R.Vidya

# Complex Variables (3M/15M)

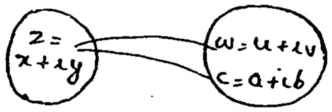
(i) Analyticity
(ii) Complex Integration
(iii) Complex Power Series
(iv) Zeroes & Types of singularities (Poles)
(v) Residues

## ANALYTICITY

i) Complex Variable : If $x$, $y$ are real variables $x+iy$ is a comp variable where $i$ is an imaginary unit such that $i^2 = -1$

$\boxed{z = x+iy}$ where $x = Re(z)$ ; $y = Im(z)$

ii) Complex Function : Function with complex variables. If every element $(z)$ of set A is associated with a unique element $(w)$ of set such an association is a complex function or single valued function. A, B are the set of all complex nos.




multivalued fn : $\underset{u+iv}{\underset{\downarrow}{w}} = \underset{x+iy}{\underset{\downarrow}{f(z)}}$

Eg: (i) $w = f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$

(ii) $w = f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$

iii) Neighbourhood of a point $z_0$ :

The set of all points within the circle having centre at $z$ but not on the circle is called neighbourhood of a point $z_0$. also called open circular disc; is denoted by

$$N_d(z_0) \text{ or } N(d, z_0) = \{z : \underbrace{|z - z_0| < d}_{\text{represents interior of } \odot}\}$$

iv) Analytic Function : For real valued function $y = f(x)$, if

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a),$$ then the function is said to be differentiable.

If a complex function $f(z)$ is differentiable at a pt $z_0$ ie

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$$ & also differentiable in some neighbourhood of the pt $z_0$, then the fn $f(z)$ is called analytic f

Analytic — $z_0$ ✓, neighbourhood ✓ | Not analytic / Differentiable — $z_0$ ✓, neighbourhood × | Not analytic / Diff — $z_0$ ×, neighbourhood ✓

v) If a fn $f(z)$ is not defined or not differentiable or no analytic at $z_0$, then pt, $z_0$ is called singular pt of $f(z)$

(i) $f(z) = \dfrac{z-6}{z-7}$ $z = 7$ singular point.

(ii) $f(z) = \sqrt{z-7}$ ; Defined at all values of $z$

$f'(z) = \dfrac{1}{2\sqrt{z-7}}$ ; Not defined at $z = 7 \rightarrow$ singular point.

vi) Entire Function If a fn $f(z)$ is differentiable or analytic at every point within a finite complex plane, $f(z)$ is called an entire function.

Theorem : 1 : Necessary conditions for a fn $f(z)$ to be analytic

If $f(z) = u(x,y) + i\,v(x,y)$ is analytic at $z_0$, then $u$ & $v$ satisfy the Cauchy Riemann equations

$u_x = v_y$

$v_x = -u_y$ at every pt in some neighbourhood of a point $z_0$ provided $u_x, u_y, v_x, v_y$ exists

Theorem : 2 : Sufficient condition for a fn $f(z)$ to be analytic

If (i) $f(z) = u(x,y) + i\,v(x,y)$ is defined at every pt in some neighbourhood of a point $z_0$.

(ii) $u$ & $v$ satisfy CR equations $u_x = v_y$ ; $v_x = -u_y$ at every pt in some neighbourhood of a pt $z_0$.

(iii) $u, v, u_x, u_y, v_x, v_y$ are continuous at every pt in some neighbourhood of a point $z_0$.

Then the fn $f(z)$ is analytic at $z_0$. & $f'(z) = u_x + i v_x$

Model no ①: Test whether the function $f(z)$ is analytic or not.

a) $f(z) = e^x \cos y + i e^x \sin y = u + iv$

$u = e^x \cos y$ $v = e^x \sin y$

$u_x = e^x \cos y$ $v_y = e^x \cos y$

$u_y = -e^x \sin y$ $v_x = e^x \sin y$

$\therefore u_x = v_y$ & $v_x = -u_y$

· Here CR eqns are satisfied at every point

~~$e^x$, $\cos x$, $\sin x$, $\sinh x$, $\cosh x$, $a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$~~ ($a_n \neq 0$) are 6 basic functions defined & continuous everywhere

$\therefore$ Continuity condition also satisfied at all points for $u, v, u_x, u_y, v_x, v_y$ at every point.

$f(z) = u + iv$ analytic at all points. It is also an entire function

$$f'(z) = e^x \cos y + i e^x \sin y$$

Note i): $e^{i\theta} = \cos\theta + i\sin\theta = (\cos\theta, \sin\theta)$

ii) $f(z) = e^x \cos y + i e^x \sin y$

$= e^x [\cos y + i \sin y] = e^x \cdot e^{iy} = e^{x+iy} = e^z \rightarrow$ Standard entire Function

iii) $e^z$, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, $a_0 + a_1 z + \dots a_n z^n$ ($a_n \neq 0$) are standard entire functions.

Eg: a) $e^z + \sin z + z^2 \rightarrow$ Analytic

b) $\sin(z^3) + e^{z^2} + z^4 \rightarrow$ Analytic.

Analytic (analytic) $\rightarrow$ Analytic

$f(z) = \bar{z} = x - iy = u + iv$

$u = x \quad v = -y$

$u_x = 1 \quad v_y = -1 \quad u_x \neq v_y \rightarrow$ Not analytic at any point

$u_y = 0 \quad v_x = 0$

) $f(z) = |z|^2 = |\sqrt{x^2+y^2}|^2 = x^2 + y^2 + i(0)$

$u = x^2 + y^2 \quad v = 0$

$u_x = 2x \quad u_y = 2y \quad v_x = 0 \quad v_y = 0$

$u_x \neq v_y$ & $v_x \neq -u_y$ $\therefore$ Not analytic fn.

Note i) $f(z) = |z|^2$ is differentiable only at $(0,0)$. Hence function is not analytic

ii) $f(z) = |z|^2 = z\bar{z} \rightarrow$ Product of analytic & non analytic $\rightarrow$ Non Analytic

iii)

| | Real | Complex |
|---|---|---|
| $\log 7$ | ✓ | |
| $\log(-7)$ | × | ✓ |
| $\log(0)$ | × | × |

$$\boxed{\log(-N) = i\pi + \log N}$$

Model No ②: Finding unknown values in the analytic fn.

Note (i) CR eqns in polar form

$u_x = v_y \qquad -u_y = v_x$

In polar form $(x, y) \longrightarrow (r, \theta)$

$u_r = \frac{1}{r} v_\theta \qquad \frac{-1}{r} u_\theta = v_r$

(ii) $f'(z) = (u_r + i v_r)\, e^{-i\theta}$

a) Find P such that function $f(z) = r^2 \cos(2\theta) + i\, r^2 \sin(P\theta)$ is analytic

$u_r = 2r \cos 2\theta \qquad v_r = 2r \sin P\theta$

$u_\theta = -2r^2 \sin 2\theta \qquad v_\theta = +Pr^2 \cos P\theta$

Since fn is analytic,

$u_r = \frac{1}{r} v_\theta \rightarrow 2\cancel{r} \cos 2\theta = +(Pr^{\cancel{2}} \cos P\theta)/r$

$\underline{\underline{P = +2}}$

(OR) $f(z) = r^2 [\cos 2\theta + i \sin P\theta] = r e^{i(2\theta)} \longrightarrow \underline{\underline{P = 2}}$

b) Find a, b, c such that $f(z) = (x^2 - xy - y^2) + i(ax^2 + bxy + cy^2)$ is analytic

$u_x = 2x - y \qquad v_x = 2ax + by$

$u_y = -x - 2y \qquad v_y = bx + 2cy$

For analytic $\quad 2x - y = bx + 2cy$

$\underline{\underline{b = 2}} \qquad 2c = -1 \rightarrow \underline{\underline{c = -1/2}}$

$2ax + by = x + 2y$

$2a = 1 \rightarrow \underline{\underline{a = 1/2}}$

Model No ③: Construction of analytic function.

