3\left[-\frac{5}{7}\right] + x^2\left[\frac{45}{7} + \frac{6}{7} + \cancel{\frac{5}{7} + \frac{13}{7}}\right]$$

$$+ x\left[-135 - \frac{24}{7} + \frac{5}{7} + \frac{13}{7}\right] + \frac{135}{7} + \frac{24}{7} - \frac{15}{7}$$

$$-\frac{26}{7}$$

$$S(x) = -\frac{5}{7}x^3 + \frac{51}{7}x^2 - \frac{951}{7} + \frac{118}{7}, \quad 2 < x < 3$$

case (iii) $3 < x < 4$

Put $i = 3$

$$S(x) = \frac{1}{6}\left[(x_3 - x)^3 M_2 - (x_2 - x) M_3\right]$$

$$+ (x_3 - x)\left[y_2 - \frac{1}{6}M_2\right] - (x_2 - x)\left[y_3 - \frac{1}{6}M_3\right]$$

$$= \frac{1}{6}\left[(4-x)^3\left(-\frac{36}{7}\right) - (3-x)^3\left(\frac{30}{7}\right)\right]$$

$$+ (4-x)\left[1 - \frac{1}{6}\left(-\frac{36}{7}\right)\right]$$

$$-(3-x)\left[0 - \frac{1}{6}\left(\frac{30}{7}\right)\right]$$

$$= \frac{1}{6}\Big[-\frac{36}{7}[64 - 48x + 12x^2 - x^3]$$

$$-\frac{30}{7}[27 - 27x + 9x^2 - x^3]\Big]$$

$$+ (4-x)\left(1 + \frac{6}{7}\right) - (3-x)\left(-\frac{5}{7}\right)$$

$$= \cancel{\Big[\frac{-384}{7}}\quad {}^{13}/_{7}$$

$$= \frac{1}{7}\Big[-384 + 288x - 72x^2 + 6x^3 - 810$$

$$+ 810x - 270x^2 + 30x^3.$$

$$+ 52 - 13x + 15 - 5x\Big]$$

$$= \frac{1}{7}\Big[x^3[30+6] + x^2[-72-270]$$

$$+ x[288 + 810 - 13 - 5] +$$

$$[-384 - 810 + 52 + 15]\Big]$$

$$S(x) = \frac{1}{7}[36x^3 - 342x^2 + 1080x - 1127, \quad 3 \le x \le 4$$

case (v) $4 < x < 5$

Put $i = 4$.

$$S(x) = \frac{1}{6}[(x_3 - x)^3 M_3 - (x_2 - x) M_4]$$

$$+ (x_4 - x)\left[y_3 - \frac{1}{6}M_3\right]$$

$$- (x_3 - x)\left[y_4 - \frac{1}{6}M_4\right]$$

$$= \frac{1}{6}\left[(5-x)^3\left(\frac{30}{7}\right) - 0\right] + (5-x)\left[0 - \frac{1}{6}\left(\frac{30}{7}\right)\right]$$

$$+ (x-4)[1-0]$$

4) Find the cubic spline for the data

x	1	2	3
y	-6	-1	16

Hence evaluate $y(1.5)$ given that $y_0'' = y_2'' = 0$.

Soln

Given $h = 1$ $M_0 = M_2 = 0$

W.K.T

$$M_{i-1} + 4M_i + M_{i+1} = 6[y_{i-1} - 2y_i + y_{i+1}]$$

Put $i = 1$

$$M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$$

$$4M_1 = 6[-6 - 2(-1) + 16]$$

$$4M_1 = 72$$

$$\boxed{M_1 = 18}$$

The cubic spline polynomial is

$$S(x) = \frac{1}{6}[(x_i - x)^3 M_{i-1} - (x_{i-1} - x)^3 M_i]$$
$$+ (x_i - x)[y_{i-1} - \frac{1}{6}M_{i-1}]$$
$$- (x_{i-1} - x)[y_i - \frac{1}{6}M_i]$$

Case (i) $1 \le x \le 2$

Put $i = 1$

$$S(x) = \frac{1}{6}[(x_1 - x)^3 M_0 - (x_0 - x)^3 M_1]$$

$$= \frac{1}{6}\left[(2-x)^3(0) + (x-1)^3(18)\right] + (2-x)\left[-6-\frac{1}{6}(0)\right] + (x-1)\left[-1-\frac{1}{6}(18)\right]$$

$$= \frac{1}{6}\left[(x-1)^3(18)\right] + (2-x)(-6-0) + (x-1)(-1-3)$$

$$= 3(x^3-3x^2+3x-1) - 12 + 6x - 4x + 4$$

$$S(x) = 3x^3 - 9x^2 + 11x - 11$$

Case (ii) $\quad 2 \leq x \leq 3$

Put $i = 2$.

$$S(x) = \frac{1}{6}\left[(x_2-x)^3 M_1 - (x_1-x)^3 M_2\right] + (x_2-x)\left[y_1 - \frac{1}{6}M_1\right] - (x_1-x)\left[y_2-\frac{1}{6}M_2\right]$$

$$= \frac{1}{6}\left[(3-x)^3\, 18 - (2-x)^3(0)\right] + (3-x)\left[-1-\frac{1}{6}(18)\right] - (x-2)\left[16-\frac{1}{6}(0)\right]$$

$$= \frac{18}{6}\left[27 - 27x + 9x^2 - x^3\right] - 12 + 4x + 16x - 32$$

$$S(x) = -3x^3 + 27x^2 - 61x + 37$$

$$y = S(x) = \begin{cases} 3x^3 - 9x^2 + 11x - 11, & 1 \le x \le 2 \\ -3x^3 + 27x^2 - 61x + 37, & 2 \le x \le 3 \end{cases}$$

To find $y(1.5)$

$$S(1.5) = 3(1.5)^3 - 9(1.5)^2 + 11(1.5) - 11$$
$$= -4.625$$

Newton's forward interpolation formula (equal intervals).

$$y = f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

where $u = \frac{x-x_0}{h}$

(1) Using Newton's forward interpolation formula, find the polynomial $f(x)$ satisfying the following data. Hence evaluate y at $x=5$.

x	4	6	8	10
y	1	3	8	10

Soln

$u = \frac{x-x_0}{h}$, $h=2$

$u = \frac{x-4}{2}$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		-6
8	8		-3	
		2		

The Newton's forward interpolation form. is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + \left(\frac{x-4}{2}\right)(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)}{2} \times 3$$

$$+ \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)\left(\frac{x-4}{2}-2\right)}{3!} \times -6$$

$$= 1 + (x-4) + \frac{3(x-4)(x-6)}{8} - \frac{(x-4)(x-6)(x-8)}{8}$$

$$= \frac{1}{8}\left[8 + 8x - 32 + 3\left[x^2 - 10x + 24\right] - \left[x^3 - 18x^2 + 104x - 192\right]\right]$$

$$y = \frac{1}{8}\left[-x^3 + 21x^2 - 126x + 240\right].$$

Put $x = 5$

$$y(5) = \frac{1}{8}\left[-5^3 + 21 \times 5^2 - 126 \times 5 + 240\right]$$

$$\boxed{y(5) = 1\cdot 25}$$

(2) Fit a polynomial, by using Newtons forward interpolation formula to the data given below.

x	0 x_0	1 x_1	2 x_2	3 x_3
y	1 y_0	2 y_1	1 y_2	10 y_3

Soln

$$u = \frac{x - x_0}{h}, \quad h = 1$$

$$u = \frac{x - 0}{} = x$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
			−2	
1	2			12
		−1		
			10	
2	1			
		9		
3	10			

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + \frac{x}{1!}(2) + \frac{x(x-1)}{2!}(-1) + \frac{x(x-1)(x-2)}{3!}(10)$$

$$= 1 + 2x + \frac{(x^2 - x)}{2} + \frac{10}{6}\left[(x^2 - x)(x-2)\right]$$

$$= 1 + 2x + \frac{x^2}{2} - \frac{x}{2} + \frac{5}{3}\left[x^3 - 2x^2 - x^2 + 2x\right]$$

$$= \frac{5}{3}x^3 + x^2\left[\frac{1}{2} - \frac{10}{3}\right] + x\left[2 - \frac{1}{2} + \frac{10}{3}\right] + 1$$

(3) From the data given below find the number of students whose weight is between 60 to 70.

Weight in kg's	0-40	40-60	60-80	80-100	100-120
No. of Students	250	120	100	70	50

Soln

$$u = \frac{x - x_0}{h}, \quad h = 20$$

$$u = \frac{x-40}{20}$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		-20		
		100		-10	
Below 80	470		-30		20
		70		10	
Below 100	540		-20		
		50			
Below 120	590				

The Newton's forward interpolation formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{} \Delta^4 y_0 + \cdots$$

$$y = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 +$$

$$= 250 + \left(\frac{x-40}{20}\right)120 + \frac{\left(\frac{x-40}{20}\right)\left(\frac{x-40}{20}-1\right)}{2} \times -20$$

$$+ \frac{\left(\frac{x-40}{20}\right)\left(\frac{x-40}{20}-1\right)\left(\frac{x-40}{20}-2\right)}{6} \times -10$$

$$+ \frac{\left(\frac{x-40}{20}\right)\left(\frac{x-40}{20}-1\right)\left(\frac{x-40}{20}-2\right)\left(\frac{x-40}{20}-3\right)}{24} \times 20$$

$$y = 250 + 6(x-40) - 10\left(\frac{x-40}{20}\right)\left(\frac{x-60}{20}\right)$$

$$-\frac{5}{3}\left(\frac{x-40}{20}\right)\left(\frac{x-60}{20}\right)\left(\frac{x-80}{20}\right)$$

$$+\frac{5}{6}\left(\frac{x-40}{20}\right)\left(\frac{x-60}{20}\right)\left(\frac{x-80}{20}\right)\left(\frac{x-100}{20}\right)$$

$$y(70) = 250 + 6(70-40) - 10\left(\frac{70-40}{20}\right)$$

$$\left(\frac{70-60}{20}\right)\Big\} - \frac{5}{3}\left(\frac{70-40}{20}\right)\left(\frac{70-60}{20}\right)\left(\frac{70-80}{20}\right)$$

$$+\frac{5}{6}\left(\frac{70-40}{20}\right)\left(\frac{70-60}{20}\right)\left(\frac{70-80}{20}\right)\left(\frac{70-100}{20}\right)$$

$$= 250 + 180 - \frac{15}{2} + \frac{5}{8} + \frac{15}{32}$$

$$y(70) = 423.59 \simeq 424.$$

$$y(60) = 370.$$

No. of Students whose weight between 60–70 $\Big\} = y(70) - y(60)$

$= 424 - 370$

Newton's Backward Interpolation formula

$$y = y_0 + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$

Where $v = \frac{x - x_n}{h}$

(1) Use Newton's backward difference formula to construct an interpolating polynomial of degree 3 for the data.

$f(-0.75) = -0.07181250$, $f(-0.5) = -0.024750$

$f(-0.25) = 0.33493750$, $f(0) = 1.10100$.

Hence find $f(-\frac{1}{3})$.

Soln.

$v = \frac{x - x_n}{h}$ & $h = 0.25$

$v = \frac{x - 0}{0.25} = \frac{x}{0.25}$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
-0.75	-0.07181250			
		0.0470625		
-0.50	-0.024750		0.312625	
		0.3596875		0.09375
-0.25	0.33493750		0.40625	
		0.7660625		
0	1.10100			

The Newton's backward interpolation formula is

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

$$= 1.10100 + \left(\frac{x}{0.25}\right)(0.7660625) + \left(\frac{x}{0.25}\right)\left(\frac{x}{0.25}+1\right)(0.406375) + \frac{\left(\frac{x}{0.25}\right)\left(\frac{x}{0.25}+1\right)\left(\frac{x}{0.25}+2\right)}{3!}(0.09375)$$

$$= 1.10100 + (-1.33333)(0.7660625) + \frac{(-1.33333)(-0.33333)}{2}(0.406375) + \frac{(-1.33333)(-0.33333)(-0.66666)}{6}(0.09375)$$

$$= 1.10100 - 1.021414 + 0.090304426 + 0.0046295$$

$$y(-1/3) = 0.165260.$$

② The amount A of a substance remaining in a reacting system after an interval of time t in a certain chemical experiment

T (min)	2	5	8	11
A (gm)	94.8	87.9	81.3	75.1

Obtain the value of A where $t = 9$ mins using Newton's interpolation formula.

Soln

T x	A y	∇y	$\nabla^2 y$	$\nabla^3 y$
2	94.8			
		-6.9		
			0.3	
5	87.9			0.1
		-6.6		
			0.4	
8	81.3			
		-6.2		
11	75.1			

$$v = \frac{x - x_n}{h}, \quad h = 3$$

The Newton's Backward interpolation formula is

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

$$y = 75.1 + \left(\frac{x-11}{3}\right)(-6.2) + \frac{\left(\frac{x-11}{3}\right)\left(\frac{x-11}{3}+1\right)}{}(0.4)$$

$$y = 75.1 - 6.2\left(\frac{x-11}{3}\right) + \left(\frac{(x-11)(x-8)}{8}\right) \times 0.4$$

$$+ \frac{(x-11)(x-8)(x-5)}{162} \times 0.1$$

Put $x = 9$

$$y(9) = 75.1 - \frac{6.2(9-11)}{3} + \frac{(9-11)(9-8)}{18} \times 0.5$$

$$+ \frac{(9-11)(9-8)(9-5)}{162} \times 0.1$$

$$= 75.1 + \frac{62}{15} - \frac{2}{45} - \frac{2}{405}$$

$$y(9) = 79.1839.$$

8015290573

UNIT - 3

Numerical Differentiation and Integration

Numerical differentiation :

It is the Process of finding the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ & $\frac{d^3y}{dx^3}$, ... for some particular value of x.

① find the first derivatives of $f(x)$ at $x=2$ for the data $f(-1)=-21$, $f(1)=15$, $f(2)=12$, $f(3)=3$. using Newtons divided difference formula.

