Module 3 -Posets and Lattice

CE– SE–DSGT

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Module 3 -Posets and Lattice

- Partial Order Relations,
- Poset, Hasse Diagram,
- Chain and Antichains, Lattice, Types of Lattice, Sub lattic

Partial Ordered Relations

A relation R on a set A is a partial ordered relation ,iff , it is reflexive, antisymmetric and transitive

Partial Ordered Sets

A relation R on a set A is called a **partial order** if R is reflexive, antisymmetric, and transitive. The set A together with the partial order R is called a **partially ordered set**, or simply a **poset**, and we will denote this poset by (A, R). If there is no possibility of confusion about the partial order, we may refer to the poset simply as A, rather than (A, R).

Example 1. Let A be a collection of subsets of a set S. The relation $\subseteq$ of set inclusion is a partial order on A, so $(A, \subseteq)$ is a poset. ♦

Example 2. Let Z^+ be the set of positive integers. The usual relation $\leq$ (less than or equal to) is a partial order on Z^+, as is $\geq$ (greater than or equal to). ♦

Example 3. The relation of divisibility ($a\ R\ b$ if and only if $a \mid b$) is a partial order on Z^+. ♦

Example 4. Let $\mathcal{R}$ be the set of all equivalence relations on a set A. Since $\mathcal{R}$ consists of subsets of $A \times A$, $\mathcal{R}$ is a partially ordered set under the partial order of set containment. If R and S are equivalence relations on A, the same property may be expressed in relational notation as follows.

$$R \subseteq S \text{ if and only if } x\,R\,y \quad \text{implies} \quad x\,S\,y \text{ for all } x, y \text{ in } A.$$

Then $(\mathcal{R}, \subseteq)$ is a poset. ♦

Example 5. The relation $<$ on Z^+ is not a partial order, since it is not reflexive. ♦

Example 6

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

In the poset $(\mathbf{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution: The integers 3 and 9 are comparable, because $3 \mid 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$.

The adjective "partial" is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering**.

Theorem 1. *If $(A, \leq)$ and $(B, \leq)$ are posets, then $(A \times B, \leq)$ is a poset, with partial order $\leq$ defined by*

$$(a, b) \leq (a', b') \qquad \text{if } a \leq a' \text{ in } A \quad \text{and} \quad b \leq b' \text{ in } B.$$

Note that the symbol $\leq$ is being used to denote three distinct partial orders. The reader should find it easy to determine which of the three is meant at any time.

Proof: If $(a, b) \in A \times B$, then $(a, b) \leq (a, b)$ since $a \leq a$ in A and $b \leq b$ in B, so $\leq$ satisfies the reflexive property in $A \times B$. Now suppose that $(a, b) \leq (a', b')$ and $(a', b') \leq (a, b)$, where a and $a' \in A$ and b and $b' \in B$. Then

$$a \leq a' \quad \text{and} \quad a' \leq a \qquad \text{in } A$$

and

$$b \leq b' \quad \text{and} \quad b' \leq b \qquad \text{in } B.$$

Since A and B are posets, the antisymmetry of the partial orders in A and B implies that

$$a = a' \quad \text{and} \quad b = b'.$$

Hence $\leq$ satisfies the antisymmetric property in $A \times B$.

Finally, suppose that

$$(a, b) \leq (a', b') \quad \text{and} \quad (a', b') \leq (a'', b''),$$

where $a, a', a'' \in A$, and $b, b', b'' \in B$. Then

$$a \leq a' \quad \text{and} \quad a' \leq a'',$$

so $a \leq a''$, by the transitive property of the partial order in A. Similarly,

$$b \leq b' \quad \text{and} \quad b' \leq b'',$$

so $b \leq b''$, by the transitive property of the partial order in B. Hence

Consequently, the transitive property holds for the partial order in $A \times B$, and we conclude that $A \times B$ is poset. ◆

Hasse Diagrams

Many edges in the directed graph for a finite poset do not have to be shown because they must be present. For instance, consider the directed graph for the partial ordering $\{(a, b) \mid a \leq b\}$ on the set $\{1, 2, 3, 4\}$, shown in Figure 2(a). Because this relation is a partial ordering, it is reflexive, and its directed graph has loops at all vertices. Consequently, we do not have to show these loops because they must be present; in Figure 2(b) loops are not shown. Because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity. For example, in Figure 2(c) the edges (1, 3), (1, 4), and (2, 4) are not shown because they must be present. If we assume that all edges are pointed "upward" (as they are drawn in the figure), we do not have to show the directions of the edges; Figure 2(c) does not show directions.

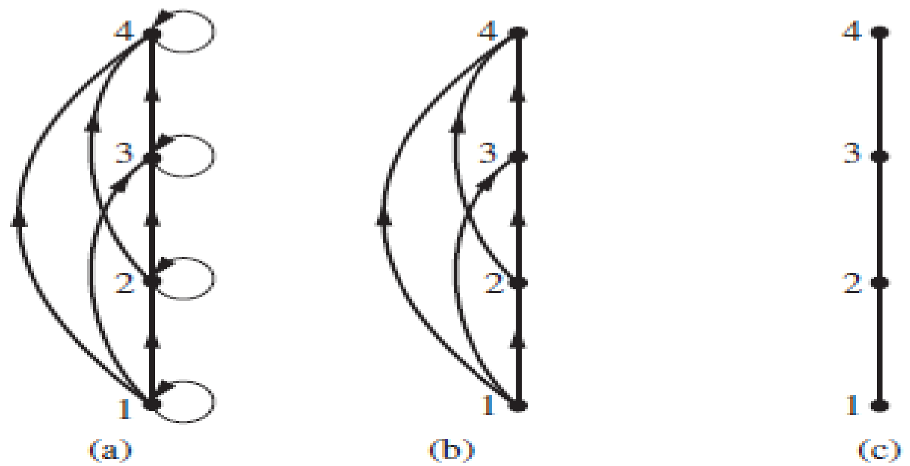


FIGURE 2 **Constructing the Hasse diagram for ({1, 2, 3, 4}, ≤).**

In general, we can represent a finite poset $(S, \preccurlyeq)$ using this procedure: Start with the directed graph for this relation. Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops. Next, remove all edges that must be in the partial ordering because

of the presence of other edges and transitivity. That is, remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec x$. Finally, arrange each edge so that its initial vertex is below its terminal vertex (as it is drawn on paper). Remove all the arrows on the directed edges, because all edges point "upward" toward their terminal vertex.

Example 1

Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

Solution: Begin with the digraph for this partial order, as shown in Figure 3(a). Remove all loops, as shown in Figure 3(b). Then delete all the edges implied by the transitive property. These are (1, 4), (1, 6), (1, 8), (1, 12), (2, 8), (2, 12), and (3, 12). Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram. The resulting Hasse diagram is shown in Figure 3(c). ◀

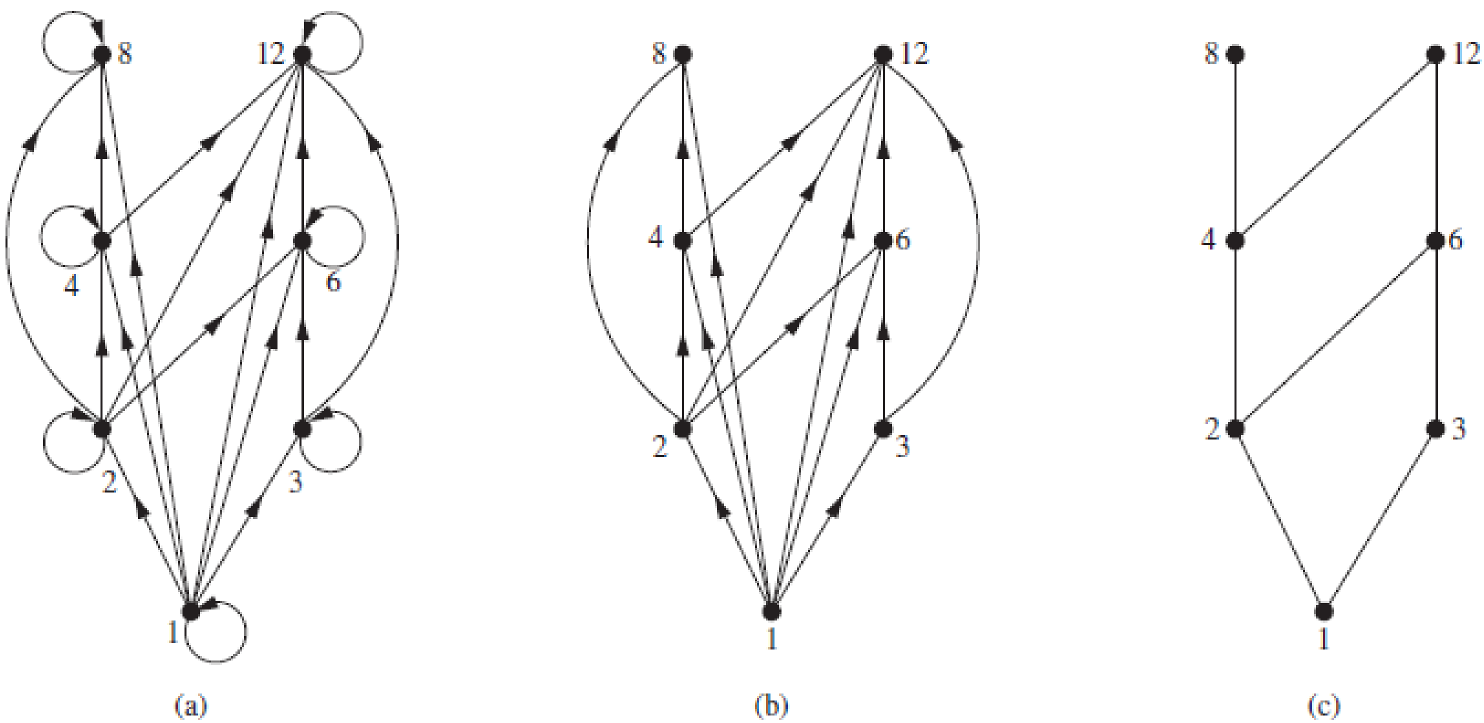


FIGURE 3 **Constructing the Hasse diagram of ({1, 2, 3, 4, 6, 8, 12}, |).**

Example 2

Draw the Hasse diagram for the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set $P(S)$, where $S = \{a, b, c\}$.

Solution: The Hasse diagram for this partial ordering is obtained from the associated digraph by deleting all the loops and all the edges that occur from transitivity, namely, $(\emptyset, \{a, b\})$, $(\emptyset, \{a, c\})$, $(\emptyset, \{b, c\})$, $(\emptyset, \{a, b, c\})$, $(\{a\}, \{a, b, c\})$, $(\{b\}, \{a, b, c\})$, and $(\{c\}, \{a, b, c\})$. Finally, all edges point upward, and arrows are deleted. The resulting Hasse diagram is illustrated in Figure 4.

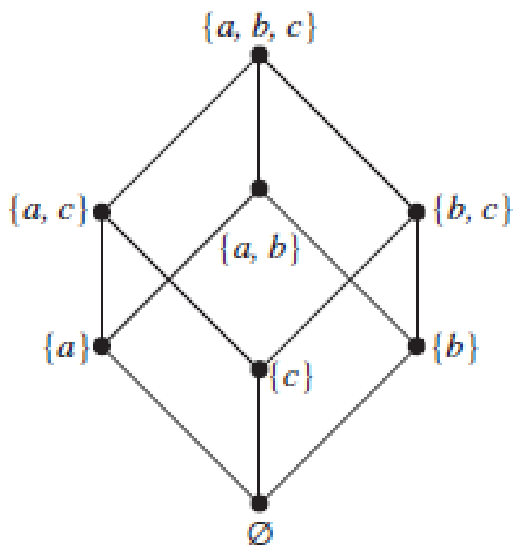


FIGURE 4 The Hasse diagram of $(P(\{a, b, c\}), \subseteq)$.

