Linear differential equations with constant coefficients
and
variable coefficient of Higher order

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Module 2[Engineering Mathematics – II]

Prof.Vaishali A.Bokade

Linear differential equations with constant coefficients

- **TOPICS**

- Linear Differential Equations
- Method of variation of parameters
- Cauchy’s linear Equations
- Legendre’s linear Equations

Module at a Glance

An equation of the form

$$D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + \cdots \ldots \ldots + P_n y = X$$

i.e. f(D)y=X

Solution of f(D)y =X

General Solution = Complementary function + Particular Integral

When X = 0

General Solution = Complementary function

To solve f(D)y=0,

General Solution is $y = C_1e^{m_1x} + C_2e^{m_2x} + \cdots \dots \dots \dots \dots . +C_ne^{m_nx}$,

Here $m_1, m_2, m_3 \dots \dots \dots \dots \dots \dots . . m_n$ are roots of auxiliary equation f(D)=0

A)If all roots of the auxiliary equation are real and distinct,

then,

Complementary function $=C_1e^{m_1x} + C_2e^{m_2x} + \cdots \dots \dots \dots \dots . +C_ne^{m_nx}$

B)If the roots of the auxiliary equation are repeated, then

$y = (C_1 + C_2x)\, e^{m_1x}$

$y = (C_1 + C_2x + C_3x^2)\, e^{m_1x}$, and so on.

Roots ofAuxillary equqtions

C)If the roots of the auxiliary equation are imaginary, then$\propto \pm i\ \beta$

$$y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$$

D)If the imaginary roots are repeated,

$$y = e^{\alpha x}[(C_1 + C_2x)\cos\beta x + (C_3 + C_4x)\sin\beta x$$

e) If the roots of the auxiliary equation are irrational then$\propto \pm\sqrt{\beta}$

y= $e^{\alpha x}(C_1\cosh\sqrt{\beta}x + C_2\sinh\sqrt{\beta}x)$

F)If the irrational roots are repeated then

$$y = e^{\alpha x}[(C_1 + C_2x)\cosh\sqrt{\beta}x + (C_3 + C_4x)\sinh\sqrt{\beta}x)]$$

Q.Solve $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 0$

Solution : Given D.E. is $(D^3 - 4D)y = 0$

- F(D)y=0
- F(D) = $(D^3 - 4D)$
- A.E. for given D.E. is,
- $(m^3 - 4m)$=0
- m$(m^2 - 4)$=0
- m(m-2)(m+2)=0
- m=0,2,-2
- Complementary function = C.F. = $(c_1e^{0x} + c_2e^{2x} + c_3e^{-2x})$

Q. Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

Solution : Given D. E is $(D^2+4D + 4)y = 0$

f(D)y = 0

- f(D)= $(D^2+4D + 4)$
- Auxillary equation is $(m^2+4m + 4)$=0
- $(m + 2)^2 = 0$
- M = -2,-2
- Complementary function = C.F. = $(c_1 + c_2\mathrm{x})e^{-2x}$
- General Solution = C.F. = $(c_1 + c_2\mathrm{x})e^{-2x}$

Q.Solve $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x - \cos 2x$

Solution : Given D.E. is F(D)y = X/Q

- $F(D) = D^2 + 2$
- A.E.is $(m^2 + 2) = 0.$
- m = $\pm\sqrt{2}$ i=0 $\pm\sqrt{2}$ i
- C.F.is y=$e^{0x}(C_1 \cos\sqrt{2}x + C_1 \sin\sqrt{2}x)$

