Mandavi
5
~~Dubey~~ Tutorial-5

1. Evaluate $\int_0^\infty x^{m-1} \cos(ax)\, dx$

Solution $\int_0^\infty x^{m-1} [\text{R.P. of } e^{-iax}]\, dx$

$ax = t \Rightarrow x = \frac{t}{a} \Rightarrow \frac{dx}{dt} = \frac{1}{a}$

$dx = \frac{dt}{a}$

$\int_0^\infty x^{m-1} e^{-it}\, dx$

$I = \int_0^\infty x^{m-1} e^{-it} \frac{dt}{a}$

$= \int_{t=0}^\infty \text{Real part of} \left(\frac{t}{a}\right)^{m-1} e^{-it} \frac{dt}{a}$

$= \text{Real part of } \frac{1}{a^m} \int t^{m-1} e^{-it} \cdot dt$

$it = m, \quad i\,dt = dm$

$I = \text{Real part of } \frac{1}{a^m} \int_0^\infty e^{-m} \left(\frac{m}{i}\right)^{m-1} dt$

$\text{R.P. of } \frac{1}{a^m} \frac{\Gamma 1}{i^{m-1}}$

$I = \text{R.P. of } \frac{1}{a^m} \frac{\Gamma 1}{i^{m-1}}$

$\text{R.P. of } \frac{1}{a^m} \frac{\Gamma 1 + 1}{\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)^{m-1}}$

$\text{R.P. of } \left(\cos \frac{m\pi}{2} + i \sin \frac{m\pi}{2}\right) \frac{\Gamma 1 + 1}{a^m}$

Mandavi

5

Dubey

$$I = \cos\frac{m\pi}{2} \frac{\Gamma 1 + 1}{a^m} = \cos\frac{m\pi}{2} \frac{2}{a^m}$$

2. Evaluate $\int_0^\infty e^{-ax} x^{m-1} \sin(bx)\, dx$

Solution $\int_0^\infty e^{-ax} x^{m-1} [\text{I.P. of } e^{-ibx}]\, dx$

By substituting $x = ay$ in $\Gamma(m)$

$= \int_0^\infty e^{-x} x^{m-1} dx$, we obtain

$\int_0^\infty e^{-ay} y^{m-1} dy = \frac{\Gamma m}{a^m}$. Thus

$$\int_0^\infty e^{-ax} x^{m-1} \sin bx\, dx = IP \int_0^\infty e^{-(a-ib)x} x^{m-1} dx$$

$$= IP \frac{\Gamma m}{(a-ib)^m}$$

using $a = r\cos\theta$, $b = r\sin\theta$,

$r^2 = a^2 + b^2$.

$$= \frac{\Gamma m}{(a^2+b^2)^{m/2}} \sin m\theta$$

Mandavi S Dubey

Date ______
Page ______

3) a) Evaluate $\int_0^1 \frac{dx}{\sqrt{x\log(1/x)}}$

Solution: Let $I = \int_0^1 \frac{dx}{\sqrt{x\log(1/x)}}$ — (A), Put $\log\frac{1}{x} = t$

$\frac{1}{x} = e^{+t} \Rightarrow x = e^{-t}$ — (1)

$\frac{dx}{dt} = -e^{-t} \Rightarrow dx = -e^{-t}dt$

we now check limit points from (1)

when $x=0$, $e^{-t} = 0$, $t = \infty$

$x = 1$, $e^{-t} = 1$, $t = 0$

x	0	1
t	∞	0

$\therefore$ integral (A) becomes

$I = \int_\infty^0 \frac{-dt\, e^{-t}}{\sqrt{e^{-t}t}} = -\int_\infty^0 e^{-t} e^{t/2} t^{1/2} dt$

$= -\int_\infty^0 e^{-t} t^{1/2} e^{t/2} dt = \int_0^\infty e^{-t/2} t^{-1/2} dt$

(use property of integration)

using $\int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma n}{k^n}$

$= \int_0^\infty e^{-1/2 y} t^{1/2 - 1} dt \Rightarrow n = 1/2,\ k = 1/2$

$= \frac{\Gamma n}{k^n} = \frac{\Gamma 1/2}{(1/2)^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{1/2}} = \sqrt{2\pi}$

Mandavi
5
Dubey

Date
Page

(k) $\int_0^1 \sqrt{\log(1/x)}\, dx$

Solution Put $\log \frac{1}{x} = t$

$\frac{1}{x} = e^t$ $\qquad x = e^{-t}$ — (1)