Elements of posets that have certain extremal properties are important for many applications. An element of a poset is called maximal if it is not less than any element of the poset. That is, a is **maximal** in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$. Similarly, an element of a poset is called minimal if it is not greater than any element of the poset. That is, a is **minimal** if there is no element $b \in S$ such that $b \prec a$. Maximal and minimal elements are easy to spot using a Hasse diagram. They are the "top" and "bottom" elements in the diagram.

Example 3

Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and which are minimal?

Solution: The Hasse diagram in Figure 5 for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element. ◀

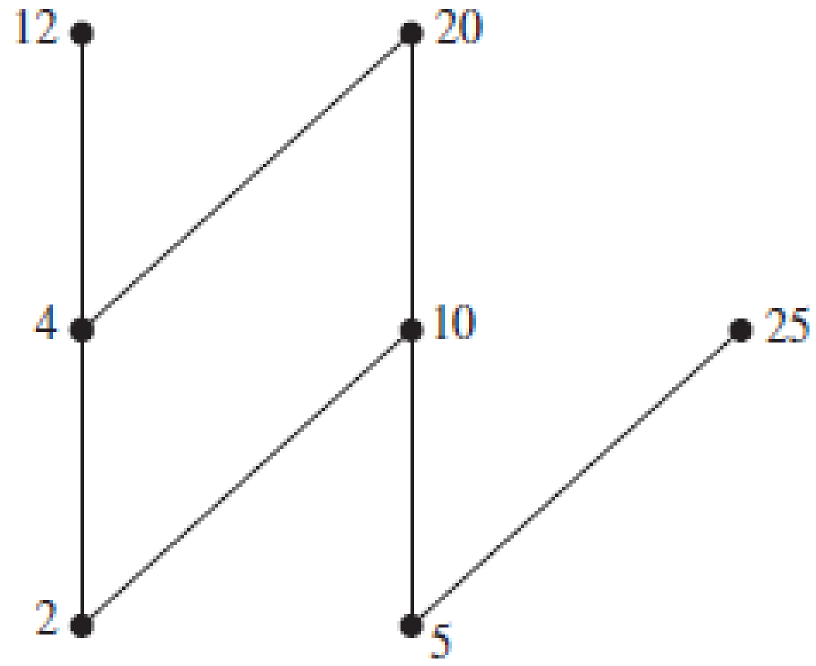


FIGURE 5 **The Hasse diagram of a poset.**

Sometimes there is an element in a poset that is greater than every other element. Such an element is called the greatest element. That is, a is the **greatest element** of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$. The greatest element is unique when it exists [see Exercise 40(a)]. Likewise, an element is called the least element if it is less than all the other elements in the poset. That is, a is the **least element** of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$. The least element is unique when it exists [see Exercise 40(b)].

Example 4

Determine whether the posets represented by each of the Hasse diagrams in Figure 6 have a greatest element and a least element.

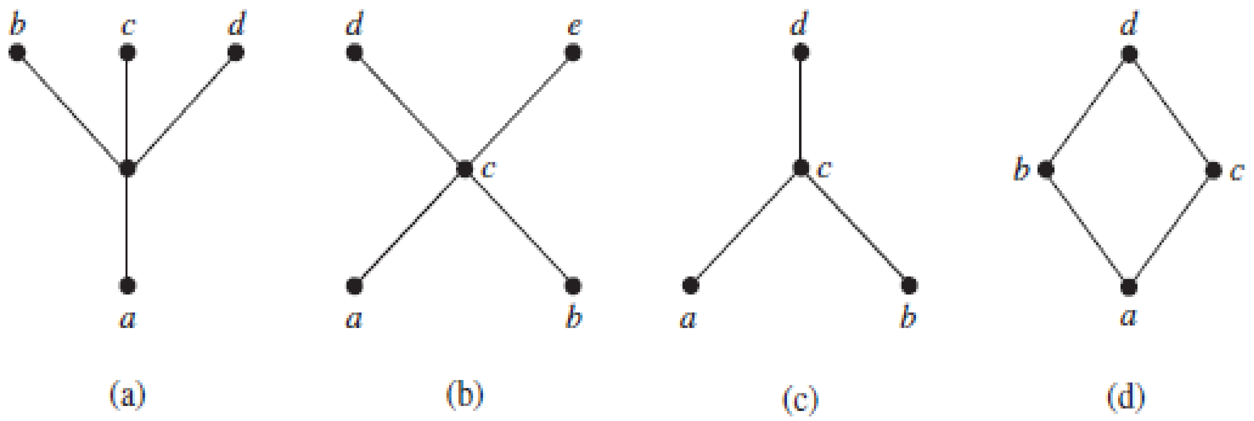


FIGURE 6 **Hasse diagrams of four posets.**

Solution: The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element. The poset with Hasse diagram (b) has neither a least nor a greatest element. The poset with Hasse diagram (c) has no least element. Its greatest element is d. The poset with Hasse diagram (d) has least element a and greatest element d. ◀

Example 5

Let S be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.

Solution: The least element is the empty set, because $\emptyset \subseteq T$ for any subset T of S. The set S is the greatest element in this poset, because $T \subseteq S$ whenever T is a subset of S. ◀

Example 6

Is there a greatest element and a least element in the poset $(\mathbf{Z}^+, |)$?

Solution: The integer 1 is the least element because $1|n$ whenever n is a positive integer. Because there is no integer that is divisible by all positive integers, there is no greatest element. ◀

Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an **upper bound** of A. Likewise, there may be an element less than or equal to all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** of A.

Example 7

Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

Solution: The upper bounds of $\{a, b, c\}$ are e, f, j, and h, and its only lower bound is a. There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e, and f. The upper bounds of $\{a, c, d, f\}$ are f, h, and j, and its lower bound is a. ◀

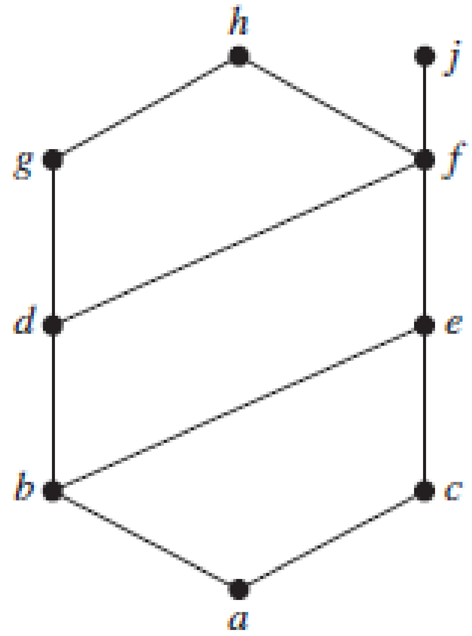


FIGURE 7 **The Hasse diagram of a poset.**

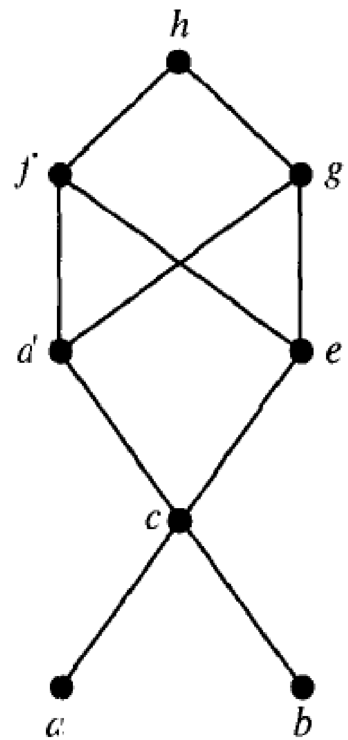
Example 9

Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

Solution: An integer is a lower bound of $\{3, 9, 12\}$ if 3, 9, and 12 are divisible by this integer. The only such integers are 1 and 3. Because $1 \mid 3$, 3 is the greatest lower bound of $\{3, 9, 12\}$. The only lower bound for the set $\{1, 2, 4, 5, 10\}$ with respect to $|$ is the element 1. Hence, 1 is the greatest lower bound for $\{1, 2, 4, 5, 10\}$.

An integer is an upper bound for $\{3, 9, 12\}$ if and only if it is divisible by 3, 9, and 12. The integers with this property are those divisible by the least common multiple of 3, 9, and 12, which is 36. Hence, 36 is the least upper bound of $\{3, 9, 12\}$. A positive integer is an upper bound for the set $\{1, 2, 4, 5, 10\}$ if and only if it is divisible by 1, 2, 4, 5, and 10. The integers with this property are those integers divisible by the least common multiple of these integers, which is 20. Hence, 20 is the least upper bound of $\{1, 2, 4, 5, 10\}$. ◀

Example 8. Consider the poset $A = \{a, b, c, d, e, f, g, h\}$, whose Hasse diagram is shown in Figure 7.24. Find all upper and lower bounds of the following subsets of A: (a) $B_1 = \{a, b\}$; (b) $B_2 = \{c, d, e\}$.



Solution

(a) B_1 has no lower bounds; its upper bounds are c, d, e, f, g, and h.

(b) The upper bounds of B_2 are f, g, and h; its lower bounds are c, a, and b.

◆

LUB and GLB

The element x is called the **least upper bound** of the subset A if x is an upper bound that is less than every other upper bound of A. Because there is only one such element, if it exists, it makes sense to call this element *the* least upper bound [see Exercise 42(a)]. That is, x is the least upper bound of A if $a \preccurlyeq x$ whenever $a \in A$, and $x \preccurlyeq z$ whenever z is an upper bound of A. Similarly, the element y is called the **greatest lower bound** of A if y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A. The greatest lower bound of A is unique if it exists [see Exercise 42(b)]. The greatest lower bound and least upper bound of a subset A are denoted by glb(A) and lub(A), respectively.

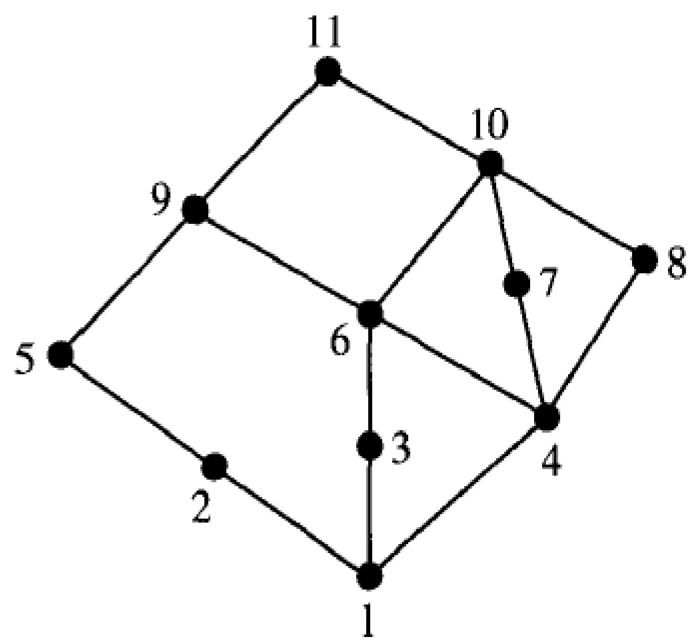
Example 8

Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist, in the poset shown in Figure 7.

Solution: The upper bounds of $\{b, d, g\}$ are g and h. Because $g \prec h$, g is the least upper bound. The lower bounds of $\{b, d, g\}$ are a and b. Because $a \prec b$, b is the greatest lower bound.