Soln

x	-1	1	2	3
y	-21	15	12	3

The Newton's divided difference formula is

$$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + \cdots$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

$$y = -21 + (x+1)\,18 + (x+1)(x-1)(-7) + (x+1)(x-1)(x-2)\,(1)$$

$$= -21 + 18x + 18 - 7(x^2-1) + (x^2-1)(x-2)$$

$$= -21 + 18x + 18 - 7x^2 + 7 + x^3 - 2x^2 - x + 2$$

$$y = x^3 - 9x^2 + 17x + 6$$

$$y' = 3x^2 - 18x + 17$$

$$y'(2) = -7$$

(2) find $f'(10)$ from the following data

x	3	5	11	27	34
$f(x)$	-13	23	899	17315	35606

The newton's divided difference formula is

$$y = f(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + \cdots$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
3	-13				
		18			
5	23		16		
		146		1	
11	899		40		0
		1026		1	
27	17315		69		
		2613			
34	35606				

$$y = f(x) = -13 + 18(x-3) + 16(x-3)(x-5) + (x-3)(x-5)(x-11)$$

$$= -13 + 18x - 54 + 16[x^2 - 8x + 15] + (x^2 - 8x + 15)(x-11)$$

$$= -13 + 18x - 54 + 16x^2 - 128x + 240 + x^3 - 11x^2 - 8x^2 + 88x + 15x - 165$$

$$f(x) = x^3 - 3x^2 - 7x + 8$$

$$f'(x) = 3x^2 - 6x - 7$$

$$f'(10) = 233.$$

Newton's forward formula for derivatives

$$y = f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$

$$y' = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \cdots \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$y'' = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(6u-6)}{3!}\Delta^3 y_0 + \frac{(12u^2-36u+22)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$y''' = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{(24u-36)}{4!}\Delta^4 y_0 + \cdots\right]$$

(1) Find the first three derivatives of $f(x)$ at $x = 1.5$ & at $x = 4.0$ using Newton's forward interpolation formula to the data given below.

x	1.5	2	2.5	3	3.5	4
y	3.375	7	13.625	24	38.875	59

Soln

$$f'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$f''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(6u-6)}{3!}\Delta^3 y_0 + \frac{(12u^2-36u+22)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$f'''(x) = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{(24u-36)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$u = \frac{x-x_0}{h} = \frac{x-1.5}{0.5}$$

When $x = 1.5$ $\boxed{u=0}$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.375					
		3.625				
2	7		3			
		6.625		0.75		
2.5	13.625		3.75		0	
		10.375		0.75		0
3	24		4.5		0	
		14.875		0.75		
3.5	38.875		5.25			
		20.125				
4	59					

$$f'(1.5) = \frac{1}{0.5}\left[3.625 + (0-1)\cdot\frac{3}{2} + \frac{2}{6}(0.75)\right]$$
$$= \frac{1}{0.5}\left[3.625 - 1.5 + 0.25\right]$$
$$= 4.75$$

$$f''(1.5) = \frac{1}{0.5^2}\left[3 + (-6)\times\frac{0.75}{6}\right]$$
$$= \frac{1}{0.5^2}\left[3 - 0.75\right] = 9$$

$$f'''(1.5) = \frac{1}{0.5^3}\left[0.75\right] = 6$$

Newton's Backward Interpolation formula

$$y' = \frac{1}{h}\left[\nabla y_n + \frac{(2v+1)}{2!}\nabla^2 y_n + \frac{(3v^2+6v+2)}{3!}\nabla^3 y_n + \frac{(4v^3+18v^2+22v+6)}{4!}\nabla^4 y_n + \cdots\right]$$

$$y'' = \frac{1}{h^2}\left[\nabla^2 y_n + (6v+6)\frac{\nabla^3 y_n}{3!} + \frac{(12v^2+36v+22)}{4!}\nabla^4 y_n + \cdots\right]$$

$$y''' = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{(24v+36)}{4!}\nabla^4 y_n + \cdots\right]$$

$$v = \frac{x - x_n}{h} = \frac{x-4}{0.5}$$

When $x = 4 \Rightarrow \boxed{v=0}$

$$f''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(6u-6)}{3!}\Delta^3 y_0 + \frac{(12u^2-36u+22)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$f'''(x) = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{(24u-36)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$u = \frac{x - x_0}{h} = \frac{x - 1.5}{0.5}$$

When $x = 1.5$ $\boxed{u = 0}$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.375					
		3.625				
2	7		3			
		6.625		0.75		
2.5	13.625		3.75		0	
		10.375		0.75		0
3	24		4.5		0	
		14.875		0.75		
3.5	38.875		5.25			
		20.125				
4	59					

$$y' = \frac{1}{0.5}\left[20.125 + \frac{1}{2}\times 5.25 + \frac{2}{6}\times 0.75\right]$$

$$= 46$$

$$y'' = \frac{1}{0.5^2}\left[5.25 + 6\times\frac{0.75}{6}\right] = 24$$

$$y''' = \frac{1}{0.5^3}\left[0.75\right] = 6.$$

(2) For the given data, find the first two derivatives at $x = 1.1$

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Soln

$$y' = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$y'' = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(6u-6)}{3!}\Delta^3 y_0 + \frac{(12u^2-36u+22)}{4}\Delta^4 y_0 + \cdots\right]$$

$$u = \frac{x-x_0}{h} = \frac{x-1.0}{0.1}$$

At $x = 1.1$ $\quad u = \frac{1.1-1.0}{0.1} = 1.$

$y' = \frac{1}{0.1}[0.4$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	Δ^6
1.0	7.989						
		0.4140					
1.1	8.403		-0.0360				
		0.3780		0.0060			
1.2	8.781		-0.03		-0.0020		
		0.3480		0.0040		0.001	
1.3	9.129		-0.0260		-0.0010		
		0.3220		0.003		0.003	
1.4	9.451		-0.0230		0.002		
		0.2990		0.0050			
1.5	9.750		-0.0180				
		0.2810					
1.6	10.031						0.00

$$y'(1.1) = \frac{1}{0.1}\left[0.414 + \frac{(2-1)}{2}(-0.0360) + \frac{(3-6+2)}{6}(0.006) + \frac{(4-18+22-6)}{24}(-0.002)\right]$$

$$= \frac{1}{0.1}\left[0.414 - 0.0180 - 0.0010 - 0.0002\right]$$

$$= 3.9480$$

$$y''(1.1) = \frac{1}{(0.1)^2}\left[(-0.0360) + \left(\frac{6-6}{6}\right)(0.0060) + \left(\frac{12-36+22}{24}\right)(-0.0020)\right]$$

$$= 100\left[-0.0360 + 0\right] + \frac{(-2)}{24}(-0.0020)\Big]$$

$= -36 + 0.00016$

$= \cancel{-35.9998}\ -3.584$

(3) find the first two derivatives of $x^{1/3}$ at $x = 50$ and $x = 56$ for the given data

x	50	51	52	53	54	55	56
$y = x^{1/3}$	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Soln

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
50	3.6840						
		0.0244					
51	3.7084		-0.0003				
		0.0241		0			
52	3.7325		-0.0003		0		
		0.0238		0		0	
53	3.7563		-0.0003		0		0
		0.0235		0		0	
54	3.7798		-0.0003		0		
		0.0232		0			
55	3.8030		-0.0003				
		0.0229					
56	3.8259						

Newton's forward formula:

$$y' = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0 + \cdots\right]$$

$= -36 + 0.00016$

$= \cancel{-35.9998}\ -3.584$

(3) find the first two derivatives of $x^{1/3}$ at $x = 50$ and $x = 56$ for the given data

x	50	51	52	53	54	55	56
$y = x^{1/3}$	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Soln

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
50	3.6840						
		0.0244					
51	3.7084		-0.0003				
		0.0241		0			
52	3.7325		-0.0003		0		
		0.0238		0		0	
53	3.7563		-0.0003		0		0
		0.0235		0		0	
54	3.7798		-0.0003		0		
		0.0232		0			
55	3.8030		-0.0003				
		0.0229					
56	3.8259						

Newton's forward formula:

$$y' = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$y'' = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{(6u-6)}{3!}\Delta^3 y_0 + \frac{(12u^2-36u+22)}{4!}\Delta^4 y_0 + \cdots\right]$$

$$u = \frac{x-x_0}{h} = \frac{50-50}{1} = 0$$

$$y' = \frac{1}{1}\left[0.02414 + \frac{(-1)}{2}(-0.0003)\right]$$

$$= 0.0244 + 0.0002$$

$$= 0.0246$$

$$y'' = \frac{1}{1}\left[-0.0003\right] = -0.0003$$

Newton's Backward Interpolation formula

$$y' = \frac{1}{h}\left[\nabla y_n + \frac{(2v+1)}{2!}\nabla^2 y_n + \frac{(3v^2+6v+2)}{3!}\nabla^3 y_n + \frac{(4v^3+18v^2+22v+6)}{4!}\nabla^4 y_n + \cdots\right]$$

$$y'' = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{(6v+6)}{3!}\nabla^3 y_n + \frac{(12v^2+36v+22)}{4!}\nabla^4 y_n + \cdots\right]$$

$$v = \frac{x-x_n}{h} = \frac{x-56}{0.5}$$

$$v = \frac{56-56}{0.5} = 0$$

$$y' = \frac{1}{0.5}\left[0.0299 + \frac{(0+1)}{2!}(-0.0003) + \frac{2}{3!}(0) + 0\right]$$

$$= \frac{1}{0.5}\left[0.0299 - \frac{0.0003}{2} + 0\right]$$

$$= 0.0595$$

$$y'' = \frac{1}{0.5^2}\left[-0.0003\right] = -0.0012$$

Numerical Integration

Trapezoidal rule

$$I = \int_a^b f(x)\,dx = \frac{h}{2}\left[(\text{sum of first and last ordinate}) + 2(\text{sum of remaining ordinates})\right]$$

$$h = \frac{b-a}{n}$$

Simpson's 1/3 rule

$$I = \int_a^b f(x)\,dx = \frac{h}{3}\left[(\text{first} + \text{last}) + 4(\text{sum of odd ordinates}) + 2(\text{sum of even ordinates})\right]$$

$$h = \frac{b-a}{n} - [\text{multiples of } 2]$$

Simpson's 3/8 rule

$$I = \frac{3h}{8}\left[(\text{first last}) + 2(\text{sum of multiples of } 3) + 3(\text{sum of non-multiples of } 3)\right]$$

$$h = \frac{b-a}{n} \quad [\text{multiples of } 3]$$

(1) Using Trapezoidal rule, evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ taking 8 intervals.

Soln

$$h = \frac{b-a}{n} = \frac{1+1}{8} = \frac{2}{8} = 0.25$$

x	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
y	0.5	0.65	0.8	0.9412	1	0.9412	0.8	0.64	0.5

$$I = \frac{h}{2}\left[(y_0+y_8) + 2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$= \frac{0.25}{2}\left[(0.5+0.5) + 2(0.65+0.8+0.9412+1+0.9412+0.8+0.64)\right]$$

$$= \frac{0.25}{2}\left[1 + 2(5.7624)\right]$$

$$= \frac{0.25}{2}\left[12.5248\right]$$

$$= 1.5656$$

2) Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ with $h = 1/6$ by Trapezoidal rule.

Soln

$$f(x) = \frac{1}{1+x^2} \qquad h = 1/6$$

x	0	1/6	2/6	3/6	4/6	5/6	1
y	1	36/37	9/10	4/5	9/13	36/61	1/2

$$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{(1/6)}{2}\left[(1+1/2) + 2\left(\frac{36}{37}+\frac{9}{10}+\frac{4}{5}+\frac{9}{13}+\frac{36}{61}\right)\right]$$

$$= \frac{1}{12}\left[\frac{3}{2} + 2(3.9554)\right]$$

$$= \frac{1}{12}\left[\frac{3}{2} + 7.9108\right]$$

$$= 0.7842$$

(3) Evaluate $\int_0^6 \frac{1}{1+x^2}\,dx$ by Trapezoidal rule. Also check up the results by actual integration.

Soln

$$f(x) = \frac{1}{1+x^2}, \quad h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

x	0	1	2	3	4	5	6
y	1.00	0.500	0.200	0.100	0.058824	0.038426	0.27026

$$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027027) + 2(0.5+0.2+0.1+0.058824+0.038462)\right]$$

$$= 1.41079950$$

By actual Integration

$$I = \int_0^6 \frac{1}{1+x^2}\,dx = [\tan^{-1}x]_0^6 = \tan^{-1}6 - \tan^{-1}0$$

$$= 1.4056476 5$$

(4) Evaluate $\int_{1.0}^{1.3} \sqrt{x}\,dx$ taking $h = 0.05$ by trapezoidal rule

Soln

$f(x) = \sqrt{x}$

$h = \frac{b-a}{n} = 0.05$

x	1.0	1.05	1.1	1.15	1.2	1.25	1.3
y	1	1.0247	1.0488	1.0724	1.0954	1.1180	1.1402

$$I = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.05}{2}\left[(1 + 1.402) + 2(1.0247 + 1.0488 + 1.0724 + 1.0954 + 1.1180)\right]$$

$$= 0.1\,[2.1402 + 2(5.3593)]$$

$$= 0.1\,[2.1402 + 10.7186]$$

$$= \cancel{0.1}\ 0.025\,[12.8588]$$

$$= \cancel{1.28588}\ 0.3214$$

$$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{(1/6)}{2}\left[(1+1/2) + 2\left(\frac{36}{37}+\frac{9}{10}+\frac{4}{5}+\frac{9}{13}+\frac{36}{61}\right)\right]$$

$$= \frac{1}{12}\left[\frac{3}{2} + 2(3.9554)\right]$$

$$= \frac{1}{12}\left[\frac{3}{2} + 7.9108\right]$$

$$= 0.7842$$

(3) Evaluate $\int_0^6 \frac{1}{1+x^2}\,dx$ by Trapezoidal rule. Also check up the results by actual Integration

Soln

$f(x) = \frac{1}{1+x^2}$, $h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

x	0	1	2	3	4	5	6
y	1.00	0.500	0.200	0.100	0.058824	0.038426	0.27026

$$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027027) + 2(0.5+0.2+0.1+0.058824+0.038462)\right]$$

$$= 1.41079950$$

By actual Integration

$$I = \int_0^6 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^6 = \tan^{-1}6 - \tan^{-1}0$$

$$= 1.40564765$$

(4) Evaluate $\int_{1.0}^{1.3} \sqrt{x}\,dx$ taking $h = 0.05$ by trapezoidal rule

Soln

$f(x) = \sqrt{x}$

$h = \frac{b-a}{n} = 0.05$

x	1.0	1.05	1.1	1.15	1.2	1.25	1.3
y	1	1.0247	1.0488	1.0724	1.0954	1.1180	1.1402

$$I = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.05}{2}\left[(1 + 1.402) + 2(1.0247 + 1.0488 + 1.0724 + 1.0954 + 1.1180)\right]$$

$$= 0.1\left[2.1402 + 2(5.3593)\right]$$

$$= 0.1\left[2.1402 + 10.7186\right]$$

$$= \cancel{0.1}\ 0.025\,(12.8588)$$

$$= \cancel{1.28588}\ 0.3214$$

(5) Dividing the range into 10 equal parts find the value of $\int_0^{\pi/2} \sin x \, dx$ by Simpson's $1/3$ rule.

Soln

$f(x) = \sin x \qquad h = \frac{b-a}{n} = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$

x	0	$\pi/20$	$2\pi/20$	$3\pi/20$	$4\pi/20$	$5\pi/20$	$6\pi/20$	$7\pi/20$	$8\pi/20$
$f(x)$	0	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9511

$$I = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{\pi/20}{3}\left[(0+1) + 4(0.1564 + 0.4540 + 0.7071 + 0.8910) + 2(0.3090 + 0.5878 + 0.8090)\right]$$

$$= \pi/60 * 19.0986 = 1$$

(6) The velocity v of a particle at a distance s from a point on its path is gn by the table below.

s	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by Simpson's $1/3$ rule.

Soln

$$\text{Velocity} = \frac{\text{distance}}{\text{time}}$$

$$v = \frac{ds}{dt}$$

$$dt = \frac{1}{v}\, ds$$

$$t = \int_0^{60} \frac{1}{v}\, ds \; , \; h = 10$$

$$I = \int_0^{60} \frac{1}{v}\, ds = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right]$$

v	47	58	64	65	61	52	38
$1/v$	0.02127	0.01724	0.015625	0.01538	0.01639	0.01923	0.026316

$$I = \frac{10}{3}\left[(0.02127 + 0.026316) + 4(0.07124 + 0.01538 + 0.01923) + 2(0.015625 + 0.01639) \right]$$

$$I = 1.06338$$

(7) Compute $\int_0^{\pi/2} \sin x \, dx$ using Simpson's $3/8^{th}$ rule of numerical integration.

Soln

$$I = \int_0^{\pi/2} \sin x \, dx$$

$f(x) = \sin x \qquad h = \frac{\pi/2 - 0}{9} = \frac{\pi}{18}$

x	0	$\pi/18$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$5\pi/18$	$6\pi/18$	$7\pi/18$	$8\pi/18$	$9\pi/18$
$f(x)$	0	0.1736	0.3420	0.50	0.6428	0.7660	0.8660	0.9397	0.9848	1

$$I = \frac{3h}{8}\left[(y_0 + y_9) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8) + 2(y_3 + y_6)\right]$$

$$= \frac{3\pi}{8 \times 18}\left[(0 + 1) + 3(0.1736 + 0.3428 + 0.6428 + 0.7660 + 0.9397 + 0.9848) + 2(0.5 + 0.8660)\right]$$

$I = 0.99998857$

$I \sim 1$

(7) The velocities of a car running on a straigh road at intervals of 2 minutes are gn below

Time(min)	0	2	4	6	8	10	12
Velocity (km/hr)	0	22	30	27	18	7	0

using Simpsons 1/3 rule find the distance covered by the car.

