Mandavi
5
Dubey

Date
Page

Assignment 2

1\. 1) Choose the correct answer from the given options below.

1\. The roots of the auxillary eq^n ~~are~~ are imaginary and repeated $(m = \alpha \pm i\beta)$ in the differential eq^n $f(D)y = 0$ then its solution is

Ans: b) $y = e^{\alpha x}\left[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x\right]$

2\. $e^{-x}(C_1\cos\sqrt{3x} + C_2\sin\sqrt{3x}) + C_3 e^{2x}$ is the general solution of

Ans: d) $\frac{d^3y}{dx^3} + \frac{2d^2y}{dx^2} y + \frac{dy}{dx} = 0$

3\. The D.E. whose auxiliary equation has the roots $0, -1, -1$ is -

Ans: d) $\frac{d^3y}{dx^3} + \frac{2d^2y}{dx^2} y + \frac{dy}{dx} = 0$

4\) If $f(D) = D^2 - 2$, $\frac{1}{f(D)} e^{2x} = \dots$

mandavi

5

Ans: d) $2e^{2x}$

5) If $f(D) = (D-a)^r \phi(D)$ then $\frac{1}{f(D)} e^{ax} =$

Ans: c) $\frac{1}{f(a)} e^{ax}$

2. Match the following

1. $\frac{1}{f(D)} e^{ax}$ — f) $\frac{x^r}{r!} \frac{1}{\phi(a)} e^{ax}$
2. $\frac{1}{f(D)} a^x$ — e) $\frac{1}{f(\log a)} a^x$
3. $\frac{1}{(D^2+a^2)^2} \cos\alpha x$ — c) $-\frac{x\cos ax}{2a}$
4. $\frac{1}{f(D)} e^{-ax} \cdot V$ — i) $e^{-ax} \frac{1}{f(D-a)} V$
5. U (variation of parameter) — a) $\int \frac{y_1 x}{W} dx$
6. V (variation of parameter) — b) $-\int \frac{y_2 x}{W} dx$
7. W — d) $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$
8. $\frac{\sin ax}{f(D^2+a^2)}$ — h) $\frac{-x^2 \sin ax}{4a^2}$
9. $\frac{X}{(D-a)}$ — f) $e^{ax} \int e^{-ax} x \, dx$

Mandavi
5

3. Fill in the blanks

1. P.I. of $(D^2+4)y = \sin 3x$ is $\frac{\sin 3x}{9}$.

2. P.I. of $(D^2-2D+1)y = e^x$ is e^x.

3. If the characteristic equation of D.E. $\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$. Have two equal roots, then $\alpha = \frac{1}{2}$

4. Define the following

1. Complementary function

Ans: A complementary function is one part of the solution to a linear, atonomous differential equation.

2. Particular Integral

Ans: It is a form of the solution of differential equation with specific values assigned to the arbitrary constants.

Mandavi
5

Date
Page

5. State true or False

i) The differential eq^n $y'' - \sin y\, y' + 2y = 0$ is a linear eq^n with constant coefficient.

Ans: True

ii) The method of variation of parameters can be used to solve the equation $y'' + e^t y' + t^2 y = \sin t$

Ans: True

(iii) General solution of the differential equation $\frac{d^3y}{dx^3} + 4y = 0$ must contain four arbitrary constants.

Ans: False

6. i. Solve $(D^2 - 2D + 1)y = e^x + 1$

Ans: A.E. is $m^2 - 2m + 1 = 0$

$m^2 - m - m + 1 = 0$

$m(m-1) - (m-1) = 0$

$m = 1, 1$

Mandavi

5

Date
Page

P.I. = ~~1~~

C.F. = $C_1 e^{x} + C_2 e^{x}$

P.I. = $\frac{1}{f(D)} X$, $X = e^{x} + e^{0}$

$f(D) = D^2 - 2D + 1$

$PI = \frac{1}{f(D)} X = \frac{e^{x} + e^{0}}{D^2 - 2D + 1}$

$= \frac{e^{x}}{(D-1)(D-1)} + \frac{e^{0}}{(D-1)(D-1)}$

$= 0 + \frac{1}{\cancel{1}} = 1$

Complete solution is C.F. + P.I.