Q. Solve $\frac{d^4y}{dx^4} + K^4y = 0$

Given D.E. is $(D^4 + k^{4\cdot})y = 0$
where f(D) = $(D^4 + k^4\ )$

- A.E.$(\ m^4 + K^4) = 0$
- .$(\ m^4 + K^4 + 2m^2K^2 - 2m^2K^2) = 0$
- $(m^2 + K^2)^2 - (\sqrt{2}.mK)^2 = 0$
- $(m^2 + K^2 + \sqrt{2}.mK)(m^2 + K^2 - \sqrt{2}.mK) = 0$
- m= $\frac{-k}{\sqrt{2}} \pm \frac{ki}{\sqrt{2}}, \frac{k}{\sqrt{2}} \pm \frac{ki}{\sqrt{2}}$
- The Complementary function is
- General Solution, y= $e^{\frac{-k}{\sqrt{2}}x}\left(C_1 cos\frac{k}{\sqrt{2}}x + C_2 sin\frac{k}{\sqrt{2}}x\right) + e^{\frac{k}{\sqrt{2}}x}\left(C_1 cos\frac{k}{\sqrt{2}}x + C_2 sin\frac{k}{\sqrt{2}}x\right)$

Q. Solve $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

Solution:Given D.E. is $(D^4 + 8D^{2\cdot} + 16)y = 0$

where f(D) = $(D^4 + 8D^{2\cdot} + 16)$

- $(m^4 + 8m^{2\cdot} + 16) = 0$
- $(m^2 + 4)^2 = 0$
- $m^2 = -4 \,, m^2 = -4$
- m = $0 \pm 2i \,, 0 \pm 2i$
- The Complementary function is
- General Solution, y= $e^{0x}\left((C_1 + C_2x)cos2x + (C_3 + C_4x)sin2x\right)$

Particular Integral,

$$P.I. = \frac{1}{f(D)} X$$

$X = e^{ax}$

$= \cos ax \,/\, \sin ax$

$= e^{ax} . V$

$= x^m$

$= x.V$

Note : 1) $\frac{1}{D} X = \int X\, dx$

2) $\frac{1}{D-a} X = e^{ax} \int e^{-ax} X\, dx$

3) $\frac{1}{D+a} X = e^{-ax} \int e^{ax} X\, dx$

1]. If $X = e^{ax}$ then

$$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$$

- a)If $(D-a)$is factor of $f(D)$then $\frac{1}{f(D)}e^{ax} = x.\frac{1}{\varphi(a)}e^{ax}$
- b) If $(D-a)$is factor of $f(D)$repeated r times , then $\frac{1}{f(D)}e^{ax} = \frac{x^r}{r!}.\frac{1}{\varphi(a)}e^{ax}$
- Exa. Solve $(D^4-1)y = e^x$
- Solution : Given D.E. is $(D^4-1)y = e^x$
- F(D) = (D^4-1)
- Auxillary Equation is , (m^4-1)=0

(m^4-1)=0

$(m^2-1)\ (m^2+1)$=0

(m-1)(m+1)(m+i)(m-i)=0

- M = 1,-1,i,-I

The Complementary function is y= $C_1\ e^{1x}$+ $C_2\ e^{-1x}$+

- $e^{0x}\ (C_3 cosx + C_4 sinx)$

- Particular Integral,

  P.I.= $\frac{1}{\mathrm{f(D)}}$X ,X = e^{1x},f(D)= (D^4-1)

- = $\frac{1}{(D^4-1)}\ e^{1x}$= $\frac{1}{(1^4-1)}\ e^{1x}$.......................................N0

- = x$\frac{1}{(4D^3-1)}\ e^{1x}$----------------- $\frac{1}{\mathrm{f(D)}}\mathrm{e}^{\mathrm{ax}} = \mathrm{x}.\frac{1}{\mathrm{f'(a)}}\mathrm{e}^{\mathrm{ax}}$

- = x$\frac{1}{(4 1^3-1)}\ e^{1x}$ = x$\frac{1}{(3)}\ e^{x}$

- Complete Solution = C.F. +P.I.