Soln

$$\text{Velocity} = \frac{dx}{dt} \quad (\text{ie}) \quad v = \frac{dx}{dt}$$

$$dx = v\,dt$$

$$x = \int v\,dt$$

t	0	2	4	6	8	10	12
v	0	$\frac{22}{60}$	$\frac{30}{60}$	$\frac{27}{60}$	$\frac{18}{60}$	$\frac{7}{60}$	0

$$I = \frac{h}{3}\left[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)\right]$$

$$= \frac{2}{3}\left[0 + 0 + 2\left(\frac{30}{60} + \frac{18}{60}\right) + 4\left(\frac{22}{60} + \frac{27}{60} + \frac{7}{60}\right)\right]$$

$$= 3.5556\ \text{Km}$$

Romberg Method

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right)$$

I_1 — Value of integral with $h = \frac{b-a}{2}$

I_2 — Value of integral with $h = \frac{b-a}{4}$

I_3 — " " " $h = \frac{b-a}{8}$.

① Compute $I = \int_0^{1/2} \frac{x}{\sin x}\,dx$, using Simpsons rule with $h = 1/4, 1/8, 1/16$ and then Romberg's Method.

Soln

$$I = \int_0^{1/2} \frac{x}{\sin x}\,dx$$

$$f(x) = \frac{x}{\sin x}$$

i) Take $h = 1/4$

x	0	1/4	1/2
$f(x)$	y_0 1	y_1 1.0105	y_2 1.0429

By Simpsons 1/3 rule,

$$I_1 = \frac{h}{3}\left[(y_0 + y_2) + 4(y_1) + 0\right]$$

$$= \frac{1}{12}\left[(1 + 1.0429) + 4(1.0105)\right]$$

$$I_1 = 0.507075$$

(ii) Take $h = 1/8$

x	0	1/8	2/8	3/8	4/8
$f(x)$	1 (y_0)	1.0026 (y_1)	1.0105 (y_2)	1.0238 (y_3)	1.0429 (y_4)

$$I_2 = \frac{h}{3}\left[(y_0+y_4) + 4(y_1+y_3) + 2(y_2)\right]$$

$$= \frac{1}{24}\left[(1 + 1.0429) + 4(1.0026 + 1.0236) + 2(1.01051)\right]$$

$$I_2 = 0.5070625$$

(iii) Take $h = \frac{1}{16}$

x	0	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{5}{16}$	$\frac{6}{16}$	$\frac{7}{16}$	$\frac{8}{16}$
$f(x)$	1 (y_0)	1.0007 (y_1)	1.0026 (y_2)	1.0059 (y_3)	1.0105 (y_4)	1.0165 (y_5)	1.0238 (y_6)	1.0326 (y_7)	1.0429 (y_8)

$$I_3 = \frac{h}{3}\left[(y_0+y_8) + 4(y_1+y_3+y_5+y_7) + 2(y_2+y_4+y_6)\right]$$

$$= \frac{1}{48}\left[(1+1.0429) + 4(1.0007+1.0059 + 1.0165 + 1.0326) + 2(1.0026 + 1.0105 + 1.0238)\right]$$

$$I_3 = 0.507729$$

for I_1, I_2

Romberg formula is

$$I_4 = I_2 + \left(\frac{I_2 - I_1}{3}\right)$$

$$= 0.5070625 + \left(\frac{0.5070625 - 0.507075}{3}\right)$$

$$I = 0.507058$$

for I_2, I_3

$$I_5 = I_3 + \left(\frac{I_3 - I_2}{3}\right)$$

$$= 0.5070729 + \left(\frac{0.5070729 - 0.5070625}{3}\right)$$

$$= 0.50707636 6.$$

Romberg for I_4 & I_5

$$I = I_5 + \left(\frac{I_5 - I_4}{3}\right)$$

(3) Evaluate $I = \int_0^1 \frac{dx}{1+x^2}$ by using Romberg's method. Hence deduce an approximate value of π

Soln

$a = 0 \; ; \; b = 1$

$f(x) = \frac{1}{1+x^2}$

I $h = \frac{b-a}{2} = \frac{1-0}{2} = 0.5$

x	0	0.5	1
$f(x)$	1	0.8	0.5

$I_1 = \frac{h}{2}\left[(y_0 + y_2) + 2(y_1)\right]$

$= \frac{0.5}{2}\left[(1 + 0.5) + 2 \times 0.8\right]$

$I_1 = 0.7750$

II $h = \frac{b-a}{4} = \frac{1-0}{4} = 0.25$

x	0	0.25	0.5	0.75	1
$f(x)$	1	0.9412	0.8	0.64	0.5

(5) Evaluate $I = \int_0^1 \frac{dx}{1+x^2}$ by using Romberg's method. Hence deduce an approximate value of π

Soln

$a = 0 \; ; \; b = 1$

$f(x) = \frac{1}{1+x^2}$

I $h = \frac{b-a}{2} = \frac{1-0}{2} = 0.5$

x	0	0.5	1
$f(x)$	1	0.8	0.5

$I_1 = \frac{h}{2}\left[(y_0 + y_2) + 2(y_1)\right]$

$= \frac{0.5}{2}\left[(1+0.5) + 2 \times 0.8\right]$

$I_1 = 0.7750$

II $h = \frac{b-a}{4} = \frac{1-0}{4} = 0.25$

x	0	0.25	0.5	0.75	1
$f(x)$	1	0.9412	0.8	0.64	0.5

$$I_2 = \frac{0.25}{2}\left[(1+0.5) + 2(0.9412 + 0.8 + 0.64)\right]$$

$$\boxed{I_2 = 0.7828}$$

(iii) $h = \frac{b-a}{8} = \frac{1-0}{8} = 0.125$

x	0	0.12	0.25	0.375	0.5	0.625	0.75	0.875	1
$f(x)$	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

$$I_3 = \frac{0.5}{2}\left[(1+0.5) + 2(0.9846 + 0.9412 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5664)\right]$$

$$\boxed{I_3 = 0.7848}$$

Romberg for I_1, I_2

$$I_4 = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.7854$$

Romberg for I_2, I_3

$$I_5 = I_3 + \left(\frac{I_3 - I_2}{3}\right) = 0.7855$$

Romberg for I_4, I_5

$$I = I_5 + \left(\frac{I_5 - I_4}{3}\right) = 0.7855$$

$$I = \int_0^1 \frac{dx}{1+x^2}$$

$$0.7855 = \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$\frac{\pi}{4} = 0.7855$$

$$\pi = 3.1420.$$

(3) Using Romberg Integration, evaluate $\int_0^1 \frac{dx}{1+x}$

Soln

I) Here $a=0,\ b=1$

x	0	0.5	1
$\frac{1}{1+x}$	1	0.6667	0.5

$$I_1 = \frac{h}{2}\left[(y_0 + y_2) + 2(y_1)\right]$$

$$= \frac{0.5}{2}\left[(1+0.5) + 2(0.6667)\right]$$

$$\boxed{I_1 = 0.7084}$$

II) $h = \frac{b-a}{4} = \frac{1-0}{4} = 0.25$

x	0	0.25	0.5	0.75	1
$f(x)$	1	0.8	0.6667	0.5714	0.5

$$I_2 = \frac{0.25}{2}\left[(1+0.5) + 2(0.8 + 0.6667 + 0.5714)\right]$$

$$\boxed{I_2 = 0.6970}$$

III) $h = \frac{b-a}{8} = \frac{1-0}{8} = 0.125$

x	0	0.125	0.25	0.375	0.5
$f(x)$	1	0.8889	0.8	0.7273	0.6667
		0.625	0.75	0.875	1
		0.6154	0.5714	0.5333	0.5

$$I_3 = \frac{0.125}{2}\left[(1+0.5) + 2(0.8889 + 0.8 + 0.7273 + 0.6667 + 0.6154 + 0.5714 + 0.5333)\right]$$

$$\boxed{I_3 = 0.6941.}$$

Romberg for I_1, I_2

$$I_4 = I_2 + \left(\frac{I_2 - I_1}{3}\right)$$

$$= 0.6970 + \left(\frac{0.6970 - 0.7084}{3}\right)$$

$$\boxed{I_4 = 0.6932}$$

Romberg for I_2, I_3

$$I_5 = I_3 + \left(\frac{I_3 - I_2}{3}\right)$$

$$= 0.6941 + \left(\frac{0.6941 - 0.6970}{3}\right).$$

$$\boxed{I_5 = 0.6931}$$

Romberg for I_4, I_5

$$I_6 = I_5 + \left(\frac{I_5 - I_4}{3}\right)$$

$$\boxed{I_6 = 0.6931}$$

Gauss Quadrature formula

Quadrature:

The Process of finding a definite integral from a tabulated values of a function is known as Quadrature.

Gaussian two point Quadrature formula

$$\text{Let} \quad I = \int_a^b f(x)\, dx.$$

$$\text{Take} \quad x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right) t$$

$$dx = \left(\frac{b-a}{2}\right) dt.$$

By using this transformation

$$I = \int_{-1}^{1} g(t)\, dt = g\left(\tfrac{-1}{\sqrt{3}}\right) + g\left(\tfrac{1}{\sqrt{3}}\right)$$

① Evaluate $\int_{-1}^{1} e^{-x^2} \cos x \, dx$ by Gauss two Point Quadrature formula.

Soln

$$I = \int_{-1}^{1} e^{-x^2} \cos x \, dx$$

$$f(x) = e^{-x^2} \cos x$$

$$I = f\left(\tfrac{-1}{\sqrt{3}}\right) + f\left(\tfrac{1}{\sqrt{3}}\right)$$

$$= e^{-\left(\frac{-1}{\sqrt{3}}\right)^2} \cos\left(\tfrac{-1}{\sqrt{3}}\right) + e^{-\left(\frac{1}{\sqrt{3}}\right)^2} \cos\left(\tfrac{1}{\sqrt{3}}\right)$$

$$= e^{-1/3} \cos\left(\tfrac{-1}{\sqrt{3}}\right) + e^{-1/3} \cos\left(\tfrac{1}{\sqrt{3}}\right)$$

$$= e^{-1/3} \left[\cos\left(-1/\sqrt{3}\right) + \cos\left(1/\sqrt{3}\right) \right]$$

$$I = 1.2008$$

② Apply Gauss two point formula to evaluate $\int_{-1}^{1} \frac{1}{1+x^2} dx$

Soln

$$I = \int_{-1}^{1} \frac{1}{1+x^2} dx$$

$$f(x) = \frac{1}{1+x^2}$$

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{1+\left(\frac{-1}{\sqrt{3}}\right)^2} + \frac{1}{1+\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{3}{4} + \frac{3}{4}$$

$$= \frac{6}{4}$$

$$= 1.5$$

(3) Evaluate the integral $I = \int_1^2 \frac{2x}{1+x^4}\,dx$ using Gaussian two point formula

Soln

$$I = \int_1^2 \frac{2x}{1+x^4}\,dx$$

$$f(x) = \frac{2x}{1+x^4}, \quad a = 1, \quad b = 2$$

$$x = \frac{a+b}{2} + \left(\frac{b-a}{2}\right)t$$

$$x = \frac{3}{2} + \frac{1}{2}t$$

$$dx = \tfrac{1}{2}\,dt$$

$$I = \int_{-1}^{1} \frac{\cancel{2}\left(\frac{3}{2}+\frac{1}{2}t\right)}{1+\left(\frac{3}{2}+\frac{1}{2}t\right)^4} \cdot \frac{dt}{\cancel{2}}$$

$$= \int_{-1}^{1} \frac{\left(\frac{3+t}{2}\right)}{1+\left(\frac{3+t}{2}\right)^4}\, dt$$

$$g(t) = \frac{\frac{3+t}{2}}{1+\left(\frac{3+t}{2}\right)^4}$$

$$I = g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\frac{3-\frac{1}{\sqrt{3}}}{2}}{1-\left(\frac{3-\frac{1}{\sqrt{3}}}{2}\right)^4} + \frac{\frac{3+\frac{1}{\sqrt{3}}}{2}}{1+\left(\frac{3+\frac{1}{\sqrt{3}}}{2}\right)^4}$$

$$= \frac{1.2113}{3.1530} + \frac{1.7887}{11.2359}$$

$$= 0.3842 + 0.1592$$

$$= 0.5434.$$

$= \frac{\pi}{4}[0.3259 + 0.9454]$

$= 0.9985$

Gaussian Three Point Quadrature formula:

$$I = \int_a^b f(x)\,dx$$

Take $x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t$

$$dx = \left(\frac{b-a}{2}\right)dt$$

$$I = \int_{-1}^{1} g(t)\,dt = \frac{5}{9}\left[g\left(-\sqrt{\tfrac{3}{5}}\right) + g\left(\sqrt{\tfrac{3}{5}}\right)\right] + \frac{8}{9}g(0)$$

① Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using 3 point Quadrature formula.

Soln

$$I = \int_0^1 \frac{dx}{1+x^2}$$

$f(x) = \frac{1}{1+x^2}$, $a = 0$, $b = 1$

Take $x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t$

$$dx = \left(\frac{b-a}{2}\right) dt$$

$$\Rightarrow x = \frac{1}{2} + \frac{1}{2}t$$

$$dx = \frac{1}{2} dt$$

$$I = \int_{-1}^{1} \frac{\frac{1}{2} dt}{1 + \left(\frac{1+t}{2}\right)^2} = \frac{1}{2}\int_{-1}^{1} \frac{dt}{1 + \left(\frac{1+t}{2}\right)^2}$$

$$\therefore \quad g(t) = \frac{1}{1 + \left(\frac{1+t}{2}\right)^2}$$

$$I = \frac{1}{2}\left[\frac{5}{9}\left[g\left(-\sqrt{\tfrac{3}{5}}\right) + g\left(\sqrt{\tfrac{3}{5}}\right)\right] + \frac{8}{9} g(0)\right]$$

$$= \frac{1}{2}\left[\frac{5}{9}\left(\frac{1}{1+\left(\frac{1+(-\sqrt{3/5})}{2}\right)^2} + \frac{1}{1+\left(\frac{1+(\sqrt{\frac{3}{5}})}{2}\right)^2}\right) + \frac{8}{9}\left[\frac{1}{1+\left(\frac{1}{2}\right)^2}\right]\right]$$

$$= \frac{1}{2}\left[\frac{5}{9}\left(0.9875 + 0.5595 + 0.7111\right]\right.$$

$$= 0.7853.$$

(2) Apply three point Gaussian Quadrature formula to evaluate $\int_0^1 \frac{\sin x}{x} dx$

Soln

$$I = \int_0^1 \frac{\sin x}{x} dx$$

$$f(x) = \frac{\sin x}{x}, \quad a = 0, \quad b = 1$$

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right) t$$

$$dx = \left(\frac{b-a}{2}\right) dt$$

$$\Rightarrow x = \tfrac{1}{2} + \tfrac{1}{2} t = \tfrac{1}{2}(1+t)$$

$$dx = \tfrac{1}{2} dt$$

$$I = \int_{-1}^{1} \frac{\sin \frac{1}{2}(1+t)}{\cancel{\frac{1}{2}}(1+t)} \cdot \cancel{\tfrac{1}{2}}\, dt$$

$$= \int_{-1}^{1} \frac{\sin \frac{1}{2}(1+t)}{(1+t)} dt$$

$$\therefore g(t) = \frac{\sin \frac{1+t}{2}}{1+t}$$

$$g(0) = \sin \tfrac{1}{2} = 0.47943$$

$$g\left(\sqrt{\tfrac{3}{5}}\right) = \sin\left[\frac{\sqrt{\frac{3}{5}}+1}{2}\right] \Big/ \sqrt{\tfrac{3}{5}}+1 = \frac{0.7754}{1.7746} = 0.437$$

$$g\left(-\sqrt{\tfrac{3}{5}}\right) = \frac{\sin\left[\frac{-\sqrt{\frac{3}{5}}+1}{2}\right]}{-\sqrt{\frac{3}{5}}+1} = \frac{0.1125}{0.2254} = 0.499$$

$$I = \frac{5}{9}\left[g\left(-\sqrt{\tfrac{3}{5}}\right) + g\left(\sqrt{\tfrac{3}{5}}\right)\right] + \frac{8}{9}g(0)$$

$$= \frac{5}{9}\left[0.499 + 0.437\right) + \frac{8}{9}(0.47943)$$

$$= 0.52 + 0.42616$$

$$= 0.94616$$

③ Evaluate $\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}\,dx$ by Gaussian three point formula.