Example 10. Let $A = \{1, 2, 3, 4, 5, \ldots, 11\}$ be the poset whose Hasse diagram is shown in Figure 7.25. Find the LUB and the GLB of $B = \{6, 7, 10\}$, if they exist.

Solution: Exploring all upward paths from vertices 6, 7, and 10, we find that LUB $(B) = 10$. Similarly, by examining all downward paths from 6, 7, and 10, we find that GLB $(B) = 4$. ◆



Example 9

Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

Solution: An integer is a lower bound of $\{3, 9, 12\}$ if 3, 9, and 12 are divisible by this integer. The only such integers are 1 and 3. Because $1 \mid 3$, 3 is the greatest lower bound of $\{3, 9, 12\}$. The only lower bound for the set $\{1, 2, 4, 5, 10\}$ with respect to $|$ is the element 1. Hence, 1 is the greatest lower bound for $\{1, 2, 4, 5, 10\}$.

An integer is an upper bound for $\{3, 9, 12\}$ if and only if it is divisible by 3, 9, and 12. The integers with this property are those divisible by the least common multiple of 3, 9, and 12, which is 36. Hence, 36 is the least upper bound of $\{3, 9, 12\}$. A positive integer is an upper bound for the set $\{1, 2, 4, 5, 10\}$ if and only if it is divisible by 1, 2, 4, 5, and 10. The integers with this property are those integers divisible by the least common multiple of these integers, which is 20. Hence, 20 is the least upper bound of $\{1, 2, 4, 5, 10\}$. ◀

Lattices

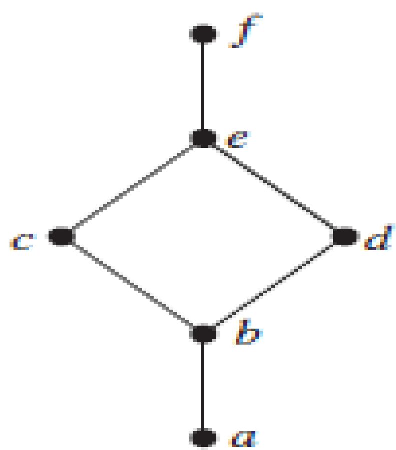
A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice.**

Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

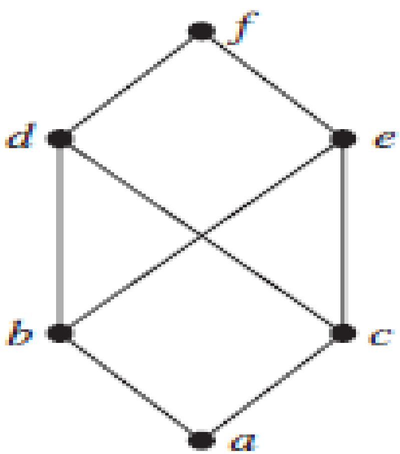
Example:

Determine whether the posets represented by each of the Hasse diagrams in Figure are lattices.

every pair of elements has both a least upper bound and a greatest lower bound, as the reader should verify. On the other hand, the poset with the Hasse diagram shown in (b) is not a lattice, because the elements b and c have no least upper bound. To see this, note that each of the elements d, e, and f is an upper bound, but none of these three elements precedes the other two with respect to the ordering of this poset.



(a)



(b)

Example:

Is the poset $(\mathbf{Z}^+, |)$ a lattice?

Solution: Let a and b be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively, as the reader should verify. It follows that this poset is a lattice. ◀

Example:

Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Solution: Because 2 and 3 have no upper bounds in $(\{1, 2, 3, 4, 5\}, |)$, they certainly do not have a least upper bound. Hence, the first poset is not a lattice.

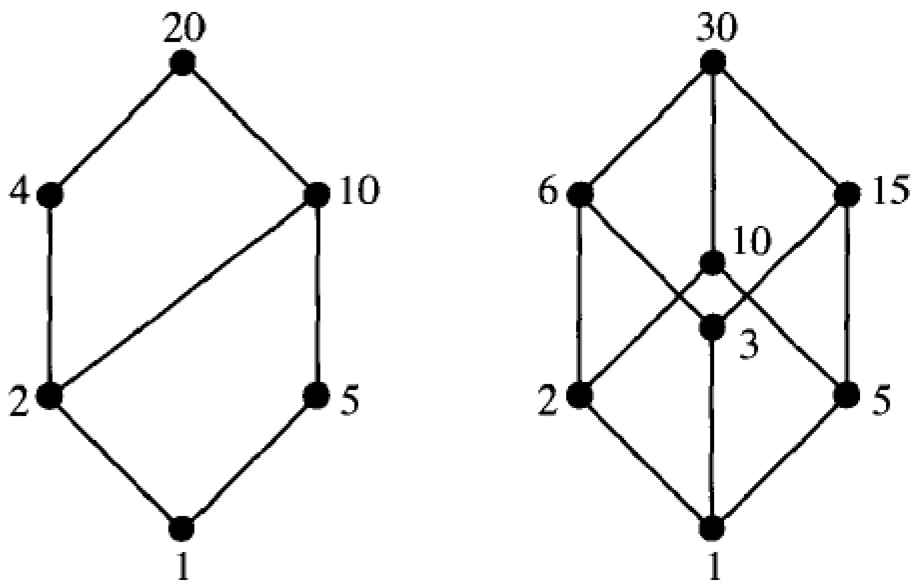
Every two elements of the second poset have both a least upper bound and a greatest lower bound. The least upper bound of two elements in this poset is the larger of the elements and the greatest lower bound of two elements is the smaller of the elements, as the reader should verify. Hence, this second poset is a lattice. ◀

Example:

Determine whether $(P(S), \subseteq)$ is a lattice where S is a set.

Solution: Let A and B be two subsets of S. The least upper bound and the greatest lower bound of A and B are $A \cup B$ and $A \cap B$, respectively, as the reader can show. Hence, $(P(S), \subseteq)$ is a lattice. ◀

Example 3. Let n be a positive integer and let D_n be the set of all positive divisors of n. Then D_n is a lattice under the relation of divisibility as considered in Example 2. Thus, if $n = 20$, we have $D_{20} = \{1, 2, 4, 5, 10, 20\}$. The Hasse diagram of D_{20} is shown in Figure 7.39(a). If $n = 30$, we have $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$. The Hasse diagram of D_{30} is shown in Figure 7.39(b). ◆



D_{20} D_{30}

(a) (b)

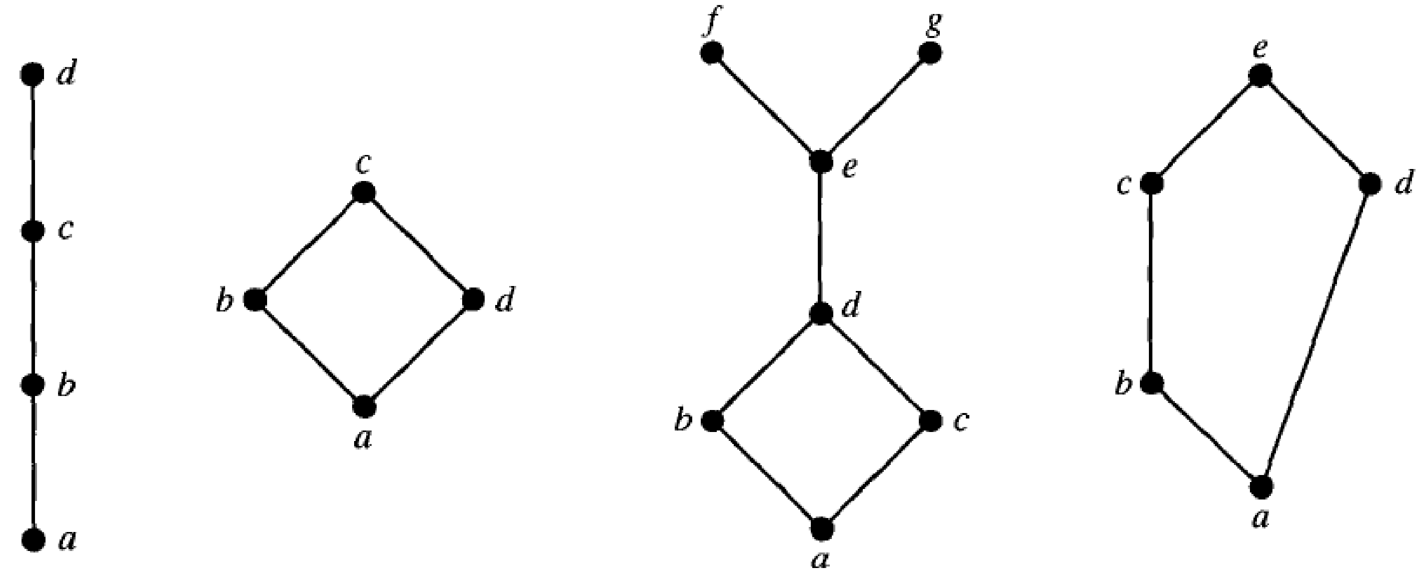
Lattices

A **lattice** is a poset $(L, \leq)$ in which every subset $\{a, b\}$ consisting of two elements has a least upper bound and a greatest lower bound. We denote LUB $(\{a, b\})$ by $a \vee b$ and call it the **join** of a and b. Similarly, we denote GLB $(\{a, b\})$ by $a \wedge b$

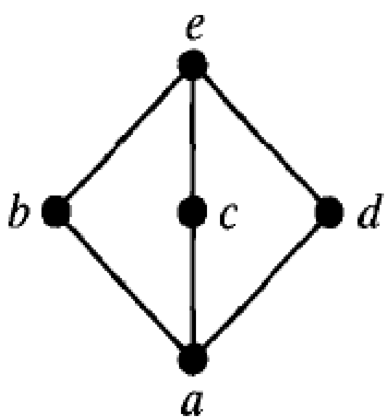
and call it the **meet** of a and b.

Example 4. Which of the Hasse diagrams in Figure 7.40 represent lattices?

