Rectification

FE-I(CBCS)

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Rectification(Arc Length):

The method of finding the length of the arc of a curve is called the rectification.

For arc length, the function and its derivative must both be continuous on the closed interval

- **Arc length with Cartesian coordinate**

- If y = f (x) and f' (x) are continuous on [a, b], then the arc length (L) of f (x) on [a,b] is given by

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

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- Similarly for, x = f (y) and f' (y) are continuous on [a,b], then the arc length (L) of f (y) on [a,b] is given by

$$L = \int_a^b \sqrt{1+[f'(y)]^2}\,dy = \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Arc length with Polar Coordinates formula

Polar co-ordinates- Here we describe a co-ordinate system introduced by Newton,called the polar co-ordinate systemWe now need to move into the Calculus II applications of integrals and how we do them in terms of polar coordinates. In this section we'll look at the arc length of the curve given by

- $r = f(\theta)$ and $\alpha \le \theta \le \beta$
- S=$\int_{\theta_1}^{\theta_2} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$ or S=$\int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$
- Smilarly if $\theta = \emptyset(r)$and $r_1 \le r \le r_2$
- S=$\int_{r_1}^{r_2} \sqrt{1 + r^2(\frac{d\theta}{dr})^2}\, dr$

Arc length with parametric formula

Consider the curve which is given by a parametric equation

- $x = f(t)$ and $y = g(t)$
- $S=\int_{t_1}^{t_2} \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\, dt$

Example 1 Find the length of the arc of y= (x) = $x^{3/2}$ on [0, 5].

Solution : $f(x) = x^{3/2} \quad \therefore f'(x) = \frac{3}{2} x^{1/2}$

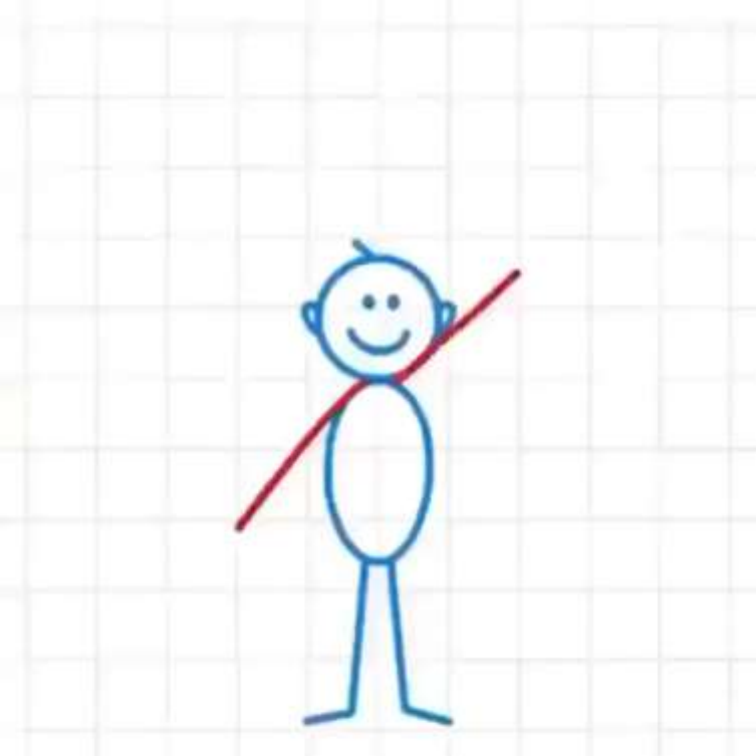
both are continuous on [0, 5].

$f(x) = \int_0^5 \sqrt{1 + f^1(x)^2}\, dx = \int_0^5 \sqrt{1 + \frac{9}{4}x}\, dx$

$= 1/2\int_0^5 \sqrt{4 + 9x} = ½ \, [\frac{(4+9x)^{\frac{3}{2}}}{9.\frac{3}{2}}]_{x=0}^5 = \frac{1}{27}[49^{3/2} - 4^{3/2}]$

$= \frac{1}{27}[7^3 - 2^3] = \frac{1}{27}[147\text{-}8] = \frac{139}{27}$

Types of curves in Cartesian form



Exa. Evaluate length of curve x=$\frac{y^3}{3}+\frac{1}{4y}$ from y=1 to y=2

- Exa.Find the length of the loop 9a $y^2 = x(x-3a)^2$
- Solution.Loop = closed curve

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

- 9a $y^2 = x(x-3a)^2$
- Diff. w.r.to x 18ay $\frac{dy}{dx}$=x.2(x-3a)+ $(x-3a)^2$=(2x+x-3a)(x-3a)
- $\frac{dy}{dx} = \frac{(3x-3a)(x-3a)}{18ay}$
- $(\frac{dy}{dx})^2 = \frac{9(x-a)^2(x-3a)^2}{324.a^2y^2} = \frac{9(x-a)^2(x-3a)^2.9a}{324.a^2x(x-3a)^2} = \frac{(x-a)^2.}{4.a\,x}$
- 1+ $(\frac{dy}{dx})^2$ = 1+ $\frac{(x-a)^2.}{4.a\,x}$

