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Assignment 4
(Complex Integration)

Q1) Using Cauchy's Residue theorem evaluate
$$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx$$

Soln: (i) Considering the contour as above & nothing that $z f(z) \to 0$ as $|z| \to \infty$, we find the poles.

(ii) $z^6+1=0$ gives $z^6 = e^{(2n+1)\pi i}$

$\therefore z = e^{(2n+1)/\pi/6}$; $n = 0, 1, 2, 3, 4, 5.$

$\therefore z = e^{(i\pi)/6}, e^{(i\pi)/2}, e^{(5i\pi)/6}, e^{(7i\pi)/6}, e^{(9i\pi)/2}, e^{(11i\pi)/6}$

(iii) Of these first 3 poles $\alpha_1 = e^{i\pi/6}$, $\alpha_2 = e^{i\pi/2}$, $\alpha_3 = e^{5i\pi/6}$ lie in the upper half plane.
Let α be one of the three poles.

(iv) Residue (at $z=\alpha$) $= \lim_{z\to\alpha} \frac{(z-\alpha)(z^2)}{z^6+1}$ [form $\frac{0}{0}$]

Residue (at $z=\alpha$) $= \lim_{z\to\alpha} \frac{(z-\alpha)2z+z^2}{6z^5}$ [L'Hospital's Rule]

$$= \frac{\alpha^2}{6\alpha^5} = \frac{\alpha^3}{6\alpha^6} = -\frac{\alpha^3}{6} \quad [\because \alpha^6 = -1]$$

$\therefore$ Sum of the residues $= -\frac{1}{6}[\alpha_1^3+\alpha_2^3+\alpha_3^3]$

$$= -\frac{1}{6}\left[e^{i\pi/2} + e^{3i\pi/2} + e^{5i\pi/2}\right]$$

$$= -\frac{1}{6}\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) + \cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right)\right]$$

$$= -\frac{1}{6}[i - i + i] = -\frac{i}{6}$$

(v) $$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx = 2\pi i\left(-\frac{i}{6}\right) = \pi/3$$

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Q3) Evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$

Sol^n: Let $z = e^{i\theta}$ $\therefore dz = ie^{i\theta}.d\theta = iz\,d\theta$ $\therefore d\theta = \frac{dz}{iz}$,

$\cos\theta = \frac{z^2+1}{2z}$

$\therefore I = \int_c \frac{1}{5+3\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz} = \int_c \frac{2dz}{(bz^2+2az+b)i}$

where C is the circle $|z|=1$. Now, the poles of $f(z)$ are given by $z = \frac{-10 \pm \sqrt{100-36}}{6}$

$\alpha = \frac{-10+8}{6}$ $\quad \beta = \frac{-10-8}{6}$

$= -\frac{1}{3}$ $\quad = -3$ which are simple poles.

$\because a > b > 0$, α lies inside & β lies outside the circle $|z|=1$

$\therefore$ Residue of $f(z)$ (at $z=\alpha$) $= \lim_{z\to\alpha} (z-\alpha) . \frac{2}{b(z-\alpha)(z-\beta)i}$

$= \frac{2}{bi(\alpha-\beta)}$

But $\alpha - \beta = \frac{2\sqrt{a^2-b^2}}{b} = \frac{8}{3}$

$\therefore$ Residue of $f(z)$ (at $z=\alpha$) $= \frac{2}{bi . \frac{2\sqrt{a^2-b^2}}{b}} = \frac{1}{i\sqrt{a^2-b^2}}$

$= \frac{1}{4i}$

$\therefore I = 2\pi i \left(\frac{1}{i\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}} = \frac{2\pi}{4}$

$= \frac{\pi}{2}$

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Q6) Evaluate $\int_C \frac{z^2}{(z-1)^2(z+1)}\,dz$ where C is $|z|=2$ using Residue theorem.

Sol^n: $f(z)$ has a simple pole at $z=-1$ & a pole of order two at $z=1$

(i) If C is $|z|=1/2$, both poles lie outside C & hence by Cauchy's theorem.

$$\int_C f(z)\,dz = \int_C \frac{z^2}{(z-1)^2(z+1)}\,dz = 0$$

(ii) If C is $|z|=2$, both poles lie inside C.

