EE1060
Differential Equations And Transform Techniques
Quiz 06

Submission by Team 08

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Question

Compute the convolution of a given signal $f(t)$ with a rectangular kernel $h(t)$, analytically. The rectangular kernel is defined as:

$$\text{h(t)} = \begin{cases} 1, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

Derive the convolution expression $y(t) = (f * h)(t)$ in terms of known functions, and analyse the system's behaviour for various values of the kernel duration T and the input signal $f(t)$. Additionally, investigate the following scenarios:

(a) Modify the kernel to only consider the part of the kernel for $t > 0$. How does this affect the convolution result?

(b) Shift the kernel by a time τ_0. Analyse how the shift impacts the convolution output and discuss the significance of this shift in the context of time-delayed systems.

You are free to choose the form of the input signal $f(t)$, but make sure it is well-defined and appropriate for convolution. A step or sinusoidal signal would work well for this analysis.

Contents

1 The Theory Behind Convolution

1.1 Impulse Signal Definition

An impulse signal is idealised as an infinitesimally brief pulse. The continuous delta function (denoted $\delta(t)$) embodies this concept with three defining traits:

- **Zero duration:** The pulse occurs instantaneously.
- **Occurs at $t = 0$:** Centred at the origin.
- **Unit area:** Integrates to 1, implying infinite amplitude.

A system's output to a delta function is termed the impulse response, denoted $h(t)$ in continuous systems or $h[n]$ in discrete cases.

1.2 Impulse Decomposition

Impulse decomposition divides a signal into N constituent signals, each retaining one sample from the original while setting all others to zero. This method isolates individual samples, enabling analysis of signals point-by-point.
Systems are characterised by their reaction to impulses. By understanding how a system responds to an impulse, its output for any input can be determined through a process called convolution.

1.3 Input Signal Representation Using Impulses

For linear systems, the response to any arbitrary input can be determined using the principle of superposition. This principle states that if the input is expressed as a sum of signals, then the total response is the sum of the individual responses.
To apply this to an arbitrary input signal $f(t)$, we decompose $f(t)$ into a sum of rectangular pulses of width Δ. Each pulse has a height equal to $f \times (k\Delta)$ and is centered at $t = k\Delta$. This gives an approximation of the signal using:

$$f(t) \approx \sum_{k=-\infty}^{\infty} f \times (k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

Here, $\delta_{\Delta}(t - k\Delta)$ represents a rectangular approximation of an impulse centered at $t = k\Delta$. As the width $\Delta \to 0$, the rectangular pulses become ideal impulses, and the summation becomes an integral:

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)\, d\tau$$

This is known as the **sifting property** of the delta function, where the integral evaluates to $f(t)$. It is important to note that this impulse representation does *not* define the delta function itself. Rather, it is a method to synthesise any input signal by a weighted sum of time-shifted impulses, where the weights are the values of the signal $f(\tau)$ at each instant τ.

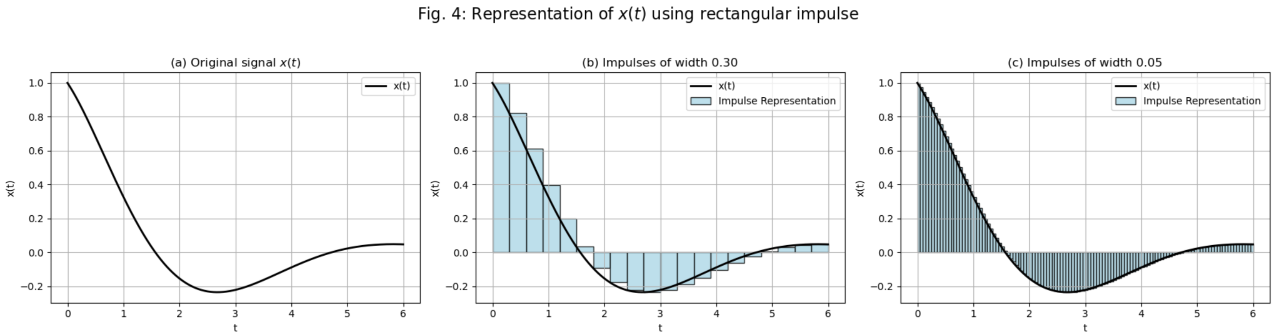


Figure 1: Impulse Decomposition Demonstration

1.4 Outline of the Convolution Process

Convolution mathematically combines two signals (typically input and impulse response) to produce a third signal (output). Its steps are:

1. **Decompose input:** Represent the input as scaled/shifted impulses by performing impulse decomposition.
2. **Scale and shift responses:** Each impulse generates a scaled/shifted version of $h(t)$.
3. **Superposition:** Sum all scaled/shifted responses to obtain the output.

In other words, if we know a system's impulse response, then we can calculate the output for any possible input signal. This means we know everything about the system.

1.5 About Kernels

The impulse response goes by different names in different applications. Filter kernel (in filters), convolution kernel, and point spread function (in image processing) are some examples.
While these terms are used in slightly different ways, they all mean the same thing: the signal produced by a system when the input is a delta function.
It is interesting to note here that convolution can also be defined for any two signals $x(t)$ and $h(t)$, as long as $h(t)$ is chosen such that it transforms $x(t)$ in a useful manner. Such functions $h(t)$ are called kernels.

1.6 Causal Systems

A system is causal if it never responds to future inputs. That is, the output at time t depends only on inputs at time t and earlier (i.e., from $-\infty$ to t), and not on any inputs from time $t' > t$.
Causality is important in real-time or physical systems like circuits, control systems, or audio filters because it is impossible to react to input data that hasn't arrived yet.

1.7 Linear Time-Invariant (LTI) Systems

LTI systems adhere to two principles:

- **Linearity:** Outputs scale with inputs ($af(t) \to ay(t)$) and obey superposition ($f_1 + f_2 \to y_1 + y_2$).
- **Time invariance:** Delaying the input delays the output identically ($f(t-T) \to y(t-T)$).

Systems described by linear differential equations with constant coefficients are inherently time-invariant. This class of differential equations will be used for the analysis in this report. Convolution provides a complete characterisation of LTI behaviour.

1.8 Convolution for LTI Systems

To determine the output $y(t)$ of a linear time-invariant (LTI) system for an arbitrary input $f(t)$, we apply the principle of superposition using the impulse decomposition of the input. As discussed earlier, any signal $f(t)$ can be represented as a continuous sum (integral) of scaled and shifted delta functions:

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,d\tau$$

In an LTI system, the response to an impulse $\delta(t-\tau)$ is a time-shifted version of the system's impulse response, namely $h(t-\tau)$. By the superposition principle, the overall output is the sum of the responses to each of these impulses, scaled by the value of $f(\tau)$ at that time. This yields the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

This operation is denoted by the convolution operator $*$, giving:

$$y(t) = (f * h)(t)$$

Thus, the convolution integral captures the complete output response by summing all shifted and scaled copies of the impulse response. Each copy is scaled by the corresponding value of the input $f(\tau)$ and shifted by time τ. This provides a powerful and general method for analyzing how a system transforms any input signal.

1.9 General Convolution Expression

For a rectangular kernel defined as:

$$h(t) = \begin{cases} 1, & \text{for } -T \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

The convolution $y(t) = (f * h)(t)$ is given by:

$$y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau \tag{2}$$

The integration limits can be simplified by noting that $h(t-\tau) = 1$ when $-T \le t-\tau \le T$, which is equivalent to $t - T \le \tau \le t + T$:

$$y(t) = \int_{t-T}^{t+T} f(\tau)\,d\tau \tag{3}$$

This represents a moving average over a window of width $2T$ centred at time t. The limits of the integral may further reduce depending on $f(t)$.

1.10 Properties of Convolution

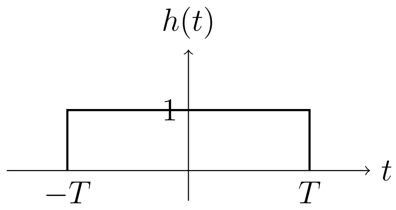
- Commutative: $f(t) * h(t) = h(t) * f(t)$
- Associative: $f(t) * (h_1(t) * h_2(t)) = (f(t) * h_1(t)) * h_2(t)$
- Distributive: $f(t)(h_1(t) + h_2(t)) = f(t) * h_1(t) + f(t) * h_2(t)$
- Scaling: $(af(t)) * h(t) = a(f(t) * h(t))$

1.11 The Given Kernel $h(t)$

The given kernel is defined as:

$$\text{h(t)} = \begin{cases} 1, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

The signal looks like so:



1.12 The Significance of a Rectangular Kernel

Using a rectangular kernel has the following advantages:

- **Averaging Effect:** A rectangular kernel is like an averaging filter. When a signal is convolved with a rectangular pulse like $h(t)$, it is similar to taking the local average of the signal over the width $2T$ (width of the kernel).
- **Low Pass Filter Behaviour:** Due to the averaging property, any high-frequency / quickly fluctuating oscillations are suppressed. Slowly varying things are allowed to "pass".
- **Local Integration:** Mathematically, a rectangular kernel calculates the area under the input signal $f(t)$ between $t = -T$ and $t = T$. In the context of signal processing, it can be thought of as representing how much "energy" or "weight" the function has around t.
- **Simplicity:** A rectangular kernel is just 1's and 0's. The integration to be computed to find $y(t)$ is significantly reduced if $f(t)$ is easily integrable.
- **Making of More Complex Kernels:** A rectangular kernel is the most basic signal to consider, and hence other signals can be modelled as a summation of shifted and scaled rectangular signals.

1.13 Choosing functions $f(t)$ for Analysis

The following functions have been selected for analysis of the given kernel $h(t)$:

1. $f(t) = u(t)$

2. $f(t) = cos(\omega t + \phi)$
3. $f(t) = \begin{cases} 1, & \text{for } -S \leq t \leq S \\ 0, & \text{otherwise} \end{cases}$
4. $f(t) = sinc(t)$
5. $f(t) = e^{-\alpha t}$ for $\alpha > 0$
6. $f(t) = e^{st}$ for$s \in \mathbb{C}$
7. $f(t) = e^{-\frac{t^2}{2\sigma^2}}$
8. $f(t) = tu(t)$
9. $f(t) = t^n u(t)$
10. $f(t) = \delta(t)$

2 Input Signal $f(t) = u(t)$

2.1 Analytical Convolution

Let the rectangular kernel be defined as:

$$h(t) = \begin{cases} 1, & -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

We compute the convolution:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

Let $f(\tau) = u(\tau)$, the unit step function. Then:

$$y(t) = \int_{0}^{\infty} h(t-\tau)\,d\tau$$

Since $h(t-\tau) = 1$ for $t - T \le \tau \le t + T$, the limits become:

$$\tau \in [\max(0, t-T), t+T]$$

Hence:

$$y(t) = \int_{\max(0,t-T)}^{t+T} 1\,d\tau = (t+T) - \max(0, t-T)$$

So the final result is:

$$y(t) = \begin{cases} 0, & t < -T \\ t+T, & -T \le t < T \\ 2T, & t \ge T \end{cases}$$

2.2 The Effect of Varying T

Let the modified (causal) rectangular kernel be defined as:

$$h(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

We compute the convolution:

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\,d\tau$$

Since $u(\tau) = 0$ for $\tau < 0$, and $h(t-\tau) = 1$ for $t - T \le \tau \le t$, the effective limits are:

$$\tau \in [\max(0, t-T), t]$$

Thus:

$$y(t) = \int_{\max(0,t-T)}^{t} 1\,d\tau = t - \max(0, t-T)$$

Therefore:

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \leq t < T \\ T, & t \geq T \end{cases}$$

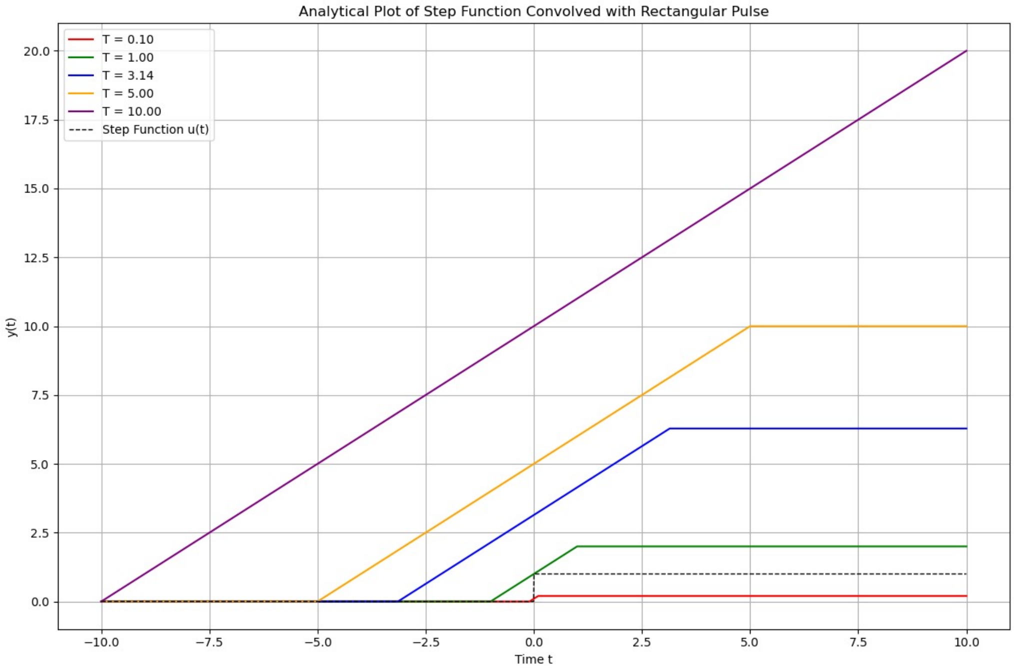


Figure 2: Convolution Result $y(t)$ for Varying T

2.3 Analysis for Causal Kernel

Property	**Causal Kernel**
Definition of $h(t)$	$\begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$
Limits of Convolution	$[t-T, t]$
Output with Step Function	$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \leq t < T \\ T, & t \geq T \end{cases}$
Causality	Causal (uses only current and past values)

Table 1: Causal rectangular kernel convolution with step input

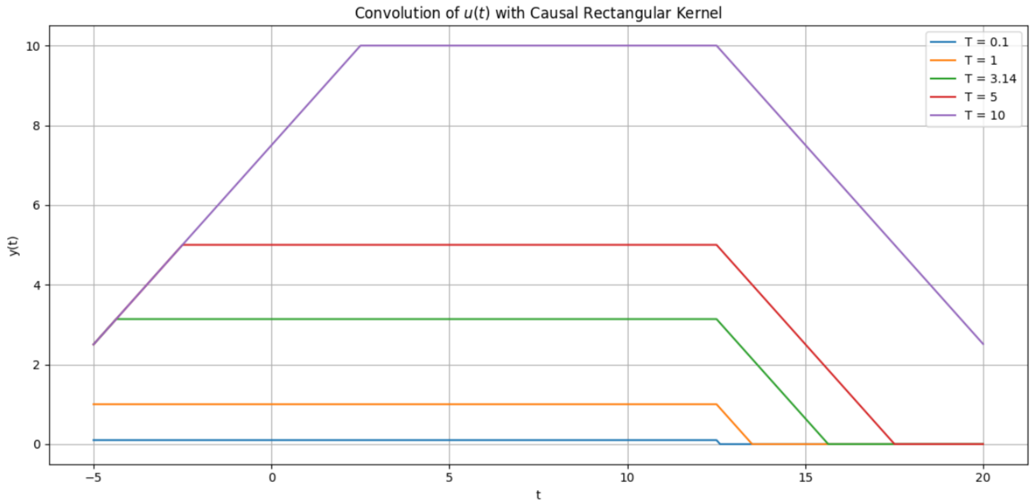


Figure 3: Convolution Result $y(t)$ for Causal $h(t)$

2.4 Analysis for Shifted Kernel

Suppose the original kernel is $h(t)$, and we define a shifted version:

$$h_{\text{shifted}}(t) = h(t - \tau_0)$$

Then the convolution becomes:

$$y(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t - \tau - \tau_0)\, d\tau$$

Let $u = \tau + \tau_0 \Rightarrow \tau = u - \tau_0$. Substituting:

$$y(t) = \int_{-\infty}^{\infty} f(u - \tau_0)\, h(t - u)\, du = (f(t - \tau_0) * h(t))$$

Conclusion: Shifting the kernel by τ_0 leads to a shift in the convolution output by the same amount. This is interpreted as:

- $\tau_0 > 0$: Delays the output
- $\tau_0 < 0$: Advances the output

Property	**Symmetric Kernel with Shift** τ_0
Shifted Kernel	$h_{\text{shifted}}(t) = h(t - \tau_0)$
Output Expression	$y(t) = f(t) * h(t - \tau_0) = f(t - \tau_0) * h(t)$
Output with Step Input	$y(t) = \begin{cases} 0, & t < -T + \tau_0 \\ t + T - \tau_0, & -T + \tau_0 \leq t < T + \tau_0 \\ 2T, & t \geq T + \tau_0 \end{cases}$
Interpretation	$\tau_0 > 0$: Output is delayed $\tau_0 < 0$: Output is advanced

Table 2: Effect of symmetric kernel shift on convolution with step input

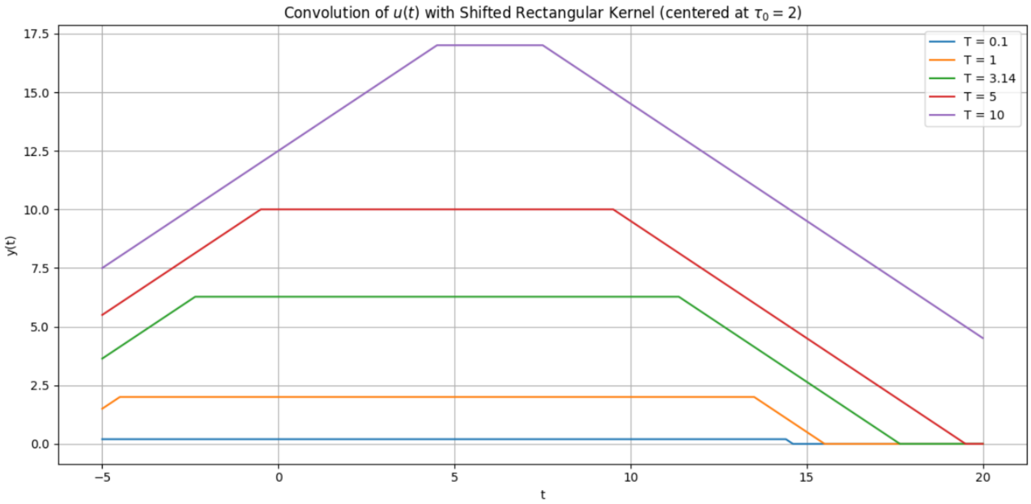


Figure 4: Convolution Result $y(t)$ for Shifted $h(t)$

2.5 System Analysis

For the system defined by convolution with a causal rectangular kernel $h(t)$ and step input $u(t)$, the output is:

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \leq t < T \\ T, & t \geq T \end{cases}$$

1. The total energy of the output signal is:

$$E = \int_{-\infty}^{\infty} |y(t)|^2 \, dt = \frac{4T^3}{3}$$

   Thus, the energy increases with the kernel width T.