$= C_1 e^{x} + C_2 e^{x} + 1$

$= e^{x}(C_1 + C_2) + 1$

ii. Solve $(D^4 + 1) y = \cosh 4x \cdot \sinh 3x$

Sol^n

$\cosh 4x = (e^{4x} + e^{-4x})/2$, $\sinh 3x = (e^{3x} + e^{-3x})/2$

$m^4 + 1 = 0$

$m^4 = -1$

$m = \pm i$

C.F. = $e^{x}(C_1 \cos x + C_2 \sin x)$

P.I. = $\frac{1}{D^4 + 1} \cosh 4x \cdot \sinh 3x$

Mandavi

5

Date ______
Page ______

$$= \frac{1}{D^4+1}\left(\frac{e^{4x}+e^{-4x}}{2}\right)\left(\frac{e^{3x}+e^{-3x}}{2}\right)$$

$$= \cancel{\frac{1}{4}\left(\frac{e^{4x}+e^{-4x}}{D^4+1}\right)+\left(\frac{e^{4x}}{D^4+1}\right)+\left(\frac{e^{3x}}{D^4+1}\right)+\left(\frac{e^{-3x}}{D^4+1}\right)}$$

$$= \frac{1}{4}\left(\frac{e^{7x}+e^{x}+e^{-x}+e^{-7x}}{D^4+1}\right)$$

$$= \frac{1}{4}\left(\frac{e^{7x}+e^{x}+e^{-x}+e^{-7x}}{2}\right)$$

$= \cancel{\frac{1}{}}$ $y = C.F. + P.I.$

$$y = e^{x}(c_1\cos x + c_2\sin x) + \frac{1}{8}\left(e^{7x}+e^{x}+e^{-x}+e^{-7x}\right)$$

(iii) Solve: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^{x} + \cos 2x$

Sol^n:

$D^2 - 4D + 4 = 0$

$\cancel{D^2 - 2D}$ $m^2 - 4m + 4 = 0$

$m^2 - 2m - 2m + 4 = 0$

$m(m-2) - 2(m-2) = 0$

$m = 2$ or $m = 2$

$\therefore$ C.F. $= (c_1 + c_2x)e^{2x}$

$$P.I. = \frac{x^2}{D^2-4D+4} + \frac{e^{x}}{D^2-4D+4} + \frac{\cos 2x}{D^2-4D+4}$$

$= PI_1 + PI_2 + PI_3$

$$PI_1 = \frac{x^2}{D^2-4D+4} \quad (\because D \to 2) = \cancel{\frac{x^2}{4}}\ \frac{1}{4}\left[x^2+2x+3/2\right]$$

$$PI_2 = \frac{e^{x}}{D^2-4D+4} = \cancel{\frac{x e^{2x}}{2D = 4}} = \cancel{\frac{x^2}{2}} e^{2x}$$

nandani

5

Date

Page

$$PI_3 = \frac{\cos 2x}{D^2-4D+4} = \frac{\cos 2x}{-4D} \times \frac{D}{D} = \frac{-\sin 2x \cdot 2}{4D^2}$$

$$= \frac{-\sin 2x}{8}$$

$$y = CF + PI = (c_1 + c_2 x) e^{2x} + \cancel{\frac{x^2}{8}}$$

$$\cancel{\frac{e^{2x}}{8}} - \frac{\sin 2x}{8} + \frac{1}{4}\left[x^4 + 2x + \frac{3}{2}\right]$$

(iv) Solve $(D^2+2)y = e^x \cos x + x^2 e^{3x}$

Sol^n: For complementary sol^n, $f(D)=0$

$\therefore (D^2+2)=0$

Roots are: $\sqrt{2}i, -\sqrt{2}i$

Roots of D.E. are complex.

The complementary sol^n of D.E. is given by, $y_c = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2x}$

$$P.I. = \frac{1}{f(D)} x = \frac{1}{D^2+1} e^x \cos x + \frac{1}{D^2+1} x^2 e^{3x}$$

$$= \frac{e^x \cos x}{(D+1)^2+1} + \frac{x^2 e^{3x}}{D^2+1}$$

$$= \frac{e^x \cos x}{D^2+2D+3} + \frac{e^{3x} x^2}{(D+3)^2+2}$$

$$= e^x \frac{1}{2} \frac{D-1}{D^2+1} \cos x + e^{3x} \frac{1}{D^2+6D+11} x^2$$

Mandavi
5

Date
Page

$$= e^{x}\frac{1}{4}(\sin x \cos x) + \frac{e^{3x}}{11}\left[1+\frac{6D+D^2}{11}\right]^{-1} x^2$$

$$= e^{x}\frac{1}{4}(\sin x \cos x) + \frac{e^{3x}}{11}\left\{1 + \frac{6D+D^2}{11} + \frac{36 D^2}{121} + \cdots\right\} x^2$$

$$\Rightarrow y_p \quad P.I. = e^{x}\frac{1}{4}(\sin x + \cos x) + \frac{e^{3x}}{11}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right]$$

general Solution = C.F. + P.I.