Method (Cartesian Form)

Step (i) If $u(x, y)$ (or $v(x, y)$) is given to find $f(z)$, then

$f'(z) = u_x + i v_x$

~~Replace $x$ by $z$ & $y$ by $0$.~~

i) Integrate $f'(z)$ w.r.t $z$. to get $f(z)$

ie $f(z) = \int g(z)\, dz + C$ where $C = C_1 + iC_2$

.) If $u(x,y) = x^2 - y^2 - y - 2$ is a real part of analytic fn $f(z) = u + iv$, then find $f(z)$

$u_x = 2x \qquad u_y = -v_x = -2y - 1 \rightarrow v_x = 2y + 1$

$$f'(z) = u_x + i v_x$$
$$= 2x + i(2y+1)$$
$$= 2z + i$$
$$f(z) = \int 2z\,dz + i\int dz$$
$$= \frac{2z^2}{2} + iz + C$$

$\boxed{f(z) = z^2 + iz + C}$ where $C = C_1 + iC_2$

$x + iy$

$C_1 \rightarrow$ Real part of $u(x,y)$ constant $= -2$

$$f(z) = z^2 + iz - 2 + iC_2$$

Check: $f(z) = (x+iy)^2 + i(x+iy) - 2 + iC_2$

$$f(x,y) = x^2 - y^2 + 2xyi + ix - y - 2 + iC_2$$
$$= x^2 - y^2 - y - 2 + i(x + 2xy + C_2)$$

same as in question $\therefore$ Answer correct.

Sub $y = 0$ we'll get $f(z)$

b) If $v = r^3 \sin 3\theta + r\sin\theta + 2$ is imaginary part of $f(z)$, then find $f(z)$

$$v_r = 3r^2 \sin 3\theta + \sin\theta$$
$$v_\theta = 3r^3 \cos 3\theta + r\cos\theta$$
$$f(z) = (u_r + i v_r)\, e^{-i\theta}$$

~~$u_r = \frac{1}{r} v_\theta = 3r^2\cos 3\theta + \cos\theta$~~

$$f(z) = [3r^2\cos 3\theta + \cos\theta + i(3r^2 \sin 3\theta + \sin\theta)]\, e^{-i\theta}$$

Replace $r = z$ & $\theta = 0$

$$f'(z) = \left[3z^2 \cos 0 + \cos 0 \quad + i\left[3z^2 \sin 0 + \sin 0\right]\right] e^{-i0}$$

$$= 3z^2 + 1$$

$$f(z) = z^3 + z + C_1 + iC_2$$

Now from question $C_2 = 2$

$$\underline{\underline{f(z) = z^3 + z + C_1 + i2}}$$

c) If $u - v = e^x[\cos y - \sin y]$, then find A.F $f(z) = u + iv$

$$f(z) = u + iv$$

$$if(z) = iu - v$$

$$(1+i)f(z) = \cancel{u(1+i)}\ u - v + i(u+v)$$

$$F(z) = U + iV$$

$$F'(z) = U_x + iV_x = U_x - iU_y$$

$$F'(z) = [e^x \cos x - e^x \sin x + i\, e^x \cos y]x$$

$$= e^x[\cos y - \sin y] + ie^x[\sin y + \cos y]$$

$$= e^z[1 - 0] + ie^z[0 - 1]$$

$$F'(z) = e^z[1+i) + c$$

$$(1+i)F(z) = e^z[1+i] + c$$

$$f(z) = e^z + \frac{c \cdot (1-i)}{(1+i)(1-i)}$$

$$\boxed{f(z) = e^z + k} \qquad k = \frac{(1-i)c}{2}$$

<u>Model NO : 4</u> : Construction of harmonic conjugate function

$$\nabla^2(\ ) = 0$$

$$\frac{\partial^2(\ )}{\partial x^2} + \frac{\partial^2(\ )}{\partial y^2} = 0$$

For a real valued fn $u(x,y) \to \left.\begin{matrix} u_x, u_y \\ u_{xx}, u_{yy}\end{matrix}\right\}$ real & continuous

If $u_x, u_y, u_{xx}, U_{yy}$ are continuous fns of a function $u(x,y)$, ~~find~~ & $u_{xx} + u_{yy} = 0$. then $u(x,y) \to$ Harmonic fn.

$u_{xx} + u_{yy} = 0 \to$ Laplace eqn satisfied $\therefore$ Harmonic fn

$\downarrow$ Laplace eqn $(\nabla^2 u = 0)$

~~(1) $u(x,y) = 2xy$~~

$u_x = 2y \quad u_{xx} = 0 \quad u_y = 2x \quad u_{yy} = 0$

$u_{xx} + u_{yy} = 0$ All four are continuous also

$\therefore$ Harmonic fn.

<u>Note</u> i) If $f(z) = u + iv$ is analytic function; then $u$ & $v$ are harmonic functions.

ii) If $u$ & $v$ are harmonic functions, then $u + iv$ may or may not be analytic function.

Eg: (a) $u(x, y) = x^2 - y^2 \qquad v(x, y) = 0$

$u_x = 2x \quad u_{xx} = 2 \qquad u_x, v_y, v_{xx}, v_{yy} = 0$

$u_y = -2y \quad u_{yy} = -2 \qquad \rightarrow$ Harmonic fn

$u_{xx} + u_{yy} = 0$

$\rightarrow$ Harmonic fn

Analyticity check: $u_x = 2x \qquad v_y = 0$

$\therefore$ Not analytic fn.

Here $u, v$ are harmonic & $u + iv$ is not analytic

Eg: (b) $u(x, y) = x^2 - y^2 \qquad v(x, y) = 2xy$

$u_x = 2x \quad u_y = -2y \qquad v_x = 2y \quad v_y = 2x$

$u_{xx} = 2 \quad u_{yy} = -2 \qquad v_{xx} = 0 \quad v_{yy} = 0$

$u_{xx} + u_{yy} = 0$ Harmonic $\qquad v_{xx} + v_{yy} = 0$ Harmonic

Analyticity check: $\left.\begin{array}{l} u_x = 2x = v_y \\ u_y = -2y = -v_x \end{array}\right\}$ All are continuous

Analytic

Here, $u, v$ are harmonic & $u + iv$ is analytic

<u>Harmonic Conjugate Function</u>

If $u$ & $v$ are harmonic fns & $u + iv$ is analytic, then $v$ is called the harmonic conjugate function of $u$, <u>or</u> vice versa.

<u>Note</u> (i) If $v$ is a harmonic conjugate fn of $u$, then $u$ is a harmonic conjugate fn of $-v$.

