Cauchy's Theorem

Simply and Multiple connected Regions :-

If a closed curve does not intersect itself, it is called a simple closed curve or a Jordan curve. If a closed curve intersect itself, it is called a multiple curve.

Cauchy's Integral Theorem :-

If $f(z)$ is an analytic function and if its derivative $f'(z)$ is continuous at each point within and on a simple closed curve c then the integral of $f(z)$ along the closed curve c is zero.

i.e, $\oint f(z)\,dz = 0$.

Cauchy-Goursat theorem:

If $f(z)$ is analytic in and on a closed curve c then the integral of $f(z)$ along a closed curve,

$$\oint_c f(z)\,dz = 0.$$

1. Evaluate $\int_c \frac{z+3}{z^2-2z+5}\,dz$ where c is the circle $|z-1| = 1$

c is the circle; ie, c is a closed curve.

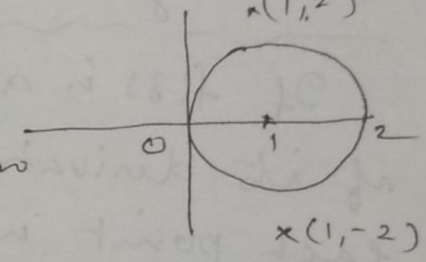
$|z-1| = 1$ $\qquad\qquad$ $|x+iy| = r$

$|x+iy-1| = 1$ $\qquad\qquad$ $x^2+y^2 = r^2$

$|(x-1)+iy| = 1$

$(x-1)^2+y^2 = 1^2$

This is a circle with centre $(1,0)$ and radius $r=1$



Let the denominator be zero

$z^2-2z+5=0$

$z = 1 \pm 2i$

$z = 1+2i$, $\quad z = 1-2i$

$x=1$, $y=2$ $\;(1,2)$ $\qquad$ $x=1$, $y=-2$ $\;(1,-2)$

Both the points are outside the curve. and $f(z)$ is analytic in and on c.

$\therefore$ By Cauchy's theorem,

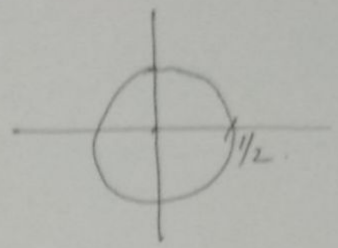
$\oint f(z)\,dz = 0$

$\therefore \oint \frac{z+3}{z^2-2z+5}\,dz = 0$

2. Evaluate $\int_C \tan z\, dz$, where $|z| = 1/2$

$|z| = 1/2$ is a circle with centre $(0,0)$ and radius $r = 1/2$.

$$\tan z = \frac{\sin z}{\cos z}$$



Let $\cos z = 0$

$z = \pm \pi/2$

Here $z = \pm \pi/2$ lies outside the circle.

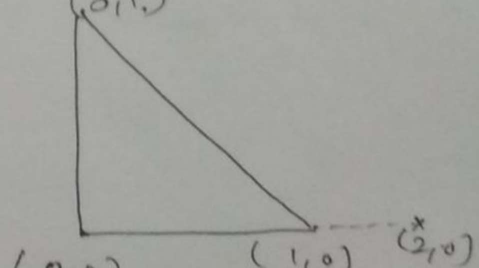
Hence, $f(z)$ is analytic in and on C.

Hence, by Cauchy's theorem,

$$\int_C f(z)\, dz = 0$$

$$\therefore \int_C \tan z\, dz = 0.$$

Q. 3. Evaluate ~~$\oint \frac{1}{z}\, dz$~~ $\oint_C \frac{dz}{z-2}$ around the triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$.



$z - 2 = 0$

$z = 2$

$x + iy = 2 \quad (2,0)$

The point $(2,0)$ is outside the triangle.

$f(z)$ is analytic in and on C.

Hence. $$\oint_C \frac{dz}{z-2} = 0$$

Cauchy's Integral Formula [Fundamental Formula].

If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if z_0 is any point within C then

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0).$$

Corollary: $\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0).$

Q. 1. Evaluate $\int_C \frac{\cot z}{z}\,dz$ where C is the ellipse $9x^2+4y^2=1$

$$\int_C \frac{\cot z}{z}\,dz = \int_C \frac{\cos z}{z \cdot \sin z}\,dz$$

$z \sin z = 0$.

$z=0$ lies ~~in it~~ inside the ellipse.