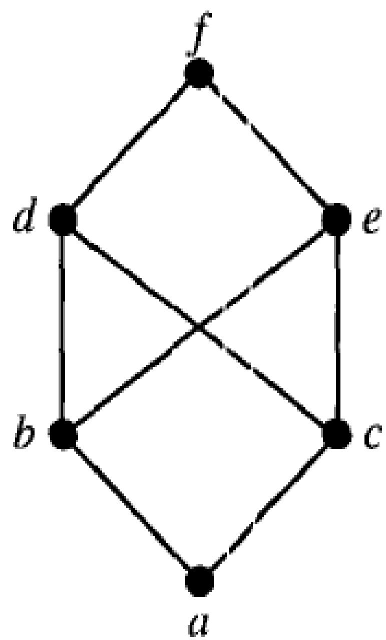
Solution: Hasse diagrams (a), (b), (d), and (e) represent lattices. Diagram (c) does not represent a lattice because $f \vee g$ does not exist. Diagram (f) does not represent a lattice because neither $d \wedge e$ nor $b \vee c$ exist. Diagram (g) does not represent a lattice because $c \wedge d$ does not exist. ◆



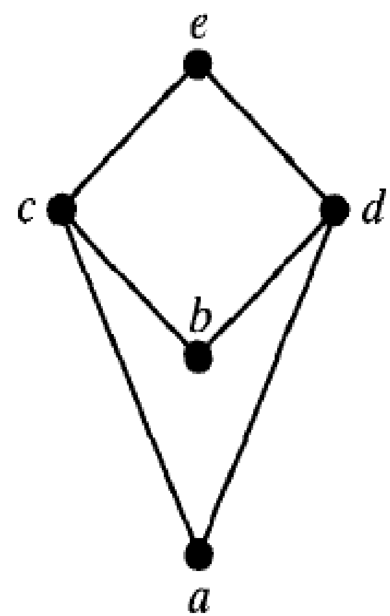
(a) (b) (c) (d)



(e)



(f)



(g)

Theorem 1. *If* $(L_1, \leq)$ *and* $(L_2, \leq)$ *are lattices, then* $(L, \leq)$ *is a lattice, where* $L = L_1 \times L_2$, *and the partial order* $\leq$ *of* L *is the product partial order.*

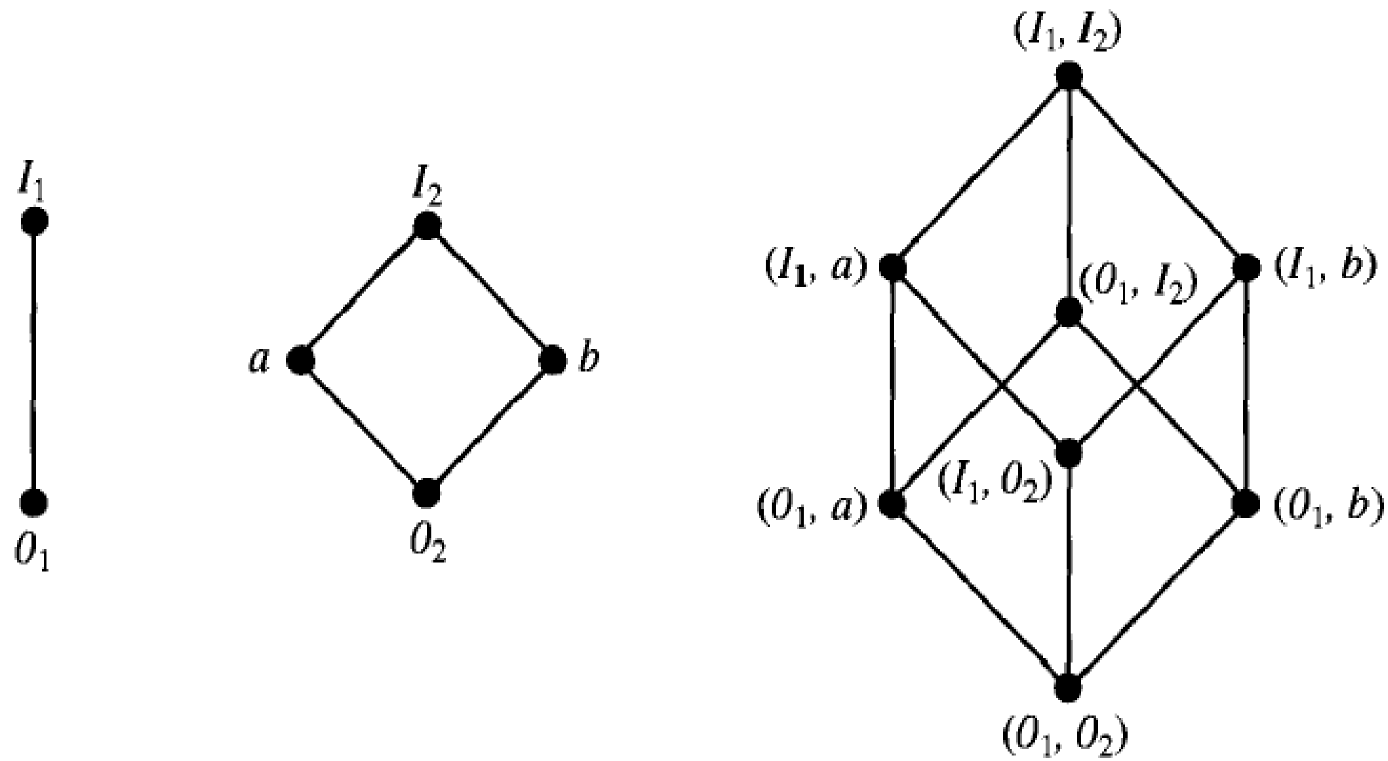
Proof: We denote the join and meet in L_1 by $\vee_1$ and $\wedge_1$, respectively, and the join and meet in L_2 by $\vee_2$ and $\wedge_2$, respectively. We already know from Theorem 1 of Section 7.1 that L is a poset. We now need to show that if (a_1, b_1) and $(a_2, b_2) \in L$, then $(a_1, b_1) \vee (a_2, b_2)$ and $(a_1, b_1) \wedge (a_2, b_2)$ exist in L. We leave it as an exercise to verify that

$$(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee_1 a_2, b_1 \vee_2 b_2)$$

$$(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge_1 a_2, b_1 \wedge_2 b_2).$$

Thus L is a lattice. ◆

Example 7. Let L_1 and L_2 be the lattices shown in Figure 7.41(a) and (b) respectively. Then $L = L_1 \times L_2$ is the lattice shown in Figure 7.41(c).



L_1 L_2 $L = L_1 \times L_2$

Chain and AntiChain

Let (S, ≤) be the poset. A subset of A is called a chain if every two elements in the subset are related. The number of elements in the chain is called length of chain.

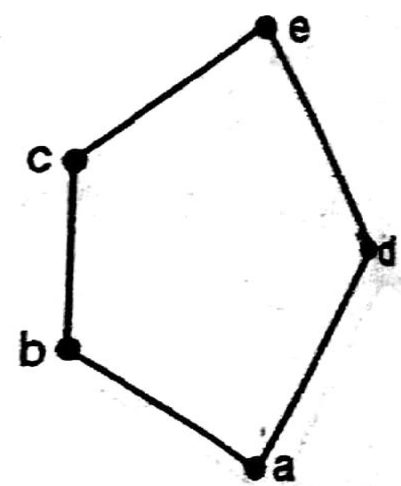
A subset of A is called an antichain if no every two elements in the subset are related.

A Poset (S, ≤) is called as **"totally ordered set"** if A is a chain. In this case binary relation ≤ is called a totally ordered relation.

Totally ordered set is alsocalled as linearly ordered set.

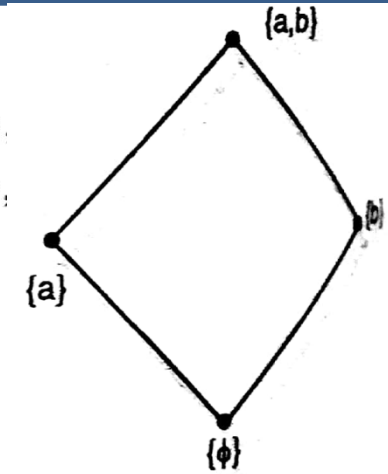
Example 1 :

For the partially ordered set mentioned above A = {a, b, c, d, e}, we have {a, b, c, e} {a, b, c}, {a, b}, {a, d, e} · {a, d}, {a} are chains and {b, d} {c, d}, are antichains.



Example 2 :

Let A = {a, b} and consider its poset {P(a), ⊆) Then {{ϕ}, {a}, {a, b}}, {{ϕ}, {b}, {a, b}}, {{ϕ}, {a}}, {{ϕ}, {b}}, {{a}, {a, b}}, {{b}, {a, b}} are chains and {{a}, {b}} is antichain.



Example 3 :

Draw the Hasse diagram of the following sets under partial ordering relation "divides" and indicate those which are chains.

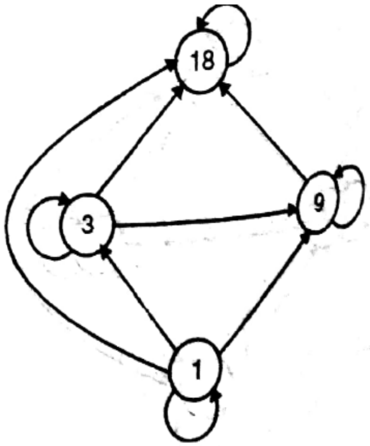
(a) {1, 3, 9,18} (b) {3, 5, 30} (c) {1, 2, 5, 10, 20}

Solution :

(a) Partial ordered relation "divides" for the given set {1, 3, 9, 18} is

$$R = \{(1, 1), (1, 3), (1, 9), (1, 18), (3, 3), (3, 9), (3, 18), (9, 9), (9, 18), (18, 18)\}$$

Digraph for the above relation is as shown

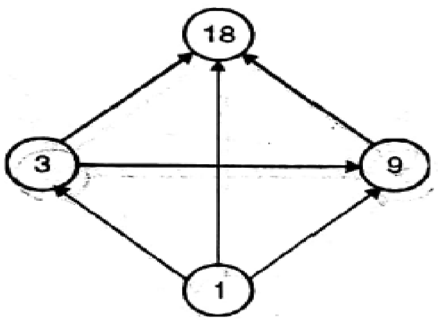


Matrix :

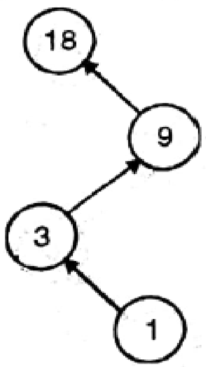
$$M_R = \begin{array}{c} \\ 1 \\ 3 \\ 9 \\ 18 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 9 & 18 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

To convert this diagraph into hasse diagram.

Step 1 : Remove all cycles.

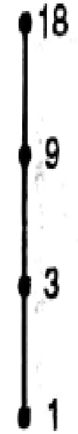


Step 2 : Remove transitive edges (1, 18), (3,18), (1, 9).



Step 3 : All edges are pointing upwords. Now replace circles by dots and remove all arrows from edges.

Hasse **Diagram :**



This poset is chain as every two elements are related

(b) Partial ordered relation "divides" for the given set $\{3, 5, 30\}$ is

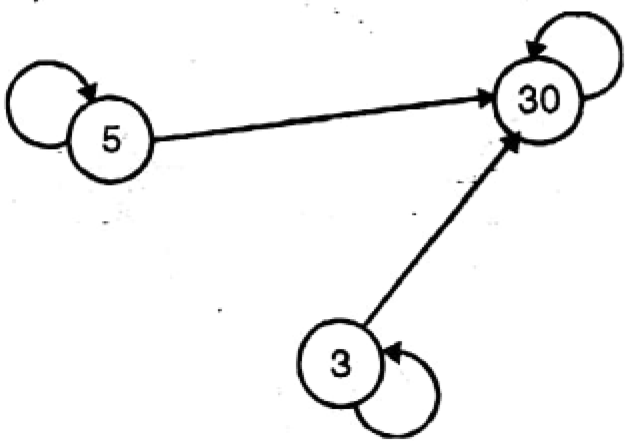
$$R = \{(3, 3), (3, 30), (5, 30), (30, 30), (5, 5)\}$$

Digraph for the above relation is

Matrix :

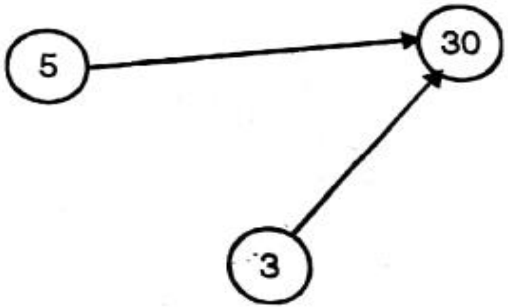
$$M_R = \begin{array}{c} \\ 3 \\ 5 \\ 30 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 5 & 30 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Diagraph for the above matrix is,



To convert this digraph into hasse diagram

Step 1 : Remove all cycles.

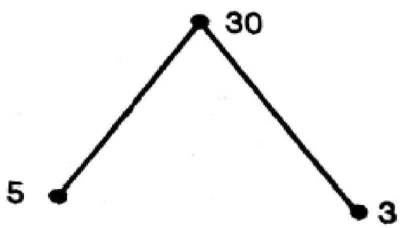


Step 2 : Remove transitive edges

But the digraph has no transitive edges.

