

# Ellipse Question

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## Question:

Given that the length of the major axis of a conic is 16 units and the coordinates of the foci are  $(0, \pm 6)$ , find the equation of the conic.

## Solution:

Variable	Description	Value
$2k$	Length of major axis	16
$\mathbf{F}_1$	One of the foci of the conic	$\begin{pmatrix} 0 \\ 6 \end{pmatrix}$
$\mathbf{F}_2$	Other foci of the conic	$\begin{pmatrix} 0 \\ -6 \end{pmatrix}$

TABLE 0: Given Information

2 foci exist for this conic. Therefore, it must be an ellipse or a hyperbola. The general equation of a conic with directrix  $\mathbf{n}^\top \mathbf{x} = c$  can be written as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f \quad (0.1)$$

$$\mathbf{u} = \frac{\mathbf{F}_1 + \mathbf{F}_2}{2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{n} \equiv \mathbf{F}_1 - \mathbf{F}_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (0.2)$$

From (0.2), we can conclude that we have a standard conic, with its center at the origin, foci on a coordinate axis (y-axis), and directrices parallel to a coordinate axis (x-axis). Therefore conic equation is either

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad (0.3)$$

Given that  $e \left( = \sqrt{1 \pm \frac{a^2}{b^2}} \right)$  represents the eccentricity of the conic, comparing (0.3) and (0.1) gives the relation

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \quad (0.4)$$

We know that the major axis is the y-axis. Assume some point  $\begin{pmatrix} 0 \\ k \end{pmatrix}$  on the y-axis that

satisfies (0.1). This gives

$$k = \sqrt{\frac{|f|}{|1 - e^2|}} \quad (0.5)$$

$$\implies 2k = 16 = 2\sqrt{\frac{|f|}{|1 - e^2|}} \quad (0.6)$$

Rearranging (0.6) gives

$$8\sqrt{|1 - e^2|} = \sqrt{|f|} \quad (0.7)$$

Now,  $\mathbf{F} = \frac{ce^2\mathbf{n}-\mathbf{u}}{\lambda_2}$  and  $\lambda_2 = \|\mathbf{n}\|^2$ .

$$\implies \mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\|\mathbf{n}\|^2} \quad (0.8)$$

Substituting values of  $\mathbf{F}_1$ ,  $\mathbf{F}_2$ ,  $\mathbf{n}$ , and  $\mathbf{u}$  and simplifying reduces (0.8) to

$$\pm ce^2 = 6 \quad (0.9)$$

Consider another equation

$$c = \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2 (\mathbf{u}^\top \mathbf{n})^2 - \lambda_2 (e^2 - 1) (\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e (e^2 - 1)}$$

Substituting known values gives

$$c = \pm \frac{1}{e} \sqrt{\frac{|f|}{|e^2 - 1|}} \quad (0.10)$$

We have 3 equations ((0.7), (0.9), (0.10)) and 3 variables ( $c$ ,  $f$ ,  $e$ ).

Upon solving, we get  $e = \frac{3}{4}$ ,  $c = \pm \frac{32}{3}$ , and  $|f| = 28$

However,  $f$  can only have one value (either +28 or -28).

To find this, assume some point  $(0, \alpha)$  to lie on the major axis of the conic. Substituting this point in the general conic equation (0.1) along with values of  $\mathbf{V}$  and  $\mathbf{u}$  gives

$$\frac{7\alpha^2}{16} + f = 0$$

This implies that  $f$  must be negative, and therefore  $f = -28$ .

The equation of the conic thus becomes

$$g(\mathbf{x}) = \mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{7}{16} \end{pmatrix} \mathbf{x} - 28$$

Code for Figure (0.1) can be found at:

codes/code.py

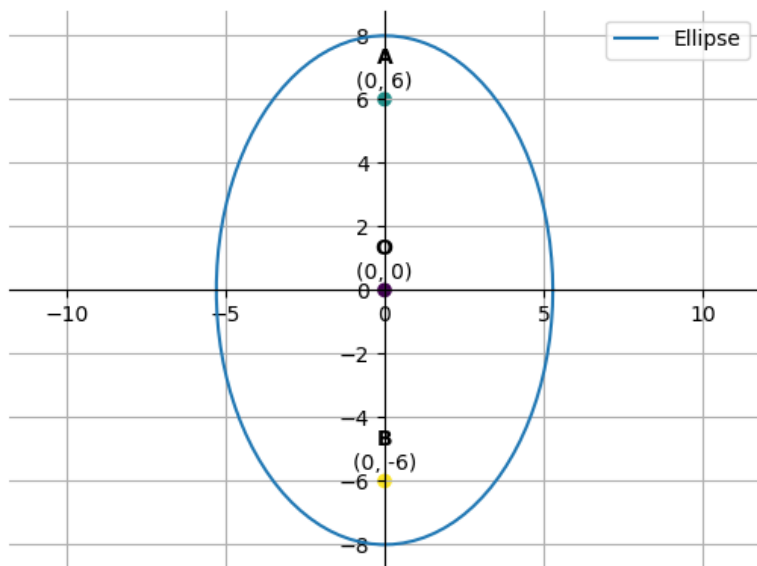


Fig. 0.1: Plot of conic