Assignment 1

1. If the two eigen values of $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$ are 2 & 3, find the value of a.

Solⁿ Determinant of matrix $A = |A| = 2(4) - 0 + 1(0 - 2a) = 8 - 2a$. We have two eigen values i.e. 2 & 3. Let the third eigen value be x. Addition of diagonal element = Addition of three eigen values.

$\therefore 2+2+2 = 2+3+x$

$1 = x$

As we know, product of eigen values = Determinant of matrix

$\therefore 2 \times 3 \times 1 = 8 - 2a$

$\therefore 6 = 8 - 2a$

$\boxed{a = 1}$

2. If A is a singular matrix of order 3 & two eigen values of matrix A are 2, 3, find the third eigen value.

Solⁿ: A is a singular matrix. We know that the determinant is the product of the eigen values. $\because$ A is a singular matrix so det $A = 0$

$\because$ two given eigen values are 2 and 3.
So two given eigen values are non zero.
Hence third eigen values must be zero.

3. Verify that the sum of the elements in the diagonal elements of the matrix

$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ is equal to the sum of the eigen values.

Solution Sum of diagonal elements $= 3+5+3 = 11$.

The characteristic equation is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\therefore (3-\lambda)[(5-\lambda)(3-\lambda)-1] + 1[-1(3-\lambda)+1] + 1[1-(5-\lambda)] = 0$

$\therefore \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$\therefore \lambda^3 - 2\lambda^2 - 9\lambda^2 + 18\lambda + 18\lambda - 36 = 0$

$\therefore (\lambda-2)(\lambda^2 - 9\lambda + 18) = 0$

$\therefore (\lambda-2)(\lambda-3)(\lambda-6) = 0$

$\therefore \lambda = 2, 3, 6$

Sum of eigen values $= 2+3+6 = 11$.

$\therefore$ It is verified that sum of diagonal elements is equal to sum of eigen values.

4. Find the eigen values and eigen vectors of the following matrices.

a. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Solution: The charateristic eqn is

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)[(2-\lambda)(3-\lambda)-2] - 0 - 1[2-2(2-\lambda)] = 0$

$\therefore (1-\lambda)[6-5\lambda+\lambda^2-2] + 2[1-\lambda] = 0$

$\therefore (1-\lambda)(\lambda^2-5\lambda+6) = 0$

$\therefore (1-\lambda)(\lambda-2)(\lambda-3) = 0$

$\therefore \lambda = 1, 2, 3$

(i) For $\lambda = 1$,

$[A - \lambda_1 I] = 0$ gives

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 2R_2$ $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore x_3 = 0,\ x_1 + x_2 + x_3 = 0,\ \therefore x_1 + x_2 = 0$

Let $x_2 = -1,\ x_1 = 1$ $\quad\therefore X_1 = [1 \quad -1 \quad 0]$

(ii) For $\lambda = 2$,

$[A - \lambda_2 I]X = 0$ gives

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 + R_1$, By $R_2 + 2R_1$ $\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore -x_1 - x_3 = 0,\ 2x_2 - x_3 = 0$

Let $x_2 = -1,\ x_3 = 2$ $\quad\therefore x_1 = -x_3 = -2$

$\therefore X_2 = [-2 \quad 1 \quad 2]$

(iii) For $\lambda = 3$,

$[A - \lambda_3 I]X = 0$ gives

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -2x_2 + x_3 = 0,\ x_1 - x_2 + x_3 = 0$

Let $x_2 = 1$ $\therefore x_3 = 2$ $\therefore x_1 - 1 + 2 = 0$

$x_1 = -1$

$\therefore X_3 = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}$

b. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution The charateristic eq^n is

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$\lambda^3 -$ (sum of diagonal element) $\lambda^2 +$ (sum of minor D.E.) $\lambda - |A| = 0$

$\lambda^3 - (-1)\lambda^2 + (-21)\lambda - 45 = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$\boxed{\lambda = 5, -3, -3}$

$\therefore$ Eigen values are $5, -3, 3$

(i) For $\lambda = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -7 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0$