Soln

$$I = \int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}$$

$$f(x) = \frac{x^2+2x+1}{1+(x+1)^4}, \quad a=0, \ b=2$$

$$x = \frac{b+a}{2} + \left(\frac{b-a}{2}\right)t$$

$$dx = \left(\frac{b-a}{2}\right)dt$$

$\Rightarrow \quad x = 1 + t$

$dx = dt$

$$I = \int_{-1}^{1} \frac{(z+1)^2 + 2(z+1)+1}{1+[(z+1)+1]^4}\, dz$$

$$g(t) = \frac{(z+1)^2 + 2(z+1)+1}{1+[(z+1)+1]^4}$$

$$= \frac{z^2+2z+1+2z+2+1}{1+(z+2)^4}$$

$$g(t) = \frac{(z+2)^2}{1+(z+2)^4}$$

$$g(0) = \frac{4}{17}$$

$$g\left(-\sqrt{\tfrac{3}{5}}\right) = \frac{\left(-\sqrt{\tfrac{3}{5}}+2\right)^2}{1+\left(-\sqrt{\tfrac{3}{5}}+2\right)^4} = \frac{1.50161}{3.2548} = 0.4614$$

$$g\left(\sqrt{\tfrac{3}{5}}\right) = \frac{\left(\sqrt{\tfrac{3}{5}}+2\right)^2}{1+\left(\sqrt{\tfrac{3}{5}}+2\right)^4} = \frac{7.69839}{60.2652} = 0.12774$$

$$I = \frac{5}{9}\left[g\left(-\sqrt{\tfrac{3}{5}}\right) + g\left(\sqrt{\tfrac{3}{5}}\right)\right] + \frac{8}{9}\, g(0).$$

$$= \frac{5}{9}\left[0.4614 + 0.12774\right] + \frac{8}{9}\left(\frac{4}{17}\right)$$

$$= \frac{5}{9}\left[0.4614\right.$$

$$= 0.5364$$

Double Integration

Trapezoidal rule:

$$I = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$I = \frac{hk}{4}\Big[\text{Sum of four Corners} + 2(\text{Sum of remaining boundary values}) + 4(\text{Sum of interior Values})\Big]$$

Simpson's rule

$$I = \frac{hk}{9}\Big[\text{Sum of four corners} + \underbrace{2(\text{Sum of odd position values}) + 4(\text{Sum of even position values})}_{\text{Boundary}}$$

$$+ \underbrace{4(\text{Sum of odd position values}) + 8(\text{Sum of even position values})}_{\text{odd rows}}$$

$$+ \underbrace{8(\text{Sum of odd position values}) + 16(\text{Sum of even position values})}_{\text{even rows}}\Big]$$

$\cancel{I} = \frac{hk}{4}\Big[$ ~~Sum of four corn~~

$$I = \frac{0.1\times 0.1}{4}\Big[0.5 + 0.4167 + 0.4545 + 0.3846$$
$$+ 2(0.4762 + 0.4545 + 0.4348 + 0.4762$$
$$+0.4 + 0.4348 + 0.4167 + 0.4)$$
$$+ 4(0.4545 + 0.4348 + 0.4167)\Big]$$

$$= \frac{0.1\times 0.1}{4}\left[1.7558 + 6.9864 + 5.2240\right]$$

$$= \frac{0.1\times 0.1}{4} \times 13.9662 = 0.0349$$

(2) Evaluate $\int_1^2\int_1^2 \frac{1}{x^2+y^2}\,dx\,dy$, numerically with $h=0.2$, along x-direction and $k=0.25$ along y-direction

Soln

$$I = \int_1^2\int_1^2 \frac{1}{x^2+y^2}\,dx\,dy$$

$$f(x,y) = \frac{1}{x^2+y^2}$$

By Trapezoidal

$$I = \frac{hk}{4}\Big[\text{Sum of four corners} + 2(\text{Sum of remaining boundary}) + 4(\text{Sum of interiors})\Big]$$

y \ x	1	1.2	1.4	1.6	1.8	2
1	0.5	0.4098	0.3378	0.2809	0.2359	0.2
1.25	0.3902	0.3331	0.2839	0.2426	0.2082	0.1798
1.5	0.3077	0.2710	0.2375	0.2079	0.1821	0.16
1.75	0.2462	0.2221	0.1991	0.1779	0.1587	0.1416
2	0.2	0.1838	0.1679	0.1524	0.1381	0.125

$$I = \frac{(0.2)(0.25)}{4}\Big[0.5+0.2+0.2+0.125 + 2(0.4098+0.3378+0.2809+0.2359 + 0.1798+0.16+0.1416 + 0.1381+0.1524+0.1679+0.1838 + 0.2462+0.2710+0.3331) + 4(0.3331+0.2839+0.2426 + 0.2082+0.2710+0.2375 + 0.2079+0.1821+0.2221 + 0.1991+0.1779+0.1587)\Big]$$

$$= \frac{(0.2)(0.25)}{4}\left[1.025+6.6642+10.8964)\right]$$

$$= 0.2323.$$

3. Evaluate $I = \int_0^{1/2}\int_0^{1/2} \frac{\sin(xy)}{1+xy}\,dx\,dy$ using Simpson's rule with $h = k = 1/4$

Soln

$$I = \int_0^{1/2}\int_0^{1/2} \frac{\sin(xy)}{1+xy}\,dx\,dy$$

$$f(x,y) = \frac{\sin xy}{1+xy}$$

By Simpsons 1/3 rule,

$$I = \frac{hk}{9}\Big[\text{Sum of four corners} + 2(\underbrace{\text{Sum of odd position}) + 4(SEP)}_{\text{Boundary}}$$

$$+ \underbrace{4(SOP) + 8(SEP)}_{\text{odd rows}}$$

$$+ \underbrace{8(SOP) + 16(SEP)}_{\text{even rows}}\Big]$$

	y \ x	0	1/4	1/2
I	0	0	0	0
II	1/4	0	0.0588	0.1108
III	1/2	0	0.1108	0.1979

$$I = \frac{0.1 \times 0.1}{9}\Big[0.5 + 0.4167 + 0.3571 + 0.2976$$
$$+ 2[0.4545 + 0.4167 + 0.3247 + 0.3472]$$
$$+ 4[0.4762 + 0.4348 + 0.3788 + 0.3205$$
$$+ 0.3401 + 0.3106 + 0.4545 + 0.3816]$$
$$+ 4(0.3788) + 8(0.3968 + 0.3623)$$
$$+ 8(0.4132 + 0.3497)$$
$$+ 16(0.4329 + 0.3953$$
$$+ 0.3663 + 0.3344)\Big]$$

$$= \frac{0.1 \times 0.1}{9}\Big[1.5714 + 3.0862 + 12.4004$$
$$+ 1.5152 + 6.0728 + 6.1032$$
$$+ 24.4624\Big]$$

$I = 0.0613$

5 Evaluate $\int_0^2 \int_0^1 4xy \, dx \, dy$ using Simpsons rule by taking $h = 1/4$ & $k = 1/2$

Soln

$$I = \int_0^2 \int_0^1 4xy \, dx \, dy$$

Here $f(x, y) = 4xy$

$h = 0.25 \quad k = 0.5$

y \ x	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.5	0	0.5	1	1.5	2
1	0	1	2	3	4
1.5	0	1.5	3	4.5	6
2	0	2	4	6	8

$$I = \frac{0.25 \times 0.5}{9}\left[8 + 16 + 64 + 8 + 32 + 32 + 128\right]$$

$$I = 4.$$

Unit - IV

Initial Value Problem for ordinary differential Equation

Method - 1

Taylor Series:

The taylor Series formula is

$$y = y_0 + (x-x_0)\frac{y_0'}{1!} + (x-x_0)^2\frac{y_0''}{2!} + (x-x_0)^3\frac{y_0'''}{3!} + \cdots$$

1. Use taylor series method to find $y(0.1)$ and $y(0.2)$. Given that $\frac{dy}{dx} = 3e^x + 2y$; $y(0) = 0$;

Soln: Given $\frac{dy}{dx} = y' = 3e^x + 2y$; $y(0) = 0$;

The taylor series formula is,

$$y = y_0 + (x-x_0)\frac{y_0'}{1!} + (x-x_0)^2 \frac{y_0''}{2!} + (x-x_0)^3 \frac{y_0'''}{3!} + \frac{(x-x_0)^4}{}$$

x	0	x_0
y	0	y_0
$y' = 3e^x + 2y$	3	y_0'
$y'' = 3e^x + 2y'$	9	y_0''
$y''' = 3e^x + 2y''$	21	y_0'''
$y^{iv} = 3e^x + 2y'''$	45	y_0^{iv}

$$y = 0 + (x-0) \cdot 3/1 + (x-0)^2 \cdot 9/2 + (x-0)^3 \frac{21}{6} +$$

$$(x-0)^4 \cdot \frac{45}{24};$$

$$y = 3x + 9/2\, x^2 + 7/2\, x^3 + 15/8\, x^4$$

$y(0.1) = 0.3487$.

$y(0.2) = 0.8110$.

2. use taylor series method, solve $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$ at $x = 0.1, 0.2, 0.3$.

Soln:

The taylor series formula is,

$$y = y_0 + (x-x_0)\frac{y_0'}{1!} + (x-x_0)^2\frac{y_0''}{2!} + (x-x_0)^3\frac{y_0'''}{3!} + (x-x_0)^4\frac{y_0^{iv}}{4!}$$

$y' = x^2 - y$; $y(0) = 1$

x	0	x_0
y	1	y_0
$y' = x^2 - y$	-1	y_0'
$y'' = 2x - y'$	1	y_0''
$y''' = 2 - y''$	1	y_0'''
$y^{iv} = -y'''$	-1	y_0^{iv}

$$y = 1 + (x-0)\left(\frac{-1}{1!}\right) + (x-0)^2\frac{1}{2!} + (x-0)^3 \times \frac{1}{3!} + (x-0)^4\left(\frac{-1}{4!}\right)$$

$$y = 1 - x + x^2/2 + x^3/6 - \frac{x^4}{24}.$$

$y(0.1) = \cancel{0.9100}$. 0.9052

$= \cancel{7/6\, x^4 + 4/3\, x^3}$

$y = 7/6\, x^4 + 4/3\, x^3 + x^2 + x + 1$

$y(0.1) = 1.1115$

$y(0.2) = 1.2525$

4. Obtain y by taylor series method given that $y' = xy + 1$; $y(0) = 1$; for $x = 0.1$; $x = 0.2$; Correct to four decimal places.

Soln: The formula is,

$$y = y_0 + (x - x_0)\frac{y_0'}{1!} + (x - x_0)^2 \frac{y_0''}{2!} + (x - x_0)^3 \frac{y_0'''}{3!} + (x - x_0)^4 \frac{y_0^{iv}}{4!} + \dots$$

x	0	x_0
y	1	y_0
$y' = xy + 1$	1	y_0'
$y'' = y + xy'$	1	y_0''
$y''' = y' + y' + xy''$	2	y_0'''
$y^{iv} = y'' + y'' + y'' + xy'''$	3	y_0^{iv}

$$y = 1 + (x - 0)\frac{1}{1!} + (x - 0)^2 \frac{1}{2} + (x - 0)^3 \frac{2}{6} + (x - 0)^4 \frac{3}{\ }$$

$$y = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{8}x^4.$$

$y(0.1) = 1.1053$

$y(0.2) = 1.2229.$

5. G.T $y'' + xy' + y = 0$; $y(0) = 1$; $y'(0) = 0$ Obtain the value of y for $x = 0.1$ & $x = 0.2$; 0.3 by taylor series method.

Soln:

The taylor series formula is,

$$y = y_0 + (x-x_0)\frac{y_0'}{1!} + (x-x_0)^2\frac{y_0''}{2!} + (x-x_0)^3\frac{y_0'''}{3!} + (x-x_0)^4\frac{y_0^{iv}}{4!} + \cdots$$

x	0	x_0
y	1	y_0
y'	0	y_0'
$y'' = -xy' - y.$	-1	y_0''
$y''' = -xy'' - y' - y'$	0	y_0'''
$y^{iv} = -xy''' - y'' - y'' - y''$	$+3$	y_0^{iv}

$$y = 1 + (x-0)\frac{0}{1!} + (x-0)^2\frac{-1}{2} + (x-0)^3\frac{0}{6} + (x-0)^4\frac{+3}{24}$$

$$y = 1 - x^2/2 + x^4/8$$

$y(0.1) = 0.9950$

$y(0.2) = 0.9802$

$y(0.3) = 0.9560$

Method - II : Euler's method :

Consider $\frac{dy}{dx} = f(x, y)$

The Euler's formula is,

$$y_{n+1} = y_n + h f(x_n, y_n) \text{ (or)}$$

$$y_{n+1} = y_n + h y_n'$$

1. Solve $y' = \frac{y-x}{y+x}$, $y(0) = 1$ at $x = 0.1$ by taking $h = 0.02$; by using Euler's method.

Soln :

$$y' = \frac{y-x}{y+x} \; ; \; y(0) = 1$$

The Euler's formula is,

$$y_{n+1} = y_n + h f(x_n, y_n)$$

(or)

$$y_{n+1} = y_n + h \cdot y_n'$$

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x	0	0.02	0.04	0.06	0.08	0.
y	1	1.02	1.0392	1.0577	1.0756	1.
$y' = \frac{y-x}{y+x}$	1	0.9615	0.9259	0.8926	0.8615	0

$n=0$;

$y_1 = y_0 + h y_0' = 1 + 0.02 \times 1 = 1.02$.

$n=1$;

$y_2 = y_1 + h y_1' = 1.02 + 0.02 \times 0.9615 = 1.0392$.

$n=2$;

$y_3 = y_2 + h \cdot y_2' = 1.0392 + 0.02 \times 0.9259 = 1.0577$.

$n=3$;

$y_4 = y_3 + h y_3' = \cancel{0.9} \cdot 1.0577 + 0.02 \times 0.8926$

$y_4 = 1.0756$

$n=4$;

$y_5 = y_4 + h y_4' = 1.0756 + 0.02 \times 0.8615$

$y_5 = 1.0928$.

~~$n=5$;~~

~~$y_6 = y_5 + h y_5' = 1.0928 + 0.$~~

2\. Using Euler's method to find $y(0.4)$ Gi[ven]

$\frac{dy}{dx} = x+y$, $y(0) = 1$. Taking $h = 0.2$.

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Soln:

Given $\frac{dy}{dx} = x+y$, $y(0)=1$.