- = $C_1\ e^{1x}$+ $C_2\ e^{-1x}$+ $e^{0x}\ (C_3 cosx + C_4 sinx)$+ $\frac{\mathrm{X}e^{x}}{(3)}$

Q.Solve $6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = e^{-\frac{3}{2}x} + 2^x$

Solution:Given D.E. is $(6D^2 + 17D + 12)y = e^{-\frac{3}{2}x} + 2^x$

- A.E. is $(6m^2 + 17m + 12)$=0
- $(6m^2 + 9m + 8m + 12)$=0
- 3m(2m+3)+4(2m+3)=0
- (3m+4)(2m+3)=0
- M= $-\frac{4}{3}, -\frac{3}{2}$
- The Complementary function is y= $C_1 e^{-\frac{4}{3}x} + C_2 e^{-\frac{3}{2}x}$
- Particular Integral,

  P.I.= $\frac{1}{\mathrm{f(D)}}$X ,X = $e^{-\frac{3}{2}x} + e^{log2^x}$,f(D)= $(6D^2 + 17D + 12)$
- X = $e^{-\frac{3}{2}x} + e^{x(log2)}$.

$$\text{P.I.} = \frac{1}{f(D)} X = \frac{1}{(6\,D^2+17D+12)}\, e^{-\frac{3}{2}x} + e^{x(\log 2)}$$

- $= \frac{1}{(3D+4)(2D+3)}\, e^{-\frac{3}{2}x} + \frac{1}{(3D+4)(2D+3)}\, e^{x(\log 2)}$

$$= \frac{1}{(3(-\frac{3}{2})+4)(2D+3)}\, e^{-\frac{3}{2}x} + \frac{1}{(6\,(\log 2)^2+17(\log 2)+12)}\, e^{x(\log 2)}$$

$$= \frac{1}{(-\frac{1}{2})(2D+3)}\, e^{-\frac{3}{2}x} + \frac{1}{(6\,(\log 2)^2+17(\log 2)+12)}\, e^{x(\log 2)}$$

$$= x\frac{1}{(-\frac{1}{2})(2)}\, e^{-\frac{3}{2}x} + \frac{1}{(6\,(\log 2)^2+17(\log 2)+12)}\, e^{x(\log 2)} = -x\, e^{-\frac{3}{2}x} + \frac{1}{(6\,(\log 2)^2+17(\log 2)+12)}\, 2^x$$

Complete Solution = C.F. +P.I.

$$= C_1\, e^{-\frac{4}{3}x} + C_2\, e^{-\frac{3}{2}x} - x\, e^{-\frac{3}{2}x} + \frac{1}{(6\,(\log 2)^2+17(\log 2)+12)}\, 2^x$$

2]. If $X = \sin(ax+b)$ then

$\frac{1}{f(D)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)$.....replace each $D^2$by $-a^2$ from f(D)

Also, If $X = \cos(ax+b)$ then

$\frac{1}{f(D)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b)$.....replace each $D^2$by $-a^2$ from f(D)

Q. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y$ =sinx

Solution Given D.E. is $(D^2 + D - 2)y$ =sinx

A.E. is $(m^2 + m - 2)y$ =0

(m+2)(m-1)=0

M = -2,1

- The Complementary function is y= $C_1\, e^{-2x} + C_2\, e^{x}$
- Particular Integral,

  P.I.= $\frac{1}{f(D)}X$,X=sinx,f(D)= $(D^2 + D - 2)$

$$\text{P.I.} = \frac{1}{f(D)} X \text{ ,X=sinx,f(D)= } (D^2 + D - 2)$$

$$= \frac{1}{(D^2+D-2)} \sin(1)X$$

- D^2 =- 1^2= -1
- P.I. $= \frac{1}{(-1+D-2)} \sin(1)X$
- $= \frac{1}{(D-3)} \sin(1)X = \frac{D+3}{(D^2-9)} \sin(1)X$-------------Replace D^2 by - 1^2=-1
- $= \frac{D+3}{(-10)} \sin(1)X$
- $= \frac{1}{(-10)}$ [cosx+3sinx]…………….Dsinx = cosx
- Complete Solution = C.F. +P.I.
- = $C_1\ e^{-2x}$+ $C_2\ e^{x}$- $\frac{1}{(10)}$ [cosx+3sinx]