## Method (Cartesian form)

Step (i): If $v(x,y)$ is given to find its H.C fn $u(x,y)$ then consider $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = (u_x)\,dx + (u_y)\,dy$

(ii) $du = (V_y)\,dx - (V_x)\,dy$

(iii) Integrate $u = \underbrace{\int (V_y)\,dx}_{\text{Treating } y \text{ const}} + \underbrace{\int (-V_x)\,dy}_{\text{Term free from } x} + k$

a) If $u = x^3 - 3xy^2 + 7$, then find its Harmonic conjugate.

$dv = v_x\,dx + v_y\,dy$

$= (-u_y)\,dx + (u_x)\,dy$

$u_x = 3x^2 - 3y^2$

$u_y = -3x \times 2y = -6xy$

$dv = 6xy\,dx + (3x^2 - 3y^2)\,dy$

$v = \frac{6yx^2}{2} + -\frac{3y^3}{3} + k = \underline{\underline{3x^2y - y^3 + k}}$ → real const

b) If $v = r^3 \sin 3\theta$, then find its harmonic conjugate fn.

$du = u_r\,dr + u_\theta\,d\theta = (\frac{1}{r} V_\theta)\,dr + (-r V_r)\,d\theta$

$V_r = 3r^2 \sin 3\theta \qquad V_\theta = 3r^3 \cos 3\theta$

$du = 3r^2 \cos 3\theta\,dr - 3r^3 \sin 3\theta\,d\theta + k$

$\underline{\underline{u = r^3 \cos 3\theta + k}}$

c) If $f(z) = u + iv$ is analytic fn, such that $Re\{f'(z)\} = 2y$ & $f[1+i] = 2$, then find $Im(f(z))$?

a) $x^3 - y^3$ b) $2x + y^2$ c) $x^2 - y^2$ d) $y^2 - x^2$

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$Re[f'(z)] = 2y \qquad f'(z) = u_x + iV_x$

$u_x = 2y = V_y$ — (1)

$f(1+i) = 2$

$u(1,1) + i\,v(1,1) = 2 + i\,0$

$u(1,1) = 2$

$v(1,1) = 0$ — (2)

Sub: condns (1), (2) in choices d) is correct

d) which of the following is not a real part of some analytic fn ?

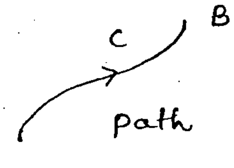
a) $e^x \sin y$ b) $\cos x \sinh y$ c) $3x^2 - y^3$ ✓d) $x^2 + y^2$

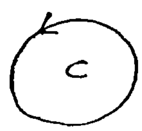
Check one by one for non harmonicity. Not harmonic is d

$\left[ d/dx \sinh x = \cosh x \quad d/dx \cosh x = \sinh x \right]$

COMPLEX INTEGRATION

(1) Complex Line Integral : If $f(z)$ is defined at every pt on the curve C, then the evaluation of integral of complex fn $f(z)$ along any curve C or any path C is called ~~com~~ line integral of a complex fn $f(z)$ & it is denoted by $\int_C f(z)\,dz$ where C is the path of integration.




If path is closed, integral is evaluated in anticlock (+ve direction) direction unless otherwise specified. $\oint_C f(z)\,dz$

(2) Relation b/w real line integral & complex line integral

If $f(z) = u + iv$ & $dz = dx + i\,dy$ where $z = x + i$

then $\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$

$$= \int_C (u\,dx - v\,dy) + i \int_C v\,dx + u\,dy$$

a) Evaluate $\int_0^{1+i} (x^2 - iy)\,dz$ along a curve c ; c is (i) $y = x$ (ii) $y = x^2$

$\int_0^{1+i} (x^2 - iy)\,dz$ ⟶ along 2 open paths $y = x$ & $y = x^2$

(OR)

$$I = \int_0^1 (x^2 - ix)(1+i)\,dx$$

$$= \left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1 [1+i] = \left[\frac{1}{3} - \frac{i}{2}\right][1+i] = \frac{5-i}{6}$$

$$I = \int_0^1 (x^2 - ix^2)(dx + i\,2x\,dx)$$

$$= (1-i)\int_0^1 (x^2 + i\,2x^3)\,dx$$

$$= (1-i)\left(\frac{x^3}{3} + i\frac{x^4}{2}\right]_0^1 = \underline{\underline{\frac{5+i}{6}}}.$$

Note:

Since $dz = dx + i\,dy$

$= dx + i\dfrac{d}{dx}(x^2)$

$= \underline{\underline{dx + 2x\,i\,dx}}.$

b) Evaluate $\int_0^{1+i} z\,dz$ along (i) $y = x$ (ii) $y = x^2$.

$$\int_0^{1+i} z\,dz = \int_0^{1+i} (x+iy)(dx + i\,dy)$$

$\boxed{y = x}$ $\quad I = \int_0^1 (x + ix)(dx + i\,dx)$

$$= (1+i)^2 \int_0^1 x\,dx$$

$$= 2i\left[\frac{x^2}{2}\right]_0^1 = \underline{\underline{i}}.$$

$y = x^2$ $\quad I = \int (x + ix^2)(dx + i\,2x\,dx)$

$$= \cancel{(1+i)}\ \cancel{(1+2i)} \int_0^1 (x + ix^2)(dx + i2x\,dx)$$

$$= \int_0^1 (x + 3ix^2 - 2x^3)\,dx = \left[\frac{x^2}{2} + ix^2 - \frac{x^4}{2}\right]_0^1$$

$$= \underline{\underline{i}}.$$

Note (i)

Depending upon analyticity, value of integral along any path spanning 2 pts → change

Analytic → same

Non analytic → Different

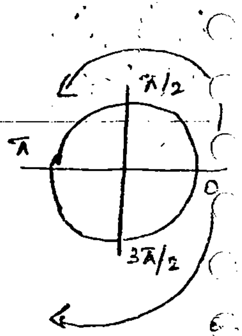
(i) $x^2 + y^2 = r^2 \longrightarrow |z| = r$

$x = r\cos\theta \; ; \; y = r\sin\theta \qquad z = r e^{i\theta}$

(ii) $(x - x_0)^2 + (y - y_0)^2 = r^2 \qquad |z - z_0| = r$

$x = x_0 + r\cos\theta \qquad z = z_0 + r e^{i\theta}$

$y = y_0 + r\sin\theta$




Apply limits as per the drn & the quadrant.

f) $\int_c \frac{2z+4}{z}\, dz$ along a curve c where $c = |z| = 2$

$|z| = 2$

$z = 2e^{i\theta}$

$dz = 2ie^{i\theta} d\theta$

$\theta \rightarrow$ varies 0 to $2\pi$

$$I = \int_0^{2\pi} \frac{2\times 2e^{i\theta} + 4}{\cancel{2}\cancel{e^{i\theta}}} \times \cancel{2} i \cancel{e^{i\theta}}\, d\theta$$

$$= 4i\left[\int_0^{2\pi} e^{i\theta}\, d\theta + \int_0^{2\pi} 4\, d\theta\right] = 4i\left[\left[\frac{e^{i\theta}}{i} + \theta\right]_0^{2\pi}\right]$$

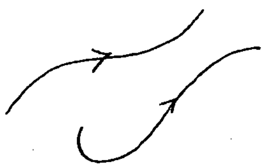
$$= 4i\left[\frac{e^{i2\pi}}{i} + 2\pi - \frac{1}{i}\right]$$

$$= 4i\left[\frac{1}{i} + 2\pi - \frac{1}{i}\right] = \underline{\underline{8\pi i}}$$

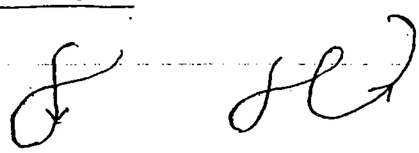
$\int_c \frac{(x+iy)2 + 4}{(x+iy)}(dx + i dy) \longrightarrow$ Complicated Don't do.

