Parabola Question

EE24BTECH11040 - Mandara Hosur

Question:
Given that the length of the major axis of a conic is 16 units and the coordinates of the foci are $(0, \pm 6)$, find the equation of the conic.

Solution:
2 foci exist for this conic. Therefore, it must be an ellipse or a hyperbola.
The general equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$ can be written as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f$$

Let focus $\mathbf{F_1}$ be $(0, 6)$ and focus $\mathbf{F_2}$ be $(0, -6)$
$\mathbf{u} = \frac{\mathbf{F_1}+\mathbf{F_2}}{2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{n} \equiv \mathbf{F_1} - \mathbf{F_2} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
Given that e represents eccentricity of the conic, $\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix}$
Length of major axis $= 16 = 2\sqrt{\frac{|f|}{|1-e^2|}}$
Rearranging this equation gives

$$8\sqrt{|1 - e^2|} = \sqrt{|f|} \tag{0.1}$$

Now, $\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2}$ and $\lambda_2 = ||n||^2$.

$$\implies \mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{||n||^2}$$

Substituting values of $\mathbf{F_1}$, $\mathbf{F_2}$, $\mathbf{n}$, and $\mathbf{u}$ and simplifying reduces the above equation to

$$\pm ce^2 = 6 \tag{0.2}$$

Consider another equation

$$c = \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_2\left(e^2 - 1\right)\left(||\mathbf{u}||^2 - \lambda_2 f\right)}}{\lambda_2 e\left(e^2 - 1\right)}$$

Substituting known values gives

$$c = \pm\frac{1}{e}\sqrt{\frac{|f|}{|e^2 - 1|}} \tag{0.3}$$

We have 3 equations and 3 variables (c, f, e).

Upon solving, we get $e = \frac{3}{4}$, $c = \pm\frac{32}{3}$, and $|f| = 28$

However, f can only have one value (either $+28$ or -28).
To find this, assume some point $(0, \alpha)$ to lie on the major axis of the conic. Substituting this point in the general conic equation $g(\mathbf{x}) = 0$ along with values of $\mathbf{V}$ and $\mathbf{u}$ gives

$$\frac{7\alpha^2}{16} + f = 0$$

This implies that f must be negative, and therefore $f = -28$.

The equation of the conic thus becomes

$$g(\mathbf{x}) = \mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{7}{16} \end{pmatrix} \mathbf{x} - 28$$