2. For this LTI system:
   - **Shifting the Input**: Shifting the input by τ_0 shifts the output by the same amount.
   - **Scaling the Input**: Scaling the input by S scales the output by S.
   - **Output Shape**: The output depends on the overlap between the input and the kernel.

3. The convolution acts as a moving integrator, smoothing or averaging the input over the kernel width T, resulting in an output that increases linearly to T and then stays constant.

4. In summary, the energy increases with T. The system is linear, time-invariant, and behaves as a moving integrator, smoothing the input signal.

2.6 Conclusions

We analyzed the convolution of a step input signal with different types of rectangular kernels: symmetric, shifted, and causal. The resulting output is piecewise linear and dependent on the kernel width T.

1. The symmetric kernel produces a symmetrical linear ramp in the output, starting at $t = 0$ and reaching a maximum value of T.
2. The causal kernel introduces a delay in the output, starting at $t = 0$ and increasing linearly until it reaches T, after which it remains constant at T.
3. The shifted kernel delays the output by τ_0, with the output shape remaining same but shifted along the time axis.

Varying T shows how the width of the kernel controls the rate at which the output increases. As T increases, the output becomes smoother, and the convolution acts more like a low-pass filter, averaging the input signal over a broader region. This illustrates the importance of the rectangular convolution kernel in shaping the response of systems to step inputs, particularly in filtering and smoothing applications.

3 Input Signal $f(t) = cos(\omega t + \phi)$

3.1 Analytical Convolution

Let the input be $f(t) = \cos(\omega t + \phi)$, and the kernel:

$$h(t) = \begin{cases} 1, & -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

The convolution is:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} \cos(\omega\tau + \phi)\, h(t - \tau)\, d\tau$$

Since $h(t - \tau) = 1$ for $\tau \in [t - T, t + T]$, we get:

$$y(t) = \int_{t-T}^{t+T} \cos(\omega\tau + \phi)\, d\tau$$

Integrating:

$$\int \cos(\omega\tau + \phi)\, d\tau = \frac{1}{\omega} \sin(\omega\tau + \phi)$$

So:

$$y(t) = \frac{1}{\omega} \left[\sin(\omega(t + T) + \phi) - \sin(\omega(t - T) + \phi)\right]$$

Using the identity:

$$\sin A - \sin B = 2 \cos\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right)$$

We obtain:

$$y(t) = \frac{2}{\omega} \cos(\omega t + \phi) \cdot \sin(\omega T)$$

T	Amplitude $A(T)$	Output $y(t)$
0.1	0.1996	0.1996 $\cos(t + \phi)$
1	1.683	1.683 $\cos(t + \phi)$
3.14	0.0032	0.0032 $\cos(t + \phi)$
10	-1.088	-1.088 $\cos(t + \phi)$
100	-1.0128	-1.0128 $\cos(t + \phi)$

3.2 The Effect of Varying T

Let the modified (causal) rectangular kernel be defined as:

$$h(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

We compute:

$$y(t) = \int_{t-T}^{t} \cos(\omega\tau + \phi)\, d\tau$$

$$y(t) = \frac{1}{\omega}\left[\sin(\omega t + \phi) - \sin(\omega(t - T) + \phi)\right]$$

Using the identity $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$, we get:

$$y(t) = \frac{2}{\omega}\cos\left(\omega t - \frac{\omega T}{2} + \phi\right)\cdot\sin\left(\frac{\omega T}{2}\right)$$

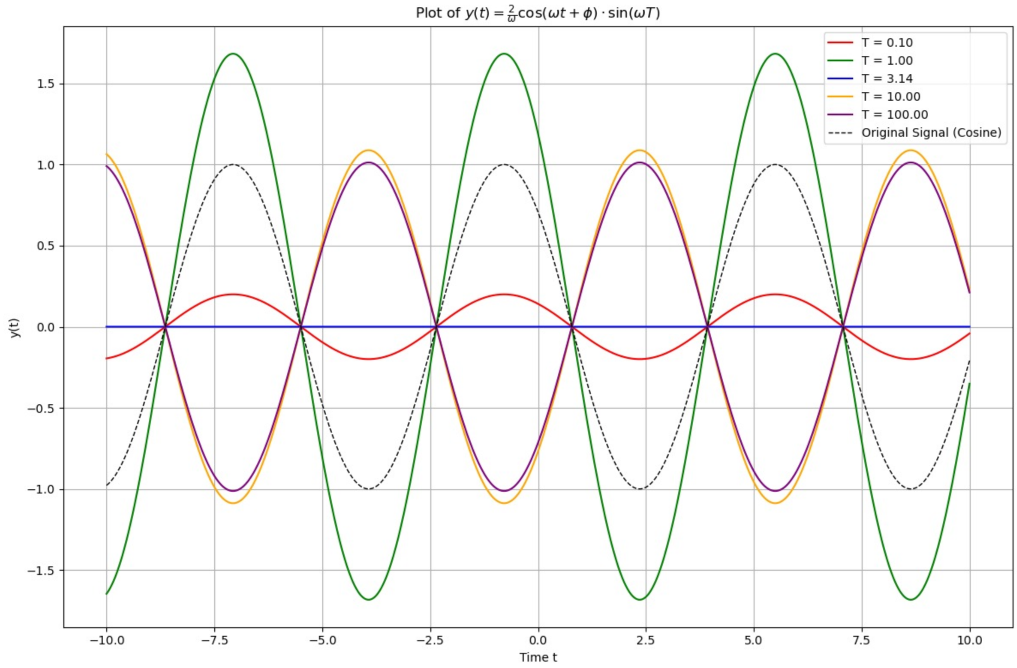


Figure 5: Convolution Result $y(t)$ for Varying T

3.3 Analysis for Causal Kernel

Property	**Causal Kernel**
Definition of $h(t)$	$\begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$
Limits of Convolution	$[t - T, t]$
Output with Sinusoidal Function	$y(t) = \frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega}\cdot\cos\left(\omega t - \frac{\omega T}{2} + \phi\right)$
Amplitude Scaling	$\frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega}$
Phase Shift	$-\frac{\omega T}{2}$
Causality	Causal (uses only current and past values)

Table 3: Causal rectangular kernel convolution with sinusoidal input

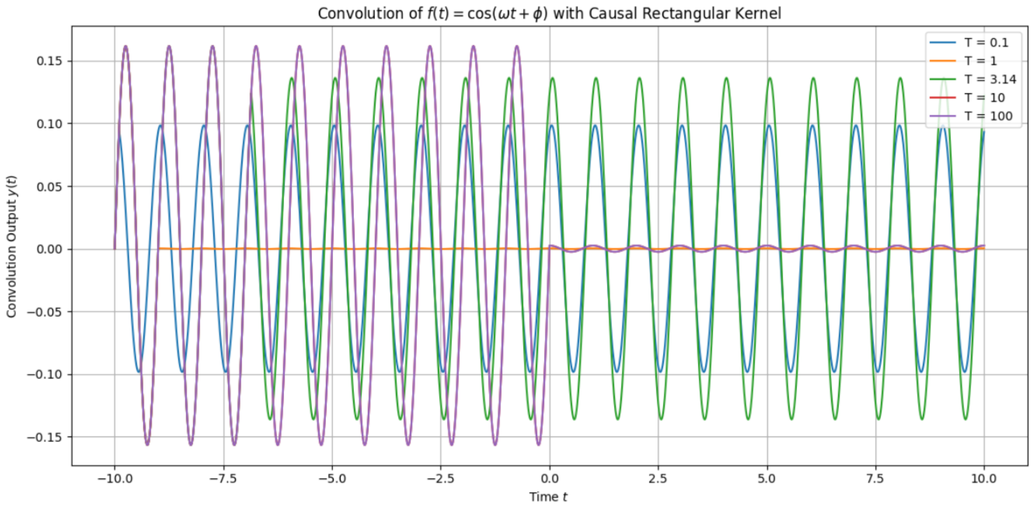


Figure 6: Convolution Result $y(t)$ for Causal $h(t)$

3.4 Analysis for Shifted Kernel

Property	Symmetric Kernel with Shift τ_0
Shifted Kernel	$h_{\text{shifted}}(t) = h(t - \tau_0)$
Output Expression	$y(t) = f(t - \tau_0) * h(t)$
Output with Sinusoidal Input	$y(t) = \dfrac{2\sin(\omega T)}{\omega} \cdot \cos(\omega t - \omega\tau_0 + \phi)$
Amplitude Scaling	$\dfrac{2\sin(\omega T)}{\omega}$
Additional Phase Shift Due to Shift	$-\omega\tau_0$
Interpretation	$\tau_0 > 0$: Output phase is delayed
	$\tau_0 < 0$: Output phase is advanced

Table 4: Effect of symmetric kernel shift on convolution with sinusoidal input

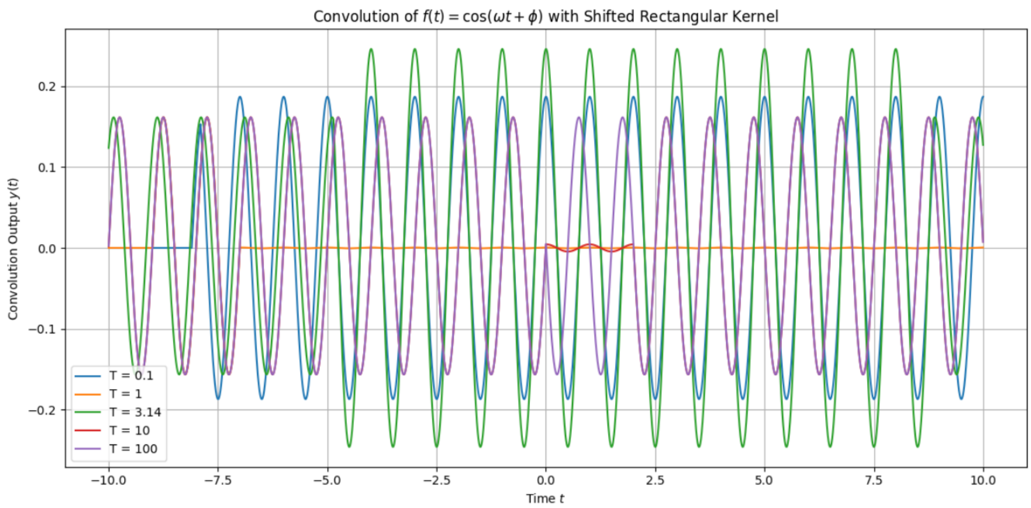


Figure 7: Convolution Result $y(t)$ for Shifted $h(t)$

3.5 System Analysis

The output of convolving a sinusoidal input $f(t) = \sin(\omega t)$ with a rectangular kernel $h(t)$ is sinusoidal, but its amplitude and phase depend on the kernel type.

1. The output energy is finite and can be computed for a sinusoidal signal, showing the signal remains bounded.
2. For this LTI system:
   - Shifting the input shifts the output by the same amount.
   - Scaling the input scales the output by the same factor.
   - The output is sinusoidal, with modulated amplitude and phase.
3. The output sinusoid retains the same frequency but its amplitude is scaled by:

$$\frac{2\sin(\omega T)}{\omega} \quad \text{(symmetric kernel)} \quad \text{and} \quad \frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega} \quad \text{(causal kernel)}.$$

   The symmetric kernel causes no phase shift, while the causal kernel introduces a phase shift of $-\frac{\omega T}{2}$.
4. The convolution smooths high-frequency components of the input sinusoid, acting as a low-pass filter.
5. In summary the convolution of a sinusoidal signal with a rectangular kernel results in a sinusoid with:
   - Amplitude scaling and phase shift.
   - Frequency unchanged, but the amplitude is attenuated by the kernel width T.

3.6 Conclusions

We analyzed the convolution of a sinusoidal input signal with different types of rectangular kernels: symmetric, shifted, and causal. The resulting output is sinusoidal, with changes in amplitude and phase depending on the kernel type and width T.

1. The symmetric kernel produces a sinusoidal output with no phase shift and amplitude scaled by $\frac{2\sin(\omega T)}{\omega}$.
2. The causal kernel causes a phase shift of $-\frac{\omega T}{2}$ and scales the amplitude by $\frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega}$.
3. A shifted kernel delays the output without changing the frequency or amplitude, only shifting the waveform in time.

Varying T influences the amplitude scaling and frequency response. As T increases, the amplitude of the output decreases, and the system acts more like a low-pass filter, smoothing high-frequency components. This makes the rectangular convolution kernel crucial for filtering and shaping sinusoidal signals in time-domain signal processing.

4 Input Signal $f(t) = \begin{cases} 1, & \textbf{for } -S \le t \le S \\ 0, & \textbf{otherwise} \end{cases}$

4.1 Analytical Convolution

Let the input signal be a rectangular pulse:

$$f(t) = \begin{cases} 1, & \text{for } -S \le t \le S \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

We convolve it with a symmetric rectangular kernel of width $2T$:

$$h(t) = \begin{cases} 1, & \text{for } -T \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

The convolution is defined as:

$$y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\, d\tau \tag{6}$$

$$= \int_{t-T}^{t+T} f(\tau)\, d\tau \tag{7}$$

This integral is non-zero only over the region where the supports of $f(\tau)$ and $h(t-\tau)$ overlap. We identify five distinct cases based on the value of t.
Case 1: No Overlap: $t + T \le -S$ or $t - T \ge S$

$$y(t) = 0 \tag{8}$$

Case 2: Partial Overlap (Left): $t - T < -S < t + T \le S$

$$y(t) = \int_{-S}^{t+T} 1\, d\tau = t + T + S \tag{9}$$

Case 3: Full Overlap: $-S + T \le t \le S - T$

$$y(t) = \int_{t-T}^{t+T} 1\, d\tau = 2T \tag{10}$$

Case 4: Partial Overlap (Right): $-S \le t - T < S < t + T$

$$y(t) = \int_{t-T}^{S} 1\, d\tau = S + T - t \tag{11}$$

Case 5: Kernel Contains Pulse: $t - T < -S$ and $t + T > S$

$$y(t) = \int_{-S}^{S} 1\, d\tau = 2S \tag{12}$$

Combining all cases:

$$y(t) = \begin{cases} 0, & t < -(S+T) \\ t+S+T, & -(S+T) \leq t \leq -(S-T) \\ 2T, & -(S-T) \leq t \leq (S-T) \\ S+T-t, & (S-T) \leq t \leq (S+T) \\ 0, & t > (S+T) \end{cases} \tag{13}$$

The output $y(t)$ is a trapezoid symmetric about $t = 0$ with a flat top when $2T \leq 2S$.

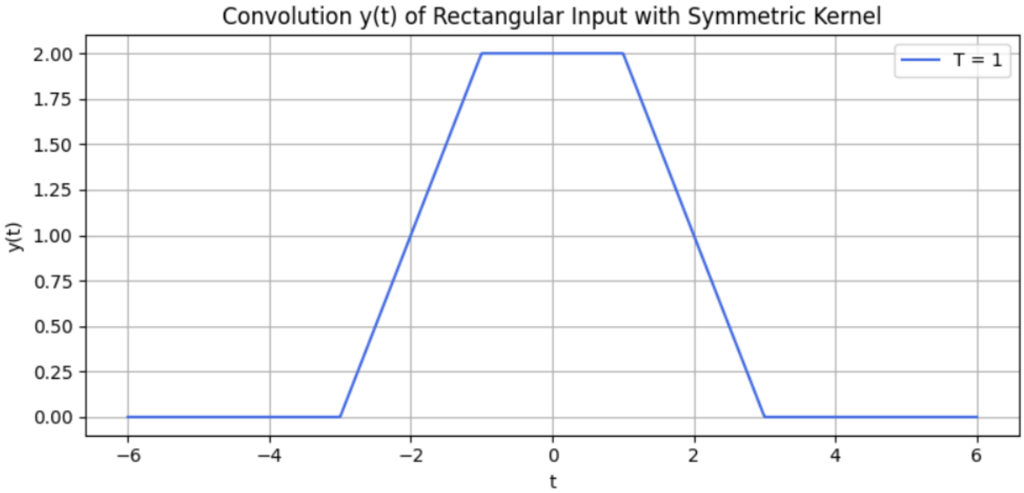


Figure 8: Convolution Result $y(t)$

4.2 The Effect of Varying T

As T increases:

- The width of the output signal increases linearly.
- The peak value remains at $2T$ as long as $T \leq S$.
- For $T > S$, the kernel fully covers the input, so $y(t) = 2S$ during the flat region, and the output's maximum no longer increases.
- The slope regions (rising and falling) extend further outward as T increases.