$$= C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x + e^{x}\frac{1}{4}(\sin x + \cos x) + \frac{e^{3x}}{11}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right]$$

(v) Solve $\frac{d^3y}{dt^3} + \frac{dy}{dt} = \cos t + t^2 + 3$

Solⁿ AE is $D(D^2+1) = 0 \therefore D = 0, i, -i$

$C.F. = C_1 + C_2 \cos t + C_3 \sin t$

$$PI = \frac{1}{D+D^3}(\cos t + t^3 + 3)$$

$$\frac{1}{D+D^3}\cos t = \frac{1}{D(1+D^2)}\cos t = \frac{1}{D}\frac{t}{2}\sin t$$

Mamdavi

5

Date

Page

$= \frac{1}{2}\int t \sin t\,dt = \frac{1}{2}\left[-t\cos t + \sin t\right]$

$\Rightarrow \frac{1}{D+D^3} t^2 = \frac{1}{D(1+D^2)} t^2 = \frac{1}{D}(1-D^2+\cdots) t^2$

$= \frac{1}{D}\left[t^2-2\right] = \int\left[t^2-2\right]dt$

$= t^3/3 - 2t$

$\frac{1}{D+D^3} 3 = \frac{3 e^{0t}}{D(1+D^2)} = \frac{3}{D} = 3\int dt = 3t$

$y = C_1 + C_2\cos t + C_3\sin t + \frac{1}{2}\left[-t\cos t + \sin t\right] + \frac{t^3}{3} + t$

7. 1. Solve: $(D^2 - D - 2)y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$

Sol^n

$m^2 - m - 2 = 0$

$m^2 - 2m + m - 2 = 0$

$m(m-2) + 1(m-2) = 0$

$m = 2, -1$

$C.F. = C_1 e^{2x} + C_2 e^{-x}$

$P.I. = \frac{2\log x}{D^2-D-2} + \frac{1/x}{D^2-D-2} + \frac{1/x^2}{D^2-D-2}$

$PI = \frac{1}{(D-2)(D+1)}\left\{2\log x + \frac{1}{x} + \frac{1}{x^2}\right\}$

$= \frac{1}{(D-2)} e^{-x}\int e^{x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)dx$

$= \frac{1}{(D-2)} e^{-x}\left[\int e^{x}\left(2\log x + \frac{2}{x}\right)dx + \int e^{x}\left(-\frac{1}{x} + \frac{1}{x^2}\right)dx\right]$

$$= \frac{1}{(D-2)} e^{-x}\left[e^{x} 2\log x\, e^{x} \frac{1}{x}\right]$$

$$= \frac{1}{(D-2)}\left[2\log x - \frac{1}{x}\right]$$

$$= e^{2x}\int e^{-2x}\left(2\log x - \frac{1}{x}\right)dx$$

$$= e^{2x}\left[2\log x\left(\frac{-e^{-2x}}{2}\right) - \int\left(\frac{-e^{-2x}}{2} \times \frac{2}{x}\right)\right.$$

$$\left. dx - \int e^{-2x}\frac{1}{x}\,dx\right]$$

Mandavi
5

Date
Page

$= -e^{2x} e^{-2x} \log x = -\log x$

$y = C_1 e^{-x} + C_2 e^{2x} - \log x.$

Mandavi

5

(ii) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

Sol^n: A.E is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$C.F. = (C_1 + C_2 x)e^x$

$P.I. = \frac{1}{D^2 - 2D + 1} xe^x \sin x$

$= \frac{1}{(D-1)^2} xe^x \sin x \quad (D \rightarrow D+1)$

$= \frac{e^x}{D^2} x \sin x$

Now $\frac{1}{D^2}(x \sin x) = \frac{1}{D}\int x \sin x \, dx$

$\because \frac{1}{D}$ is the integration

$= \frac{1}{D}[x(-\cos x) - 1(-\sin x)]$

On integration by parts

$= \frac{1}{D}(-x\cos x + \sin x)$

$= -\int x \cos x \, dx + \int \sin x \, dx$

$= -\{x \sin x - 1(-\cos x)\} - \cos x$

$= -x \sin x - 2\cos x$

Mandavi

5

Date
Page

$= -(x\sin x + 2\cos x)$

Hence P.I. $= -e^x(x\sin x + 2\cos x)$

$\therefore$ The complete solⁿ is

$y = C.F. + P.I.$

$= (C_1 + C_2x)e^x - e^x(x\sin x + 2\cos x)$

(iii) Solve by method of variation of parameters $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$

Solⁿ $(D^2+1)y = 1/(1+\sin x)$

AE is $m^2+1=0$ & hence $m = \pm i$

$\therefore C.F. = C_1\cos x + C_2\sin x$

$y = A(x)\cos x + B(x)\sin x$ — ①

be the complete solⁿ of the given D.E. where $A(x)$ and $B(x)$ are to be found.