Step 3 : All edges are pointing upwords. Now replace circles by dots. and remove arrows edges.

Hasse Diagram :



Given poset is not chain as 5 not related to 3 or 3 not related to 5.

(c) Partial ordered relation "divides" for the given set {1, 2, 5, 10, 20} is

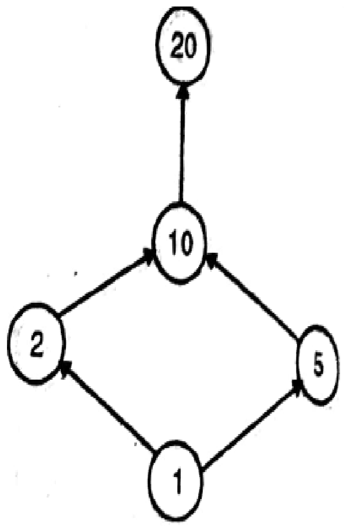
$$R = \{(1, 1), (1, 2), (1, 5), (1, 10), (1, 20), (2, 2), (2, 10), (2, 20), (5, 5), (5, 10), (5, 20), (10,10), (10, 20), (20, 20)\}$$

Matrix for the above relation is

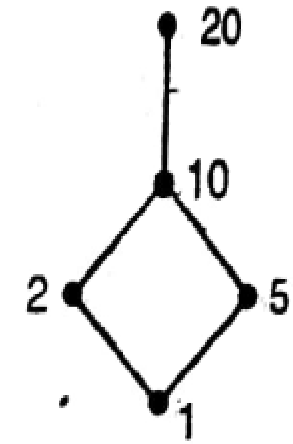
$$M_R = \begin{array}{c|ccccc} & 1 & 2 & 5 & 10 & 20 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 1 \\ 5 & 0 & 0 & 1 & 1 & 1 \\ 10 & 0 & 0 & 0 & 1 & 1 \\ 20 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Step 2 : Remove transitive edges (1, 20), (1, 10), (2, 20)

Step 3 : All edges are pointing upwords. Now replace circles by dots, and remove arrows from edges.



Hasse diagram :



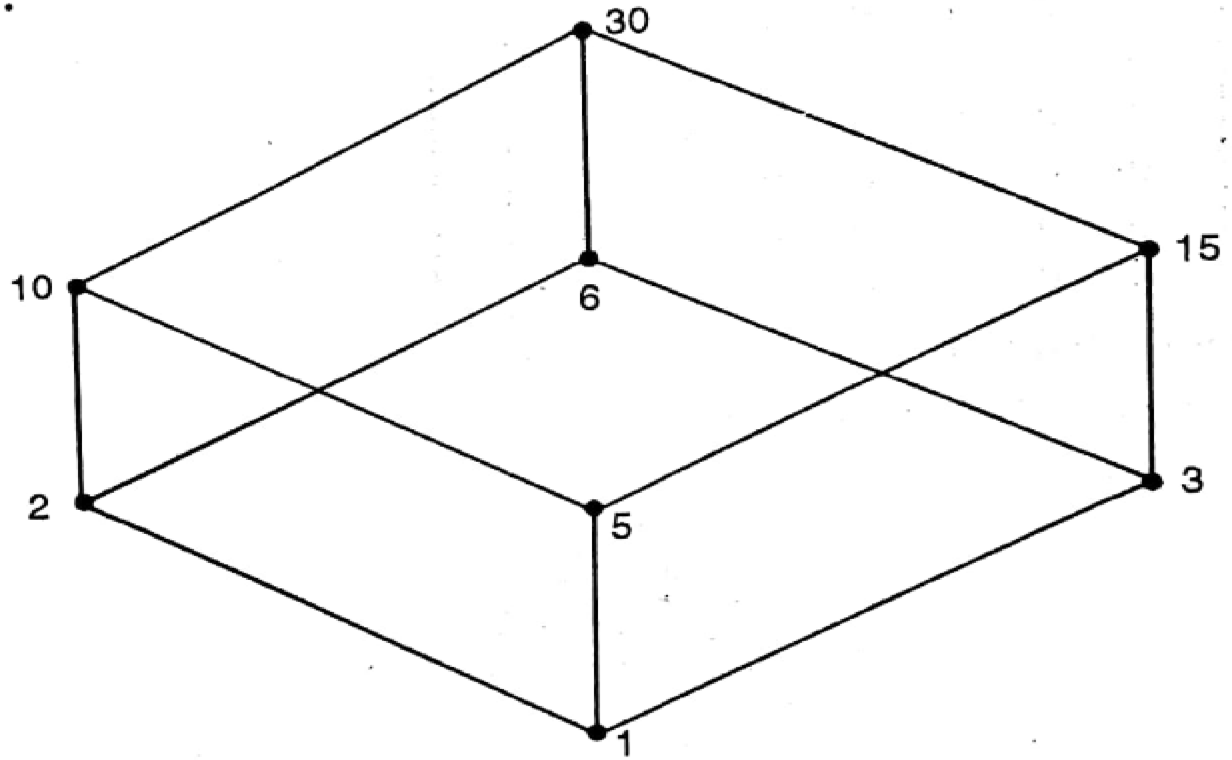
Above poset is not a chain because $2 \not R 5$ or $5 \not R 2$.

Example 4 :

Determine the matrix of the partial order of divisibility on the set A. Draw the Hasse diagram of the poset. Indicate those. which are chains.

(a) A = {1, 2, 3, 5, 6, 10, 15, 30} (b) A = {3, 6, 12, 36, 72} (c) A = {2, 4, 8, 16, 32}

Hasse Diagram :



Above poset is not chain or linearly ordered set, because 5 is not divisible by 3, 6 is not divisible by 5.

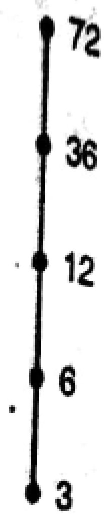
(b) Given set A is {3, 6, 12, 36, 72}

Partial order relation of divisibility on the set A is

$$R = \{(3, 3), (3, 6), (3, 12), (3, 36), (3, 72), (6, 6), (6, 12), (6, 36), (6, 72), (12, 12), (12, 36), (12, 72), (36, 36), (36, 72), (72, 72)\}$$

Matrix for the above relation is

$$M_R = \begin{array}{c} \\ 3 \\ 6 \\ 12 \\ 36 \\ 72 \end{array} \begin{array}{c} \begin{array}{ccccc} 3 & 6 & 12 & 36 & 72 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



Above poset is a chain.

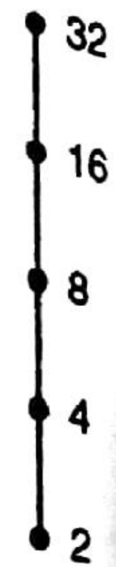
(c) Given set is {2, 4, 8, 16, 32}

Partial order relation of divisibility on the given set A is

$$R = \{(2,2), (4,4), (8,8), (16,16), (32,32), (2,4), (2,8), (2,16), (2,32), (4,8), (4,16), (4,32), (8,16), (8,32), (16,32)\}$$

Matrix for the above relation set is

$$M_R = \begin{array}{c} \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 4 & 8 & 16 & 32 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



Above poset is a chain.

Example 5 :

Draw the Hasse Diagram of the following sets under the partial ordering relation divides and indicate those which are chains. **(May 1997)**

(i) A = {2, 4, 12. 24}

(ii) A = {1. 3, 5. 15, 30}

Solution :

(i) Given set A is {2, 4, 12. 24}

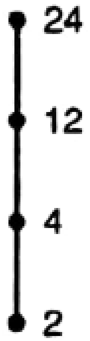
Partial ordering relation of divisibility on set A is as follows

R = {(2, 2), (4. 4), (12. 12). (24, 24). (2. 4). (2. 12). (2. 24). (4. 12). (4. 24), (12, 24)}

Matrix of the above relation is

$$M_R = \begin{array}{c} \\ 2 \\ 4 \\ 12 \\ 24 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 4 & 12 & 24 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right] \end{array}$$

All edges are pointing upwards. Now remove arrows from edges and replace circles by dots.



Above poset is a chain.

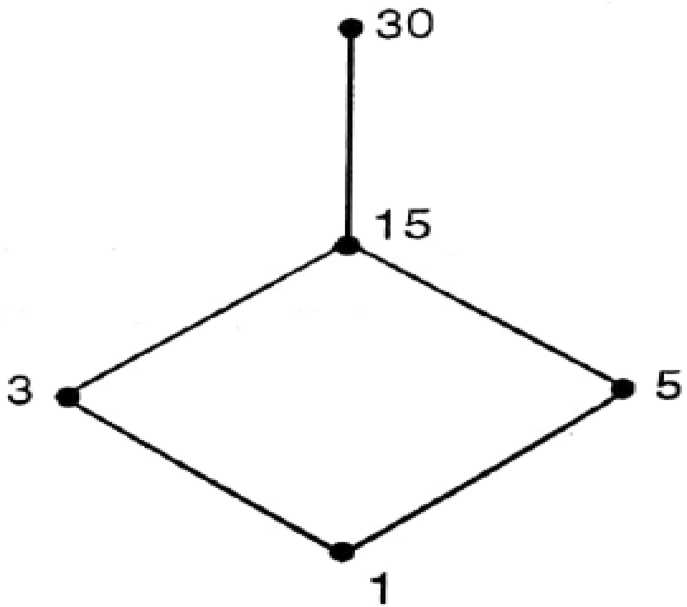
(ii) Given set A is {1, 3, 5, 15, 30}. Partial ordering relation of divisibility on set A is

R = {(1, 1), (3, 5), (5, 5), (15, 15), (30, 30), (1, 5), (1, 15), (1, 30), (3, 15), (3,30), (5, 15), (5,30), (15,30)}

Matrix of the above relation is as follows :

$$M_R = \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 15 \\ 30 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 3 & 5 & 15 & 30 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

All edges are pointing upwards. Now remove arrows from edges, and replace circles by dots.



This poset is not chain because $3 \not R \; 5$

($\because$ 5 is not divisible by 3)

Example 7 :

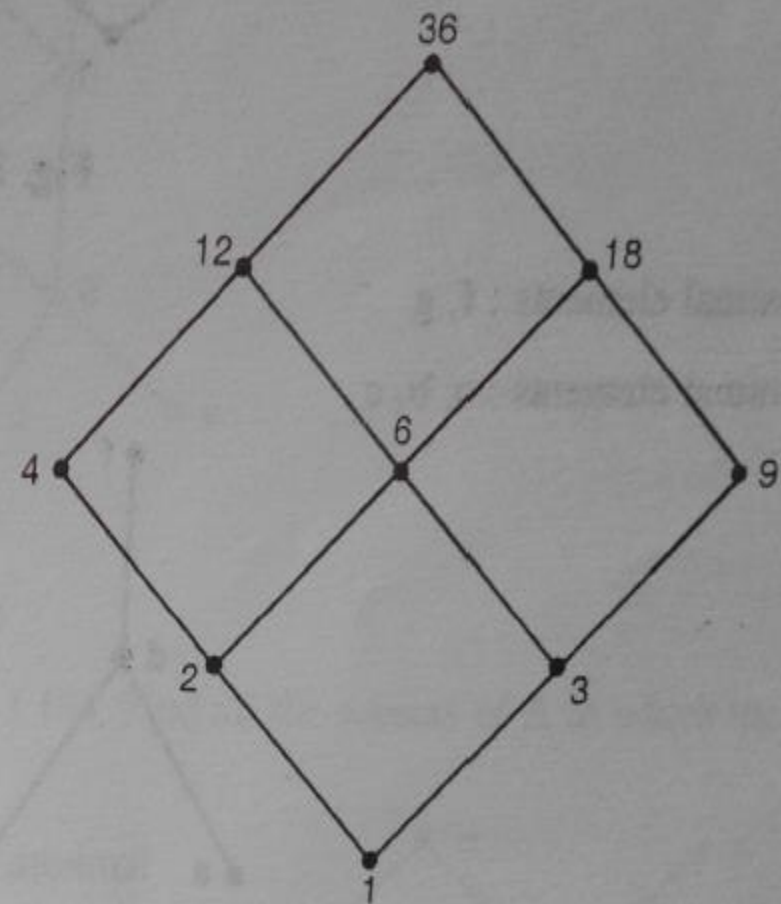
Draw the Hasse diagram of D_{36}.