Thus, T controls both the width and the height (up to a limit) of the output trapezoid. For very large T, the convolution result approaches a flat-topped function centred at $t = 0$ with height $2S$ and width $2T - 2S$.

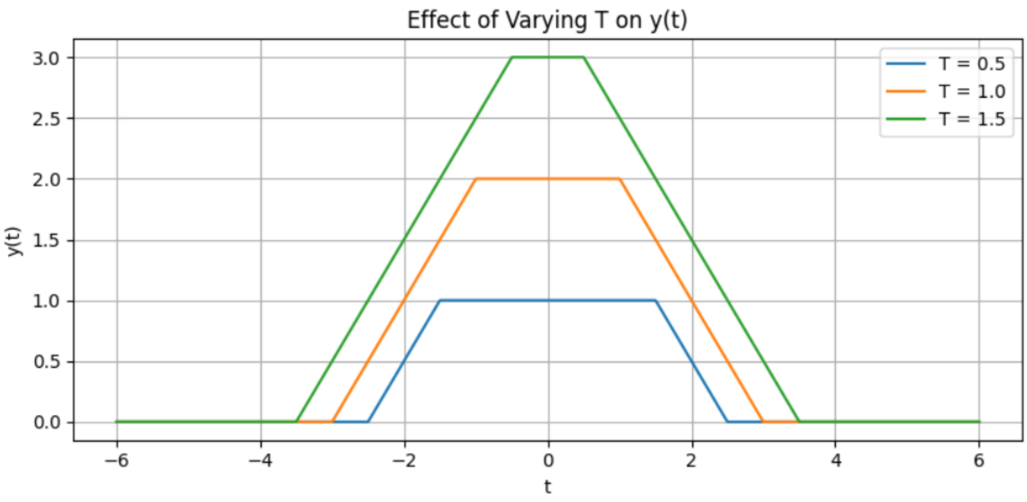


Figure 9: Convolution Result $y(t)$ as T is Varied

4.3 Analysis for Causal Kernel

Consider a causal rectangular kernel:

$$h_{\text{causal}}(t) = \begin{cases} 1, & 0 \leq t \leq 2T \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

The convolution becomes:

$$y(t) = \int_{-\infty}^{\infty} f(\tau) h_{\text{causal}}(t-\tau)\, d\tau \tag{15}$$

$$= \int_{t-2T}^{t} f(\tau)\, d\tau \tag{16}$$

This shifts the support of the output to $[2T - S, 2T + S]$ and breaks the symmetry of $y(t)$, introducing a delay. The output remains trapezoidal but is right-shifted by approximately T compared to the symmetric case.

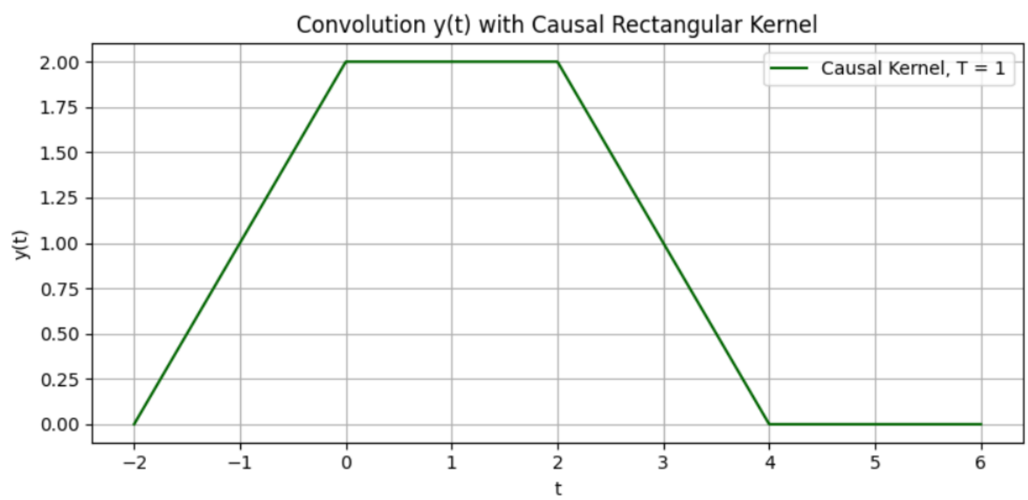


Figure 10: Convolution Result $y(t)$ for Causal Kernel

4.4 Analysis for Shifted Kernel

Let the symmetric kernel be shifted by τ_0:

$$h_{\text{shifted}}(t) = \begin{cases} 1, & -T + \tau_0 \le t \le T + \tau_0 \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

Then:

$$y(t) = \int_{-\infty}^{\infty} f(\tau) h_{\text{shifted}}(t - \tau)\, d\tau \tag{18}$$

$$= \int_{t-T-\tau_0}^{t+T-\tau_0} f(\tau)\, d\tau \tag{19}$$

This results in the same trapezoidal shape as the original convolution, but shifted by τ_0 to the right. Hence, a time shift in the kernel directly translates to a shift in the output signal.

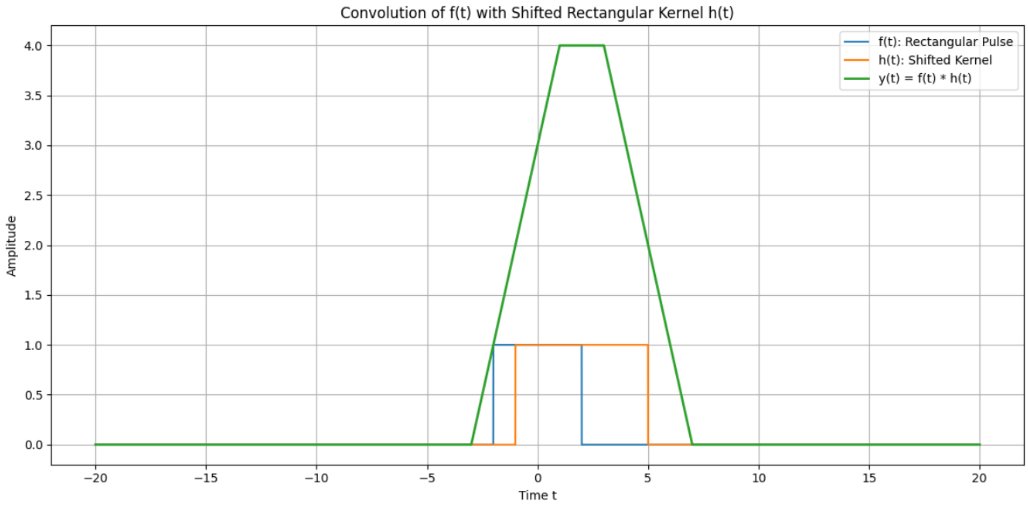


Figure 11: Convolution Result $y(t)$ for Shifted Kernel $h(t)$

4.5 System Analysis

The total energy of the output signal is:

$$E = \int_{-\infty}^{\infty} |y(t)|^2\, dt \tag{20}$$

Since $y(t)$ is a piecewise linear trapezoid, the energy increases with both T and S. If the system defined by convolution with $h(t)$ is linear and time-invariant (LTI), then:

1. Shifting the input shifts the output by the same amount.
2. Scaling the input scales the output linearly.
3. The output shape depends only on the overlap between the input and kernel.

The convolution operation acts as a moving integrator, smoothing or averaging the input signal.

4.6 Conclusions

We analyzed the convolution of a rectangular input signal with different types of rectangular kernels: symmetric, shifted, and causal. The analytical solution reveals a piecewise trapezoidal output function, whose shape depends on the kernel width T and signal width S.

1. The symmetric kernel produces a centered trapezoid.
2. The causal kernel skews the output and ensures causality.
3. A shifted kernel delays the output without affecting its shape.

Varying T provides intuitive insight into the smoothing behavior of convolution. As T increases, the convolution averages the input over a broader region, effectively filtering out rapid transitions. This makes the rectangular convolution kernel a fundamental building block in time-domain signal analysis and filtering.

5 Input Signal $f(t) = sinc(t)$

5.1 Analytical Convolution

Let the input signal be the sinc function:

$$f(t) = \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Let the impulse response $h(t)$ be a rectangular pulse of width $2T$ centered at zero:

$$h(t) = \begin{cases} 1, & -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

The output of the system is given by the convolution of $f(t)$ and $h(t)$:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

Because $h(t-\tau)$ is only non-zero when $t-\tau \in [-T, T]$, i.e., $\tau \in [t-T, t+T]$, we can simplify the limits of integration:

$$y(t) = \int_{t-T}^{t+T} f(\tau)\,d\tau = \int_{t-T}^{t+T} \frac{\sin(\pi\tau)}{\pi\tau}\,d\tau$$

This integral does not have a closed-form expression but can be evaluated numerically for visualization.

5.2 The Effect of Varying T

The parameter T controls the width of the rectangular kernel and hence the integration window in the convolution. As T increases, the convolution integral covers more of the sinc(t) waveform, accumulating more of its area. This leads to a smoother and broader output $y(t)$.

- For small T, the integral captures only a narrow slice around t, and $y(t)$ closely follows the local value of $f(t)$.
- As $T \to \infty$, the rectangular kernel approaches an infinite window, and $y(t) \to$ constant, depending on the total area under $f(t)$.

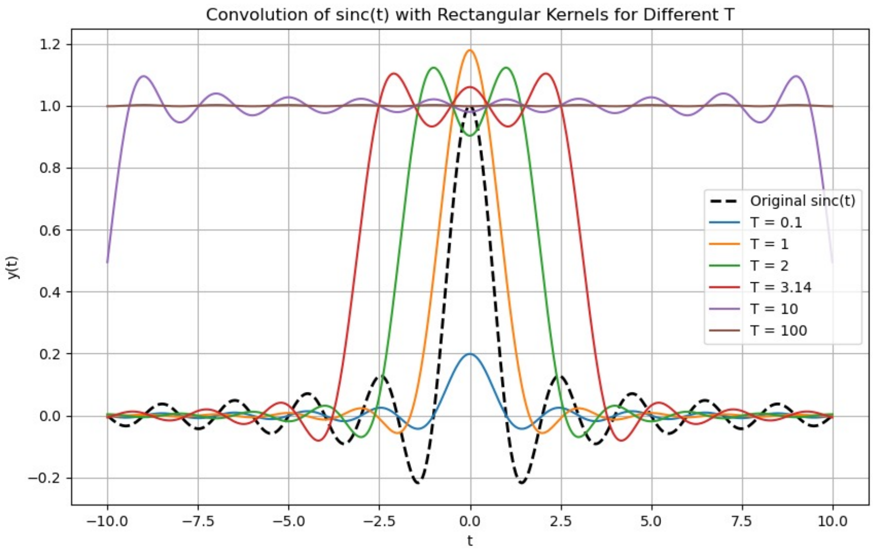


Figure 12: $y(t)$ as T Varies

5.3 Analysis for Causal Kernel

If the kernel is causal:

$$h_{\text{causal}}(t) = \begin{cases} 1, & 0 \le t \le 2T \\ 0, & \text{otherwise} \end{cases}$$

Then the convolution becomes:

$$y(t) = \int_{-\infty}^{\infty} f(\tau) h_{\text{causal}}(t-\tau)\, d\tau = \int_{t-2T}^{t} f(\tau)\, d\tau$$

This implies that the output at time t depends only on past values of the input from $t - 2T$ to t, making the system causal.

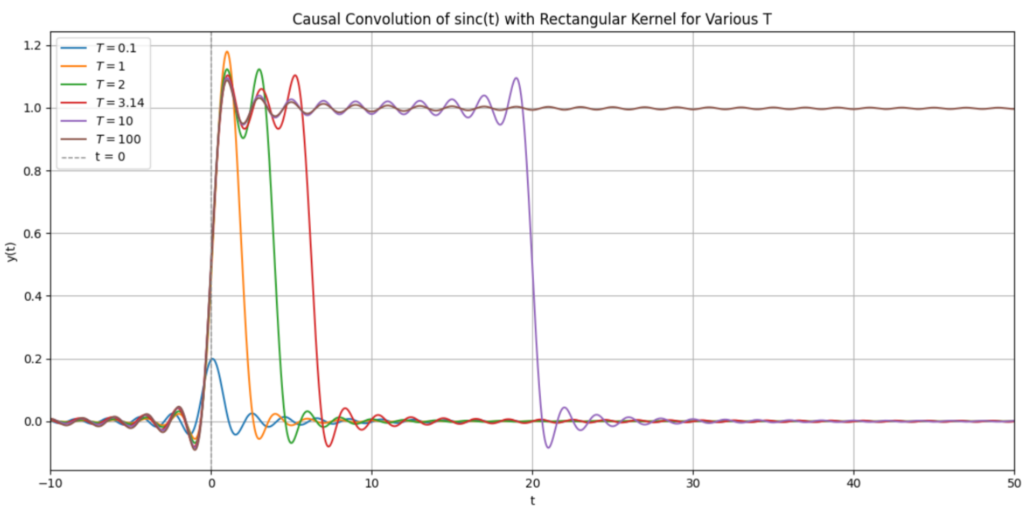


Figure 13: $y(t)$ for Causal $h(t)$

5.4 Analysis for Shifted Kernel

If the rectangular kernel is shifted by τ_0, then:

$$h_{\text{shifted}}(t) = \begin{cases} 1, & -T + \tau_0 \leq t \leq T + \tau_0 \\ 0, & \text{otherwise} \end{cases}$$

Then the convolution becomes:

$$y(t) = \int_{t-T-\tau_0}^{t+T-\tau_0} f(\tau)\, d\tau$$

This is equivalent to the original convolution shifted in time by τ_0. The output waveform will appear shifted to the right by τ_0.

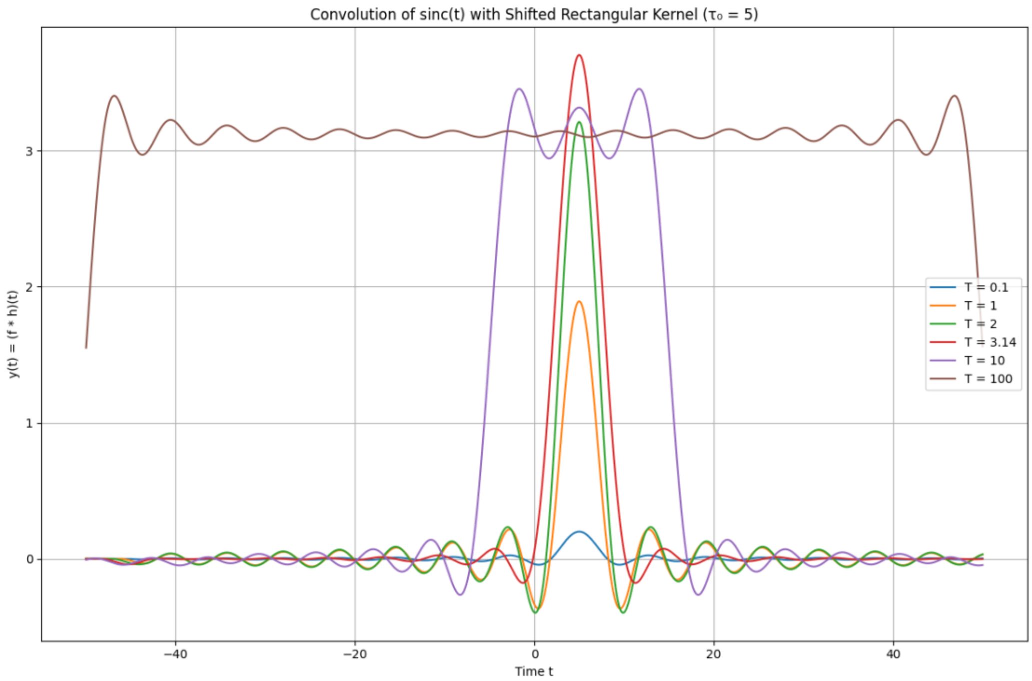


Figure 14: $y(t)$ for Shifted $h(t)$

5.5 System Analysis

The system defined by convolution with a rectangular kernel is:

- **Linear:** Since convolution is linear.
- **Time-Invariant:** Shifting the input shifts the output by the same amount.
- **Causal or Non-Causal:** Depends on the kernel. A symmetric kernel is non-causal, while a right-sided kernel is causal.
- **Smoothing Filter:** The rectangular convolution acts as a low-pass filter. Convolution with a sinc signal attenuates its high-frequency oscillations, resulting in a smoother output.

5.6 Conclusions

Convolving $f(t) = \text{sinc}(t)$ with a rectangular kernel yields a smoothed version of the input. The output is determined by the local average of $f(t)$ over an interval of length $2T$, centered or shifted depending on the kernel. Varying T controls the degree of smoothing. Causal and shifted kernels affect the timing and structure of the output, demonstrating how convolution can be used to design various signal-processing behaviors.