$dx = -e^{-t}\, dt$

From (1), $x = 0$, $e^{-t} = 0$, $t = \infty$

$x = 1$, $e^{-t} = 1$, $t = 0$

$\int_\infty^0 -e^{-t} t\, dt = \int_\infty^0 -e^{-t} t^{1-1}\, dt$

using property of integration

$\int_0^\infty e^{-ky} y^{n-1}\, dy = \frac{\Gamma n}{k^n}$

$\Rightarrow$ $n = 1$, $k = 1$

$\frac{\Gamma n}{k^n} = \frac{\Gamma 1}{1^1} = \boxed{1}$

Mandavi
5
Dubey

4) Evaluate $\int_0^\infty x^2 e^{-x^4} dx \cdot \int_0^\infty e^{-x^4} dx$

Sol^n Solving I^{st} part i.e. $\int_0^\infty x^2 e^{-x^4} dx$

Let $x^4 = t$

$x = \sqrt[4]{t}$

$\frac{dx}{dt} = \frac{1}{4} t^{-3/4}$

$dx = \frac{1}{4} t^{-3/4} dt$

$= \int_0^\infty \sqrt{t}\, e^{-t} \frac{1}{4} t^{-3/4} dt$

$= \frac{1}{4} \int_0^\infty t^{-1/4} e^{-t} dt$

$= \frac{1}{4} \cdot \Gamma\frac{-1}{4} = \Gamma\frac{3}{4} / -1 = -\sqrt{2}$

Solving II^{nd} part i.e. $\int_0^\infty e^{-x^4} dx$

Let $x^4 = t$

$x = \sqrt[4]{t}$

$dx = \frac{1}{4} t^{-3/4} dt$

$= \int_0^\infty e^{-t} \frac{1}{4} t^{-3/4} dt$

$= \frac{1}{4} \int_0^\infty e^{-t} t^{-3/4} dt$

$= \frac{1}{4} \Gamma\frac{-3}{4} = \Gamma\frac{1}{4} / -3 = \frac{\pi}{-3}$

Mandavi
5
Nbuleey

Answer is $\sqrt{2} \times \frac{\pi}{-3} = \boxed{\frac{\pi\sqrt{2}}{3}}$

5) Evaluate $\int_0^\infty 7^{-7x^2}\,dx$

<u>Solution</u> Let $7^{-7x^2} = e^{-t}$, $-7x^2 \log 7 = -t$,

$7x^2 \log 7 = t$

$x = \frac{\sqrt{t}}{2\log 7}$; $dx = \frac{1}{2\sqrt{\log 7}} \frac{1}{2\sqrt{t}}\,dt$

$\int_0^\infty 7^{-7x^2}\,dx = \int_0^\infty e^{-t} \frac{1}{4\sqrt{\log 7}\sqrt{t}}\,dt$

$= \frac{1}{4\sqrt{\log 7}} \int_0^\infty e^{-t}\, t^{-1/2}\,dt$

$= \frac{1}{4\sqrt{\log 7}} \Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{4\log 3}$ (Answer)

6) Evaluate $\int_0^1 \sqrt{1-x^6}\,dx$

<u>Solution</u> $\int_0^1 (1-x^6)^{1/2}\,dx = \int_0^1 (1-x^3)\,dx$

Put $x^3 = t$

$x = t^{1/3}$

$\frac{dx}{dt} = \frac{1}{3} t^{-2/3}$

$dx = \frac{1}{3} t^{-2/3}\,dt$

Mandavi
5
~~Nullley~~

$$= \int_0^1 (1-t) \frac{1}{3} t^{-2/3} dt$$

$$= \frac{1}{3} \int_0^1 t^{-2/3} (1-t) dt$$

$$= \cancel{B\left(\frac{1}{3}\right.} \frac{1}{3} \int_0^1 t^{\frac{1}{3}-1} (1-t)^{2-1} dt)$$

~~Let~~ $B\left(\frac{1}{3}, 2\right)$

Let $t = \sin^2\theta$

$$= \frac{1}{3} \int_0^{\pi/2} \sin^{2/3-1}\theta \cos^{2-1}\theta \, d\theta$$

$$p = -\frac{1}{3}, q = 1$$

$$= \frac{1}{3} \beta\left(\frac{-\frac{1}{3}+1}{2}, \frac{1+1}{2}\right)$$

$$= \frac{1}{3} \beta\left(\frac{1}{3}, 1\right)$$