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GAMMA FUCTIONS

FE-SEM-I(CBCS)-C-SCHEME

2 THE IMPROPER INTEGRAL $\int_0^\infty e^{-x}x^{n-1}dx$ FOR N > 0 IS CALLED GAMMA FUNCTION AND DENOTED BY Γn

I.E. $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$ OR $\Gamma(N+1) = \int_0^\infty e^{-x}x^{n}dx$

- Properties:
- (i) $\Gamma(n+1) = n\,\Gamma n$,if n positive real number.
- (ii) $\Gamma(n+1) = n!$ if n is positive integer
- (iii) $\Gamma 0 = \infty$
- (iv) $\Gamma 1 = 1$
- (v) $\Gamma 1/2 = \sqrt{\pi}$
- (vi) $\Gamma n = \infty$ if n is negative integer
- (vii) $\Gamma n = \frac{\Gamma(n+1)}{n}$ if n is positive or negative fraction n
- (viii) $\Gamma n = 2\int_0^\infty e^{-x^2}\, x^{2n-1}\, dx$

3 NOTE

- $\Gamma p \Gamma 1 - p = \frac{\Pi}{sinp\Pi}$
- $\Gamma \frac{1}{2} = \sqrt{\Pi}$
- $\Gamma \frac{1}{4} \Gamma \frac{3}{4} = \Pi\sqrt{2}$
- $\Gamma \frac{1}{3} \Gamma \frac{2}{3} = \frac{2\pi}{\sqrt{3}}$
- $\Gamma \frac{1}{6} \Gamma \frac{5}{6} = 2\Pi$

4 TYPES

- I. $\int_0^\infty e^{-kx^n} x^n dx$

$put\ kx^n = \mathrm{t}$

- II. $\int_0^1 x^m (logx)^n dx$ or $\int_0^1 x^m (log\frac{1}{x})^n dx$

put $logx = -t,\ log\frac{1}{x} = t$

- III. $\int_0^\infty \frac{x^a}{b^x} dx$ or $\int_0^\infty a^{-bx^2} dx$
- Put $b^x = e^t, a^{-bx^2} = e^{-t}$

5 EVALUATE $\int_0^\infty x^n e^{-\sqrt{ax}}$ DX

- Solution:
- Step.1: Let $\sqrt{ax} = t$, x = $\frac{t^2}{a}$, dx = $\frac{2t}{a}\, dt$
- Limits of t ?
- Step.2: $\int_0^\infty x^n e^{-\sqrt{ax}}$ dx = $\int_0^\infty (\frac{t^2}{a})^n\, e^{-t}\, \frac{2t}{a}\, dt$
- Step 3: $= \frac{2}{a^{n+1}} \int_0^\infty e^{-t} t^{2n+1}\, dt$
- Step 4: $= \frac{2}{a^{n+1}} \Gamma 2n + 2$ (Answer)

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EVALUATE $\int_0^1 (\log x)^5 \, dx$

SOLUTION:

- $Step$ 1: $Let\ lo\,g\,x = -t\ , x\ =\ e\ -^{t}, dx\ =\ -e\,-^{t}\,dt$
- Limits of t ?
  - Step 2 : $\int_0^1 (log\ x)^5\ dx = \int_\infty^0 (-t)^5(-e^{-t}\ )dt$
  - Step 2: $= -\int_0^\infty e^{-t} t^5\ dt$
  - Step 3: $= -\Gamma 6$
  - Step 4: $= -120$ (Answer)

7 EVALUATE $\int_0^{\infty} 3^{-4x^2}\, dx$

- Step 1: Let $3^{-4x^2} = e^{-t}$, -4 x ^{2} $log3 = -t$, 4 x ^{2} $log3 = t$
- x = $\frac{\sqrt{t}}{2\sqrt{log3}}$; $dx = \frac{1}{2\sqrt{log3}} \frac{1}{2\sqrt{t}} dt$ ....Limits of t ?
- Step2: $\int_0^{\infty} 3^{-4x^2}\, dx$ = $\int_0^{\infty} e^{-t} \frac{1}{4\sqrt{log3}} \frac{1}{\sqrt{t}}\, dt$
- Step.3: $= \frac{1}{4\sqrt{log3}} \int_0^{\infty} e^{-t} t^{-1/2}\, dt$
- Step4 : $= \frac{1}{4\sqrt{log3}} \Gamma 1/2$
- Step5: $= \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$ (Answer)