$\therefore$ By Cauchy's Integral formula,

$$\int_C \frac{\cot z}{z}\,dz = \int_C \frac{\cos z}{z \sin z}\,dz =$$

$\boxed{f(z) = \cos z \quad z_0 = 0}$

$= 2\pi i\, f(z_0)$

$= 2\pi i\, f(0) = 2\pi i \cdot \cos 0$

$= 2\pi i$

Q. 2. Evaluate $\int_C \frac{e^{3z}}{z-i}\,dz$, where C is the curve $|z-2|+|z+2|=6$.

$$|z-2| + |z+2| = 6.$$

$$|x-2+iy| + |x+2+iy| = 6.$$

$$\sqrt{(x-2)^2+y^2} + \sqrt{(x+2)^2+y^2} = 6$$

Put $y=0$, $\quad x-2+x+2 = 6.$

$$2x = 6$$

$$\boxed{x = 3}$$

Put $x=0$, $\quad \sqrt{y^2+4} + \sqrt{y^2+4} = 6$

$$2\sqrt{y^2+4} = 6$$

$$\sqrt{y^2+4} = 3$$

$$y^2+4 = 9 \qquad y^2 = 5$$

$$\boxed{y = \pm\sqrt{5}}$$

The curve $|z-2| + |z+2| = 6$ is an ellipse with foci at $(-2,0)$, $(2,0)$ and intersecting the real axis in $(-3,0)$, $(3,0)$ and imaginary axis in $(0,\sqrt{5},)$ $(0,-\sqrt{5})$

$$z - i = 0$$

$$z = i \quad \boxed{(0,1)}$$

$z=i$ lies inside C and $f(z) = e^{3z}$ is analytic in and on C. Hence by Cauchy's Integral formula

$$\int_C \frac{e^{3z}}{z-i}\,dz = 2\pi i\, e^{3i} = 2\pi i\,(\cos 3 + i \sin 3)$$

Q.3. Evaluate $\int_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3} dz$

where C is $|z|=1$

$|z|=1$ is a circle with centre $(0,0)$ and $r=1$

Hence the point $z=\pi/6$ lies inside C. $f(z) = \sin^6 z$ is analytic in and on C.

By Corollary of Cauchy's Integral formula,

$$\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

where $f(z) = \sin^6 z$, $z_0 = \pi/6$, $n=3$.

$f'(z) = 6\sin^5 z \cos z$

$f''(z) = 6[5\sin^4 z \cdot \cos^2 z - \sin^6 z)$

$f''(z_0) = f''(\pi/6) = 6[5\sin^4 \pi/6 \cdot \cos^2 \pi/6 - \sin^6 \pi/6]$

$\cos \pi/6 = \sqrt{3}/2$
$\sin \pi/6 = 1/2$

$= 6\left[5 \cdot \frac{1}{16} \cdot \frac{3}{4} - \frac{1}{64}\right]$

$= 21/16.$

$$\int_C \frac{\sin^6 z}{(z-\pi/6)^3} dz = \frac{2\pi i}{2!} \times \frac{21}{16} = \frac{21 \cdot \pi i}{16}$$

Q. 4. Evaluate $\int_c \frac{3z^2+z}{z^2-1}\,dz$, where c is the circle $|z|=2$

$|z|=2$ is a circle with centre $(0,0)$ and $r=2$

$$z^2-1=0$$
$$z^2=1$$
$$z=\pm 1$$

Here $z=\pm 1$ lies inside the circle.

$$\frac{1}{z^2-1} = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$$

$f(z) = 3z^2+z$ is analytic in and on C.

$$\therefore \int_c \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\int_c \frac{3z^2+z}{z-1}\,dz - \frac{1}{2}\int_c \frac{3z^2+z}{z+1}\,dz$$

$$= \frac{1}{2}\,2\pi i\, f(1) - \frac{1}{2}\,2\pi i\, f(-1).$$

$$= \frac{1}{2}\,2\pi i\,(4) - \frac{1}{2}\,2\pi i\,(2).$$ $f(z)=3z^2+z$, $z_0=1,-1$

$$= \underline{\underline{2\pi i}}.$$

Q. 5. Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)}\,dz$ where c is the circle $|z|=3$.

$|z|=3$ is the circle with centre $(0,0)$ and radius $r=3$.

$(z-1)(z-2)=0$

$z=1,2$ lies inside the circle.

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

and $f(z) = e^{2z}$ which is analytic in C

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = \int_C \frac{e^{2z}}{z-2}\,dz - \int_C \frac{e^{2z}}{z-1}\,dz$$

$f(z) = e^{2z}$

$z_0 = 1, 2$

$$= 2\pi i\, f(2) - 2\pi i\, f(1)$$

$$= 2\pi i\, e^{4} - 2\pi i\, e^{2}$$

$$= \underline{\underline{2\pi i\, e^{2}(e^{2}-1)}}.$$

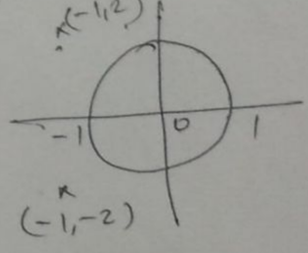
Q.6. Evaluate $\int_C \frac{z+3}{z^2+2z+5}\,dz$, where c is the circle (1) $|z|=1$ (2) $|z+1-i|=2$.

$$z^2+2z+5 = (z+1)^2+2^2 = 0.$$

$$(z+1+2i)(z+1-2i)=0$$

$$z=-1-2i,\quad z=-1+2i$$

(1) $|z|=1$ is circle with centre $(0,0)$ and radius 1.