Q.Solve $\frac{d^2y}{dx^2} + 2y = e^{3x} + e^x - \cos 2x$

Solution : Given D.E. is F(D)y = X/Q

- $F(D) = D^2 + 2$
- A.E.is $(m^2 + 2) = 0.$
- m = $\pm\sqrt{2}$ i=0 $\pm\sqrt{2}$ i
- C.F.is y=$e^{0x}(C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x)$
- P.I.= $\frac{1}{\mathrm{f(D)}}$X ,X= $e^{3x} + e^x - \cos 2x$,f(D)= ($D^2 + 2$)

  =$\frac{1}{(D^2+2)}[e^{3x}+e^x - \cos 2x]$

$$\text{P.I.}=\frac{1}{(D^2+2)}[e^{3x}+e^{x}-\cos 2x]$$

$$=\frac{1}{(D^2+2)}e^{3x}+\frac{1}{(D^2+2)}e^{x}-\frac{1}{(D^2+2)}\cos 2x$$

- $\frac{1}{\mathrm{f(D)}}\mathrm{e^{ax}}=\frac{1}{\mathrm{f(a)}}\mathrm{e^{ax}}$ and $\frac{1}{\mathrm{f(D)}}\cos(\mathrm{ax+b})=\frac{1}{\mathrm{f}(-a^2)}\cos(\mathrm{ax+b})$
- P.I. = $\frac{1}{(3^2+2)}e^{3x}$[replace D by 3] $+\frac{1}{(1^2+2)}e^{x}$ [replace D by 1]
- $-\frac{1}{(-2^2+2)}\cos 2x$[[replace $D^2$by by -2^2 i.e. -4]

P.I. = $\frac{1}{(11)}e^{3x}+\frac{1}{(3)}e^{x}-\frac{1}{(-2)}\cos 2x$

Complete Solution = C.F. +P.I.

$$= e^{0x}(C_1\cos\sqrt{2}x+C_2\sin\sqrt{2}x)+\frac{1}{(11)}e^{3x}+\frac{1}{(3)}e^{x}-\frac{1}{(-2)}\cos 2x$$

$$= (C_1\cos\sqrt{2}x+C_2\sin\sqrt{2}x)+\frac{1}{(11)}e^{3x}+\frac{1}{(3)}e^{x}+\frac{1}{2}\cos 2x$$

Q. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = cos^2x$

Solution : Solution Given D.E. is $(D^2 + 2D + 1)y$ = cos^2x

A.E. is $(m^2 + 2m + 1)y$ =0

- $(m + 1)^2 = 0$
- M = -1,-1
- The Complementary function is y= $(C_1 + C_2x)\ e^{-x}$
- Particular Integral,

  P.I.= $\frac{1}{\text{f(D)}}$X ,X=cos^2x,f(D)= ($D^2 + 2D + 1$)

- X=cos^2x=$\frac{(1+cos2x)}{2} = \frac{1}{2}e^{0x} + \frac{1}{2}cos2x$

P.I.= $\frac{1}{\text{f(D)}}\text{X} = \frac{1}{(\ D^2+2D+1)}\frac{1}{2}e^{0x} + \frac{1}{2}\frac{1}{(\ D^2+2D+1)}\ cos2x$

P.I.= $\frac{1}{\text{f(D)}}\text{X} = \frac{1}{(\ D^2+2D+1)}\frac{1}{2}e^{0x}(replace\ D\ by\ 0) + \frac{1}{2}\frac{1}{(\ D^2+2D+1)}\ cos2x(replace\ D^2$ by $-2^2)$

$$\text{P.I.}=\frac{1}{f(D)}X=\frac{1}{(D^2+2D+1)}\frac{1}{2}e^{0x}(replace\ D\ by\ 0)+\frac{1}{2}\frac{1}{(D^2+2D+1)}cos2x(replace\ D^2\ \text{by}\ -2^2)$$