(I) Types of Curves

a) Simple Curve

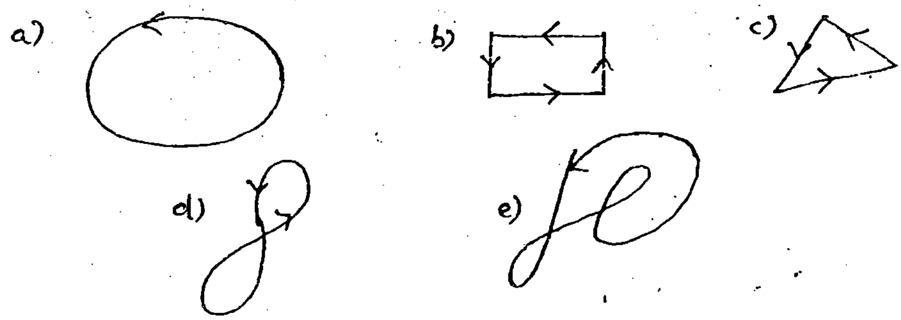


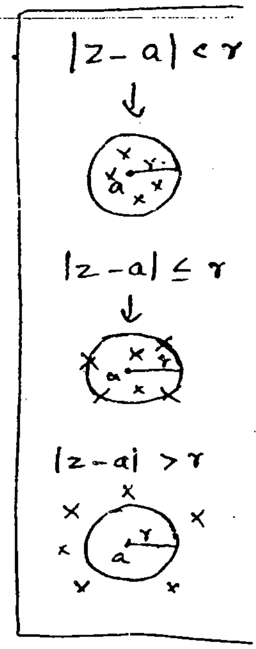
b) multiple curve



c) Closed curve

Initial & final points same → closed curve

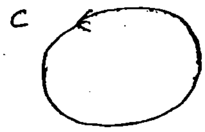



d) Simple closed curve : a), b) & c) above

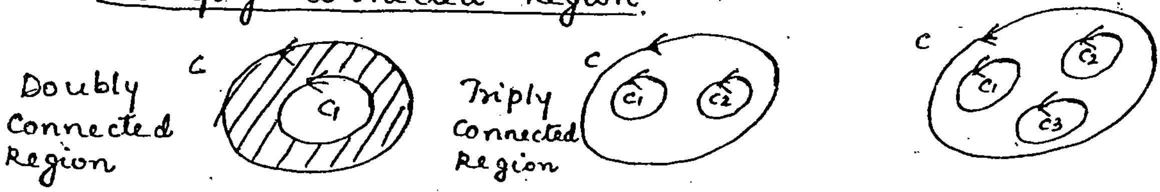
) Types of Regions

a) Simply connected region



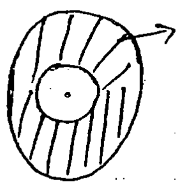

Region bounded by a simple curve.

b) multiply Connected Region.




Region obtained by drawing more than 1 closed curve

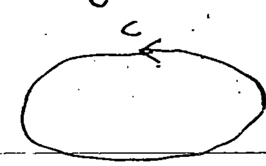
c) Region b/w 2 concentric circles



→ annular/annulus / ring shaped region

Theorem 1: Cauchy's Integral Theorem (C.I.T / C.T)

If $f(z)$ is analytic at every point within on a simple closed curve C, then

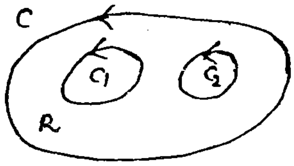



$$\oint_c f(z)\, dz = 0$$

Theorem 2: Cauchy's Integral Theorem for multiply connected Reg

If ~~f(z) is~~ R is a triply connected region, wh outer & inner boundary curves are $C, C_1, C_2$, ~~then~~ and $f(z)$ is analytic within $R$ and on $C, C_1, C_2$ but not within $C_1, C_2$, then

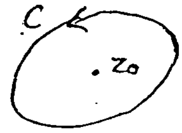
$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz$$




**Theorem 3:** Cauchys' Integral Formula (CIF)

If $f(z)$ is analytic at every point within & on a simple closed curve $C$, & $z_0$ is any point within $C$ then

(i) $\displaystyle\oint_C \frac{f(z)}{(z-z_0)}\,dz = 2\pi i\, f(z_0)$




(ii) $\displaystyle\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n'}(z_0) \rightarrow$ Obtained by differentiating (i)

$\int (\ )\,dz$ $\quad \phi(z) \rightarrow$ See if any poles (singular points) lie within $C$. ~~else~~ If no I
(Theorems)

① Evaluate $\displaystyle\int_C \frac{e^z + \sin(z^3) + \cosh z}{(z-3)^5 (z-6)^4 (z-5)^3}\,dz \qquad |z| = 3/2$

$e^z + \sin(z^3) + \cosh z \rightarrow$ Analytic.

$$\phi(z) = \frac{e^z + \sin(z^3) + \cosh z}{(z-3)^5 (z-6)^4 (z-5)^3}$$

Singular Points $\rightarrow 3, 5, 6 \rightarrow$ do not lie within $|z| = 3/2$
All lie outside the circle.

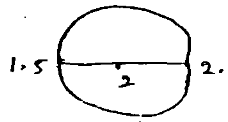
Accdg to C.I.T $\rightarrow \int \phi(z)\,dz = 0$.

~~② $\displaystyle\int_C \frac{2z+4}{z}\,dz \qquad C = |z| = 2$~~

By CIF

$z = 0$ lies within $|z| = 2 \qquad = 2\pi i\, f(0) = 2\pi i \times 4 = \underline{8\pi i}$

③ Evaluate $\displaystyle\int_c \frac{z}{(z-1)(z-2)^3}\, dz$ where $c$ is $|z-2| = 1/2$.




$$\int_c \frac{z/z-1}{(z-2)^3} = \frac{2\pi i}{2}\, f''(2)$$

analytic in $|z-2| = 1/2$ $\leftarrow$ $f(z) = \dfrac{z}{z-1}$ $\qquad f'(z) = \dfrac{1}{z-1} - \dfrac{z}{(z-1)^2} = \dfrac{z-1-z}{(z-1)^2} = \dfrac{-}{(z}$

$$f''(z) = \frac{d}{dz}\left(-(z-1)^{-2}\right)$$

$$= \frac{-1 \times -2}{(z-1)^3} = \frac{2}{(z-1)^3}.$$

$$\int_c \frac{z/z-1}{(z-2)^3} = \frac{2\pi i}{\cancel{2}}\; \frac{\cancel{2}}{1} = 2\pi i.$$