$y_1 = \cos x \qquad y_2 = \sin x$

$y_1' = -\sin x \qquad y_2' = \cos x$

$W = y_1y_2' - y_2y_1' = 1$ also $\phi(x) = 1/1+\sin x$

$A' = \frac{-y_2\phi(x)}{W}$ & $B' = \frac{y_1\phi(x)}{W}$

$A' = \frac{-\sin x}{1+\sin x}$ & $B' = \frac{\cos x}{1+\sin x}$

$A' = -\frac{(1+\sin x - 1)}{1+\sin x} = -1 + \frac{1}{\sin x}$

$$A = \int \left[-1 + \frac{1}{1+\sin x} \right] dx + k_1$$

$$= -x + \int \frac{1-\sin x}{\cos^2 x}\, dx + k_1$$

$$= -x + \int (\sec^2 x - \sec x \tan x)\, dx + k_1$$

$$A = -x + \tan x - \sec x + k_1 \quad \text{— (2)}$$

$$B' = \frac{\cos x}{1+\sin x} = \frac{\cos x(1+\sin x)}{\cos^2 x} = \frac{1-\sin x}{\cos x}$$

$$B = \int \frac{1-\sin x}{\cos x}\, dx + k_2$$

$$= \int (\sec x - \tan x)\, dx + k_2$$

$$= \log(\sec x + \tan x + \log(\cos x) + k_2$$

$$= \log\left(\frac{1+\sin x}{\cos x}\right) + \log(\cos x) + k_2$$

$$= \log(1+\sin x) - \log(\cos x) + \log(\cos x) + k_2$$

$$B = \log(1+\sin x) + k_2 \quad \text{— (3)}$$

Mandavi
5

Date ______
Page ______

Using eq^n (2) & (3) in (1), we have

$y = [-x + \tan x - \sec x + k_1] \cos x$
$+ [\log(1+\sin x) + k_2] \sin x$

i.e., $y = k_1 \cos x + k_2 \sin x - x \cos x$
$+ \sin x - 1 + \sin x \log(1+\sin x)$

The term $\sin x$ can be neglected in view of them $k_2 \sin x$ present in the sol^n. Thus

$y = k_1 \cos x + k_2' \sin x - (x \cos x + 1) +$
$\sin x \log(1+\sin x)$

mandavi
5

Date
Page

(iv) Solve by method of variation of parameters $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$.

Sol^n AE: $D^2 + 3D + 2 = 0$

$D = -1, -2$

$C.F. = C_1 e^{-x} + C_2 e^{-2x}$

$PI = \frac{e^{e^x}}{D^2 + 3D + 2} = \frac{e^{-x}(e^x e^{e^x})}{D^2 + 3D + 2}$

(Comparable to $\frac{1}{f(D)} e^{ax} V(x)$)

$= e^{ax} \frac{1}{f(D+a)} V(x)$ with $a = -1$ & $V(x) = e^x e^{e^x}$)

$= e^{-x} \frac{1}{(D-1)^2 + 3(D-1) + 2} (e^x e^{e^x})$

$= e^{-x} \frac{1}{D^2 + D} D(e^{e^x})$ [as $D(e^{e^x}) = e^{e^x} D(e^x) = e^{e^x} e^x$]

$= e^{-x} \frac{1}{(D+1)} \frac{1}{D} D(e^{e^x})$

$= e^{-x} \left[\frac{1}{(D+1)} e^{(e^x)}\right] = e^{-x} \left[e^{-x} \int e^x e^{e^x} dx\right]$

(using $\frac{1}{(D-a)} X(x) = e^{ax} \int e^{-ax} X dx$)

$= e^{-x} \left[e^{-x} \int d(e^{e^x}) dx\right] = e^{-x} \left[(e^{-x} e^{e^x})\right]$

$$= e^{-2x} e^{e^x}$$

$$y = CF + PI = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$$