The Euler's formula is $y_{n+1} = y_n + h y_n'$

x	0	0.2	0.4
y	1	1.2	1.48
$y' = x+y$	1	1.4	1.88

$n=0 \Rightarrow y_1 = y_0 + h y_0' = 1 + (0.2 \times 1) = 1.2$

$n=1 \Rightarrow y_2 = y_1 + h y_1' = 1.2 + (0.2 \times 1.4) = 1.48$

3. Using Euler's method find the solution of the initial value problem (IVP) $\frac{dy}{dx} = \log_e(x+y)$, $y(0)=2$ at $x=0.6$ by assuming $h=0.2$.

Soln:

Given $y' = \log_{10}(x+y)$; $y(0)=2$.

The Euler's formula is $y_{n+1} = y_n + h y_n'$

x	0	0.2	0.4	0.6
y	2	2.0602	2.1810	2.2117
$y' = \log_{10}(x+y)$	0.3010	0.3541	0.4033	0.4490

$n=0 \Rightarrow y_1 = y_0 + h y_0' = 2 + (0.2 \times 0.3010) = 2.0602$

$n=1 \Rightarrow y_2 = y_1 + h y_1' = 2.0602 + (0.2 \times 0.3541) = 2.1810$

$n=2 \Rightarrow y_3 = y_2 + h y_2' = 2.1810 + (0.2 \times 0.4033) = 2.2117$

4. Using Euler's method, find $y(4.1)$ & $y(4.2)$ if $5x\frac{dy}{dx} + y^2 - 2 = 0$; $y(4)=1$

Soln :

Given $5\frac{dy}{dx}+y^2-2=0$; $y(4)=1$

$$\frac{dy}{dx}=\frac{-y^2+2}{5x}$$

The Euler's formula is $y_{n+1}=y_n+hy_n'$

x	4	4.1	4.2
y	1	1.0050	1.0098
$y'=\frac{-y^2+2}{5x}$	0.05	0.0483	0.0467.

$n=0 \Rightarrow y_1=y_0+hy_0'=1+0.1(0.05)$

$=1.0050.$

$n=1 \Rightarrow y_2=y_1+hy_1'=1.005+0.1(0.0483)$

$=1.0098$ //

5. find $y(0.2)$ for $y'=y+e^x$, $y(0)=0$ by Euler's method. Take $h=0.1$

Soln :

Given $y'=y+e^x$, $y(0)=0$

The Euler's formula is $y_{n+1}=y_n+hy$

x	0	0.1	0.2
y	0	0.1	0.2205
			1.4415

$n=0 \Rightarrow$

$$y_1 = y_0 + h y_0' = 0 + 0.1(1) = 0.1$$

$n=1 \Rightarrow$

$$y_2 = y_1 + h y_1' = 0.1 + 0.1 \times (1.2052) = 0.2205.$$

Fourth order Range-kutta method.

Consider $g(x, y, y') = 0$.

$$y' = f(x, y)$$

$$k_1 = h f(x, y)$$

$$k_2 = h f(x + h/2, y + k_1/2)$$

$$k_3 = h f(x + h/2, y + k_2/2)$$

$$k_4 = h f(x+h, y+k_3)$$

$$y = y_0 + 1/6 (k_1 + 2k_2 + 2k_3 + k_4)$$

7. using Runge-kutta method of order 4; find y value when $x = 1.2$ in steps of 0.1 given that $y' = x^2 + y^2$, $y(1) = 1.5$.

soln:

The Runge-kutta formula is

$$k_1 = h \cdot f(x, y)$$

$$k_2 = h \cdot f(x + h/2, y + k_1/2)$$

$$k_3 = h \cdot f(x + h/2, y + k_2/2)$$

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$k_4 = h \cdot f(x+h, y+k_3)$

Given $y' = x^2+y^2$

here, $f(x,y) = x^2+y^2$; $h = 0.1$

x	1	1.1	1.2
y	1.5	y_1 1.8955	y_2 2.5044.

To find y_1

$x = 1$; $y = 1.5$.

$k_1 = h\,f(x,y) = 0.1 \times f(1, 1.5)$.

$= 0.1 \times 3.25 = 0.325$.

$k_2 = h \cdot f(x+h/2, y+k_1/2) = 0.1 \times f(1.05, 1.662)$

$= 0.1 \times 3.8664 = 0.3866$.

$k_3 = h \cdot f(x+h/2, y+k_2/2) = 0.1\,f(1.05, 1.6933)$

$= 0.1 \times 3.9698 = 0.3970$.

$k_4 = h \cdot f(x+h, y+k_3) = 0.1 \times f(1.1, 1.8970)$

$= 0.4809$.

$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$= 1.5 + \frac{1}{6}\left[0.325 + 2 \times 0.3866 + 2 \times 0.3970 + 0.4809\right]$

$y_1 = 1.8955$

$f(x,y) = x^2 + y^2$

$k_1 = h \cdot f(x,y) = 0.1 \times f(1.1, 1.8955)$

$= 0.1 \times 4.8029 = 0.4803$

$k_2 = h \, f(x + h/2, y + k_1/2)$ $= \cancel{0.1 \times}$

$= 0.1 \, f(1.150, 2.1357)$

$= 0.1 \times 5.8837 = 0.5884$

$k_3 = h \, f(x + h/2, y + k_2/2)$

$= 0.1 \times f(1.150, 2.1097)$

$= 0.1 \times 6.1178 = 0.6117$

$k_4 = h \, f(x+h, y+k_3)$

$= 0.1 \times f(1.2, 2.5072)$

$= 0.7726$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.8955 + \frac{1}{6}[0.4803 + 2 \times 0.5884 + 2 \times 0.6117 + 0.7726]$

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2. Find $y(0.7)$ & $y(0.8)$ given that $y' = y - x^2$ $y(0.6) = 1.7379$ by using RK method of 4th order.

Soln:

$k_1 = h \cdot f(x, y)$

$k_2 = h \cdot f(x + h/2, y + k_1/2)$

$k_3 = h \cdot f(x + h/2, y + k_2/2)$

$k_4 = h \cdot f(x + h, y + k_3)$

given $y' = y - x^2$.

Here $f(x,y) = y - x^2$; $h = 0.1$

	x_0	x_1	x_2
x	0.6	0.7	0.8
y	1.7379	1.8763	2.0145.

To find y_1 :

$x = 0.6$; $y = 1.7379$.

$k_1 = h \cdot f(x, y) = 0.1 \times f(0.6, 1.7379)$

$= 0.1378$.

$k_2 = 0.1 \times f(0.6 + 0.1/2, 1.7379 + 0.1378/2)$

$k_2 = \cancel{0.0240}\ 0.1384$

$k_3 = 0.1 \times f\left[0.6 + \frac{0.1}{2}, 1.7879 + 0.1384/2\right]$

$= 0.1 \times f(0.65, 1.8071)$

$= 0.1385$

$k_4 = 0.1 \times f(0.7, 1.8764)$

$= 0.1386$

$y_1 = \overset{1.7879}{\cancel{y_0}} + 1/6\ (0.1378 + 0.1384 \times 2 + 0.1385 \times 2 + 0.1386)$

$= 1.8763$

To find y_2.

$x = 0.7\ ;\ y = 1.8763$

$k_1 = 0.1 \times f(0.7, 1.8763) = 0.1386$

$k_2 = 0.1 \times f(0.75, 1.9456) = 0.1383$

$k_3 = 0.1 \times f(0.75, 1.9455) = 0.1383$

$k_4 = 0.1 \times f(0.8, 2.0146) = 0.1375$

$y_2 = y_1 + 1/6\ [k_1 + 2k_2 + 2k_3 + k_4]$

$= 1.8763 + 1/6\ (0.1386 + 2 \times 0.1383 + 2 \times 0.1383 + \overset{0.1375}{\cancel{0.1402}})$

8. using R.K method to find $y(0.2)$, $y(0.4)$. Given $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$, $y(0)=1$

Soln:

$$y' = \frac{y^2-x^2}{y^2+x^2}$$

Here, $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$; $h=0.2$

x	0	0.2	0.4
y	1	1.1960	

To find y_1:

$x=0$; $y=1$

$k_1 = h \cdot f(x,y) = 0.2 \times f(0,1)$

$= 0.2$

$k_2 = 0.2 \times f(0.1, 1.1) = 0.1967.$

$k_3 = 0.2 \times f(0.1, 1.0984) = 0.1967.$

$k_4 = 0.2 \times f(0.2, 1.1967) = 0.1891$

$y_1 = 1 + \frac{1}{6}(0.2 + 2\times 0.1967 + 2\times 0.1967 + 0.1891)$

$= 1.1960$

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To find y_2 :-

$x = 0.2 ; \ y = 1.1960$

$k_1 = 0.2 \times f(0.2, 1.1960) = 0.1891$

$k_2 = 0.2 \times f(0.3, 1.2906) = 0.1795$

$k_3 = 0.2 \times f(0.3, 1.2842) = 0.1798$

$k_4 = 0.2 \times f(0.4, 1.3753) = 0.1688$

$y_2 = 1.1960 + \frac{1}{6}(0.1891 + 2 \times 0.1763 + 0.1793 \times 2 + 0.1688)$

$= 1.3753$

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17/3/14. Using R.K method for solving simultaneous Equation:

consider,

$$\frac{dy}{dx} = f(x, y, z); \quad \frac{dz}{dx} = g(x, y, z)$$

$f(x,y,z)$	$g(x,y,z)$
$k_1 = h \cdot f(x, y, z)$	$l_1 = h \cdot g(x, y, z)$
$k_2 = h \cdot f(x+h/2, y+k_1/2, z+l_1/2)$	$l_2 = h \cdot g(x+h/2, y+k_1/2, z+l_1/2)$
$k_3 = h \cdot f(x+h/2, y+k_2/2, z+l_2/2)$	$l_3 = h \cdot g(x+h/2, y+k_2/2, z+l_2/2)$
$k_4 = h \cdot f(x+h, y+k_3, z+l_3)$	$l_4 = h \cdot g(x+h, y+k_3, z+l_3)$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$z_1 = z_0 + 1/6[l_1 + 2l_2 + 2l_3 + l_4]$$

1. Solve for $y(0.1)$ and $z(0.1)$ from the simultaneous equation $\frac{dy}{dx} = 2y + z$; $\frac{dz}{dx} = y - 3z$ $y(0) = 0$; $z(0) = 0.5$; using R.K method of order 4.

Soln:

Given, $\frac{dy}{dx} = y - 3z$; $g(x, y, z) = y - 3z$

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x	0	0.1
y	0	0.0481
z	0.5	0.3726

$h = 0.1$

$f(x,y,z) = 2y+z$	$g(x,y,z) = y-3z$
$k_1 = h.f(x,y,z)$ $= 0.1 \times f(0,0,0.5)$ $k_1 = 0.05$	$l_1 = 0.1 \times g(0,0,0.5)$ $l_1 = -0.15$
$k_2 = h.f(x+h/2, y+k_1/2, z+l_1/2)$ $= 0.1 \times f(0.05, 0.025, 0.425)$ $k_2 = 0.0475$	$l_2 = 0.1 \times g(0.05, 0.025, 0.4$ $l_2 = -0.125$
$k_3 = h.f(x+h/2, y+k_2/2, z+l_2/2)$ $= 0.1 \times f(0.05, 0.0238, 0.4375)$ $k_3 = 0.0485$	$l_3 = 0.1 \times g(0.05 \times 0.238 \times 0.4375)$ $l_3 = -0.1289$
$k_4 = h.f(x+h, y+k_3, z+l_3)$ $= 0.1 \times f(0.1, 0.0485, 0.3711)$ $= 0.0468$	$l_4 = 0.1 \times g(0.1, 0.0485, 0.37$ $= -0.1065$

$y_1 = y_0 + 1/6\,(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0 + 1/6\,(0.05 + 2\times 0.0475 + 2\times 0.0485 + 0.0460)$

$= 0.0481$

$z_1 = 0.5 + 1/6\,(-0.15 - 2\times 0.125 - 2\times 0.1289 - 0.1065)$

$= 0.3726$

R.K method for solving second order equation.

Consider, $\varphi(x, y, y', y'') = 0$ —— ①

take $y' = z$ —— ②

By using ② in ①, we get

$$z' = g(x, y, z)$$

1. Given $y'' + xy' + y = 0$; $y(0) = 1$; $y'(0) = 0$; find the value of $y(0.1)$ by using R.K method

Soln:

Given, $y'' + xy' + y = 0$ —— ①

Take $y' = z$;

$$z' + xz + y = 0.$$

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$z' = -xz - y$.

x	0	0.1
y	1	0.9950
$z = y'$	0	-0.0995

$h = 0.1$.

$f(x,y,z) = z$	$g(x,y,z) = -xz - y$.
$k_1 = h \cdot f(x,y,z)$ $= 0.1 \times f(0,1,0)$ $k_1 = 0$.	$l_1 = 0.1 \times g(0,1,0)$ $= -0.1$
$k_2 = h \cdot f(x + h/2, y + k_1/2, z + l_1/2)$. $= 0.1 \times f(0.05, 1, -0.05)$ $= -0.005$.	$l_2 = 0.1 \times g(0.05, 1, -0.05)$ $= -0.0998$.
$k_3 = h \cdot f(x + h/2, y + k_2/2, z + l_2/2)$. $= 0.1 \times f(0.05, 0.9975, -0.0499)$ $= -0.005$.	$l_3 = 0.1 \times g(0.05, 0.9975, -0.0499)$ $= -0.0995$.
$K_4 = h \cdot f(x+h, y+k_3, z+l_3)$ $= 0.1 \times f(0.1, 0.9950, -0.0995)$ $= -0.0100$	$l_4 = 0.1 \times g(0.1, 0.9950, -0.0995)$ $= \cancel{0.1015} = -0.0985$.

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$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + 1/6\,(0 - 2\times 0.005 - 2\times 0.005 - 0.01)$$

$$= 0.9950 \;//.$$

$$z_1 = 0 + 1/6\,(-0.1 - 2\times 0.0998 - 2\times 0.0997 - 0.0985)$$

$$= -0.0995 \;//.$$

2 Consider the 2^{nd} Order initial value pbm: $y'' - 2y' + 2y = e^{2t}\sin t$; $y(0) = -0.4$; $y'(0) = -0.6$ using R 4th order R.K method find $y(0.2) = ?$

Soln:

Given $y'' - 2y' + 2y = e^{2x}\sin x$

Take $y' = z$.

$$f(x, y, z) = z.$$

$$z' - 2z + 2y = e^{2x}\sin x.$$

$$z' = e^{2x}\sin x - 2y + 2z.$$

$$g(x, y, z) = e^{2x}\sin x - 2y + 2z.$$

x	0	0.2
y	-0.4	
$z = y'$	-0.6	

$h = 0.2$

$f(x,y,z) = z$	$g(x,y,z) = e^{2x}\sin x - 2y + 2z$
$k_1 = h \cdot f(x,y,z)$ $= 0.2 \times f(0,-0.4,-0.6)$ $= -0.12$	$l_1 = 0.2 \times g(0,-0.4,-0.6)$ $l_1 = -0.08$
$k_2 = h \cdot f(x+h/2, y+k_1/2, z+l_1/2)$ $= 0.2 \times f(0.1,-0.46,-0.64)$ $= -0.1280$	$l_2 = 0.2 \times g(0.1,-0.46,-0.6)$ $= \cancel{-0.0592}$ -0.0476
$k_3 = h \cdot f(x+h/2, y+k_2/2, z+l_2/2)$ $= 0.2 \times f(0.1, \cancel{-0.9360}\ -0.4640, \cancel{-0.6288}\ -0.6238)$ $= \cancel{-0.1257}$ -0.1248	$l_3 = 0.2 \times g(0.1, -0.4640, -0.6286)$ $= \cancel{-0.0588}$ -0.0395
$k_4 = h \cdot f(x+h, y+k_3, z+l_3)$ $= 0.2 \times f(0.2, \cancel{-0.5257}\ -0.5248, \cancel{-0.6611}\ -0.6395)$ $= \cancel{-0.1302}$ -0.1279	$l_4 = 0.2 \times g(0.2, \cancel{-0.5257}\ -0.5248, \cancel{-0.6611}\ -0.6395)$ $= \cancel{+0.0086}$ -0.0136

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$y_1 = y_0 + 1/6\,(k_1 + 2k_2 + 2k_3 + k_4)$

$= -0.47\ 1/6\,(-0.12 - 2\times 0.1280 - 2\times \overset{0.1248}{\cancel{0.1257}} \overset{-0.1279}{\cancel{-0.1302}})$

$= \cancel{-0.5263} \quad -0.5256$ //

$z_1 = -0.6 + 1/6\,(-0.08 - 2\times \overset{0.0476}{\cancel{0.0572}} - 2\times \overset{0.0395}{\cancel{0.0511}} + \underset{-0.0134}{\cancel{0.0086}})$

$= -0.6480$ //

$= -0.6401$ //

13/3/14. Milne's Predictor - corrector Method.