$2x_1 - 4x_2 - 6x_3 = 0$

By Cramer's rule

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-24} = \frac{-x_2}{48} = \frac{x_3}{24}$$

$$\frac{x_1}{-1} = \frac{-x_2}{2} = x_3$$

$X_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

(ii) ~~Put~~ For $\lambda = 3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\frac{R_2}{2}, -R_3$ $\begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$r = 1$, $n = 3$

$x_1 + 2x_2 - 3x_3 = 0$

put $x_2 = s$, $x_3 = t$

$x_1 + 2s - 3t = 0$

$x_1 = -2s + 3t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s+3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} t$$

$X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Eigen vectors are $X_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Sol^n. The characteristic eq^n is

$$\begin{vmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{vmatrix} = 0$$

$\therefore \lambda^2(\lambda^2-1)-1(-\lambda-1)+1(1+\lambda)=0$

$\therefore -\lambda^3+\lambda+\lambda+1+\lambda+1=0$

$\therefore \lambda^3-3\lambda-2=0$

$\therefore \lambda^3+\lambda^2-\lambda^2-\lambda-2\lambda-2\lambda=0$

$\therefore (\lambda^2-\lambda-2)(\lambda+1)=0$

$\therefore (\lambda-2)(\lambda+1)(\lambda+1)=0$

$\therefore \lambda=-1,-1,2$

As seen above the A.M. of $\lambda=-1$ is 1.

(i) For $\lambda=-1$, $[A-\lambda_1 I]X=0$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By R_2-R_1, R_3-R_1

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because$ there are three variables and rank is 1, there are $3-1=2$ independent eigenvectors. Now, we have $x_1+x_2+x_3=0$. Putting $x_2=-s$, $x_3=-t$, we get $x_1=-x_2-x_3=s+t$

$$\therefore X = \begin{bmatrix} s+t \\ -s+0 \\ 0-t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore X_1=[1,-1,0]'$ and $X_2=[1,0,-1]'$

$\because$ there are two linearly independent eigen vectors corresponding to $\lambda=-1$, the G.M. of $\lambda=1$ is 2.

(ii) For $\lambda=2$, $[A-\lambda_2 I]X=0$ gives

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By R_2

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By R_2+2R_1, R_3-R_1

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By R_3+R_2, $-\frac{R_2}{3}$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because$ there are three variables and the rank is 2, there is $3-2=1$ eigenvector.

$\therefore x_1-2x_2+x_3=0$, $x_2-x_3=0$

Putting $x_3=t$, we get $x_2=t$, $x_1=2x_2-x_3=2t-t$

$$\therefore X = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \therefore X=[1\ 1\ 1]'$$

5. Use Cayley Hamilton theorem to find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & -3 \\ -2 & -4 & 4 \end{bmatrix}$.

Soln The characteristic eqn of matrix A is

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 2 & 3-\lambda & -3 \\ -2 & -4 & 4-\lambda \end{vmatrix} = 0$$

λ^3- (Sum of diagonal elements) λ^2+ (sum of minor D.E.) $\lambda-|A|=0$

$\lambda^3-8\lambda^2+11\lambda-(-8)=0$

$\lambda^3-8\lambda^2+11\lambda+8=0$

By Cayley Hamilton theorem

$A^3-8A^2+11A+8I$

$$= \begin{bmatrix} -43 & -75 & 63 \\ 90 & 143 & -87 \\ -122 & -196 & 124 \end{bmatrix} - 8\begin{bmatrix} -3 & -8 & 12 \\ 14 & 23 & -15 \\ -18 & 30 & 22 \end{bmatrix} + 11\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & -3 \\ -2 & -4 & 4 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -43 & -75 & 63 \\ 90 & 143 & -87 \\ -122 & -196 & 124 \end{bmatrix} - \begin{bmatrix} -24 & -64 & 96 \\ 112 & 184 & -120 \\ -144 & -240 & 176 \end{bmatrix} + \begin{bmatrix} 11 & 11 & 33 \\ 22 & 33 & -33 \\ -22 & -44 & 44 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ Cayley Hamilton theorem is verified.