6 Input Signal $f(t) = e^{-\alpha t}$

6.1 Analytical Convolution

We are given:

$$f(t) = e^{-\alpha t} \qquad (\alpha > 0) \tag{21}$$

$$h(t) = \begin{cases} 1, & -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \tag{22}$$

The convolution $y(t) = (f * h)(t)$ is defined as:

$$y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau \tag{23}$$

From the commutative property of convolution, we have $y(t) = (f * h)(t) = (h * f)(t)$:

$$y(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)\,d\tau \tag{24}$$

Since $h(\tau)$ is nonzero only when $-T \leq \tau \leq T$, we can reduce the integral to:

$$y(t) = \int_{-T}^{T} e^{-\alpha(t-\tau)}\,d\tau \tag{25}$$

Evaluating:

$$y(t) = \left[\frac{e^{-\alpha t}}{\alpha} e^{\alpha \tau}\right]_{-T}^{T} \tag{26}$$

$$= \frac{e^{-\alpha t}}{\alpha}\left(e^{\alpha T} - e^{-\alpha T}\right) \tag{27}$$

$$\boxed{y(t) = \frac{e^{-\alpha t}}{\alpha}\left(e^{\alpha T} - e^{-\alpha T}\right)}$$

6.2 The Effect of Varying T

As we increase T, the rectangular window $h(t)$ becomes wider.

- For small T, $y(t)$ closely tracks the original signal $f(t)$.
- As T grows, $y(t)$ becomes more scaled and is substantially larger than $f(t)$ for $t < 0$.
- For $t > 0$, all $y(t)$ for higher T follow $f(t)$ relatively better.

This makes sense because a wider kernel averages over a larger neighbourhood. The following image demonstrates what happens when $\alpha = 1$.

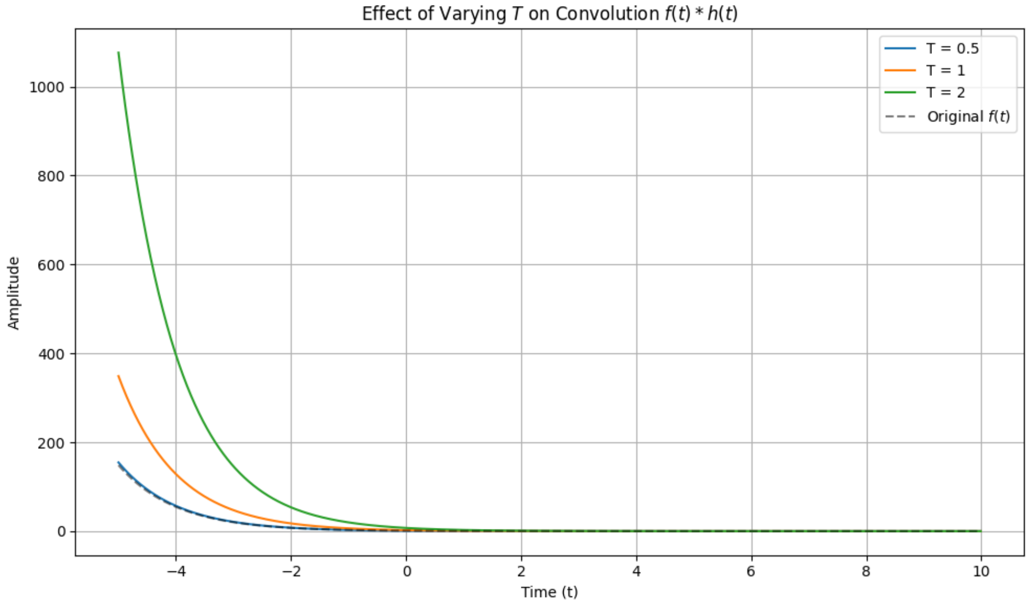


Figure 15: Effect of Varying T

The convolution can be defined for $\alpha < 0$ as well, although not practically useful. Here, $\alpha = -1$.

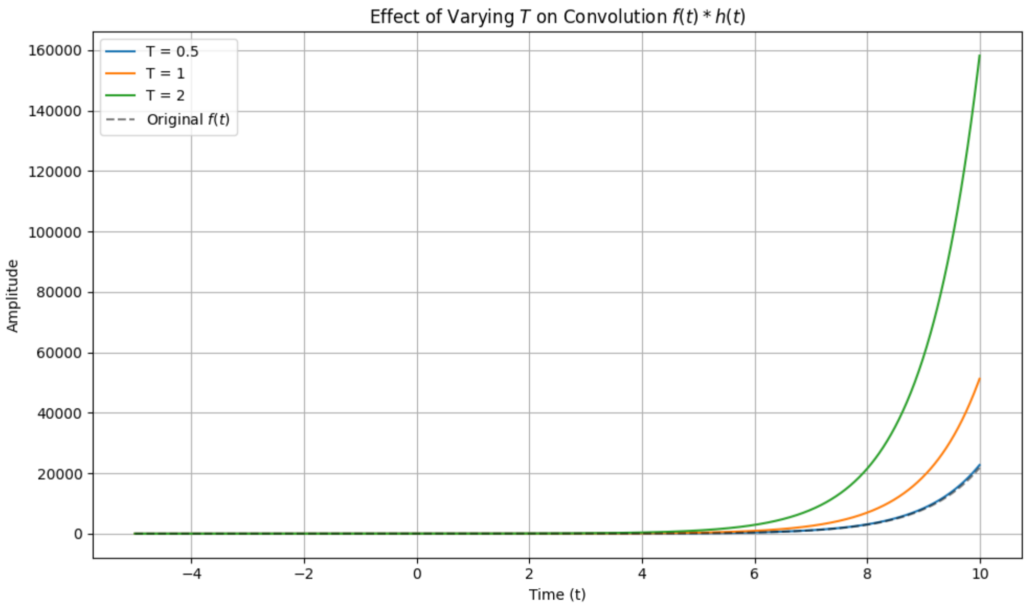


Figure 16: Effect of Varying T for $\alpha < 0$

6.3 Analysis for Causal Kernel

Now, consider a causal rectangular kernel:

$$h_c(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

The convolution becomes:

$$y_c(t) = \int_{-\infty}^{\infty} f(\tau)h_c(t-\tau)\,d\tau \tag{29}$$
$$= \int_{-\infty}^{\infty} f(t-\tau)h_c(\tau)\,d\tau \tag{30}$$
$$= \int_{0}^{T} e^{-\alpha(t-\tau)}\,d\tau \tag{31}$$

Evaluating:

$$y_c(t) = \left[\frac{e^{-\alpha t}}{\alpha}e^{-\alpha\tau}\right]_0^T \tag{32}$$
$$= \frac{e^{-\alpha t}}{\alpha}\left(e^{\alpha T}-1\right) \tag{33}$$

$$\boxed{y_c(t) = \frac{e^{-\alpha t}}{\alpha}\left(e^{\alpha T}-1\right)}$$

For the symmetric kernel, we integrated $f(t-\tau)$ from $-T$ to T. This is equivalent to integrating $f(\tau)$ from $t-T$ to $t+T$. So, the symmetric kernel looks both "into the past and future" to compute the convolution.
In contrast, the causal kernel "only looks at the past", as we integrate $f(\tau)$ from $t-T$ to t. This makes the system causal – depending only on past inputs.

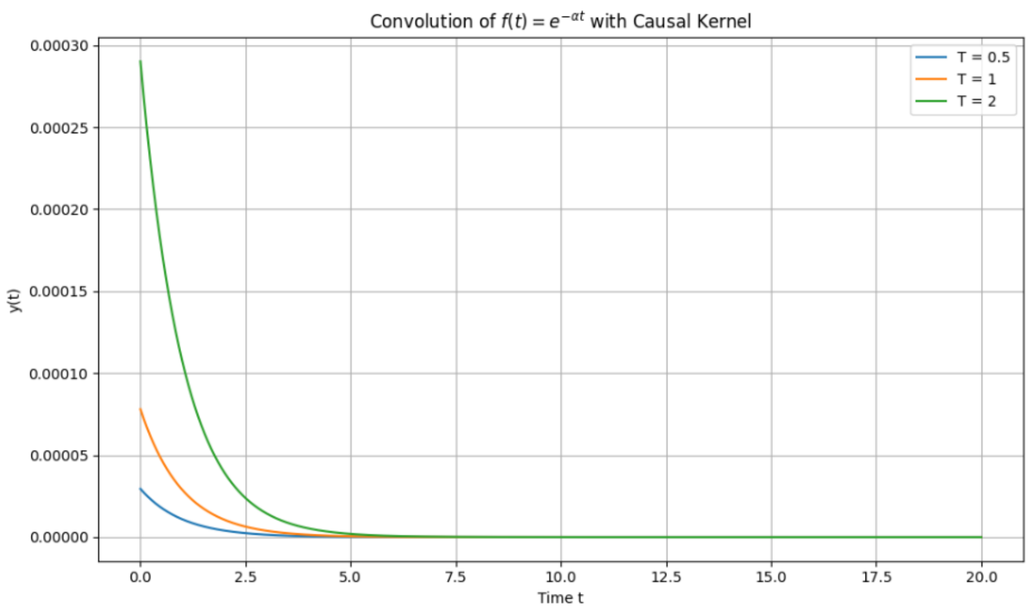


Figure 17: Effect of Causal $h(t)$

6.4 Analysis for Shifted Kernel

Suppose we shift the kernel by τ_0:

$$h_s(t) = h(t-\tau_0) \tag{34}$$

Then the convolution becomes:

$$\begin{aligned} y_s(t) &= (f * h_s)(t) && (35) \\ &= (f * h)(t - \tau_0) && (36) \end{aligned}$$

This simply delays the output by τ_0 units:

$$\boxed{y(t) = \frac{e^{-\alpha(t-\tau_0)}}{\alpha}\left(e^{\alpha T} - e^{-\alpha T}\right)}$$

Thus, shifting the kernel introduces a corresponding time delay in the output, which is crucial in modelling time-delayed systems like communication channels or physical transport phenomena.

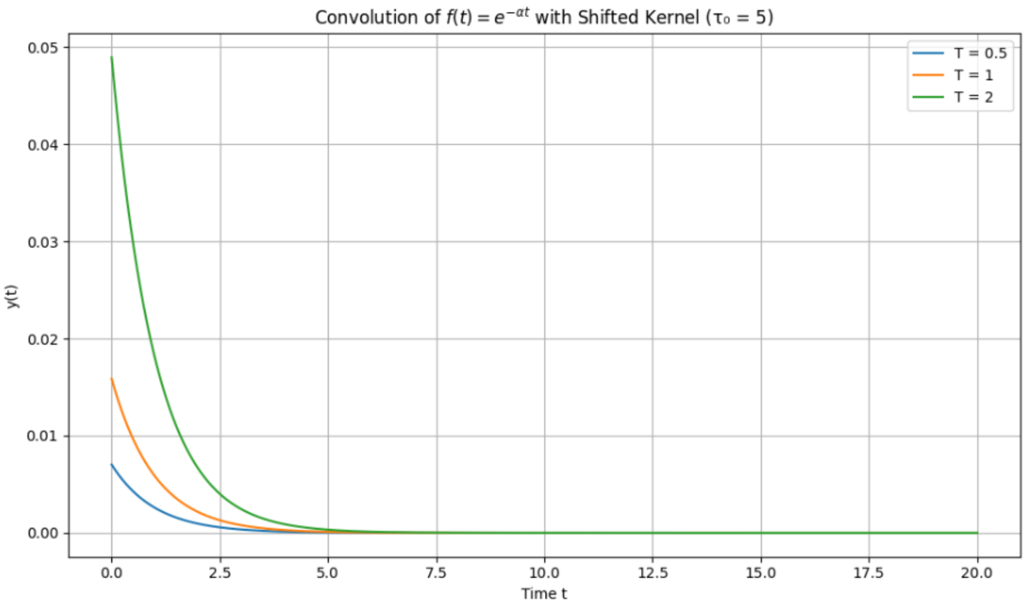


Figure 18: Effect of Shifted $h(t)$

6.5 System Analysis

The rectangular kernel acts like a local integrator:

- It computes the local "energy" or "area" of the exponential signal around each point t.
- For large positive α, $f(t)$ decays quickly, so only values near t contribute.
- For small positive α, the signal decays slowly, and distant values contribute more.
- It is interesting to note that $y(t)$ is a scaled version of the input $f(t)$. Thus $f(t)$ is called the eigenfunction of the LTI system, and the scaling factor is called the eigenvalue.

This system behaves like a low-pass filter. It smooths out sharp features and noise while preserving slow changes.

6.6 Conclusions

1. The rectangular kernel smooths and averages the signal.

2. A causal kernel only uses past information – ideal for real-time systems.
3. Shifting the kernel delays the response.
4. The case of ($\alpha < 0$) was not considered explicitly, as that would mean $f(t)$ and $y(t)$ would grow exponentially with time, which usually isn't physically realistic for most systems (unless an unstable system is specifically wanted).
5. However, if we closely study the $y(t)$ obtained for the given kernel, it can be seen that the obtained $y(t)$ just flips about y-axis if the sign of α is reversed.

7 Input Signal $f(t) = e^{st}$

7.1 Analytical Convolution

We are given:

$$f(t) = e^{st} \quad (s \in \mathbb{C}) \tag{37}$$

$$h(t) = \begin{cases} 1, & -T \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{38}$$

The convolution $y(t) = (f * h)(t)$ is defined as:

$$y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\, d\tau \tag{39}$$

Using the commutative property, we write:

$$y(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)\, d\tau \tag{40}$$

Since $h(\tau)$ is nonzero only for $-T \le \tau \le T$, we restrict the limits of integration:

$$y(t) = \int_{-T}^{T} e^{s(t-\tau)}\, d\tau \tag{41}$$

Pulling out the e^{st} term:

$$y(t) = e^{st} \int_{-T}^{T} e^{-s\tau}\, d\tau \tag{42}$$

Evaluating the integral:

$$\int_{-T}^{T} e^{-s\tau}\, d\tau = \left[\frac{e^{-s\tau}}{-s}\right]_{-T}^{T} = \frac{1}{s}\left(e^{sT} - e^{-sT}\right) \tag{43}$$

Thus, the convolution result is:

$$y(t) = \frac{e^{st}}{s}\left(e^{sT} - e^{-sT}\right) \tag{44}$$

$$\boxed{y(t) = \frac{e^{st}}{s}\left(e^{sT} - e^{-sT}\right)}$$

As s varies from purely real to purely imaginary to a combination of both, the result $y(t)$ varies as follows:

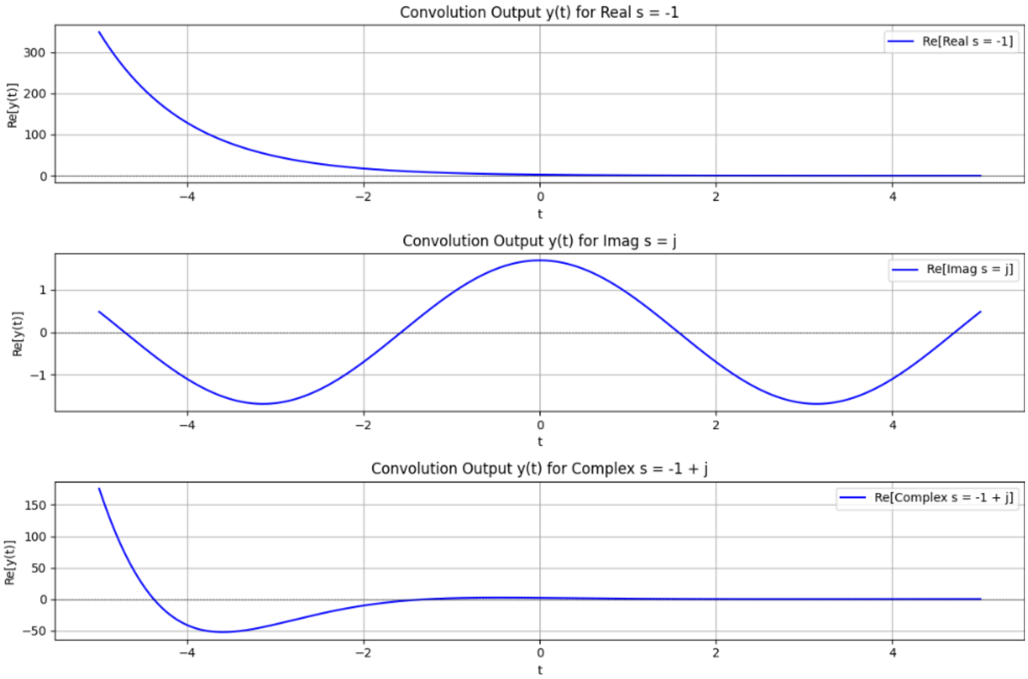


Figure 19: Result of Convolution $y(t)$

It is to be noted that the first case ($s = -1$) follows from the previous section.

7.2 The Effect of Varying T

As T increases, the rectangular kernel becomes wider. This means more of the exponential signal contributes to the convolution at each point.

- For small T, $y(t)$ closely resembles a scaled version of $f(t)$.
- As T increases, $y(t)$ is scaled by a larger factor.
- The scaling factor depends on both the real and imaginary parts of s.
- For instance, when $s = \sigma$ (real), the convolution amplifies or attenuates based on whether $\sigma > 0$ (growth) or $\sigma < 0$ (decay).
- When $s = j\omega$, $f(t)$ is a pure sinusoid. Then, $y(t)$ becomes a sinusoid scaled by the integral of a complex exponential over a symmetric window, which can be interpreted as a frequency-domain filter.