Consider $\frac{dy}{dx} = f(x, y)$

$$P:\quad y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$$C:\quad y_{n+1} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

① By using Milne's predictor - corrector formula to find $y(0.4)$ & $y(0.5)$; G.T $\frac{dy}{dx} = \frac{(1+x^2)y^2}{2}$, $y(0) = 1$; $y(0.1) = 1.06$; $y(0.2) = 1.12$; $y(0.3) = 1.21$

Soln: The Milne's predictor - corrector formula is,

$$P: \; y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right] \quad \text{--- (1)}$$

$$C: \; y_{n+1} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right] \quad \text{--- (2)}$$

x	0 x_0	0.1 x_1	0.2 x_2	0.3 x_3	0.4 x_4		0.5 x_5
y	1 y_0	1.06 y_1	1.12 y_2	1.21 y_3	1.2771 y_4	1.2797	$1.$
$y' = \frac{(1+x^2)y^2}{2}$	y'_0 0.5	y'_1 0.5674	y'_2 0.6523	y'_3 0.7979	0.9460	0.9478	$1.$

Put $n=3$ in (1).

$$P: \; y_4 = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right)$$

$$= 1 + \frac{4 \times 0.1}{3}\left(2 \times 0.5674 - 0.6523 + 2 \times 0.7979\right)$$

$$P: \; y_4 = 1.2771.$$

put $n=3$ in eqn (2).

$$C: \; y_4 = y_2 + h/3\left(y'_2 + 4y'_3 + y'_4\right)$$

$$= 1.12 + \frac{0.1}{3}\left(0.6523 + 4 \times 0.7979 + 0.9460\right]$$

$$C: y_4 = 1.2797.$$

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put $n=4$ in ①,

$$P: \; y_5 = y_1 + \frac{4h}{3}\left[2y_2' - y_3' + 2y_4'\right]$$

$$= 1.06 + \frac{4\times 0.1}{3}\left[2\times 0.6523 - 0.7979 + 2\times 0.9498\right]$$

$$P: \; y_5 = 1.8808.$$

put $n=4$ in ②,

$$C: \; y_5 = y_3 + \frac{h}{3}\left(y_3' + 4y_4' + y_5'\right)$$

$$= 1.21 + \frac{0.1}{3}\left(0.7979 + 4\times 0.9498 + 1.1916\right)$$

$$y_5 = 1.4030.$$

② Given $y' = \frac{1}{x+y}$; $y(0) = 2$; $y(0.2) = 2.0933$; $y(0.4) = 2.1755$, $y(0.6) = 2.2493$. Find $y(0.8)$ by using Milne's method.

Soln:

The Milne's formula is,

$$P: \; y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y_{n-2}' - y_{n-1}' + 2y_n'\right] \quad — ①$$

$$C: \; y_{n+1} = y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + y_{n+1}'\right] \quad — ②$$

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	x_0	x_1	x_2	x_3	x_4
x	0	0.2	0.4	0.6	0.8
y	y_0 2	y_1 2.0933	y_2 2.1755	y_3 2.2493	2.3162
$y' = \frac{1}{x+y}$	y_0' 0.5	y_1' 0.4861	y_2' 0.3883	y_3' 0.3510	0.3209

put $n=3$ in ①

$$P: y_4 = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 2 + \frac{4\times0.2}{3}\left[2\times0.4861 - 0.3883 + 2\times0.3510\right]$$

$P: y_4 = 2.3162.$

put $n=3$ in ②

$$C: y_4 = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 2.1755 + \frac{0.2}{3}\left[0.3883 + 4\times0.3510 + 0.3209\right]$$

$C: y_4 = 2.3164$ //.

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3. Given $y' = xy + y^2$, $y(0) = 1$; $y(0.1) = 1.1169$; $y(0.2) = 1.2774$. using R.K method of 4th order, find $y(0.3)$. Continue the solution at $x = 0.4$ using milne's method.

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soln:

x	0	0.1	0.2	0.3
y	1	1.1169	1.2774	1.5042.

Here, $h = 0.1$;

$y' = xy + y^2$

$f(x,y) = xy + y^2$

To find y_3;

$x = 0.2$; $y = 1.2774$.

$k_1 = h.f(x.y) = 0.1 \times f(0.2, 1.2774) = 0.1887$.

$k_2 = h \cdot f(x + h/2, y + \frac{k_1}{2}) = 0.1 \times f(0.25, 1.3718)$

$= 0.2225$.

$k_3 = h \cdot f(x + h/2, y + \frac{k_2}{2}) = 0.1 \times f(0.25, 1.3887)$

$= 0.2276$

$k_4 = h \cdot f(x+h, y+k_3) = 0.1 \times f(0.3, 1.5050) = 0.2717$.

$y_3 = y_2 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1.2774 + \frac{1}{6}[0.1887 + 2 \times 0.2225 + 2 \times 0.2276 + 0.2717]$

$= 1.5042$.

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Milne's formula is,

$$P: \quad y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$$C: \quad y_{n+1} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

x	x_0 0	x_1 0.1	x_2 0.2	x_3 0.3	x_4 0.4	
y	y_0 1	y_1 1.1169	y_2 1.2774	y_3 1.5042	1.8345	1.8[illegible]
$y' = xy + y^2$	y'_0 1	y'_1 1.3592	y'_2 1.8872	y'_3 2.7139	4.0992	4.[illegible]

Put $n = 3$ in (1)

$$P: \quad y_4 = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right]$$

$$= 1 + \frac{4 \times 0.1}{3}\left[2 \times 1.3592 - 1.8872 + 2 \times 2.7139\right]$$

$$= 1.8345.$$

Put $n = 3$ in (2)

$$C: \quad y_4 = y_2 + \frac{0.1}{3}\left[y'_2 + 4y'_3 + y'_4\right]$$

$$= 1.2774 + \frac{0.1}{3}\left[1.8872 + 4 \times 2.7139 + 4.09[illegible]\right.$$

$$= 1.8388$$

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4. Given that $y'' + xy' + y = 0$, $y(0) = 1$; $y'(0) = 0$ obtain y for $x = 0.1, 0.2$ and 0.3 by Taylor series method and find the soln for $y(0.4)$ by Milne's method.

soln:

The Taylor series is,

$$y = y_0 + (x - x_0)\frac{y_0'}{1!} + (x-x_0)^2 \frac{y_0''}{2!} + (x-x_0)^3 \frac{y_0'''}{3!} + (x-x_0)^4 \frac{y_0^{iv}}{4!} + \cdots$$

$$y'' + xy' + y = 0.$$

$$y'' = -xy' - y.$$

x	0	x_0
y	1	y_0
y'	0	y_0'
$y'' = -xy' - y$	-1	y_0''
$y''' = -xy'' - y' - y'$	0	y_0'''
$y^{iv} = -xy''' - y'' - y'' - y''$	3	y_0^{iv}

$$y = 1 + (x-0)\frac{0}{1} + (x-0)^2 \times \frac{-1}{2} + (x-0)^3 \frac{0}{6} +$$

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{8}$$

$$y' = -\frac{\cancel{2}x}{\cancel{2}} + \frac{\cancel{4}x^3}{\cancel{8}_2} \Rightarrow y' = -x + \frac{x^3}{2}$$

$y(0.1) = 0.9950$

$y(0.2) = 0.9802$

$y(0.3) = 0.9560$

The Milne's formula is,

$$P: \quad y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_{n}\right]$$

$$C: \quad y_{n+1} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

Soln:

x	x_0: 0	x_1: 0.1	x_2: 0.2	x_3: 0.3	x_4: 0.4	
y	y_0: 1	y_1: 0.9950	y_2: 0.9802	y_3: 0.9560	0.9232	0.9232
$y' = -x + \frac{x^3}{2}$	y_0': 0	y_1': −0.0995	y_2': −0.1960	y_3': −0.2865	−0.3680	−0.3680

put $n = 3$;

$$P: \quad y_4 = y_0 + \frac{4 \times 0.1}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 1 + \frac{0.4}{3}\left[2 \times (-0.0995) + 0.1960 + 2 \times (-0.2\ldots\right.$$

$$= 0.9232.$$

~~C: y_4~~ put $n = 3$;

$$C: \quad y_4 = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

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$= 0.9802 + \frac{0.1}{3}\left[-0.1960 - 4\times 0.2865 - 0.3680\right]$

$C: y_4 = 0.9232$

Adam's Bashforth predictor - corrector formula:

$P: y_{n+1} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$

$C: y_{n+1} = y_n + \frac{h}{24}\left[19y_n' - 5y_{n-1}' + y_{n-2}' + 9y_{n+1}'\right]$.

1. Using Adam's method find $y(1.4)$ given $y' = x^2(1+y)$, $y(1) = 1$; $y(1.1) = 1.233$; $y(1.2) = 1.548$ & $y(1.3) = 1.979$.

Soln:

The Adam's formula is,

$P: y_{n+1} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$

$C: y_{n+1} = y_n + \frac{h}{24}\left[19y_n' - 5y_{n-1}' + y_{n-2}' + 9y_{n+1}'\right]$

	x_0	x_1	x_2	x_3	x_4
x	1	1.1	1.2	1.3	1.4
y	y_0 1	y_1 1.233	y_2 1.548	y_3 1.979	y_4 2.5723 2.5749
$y' = x^2(1+y)$	y_0' 2	y_1' 2.7019	y_2' 3.6691	y_3' 5.0345	y_4' 7.0017 7.0068

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put n=3;

$$P: y_4 = y_3 + \frac{0.1}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 1.979 + \frac{0.1}{24}\left[55\times 5.0345 - 59\times 3.6691 + 37\times 2.7019 - 9\times 2\right]$$

$$P: y_4 = 2.5723.$$

put n=3 in (1)

$$C: y_4 = y_3 + \frac{h}{24}\left[19y_3' - 5y_2' + y_1' + 9y_4'\right]$$

$$= 1.979 + \frac{0.1}{24}\left[19\times 5.0345 - 5\times 3.6691 + 2.70 + 9\times 7.0017\right]$$

$$C: y_4 = 2.5749.$$

2. Use Adam's method to find $y(2)$ if $y' = \frac{x+y}{2}$, $y(0)=2$; $y(0.5)=2.636$; $y(1)=3$ and $y(1.5)=4.968$.

Soln:

The Adam's formula is,

$$P: y_{n+1} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - \right.$$

$$C: y_{n+1} = y_n + \frac{h}{24}\left[19y_n' - 5y_{n-1}' + y_{n-2}' + 9y_{n+1}'\right]$$

x	x_0 0	x_1 0.5	x_2 1	x_3 1.5	x_4 2	
y	y_0 2	y_1 2.636	y_2 3.595	y_3 4.968	6.8708	6.8731
$y' = \frac{x+y}{2}$	y_0' 1	y_1' 1.5680	y_2' 2.2975	y_3' 3.2340	4.4354	4.4366

put $n=3$ in ①

$$P: \quad y_4 = y_3 + \frac{0.5}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 4.968 + \frac{0.5}{24}\left[55 \times 3.2340 - 59 \times 2.2975 + 37 \times 1.5680 - 9 \times 1\right]$$

$$= 6.8708$$

$$C: \quad y_4 = y_3 + \frac{h}{24}\left[19y_3' - 5y_2' + y_1' + 9y_4'\right]$$

$$= 4.968 + \frac{0.5}{24}\left[19 \times 3.2340 - 5 \times 2.2975 + 1.5680 + 9 \times 4.4354\right]$$

$$= 6.8731.$$

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3. Using Adam's method find $y(0.4)$ given $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$; $y(0.1) = 1.1169$; $y(0.2) = 1.2774$; and $y(0.3) = 1.5041$

Soln: The Adam's formula is,

$$P: \quad y_{n+1} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

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$$C: y_{n+1} = y_n + \frac{h}{24}\left[19y_n' - 5y_{n-1}' + y_{n-2}' + 9y_{n+1}'\right]$$

	x_0	x_1	x_2	x_3	x_4
x	0	0.1	0.2	0.3	0.4
y	y_0 = 1	y_1 = 1.1169	y_2 = 1.2774	y_3 = 1.5041	1.8841
$\frac{dy}{dx} = xy + y^2$	y_0' = 1	y_1' = 1.3592 (0.0895)	y_2' = 1.8872 (0.1766)	y_3' = 2.7185	4.0976

put $n = 3$ in

$$P: y_4 = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 1.5041 + \frac{0.1}{24}\left[55\times 2.7185 - 59\times 1.8872 + 37\times 1.3592 - 9\times 1\right]$$

$$P: y_4 = 1.8841$$

put $n = 3$ in

$$C: y_4 = y_3 + \frac{h}{24}\left[19y_3' - 5y_2' + y_1' + 9y_4'\right]$$

$$= 1.5041 + \frac{0.1}{24}\left[19\times 2.7185 - 5\times 1.8872 + 1.3592 + 9\times 4.0976\right]$$

$$= 1.8889.$$

UNIT - V

BOUNDARY VALUE PROBLEM IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATION.

Finite difference Method :

Replace x by x_k

y by y_k

y' by $\frac{y_{k+1} - y_k}{h}$

y'' by $\frac{y_{k-1} - 2y_k + y_{k+1}}{h^2}$

where, $h = \frac{b-a}{n}$

1. Solve $y'' = x+y$ with the boundary condition $y(0) = y(1) = 0$.

Soln :

x	0	0.25	0.5	0.75	1
y	0	-0.0349	-0.0564	-0.05	0

$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25.$

$y'' = x+y.$

$$\frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} = x_k + y_k.$$

$$y_{k-1} - 2y_k + y_{k+1} = h^2 x_k + h^2 y_k$$

$$y_{k-1} - 2y_k + y_{k+1} - h^2 y_k = h^2 x_k$$

$$y_{k-1} + y_k(-2-h^2) + y_{k+1} = h^2 x_k$$

$$y_{k-1} - 2.0625\, y_k + y_{k+1} = 0.0625\, x_k$$

$k=1$;

$$y_0 - 2.0625\, y_1 + y_2 = 0.0625\, x_1$$

$$-2.0625 y_1 + y_2 = 0.0156 \quad \text{—} (1)$$

$k=2$;

$$y_1 - 2.0625 y_2 + y_3 = 0.0625 x_2$$

$$y_1 - 2.0625 y_2 + y_3 = 0.0313 \quad \text{—} (2)$$

$k=3$;

$$y_2 - 2.0625 y_3 + y_4 = 0.0625 x_3$$

$$y_2 - 2.0625 y_3 = 0.0469 \quad \text{—} (3)$$

Solve (1), (2) & (3)

$$y_1 = -0.0349\,; \quad y_2 = -0.0564\,; \quad y_3 = -0.0501;$$

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2. using a finite difference method compute $y(0.5)$. Given $y'' - 64y + 10 = 0$; $y(0) = y(1) = 0$. Sub dividing the interval into i) 4 equal parts. ii) 2 equal parts.