Now, Residue (at $z=-1$) $= \lim_{z\to -1}(z+1)f(z)$

$$= \lim_{z\to -1}\frac{z^2}{(z-1)^2} = \frac{1}{4}$$

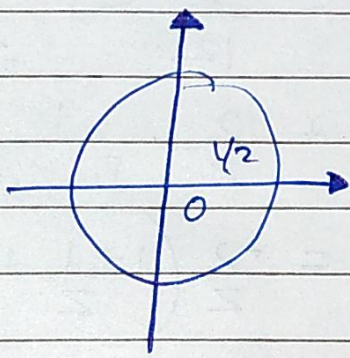
Residue (at $z=1$) $= \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 f(z)\right]$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^2}{z+1}\right] = \lim_{z\to 1}\frac{(z+1)2z-z^2}{(z+1)^2}$$

$$= \lim_{z\to 1}\frac{z^2+2z}{z+1} = \frac{3}{2}$$

$\therefore \int_C f(z)\,dz = 2\pi i\,(\text{Sum of residues}) = 2\pi i\left[\frac{1}{4}+\frac{3}{2}\right]$

$$= \frac{7\pi i}{2}$$



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Q7) Find Laurent's series of $f(z) = \frac{2}{(z-2)(z-1)}$ indicating the regions of convergence.

Solution Let $\frac{2}{(z-1)(z-2)} = \frac{a}{z-1} + \frac{b}{z-2}$ $\therefore 2 = a(z-2) + b(z-1)$

when $z=1$, $2 = -a$ $\therefore a = -2$

when $z=2$, $2 = b$

$$\therefore \frac{2}{(z-1)(z-2)} = \frac{-2}{(z-1)} + \frac{2}{z-2}$$

case (i): when $|z|<1$, clearly $|z|<2$

$$\therefore f(z) = \frac{2}{1-z} - \frac{2}{2[1-(z/2)]} = 2[1-z]^{-1} - [1-(z/2)]^{-1}$$

$$\therefore f(z) = 2[1+z+z^2+z^3+\dots] - \left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\dots\right]$$

case (ii): when $1<|z|<2$, we write

$$\frac{2}{(z-1)(z-2)} = -\frac{2}{(z-1)} + \frac{2}{(z-2)}$$ as

$$= -\frac{2}{z[1-(1/z)]} - \frac{2}{2[1-(z/2)]} = \frac{-2}{z}[1-(1/z)]^{-1} - [1-(z/2)]^{-1}$$

$$= -\frac{2}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\dots\right] - \left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\dots\right]$$

case (iii): when $|z|>2$, $\frac{|z|}{2}>1$ i.e. $\frac{2}{|z|}<1$

Also when $|z|>2$, $|z|>1$ $\therefore \frac{1}{|z|}<1$, we write $f(z) = -\frac{2}{z}\cdot\frac{1}{[1-(1/z)]} + \frac{2}{z}\cdot\frac{1}{[1-(2/z)]}$

$$= \frac{-2}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} = \frac{-2}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right) + \frac{2}{z}\left(\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\dots\right) = -2\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\dots\right) + 4\left(\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\dots\right)$$

Q8) Expand $f(z) = \frac{z^2-1}{z^2+5z+6}$ along $z=1$

Sol^n: Dividing numerator by denominator

$$f(z) = 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

$\because$, we want expansion around $z=1$, we have to obtain Laurent's series in powers of $(z-1)$

$$\therefore \frac{z^2-1}{z^2+5z+6} = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3}$$

There now arise 3 cases

Case (i): When $|z-1| < 3$, we write

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3} \text{ as}$$

$$f(z) = 1 - \frac{8}{4[1+(z-1)/4]} + \frac{3}{3[1+(z-1)/3]}$$

(When $|z-1|<3$ clearly $|z-1|<4$)

$$\therefore f(z) = 1 - \frac{8}{4}\left[1+\left(\frac{z-1}{4}\right)\right]^{-1} + \frac{3}{3}\left[1+\left(\frac{z-1}{3}\right)\right]^{-1}$$

$$\therefore f(z) = 1 - 2\left[1 - \left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \cdots\right]$$
$$+ \left[1 - \left(\frac{z-1}{3}\right) + \left(\frac{z-1}{3}\right)^2 - \left(\frac{z-1}{3}\right)^3 + \cdots\right]$$

Case (ii): When $3 < |z-1| < 4$, we write

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3} \text{ as}$$

$$= 1 - \frac{8}{4[1+(z-1)/4]} + \frac{3}{(z-1)[1+3/(z-1)]} = 1 - 2\left[1+\left(\frac{z-1}{4}\right)\right]^{-1} + \frac{3}{(z-1)}\left[1+\left(\frac{3}{z-1}\right)\right]^{-1}$$

$$\therefore f(z) = 1 - 2\left[1 - \left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \cdots\right] + \frac{3}{(z-1)}\left[1 - \left(\frac{3}{z-1}\right) + \left(\frac{3}{z-1}\right)^2 - \left(\frac{3}{z-1}\right)^3 + \cdots\right]$$

Q9) Expand $\cos z$ as Taylor's series at $z = \pi/2$.