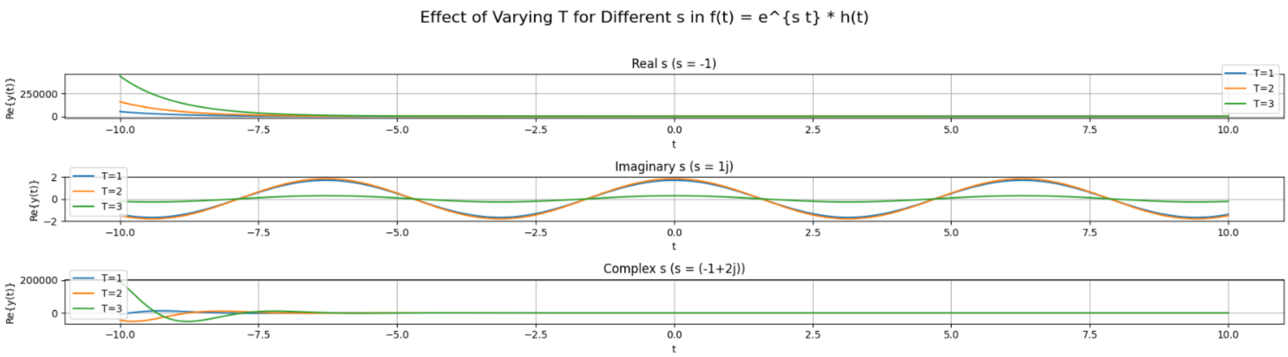


Figure 20: Effect of Varying T for Complex s

7.3 Analysis for Causal Kernel

Now consider a causal rectangular kernel:

$$h_c(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{45}$$

Then, the convolution becomes:

$$y_c(t) = \int_{-\infty}^{\infty} f(t-\tau) h_c(\tau)\, d\tau \tag{46}$$

$$= \int_0^T e^{s(t-\tau)}\, d\tau = e^{st} \int_0^T e^{-s\tau}\, d\tau \tag{47}$$

Evaluating:

$$y_c(t) = e^{st} \cdot \left[\frac{e^{-s\tau}}{-s} \right]_0^T = \frac{e^{st}}{s} \left(1 - e^{-sT}\right) \tag{48}$$

$$\boxed{y_c(t) = \frac{e^{st}}{s} \left(1 - e^{-sT}\right)}$$

This version of the system is causal because it depends only on past values (i.e., values of f from $t - T$ to t). Such kernels are ideal for real-time applications.

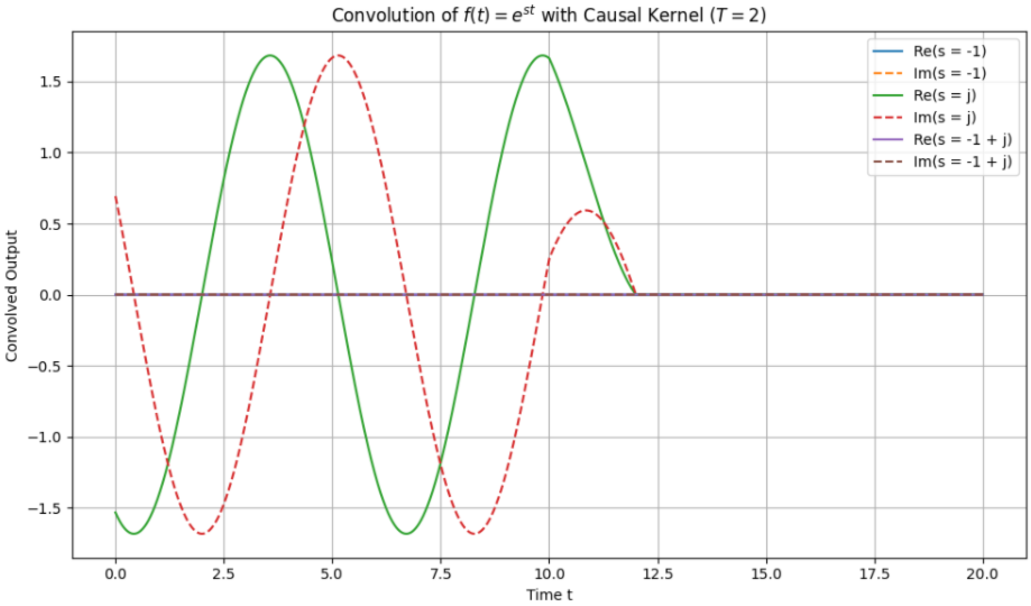


Figure 21: Result of Convolution $y(t)$ for Causal $h(t)$

7.4 Analysis for Shifted Kernel

Let us now consider the shifted kernel:

$$h_s(t) = h(t - \tau_0) \tag{49}$$

Then the convolution becomes:

$$y_s(t) = (f * h_s)(t) = (f * h)(t - \tau_0) \tag{50}$$

So the effect is to delay the output:

$$y_s(t) = \frac{e^{s(t-\tau_0)}}{s}\left(e^{sT} - e^{-sT}\right) \tag{51}$$

$$\boxed{y_s(t) = \frac{e^{s(t-\tau_0)}}{s}\left(e^{sT} - e^{-sT}\right)}$$

This shows that convolving with a time-shifted kernel simply delays the output, which is useful in modelling real systems with delay (e.g., echo, transmission latency, etc.).

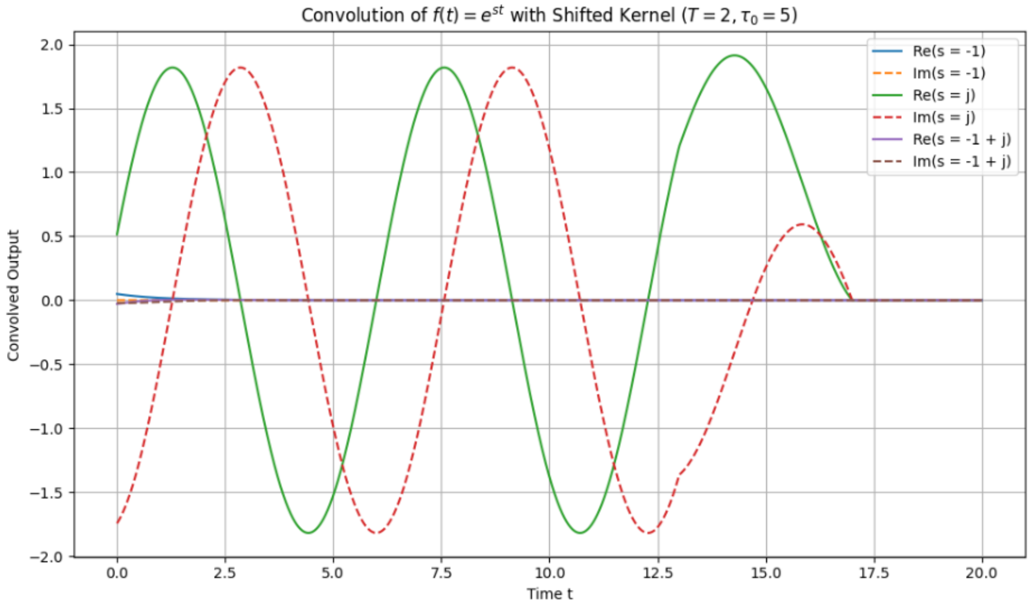


Figure 22: Result of Convolution $y(t)$ for Shifted $h(t)$

7.5 System Analysis

The rectangular kernel acts as a local integrator or averaging filter:

- When s is real and negative, $f(t)$ decays exponentially, so $y(t)$ captures mainly recent signal values.
- When s has a nonzero imaginary part, the convolution captures how well the oscillation of $f(t)$ aligns with the kernel.
- When s is purely imaginary, the result shows the filtering property of the rectangular window—essentially a band-pass behaviour centred at s.
- Similar to the previous section, it can be seen that $y(t)$ is a scaled version of the input $f(t)$. Thus $f(t)$ is called the eigenfunction of the LTI system, and the scaling factor is called the eigenvalue.

7.6 Conclusions

1. Convolution of $f(t) = e^{st}$ with a rectangular kernel gives a closed-form expression.
2. The symmetric kernel averages over both past and future. Causal kernels use only the past.
3. Shifting the kernel simply delays the output.
4. The response's magnitude is modulated by T and s; larger T captures more of the signal.
5. The system acts like a filter that can emphasise or suppress certain exponential or sinusoidal patterns.

8 Input Signal $f(t) = e^{-\frac{t^2}{2\sigma^2}}$

8.1 Analytical Convolution

We consider the convolution of a Gaussian signal $f(t) = e^{-t^2/(2\sigma^2)}$ with a rectangular kernel defined as:

$$h(t) = \begin{cases} 1, & -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

The convolution is given by:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = \int_{t-T}^{t+T} e^{-\tau^2/(2\sigma^2)}d\tau$$

This integral corresponds to a definite integral of the Gaussian function over a finite interval, and can be expressed in terms of the error function:

$$y(t) = \frac{\sigma\sqrt{2\pi}}{2}\left[\text{erf}\left(\frac{t+T}{\sqrt{2}\sigma}\right) - \text{erf}\left(\frac{t-T}{\sqrt{2}\sigma}\right)\right]$$

This result represents a smoothed version of the Gaussian, influenced by the width of the kernel T.

8.2 The Effect of Varying T

The parameter T controls the width of the rectangular window used in the convolution. Its effects are summarized as follows:

- For small $T \ll \sigma$, the output $y(t)$ closely follows the shape of the original Gaussian.
- As T increases, the result becomes more smoothed and broader, reducing the peak and increasing the width.
- As $T \to \infty$, the convolution approaches the total area under the Gaussian curve (which is $\sigma\sqrt{2\pi}$), resulting in a flat, constant output across time.

This demonstrates the low-pass filtering behavior of the convolution operation.

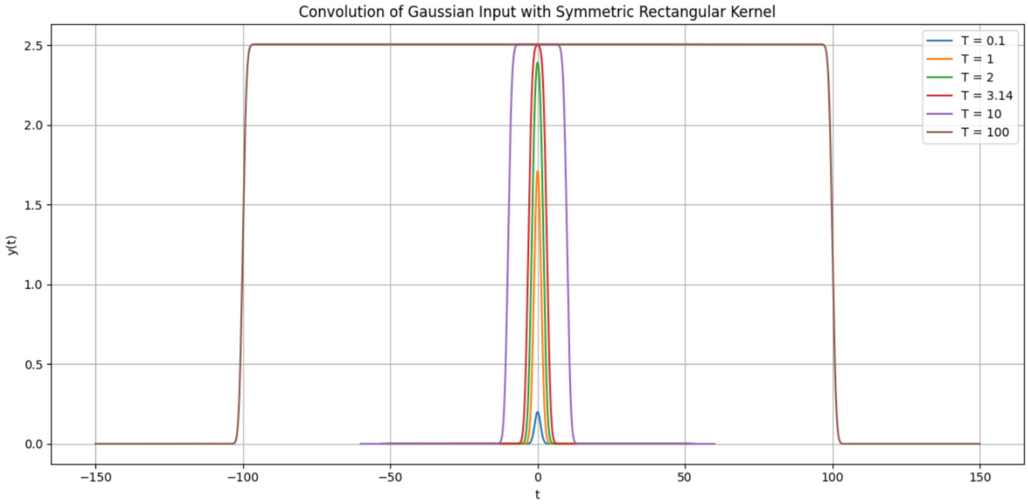


Figure 23: Result of Convolution $y(t)$ for Varying T

8.3 Analysis for Causal Kernel

When the kernel is modified to be causal, defined as:

$$h(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

the convolution becomes:

$$y(t) = \int_{t-T}^{t} e^{-\tau^2/(2\sigma^2)} d\tau$$

This version integrates only over the past values of $f(t)$, resulting in an **asymmetric smoothing**. It preserves causality, which is essential in real-time systems. However, the output is delayed relative to the symmetric case, and the peak response is also shifted in time.

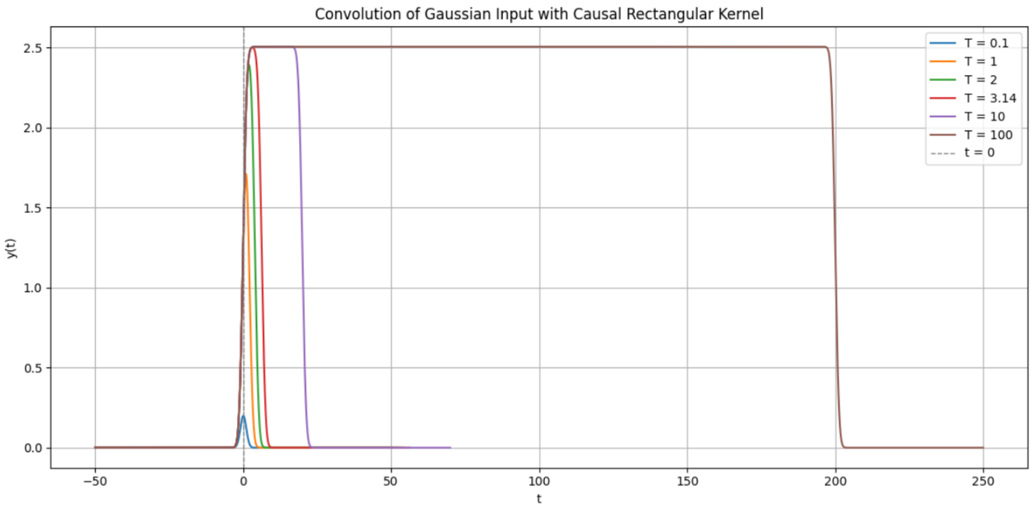


Figure 24: Result of Convolution $y(t)$ for Causal $h(t)$

8.4 Analysis for Shifted Kernel

Let the kernel be shifted by a time τ_0, giving:

$$h(t) = \begin{cases} 1, & -T + \tau_0 \le t \le T + \tau_0 \\ 0, & \text{otherwise} \end{cases}$$

Then the convolution becomes:

$$y(t) = \int_{t-T-\tau_0}^{t+T-\tau_0} e^{-\tau^2/(2\sigma^2)} d\tau$$

This result is simply a **time-shifted version** of the original convolution output. The effect is a **horizontal translation** of the output curve by τ_0, which models **time delay** in physical systems (e.g., signal propagation or response latency).

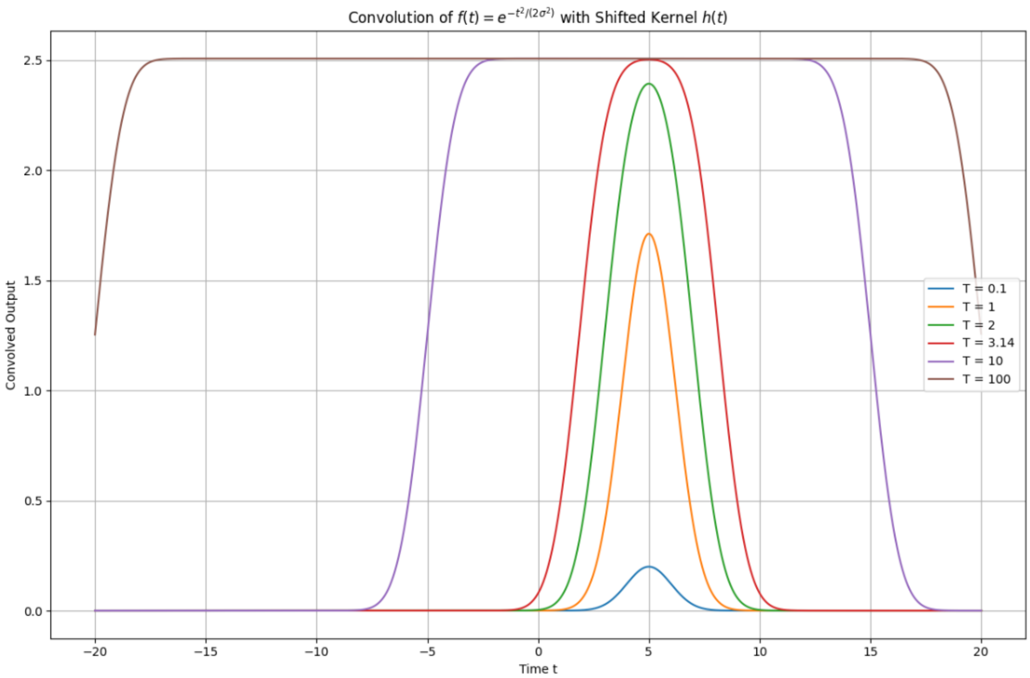


Figure 25: Result of Convolution $y(t)$ for Shifted $h(t)$

8.5 System Analysis

The convolution with a rectangular kernel acts as a **smoothing filter**, averaging the signal over a neighborhood of width $2T$. Key system characteristics include:

- **Linearity:** The system is linear, as convolution is a linear operation.
- **Time Invariance:** Shifting the input shifts the output by the same amount, confirming time invariance.
- **Causality:** The system is non-causal for symmetric kernels but becomes causal with a one-sided kernel.
- **Low-Pass Behavior:** The rectangular window suppresses high-frequency variations, smoothing out the signal.

8.6 Conclusions

The convolution of a Gaussian signal with a rectangular kernel produces a smoothed signal whose characteristics depend on the kernel width and symmetry. A symmetric kernel smooths uniformly around each point, while a causal or shifted kernel introduces asymmetry and delay. These variations are crucial for understanding system behavior in signal processing, especially in filtering, real-time control, and communications.

9 Input Signal $f(t) = tu(t)$

9.1 Analytical Convolution

The rectangular kernel $h(t)$ is defined as:

$$h(t) = \begin{cases} 1, & -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

This kernel acts as a smoothing or averaging filter when convolved with other signals.