A^{-1} By Cayley Hamilton theorem

$A^3 - 8A^2 + 11A + 8I = 0$

Multiply by A^{-1}

$A^3A^{-1} - 8A^2A^{-1} + 11AA^{-1} + 8IA^{-1} = 0$

$A^2AA^{-1} - 8AAA^{-1} + 11I + 8A^{-1} = 0$

$A^2I - 8AI + 11I + 8A^{-1} = 0$

$8A^{-1} = [-A^2I + 8A - 11I]$

Sol

$$A^{-1} = \frac{1}{8}\left\{ -\begin{bmatrix} -3 & -8 & 12 \\ 14 & 23 & -15 \\ -18 & -30 & 22 \end{bmatrix} + 8\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & -3 \\ -2 & -4 & 4 \end{bmatrix} - 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$A^{-1} = \frac{1}{8}\left\{ \begin{bmatrix} 3 & 8 & -12 \\ -14 & -23 & 15 \\ 18 & 30 & -22 \end{bmatrix} + \begin{bmatrix} 8 & 8 & 24 \\ 16 & 24 & -24 \\ -16 & -32 & 32 \end{bmatrix} - \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \right\}$$

$$= \frac{1}{8}\begin{bmatrix} 0 & 16 & 12 \\ 2 & -10 & -9 \\ 2 & -2 & -1 \end{bmatrix}$$

6. Show that the given matrices are diagonalisable. Find diagonal matrix and transforming matrix

(1) $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

Soln $[A - \lambda I]X = 0$

$$\begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^3 - (\text{Sum of D.E.})\lambda^2 + (\text{Sum of M.D.E.})\lambda - |A| = 0$

$\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$

$\boxed{\lambda = 5, 2, 1}$

Eigen value are distinct so matrix is diagonalisable.

(i) Put $\lambda = 5$

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + 2x_2 - 2x_3 = 0$

$-5x_1 - 2x_2 + 2x_3 = 0$

By Cramer's rule

$$\frac{x_1}{\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 2 \\ -5 & -2 \end{vmatrix}}$$

$$\frac{x_1}{0} = \frac{-x_2}{-12} = \frac{x_3}{12}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(ii) Put $\lambda = 2$

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + 2x_2 - 2x_3 = 0$

$-5x_1 + x_2 + 2x_3 = 0$

By Cramer's rule

$$\frac{x_1}{\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 2 \\ -5 & 1 \end{vmatrix}}$$

$$\frac{x_1}{6} = \frac{x_2}{6} = \frac{x_3}{12}$$

$$x_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(iii) Put $\lambda = 1$

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_1 + 2x_2 - 2x_3 = 0$

$-5x_1 + 2x_2 + 2x_3 = 0$

By Cramer's rule

$$\frac{x_1}{\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & -2 \\ -5 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & 2 \\ -5 & 2 \end{vmatrix}}$$

$$\frac{x_1}{8} = \frac{-x_2}{-4} = \frac{x_3}{16}$$

$$\frac{x_1}{2} = x_2 = x_3$$

$$X_3 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$D = M \cdot M^{-1} \cdot A$

$$M = [X_1 \cdot X_2 \cdot X_3] = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 3 & 2 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 3 & 2 & -2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solⁿ $[A - \lambda I]X = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$\lambda^3 - (\text{Sum of D.E.})\lambda^2 + (\text{Sum of M.D.E})\lambda - |A| = 0$

$\lambda^3 - (-1)\lambda^2 + (-21)\lambda - 45 = 0$

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$\boxed{\lambda = 5, -3, -3}$

(i) Put $\lambda = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0$

$2x_1 - 4x_2 - 6x_3 = 0$

By Cramer's rule

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-24} = \frac{-x_2}{48} = \frac{x_3}{24}$$

$$\frac{x_1}{-1} = \frac{x_2}{-2} = x_3$$

$$X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Put $\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\frac{R_2}{2}$,

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & -3 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0$

$x_2 = s, \quad x_3 = t$

$x_1 + 2s - 3t = 0$

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$\boxed{x_1 = -2s + 3t}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s + 3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} t$$

A.M. = 2, G.M. = $n - r = 3 - 1 = 2$

A.M. = G.M.