$$\text{P.I.}=\frac{1}{(0+1)}\frac{1}{2}e^{0x}+\frac{1}{2}\frac{1}{(-4+2D+1)}\ cos2x$$

$$=\frac{1}{2}+\frac{1}{2}\frac{1}{(2D-3)}\ cos2x$$

- $=\frac{1}{2}+\frac{1}{2}\frac{2D+3}{(4D^2-9)}\ cos2x(replace\ D^2$ by $-2^2)$

$$=\frac{1}{2}+\frac{1}{2}\frac{2D+3}{(-16-9)}\ cos2x$$

- $=\frac{1}{2}+\frac{2D+3}{(-50)}\ cos2x=\frac{1}{2}+\frac{1}{(-50)}\ (2D+3)cos2x$

$$=\frac{1}{2}+\frac{1}{(-50)}\ (2(-2\sin2x)+3cos2x)$$

- P.I.$=\frac{1}{2}+\frac{1}{(-50)}$ (-4sin2x+$6cos2x$)$=\frac{1}{2}+\frac{1}{(25)}$ (2sin2x-$3cos2x$)
- Complete Solution = C.F. +P.I.
- $=(C_1+C_2x)\ e^{-x}+\frac{1}{2}+\frac{1}{(25)}$ (2sin2x-$3cos2x$)

NOTE :$\frac{1}{f(D)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)$.....replace each $D^2$by $-a^2$ from f(D)
If f($-a^2$)=0 then differentiate f(D)
$x\frac{1}{f'(D)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b)$

- If f′($-a^2$)=0 then again differentiate f'(D)
- $x.x\frac{1}{f''(D)}\sin(ax+b) = x^2\frac{1}{f''(-a^2)}\sin(ax+b)$
- Same rules for X= cos(ax+b)
- Exa. P.I= $\frac{1}{D^2+9}sin3x = \frac{1}{-3^2+9}sin3x$..........................NO
- = x. $\frac{1}{2D}sin3x = x\,\frac{1}{D}$sin3x
- = x$\frac{1}{2}\frac{-c0s3x}{3} = \frac{-xcos3x}{6}$

We have $\frac{1}{1-x} = 1 + x + x^2 + x^3 ++++ +$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 ++++ +$$

- 3]P.I. = $\frac{1}{f(D)}\, x^n$
- F(D)= $D^m[1 + \emptyset(D)]\ \ OR\ D^m[1 - \emptyset(D)]$
- P.I. = $\frac{1}{f(D)}\, x^n$
- $= \frac{1}{D^m[1+\emptyset(D)]}\, x^n = \frac{1}{D^m}\frac{1}{[1+\emptyset(D)]}\, x^n$
- $= \frac{1}{D^m}[1\text{-}\emptyset(D) + (\emptyset(D))^2 - \left(\emptyset(D)\right)^3 + \cdots \ldots \ldots \ldots \ldots .]x^n$
- OR P.I. = $\frac{1}{f(D)}\, x^n$
- $= \frac{1}{D^m[1-\emptyset(D)]}\, x^n = \frac{1}{D^m}\frac{1}{[1-\emptyset(D)]}\, x^n$
- $= \frac{1}{D^m}[1+\emptyset(D) + (\emptyset(D))^2 + \left(\emptyset(D)\right)^3 + \cdots \ldots \ldots \ldots \ldots .]x^n$

Q.Solve Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3x + 1$

Solution : Solution Given D.E. is $(D^2 - 3D + 2)y$ = 3x+1

A.E. is $(m^2 - 3m + 2)$ =0

- M =2,1
- The Complementary function is y= $(C_1e^{2x} + C_2e^{x})$
- Particular Integral,

