Ellipse Question

EE24BTECH11040 - Mandara Hosur

Question:
Given that the length of the major axis of a conic is 16 units and the coordinates of the foci are $(0, \pm 6)$, find the equation of the conic.

Solution:

Variable	Description	Value
$2k$	Length of major axis	16
$\mathbf{F_1}$	One of the foci of the conic	$\begin{pmatrix} 0 \\ 6 \end{pmatrix}$
$\mathbf{F_2}$	Other foci of the conic	$\begin{pmatrix} 0 \\ -6 \end{pmatrix}$

TABLE 0: Given Information

Two foci exist for this conic. Therefore, it must be an ellipse or a hyperbola.
The general equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$ can be written as,

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f \tag{0.1}$$

$$\mathbf{u} = \frac{\mathbf{F_1} + \mathbf{F_2}}{2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.2}$$

$$\mathbf{n} \equiv \mathbf{F_1} - \mathbf{F_2} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.3}$$

From

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{0.4}$$

we get

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \tag{0.5}$$

We know that the Y - axis is the major axis. The points $\begin{pmatrix} 0 \\ k \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -k \end{pmatrix}$ on the Y - axis satisfy 0.1 as they are vertices of the ellipse. Upon substitution we get,

$$k = \sqrt{\frac{|f|}{|1 - e^2|}} \tag{0.6}$$

$$\implies 2k = 16 = 2\sqrt{\frac{|f|}{|1 - e^2|}} \tag{0.7}$$

Rearranging 0.7 gives

$$8\sqrt{|1 - e^2|} = \sqrt{|f|} \tag{0.8}$$

Now,

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{0.9}$$

$$\lambda_2 = \|n\|^2 \tag{0.10}$$

$$\implies \mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\|n\|^2} \tag{0.11}$$

Substituting values of $\mathbf{F_1}$, $\mathbf{F_2}$, $\mathbf{n}$, and $\mathbf{u}$ and simplifying 0.11, we get

$$ce^2 = \pm 6 \tag{0.12}$$

Using the below equation,

$$c = \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_2\left(e^2 - 1\right)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e\left(e^2 - 1\right)} \tag{0.13}$$

Substituting known values in (0.13) gives

$$c = \pm\frac{1}{e}\sqrt{\frac{|f|}{|e^2 - 1|}} \tag{0.14}$$

We have 3 equations 0.8, 0.12, 0.14 and 3 variables (c, f, e).
Upon solving, we get $e = \frac{3}{4}$, $c = \pm\frac{32}{3}$, and $|f| = 28$
Using the derived values, we find that,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{7}{16} \end{pmatrix} \tag{0.15}$$

Since $e < 1$, the conic is an ellipse, and since the ellipse is already in standard form, f must be negative. Thus,

$$f = -28 \tag{0.16}$$

Thus the equation of the conic becomes,

$$g(\mathbf{x}) = \mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{7}{16} \end{pmatrix} \mathbf{x} - 28 \tag{0.17}$$

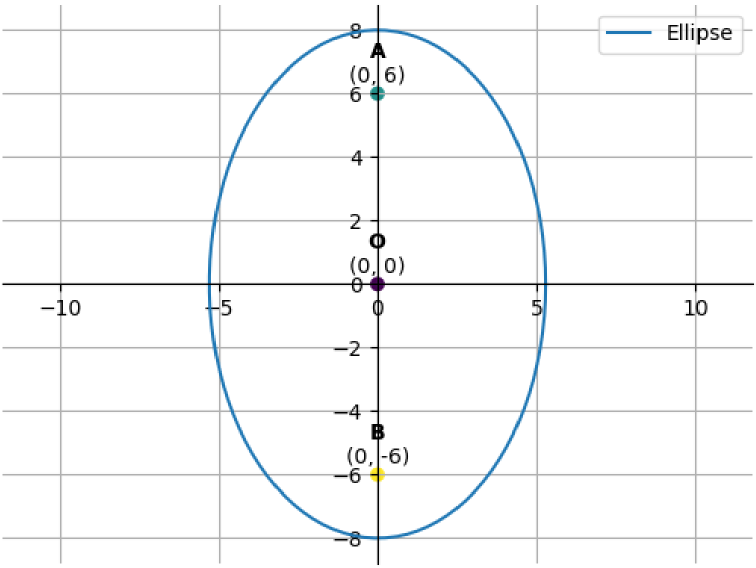

Fig. 0.1: Plot of conic

Code for Figure (0.1) can be found at:

codes/code.py