(Dec. 2006, 5 Marks)

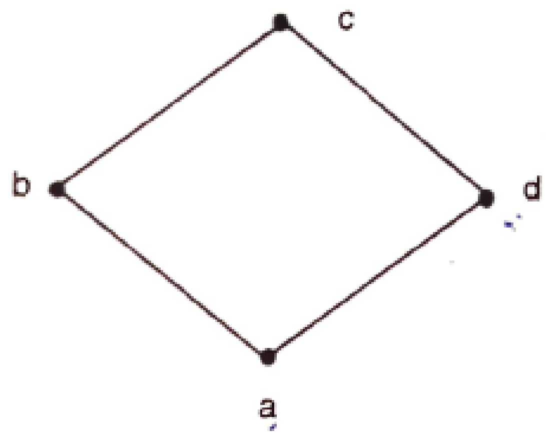
Solution :

Hasse diagram of D_{36}

$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



Example: Determine whether he following Hasse diagram represent lattice or not.



LUB :

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	c	c	d

GLB :

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

As, every subset has a least upper bound and greatest lower bound, it is a lattice.

Example: Determine whether he following Hasse diagram represent lattice or not.

As, every pair of elements has a least upper bound and greatest lower bound, it is a lattice.

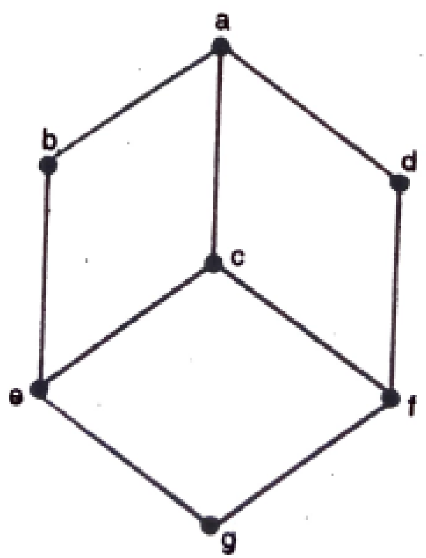
LUB :

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

GLB :

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

Example: Determine whether he following Hasse diagram represent lattice or not.



LUB :

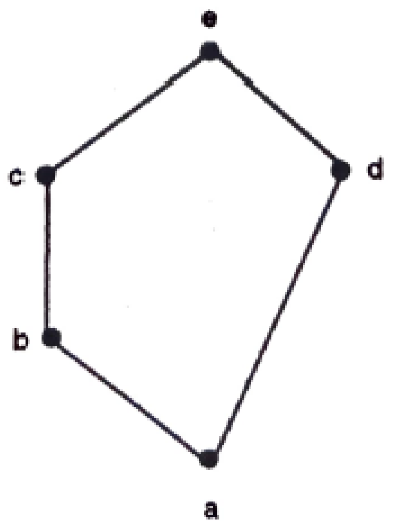
$\vee$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	a	b	a	b
c	a	a	c	a	c	c	c
d	a	a	a	d	a	d	d
e	a	b	c	a	e	c	e
f	a	a	c	d	c	f	f
g	a	b	c	d	e	f	g

GLB :

$\wedge$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	e	g	e	g	g
c	c	e	c	f	e	f	g
d	d	g	f	d	g	f	g
e	e	e	e	g	e	g	g
f	f	g	f	f	g	f	g
g	g	g	g	g	g	g	g

As, each subset of two elements has a LUB & a GLB, it is a lattice.

Example: Determine whether he following Hasse diagram represent lattice or not.



As, every pair of elements has a least upper bound and greatest lower bound, it is a lattice.

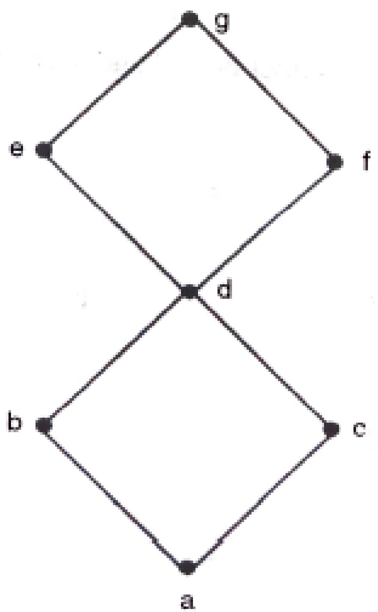
LUB :

∨	a	b	c	d	e
a	a	b	c	d	e
b	b	b	c	e	e
c	c	c	c	e	e
d	d	e	e	d	e
e	e	e	e	e	e

GLB :

∧	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	a	b
c	a	b	c	a	c
d	a	a	a	d	d
e	a	b	c	d	e

Example: Determine whether he following Hasse diagram represent lattice or not.



LUB :

∨	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	c	e	e	e	e	g	g
f	f	f	f	f	g	f	g
g	g	g	g	g	g	g	g

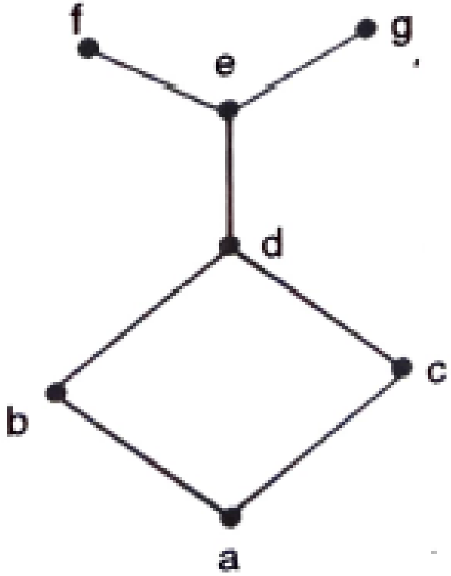
GLB :

∧	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	d	e
f	a	b	c	d	d	f	f
g	a	b	c	d	e	f	g

...ir of elements has a least upper bc

As, every pair of elements has a least upper bound and greatest lower bound, it is a lattice.

Example: Determine whether he following Hasse diagram represent lattice or not.



LUB :

$\vee$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	f	g
f	f	f	f	f	f	f	-
g	g	g	g	g	g	-	g

GLB :

$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	e	e
f	a	b	c	d	e	f	e
g	a	b	c	d	e	e	g

This is not a lattice because $f \vee g$ doesnot exist.

Example: Determine whether he following Hasse diagram represent lattice or not.

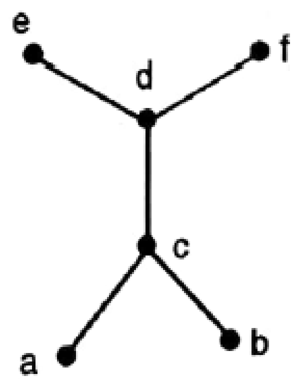


Fig. 3.120

LUB :

$\vee$	a	b	c	d	e	f
a	a	c	c	d	e	f
b	c	b	c	d	e	f
c	c	c	c	d	e	f
d	d	d	d	d	e	f
e	e	e	e	e	e	-
f	f	f	f	f	-	f

GLB :

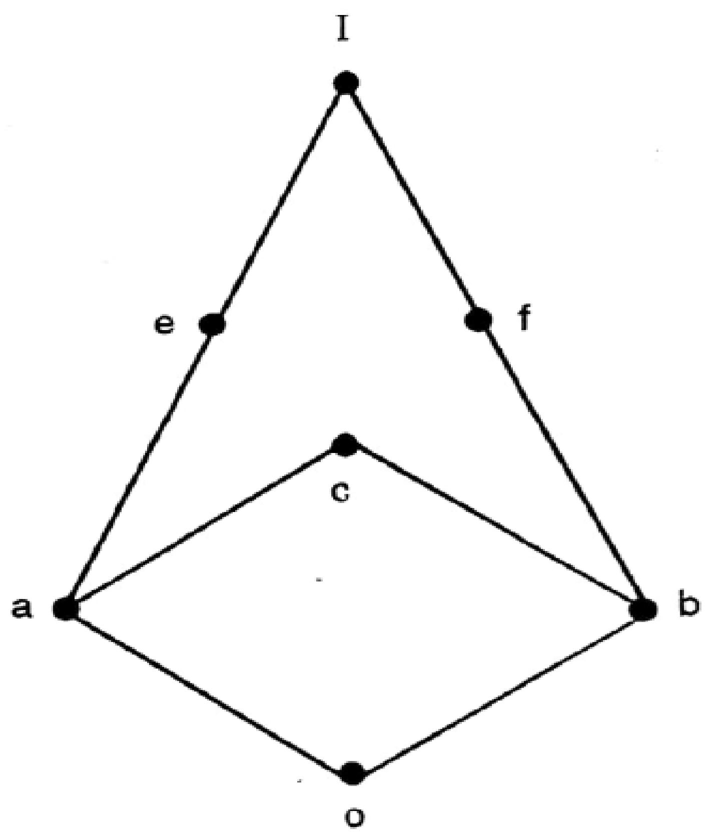
∧	a	b	c	d	e	f
a	a	-	a	a	a	a
b	-	b	b	b	b	b
c	a	b	c	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	d
f	a	b	c	d	d	f

Sublattice :

Let (L, ≤) be the lattice, a nonempty subset S of lattice L is called sublattice of L If a v b belong to S and a ∧ b belong to S whenever a belong to S and b belong to S

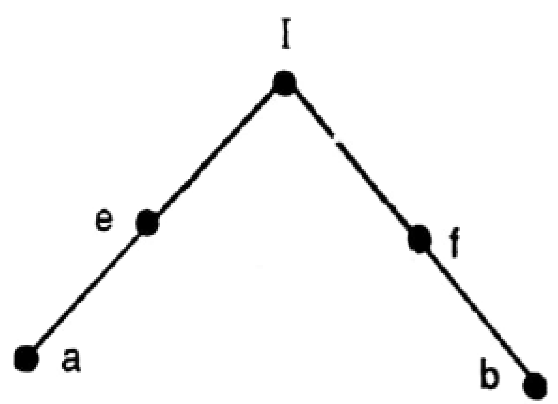
Example 1 :

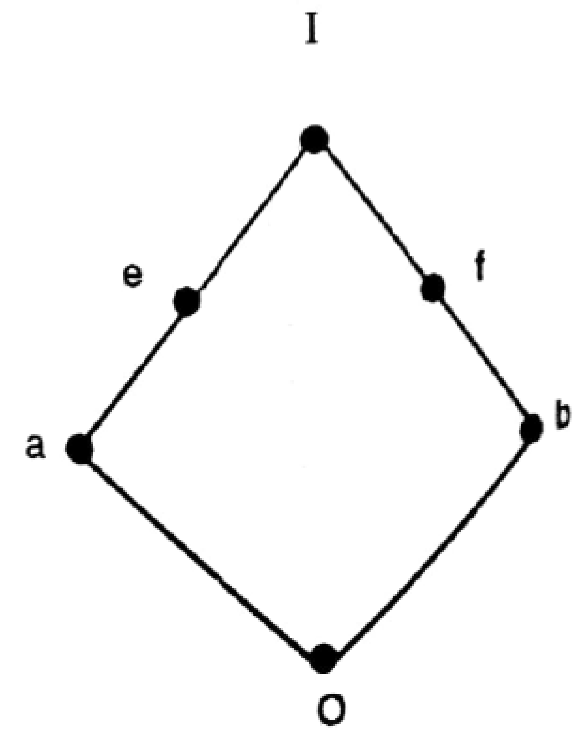
Consider the lattice shown in Fig.



