Ellipse Question

EE24BTECH11040 - Mandara Hosur

Question:
Given that the length of the major axis of a conic is 16 units and the coordinates of the foci are $(0, \pm 6)$, find the equation of the conic.

Solution:

Variable	Description	Value
$2k$	Length of major axis	16
$\mathbf{F_1}$	One of the foci of the conic	$\begin{pmatrix} 0 \\ 6 \end{pmatrix}$
$\mathbf{F_2}$	Other foci of the conic	$\begin{pmatrix} 0 \\ -6 \end{pmatrix}$

TABLE 0: Given Information

2 foci exist for this conic. Therefore, it must be an ellipse or a hyperbola.
The general equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$ can be written as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f \tag{0.1}$$

$$\mathbf{u} = \frac{\mathbf{F_1} + \mathbf{F_2}}{2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{n} \equiv \mathbf{F_1} - \mathbf{F_2} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.2}$$

From 0.2, we can conclude that we have a standard conic, with its center at the origin, foci on a coordinate axis (y-axis), and directrices parallel to a coordinate axis (x-axis). Therefore conic equation is either

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \tag{0.3}$$

Given that $e\left(= \sqrt{1 \pm \frac{a^2}{b^2}}\right)$ represents the eccentricity of the conic, comparing 0.3 and 0.1 gives the relation

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \tag{0.4}$$

We know that the major axis is the y-axis. Assume some point $\begin{pmatrix} 0 \\ k \end{pmatrix}$ on the y-axis that satisfies 0.1. This gives

$$k = \sqrt{\frac{|f|}{|1 - e^2|}} \tag{0.5}$$

$$\implies 2k = 16 = 2\sqrt{\frac{|f|}{|1 - e^2|}} \tag{0.6}$$

Rearranging 0.6 gives

$$8\sqrt{|1 - e^2|} = \sqrt{|f|} \tag{0.7}$$

Now, $\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2}$ and $\lambda_2 = \|n\|^2$.

$$\implies \mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\|n\|^2} \tag{0.8}$$

Substituting values of $\mathbf{F_1}$, $\mathbf{F_2}$, $\mathbf{n}$, and $\mathbf{u}$ and simplifying reduces 0.8 to

$$\pm ce^2 = 6 \tag{0.9}$$

Consider another equation

$$c = \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_2\left(e^2 - 1\right)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e\left(e^2 - 1\right)}$$

Substituting known values gives

$$c = \pm\frac{1}{e}\sqrt{\frac{|f|}{|e^2 - 1|}} \tag{0.10}$$

We have 3 equations (0.7, 0.9, 0.10) and 3 variables (c, f, e).

Upon solving, we get $e = \frac{3}{4}$, $c = \pm\frac{32}{3}$, and $|f| = 28$
However, f can only have one value (either $+28$ or -28).
To find this, assume some point $(0, \alpha)$ to lie on the major axis of the conic. Substituting this point in the general conic equation 0.1 along with values of $\mathbf{V}$ and $\mathbf{u}$ gives

$$\frac{7\alpha^2}{16} + f = 0$$

This implies that f must be negative, and therefore $f = -28$.

The equation of the conic thus becomes

$$g(\mathbf{x}) = \mathbf{x}^\top\begin{pmatrix}1 & 0\\ 0 & \frac{7}{16}\end{pmatrix}\mathbf{x} - 28 \tag{0.11}$$

Note that since $e < 1$, (0.11) represents the equation of an ellipse (verified by plot below).

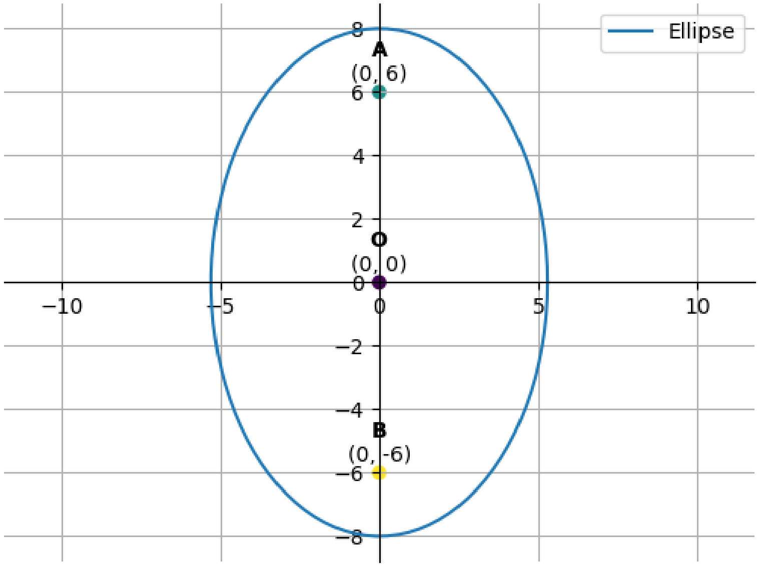

Fig. 0.1: Plot of conic

Code for Figure 0.1 can be found at:

codes/code.py