Soln:

Given $y'' - 64y + 10 = 0$

$$\frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} - 64y_k + 10 = 0$$

$$\frac{y_{k-1} - 2y_k + y_{k+1} - 64y_k h^2 + 10h^2}{h^2} = 0$$

$$y_{k-1} + y_k(-2 - 64h^2) + y_{k+1} = -10h^2 \quad \text{---} (1)$$

i) subdiving into 4parts.

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

	x_0	x_1	x_2	x_3	x_4
x	0	0.25	0.5	0.75	1
y	0	0.1287	0.1471	0.1287	0

for $h = 0.25$, (1) becomes,

$$y_{k-1} - 6y_k + y_{k+1} = -0.625 \quad \text{---} (2)$$

put $k = 1$.

$$y_0 - 6y_1 + y_2 = -0.625$$

$$-6y_1 + y_2 = -0.625 \quad \text{---} (3)$$

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put $k=2$;

$$y_1 - 6y_2 + y_3 = -0.625 \quad \text{---} \ (4)$$

put $k=3$;

$$y_2 - 6y_3 + y_4 = -0.625$$

$$y_2 - 6y_3 = -0.625 \quad \text{---} \ (5)$$

Solving by (3), (4) & (5)

$$y_1 = 0.1287 \ ; \ \boxed{y_2 = 0.1471} \ ; \ y_3 = 0.1287.$$

ii) Sub dividing to 2 parts:

$$h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5.$$

	x_1	x_2	x_3
x	0	0.5	1
	y_0	y_1	y_2
y	0	0.1389.	0.

for $h = 0.5$. Eqn (1) becomes

$$\cancel{y_{k-1} + y_k}$$

$$y_{k-1} - 18y_k + y_{k+1} = -2.5 \quad \text{---} \ (1).$$

$k=1$.

$$y_0 - 18y_1 + y_2 = -2.5.$$

$$\cancel{-}18y_1 = \cancel{-}2.5$$

$$\boxed{y_1 = 0.1389.}$$

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5. Solve by finite difference method, the BVP $y'' - y = 0$ where $y(0)=0; y(1) = 1$; take $\Delta x = 0.25$

Soln:

Given ① $y'' - y = 0$

$$\frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} - y_k = 0$$

$$\frac{y_{k-1} - 2y_k + y_{k+1} - y_k h^2}{h^2} = 0$$

$$y_{k-1} + y_k(-2-h^2) + y_{k+1} = 0 \quad \text{—①}$$

for $h = 0.25$ eqn ① becomes,

$$\cancel{y_{k-1} + y_k} \quad y_{k-1} - 2.0625\, y_k + y_{k+1} = 0 \quad \text{—②}$$

	x_0	x_1	x_2	x_3	x_4
x	0	0.25	0.5	0.75	1
y	y_0 0	0.2151	0.4437	0.7	y_4 1

$k = 1$;

$$y_0 - 2.0625\, y_1 + y_2 = 0$$

$$-2.0625\, y_1 + y_2 = 0 \quad \text{—③}$$

$k = 2$;

$$y_1 - 2.0625\, y_2 + y_3 = 0 \quad \text{—④}$$

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$k=3$;

$$y_2 - 2.0625 y_3 + y_4 = 0$$

$$y_2 - 2.0625 y_3 + 1 = 0$$

$$y_2 - 2.0625 y_3 = -1 \quad \text{---} (5)$$

Solve by (3), (4) & (5)

$$y_1 = 0.8181 \; ; \; y_2 = 0.4437 \; ; \; y_3 = 0.7000$$

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Classification of partial differential Equation

Consider,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

$B^2 - 4AC < 0$ The P.D.E is elliptic.

$B^2 - 4AC = 0$ The P.D.E is parabolic

$B^2 - 4AC > 0$ The P.D.E is hyperbolic.

One dimensional heat equation :

The One dimensional heat eqn is

$$\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial t} \quad (\text{or}) \quad U_{xx} = a\, U_t$$

$$\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0.$$

$A = 1$; $B = 0$; $C = 0$

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$b^2 - 4ac = 0 - 4 \times 1 \times 0$

$= 0$

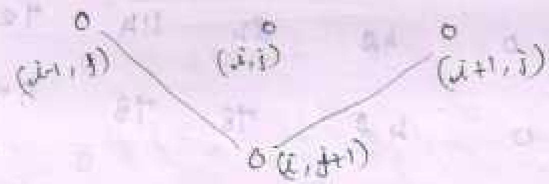
The one dimensional heat eqn is parabolic

There are two methods to solve one dimensional heat equations.

i) Bender-Schmidt formula (Explicit)

ii) Crank-Nicolson method (Implicit)

Bender-Schmidt formula:



$$u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2}$$

Here, $k = \frac{ah^2}{2}$

1. Solve $u_t = u_{xx}$ in $0<x<5$, $t>0$ given that $u(0,t)=0$, $u(5,t)=0$, $u(x,0)=x^2(25-x^2)$. Compute u upto 3sec with $\Delta x = 1$ by using Bender Schmidt formula.

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Soln:

Given $u_t = u_{xx} \Rightarrow a = 1$

$h = \Delta x = 1$

$k = \frac{ah^2}{2} = \frac{1 \times 1}{2} = 0.5$

$$u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2}$$

t \ x	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	114	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2.	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0

2. Solve $u_{xx} = 32 u_t$, $h = 0.25$ for $t \geq 0$, $0 < x < 1$, with $u(0,t) = 0$, $u(x,0) = 0$; $u(1,t) = t$.

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soln:

$u_{xx} = 32 u_t \qquad a = 32.$

$h = 0.25.$

$$k = \frac{ah^2}{2} = \frac{32 \times 0.25}{2} = 1$$

$$u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2}$$

$t \backslash x$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	0.5	2
3	0	0	0.25	1	3
4	0	0.125	0.5	1.625	4
5	0	0.25	0.875	2.25	5.

2. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ subjected to $u(0,t) = u(1,t) = 0$ and $u(x,0) = \sin(\pi x)$ using Bender Schmidt method.

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soln:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$u_{xx} = u_t$. $a = 1$

$$h = \frac{B-A}{n} = \frac{1-0}{5} = 0.2.$$

$$k = \frac{ah^2}{2} = \frac{1 \times 0.2^2}{2} = 0.02.$$

Bender schmidth formula is,

$$u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2}$$

t \ x	0	0.2	0.4	0.6	0.8
0	0	0.5878	0.9511	0.9510	0.5878
0.02	0	0.4756	0.7695	0.7695	0.4756
0.04	0	0.3848	0.6226	0.6226	0.3848
0.06	0	0.3113	0.5037	0.5037	0.3113
0.08	0	0.2519	0.4075	0.4075	0.2519
1	0	0.2028	0.3297	0.3297	0.2038

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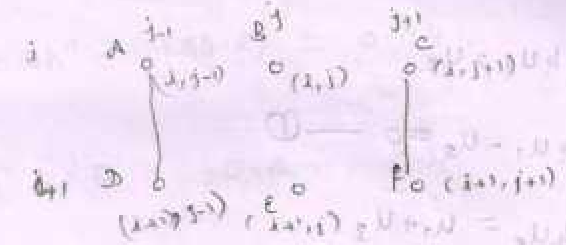
25/3/14. Crank – Nicolson's Method (Implicit method):

Consider, $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial t}$ (one dimensional heat eqn).

$k = ah^2$



$4u_E = u_A + u_C + u_D + u_F$

1. Using crank – Nicolson's scheme solve

$$16\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0<x<1,\ t>0.$$

Subjected to $u(x,0)=0$; $u(0,t)=0$; $u(1,t)=100t$. Compute u for one step in t-direction. taking $h = 1/4$

Soln:

$$16\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$a = 16.$

$h = 0.25$;

$k = ah^2 = 16\times(0.25)^2 = 1$

t/x	0	0.25	0.5	0.75	
0	0	0	0	0	0
1	0	u_1	u_2	u_3	100

$4u_1 = u_2$

$4u_1 - u_2 = 0$ —— (1)

$4u_2 = u_1 + u_3$

$u_1 - 4u_2 + u_3 = 0$ —— (2)

$4u_3 = u_2 + 100$

$-u_2 + 4u_3 = 100$ —— (3)

Solve (1), (2) & (3).

$u_1 = 1.7857$

$u_2 = 7.1429$

$u_3 = 26.7857$

2. Find $u(x,t)$ for one time step, the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ given $u(x,0) = \sin(\pi x)$; $u(0,t) = u(1,t) = 0$. Take $h = 0.2$ using implicit method.

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Soln:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$a = 1$

$h = 0.2$

$k = ah^2 = 1(0.2)^2 = 0.04$

t\x	0	0.2	0.4	0.6	0.8	1
0	0	0.5878	0.9511	0.9511	0.5878	0
0.04	0	u_1	u_2	u_3	u_4	0

$4u_1 = u_2 + 0.9511$

$4u_1 - u_2 = 0.9511$ — (1)

$4u_2 = u_1 + u_3 + 1.5389$

~~$-u_1 + 4u_2$~~ $u_1 - 4u_2 + u_3 = -1.5389$ — (2)

$4u_3 = u_2 + u_4 + 1.5389$

$u_2 - 4u_3 + u_4 = -1.5389$ — (3)

$4u_4 = 0.9511 + u_3$

~~$-u_3 + 4u_4 = 0.9511$~~ — (4)

$u_4 = \frac{u_3}{4} + 0.2378$ — (4)

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Sub (4) in (3)

$$u_2 - 4u_3 + u_4 = -1.5389.$$

$$u_2 - 4u_3 + \frac{u_3}{4} + 0.2378 = -1.5389.$$

$$u_2 - \frac{15}{4}u_3 = -1.7767.$$

$$u_2 - 3.75u_3 = -1.7767 \quad \text{---(5)}$$

Solve ~~eqn~~ (1), (2), (5).

$u_1 = 0.3993$

$u_2 = 0.6461$

$u_3 = 0.6461$

$$(4) \Rightarrow \quad u_4 = \frac{0.6461}{4} + 0.2378 = 0.3993.$$

$u_4 = 0.3993.$

27/3/14.

3\. Solve by crank nicolson's method, eqn $u_{xx} = u_t$ subjected to $u(x,0) = 0$; $u(0,t) = 0$; $u(1,t) = t$ for two time step.

Soln:

$u_{xx} = u_t$.

$a = 1$

$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$.

$k = ah^2 = 1 \times 0.25^2 = 0.0625$.

$t \backslash x$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.0625	0	$\overset{u_1}{0.0011}$	$\overset{u_2}{0.0045}$	$\overset{u_3}{0.0167}$	0.0625
0.125	0	$\overset{u_4}{0.0059}$	$\overset{u_5}{0.0191}$	$\overset{u_6}{0.0528}$	0.125

$4u_1 = u_2$.

$4u_1 - u_2 = 0$ —— (1)

$4u_2 = u_1 + u_3$

$u_1 - 4u_2 + u_3 = 0$ —— (2)

$4u_3 = u_2 + 0.0625$.

$-u_2 + 4u_3 = 0.0625$ —— (3)

solve by (1), (2), (3)

$u_1 = 0.0011$; $u_2 = 0.0045$; $u_3 = 0.0167$.

$4u_4 = u_5 + 0.0045$.

$4u_4 - u_5 = 0.0045$ —— (4)

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$4u_5 = u_4 + u_6 + 0.0178$

$u_4 - 4u_5 + u_6 = -0.0178$ —— (5)

$4u_6 = u_5 + 0.1920$

$-u_5 + 4u_6 = 0.1920$ —— (6)

solve by (4), (5), & (6)

$u_4 = 0.0059 \quad u_5 = 0.0191 \quad u_6 = 0.0528$

One dimentional wave Equation:

The One dimentional wave Equation is, $\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2}$; $k = ah$

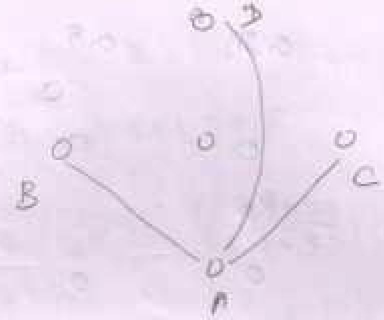
$U_{xx} = a^2 U_{tt}$

$U_{xx} - a^2 U_{tt} = 0$

$A = 1$; $B = 0$; $C = -a^2$

$B^2 - 4AC = 0 + 4a^2 = 4a^2 > 0$

The P.D.E is hyperbolic.



The formula is,

$$U_A = U_B + U_C - U_D$$

1. Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0<x<1$, $t>0$

Given $u(x,0)=0$; $\frac{\partial u}{\partial t}(x,0)=0$; $u(0,t)=0$; $u(1,t)=100\sin(\pi t)$. compute $u(x,t)$ for 4 times steps with $h=0.25$

soln:

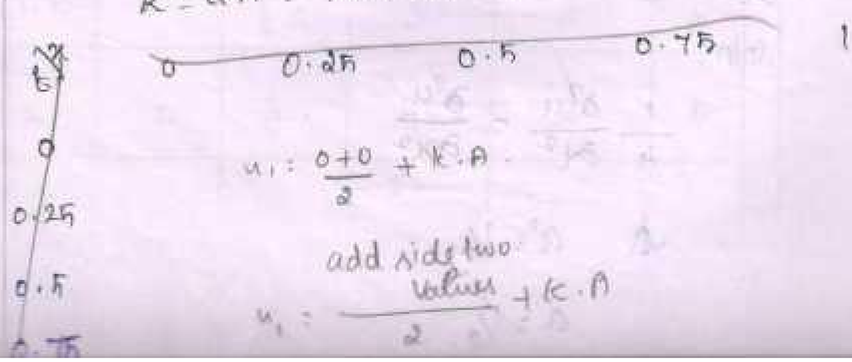
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$a^2 = 1$

$a = 1$.

$h = 0.25$.

$k = ah = 1 \times 0.25 = 0.25$.



$u_1 = \frac{0+0}{2} + k.A$

$u_1 = \frac{\text{add side two values}}{2} + k.A$

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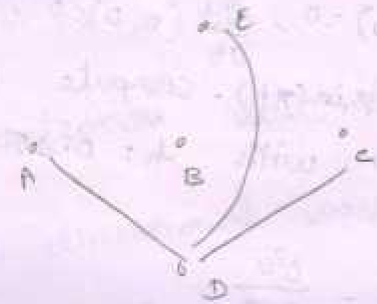
22/5/17.

t \ x	0	0.25	0.5	0.75	
0	0	0	0	0	
0.25	0	$\frac{0+0+0.0}{2}$ u_1 0	u_2 0	u_3 0	70
0.5	0	0	0	70.7107	10
0.75	0	0	70.7107	100	70.
1	0	70.7107	100	70.7107	0



$$U_D = U_A + U_C - U_E$$

2. Solve the eqn. $\frac{1}{4}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ with $u(0,t) = 0$; $u(4,t) = 0$; $u(x,0) = x(4-x)$; $\frac{\partial u}{\partial t}(x,0) = 0$; by taking $h = 1$; upto

Soln:

$$\frac{1}{4}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$a^2 = 1/4$$

$$a = 1/2.$$

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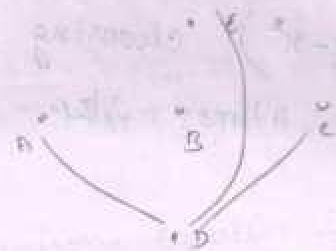
$h = 1$

$k = ah = 0.5 \times 1 = 0.5$.