④ $\displaystyle\int \frac{e^z}{(z-1)(z-2)}\, dz \qquad c : |z| = 10$

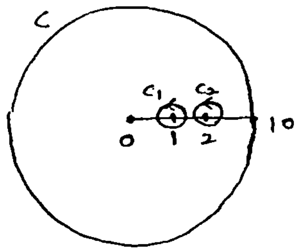
$$\left[ \frac{1}{(z-a)(z-b)} = \frac{1}{(a-b)(z-a)} - \frac{1}{(a-b)(z-b)} \right]$$

$$= \int \frac{e^z\, dz}{z-2} - \int \frac{e^z}{z-1}\, dz$$ } Using Partial Fractions

$$= 2\pi i e^2 - 2\pi i e = 2\pi i e\,(e-1).$$

Using method ①:

$$\oint_c \phi(z)\, dz = \int_{c_1} \phi(z)\, dz + \int_{c_2} \phi(z)\, dz$$




$$= \int_{c_1} \frac{e^z}{(z-1)(z-2)}\, dz + \int_{c_2} \frac{e^z\, dz}{(z-1)(z-2)}$$

$$\int_{c_1} \frac{e^z/z-2}{z-1}\, dz + \int_{c_2} \frac{e^z/z-1}{z-2}\, dz$$

$$\frac{2\pi i\, e}{-1} + \frac{2\pi i\, e^2}{2-1} = 2\pi i\,(e^2 - e).$$

5) $\int_C \frac{\sin z^2 + e^z}{(z-1)(z-2)(z-6)}$  $C: |z| = 10$

$\int_{C_1} \frac{(\sin z^2 + e^z)/(z-2)(z-6)}{z-1} + \int_{C_2} \frac{(\sin z^2 + e^z)/(z-1)(z-2)}{z-6} + \int_{C_3} \frac{(\sin z^2 + e^z)/(z-\ldots}{z-2}$

;) $\int_C \frac{\bar{z}}{z} dz$  $C: |z| = 1$

$\bar{z} \rightarrow$ Not analytic . ∴ Defn method used

$z = r e^{i\theta}$

$z = e^{i\theta}$  $dz = i e^{i\theta} d\theta$

$\int_0^{2\pi} \frac{\bar{e}^{i\theta}}{\cancel{e^{i\theta}}} i \cancel{e^{i\theta}} d\theta = i \int_0^{2\pi} e^{-i\theta} d\theta = \frac{i}{-i} [e^{-i\theta}]_0^{2\pi}$

$= -1 [e^{-2\pi i} - e^0]$

$= \underline{\underline{0}}$

). $\int_C \frac{z}{(z-1)} dz$  $C: |z| = 1$

$z = 1$ boundary pt  $z_0$ shd be interior pt.
∴ Theorem cannot be applied.

$|z| = 1$

$z = e^{i\theta}$  $dz = i e^{i\theta} d\theta$  $\theta: 0 - 2\pi$

$\int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta} - 1} i e^{i\theta} d\theta$  $e^{i\theta} = t$

$i e^{i\theta} d\theta = dt$

$\int_0^{2\pi} \frac{t}{t-1} dt$

↓

No need.

$f(z) = \frac{z}{z-1}$ is not defined at all pts on $|z| = 1$

We cannot integrate such an i...

## COMPLEX POWER SERIES

An infinite series of the form

$$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \dots a_n(z-z_0)$$

(OR)

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is called complex power series,

in powers of $(z-z_0)$ or about a pt $z = z_0$

In the above series $a_0, a_1, a_2, \dots a_n \dots$ are real or complex constants which are called coefficients of the power series, $z$ is a complex variable & $z_0$ is a fixed real or complex constant which is called centre of the power series

Eg: (i) $1 + x + x^2 + x^3 + \dots = (1-x)^{-1}$ $\quad |x| < 1$ ROC.

(ii) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$ for all values of $x$. $x \in$

(iii) $1 + z + z^2 + z^3 + \dots = (1-z)^{-1}$ $\quad |z| < 1$ within unit circle

$|z| = 1$ circle of convergence

Radius of convergence

For above series

$|z - z_0| = r$ is the circle of convergence

Converge for the neighbourhood of a pt $z_0$. $|z - z_0| < r$

ROC

Region of Convergence (ROC) : Set of values of $z$ for which the series converges is called region of convergence of a power series.

Note : (i) Radius of convergence : If $r$ is a radius of convergence of a power series $\sum_0^\infty a_n(z-z_0)^n$, then $r$ is given by

(a) $$r = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$$

(b) $$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

① Find the radius of convergence, ROC & Circle of Convergence for the following series.

$$\sum (3+4i)^n z^n$$

Comparing with general power series

$a_n = (3+4i)^n \longrightarrow$ If $[a_n]$ contains $n$ use formula

$z_0 = 0$

$$r = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}} = \lim_{n\to\infty} \frac{1}{[((3+4i)^n)]^{1/n}} = \lim_{n\to\infty} \frac{1}{5^{n\times\frac{1}{n}}} = \frac{1}{5}.$$

COC : $|z| = \dfrac{1}{5}$

ROC : $|z| < \dfrac{1}{5}$

② $\displaystyle \sum \frac{(-1)^n (z+2i)^n}{n}$

$a_n = \dfrac{(-1)^n}{n}$ $\qquad z_0 = -2i$

$\rightarrow$ Total power not $n \rightarrow$ So second one.

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(-1)^n (n+1)}{n(-1)^{n+1}}\right|$$

$$= \lim_{n\to\infty} \left| -\frac{(n+1)}{n} \right| = \lim_{n\to\infty} \left| 1+\frac{1}{n} \right|$$

$$= 1.$$

$|z+2i| = 1$ : COC

$|z+2i| < 1$ : ROC

Complex Power Series

$f(z)$ is analytic at $z_0$ $\downarrow$ Taylor series

$f(z)$ is not analytic at $z$ $\downarrow$ Laurentz series.

## TAYLOR'S THEOREM

If $f(z)$ is analytic at every point within a circle having centre at $z_0$, then for every point $z$ within the circle the function $f(z)$ can be expressed as a complex power series in positive powers of $z-z_0$ or about a point $z = z_0$.

$$\text{ie } f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \dots$$

$$\frac{(z-z_0)^n}{n!} f^{n'}(z_0) + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{n!}(z_0)$$

of the form $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$\text{where } a_n = \frac{f^{n'}(z_0)}{n!}$$

The RHS of above is called Taylors' Series about a pt z & the ROC of a Taylor series is given by $|z - z_0| < r$ where $r$ is a distance b/w the pt at which series is req & the nearest singular point of the function $f(z)$ to that

∴ Complex limit → $\lim_{z \to z_0} f(z)$ independent of the path.

