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Tutorial - 6

Q1. Evaluate $\int_0^{\pi} \frac{dx}{a-\cos x}$, $a>0$. Hence, find $\int_0^{\pi} \frac{dx}{(a-\cos x)^2}$

Solution: Put $\tan\left(\frac{x}{2}\right) = t$, $dx = \frac{dt}{1+t^2}$,

$\cos x = \frac{1-t^2}{1+t^2}$

When $x=0$, $t=0$ and $x=\pi$, $t=\infty$

$$I = \int_0^{\infty} \frac{\frac{dt}{1+t^2}}{a-\left[\frac{1-t^2}{1+t^2}\right]}, \quad a, b>0$$

$$I = \int_0^{\infty} \frac{dt}{a(1+t^2)-(1-t^2)} = \int_0^{\infty} \frac{dt}{a+at^2-1+t^2}$$

$$I = \int_0^{\infty} \frac{dt}{(a-1)+t^2(a+1)} = \frac{1}{(a+1)} \int_0^{\infty} \frac{dt}{(a-1)/(a+1)+t^2}$$

$$I = \frac{1}{\sqrt{\frac{(a-1)}{(a+1)}}(a+1)} \left[\tan^{-1} \frac{x}{\sqrt{\frac{(a-1)}{(a+1)}}}\right]_0^{\infty} = \frac{\pi}{2\sqrt{a^2-1^2}}$$

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$$\int_0^{\pi} \frac{dx}{a-\cos x} = \frac{\pi}{2\sqrt{a^2-1}}$$

differentiating w.r.t. a

$$\int_0^{\pi} \frac{\partial}{\partial a} \frac{dx}{a-\cos x} = \frac{d}{da}\left[\frac{\pi}{2\sqrt{a^2-1}}\right]$$

$$\int_0^{\pi} \frac{\partial}{\partial a} \frac{dx}{a-\cos x} = \int \frac{dx}{(a-\cos x)^2} \cancel{\cos x}$$

$$\frac{d}{da}\left[\frac{\pi}{2\sqrt{a^2-1}}\right] = \frac{\pi}{2}\left(\frac{1}{2}\right)(a^2-1)^{-3/2}(2a)$$

$$= \frac{\pi}{2(a^2-1)^{3/2}}$$

$$\int_0^{\pi} \frac{dx}{(a-\cos x)^2} = \frac{\pi}{2(a^2-1)^{3/2}}$$

$$\int_0^{\pi} \frac{dx}{(a-\cos x)^2} = \frac{\pi}{2(a^2-1)^{3/2}} = -\frac{\pi}{2(a^3-1)}$$

2. Evaluate $\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)}$, $a>0$

Hence, find $\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2}$

Solution: $\cancel{I} = \int_0^{\pi/2} \frac{\sec^4 x}{(a^2 +}$ $\quad I = \int_0^{\pi/2} \frac{dx}{\cos^2 x\left(\left(\frac{a^2\sin^2 x}{\cos^2 x}\right) + b^2\right)}$

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$$I = \int_0^{\pi/2} \frac{1/\cos^2 x}{a^2 \tan^2 x + b^2}\, dx$$

$$I = \int_0^{\pi/2} \frac{\sec^2 x}{a^2 \tan^2 x + b^2}\, dx$$

$$I = \frac{1}{a^2} \int_0^{\pi/2} \frac{\sec^2 x}{(\tan x)^2 + \left(\frac{b}{a}\right)^2}\, dx \quad \cdots (1)$$

Let us assume $y = \tan x$

Differentiating w.r.t. x on both sides we get, $d(y) = d(\tan x)$

$dy = \sec^2 x\, dx \quad \cdots (2)$

Upper limit for the definite integral:

$x = \pi/2 \Rightarrow y = \tan(\pi/2)$

$y = \infty \quad \cdots (3)$

Lower limit for the definite integral:

$x = 0 \Rightarrow y = \tan(0)$

$y = 0 \cdots (4)$

Substituting (2), (3) (4) in the eqn (1), we get $I(x) = \frac{1}{a^2} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{b}{a}\right)^2}$

we know that: $\int \frac{1}{a^2 + x^2}\, dx = \tan^{-1} x + C$

we know that: $\int_a^b f'(x)\, dx = |f(x)|_b^a$

$= f(b) - f(a)$

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$$I(x) = \frac{1}{a^2} \times \frac{1}{b/a} \times \tan^{-1}\left(\frac{x}{b/a}\right)\Big|_0^\infty$$

$$I(x) = \frac{1}{ab} \tan^{-1}\left(\frac{ax}{b}\right)\Big|_0^\infty$$

$$I(x) = \frac{1}{ab}\left(\tan^{-1}(\infty) - \tan^{-1}(0)\right)$$

$$I(x) = \frac{1}{ab}\frac{\pi}{2} = \frac{\pi}{2ab}$$

$\pi/2$

2) using differentiation under integral sign w.r.t. 'b' we have from (1)