$U_D = U_A + U_C - U_E$

$t \backslash x$	0	1	2	3	4
0	0	3	4	3	0
0.5	0	2	3	2	0
1	0	0	0	0	0
1.5	0	−2	−3	−2	0
2	0	−3	−4	−3	0
2.5	0	−2	−3	−2	0
3	0	0	0	0	0
3.5	0	+2	3	2	0
4.	0	3	4	3.	0

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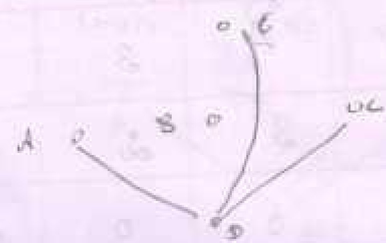
8. Solve $u_{tt} = u_{xx}$, $0<x<2$; $t>0$ subject to $u(x,0)=0$; $u(0,t)=0$; $u(2,t)=0$; $u_t(x,0) = 100(2x-x^2)$ choosing $h = 1/2$ compute 'u' for 4 times step.

soln:

$u_{tt} = u_{xx}$.

$a^2 = 1 \Rightarrow a = 1$; $h = 0.5$.

$k = ah = 1 \times 0.5 = 0.5$.



$u_D = u_A + u_C - u_E$

$t \backslash x$	0	0.5	1	1.5	2
0	0	0	0	0	0
0.5	0	37.5	50	37.5	0
1	0	50	75	50	0
1.5	0	37.5	50	37.5	0
2	0	0	0	0	0

$\frac{0+0}{2} + k \cdot 100(2x-x^2)$

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1/4/14. Laplace and poisson Equation.

The laplace Equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$U_{xx} + U_{yy} = 0$. (or) $\nabla^2 u = 0$.

The poisons Equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

(or)

$$U_{xx} + U_{yy} = f(x,y)$$

(or)

$$\nabla^2 u = f(x,y)$$

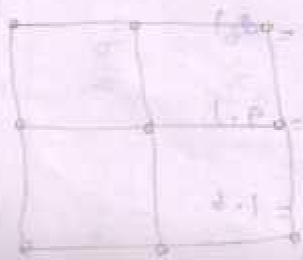
Here $A=1$; $B=0$; $C=1$

$B^2 - 4AC = 0 - 4\times1\times1$

$= -4 < 0$.

Hence, laplace and poisson equation are elliptic

Standard Diagonal five point formula,

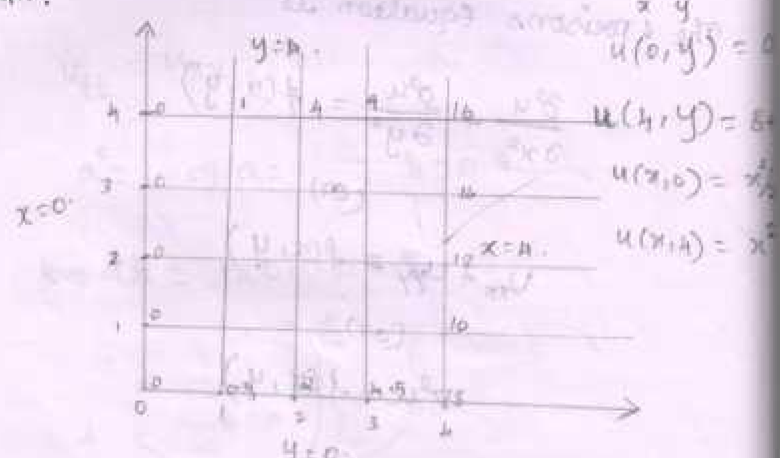


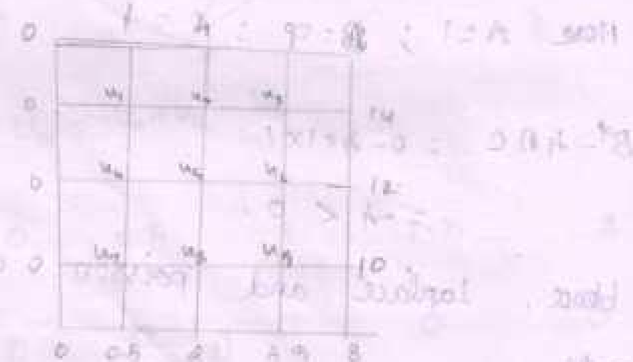
(+) SFPF: $U_E = \frac{U_B + U_D + U_F + U_H}{4}$.

(×) DFPF: $U_E = \frac{U_A + U_C + U_G + U_I}{4}$

1. By Liebmann iteration method solve $u_{xx}+u_{yy}$ over the ~~surface~~ square region of side 4 satisfying $u(0,y)=0$ $0\le y\le 4$; $u(4,y)=8+2y$; $u(x,0)=x^2/2$ $0\le x\le 4$, $u(x,4)=x^2$, $0\le x\le 4$. Com[pute] the values at the interior points with $h=k=$[1]

Soln:





Rough values:

SFPF : $u_5 = \frac{0+4+12+2}{4} = 4.5$.

DFPF : $u_1 = \frac{0+4+0+u_5}{4} = 2.1$

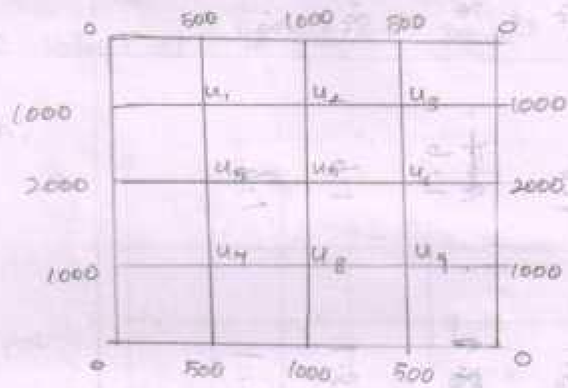
DFPF : $u_3 = \frac{4+16+12+u_5}{4} = 9.1$

DFPF : $u_7 = \frac{0+u_5+0+2}{4} = 1.6$

DFPF : $u_9 = \frac{u_5+12+2+8}{4} = 6.6$

$u_1 = \frac{1+u_2+u_4}{4}$	$u_2 = \frac{4+u_1+u_3+u_5}{4}$	$u_3 = \frac{8+u_2+u_6}{4}$	$u_4 = \frac{u_1+u_5+u_7}{4}$	$u_5 = \frac{u_2+u_4+u_6+u_8}{4}$	$u_6 = \frac{13+u_5+u_3+u_9}{4}$	$u_7 = \frac{0.5+u_4+u_8}{4}$	$u_8 = \frac{2+u_5+u_7+u_9}{4}$	$u_9 = \frac{4.5+u_6+u_8}{4}$
2.1	4.9.	9.1	2.1	4.5	8.1	1.6	3.7.	6.6
2	4.9.	9.	2	~~2.8~~ 4.7.	8.1.	~~0.7~~. 1.6	3.7.	6.6.
2.	4.9.	9	2.1	4.7.	8.1	1.6	3.7.	6.6.
2	4.9	9	2.1	4.7	8.1	1.6	3.7	6.6

2. Solve the Elliptic Eqn $U_{xx}+U_{yy}=0$ following square mesh with the bounda[ry] values are shown below



soln :

By symmetry

$u_1 = u_3$ $\quad\quad$ $u_1 = u_7$

$u_4 = u_6$ $\quad$ & $\quad$ $u_2 = u_8$

$u_7 = u_9$ $\quad\quad$ $u_3 = u_9$

Here ,

$u_1 = u_3 = u_7 = u_9$

$u_2 = u_8$

$u_4 = u_6$

Now , we find only $u_1, u_2, u_4, u_{[5]}$

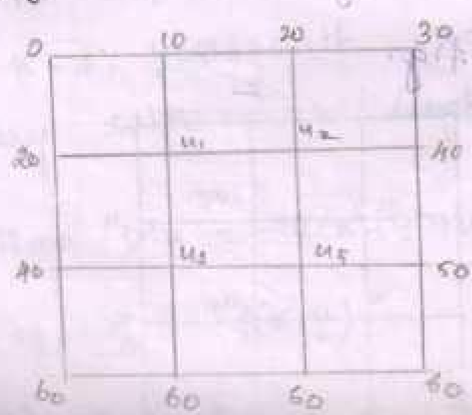
$u_1 = \frac{1500+u_2+u_4}{4}$	$u_2 = \frac{1000+2u_1+u_5}{4}$	$u_4 = \frac{2000+2u_1+u_5}{4}$	$u_5 = \frac{2u_2+2u_4+u_5}{4}$
1125	1187.5	1437.5	1500
~~1068.8~~ 1017	~~1190.9~~	~~1380.9~~	~~1234.4~~
1031.3	1140.7	1390.7	1265.4
1007.9	~~1099.0~~ 1070.3	1320.9	~~1254~~ 1195.4
992.7	1035.2	1285.2	1160.2
955.1	1017.6	1267.6	1142.6
946.3	1008.8	1258.8	1133.8
941.9	1004.4	1254.4	1129.4
939.7	1002.2	1252.2	1127.2
938.6	1001.1	1251.1	1126.1
938.1	1000.8	1250.6	1125.7
937.9	1000.4	1250.4	1125.4
937.9	1000.3	1250.3	1125.3

Rough value :

$$SFPF : u_5 = \frac{1000+2000+2000+1000}{4} = \frac{6000}{4} = 1500$$

$$DFPF : u_1 = \frac{0+1000+1500+2000}{4} = 1125.$$

3. Solve $\nabla^2 u = 0$ over the square region given by the boundary condition as in the fig. below.



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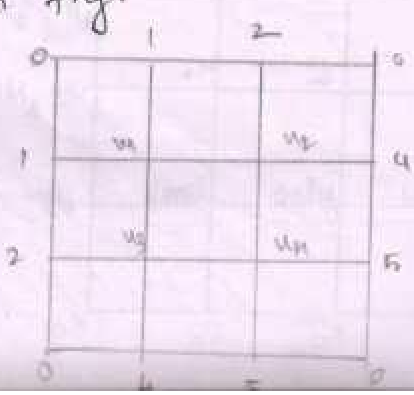
$u_1=\frac{30+u_2+u_3}{4}$	$u_2=\frac{60+u_1+u_4}{4}$	$u_3=\frac{100+u_1+u_4}{4}$	$u_4=110+$
4	18.8	28.8	0
19.4	19.9	29.9.	40
20	30.	40	45
25	32.5	42.5.	46.8
26.3	33.2	43.2.	46.6
26.6	33.3	43.3	46.7
26.7.	33.4	43.4	46.7.
26.7.	33.4	43.4	46.7.

~~Soln:~~ Rough :

$u_4=0.$

$DFPF: u_1=\frac{0+20+40+0}{4}=15.$

4. Solve $V_{xx}+V_{yy}=0$ over the squa[re] region given by the boundary condit[ions] shown in fig.



Soln:

$\because$ By symmetry, $u_3 = u_2$.

Rough; Assume, $R_4 = 0$

DFPF : $u_1 = \frac{2+2+0+0}{4} = 1$

$u_1 = \frac{2+2u_2}{4}$	$u_2 = \frac{6+u_1+u_4}{4}$	$u_4 = \frac{10+2u_2}{4}$
1	1.6	0
1.4	1.9	3.5
1.5	2.8	3.9
1.9	3	4
2	3	4
2	3	4.

3/4/14. Poisson equation problems $U_{xx} + U_{yy} = -f(x,y)$

1. Solve the equation $\nabla^2 u = -10(x^2+y^2+10)$ over the square mesh with sides $x=0$; $y=0$; $x=3$; $y=3$, with $u=0$ on the boundary with mesh length 1 unit.

Soln: given $\nabla^2 u = -10(x^2+y^2+10)$

$$U_{xx} + U_{yy} = -f(x,y)$$

$f(x,y) = 10(x^2+y^2+10)$

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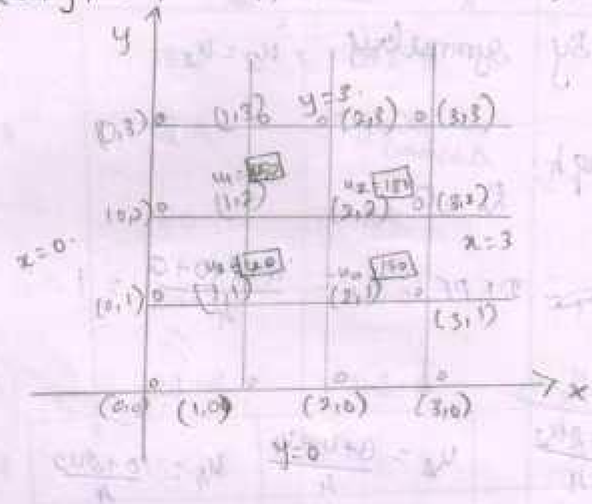
$h^2 f(x,y) = f(x,y) = 10(x^2+y^2+10)$ $\therefore h =$



By symmetry,

$u_1 = u_4$.

$u_1 = \frac{u_2+u_3+150}{4}$	$u_2 = \frac{u_1+u_4+180}{3} = \frac{2u_1+180}{4}$	$u_3 = \frac{u_1+u_4 \ldots}{4} = \frac{2u_1+\ldots}{4}$
0	0	0
37.5	68.8	48.8
65.7	77.9	62.9
72.7	81.4	66.4
74.5	82.3	67.3
74.9	82.5	67.5
75	82.5	67.5
75	82.5	67.5

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2. Solve $\nabla^2 u = 8x^2y^2$ over the square bounded by the lines $x=-2$; $x=2$, $y=-2$, $y=2$ with $u=0$ on the boundary and mesh length $=1$

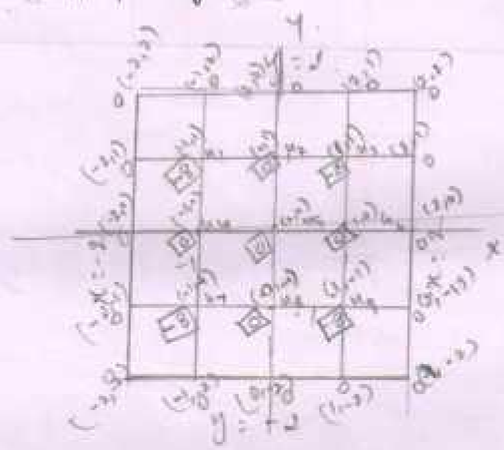
Soln :-

Given $\nabla^2 u = 8x^2y^2$

w.k.t $\nabla^2 u = -f(x,y)$

$f(x,y) = -8x^2y^2$

$h^2 f(x,y) = -8x^2y^2$; $(\because h=1)$



By symmetry :

$\cancel{u_1 = u_7}$	$\cancel{u_1 = u_3}$	$u_2 = u_4$	$\cancel{u_1 = u_9}$
$u_2 = u_8$	$u_4 = u_6$	$\cancel{u_3 = u_7}$	$u_4 = u_8$
$\cancel{u_3 = u_9}$	$\cancel{u_7 = u_9}$	$u_6 = u_8$	$u_2 = u_6$

$u_1 = u_7 = u_3 = u_9$

$u_2 = u_8 = u_6 = u_4$.

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$u_1 = \frac{u_2+u_4-8}{4} = \frac{2u_2-8}{4}$	$u_2 = \frac{u_1+u_3+u_5}{4} = \frac{2u_1+u_5}{4}$	$u_5 = \frac{u_2+u_4+u_6+u_8}{4}$, $u_5 = u_2$
0	0	0
−2	−1	−1
−2.5	−1.5	−1.5
−2.8	−1.8	−1.8
−2.9	−1.9	−1.9
−3	−2	−2
−3	−2	−2