General Problems in Complex Analysis:

(i) Find the value of $\frac{-5+10i}{3+4i}$ a) $1-2i$ ~~b)~~ $1+2i$ c) $2-i$ d) $2+$

$$= \frac{(-5+10i)(3-4i)}{25} \Rightarrow \frac{-15+40+50i}{25} = \underline{\underline{1+2i}}.$$

× by complex conjugate $(a+bi)(a-bi) = a^2+b^2$

---

i) The product of $(3-2i)(3+4i) = 9+8+12i-6i = \underline{17+6i}$

ii) $\left|\frac{3+4i}{1-2i}\right| = \frac{|3+4i|}{|1-2i|} = \frac{5}{\sqrt{5}} = \underline{\underline{\sqrt{5}}}.$

(iv) $z = x + iy$ where $x, y$ are real

$|e^{iz}|$ is

a) 1 b) $e^{\sqrt{x^2+y^2}}$ c) $e^{y}$ ~~d)~~ $e^{-y}$

$e^{iz} = e^{-y} \times e^{ix}$

$|e^{iz}| = |e^{-y}| \times |e^{ix}| = \underline{\underline{e^{-y}}}$

$|\cos x + i\sin x| \rightarrow \sqrt{\cos^2 x + \sin^2 x} = 1.$

(v) If $i = \sqrt{-1}$, then $i^{i}$ a) $\sqrt{i}$ b) $-1$ c) $\pi/2$ ~~d)~~ $e^{-\pi/2}$

$i = (-1)^{1/2}$ & $e^{i\pi/2} = \cos \pi/2 + i\sin\pi/2 = i$

$i = e^{i\pi/2} \rightarrow i^{i} = (e^{i\pi/2})^{i} = \underline{\underline{e^{-\pi/2}}}$.

(vi) $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$, then find $z^4$ a) $2\sqrt{3} + 2i$ ~~b)~~ $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$

c) $\frac{\sqrt{3}}{2} - \frac{i}{2}$ d) $\frac{\sqrt{3}}{8} + \frac{i}{8}$

$z^2 = \left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{2} + \frac{\sqrt{3}\,i}{2} = \frac{1}{4} + \frac{\sqrt{3}\,i}{2}$

Find $z^4$ & so on

Also $z = \frac{\sqrt{3}}{2} + \frac{i}{2} = e^{i\pi/6} = \underbrace{\cos\frac{\pi}{6}}_{\sqrt{3}/2} + i\underbrace{\sin\frac{\pi}{6}}_{1/2}$

※ Convert $z$ into polar coordinates.

$z^4 = (e^{i\pi/6})^4 = e^{i2\pi/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= \cos(\pi - \pi/3) + i\sin(\pi - \pi/3) = -\cos\pi/3 + i.$

$= \underline{\underline{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}}$

(vii) One of the root of $x^3 = i$ where $i = \sqrt{-1}$ is

a) $i$ ~~b)~~ $\frac{\sqrt{3}}{2} + \frac{i}{2}$ c) $\frac{\sqrt{3}}{2} - \frac{i}{2}$ d) $-\frac{\sqrt{3}}{2} - \frac{i}{2}$

$x^3 = \boxed{i = e^{i\pi/2}}$

$\rightarrow x = (e^{i\pi/2})^{1/3} = e^{i\pi/6} = \underline{\underline{\sqrt{3}/2 + i\frac{1}{2}}}$

viii) Evaluate $\int_0^{1+i} (x-y+ix^2)\,dz$ along $y=x$

$y=x \rightarrow dy = dx \quad dz = dx + i\,dx$

$\int_0^{1+i} ix^2(1+i)\,dx \Rightarrow (1+i)\times i \int_0^{1[redacted]} x^2\,dx = (i-1)\left[\frac{x^3}{3}\right]_0^{1[redacted]}$

$\Rightarrow \frac{(i-1)(\overset{1}{[redacted]})^3}{3} = \cancel{\frac{(i^2-1)(i+1)^2}{3}} = \cancel{\frac{-2}{3}(i+1)}$

$\Rightarrow \cancel{\frac{-2}{3}[2i]} = \cancel{-4i/3} = \frac{i-1}{3}$

$\dot{X}$ Convert into single variable apply $x$ limit only.

ix) $\int_c (z^2+4z)\,dz, \quad c \rightarrow |z|=4$

$\underbrace{(z^2+4z)}_{f(z)}$

$\rightarrow f(z)$ is defined at all pts in $|z|=4$ ie it is analytic

$\therefore \int_c f(z)\,dz = \int_c (z^2+4z)\,dz = 0$ in $|z|=4$

Normally only Numerator fn always analytic

Exceptions: Trigonometric Ratios $\cot x$, $\tan x$ etc.

x) $\int_c \left(\frac{z^2+z+2}{z-4}\right) dz$ where $c \Rightarrow |z|=3$.

$\downarrow$

$f(z)$ is analytic at all pts except at $z=4$

But pt of singularity $z=4$ doesn't lie in $|z|=3$

$\therefore f(z)$ is analytic in $|z|=3$

$\therefore \int_c f(z) = 0$.

(i) $\int_c \frac{z^2+2z+3}{(z-2)}\,dz$ where $|z|=3$. $\quad (z^2+2z+3) \rightarrow f(z)$

$\downarrow$

$z=2$ is a singular pt in $|z|=3$ Apply C.I.F.

$\rightarrow 2\pi i\, f(2) \Rightarrow 2\pi i\,[4+4+3] = 22\pi i$.

(ii) $\int_c \frac{z^2-z+1}{z-1}\,dz \quad c \rightarrow |z|=2$

$\Rightarrow 2\pi i\, f(1) = 2\pi i \quad$ (C.I.F).

(iii) $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$ where 'c' is $|z| = 3$. $\rightarrow$ Done earlier.

(iv) $\int_c \frac{e^{2z}}{(z-1)(z-3)}$ where $c \rightarrow |z| = 2$

$$= \int_c \frac{\overbrace{(e^{2z}/z-3)}^{f(z)}}{z-1} \rightarrow 2\pi i\, f(1) = \cancel{2}\pi i \left[\frac{e^2}{-\cancel{2}}\right] = \underline{\underline{-\pi i e^2}}$$

Since 3 doesn't lie in $|z| = 2$

(xv) $\int_c \frac{dz}{z^2 e^z}$ where 'c' is $|z| = 1$

$$= \int_c \frac{1/e^z \, dz}{(z-0)^2} \Rightarrow 2\pi i\, f'(0).$$

$$f(z) = e^{-z} \Rightarrow f'(z) = -e^{-z}$$

$$\therefore \int \frac{1/e^z \, dz}{(z-0)^2} = 2\pi i\,(-e^{-0}) = \underline{\underline{-2\pi i}}.$$

(vi) $\int_c \frac{e^{2z}}{(z-1)^2(z-3)} dz$ $\quad c \rightarrow |z| = 2$

$$\Rightarrow \int_c \frac{e^{2z}/z-3}{(z-1)^2} \rightarrow 2\pi i\, f'(1)$$

$$f(z) = \frac{e^{2z}}{z-3} \rightarrow f'(z) = \frac{2e^{2z}}{z-3} - \frac{e^{2z}}{(z-3)^2} \rightarrow f'(1) = \frac{\cancel{2}e^2}{-\cancel{2}} - \frac{e^2}{4}$$

$$= -e^2 - \frac{e^2}{4}$$

$$= -\frac{5}{4}e^2$$

$$\Rightarrow \int_c \frac{e^{2z}/z-3}{(z-1)^2} = \cancel{2}\pi i \times \frac{-5}{\cancel{4}_{\,2}} e^2 = \underline{\underline{-2.5\pi i e^2}}$$