Determine whether or not each of the following Fig. 3.121 (b), (c), (d) is a sublattice of L.

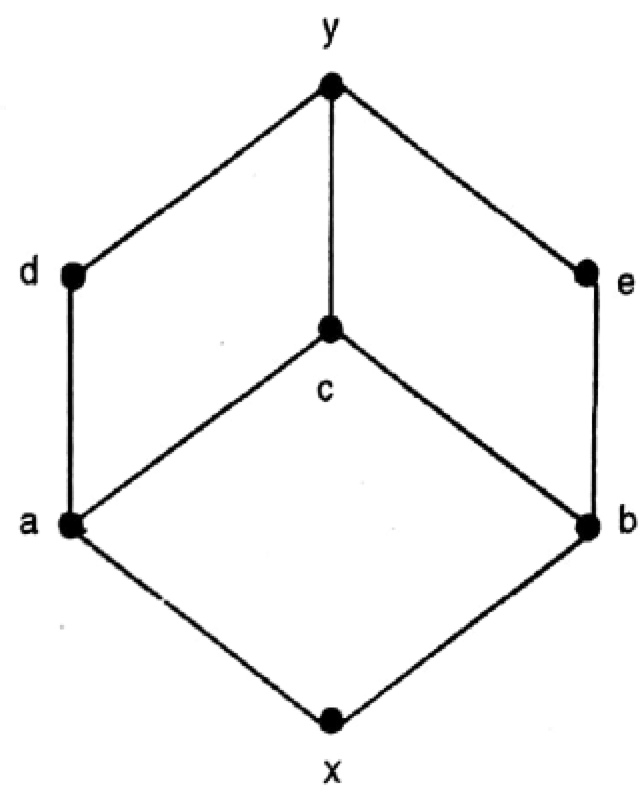
Solution :





The partially ordered subset 'Sb' shown in Fig. 3.121 (b) is not sublattice of L because $a \vee b \notin Sb$ and $a \wedge b \notin Sb$.

The partially ordered subset 'Sc' shown in Fig. 3.121 (c) is not sublattice of L because $a \vee b \notin Sc$. However Sc is a lattice when considered as a poset by istself.



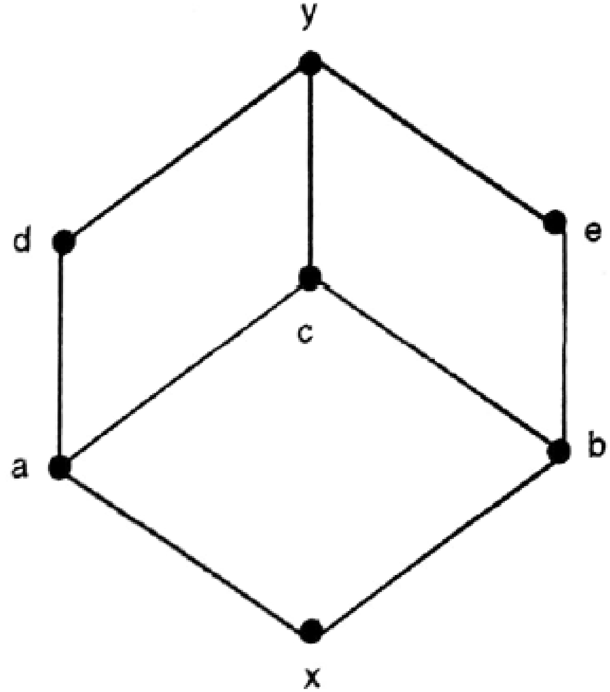
The partially ordered subset 'Sd' in Fig. 3.121 (d) is a sublattice of L because $a \vee b \in$ Sd and $a \wedge b \in$ Sd for every a, b, $\in$ Sd.

Example 2 :

Consider the lattice L in Fig. 3.122. Determine whether or not each of the following is a sublattice of L.

$L_1 = \{x, a, b, y\}$, $L_2 = \{x, a, e, y\}$

$L_3 = \{a, c, d, y\}$, $L_4 = \{x, c, d, y\}$



Solution:

A subset is sublattice if it is closed under ∨ and ∧.

L1 is not sublattice as $a \vee b = c$ which does not belong to L1.

L2 and L3 are sublattices

L4 is not sublattice as $c \wedge d = a$ which does not belong to L4.

Example 3 :

Consider the lattice D = {v, w, x, y, z}. Find all sublattics with three or more elements.

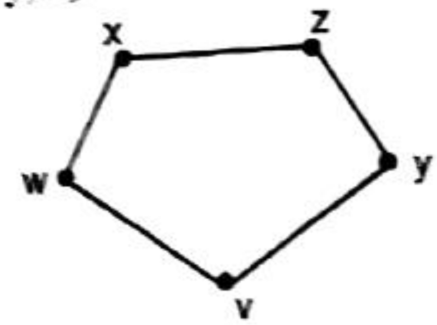


Fig. 3.123

Solution :

We construct all subsets with three or more elements where the least upper bound and the greatest lower bound, belong to the subset for every pair of elements in the subset.

They are :

$L_1 = \{v, w, x\}$ $\quad$ $L_2 = \{v, y, z\}$

$L_3 = \{v, x, z\}$ $\quad$ $L_4 = \{v, w, z\}$

$L_5 = \{w, x, z\}$ $\quad$ $L_6 = \{v, w, y, z\}$

$L_7 = \{v, x, y, z\}$ $\quad$ $L_8 = \{v, w, x, z\}$

$L_9 = \{v, w, x, y, z\}$

Properties of Lattices :

Let L be a lattice then

1. **Idempotent Properties**

   (a) $a \vee a = a$

   (b) $a \wedge a = a$

2. **Commutative Properties**

   (a) $a \vee b = b \vee a$

   (b) $a \wedge b = b \wedge a$

3. **Associative properties**

   (a) $a \vee (b \vee c) = (a \vee b) \vee c$

   (b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4. **Absorption Properties**

   (a) $a \vee (a \wedge b) = a$

   (b) $a \wedge (a \vee b) = a$

Types of Lattices

Bounded Lattice

Definition :

A lattice L is said to be ***bounded*** if it has a greatest element **I** and a least element **0**.

If L is a bounded lattice, then for all $a \in A$

$$\mathbf{0} \leq a \leq \mathbf{I}$$

$a \vee \mathbf{0} = a$, $\quad a \wedge \mathbf{0} = \mathbf{0}$

$a \vee \mathbf{I} = \mathbf{I}$, $\quad a \wedge \mathbf{I} = a$

Examples :

1. The lattice z^+ under the partial order of divisibility is not a bounded lattice since it has a least element, the number 1, but no greatest element.

2. The lattice z under the partial order $\leq$ is not bounded since it has neither a greatest nor a least element.

3. The lattice p(s) of all subsets of a set S, is bounded. Its greatest element is S and its least element is ϕ.

Distributive Lattice

A lattice L is called ***distributive*** if for any elements a, b and c in L we have the following distributive properties.

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

If L is not distributive, we say that L is non-distributive.

Example 1 :

For a set S, the lattice p(S) is distributive, since union and intersection (the join and meet respectively) each satisfy the distributive property.

Example 2 :

$$a \wedge (d \vee c) = a \wedge d = a$$

$$(a \wedge d) \vee (a \wedge c) = a \vee 0 = a$$

$$\therefore \quad a \wedge (d \vee c) = (a \wedge d) \vee (a \wedge c)$$

$$a \wedge (b \vee c) = a \wedge I = a$$

$$(a \wedge b) \vee (a \wedge c) = a \vee 0 = a$$

$$\therefore \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

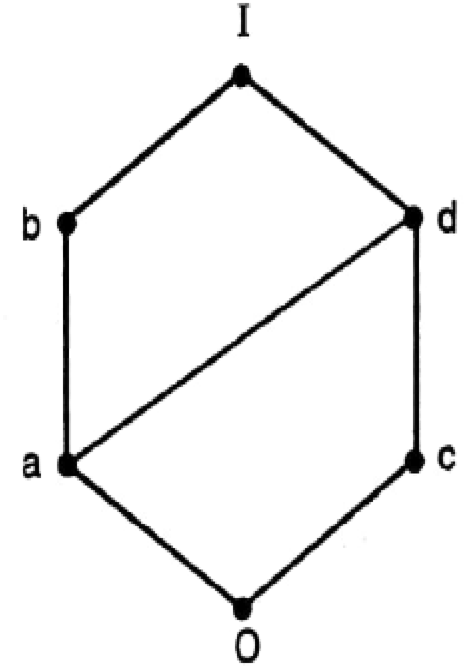
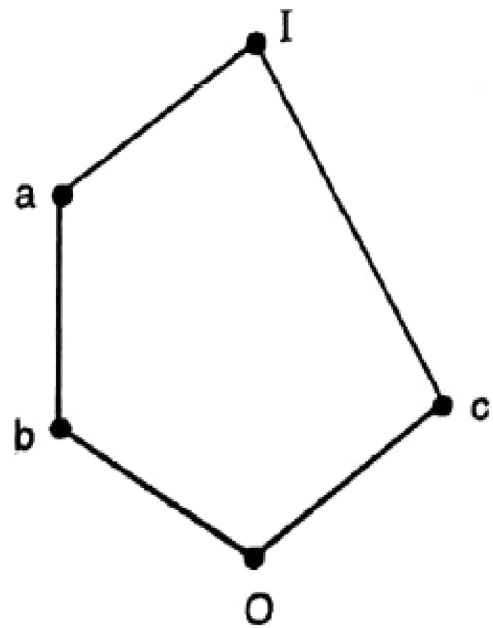
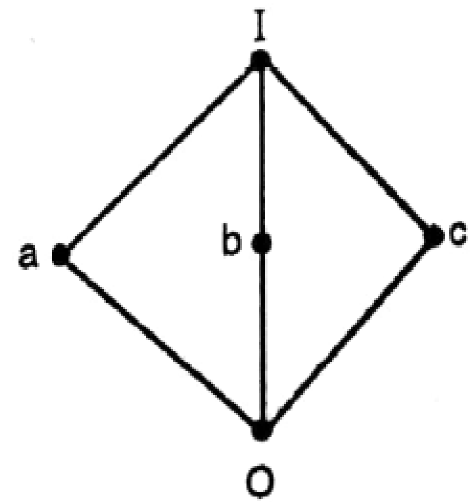


Fig. 3.125

The lattice shown in above Fig. 3.125 is distributive, as can be seen by verifying the distributive properties for all ordered triples chosen from the elements a, b, c and d.

Example 3 :





Show that the lattices pictured in Fig. 3.126 are non-distributive.

Solution :

(a) We have,

$$a \wedge (b \vee c) = a \wedge I = a$$

while $(a \wedge b) \vee (a \wedge c) = b \vee 0 = b$

so Fig. 3.126 (a) is non-distributive

(b) Observe that

$$a \wedge (b \vee c) = a \wedge I = a$$

while $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

So Fig. 3.126 (b) is non-distributive.