We define the convolution of $f(t)$ with $h(t)$ as:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)\,d\tau$$

Given the support of $h(\tau)$, the integral simplifies to:

$$y(t) = \int_{-T}^{T} f(t-\tau)\,d\tau$$

This formulation shifts the input signal and integrates over the fixed window of the kernel.

The ramp function is defined as:

$$f(t) = \begin{cases} t, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

Then the convolution becomes:

$$y(t) = \int_{-T}^{T} (t-\tau)u(t-\tau)\,d\tau$$

To evaluate this, note that $u(t-\tau) = 1$ when $\tau < t$, so the limits become:

- If $t < -T$: $y(t) = 0$
- If $-T \le t < T$: $y(t) = \int_{-T}^{t} (t-\tau)\,d\tau = \left[t\tau - \frac{\tau^2}{2}\right]_{-T}^{t}$
- If $t \ge T$: $y(t) = \int_{-T}^{T} (t-\tau)\,d\tau = 2Tt$

Evaluating for $-T \le t < T$:

$$y(t) = t(t+T) - \frac{t^2 - (-T)^2}{2} = \frac{(t+T)^2}{2}$$

Simplified expression:

$$y(t) = \begin{cases} 0, & t < -T \\ \frac{(t+T)^2}{2}, & -T \le t < T \\ 2Tt, & t \ge T \end{cases}$$

9.2 The Effect of Varying T

- For $t < -T$: $y(t) = 0$. As T increases, the point $t = -T$ moves leftward, making the nonzero region of $y(t)$ expand to the left.
- For $-T \leq t < T$: $y(t) = \frac{(t+T)^2}{2}$, a quadratic function. Increasing T:
  - Expands the width of the quadratic region.
  - Increases the magnitude of $y(t)$ because of the $(t+T)^2$ term.
- For $t \geq T$: $y(t) = 2Tt$, a linear function. Increasing T steepens the slope, since the slope is proportional to T.

In summary:

- Larger T shifts the onset of nonzero $y(t)$ leftward.
- The quadratic region becomes wider and higher.
- The linear region has a steeper slope proportional to T.

The overall physical intuition can be understood as:

- T controls the width of the smoothing (filtering) operation.
- Larger T **spreads out** $y(t)$ and **increases** accumulated values.
- Smaller T results in more **localized** filtering and sharper transitions in $y(t)$.

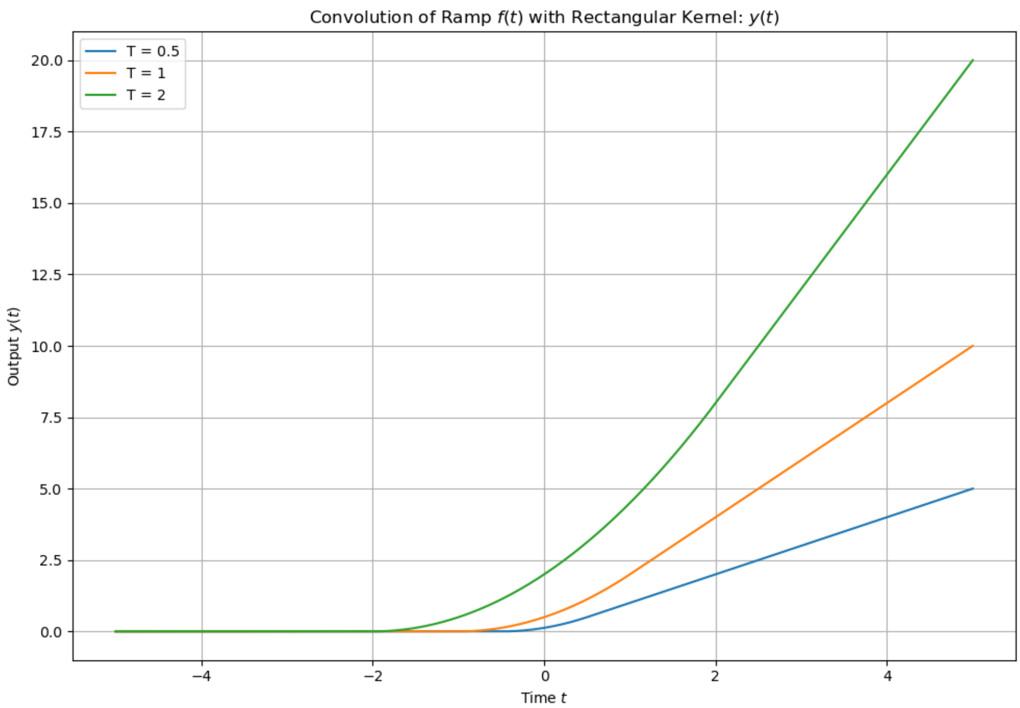


Figure 26: Convolution of $f(t)$ with $h(t)$ for different values of T

9.3 Analysis for Causal Kernel

The modified kernel is:

$$h(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

Then:

$$y(t) = \int_0^T f(t-\tau)\, d\tau$$

We consider $y(t) = \int_0^{\min(t,T)} (t-\tau)\, d\tau$

Piecewise evaluation:

- $t < 0$: The integration domain becomes empty due to $u(t-\tau) = 0 \Rightarrow y(t) = 0$
- $0 \le t < T$: $y(t) = \int_0^t (t-\tau)\, d\tau = t\tau - \frac{\tau^2}{2}\Big|_0^t = \frac{t^2}{2}$
- $t \ge T$: $y(t) = \int_0^T (t-\tau)\, d\tau = tT - \frac{T^2}{2}$

Thus,

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{t^2}{2}, & 0 \le t < T \\ tT - \frac{T^2}{2}, & t \ge T \end{cases}$$

This expression represents the convolution of a signal with a kernel that is 1 from 0 to T, and 0 otherwise.

The effect of varying T for causal $h(t)$ is as follows:

- When $t < 0$, the output $y(t) = 0$ is constant and independent of T.
- For $0 \le t < T$, the output $y(t) = \frac{t^2}{2}$ is a quadratic function of t. As T increases, the range of t for which this quadratic term applies becomes larger. This means that the quadratic growth will extend over a larger time range, increasing the slope of the curve before reaching the constant region.
- For $t \ge T$, the output $y(t) = tT - \frac{T^2}{2}$ is linear in t. As T increases, the slope of the linear portion increases, making the growth rate of $y(t)$ higher for larger T. The value $\frac{T^2}{2}$ also becomes larger, shifting the entire linear function upwards.

In summary:

- As T increases, the quadratic portion $y(t) = \frac{t^2}{2}$ extends over a larger t-interval.
- The linear growth for $t \ge T$ becomes steeper as T increases.
- The overall shape of the function becomes broader, with the transition from the quadratic to the linear regime occurring at higher values of t.

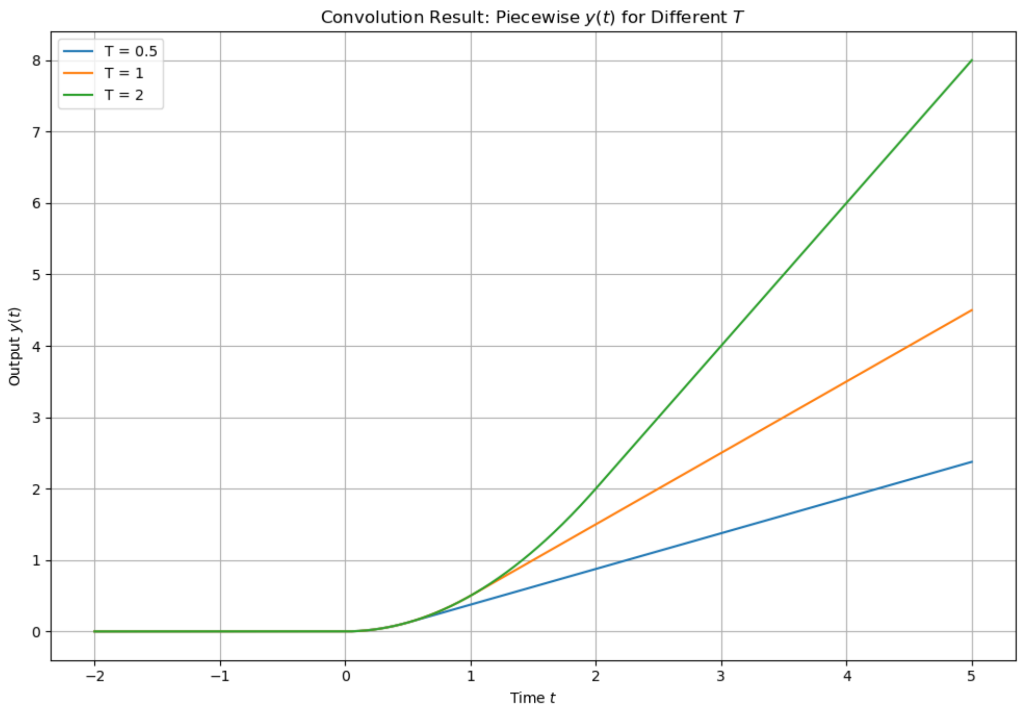


Figure 27: Convolution of $f(t)$ with causal $h(t)$ for different values of T

9.4 Analysis for Shifted Kernel

Now consider a time-shifted kernel $h(\tau - \tau_0)$:

$$y(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau-\tau_0)\, d\tau = \int_{\tau_0 - T}^{\tau_0 + T} f(t-\tau)\, d\tau$$

Substitute $s = t - \tau$:

$$y(t) = \int_{t-(\tau_0+T)}^{t-(\tau_0-T)} f(s)\, ds$$

- If $t - (\tau_0 + T) < 0$, the lower limit is clipped to 0 due to $u(s)$:

$$y(t) = \int_{\max(0,t-\tau_0-T)}^{t-\tau_0+T} s\, ds = \left[\frac{s^2}{2}\right]_{\max(0,t-\tau_0-T)}^{t-\tau_0+T}$$

We compute:

$$y(t) = \int_{\tau_0 - T}^{\tau_0 + T} (t-\tau)u(t-\tau)\, d\tau$$

Note that $u(t-\tau) = 1$ when $\tau \leq t$, and 0 otherwise. Thus:

- If $t < \tau_0 - T$: The integration interval lies completely beyond t ($\tau > t$), so $u(t-\tau) = 0$ throughout. Hence,

$$y(t) = 0$$

- If $\tau_0 - T \leq t < \tau_0 + T$: Only partial overlap; integration from $\tau_0 - T$ to t:

$$y(t) = \int_{\tau_0 - T}^{t} (t - \tau)\, d\tau$$

  Evaluating and Simplifying:

$$y(t) = \frac{(t - \tau_0 + T)^2}{2}$$

- If $t \geq \tau_0 + T$: Entire kernel is within active range; full integration over $[\tau_0 - T, \tau_0 + T]$:

$$y(t) = \int_{\tau_0 - T}^{\tau_0 + T} (t - \tau)\, d\tau$$

  Evaluating and Simplifying:

$$y(t) = 2Tt - 2\tau_0 T$$

  or equivalently:

$$y(t) = 2T(t - \tau_0)$$

Therefore, the full piecewise expression is:

$$y(t) = \begin{cases} 0, & t < \tau_0 - T \\ \frac{(t-\tau_0+T)^2}{2}, & \tau_0 - T \leq t < \tau_0 + T \\ 2T(t - \tau_0), & t \geq \tau_0 + T \end{cases}$$

The effect of time-shifted kernel on $y(t)$ is as follows:

- When $f(t)$ or $h(t)$ is shifted in time, the convolution output $y(t)$ also experiences a corresponding shift.
- If $h(t)$ is replaced with $h(t - t_0)$, then:

$$y(t) = (h(t - t_0) * f(t)) = y_{\text{original}}(t - t_0)$$

  That is, the entire output $y(t)$ is shifted by t_0 units to the right.
- Similarly, shifting $f(t)$ instead of $h(t)$ also results in a time-shifted output.

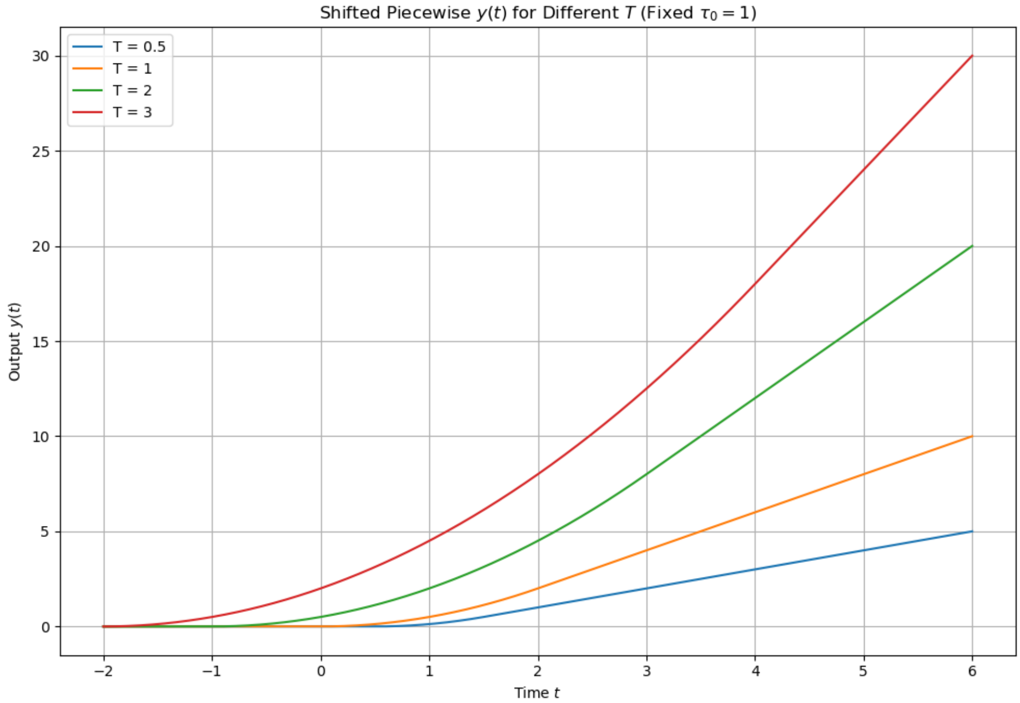


Figure 28: Convolution of $f(t)$ with shifted $h(t)$ for different values of T

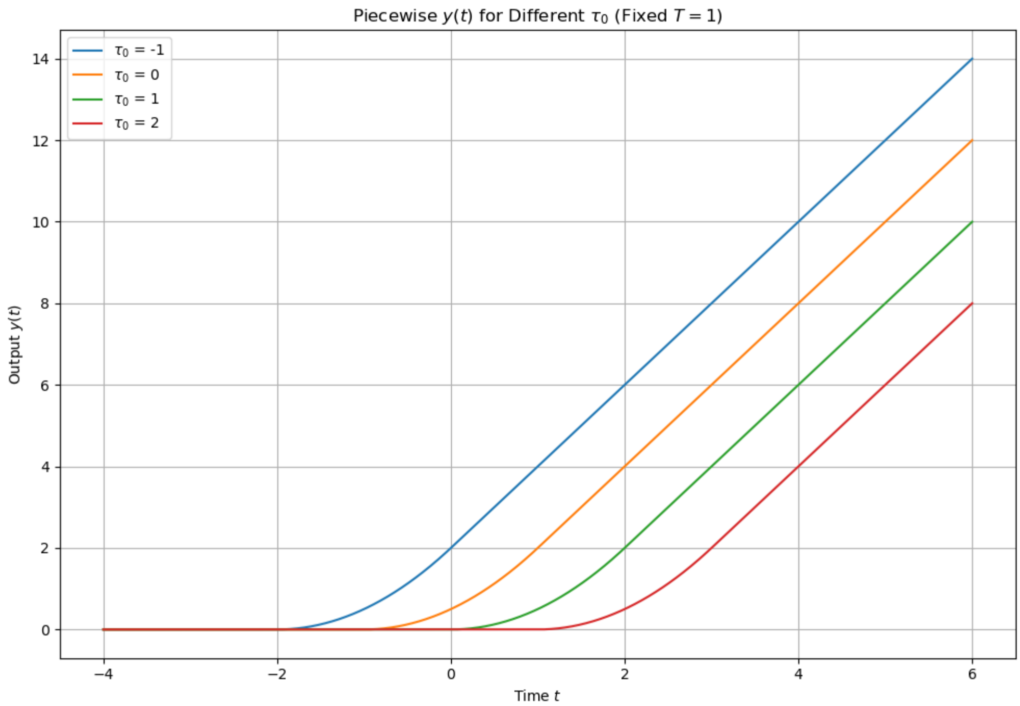


Figure 29: Convolution of $f(t)$ with shifted $h(t)$ for different values of τ_0

In practical systems (such as communication, control, or signal processing), delays are unavoidable. The significance of time-delayed systems:

- In communication systems, time delay models transmission delays.

- In control systems, delays can lead to instability if not properly accounted for.
- In filtering, delays imply phase shifts which can distort signals.

Understanding how time-shifts affect convolution output helps in designing systems that are robust to such delays. It also assists in accurately predicting system response when delays are present.

Thus, time shifts are not mere mathematical operations but reflect real-world effects crucial for system behavior analysis and design.