8 EVALUATE $\int_0^\infty \frac{x^4}{7^x} dx$

SOLUTION: I = $\int_0^\infty \frac{x^4}{7^x} dx$

- Put 7^x = e^t……..x.log7 = t
- x = $\frac{t}{log7}$, dx = $\frac{dt}{log7}$……Limits of t ?
- I = $\int_{t=0}^\infty \frac{(\frac{t}{log7})^4}{e^t} \frac{dt}{log7}$
- = $(\frac{1}{log7})^5 \int_0^\infty e^{-t} t^4 \, dt$
- = $(\frac{1}{log7})^5 \, \Gamma 5$

9 EVALUATE $\int_0^{\infty} \cos(ax^{\frac{1}{n}})\, dx$

S0LUTION : LET I = $\int_0^{\infty} \cos(ax^{\frac{1}{n}})\, dx$

- $put, ax^{\frac{1}{n}} = t,$
- $x^{\frac{1}{n}}$ = t/a , x = $\frac{t^n}{a^n}$,
- dx = n$\frac{t^{n-1}}{a^n}$ dt
- Limits of t ?
- I = $\int_0^{\infty} \cos(ax^{\frac{1}{n}})\, dx$= I = $\int_{t=0}^{\infty} \cos(t)$ n $\frac{t^{n-1}}{a^n}$ dt

I = $\int_{t=0}^{\infty} \text{COS}(t)\, \text{N} \frac{t^{n-1}}{a^n}$ Dt

- I = $\int_{t=0}^{\infty} \cos(t)\, \text{n} \frac{t^{n-1}}{a^n}$ dt
- = $\int_{t=0}^{\infty} \text{Real part of } e^{-it}.\, \text{n} \frac{t^{n-1}}{a^n}$ dt
- = Real part of $\frac{n}{a^n} \int_{t=0}^{\infty} e^{-it}. \frac{t^{n-1}}{1}$ dt.
- Put it = m , i dt = dm
- I= Real part of $\frac{n}{a^n} \int_{m=0}^{\infty} e^{-m}. \frac{m^{n-1}}{i^{n-1} i}$ dt=R.P.of $\frac{n}{a^n}. \frac{\Gamma n}{i^n}$

I= R.P.OF $\frac{n}{a^n}.\frac{\Gamma n}{i^n}$

- R.P.of $\frac{1}{a^n}.\frac{\Gamma n+1}{(cos\frac{\pi}{2}+isin\frac{\pi}{2})^n}$
- = R.P. of $(cosn\frac{\pi}{2}+isinn\frac{\pi}{2}).\frac{\Gamma n+1}{a^n}$
- I= $cosn\frac{\pi}{2}\frac{\Gamma n+1}{a^n}$

12 NOTE

- (a) $\beta(m,n) = \int_0^1 x^{m-1}(1\text{-}x)^{n-1}dx = \int_0^1 t^{m-1}(1\text{-}t)^{n-1}dt$
- (b) $\beta(m,n) = \beta(n,m)$
- (c) $\int_0^1 x^m(1\text{-}x)^n dx = \beta(m+1, n+1)$
- (d) $\beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta \cdot cos^{2n-1}\theta\, d\theta$
- (e) $\int_0^{\pi/2} sin^p\,\theta cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$
- (f) Relation between Gamma Function & Beta Function $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$
- (g) $\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx = \beta(m,n)$
- (h) $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} = \beta(m,n)$

13 $\beta(m,n) = \int_0^1 x^{m-1}$(1-X)$^{n-1}$DX

- Put , x = 1-y, dx = -dy
- When x = 0 ,y= =1
- x=1,y=0
- I = $\int_0^1 x^{m-1}$(1-x)$^{n-1}$dx= $\int_{y=1}^0 (1-y)^{m-1}$(y)$^{n-1}$(-dy)
- = -$\int_{y=1}^0 y^{n-1}$(1−$y)^{m-1}$dy = $\int_{y=0}^1 y^{n-1}$(1−$y)^{m-1}$dy = $\beta(n,m)$

14 $\beta(m,n) = \int_0^1 x^{m-1}$(1-X)$^{n-1} dx$
PUT X = $sin^2\theta\ then$

- $\beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta \cdot cos^{2n-1}\theta\ d\theta$
- Also , put 2m-1 = p , 2n – 1 = q
- We get,
- $\int_0^{\pi/2} sin^p\,\theta cos^q\theta\ d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

15 TYPES

- Following types of integrals can be transformed in to beta functions
- Type 1: $\int_0^a x^m \ (a^n\text{-}x^n)^p \ dx$
- Type 2: $\int_a^b (x-a)^m$ (b-x)^n dx
- Type 3: Using $\int_0^{\pi/2} sin^p \ \theta \ cos^q \ \theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$
- Type 4: Using $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}$dx = $\beta(m,n) \ and \ \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}$dx = $\beta(m,n)$