(vii) $\int_c \frac{\sin^2 z}{(z-\pi/6)^3}\, dz \longrightarrow c = |z| = 1$

$\Rightarrow \pi/6 = 3.14/6 \simeq 0.5$ lies inside

$\Rightarrow \int_c \frac{\sin^2 z}{(z-\pi/6)^3} = \frac{2\pi i}{2!} f''(\pi/6)$

$f(z) = \sin^2 z$

$f'(z) = 2\sin z \cos z = \sin 2z$

$f''(z) = 2\cos 2z$

$\int_c \frac{\sin^2 z}{(z-\pi/6)^3} = \pi i \; 2\cos\frac{\pi}{3} = 2\pi i \times \frac{1}{2} = \underline{\underline{\pi i}}.$

(viii) $\int_c \frac{z^3 - 2z + 1}{(z-i)^2}\, dz \longrightarrow c \; |z| = 2$

$z = i$ lies in $|z| = 2$

$\therefore \int = 2\pi i . f'(i)$

$f'(z) = 3z^2 - 2$

$f'(i) = -3 - 2 = \underline{\underline{-5}}$

$\therefore \int = 2\pi i \times -5 = \underline{\underline{-10\pi i}}.$

xix) Evaluate $f(2)$ & $f(3)$ where $f(a) = \int_c \frac{2z^2 - z - 2}{(z-a)}\, dz$

where $c$ is the circle $|z| = 2.5$

$f(2) = \int_c \frac{2z^2 - z - 2}{z-2} \Rightarrow \therefore 2\pi i\, f(2) = 2\pi i\,[8-4] = \underline{\underline{8\pi i}}$

$\downarrow$ lies in $|z| = 2.5$

$f(3) = \int_c \left(\frac{2z^2 - z - 2}{z-3}\right) dz$

$\downarrow$ analytic in $|z| = 2.5$

$\therefore$ By C.I.T $\int_c f(z)\, dz = 0.$

1

## Laurentz Series

Let $f(z)$ be analytic fn in a ring shaped region $R$ bounded by 2 concentric circles $C_1$ & $C_2$ with radius $r_1$ & $r_2$ ($r_1 < r_2$) with centre $z_0$. For any pt $z$ in $R$,

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \ldots$$
$$+ a_{-1}(z-z_0)^{-1} + a_{-2}(z-z_0)^{-2} + a_{-3}(z-z_0)^{-3} \ldots$$

ie
$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}}_{\text{Principal Part}}$$

$z = z_0$ essential singularity

where
$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

A pt at which $f(z)$ is 0 is called zero of the fn.
A pt at which $f(z)$ is not existing is called singularity.

Essential singularity: If $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$
$+ a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} \ldots$

then $z = a$ is called essential singularity.

$$e^{1/z-a} = 1 + \frac{1}{z-a} + \frac{1}{(z-a)^2 2!} + \frac{1}{3!(z-a)^3} + \ldots$$

$$\left[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \right]$$

Isolated singularity: Consider $f(z) = \dfrac{z+3}{z(z-2)}$

A pt $z=a$ is an isolated singularity if there exists a neighbourhood at $a$ in which there is no singularity other than $a$. In above $z=0$ & $z=2$ are both isolated singularity.

Pole of order 1 or Singular Pole:

If $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots a_{-1}(z-a)^{-1}$

then $z=a$ is called pole of order 1.

Pole of Order 2

If $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots a_{-1}(z-a)^{-1} + a_{-2}(z-a$

then $z=a$ is called pole of order 2.

Pole of order m

If $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-$

$+ a_{-m}(z-a)^{-m}$,

then $z=a$ is called pole of order m.

1) The singularities of $f(z) = \dfrac{z-1}{z^2+1}$ are

$z^2+1=0 \quad \underline{\underline{z=\pm i}}$. [In the total complex plan

2) $f(z) = \dfrac{e^z}{(z-3)^2(z^2+4)}$ $\quad (z-3)^2(z^2+4)=0 \to z=3, \pm 2i$

$z=3$ is a pole of order 2

$z=\pm 2i$

3) $f(z) = \dfrac{1}{\sin z - \cos z}$ @ $z=\pi/4$

@ $z=\pi/4$ $\quad \sin z - \cos z = 0$

@ $z=\pi/4$

4) $\dfrac{1-e^{2z}}{z^4}$ at $z=0$.

$z=0 \to$ Pole of order 4 $\longrightarrow$ Wrong Expand $e^{2z}$

$$f(z) = \frac{1-\left[1+2z+\frac{4z^2}{2}+\frac{8z^3}{6}+\dots\right]}{z^4} = \frac{2z+2z^2+4/3 z^3}{z^4}$$

$$= \frac{2+2z+4/3\, z^2+\dots}{z^3} \qquad \therefore z=0 \to \underline{\underline{\text{Pole of order 3}}}.$$

Residues of $f(z)$ at its poles.

i) $z = a$ is a pole of order one

Residue of $f(z)$ at $z = a$ is $= \lim_{z\to a} (z-a)\, f(z)$

ii) $z = a$ is a pole of order two

Residue of $f(z)$ at $z = a$ $= \frac{1}{1!} \lim_{z\to a} \frac{d}{dz} \left\{ (z-a)^2 f(z) \right\}$

iii) $z = a$ is a pole of order three

Residue of $f(z)$ at $z = a \Rightarrow \frac{1}{2!} \lim_{z\to a} \frac{d^2}{dz^2} \left\{ (z-a)^3 f( \right.$

In general,

If $z = a$ is a pole of order $m$,

Residue of $f(z)$ at $z = a \Rightarrow \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m \right.$

i) $f(z) = \frac{e^{2z}}{(z-1)^3}$ Residue at $z = 1$ is ?

Res $(f(z))$ at $z = 1 \rightarrow \frac{1}{2!} \lim_{z\to 1} \frac{d^2}{dz^2} \left\{ \cancel{(z-1)^3} \frac{e^{2z}}{\cancel{(z-1)^3}} \right\}$

$= \frac{1}{2!} \lim_{z\to 1} 4 e^{2z}$

$= \frac{4e^2}{2!} = \underline{\underline{2e^2}}$

2) Residue of $f(z) = \frac{1}{(z^2+1)^3}$ at $z = i$

$= \frac{1/(z+i)^3}{(z-i)^3}$

Res at $z = i \rightarrow \frac{1}{2!} \lim_{z\to i} \frac{d^2}{dz^2} \left[ \cancel{(z-i)^3} \times \frac{1}{(z+i)^3 \cancel{(z-i)^3}} \right]$

$\rightarrow \frac{1}{2!} \lim_{z\to i} 1 \times \frac{6 \times -4}{(z+i)^5}$

$\rightarrow \frac{1}{2!} \times \frac{6\times -4}{(2i)^5} = \frac{3 \times -4}{16 (i^2)^2 \times 2i} \Rightarrow \underline{\underline{\frac{3}{16}}} \times \frac{-4^2}{2i}$

$= \frac{-6}{16 i}$

1) Res. of $f(z) = \dfrac{1}{(z^2+1)^2}$ @ $z = i$ is

$$f(z) = \frac{1}{(z+i)^2(z-i)^2} \rightarrow \lim_{z\to i} \frac{d}{dz}\left\{ \cancel{(z-i)^2} \times \frac{1}{(z+i)^2\cancel{(z-i)^2}} \right\}$$

$$\lim_{z\to i} \frac{-2}{(z+i)^3} \Rightarrow \frac{-2}{(2i)^3} = \frac{-2}{8i^3} = \frac{1}{4i}$$

* $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ Calculate res. at ech of the poles.