9.5 System Analysis

We analyze the system defined by the convolution $y(t) = (f * h)(t)$, where $f(t)$ is a ramp function and $h(t)$ is a rectangular kernel. The system exhibits the following properties:

- **Linearity:** Convolution is a linear operation. If $f_1(t) \to y_1(t)$ and $f_2(t) \to y_2(t)$, then for any scalars a, b, the input $af_1(t) + bf_2(t) \to ay_1(t) + by_2(t)$.
- **Time-Invariance:** Shifting the input or the kernel results in an equivalent shift in the output. Specifically, if $h(t) \to y(t)$, then $h(t - t_0) \to y(t - t_0)$. This confirms the time-invariant nature of the system.
- **Causality:** The system is *non-causal* for the symmetric rectangular kernel defined over $[-T, T]$, because $y(t)$ depends on future values of $f(t)$. However, when using a causal kernel (non-zero only for $t \geq 0$), the system becomes causal.
- **Memory:** The system is a *finite-memory system* (or has finite impulse response) since the kernel $h(t)$ has finite support. The output $y(t)$ at any time depends only on values of $f(t)$ within a finite interval.
- **Stability:** If the input $f(t)$ is bounded and $h(t)$ is absolutely integrable, then the convolution integral yields a bounded output. Hence, the system is BIBO (Bounded Input, Bounded Output) stable.

9.6 Conclusions

The convolution of a ramp signal $f(t)$ with various forms of the rectangular kernel $h(t)$ provides a powerful demonstration of time-domain filtering and smoothing operations:

- The symmetric rectangular kernel acts as a smoothing filter centered around the current time t, averaging values of $f(t)$ in the interval $[t - T, t + T]$.
- The causal rectangular kernel introduces a delay in the smoothing operation, accumulating values only from the past. This makes the system causal and suitable for real-time applications.
- A time-shifted kernel causes a corresponding shift in the output, validating the time-invariance property of the system.
- Increasing the kernel width T increases the extent of averaging, resulting in a smoother and more spread-out output. This highlights the trade-off between responsiveness and smoothing in time-domain filtering.

Overall, the analysis demonstrates how convolution with basic kernels such as rectangular functions can be used to control the shape, delay, and smoothness of output signals. These properties are foundational in signal processing and system theory, especially in contexts like low-pass filtering, moving average filters, and real-time signal analysis.

10 Input Signal $f(t) = t^n u(t)$

10.1 Analytical Convolution

The rectangular kernel $h(t)$ is defined as:

$$h(t) = \begin{cases} 1, & -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

This kernel acts as a smoothing or averaging filter when convolved with other signals.

We define the convolution of $f(t)$ with $h(t)$ as:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)\, d\tau$$

Given the support of $h(\tau)$, the integral simplifies to:

$$y(t) = \int_{-T}^{T} f(t-\tau)\, d\tau$$

This formulation shifts the input signal and integrates over the fixed window of the kernel.

Assuming $f(t) = t^n$, we compute:

$$y(t) = \int_{-T}^{T} (t-\tau)^n\, d\tau$$

Using substitution $s = t - \tau$, $\tau = t - s$, $d\tau = -ds$:

$$y(t) = \int_{t+T}^{t-T} s^n(-ds) = \int_{t-T}^{t+T} s^n\, ds = \left[\frac{s^{n+1}}{n+1}\right]_{t-T}^{t+T}$$

$$y(t) = \frac{(t+T)^{n+1} - (t-T)^{n+1}}{n+1}$$

10.2 The Effect of Varying T

$$y(t) = \frac{(t+T)^{n+1} - (t-T)^{n+1}}{n+1}.$$

Expanding around $t = 0$ using Taylor series:

$$\begin{aligned} (t+T)^{n+1} &\approx T^{n+1} + (n+1)T^n t + \cdots \\ (t-T)^{n+1} &\approx (-1)^{n+1}T^{n+1} + (n+1)(-1)^n T^n t + \cdots \end{aligned}$$

Taking the difference:

- Constant terms may cancel (depending on parity of $n+1$).

- The linear term in t dominates initially, proportional to T^n.

Thus, increasing T:

- Increases the overall magnitude of $y(t)$.
- Makes $y(t)$ grow faster with respect to t.
- Higher powers of T lead to stronger scaling effects.

In summary:

- Larger T amplifies $y(t)$ and increases its slope near $t = 0$.
- Higher values of T cause $y(t)$ to grow much faster for large t.

The overall physical intuition can be understood as:

- T controls the width of the smoothing (filtering) operation.
- Larger T **spreads out** $y(t)$ and **increases** accumulated values.
- Smaller T results in more **localized** filtering and sharper transitions in $y(t)$.

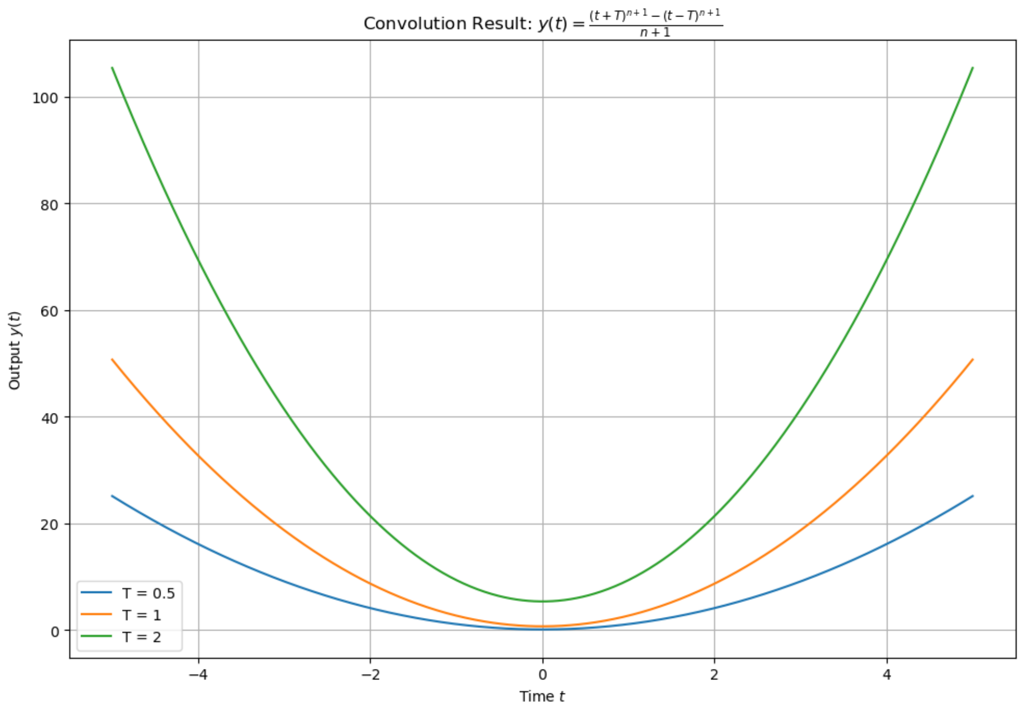


Figure 30: Convolution of $f(t)$ with $h(t)$ for different values of T

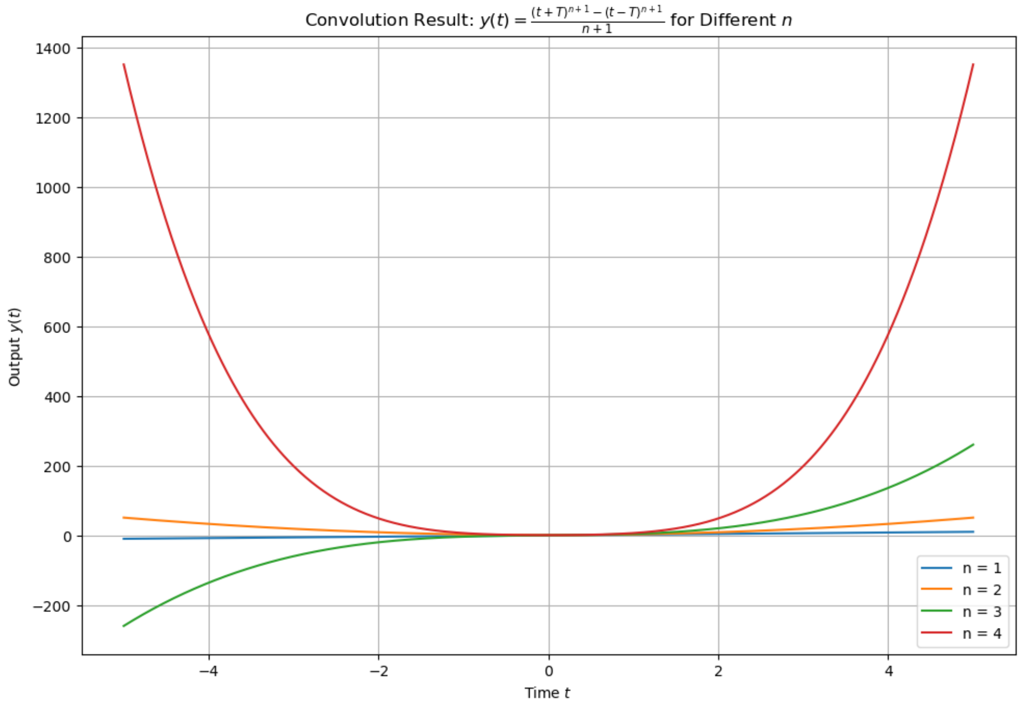


Figure 31: Convolution of $f(t)$ with $h(t)$ for different values of n

10.3 Analysis for Causal Kernel

The modified kernel is:

$$h(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

Then:

$$y(t) = \int_0^T f(t-\tau)\,d\tau$$

$$y(t) = \int_0^T (t-\tau)^n\,d\tau$$

Substitute $s = t - \tau$, $\tau = t - s$, $d\tau = -ds$:

$$y(t) = \int_{t-T}^{t} s^n\,ds = \left[\frac{s^{n+1}}{n+1}\right]_{t-T}^{t} = \frac{t^{n+1} - (t-T)^{n+1}}{n+1}$$

This result highlights causality in convolution.

Consider the convolution expression:

$$y(t) = \frac{t^{n+1} - (t-T)^{n+1}}{n+1}$$

This expression corresponds to the convolution of a signal $f(t) = t^n$ with a rectangular kernel.

The effect of varying T for causal $h(t)$:

- For smaller T, the convolution function shows a smaller difference between the terms t^{n+1} and $(t-T)^{n+1}$, leading to less pronounced features in the output $y(t)$.
- As T increases, the difference between t^{n+1} and $(t-T)^{n+1}$ becomes larger, and thus the output $y(t)$ grows more rapidly, especially for larger values of t.
- The polynomial nature of the expression means that the higher the power n, the more significant the effect of T on the output. For example, for $n = 2$, the difference between t^3 and $(t-T)^3$ grows faster than for smaller powers of n, leading to a more abrupt change in the output.

In summary:

- Increasing T amplifies the difference between t^{n+1} and $(t-T)^{n+1}$, making the output more pronounced and steeper.
- For larger T, the transition between polynomial terms becomes more noticeable, leading to a higher growth rate for the function $y(t)$.

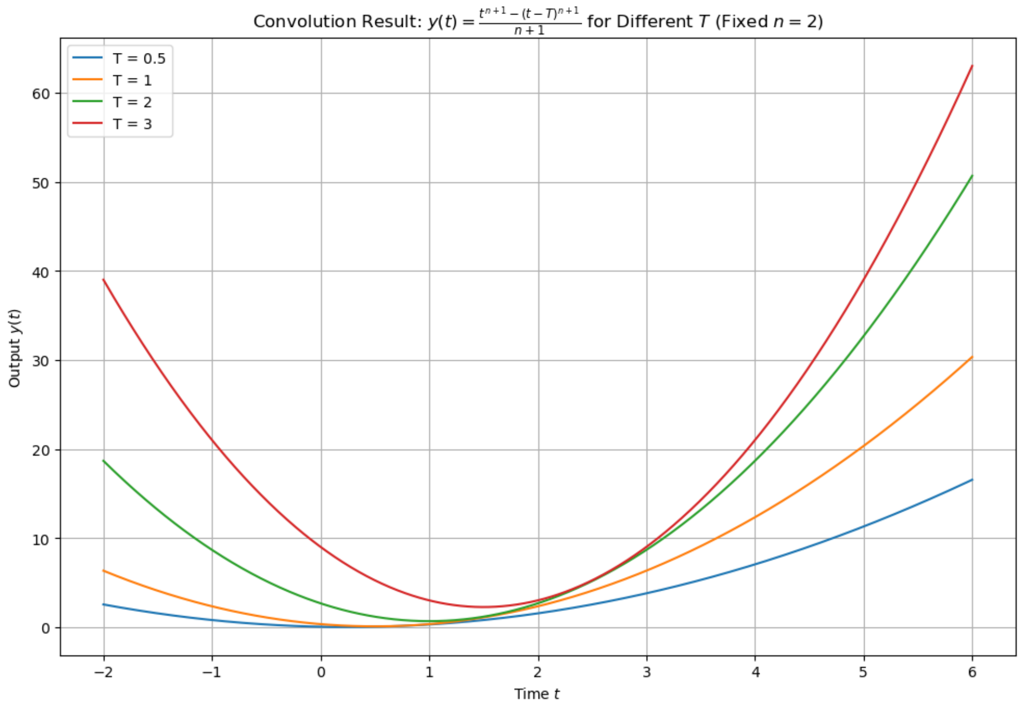


Figure 32: Convolution of $f(t)$ with causal $h(t)$ for different values of T

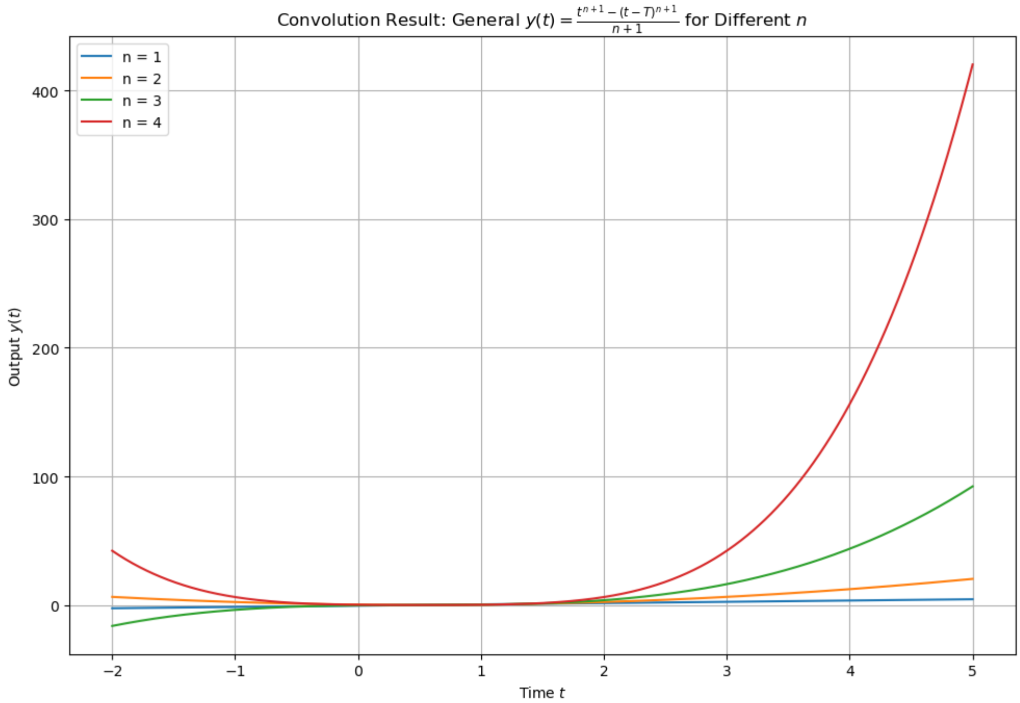


Figure 33: Convolution of $f(t)$ with causal $h(t)$ for different values of n

10.4 Analysis for Shifted Kernel

Now consider a time-shifted kernel $h(\tau - \tau_0)$:

$$y(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau-\tau_0)\,d\tau = \int_{\tau_0-T}^{\tau_0+T} f(t-\tau)\,d\tau$$

Substitute $s = t - \tau$:

$$y(t) = \int_{t-(\tau_0+T)}^{t-(\tau_0-T)} f(s)\,ds$$

$$y(t) = \int_{t-(\tau_0+T)}^{t-(\tau_0-T)} s^n\,ds = \left[\frac{s^{n+1}}{n+1}\right]_{t-(\tau_0+T)}^{t-(\tau_0-T)}$$

$$y(t) = \frac{(t-\tau_0+T)^{n+1} - (t-\tau_0-T)^{n+1}}{n+1}$$

The effect of time-shifted kernel on $y(t)$

- When $f(t)$ or $h(t)$ is shifted in time, the convolution output $y(t)$ also experiences a corresponding shift.
- If $h(t)$ is replaced with $h(t-t_0)$, then:

  $$y(t) = (h(t-t_0) * f(t)) = y_{\text{original}}(t-t_0)$$

  That is, the entire output $y(t)$ is shifted by t_0 units to the right.
- Similarly, shifting $f(t)$ instead of $h(t)$ also results in a time-shifted output.