$\therefore$ Here the given matrix is diagonalisable

$$D = M \cdot M^{-1} A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1.8 & -0.2 & 3.8 \\ -1.4 & 1.2 & 3.4 \\ 1.8 & 1.4 & 5.8 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

7) Check whether the following matrices are diagonalisable or not. If yes, find diagonal matrix & transforming matrix $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$.

Sol^n: $[A - \lambda I]X = 0$

$$\begin{bmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

$\lambda^3 - (\text{Sum of D.E.})\lambda^2 + (\text{Sum of M.D.E.})\lambda - |A| = 0$

$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

$\boxed{\lambda = 3, 2, 2}$

$\therefore$ Eigen values are 3, 2, 2.

(i) For $\lambda = 3$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 10x_2 + 5x_3 = 0$

$-2x_1 - 6x_2 - 4x_3 = 0$

By Cramer's rule

$$\frac{x_1}{\begin{vmatrix} 10 & 5 \\ -6 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 5 \\ -2 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 10 \\ -2 & -6 \end{vmatrix}}$$

$$\frac{x_1}{-10} = \frac{-x_2}{6} = \frac{x_3}{14}$$

$$\frac{x_1}{-5} = \frac{-x_2}{3} = \frac{x_3}{7}$$

$$X_1 = \begin{bmatrix} -5 \\ -3 \\ 7 \end{bmatrix}$$

(ii) For $\lambda = 2$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{R_2}{3}$, $\frac{R_3}{-5}$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 10x_2 + 5x_3 = 0$

$5x_2 + 2x_3 = 0$

$x_2 = 2t$ so $x_3 = 10t + 2x_3 = 0$

$x_3 = -5t$

$x_1 + 20t - 25t = 0$

$x_1 = 5t$

$$X = \begin{bmatrix} 5t \\ 2t \\ -5t \end{bmatrix} = t\begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$$

$\therefore$ The corresponding is $\begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix}$.

Here $n - r = 3 - 2 = 1$. G.M. $= 1$. $\lambda = 2$ is repeated. A.M. $= 2$. A.M. $\neq$ G.M.

$\therefore$ The given matrix is diagonalisable.

8. Find the symmetric matrix of order 3 having eigen values 3, 6, 9 with corresponding eigen vectors $X_1 = [1 \quad 2 \quad 2]'$, $X_2 = [-2 \quad 2 \quad -1]'$, X_3.

Sol^n. Let $X_3 = [x_1, x_2, x_3]$ be third eigen value is 9. The required matrix A is symmetric and all eigen values are distinct the three eigen vectors corresponding to three eigen vectors are orthogonal.

$\therefore x_3$ is orthogonal to x_1 and x_2.

$\therefore x_1 + 2x_2 + 2x_3 = 0$, $-2x_1 + 2x_2 - x_3 = 0$

By Cramer's rule

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}}$$

$$\frac{x_1}{-6} = \frac{-x_2}{3} = \frac{x_3}{6}$$

$$\frac{x_1}{-2} = -x_2 = \frac{x_3}{2} = t \text{ (say)}$$

$$x_3 = \begin{bmatrix} 2t \\ -t \\ -2t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$X_3 = [-2, -1, -2]$$

~~If A =~~ $\begin{bmatrix} \cancel{3} & \cancel{1} & \cancel{1} \\ \cancel{2} & \cancel{2} & \cancel{1} \\ 2 & 2 & 0 \end{bmatrix}$ ~~find A^{50}~~

Solⁿ: ∵ A is symmetric if it is orthogonally similar to a diagonal matrix D. There exists an orthogonal matrix P such that $P^{-1}AP = D$ i.e. $A = PDP^{-1} = PDP^{-1}$

∵ P is orthogonal matrix, we divide each vector by its num. Now, the num of each vector by its num. Now, the num of each vector is $\sqrt{1+4+4} = 3$

Hence, $P = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ 2/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$

(∵ P is orthogonal $P^{-1} = P'$)

$$A = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & 2/3 & -1/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}.$$