  P.I.= $\frac{1}{\mathrm{f(D)}}$X ,X=$3\mathrm{x} + e^{0x}$ (D)= $(\ D^2 - 3D + 2)$
- P.I.= $\frac{1}{(\ D^2-3D+2)}$3x+ $\frac{1}{(\ D^2-3D+2)}\ e^{0x}$
- = 3 $\frac{1}{(2-3\mathrm{D}+\ D^2)}$ x + $\frac{1}{(0-0+2)}\ e^{0x}$
- = $\frac{3}{2(1-(\frac{3D-D^2}{2}))}$ x + $\frac{1}{(0-0+2)}\ e^{0x}$
- = $\frac{3}{2(1-(\frac{3D-D^2}{2}))}$x + $\frac{1}{(2)}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 ++++ +$$

$[\frac{3}{2(1-(\frac{3D-D^2}{2}))}]$x= $\frac{3}{2}$[1+ $(\frac{3D-D^2}{2})$+ $(\frac{3D-D^2}{2})^2$+ $(\frac{3D-D^2}{2})^3$++++]x

- But Dx = 1 , D^2x= D^3x= D^4x=-------------=0
- $[\frac{1}{(1-(\frac{3D-D^2}{2}))}]$x= [1+ $(\frac{3D-D^2}{2})$+ $(\frac{3D-D^2}{2})^2$+ $(\frac{3D-D^2}{2})^3$++++]x
- = x+ $\frac{3}{2}(1) + 0 + 0 + \cdots \ldots \ldots \ldots ..$
- P.I.= x+ $\frac{3}{2}$+ $\frac{1}{2}$
- Complete Solution = C.F. +P.I.
- = $(C_1 e^{2x}$+ $C_2 e^x)$+ x+1

Exa. Solve D.E. $(D^2+2D+2)y=x^2+1$
Solution : Solution Given D.E. is $(D^2+2D+2)y=x^2+1$
A.E. is $(m^2+2m+2)=0$

- a=1,b=2,c=2
- M=-1$\pm i$
- C.F.=$e^{-x}(C_1\cos x+C_2\sin x)$
- Particular Integral,
  P.I.= $\frac{1}{\mathrm{f(D)}}$X ,X=x^2+1 f(D)= ($D^2+2D+2)$
- P.I.= $\frac{1}{(D^2+2D+2)}x^2+\frac{1}{(\ D^2-3D+2)}\ e^{0x}=\frac{1}{(2+2D+D^2)}x^2+\frac{1}{2}$
- P.I. = $\frac{1}{2}\frac{1}{(1+(\frac{2D+D^2}{2}))}\ x^2+\frac{1}{2}$
- $\frac{1}{1+x}=1-x+x^2-x^3++++++$

$$\frac{1}{(1+(\frac{2D+D^2}{2}))} x^2 = [1- (\frac{2D+D^2}{2}) + (\frac{2D+D^2}{2})^2 - (\frac{2D+D^2}{2})^3 ++++] x^2$$

But D x^2 = 2x , $D^2\ x^2 = 2, D^3\ x^2 = D^4\ x^2$ =-------------=0

- $\frac{1}{(1+(\frac{2D+D^2}{2}))} x^2 = x^2 - (\frac{2(2x)+2}{2}) + \frac{4[D^2\ x^2]}{4} - 0 + 0 - 0 +++++++ +$
- $= x^2$-(2x+1)+[2]= x^2-2x+1
- P.I. = $\frac{1}{2} \frac{1}{(1+(\frac{2D+D^2}{2}))} x^2 + \frac{1}{2} = \frac{1}{2} [x^2\text{-}2x+1] + \frac{1}{2}$
- Complete Solution = C.F. +P.I.
- $= e^{-x}(C_1 \cos x + C_2 \sin x) + \frac{1}{2} [x^2\text{-}2x+2]$