$$\int_0^{\pi/2} \frac{-2b\cos^2 x}{(a^2\sin^2 x+b^2\cos x)^2}\,dx = \frac{-\pi}{2ab^2}$$

$$\int_0^{\pi/2} \frac{\cos^2 x}{(a^2\sin^2 x+b^2\sin^2 x)}\,dx = \frac{\pi}{4ab^3} \quad (2)$$

using differentiation under integral sign w.r.t. 'a' we have from (1)

$$\int_0^{\pi/2} \frac{-2a\sin^2 x}{(a^2\sin^2 x+b^2\cos^2 x)^2}\,dx = \frac{-\pi}{2a^2b}$$

$$\int_0^{\pi/2} \frac{\sin^2 x}{(a^2\sin^2 x+b^2\cos^2 x)^2}\,dx = \frac{\pi}{4a^3b} \quad (3)$$

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On adding eq^n 2 & 3

$$\therefore \int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)} = \frac{\pi}{4ab^3} + \frac{\pi}{4a^3 b}$$

$$= \frac{\pi}{4ab}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

3. Evaluate $\int_0^{\pi} \frac{\log(1+\alpha\cos x)}{\cos x}\,dx$, $|\alpha| < 1$

<u>Solution:</u> Let $I(\alpha) = \int_0^{\pi} \frac{\log(1+\alpha\cos x)}{\cos x}\,dx$ — (1)

By using rule of DUIS $\frac{dI}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx$

$$\frac{dI}{da} = \int_0^{\pi} \frac{\partial}{\partial a} \frac{\log(1+\alpha\cos x)}{\cos x}\,dx$$

$$= \int_0^{\pi} \frac{1}{1+\alpha\cos x} \cdot (+\cos x)\,\frac{1}{\cos x}\,dx$$

3) $\int_0^{\pi} \frac{dx}{1+\alpha\cos x}$ $\quad \left[\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2} \right]$

$= \int_0^{\pi} \frac{dx}{1+\alpha\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}$ $\quad$ Put $\tan\frac{x}{2} = t$

$= \int_0^{\infty} \frac{1}{1+\alpha\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{1+t^2} dt$ $\quad \frac{x}{2} = \tan^{-1} t$

$dx = \frac{2\,dt}{1+t^2}$

$= \int_0^{\infty} \frac{2\,dt}{1+t^2+\alpha^2-\alpha t^2}$

x	0	π
t	0	∞

$= \int_0^{\infty} \frac{2\,dt}{(1+\alpha)+(1-\alpha)t^2} = \frac{1}{1-\alpha} \int_0^{\infty} \frac{dt}{\left(\frac{1+\alpha}{1-\alpha}\right)+t^2}$

$= \frac{2}{1-\alpha} \frac{1}{\sqrt{\frac{1+\alpha}{1-\alpha}}} \tan^{-1} \frac{t}{\sqrt{\frac{1+\alpha}{1-\alpha}}} \Bigg|_0^{\infty} = \frac{2\pi}{\sqrt{1-\alpha^2}\,2}$

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$$\therefore \frac{dI}{d\alpha} = \frac{\pi}{\sqrt{1-\alpha^2}}$$

Integrating both sides,

$I(\alpha) = \pi \sin^{-1}\alpha + c$ — (2)

to find c put $\alpha = 0$

$\therefore I(0) = c$

But from (1), $I(0) = 0 \therefore c = 0$

$\therefore$ from (2)

$$I(\alpha) = \int_0^{\pi} \frac{\log(1+\alpha\cos x)}{\cos x}\,dx$$

$$= \pi \sin^{-1}\alpha$$

4. Prove that $\int_0^1 \frac{(x^a - x^b)}{\log x} dx = \log\left(\frac{a+1}{b+1}\right)$

Solution: Let $I(a) = \int_0^1 \frac{x^a - x^b}{\log x} dx$ — ①

Using differentiation under integral sign w.r.t. 'a'.

$$\frac{dI}{da} = \int_0^1 \frac{\partial}{\partial a} \frac{x^a - x^b}{\log x}$$

$$= \int_0^1 \frac{x^a \log x}{\log x} dx = \int_0^1 x dx = \frac{x^{a+1}}{a+1}\Big|_0^1$$

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$$\frac{dI}{da} = \frac{1}{a+1}$$

Integrating we get, $I(a) = \log(1+a)+c$ — ②

To find c put $a = b$

$\therefore I(b) = \log(1+b)+c$

But from (1) $I(b) = 0$

$\therefore c = \log(1+b)$

$\therefore$ from (2)

$$I(a) = \int_0^1 \frac{x^a - x^b}{\log x}\,dx = \log(a+1) - \log(b+1)$$

$$= \log\left(\frac{a+1}{b+1}\right)$$

$$\boxed{\therefore \int_0^1 \frac{x^a - x^b}{\log x}\,dx = \log\left(\frac{a+1}{b+1}\right)}$$

5) Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y=1$ to $y=2$

Solution: The eqn of curve is given as $x = \frac{y^3}{3} + \frac{1}{4y}$

To find the arc length from $y=1$ to $y=2$

first find $\frac{dx}{dy}$,

$$\frac{d}{dx}(x) = \frac{d}{dy}\left(\frac{y^3}{3}\right) + \frac{1}{4}\frac{d}{dy}\left(\frac{1}{y}\right)$$

$$\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$$

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$\therefore$ the arc length of the curve from $y=1$ to $y=3$ is given by

$$I = \int_1^2 1+\left(\frac{dx}{dy}\right)^2 dy$$

$$= \int_1^2 \sqrt{1+y^4+\frac{1}{16y^4}-\frac{1}{2}}\,dy$$

$$= \int_1^2 \sqrt{y^4+\frac{1}{16y^4}+\frac{1}{2}}\,dy$$

$$= \int_1^2 \sqrt{\left(y^2+\frac{1}{4y^2}\right)^2}\,dy$$

$$= \int_1^2 \left(y^2+\frac{1}{4y^2}\right)dy = \left[\frac{y^3}{3}-\frac{1}{4y}\right]_1^2$$

$$= \left(\frac{8}{3}-\frac{1}{8}\right)-\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$= \frac{61}{24}-\frac{1}{12} = \frac{732-24}{288}$$

$$= 2.46 \text{ units}$$

$\therefore$ ~~the~~ the length of the given curve is 2.4 units.

6) Find the length of the arc of the parabola $y^2 = 8x$ cut off by the latus rectum.

Solution: length of the parabola from the vertex to the end of the

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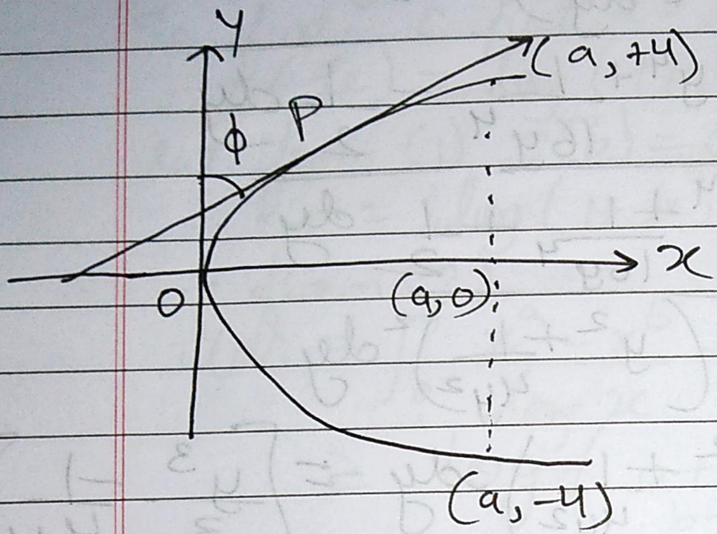
latus rectum is

$$S = \int_{y=0}^{4} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \quad \text{— (1)}$$



$$y^2 = 8x$$

$$x = \frac{y^2}{8}$$

$$\frac{dx}{dy} = \frac{2y}{8} = \frac{y}{4}$$

$$\therefore 1+\left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{16} = \frac{16+y^2}{16}$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \frac{\sqrt{16+y^2}}{4}$$

$\therefore$ from (1)

$$S = \int_0^4 \frac{\sqrt{16+y^2}}{4}\, dy \quad \text{— (2)}$$

$$= \frac{1}{4}\left[\frac{y}{2}\sqrt{y^2+16} + \frac{16}{2}\log|y+\sqrt{y^2+16}|\right]_0^4$$

$$= \frac{1}{4}\left[\frac{4}{2}\sqrt{16+16} + \frac{16}{2}\log|4+\sqrt{16+16}| - [0 + 8\log|0+\sqrt{0+16}|]\right]$$

$$= \frac{1}{4}\left[2\sqrt{32} + 8\log|4+\sqrt{32}| - [8\log 4]\right]$$

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$= \frac{1}{4}\left[8\sqrt{2} + 8\log(4+4\sqrt{2}) - 8\log 4\right]$

$= \frac{8}{4}\left[\sqrt{2} + \log 4(1+\sqrt{2}) - \log 4\right]$

$= 2\left[\sqrt{2} + \log \frac{4(1+\sqrt{2})}{4}\right]$

$$\boxed{\therefore\ S = 2\left(\sqrt{2} + \log(1+\sqrt{2})\right]}$$

$\therefore$ the length of parabola $(S) =$

$2\left[\sqrt{2} + \log(1+\sqrt{2})\right]$