Complemented Lattice

Definition :

Let L be a bounded lattice with greatest element I and least element 0, and let $a \in L$. An element $a' \in L$ is called a complement of a if. **(Dec. 99, May 2000)**

$$a \vee a' = I \text{ and } a \wedge a' = 0$$

A Lattice L is called ***complemented*** if it is bounded and if every element in L has a complement.

Example 1 :

Show that this is a complemented lattice.

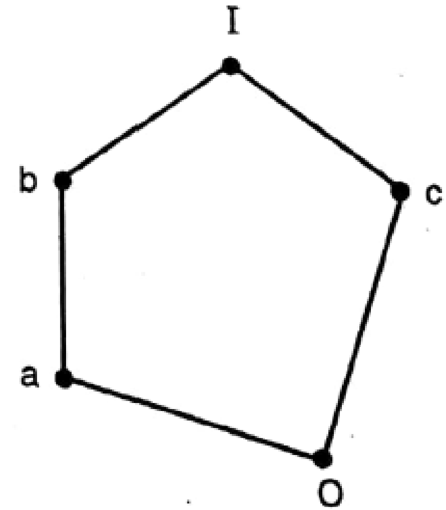


Fig. 3.127

Solution :

$a \vee c = I$

$b \vee c = I$

$a \wedge c = 0$ $\therefore$ complement of a is c

$b \wedge c = 0$ $\therefore$ complement of b is c

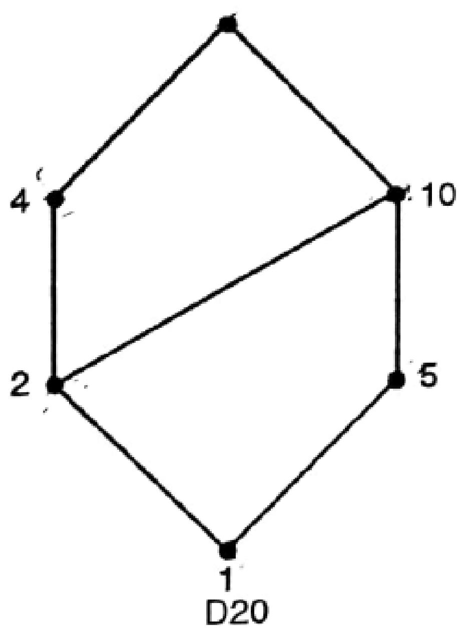
$\therefore$ Complement of c are b and a .

$\therefore$ Every element has a complement

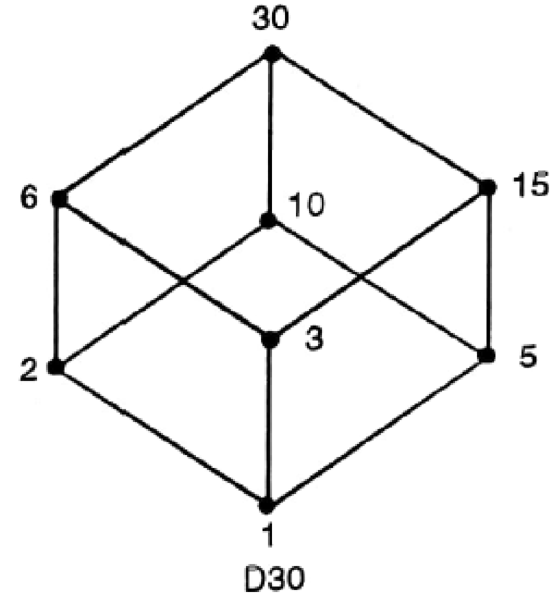
$\therefore$ Above lattice is complemented lattice.

Example 2 :

Let n be a positive integer and let Dn be the set of all positive divisors of n. Then D_n is a lattice under the relation of divisibility. Thus if n = 20, we have D_{20} = {1, 2, 4, 5, 10, 20}. The Hasse diagram of D_{20} is shown in Fig. 3.128(a). If n = 30, we have D_{30} = {1, 2, 3, 5, 6, 10, 15, 30}. The Hasse diagram of D_{30} is shown in Fig. 3.128(b). Determine whether each is complemented lattice.



(a)



(b)

Find the complement of each element in D20 and D30

Solution :

(a) In Fig. 3.128 (a), universal upper bound or greatest element is 20 and universal lower bound or least element is 1.

$$4 \vee 5 = 20 \qquad 4 \wedge 5 = 1$$

$\therefore$ Complement of 4 is 5 and Complement of 5 is 4.

But the elements 2 and 10 in D_{20} have no complements. because.

$$2 \vee 10 = 20 \qquad 2 \wedge 10 = 2$$

$$2 \vee 5 = 10 \qquad 2 \wedge 5 = 1$$

$$2 \vee 4 = 4 \qquad 2 \wedge 4 = 2$$

Similarly

$$10 \vee 5 = 10 \qquad 10 \wedge 5 = 5$$

$$10 \vee 4 = 20 \qquad 10 \wedge 4 = 2$$

$\therefore$ D_{20} is not complemented lattice.

(b) In Fig. 3.128(b) , D_{30}, Universal upper bound or greatest element is 30 and universal lower bound or least element is 1.

$$2 \vee 15 = 30 \qquad 2 \wedge 15 = 1$$

$\therefore$ Complement of 2 is 15

$6 \vee 15 = 30$ $\quad$ $6 \wedge 15 = 1$ $\quad$ $\therefore$ Complement of 6 is 15

$10 \vee 5 = 30$ $\quad$ $10 \wedge 5 = 1$ $\quad$ $\therefore$ Complement of 10 is 5

$3 \vee 10 = 30$ $\quad$ $3 \wedge 10 = 1$ $\quad$ $\therefore$ Complement of 3 is 10

$\therefore$ So every element of D_{30} has a complement.

$\therefore$ D_{30} is a complemented lattice.

Example 3 :

Show that lattice shown in Fig. 3.129 is complemented.

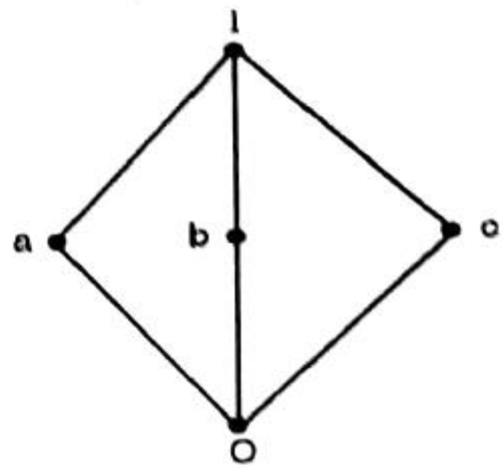


Fig. 3.129

Solution :

$a \vee b = I,$ $\quad a \wedge b = 0$ $\quad \therefore$ Complement of a is b

$\therefore$ Complement of b is a

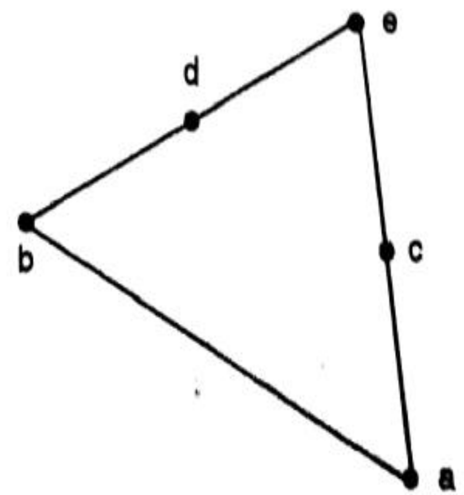
$a \vee c = I,$ $\quad a \wedge c = 0$ $\quad \therefore$ Complement of a is c

$\therefore$ Complement of c is a

$\therefore$ Every element has a complement. So above lattice is complemented.

Example 4 :

Consider the complemented lattice shown in Fig. 3.130. Give the complements of each element.



Solution :

In the given Fig. 3.130 is the greatest element or universal upper bound and a is the least element or universal lower bound.

$$b \vee c = e \qquad b \wedge c = a$$

$$d \vee c = e \qquad d \wedge c = a$$

$\therefore$ Complement of b is c

Complement of d is c

Complement of c are 'b' and 'd'

Example 5 :

Show that in a bounded distributive lattice, if a complement exists, it is unique.

(Dec. 96, May 97, May 98, May 1999, May 2000, May 2001, May 2005, Dec. 2006, May 2007)

Solution :

Let a' and a'' be complements of the element.

$a \in L$. Then

$$a \vee a' = I \qquad a \vee a'' = I$$

$$a \wedge a' = 0 \qquad a \wedge a'' = 0$$

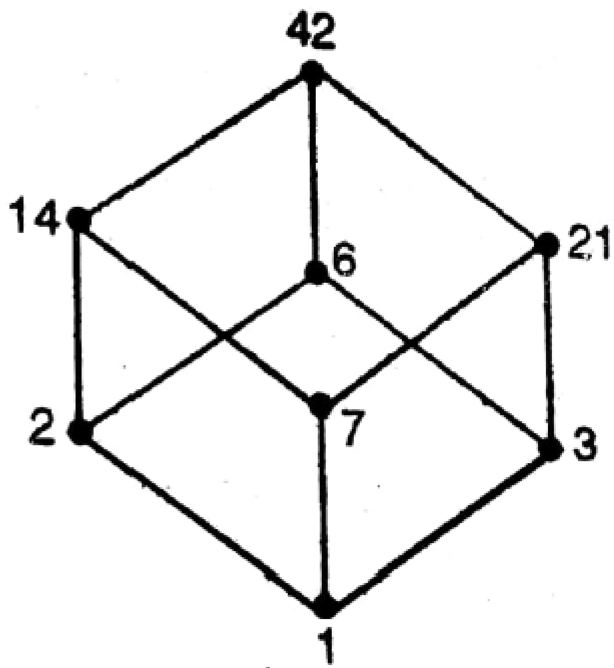
Using the distributive laws, we obtain

$$\begin{aligned} a' = a' \vee 0 &= a' \vee (a \wedge a'') \\ &= (a' \vee a) \wedge (a' \vee a'') \\ &= (a \vee a') \wedge (a' \vee a'') \\ &= I \wedge (a' \vee a'') \\ &= a' \vee a'' \end{aligned}$$

$$
\begin{aligned}
\text{Also,} \qquad a'' &= a'' \vee 0 \\
&= a'' \vee (a \wedge a') \\
&= (a'' \vee a) \wedge (a'' \vee a') \\
&= (a \vee a'') \wedge (a' \vee a'') \\
&= I \wedge (a' \vee a'') \\
&= a' \vee a''
\end{aligned}
$$

$$
\text{Hence,} \qquad a' = a''
$$

Find the complement of each element in D42



In the above Hasse diagram greatest element is 42 and least element is 1.

$1 \vee 42 = 42,$	$1 \wedge 42 = 1$	$\therefore$ Complement of 1 is 42
$42 \vee 1 = 42$	$42 \wedge 1 = 1$	$\therefore$ Complemnt of 42 is 1
$2 \vee 21 = 42$	$2 \wedge 11 = 1$	$\therefore$ Complement of 2 is 21
$21 \vee 2 = 42$	$21 \wedge 2 = 1$	$\therefore$ Complement of 21 is 2
$3 \vee 14 = 42$	$3 \wedge 14 = 1$	$\therefore$ Complement of 3 is 14
$14 \vee 3 = 42$	$14 \wedge 3 = 1$	$\therefore$ Complement of 14 is 3
$7 \vee 6 = 42$	$7 \wedge 6 = 1$	$\therefore$ Complement of 7 is 6
$6 \vee 7 = 42$	$6 \wedge 7 = 1$	$\therefore$ Complement of 6 is 7