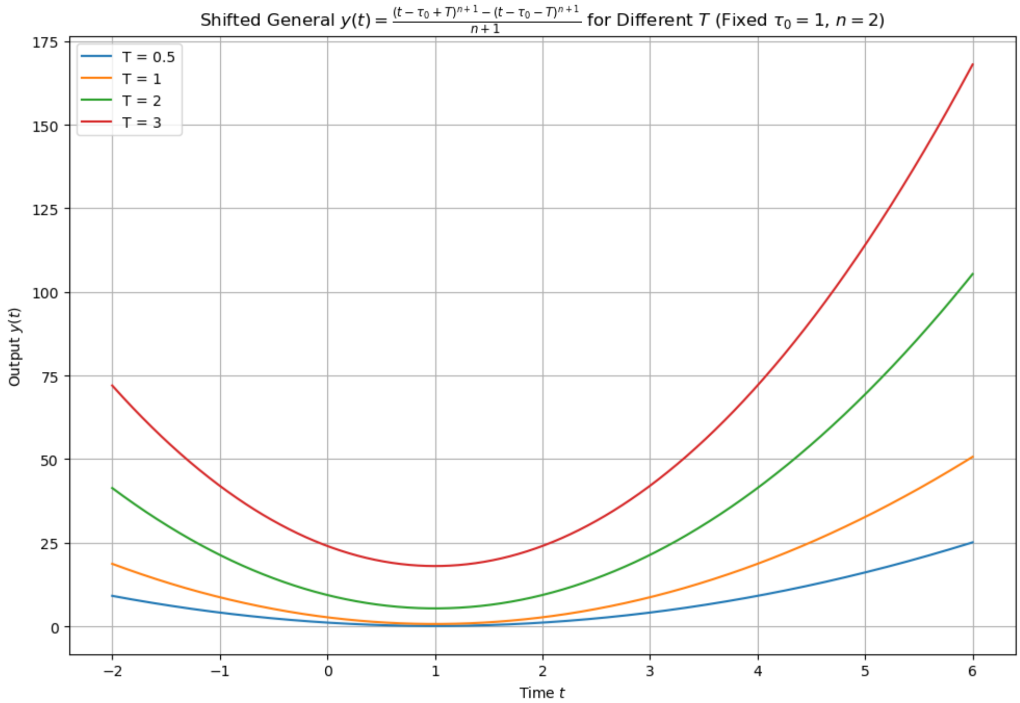


Figure 34: Convolution of $f(t)$ with shifted $h(t)$ for different values of T

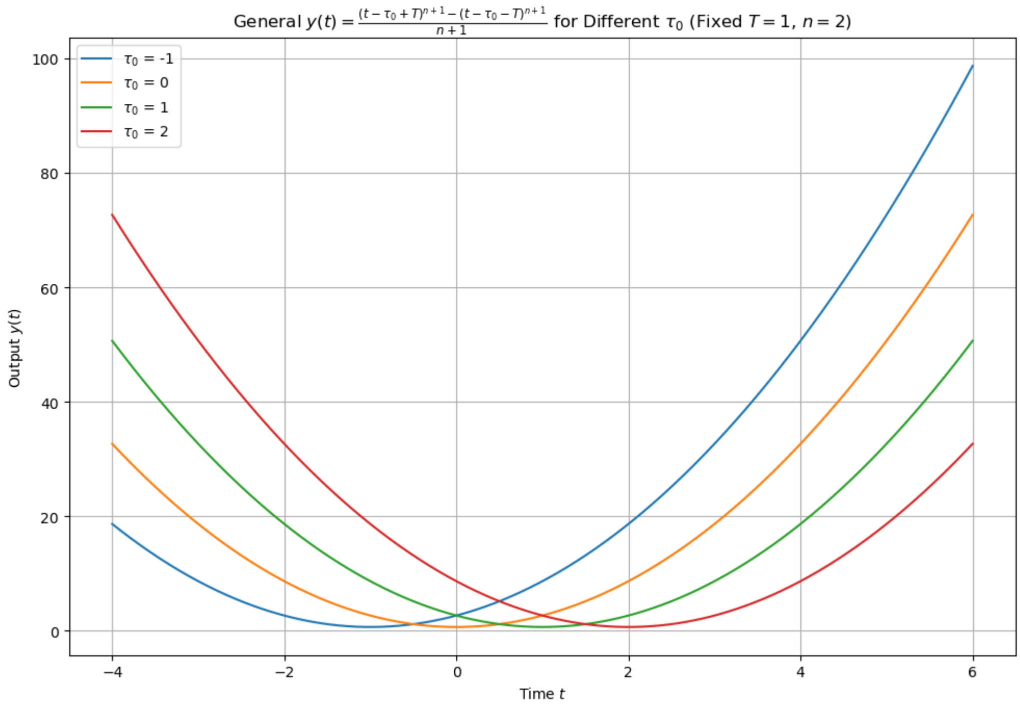


Figure 35: Convolution of $f(t)$ with shifted $h(t)$ for different values of τ_0

10.5 System Analysis

We consider a system defined by the operation:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)\,d\tau$$

where $f(t) = t^n$ is a polynomial signal and $h(t)$ is a rectangular window function. The analysis of this system under various forms of $h(t)$ reveals the following properties:

- **Linearity:** The system is linear since convolution is a linear operator. That is, for any scalars a and b,
$$af_1(t) + bf_2(t) \longrightarrow ay_1(t) + by_2(t)$$
where $y_i(t) = f_i(t) * h(t)$.
- **Time-Invariance:** The system is time-invariant. A shift in the input or kernel results in a corresponding shift in the output:
$$f(t - t_0) * h(t) = y(t - t_0), \quad f(t) * h(t - t_0) = y(t - t_0)$$
- **Causality:**
  - For the symmetric kernel defined on $[-T, T]$, the system is *non-causal*, since the output $y(t)$ depends on future values of $f(t)$.
  - For the causal kernel defined on $[0, T]$, the system is *causal*, as the output at time t depends only on past and present inputs.
- **Memory:** The system has *finite memory* (i.e., finite impulse response), since the kernel $h(t)$ has bounded support. The output depends only on a finite interval of the input signal.
- **Stability:** If $f(t)$ is bounded and $h(t)$ is absolutely integrable (which is true for a rectangular function), the system is BIBO (bounded-input, bounded-output) stable.
- **Smoothing Effect:** The rectangular kernel acts as a moving average filter. It smooths sharp variations in $f(t)$ and spreads out its growth depending on the width T of the kernel.

10.6 Conclusions

The convolution of the polynomial signal $f(t) = t^n$ with rectangular kernels $h(t)$ produces smooth output curves whose shape and growth are strongly influenced by the properties of the kernel. Our key observations are:

- **Effect of Kernel Width (T):** Larger values of T result in increased averaging and smoothing, as the convolution accumulates values over a wider window. This increases the magnitude and slope of $y(t)$, especially near the origin.
- **Effect of Polynomial Degree (n):** Higher values of n amplify the response due to the nonlinear growth of t^n, making the convolution output more sensitive to changes in T.
- **Causal vs Non-Causal Kernels:**

  - Non-causal symmetric kernels produce centered smoothing around t.
  - Causal kernels result in delayed accumulation, useful for real-time applications where future input values are not available.
- **Shifted Kernels:** Time-shifting the kernel translates the entire output curve in time, verifying the time-invariance of the system.
- **System Nature:** The system is linear, time-invariant, and (depending on the kernel) either causal or non-causal, with finite memory and BIBO stability.

These results provide insight into the behaviour of convolution with polynomial inputs and simple rectangular kernels. Such systems form the basis for many signal processing applications, such as smoothing filters, moving averages, and delay systems. The mathematical structure also illustrates how convolution encodes symmetry, scaling, and shift properties of both input and kernel.

11 Input Signal $f(t) = \delta(t)$

11.1 Analytical Convolution

Consider the input signal $f(t) = \delta(t - t_0)$, a Dirac delta function centered at t_0, and a rectangular kernel defined as:

$$h(t) = \begin{cases} 1, & \text{for } -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

The convolution becomes:

$$y(t) = \int_{t-T}^{t+T} \delta(\tau - t_0)\, d\tau$$

Using the sifting property of the delta function:

$$\int_a^b \delta(\tau - t_0)\, d\tau = \begin{cases} 1, & \text{if } t_0 \in (a, b) \\ 0, & \text{otherwise} \end{cases}$$

We obtain:

$$y(t) = \begin{cases} 1, & \text{if } t - T < t_0 < t + T \\ 0, & \text{otherwise} \end{cases} \quad \Rightarrow \quad y(t) = \begin{cases} 1, & \text{if } t \in (t_0 - T,\ t_0 + T) \\ 0, & \text{otherwise} \end{cases}$$

Thus, the output is a rectangular pulse of width $2T$, centred at t_0. The kernel is replicated and shifted to the location of the impulse.

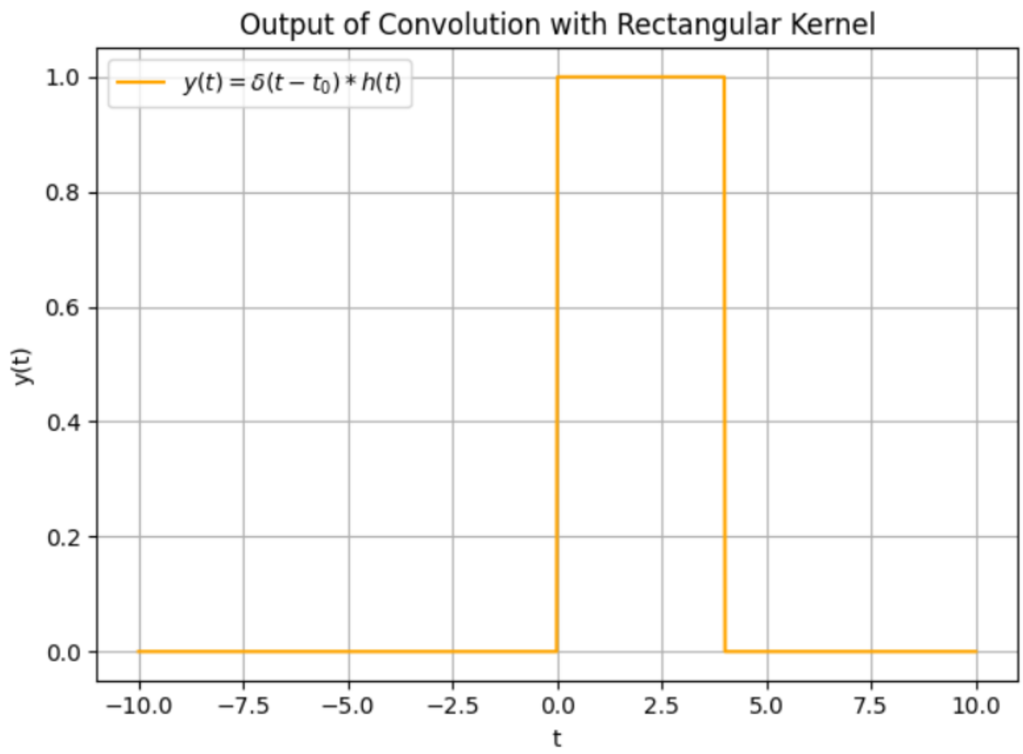


Figure 36: Convolution Result $y(t)$

11.2 The Effect of Varying T

As T increases:

- The width of the output pulse increases proportionally to $2T$.
- The height of the pulse remains constant at 1.

The delta function has unit area, and the kernel fully captures it over its support. Therefore, T controls the duration, not the amplitude, of the output.

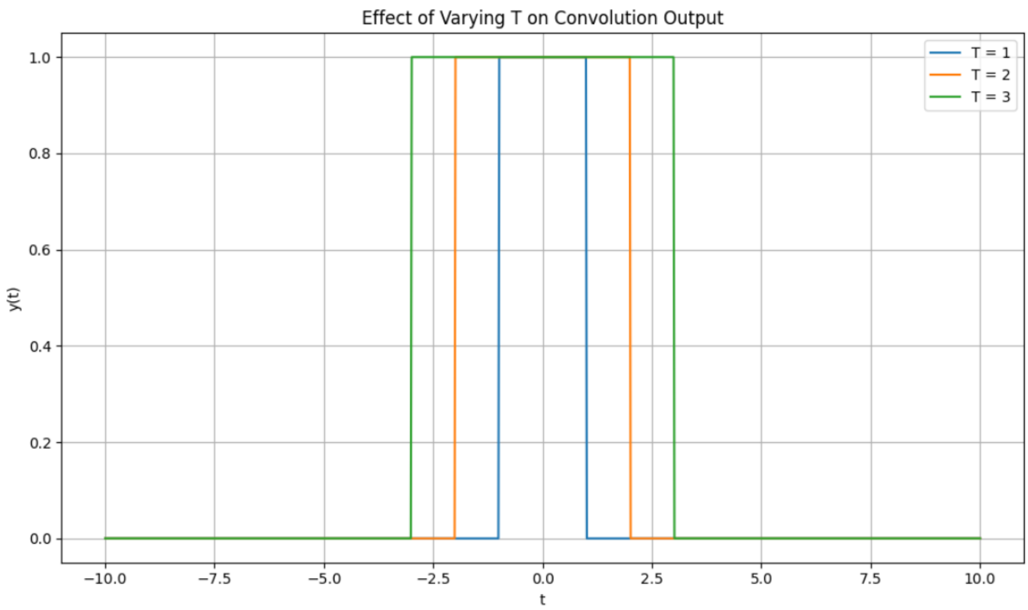


Figure 37: Convolution Result $y(t)$

11.3 Analysis for Causal Kernel

Consider a causal rectangular kernel:

$$h_{\text{causal}}(t) = \begin{cases} 1, & 0 \leq t \leq 2T \\ 0, & \text{otherwise} \end{cases}$$

Then:

$$y(t) = \int_{t-2T}^{t} \delta(\tau - t_0)\, d\tau = \begin{cases} 1, & \text{if } t - 2T < t_0 < t \\ 0, & \text{otherwise} \end{cases} \Rightarrow y(t) = \begin{cases} 1, & \text{if } t \in (t_0,\ t_0 + 2T) \\ 0, & \text{otherwise} \end{cases}$$

This produces a right-shifted rectangular pulse starting at t_0 and extending for $2T$. The system responds only after the impulse occurs (causal system).

11.4 Analysis for Shifted Kernel

Consider a shifted kernel $h_{\text{shifted}}(t) = h(t - \tau_0)$. Then:

$$y(t) = \int_{t-T-\tau_0}^{t+T-\tau_0} \delta(\tau - t_0)\, d\tau = \begin{cases} 1, & \text{if } t - T - \tau_0 < t_0 < t + T - \tau_0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow y(t) = \begin{cases} 1, & \text{if } t \in (t_0 - T + \tau_0,\ t_0 + T + \tau_0) \\ 0, & \text{otherwise} \end{cases}$$

The response is a rectangular pulse of width $2T$, centered at $t_0 + \tau_0$, reflecting the temporal shift in the kernel.

11.5 System Analysis

The delta function has unit energy:

$$\int_{-\infty}^{\infty} \delta^2(t)\, dt = \infty, \quad \text{but it integrates to 1}$$

The output $y(t)$ is a rectangular pulse of width $2T$ and height 1:

$$E_y = \int_{-\infty}^{\infty} y^2(t)\, dt = \int_{t_0-T}^{t_0+T} 1\, dt = 2T$$

Thus, the output energy scales linearly with T, while the shape remains rectangular. The system acts as a pulse-shaping filter.

11.6 Conclusions

- Convolution of $\delta(t - t_0)$ with a rectangular kernel yields a rectangular pulse centered at t_0.
- Width of the output is $2T$, and height remains 1.
- Causal and shifted kernels affect only the position of the output pulse.
- Output energy is directly proportional to T, while the system acts linearly and time-invariant.

12 Final Conclusions

12.1 Summary of Findings

Across the analysis of convolution between various input signals and rectangular kernels, several key observations have emerged:

- **Linearity and Time Invariance:** The convolution operation consistently demonstrated linearity and time-invariance. Shifting the input signal resulted in an equivalent shift in the output, and the superposition principle held for sums of inputs.
- **Effect of Kernel Width (T):** Increasing the width of the rectangular kernel ($2T$) leads to wider output responses. For ramp or step inputs, this smoothed the transition; for delta functions, this extended the duration of the output pulse. The kernel width essentially controls the "blurring" or averaging duration.
- **Causality:** Introducing causality (e.g., causal rectangular kernels) shifted output responses forward in time. Causal systems responded only after the input signal was received, producing one-sided or delayed output profiles compared to symmetric kernels.
- **Convolution with Discontinuous Inputs:** When convolving piecewise or discontinuous signals (e.g., step or ramp functions), the convolution output captured the integrated behavior over the kernel window, often resulting in linear or quadratic segments.
- **Delta Function Response:** Convolution with a delta function produced a shifted copy of the kernel itself, confirming that the system's impulse response is the kernel. This reinforces the interpretation of convolution as weighted averaging.
- **Shifting Effects:** Shifting the kernel in time led to corresponding shifts in the output. This confirms the time-invariant nature of the system and highlights how kernel position directly influences response location.
- **Energy and Amplitude Scaling:** For many cases, such as convolution with delta or step functions, the output energy scaled with kernel width, while amplitude remained constant. This reflects how the kernel accumulates or spreads signal content over time.
- **Role of Rectangular Kernels:** Rectangular kernels act as simple finite averaging filters, smoothing inputs and highlighting how localized support affects signal shaping. They provide a foundational intuition for more complex filter designs.

12.2 Implications and Applications

These findings have direct relevance in signal processing, where rectangular kernels are often used for moving average filters, low-pass smoothing, or modeling finite-duration impulse responses (FIR). Understanding their effect on various inputs helps in filter design, noise reduction, and system characterization. Additionally, the analysis reinforces core concepts in system theory such as causality, linearity, and time invariance.

12.3 Closing Remarks

The convolution operation with rectangular kernels reveals not only the structure of the output but also offers deep insights into the nature of signals and systems. By examining a range of input functions — from continuous ramps to impulsive deltas — we gain an intuitive and analytical understanding of how systems process information over time.

12.4 Codes Used For Analysis

Codes for all generated plots can be found at this GitHub repository.

References

[1] Lectures of Differential Equations and Transform Techniques, EE1060 (Jan-May 2025)

[2] The Scientist and Engineer's Guide to Digital Signal Processing by Steven W. Smith

[3] Wikipedia page on LTI Systems

[4] Wikipedia page on Convolution