$z=-1-2i \Rightarrow (-1,-2)$

$z=-1+2i \Rightarrow (-1,2)$

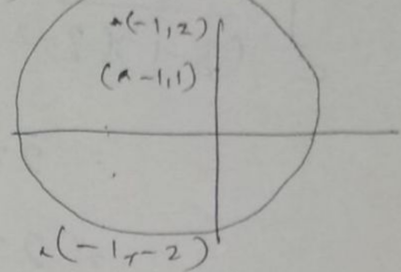
Both the points are outside the circle.

By Cauchy's Integral theorem, $\underline{\underline{\int_C f(z)\,dz = 0}}$

(ii) $|z+1-i| = 2$

$|x+iy+1-i| = 2$

$|(x+1)+i(y-1)| = 2$ is a circle with centre $(-1,+1)$ and radius $r = 2$.



Here the point $(-1,2)$ lies inside and $(-1,-2)$ lies outside the circle.

$$\int_C \frac{z+3}{z^2+2z+5}\,dz = \int_C \frac{(z+3)/(z+1+2i)}{(z^2+2z+5)/(z+1+2i)}\,dz$$

$$= \int_C \frac{(z+3)/(z+1+2i)}{z+1-2i}\,dz$$

$$= \int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0).$$

$z_0 = -2+2i$

$f(z) = \dfrac{z+3}{z+1+2i}$

$$= 2\pi i \cdot \left[\frac{-\cancel{1}+2i+3}{-\cancel{1}+2i+1+2i}\right]$$

$$= 2\pi i \left[\frac{2+2i}{4i}\right]$$

$$= \pi(1+i)$$

Q.7. Evaluate $\int_C \frac{z+6}{z^2-4}\,dz$ where C is the circle

(1) $|z|=1$ (2) $|z-2|=1$ (3) $|z+2|=1$

Q.8. Evaluate $\int_C \frac{z+2}{z^3-2z^2}\,dz$, where C is the circle $|z-2-i|=2$

$|z-2-i|=2$

$|x+iy-2-i|=2$

$|(x-2)+i(y-1)|=2$ is a circle

with centre $(2,1)$ and $r=2$

$z^3-2z^2=0$

$z^2(z-2)=0$

$z=0,\ z=2$

$z=0$ lies outside and $z=2$ lies inside the circle.

By Cauchy's Integral Formula,

$$\int_C \frac{z+2}{z^3-2z^2}\,dz = \int \frac{(z+2)/z^2}{z-2}\,dz$$

$f(z) = \frac{z+2}{z^2}$

$z_0 = 2$

$= 2\pi i\, f(z_0)$

$= 2\pi i(1) = 2\pi i$

Q. 9. Evaluate $\oint_c \frac{\sin^6 z}{(z-\pi/2)^3} dz$, c is $|z|$

$z - \pi/2 = 0$ $\quad |z| = 2$

$z = \pi/2$ $\quad (0,0), r = 2$

$z = \pi/2$ lies inside the circle.

By corollary of C.I.F

$$\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{1}{(n-1)!} 2\pi i f^{n-1}(z_0).$$

$$\oint_c \frac{\sin^6 z}{(z-\pi/2)^3} dz = \frac{1}{2!} 2\pi i f''(z_0)$$

$$= \frac{2\pi i}{2}(-6) = \underline{\underline{-6\pi i}}$$

$f(z) = \sin^6 z$

$f'(z) = 6 \sin^5 z \cos z$

$f''(z) = 30 \sin^4 z \cos^2 z + 6 \sin^5 z(-\sin z)$

Q. 10. If $f(\xi) = \int_c \frac{3z^2+2z+1}{z-\xi} dz$ c is $x^2+y^2=4$

find the values (i) $f(3)$ (2) $f'(1-i)$ (3) $f''(1-i)$

$x^2+y^2=2$

$|z|=2$

$(0,0), r=2.$

$f(z) = 3z^2+2z+1$

$f'(z) = 6z+2$

$f''(z) = 6.$

(1). $f(3)$.

$z=3$ lies outside the circle

$\therefore$ By C.I.T,

$$\underline{\int f(z)\,dz = 0}.$$

$f'(\xi) = 6\xi+2$

$f''(\xi) = 6.$

(2). $z = 1-i$ (1, −1) lies inside the circle

$$\int_c \frac{\phi(z)}{z-\xi} dz = 2\pi i\, \phi(\xi)$$

$$= 2\pi i(3\xi^2+2\xi+1) = 2\pi i(6\xi+2)$$

$$= 2\pi i(6(1-i)+2)$$

$$= \underline{\underline{2\pi i(8-6i)}}.$$

(3) $\underline{\underline{f''(1-i) = 6}}$