$z = 1, \quad z = -2$

Res at $z = 1 \rightarrow \lim_{z\to 1} \dfrac{d}{dz}\left\{ \cancel{(z-1)^2} \dfrac{z^2}{\cancel{(z-1)^2}(z+2)} \right]$

$$\Rightarrow \lim_{z\to 1} \frac{d}{dz}\left[\frac{z^2}{z+2}\right] \Rightarrow \lim_{z\to 1} \left[\frac{2z}{z+2} - \frac{z^2}{(z+2)^2}\right]$$

$$= \frac{2}{3} - \frac{1}{9} = \frac{6-1}{9} = \frac{5}{9}.$$

Res at $z = -2 \rightarrow \lim_{z\to -2} \cancel{(z+2)} \dfrac{z^2}{(z-1)^2\cancel{(z+2)}} \Rightarrow \dfrac{4}{9}.$

1) Res. of $f(z) = \dfrac{1-2z}{z(z-1)(z-2)}$ at its poles

a) $1/2, -1/2, 1$ b) $1/2\ 1/2, -1$ c) $1/2, 1, -3/2$ d) $1/2, -1, 3/2$

At $z \Rightarrow 0$ $\lim_{z\to 0} \dfrac{1-2z}{(z-1)(z-2)} = \dfrac{1}{2}.$

At $z \Rightarrow 1$ $\lim_{z\to 1} \dfrac{1-2z}{z(z-2)} = \dfrac{-1}{-1} = 1$

At $z \Rightarrow 2$ $\lim_{z\to 2} \dfrac{1-2z}{z(z-1)} = \dfrac{-3}{2}.$

* $f(z) = \dfrac{1}{(z+2)^2(z-2)^2}$ at $z=2$ is

$$\lim_{z\to 2} \frac{d}{dz}\left\{ \cancel{(z-2)^2} \times \frac{1}{(z+2)^2\cancel{(z-2)^2}} \right\} \to \lim_{z\to 2} \frac{-2}{(z+2)^3} = \frac{-2}{64}$$

$$= \frac{-1}{32}$$

Cauchys' Residue theorem.

$f(z)$ is analytic in a closed curve $c$ except at a finite no. of points, then $\int_c f(z)\,dz = 2\pi i$ { sum of residues of $f(z)$ at each of the poles

* $\oint_c \dfrac{1}{(z+2)^2(z-2)^2}\,dz \qquad |z| = 5$

$$= 2\pi i \left[ \text{Res}(2) + \text{Res}(-2) \right]$$

$$= 2\pi i \left[ \lim_{z\to 2} \frac{d}{dz} \frac{1}{(z+2)^2} + \lim_{z\to -2} \frac{d}{dz} \frac{1}{(z-2)^2} \right]$$

$$= 2\pi i \left[ \lim_{z\to 2} \frac{-2}{(z+2)^3} + \lim_{z\to -2} \frac{-2}{(z-2)^3} \right] = 2\pi i \left[ \frac{-2}{64} + \frac{2}{64} \right]$$

$$= \underline{\underline{0}}.$$

* $\oint_c \dfrac{e^z}{z^2+1}\,dz$ where $c$ is the region $|z| = 2$

$$\oint_c \frac{e^z}{(z-i)(z+i)} = 2\pi i \left\{ \text{Res}(i) + \text{Res}(-i) \right\}$$

$$= 2\pi i \left\{ \lim_{z\to i} \frac{e^z}{z+i} + \lim_{z\to -i} \frac{e^z}{z-i} \right\}$$

$$= 2\pi i \left\{ \frac{e^i}{2i} + \frac{e^{-i}}{-2i} \right] = \pi\left[ e^i - e^{-i} \right]$$

* $\int_c \dfrac{z^2+1}{z(2z+1)}\,dz \qquad |z| = 1$

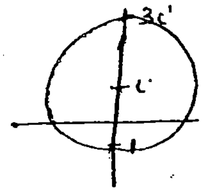
$$\frac{1}{2}\int_c \frac{z^2+1}{z(z+\frac{1}{2})} = \frac{2\pi i}{2}\left\{ \text{Res}(0) + \text{Res}(-1/2) \right\}$$

$$= \pi i \left\{ \lim_{z\to 0} \frac{z^2+1}{z+1/2} + \lim_{z\to -1/2} \frac{z^2+1}{z} \right\}$$

$\frac{5}{4}\times 2$
$\frac{-1}{2}$

$$= \pi i \left\{ 2 + -\frac{5}{2} \right] = \underline{\underline{\frac{-\pi i}{2}}}.$$

(*) $\oint_C \frac{dz}{(z^2+4)^2} \rightarrow$ c is $|z-i| = 2$




$$= \oint \frac{dz}{(z+2i)^2(z-2i)^2} = 2\pi i \lim_{z\to 2i} \frac{d}{dz} \frac{1}{(z+2i)^2}$$

$$= 2\pi i \lim_{z\to 2i} \frac{-2}{(z+2i)^3} = 2\pi i \times \frac{-2}{(4i)^3}$$

Only $z = 2i$ lies in circle.

$$= -\frac{4\pi i}{64 i^3}_{16} = \frac{-\pi}{16 i^2} = \underline{\underline{\pi/16}}$$

(*) $\oint_C \frac{z^2}{(z-1)^2(z+2)} dz$ where $c = |z| = 2.5$

$$= 2\pi i \{\text{Res}(1) + \text{Res}(-2)\}$$

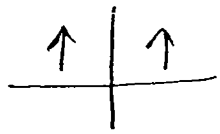
$$= 2\pi i \left\{ \lim_{z\to 1} \frac{d}{dz} \frac{z^2}{z+2} + \lim_{z\to -2} \frac{z^2}{(z-1)^2} \right\}$$

$$= 2\pi i \left\{ \lim_{z\to 1} \left\{ \frac{2z}{z+2} - \frac{z^2}{(z+2)^2} \right\} + \frac{4}{9} \right\} = 2\pi i \left\{ \frac{2}{3} - \frac{1}{9} + \frac{4}{9} \right\}$$

$$= \underline{\underline{2\pi i}}$$

(*) Evaluation of integrals of the type $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} dx$

$= 2\pi i \left\{ \text{Sum of residues of } \frac{f(z)}{F(z)} \right\}$ at its poles the upper half plane



Upper half plane → Semicircle with Rad $\infty$

(Poles lying on real axis we cannot apply this)

(*) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ $\qquad f(z) = \frac{1}{1+z^2}$

Poles at $z = \pm i$ Only $z = +i$ lies in upper

Res. of $f(z)$ at $z = i$ $= \lim_{z\to i} \frac{1}{z+i} = \underline{\underline{\frac{1}{2i}}}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \times \frac{1}{2i} = \underline{\underline{\pi}}.$$