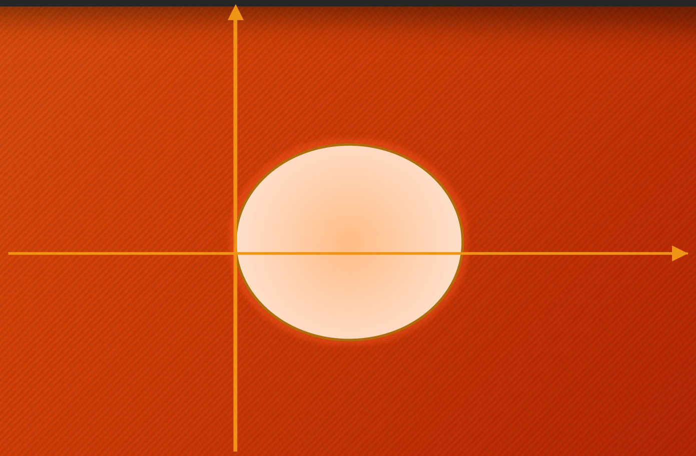
$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x-a)^2.}{4.a\,x} = \frac{(x+a)^2.}{4.a\,x}$$

- length of the loop, s= $\int_{x=}^{x=} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$
- 0<x<3a--------------???
- Total length of the loop, s= $2\int_{x=0}^{x=3a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\int_{x=0}^{x=3a} \sqrt{\frac{(x+a)^2.}{4.a\,x}}\,dx$
- s=2. $\int_{x=0}^{x=3a} (x+a)\sqrt{\frac{1.}{4.a\,x}}\,dx = \frac{1}{\sqrt{a}}\int_{x=0}^{3a}\left[\sqrt{x}+ax^{-\frac{1}{2}}\right]dx$
- S= $\frac{2}{\sqrt{a}}a^{3/2}[\sqrt{3}+\sqrt{3}]$=4a $\sqrt{3}$

Exa. Find the perimeter of the curve $r = acos\theta$.

Sol:The curve $r = acos\theta$ i.e $r^2 = arcos\theta$

i.e $x^2 + y^2 = ax$ i.e $x^2 - ax + a^2/4 + y^2 = a^2/4$



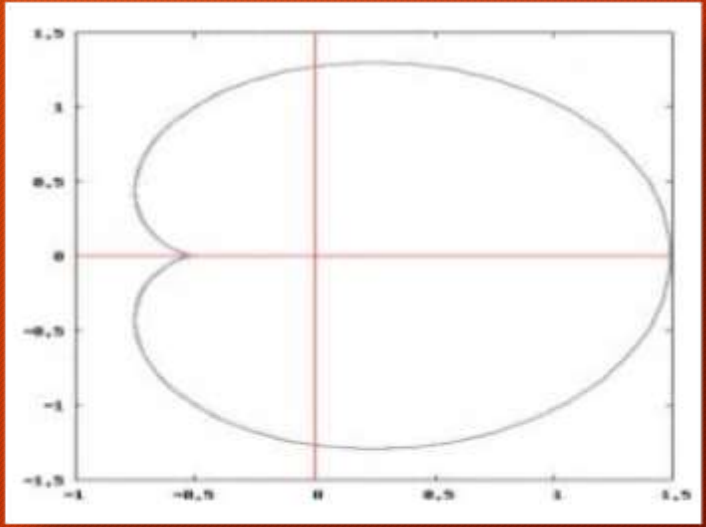
- $(x-\frac{a}{2})^2+(y-0)^2= (\frac{a}{2})^2$
- represents circle centered at $(\frac{a}{2}, 0)$ and radius $\frac{a}{2}$.
- Perimeter= $2l(arc\ AO)$
- $S=2\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$
- $r = acos\theta$, $\frac{dr}{d\theta} = -asin\theta$

$$S=2\int_{\theta=0}^{\theta=\frac{\pi}{2}}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$=2\int_{\theta=0}^{\theta=\frac{\pi}{2}}\sqrt{a^2\cos^2\theta+(a\sin\theta)^2}\,d\theta=2a.\frac{\pi}{2}=\pi a$$

- **Exa.** Prove that the perimeter of the cardiode $r=a(1+\cos\theta)$ is 8a.

