

Ellipse Question

EE24BTECH11040 - Mandara Hosur

Question:

Given that the length of the major axis of a conic is 16 units and the coordinates of the foci are $(0, \pm 6)$, find the equation of the conic.

Solution:

Variable	Description	Value
$2k$	Length of major axis	16
\mathbf{F}_1	One of the foci of the conic	$\begin{pmatrix} 0 \\ 6 \end{pmatrix}$
\mathbf{F}_2	Other foci of the conic	$\begin{pmatrix} 0 \\ -6 \end{pmatrix}$

TABLE 0: Given Information

2 foci exist for this conic. Therefore, it must be an ellipse or a hyperbola. The general equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$ can be written as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f \quad (0.1)$$

$$\mathbf{u} = \frac{\mathbf{F}_1 + \mathbf{F}_2}{2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{n} \equiv \mathbf{F}_1 - \mathbf{F}_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (0.2)$$

From

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top \quad (0.3)$$

we get

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \quad (0.4)$$

We know that the major axis is the y-axis. Assume some point $\begin{pmatrix} 0 \\ k \end{pmatrix}$ on the y-axis that

satisfies 0.1. This gives

$$k = \sqrt{\frac{|f|}{|1 - e^2|}} \quad (0.5)$$

$$\Rightarrow 2k = 16 = 2\sqrt{\frac{|f|}{|1 - e^2|}} \quad (0.6)$$

Rearranging 0.6 gives

$$8\sqrt{|1 - e^2|} = \sqrt{|f|} \quad (0.7)$$

Now, $\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2}$ and $\lambda_2 = \|\mathbf{n}\|^2$.

$$\Rightarrow \mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\|\mathbf{n}\|^2} \quad (0.8)$$

Substituting values of \mathbf{F}_1 , \mathbf{F}_2 , \mathbf{n} , and \mathbf{u} and simplifying reduces 0.8 to

$$\pm ce^2 = 6 \quad (0.9)$$

Consider another equation

$$c = \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2 (\mathbf{u}^\top \mathbf{n})^2 - \lambda_2 (e^2 - 1) (\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e (e^2 - 1)}$$

Substituting known values gives

$$c = \pm \frac{1}{e} \sqrt{\frac{|f|}{|e^2 - 1|}} \quad (0.10)$$

We have 3 equations (0.7, 0.9, 0.10) and 3 variables (c , f , e).

Upon solving, we get $e = \frac{3}{4}$, $c = \pm \frac{32}{3}$, and $|f| = 28$

However, f can only have one value (either +28 or -28).

To find this, assume some point $(0, \alpha)$ to lie on the major axis of the conic. Substituting this point in the general conic equation 0.1 along with values of \mathbf{V} and \mathbf{u} gives

$$\frac{7\alpha^2}{16} + f = 0$$

This implies that f must be negative, and therefore $f = -28$.

The equation of the conic thus becomes

$$g(\mathbf{x}) = \mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{7}{16} \end{pmatrix} \mathbf{x} - 28 \quad (0.11)$$

Note that since $e < 1$, (0.11) represents the equation of an ellipse (verified by plot below).

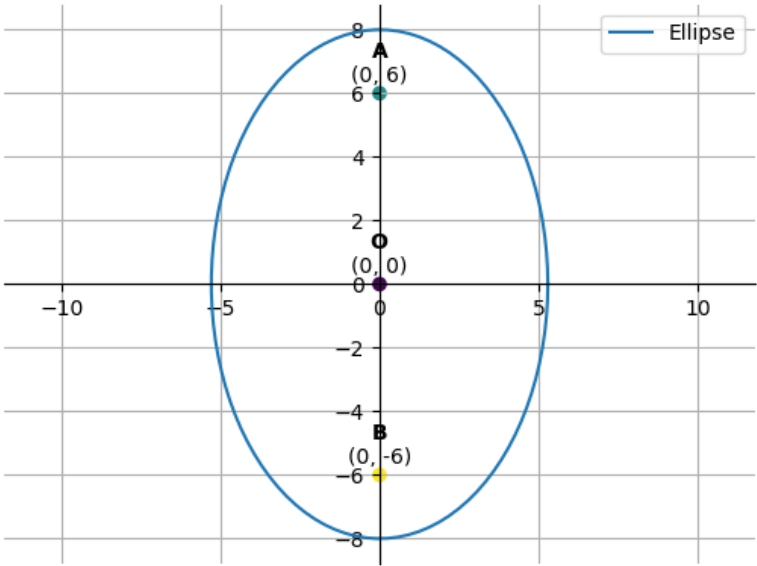


Fig. 0.1: Plot of conic

Code for Figure (0.1) can be found at:

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codes/code.py
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