4] X $= e^{ax}.V\ , V = V(x)$

$$then\ \frac{1}{f(D)}\, e^{ax}.V = e^{ax}\ \frac{1}{f(D+a)} V$$

- Exa.Solve $(D^2 - 3D + 2)y = x^2e^{2x}$
- Solution: Given D.E. is $(D^2 - 3D + 2)y = x^2e^{2x}$
  A.E. is $(m^2 - 3m + 2)$ =0
- M=1,2
- The Complementary function is y= $(C_1e^{2x} + C_2e^{x})$
- Particular Integral,
  P.I.= $\frac{1}{f(D)}X$,X=x^2e^{2x} f(D)= $(\ D^2 - 3D + 2)$
- P.I.= $\frac{1}{(\ D^2-3D+2)}\ e^{2x}x^2 = e^{2x}\ \frac{1}{(\ (D+2)^2-3(D+2)+2}\ x^2$
- $= e^{2x}\ \frac{1}{(\ (D)^2+D)}\ x^2 = e^{2x}\ \frac{1}{D\ (1+D)}\ x^2$

P.I= $e^{2x} \frac{1}{\mathrm{D}\,(1+\mathrm{D})}\, x^2 = e^{2x}\, \frac{1}{\mathrm{D}}\, \frac{1}{(1+\mathrm{D})}\, x^2$

= $e^{2x}\, \frac{1}{\mathrm{D}}$[1-D+ D^2- D^3 $+ \cdots \ldots]x^2$

= $e^{2x}\, \frac{1}{\mathrm{D}}$[$x^2$ -2x + 2 - 0+ $\cdots \ldots$]

- P.I.= $e^{2x}[\,\frac{x^3}{3} - 2\frac{x^2}{2} + 2x + 0]$
- = $e^{2x}\ [\frac{x^3}{3} - x^2 + 2x]$
- Complete Solution = C.F. +P.I.
- = $(C_1 e^{2x}$+ $C_2 e^x)$+ $e^{2x}\ [\frac{x^3}{3} - x^2 + 2x]$
- Note : x =$e^{-ax}.V\ , V = V(x)$

  $then\ \frac{1}{f(D)}\ e^{-ax}.V = e^{-ax}\ \frac{1}{f(D-a)}$V

5] X=x.V,V=V(x)

$$\frac{1}{f(D)}x.V = [X- \frac{f'(D)}{f(D)}] \frac{1}{f(D)}V$$

- Exa. Solve $(D^2+1)y$ = xsin2x
- Solution :Given D.E. is $(D^2+1)y$ = sin2x
- A.E. is (m^2+1)=0
- M= i,-i
- C.F.= $(C_1 \cos x + C_2 \sin x)$
- To find P.I., P.I.= $\frac{1}{\mathrm{f(D)}}$X ,X=$\mathrm{x.\,sin2x}$f(D)= (D^2+1)
- $\frac{1}{(D^2+1)}x.\,sin2x$ =[X- $\frac{f'(D)}{f(D)}$] $\frac{1}{f(D)}$ $\mathrm{sin2x}$ = [X− $\frac{2D}{(D^2+1)}$] $\frac{1}{(D^2+1)}$ $\mathrm{sin2x}$
- = [X− $\frac{2D}{(D^2+1)}$] $\frac{1}{(-4+1)}$ $\mathrm{sin2x}$= $\frac{1}{(-3)}$[Xsin2x− $\frac{2D}{(D^2+1)}$ sin2x]

P.I.= $\frac{1}{(-3)}$[xsin2x$-$ $\frac{2D}{(D^2+1)}$ sin2x]

$\frac{1}{(-3)}$[xsin2x$-$ $\frac{2D}{(-4+1)}$ sin2x]

- = $\frac{Xsin2x}{(-3)} - \frac{1}{(-3)}\frac{1}{(-3)}$2.2.cos2x
- = $\frac{-Xsin2x}{(3)} - \frac{4}{(9)}$cos2x
- Complete Solution = C.F. +P.I.
- = $(C_1\cos x + C_2 \sin x) - \frac{Xsin2x}{(3)} - \frac{4}{(9)}$cos2x

Q. Solve $(D^2+2D+1)y$ = $4e^{-x}$logx
Solution :Given D.E.is $(D^2+2D+1)y$ = $4e^{-x}$logx
A.E. is (m^2+2m+1)=0
m = -1,-1