Sol:Cardiode $r=a(1+\cos\theta)$ is symmetric about x-axis.



$$r=a(1+\cos\theta)$$

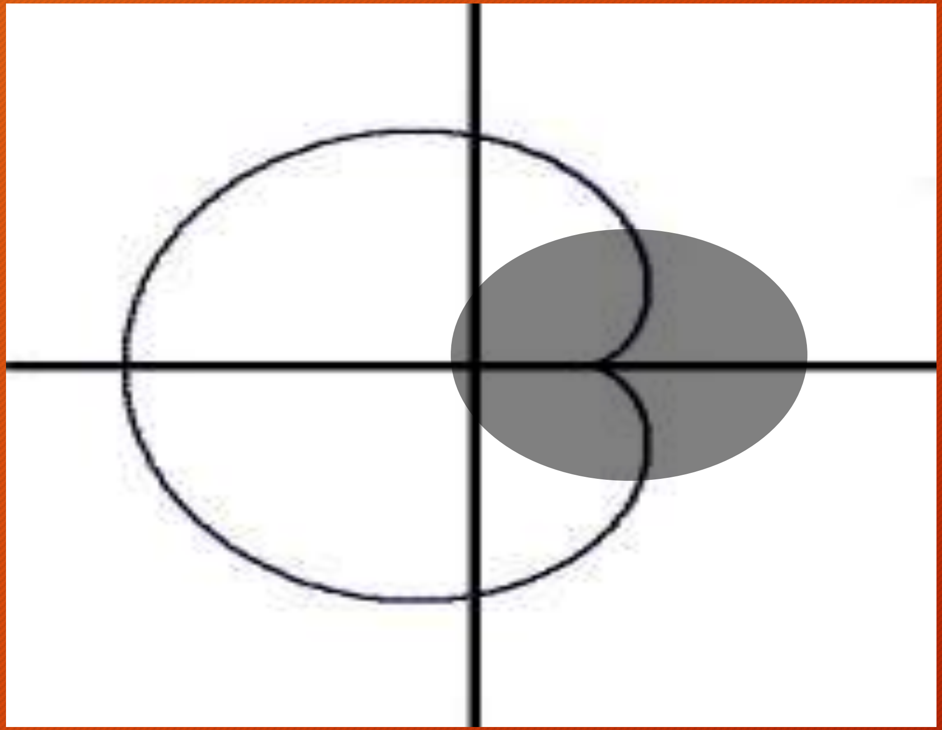
$$\frac{dr}{d\theta}=-a\sin\theta$$

$$S=2\int_{\theta=0}^{\theta=\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$=2\int_{\theta=0}^{\theta=\pi}\sqrt{a^2(1+2\cos\theta+(\cos\theta)^2+(\sin\theta)^2}\,d\theta$$

$$=2.a\int_{\theta=0}^{\theta=\pi}\sqrt{2.2(\cos\theta/2)^2 d\theta}=2a.2.2[\sin\frac{\theta}{2}]_0^{\pi}=8a$$

Find the length to the cardiode $r = a(1 - cos\theta)$ lying outside the circle $r = acos\theta$.

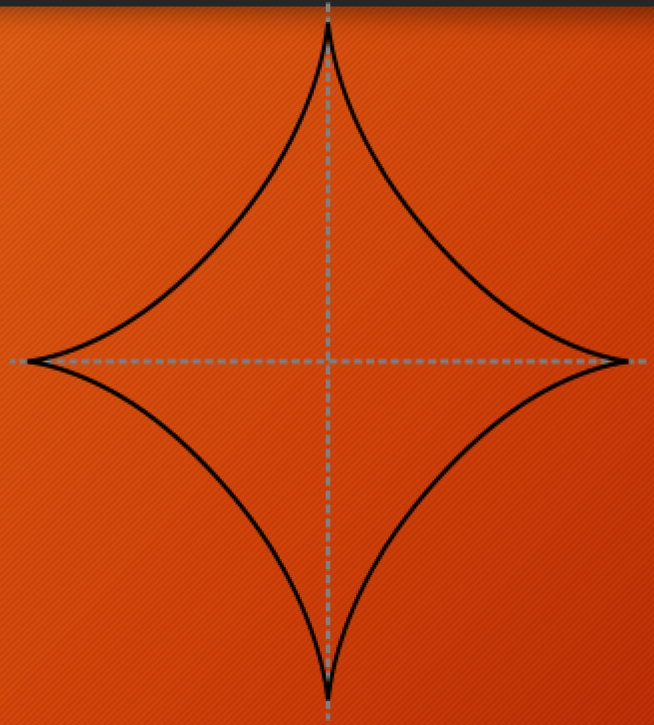


$$S=\int_{\theta=}^{\theta=} \sqrt{r^2+(\frac{dr}{d\theta})^2}\, d\theta$$

$$r=a(1-cos\theta),\ \frac{dr}{d\theta}=a\sin\theta$$

- $S=2\int_{\theta=\pi/3}^{\theta=\pi} \sqrt{r^2+(\frac{dr}{d\theta})^2}\, d\theta$
- $=2\int_{\theta=\pi/3}^{\theta=\pi} \sqrt{a^2(1-2cos\theta+(cos\theta)^2+(sin\theta)^2}\, d\theta$
- $=2.a\int_{\theta=\pi/3}^{\theta=\pi} \sqrt{2.2(sin\theta/2)^2 d\theta}$
- $=2a.2.2[\sin\frac{\theta}{2}]_{\pi/3}^{\pi}=8a(\sqrt{3}/2)$

Exa.Find the length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$



Astroid Curve

Solution : Given curve is $x^{2/3} + y^{2/3} = a^{2/3}$ is asteroid.It is symmetrical in all quadrants

- Use parametric form of curves