Solⁿ: By Taylor's series

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots$$

Here, $f(z) = \cos z$ and $a = \pi/2$ $\quad \therefore f(\pi/2) = \cos(\pi/2) = 0$

$\therefore f'(z) = -\sin z \quad \therefore f'(\pi/2) = -\sin(\pi/2) = -1$

$f''(z) = -\cos z \quad \therefore f''(\pi/2) = -\cos(\pi/2) = 0$

$f'''(z) = \sin z \quad \therefore f'''(\pi/2) = \sin(\pi/2) = 1$

$$\therefore f(z) = 0 + [z-(\pi/2)](-1) + \frac{[z-(\pi/2)]^3}{3!}(1) + \frac{[z-(\pi/2)]^5}{5!}(-1)$$

$$+ \frac{[z-(\pi/2)]^7}{7!}(1) + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{[z-(\pi/2)]^{2n+1}}{(2n+1)!}$$

Aliter: Put $z = u + (\pi/2)$

$$\therefore \cos z = \cos[u + (\pi/2)] = \cos u \cos(\pi/2) - \sin u \sin(\pi/2)$$

$$= -\sin u$$

$$= -\left[u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \dots\right] = -u + \frac{u^3}{3!} - \frac{u^5}{5!} + \frac{u^7}{7!} + \dots$$

$$= -[z-(\pi/2)] + \frac{[z-(\pi/2)]^3}{3!} - \frac{[z-(\pi/2)]^5}{5!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{[z-(\pi/2)]^{2n+1}}{(2n+1)!}$$

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Q10) Determine the nature of poles & find the residue at each pole.

(i) $\dfrac{ze^z}{(z-a)^3}$ (ii) $\dfrac{1-e^{2z}}{z^3}$

Sol^n) i) $z=a$ is pole of order 3

$\therefore$ Residue of $f(z)$ (at $z=a$)

$= \lim\limits_{z\to a} \dfrac{1}{2!} \dfrac{d^2}{dz^2}\left[(z-a)^3 f(z)\right]$

$= \lim\limits_{z\to a} \dfrac{1}{2!} \dfrac{d^2}{dz^2}\left[(z-a)^3 \dfrac{ze^z}{(z-a)^3}\right]$

$\therefore$ Residue of $f(z)$ (at $z=a$)

$= \dfrac{1}{2!} \lim\limits_{z\to a} \dfrac{d^2}{dz^2}(ze^z) = \dfrac{1}{2!} \lim\limits_{z\to a} \dfrac{d}{dz}(ze^z+e^z)$

$= \dfrac{1}{2!} \lim\limits_{z\to a} \left[ze^z+e^z+e^z\right] = \dfrac{1}{2}(ae^a+2e^a) = \dfrac{1}{2}(a+2)e^a.$

(ii) $z=0$ is a pole of order 3. $\therefore$ Residue of $f(z)$ (at $z=0$)

$= \lim\limits_{z\to 0} \dfrac{1}{2!} \dfrac{d^2}{dz^2}\left[z^3 f(z)\right] = \lim\limits_{z\to 0} \dfrac{1}{2!} \dfrac{d^2}{dz^2}\left[\dfrac{z^3(1-e^{2z})}{z^3}\right]$

$= \dfrac{1}{2!} \lim\limits_{z\to 0} \dfrac{d^2}{dz^2}(1-e^{2z}) = \dfrac{1}{2!} \lim\limits_{z\to 0} \dfrac{d}{dz}(-2e^{2z})$

$= \dfrac{1}{2!} \lim\limits_{z\to 0}(-4e^{2z}) = \dfrac{1}{2}\cdot(-4) = -2$

Alternatively

$f(z) = \dfrac{1-e^{2z}}{z^3} = \dfrac{1-\left\{1+2z+\dfrac{4z^2}{2!}+\dfrac{8z^3}{3!}+\dots\right\}}{z^3} = -\dfrac{2}{z^2}-\dfrac{2}{z}-$

$\dfrac{4}{3} - \dots$

$\therefore$ Residue (at $z=0$) $= b_1 =$ coefficient of $\dfrac{1}{z} = -2$