- C.F.= $(C_1+C_2x)e^{-x}$
- To find P.I.,= $\frac{1}{\mathrm{f(D)}}$X ,X= $4e^{-x}$logx ,f(D)= ($D^2+2D+1)$
- P.I.= $\frac{1}{\mathrm{f(D)}}$X =4$\frac{1}{D^2+2D+1}$ e^{-x}logx
- =4$\frac{1}{(D+1)^2}$ e^{-x}logx = 4 e^{-x} $\frac{1}{(D-1+1)^2}logx$
- =4 e^{-x} $\frac{1}{(D)^2}logx$= 4 e^{-x} $\frac{1}{D}\int logx.dx$
- = 4 e^{-x} $\frac{1}{D}[logx.x-\int x\frac{1}{x}dx]$= 4 e^{-x} $\frac{1}{D}[logx.x-x]$
- = 4 $e^{-x}[logx\frac{x^2}{2}-\int\frac{x^2}{2}\frac{1}{x}dx-\frac{x^2}{2}]$

P.I.=4 $e^{-x}[logx\frac{x^2}{2} - \int\frac{x^2}{2}\frac{1}{x}dx - \frac{x^2}{2}]$

= 4 $e^{-x}[logx\frac{x^2}{2} - \frac{x^2}{4}$ - $\frac{x^2}{2}]$

=4 $e^{-x}[logx\frac{x^2}{2} - \frac{3x^2}{4}]$

- Complete Solution = C.F. +P.I.
- = $(C_1 + C_2x)e^{-x}$ +4 $e^{-x}[logx\frac{x^2}{2} - \frac{3x^2}{4}]$
- Exa. Solve $(D^2 - D - 2)y$ = 2logx+$\frac{1}{x} + \frac{1}{x^2}$
- Solution : Given D.E.is $(D^2 - D - 2)y$ = 2logx $- \frac{1}{X} + \frac{2}{x} + \frac{1}{x^2}$
  A.E. is $(m^2 - m - 2)$=0
  m = 2,-1
- C.F.= $(C_1e^{2x} + C_2e^{-x})$
- To find P.I.= $\frac{1}{\mathrm{f(D)}}$X = $\frac{1}{(D^2-D-2)}$ [2logx $- \frac{1}{X} + \frac{2}{x} + \frac{1}{x^2}$]

find P.I.= $\frac{1}{f(D)}X = \frac{1}{(D^2-D-2)}$ [2logx $-\frac{1}{X}+\frac{2}{x}+\frac{1}{x^2}$]

$=\frac{1}{(D-2)(D+1)}$ [2logx $-\frac{1}{X}+\frac{2}{x}+\frac{1}{x^2}$]

$=\frac{1}{(D-2)}\frac{1}{(D+1)}$[2logx $-\frac{1}{X}+\frac{2}{x}+\frac{1}{x^2}$]

- P.I.=$\frac{1}{(D-2)}e^{-x}\int e^{x}$ [2logx $-\frac{1}{X}+\frac{2}{x}+\frac{1}{x^2}$]
- $=\frac{1}{(D-2)}e^{-x}e^{x}$[2logx $-\frac{1}{X}$]-------- $\int e^{x}$[f(x)+f'(x)]dx= e^{x}f(x)
- $=\frac{1}{(D-2)}$ [2logx $-\frac{1}{X}$]
- $= e^{2x}\int e^{-2x}$ [2logx $-\frac{1}{X}$]dx
- $= e^{2x}$[2logx.$\frac{e^{-2x}}{-2}-\int\frac{e^{-2x}}{-2}\frac{2}{X}dx$]- $\int e^{-2x}\frac{1}{X}$ dx
- $= e^{2x}$[2logx.$\frac{e^{-2x}}{-2}+\int\frac{e^{-2x}}{1}\frac{1}{X}dx$]- $\int e^{-2x}\frac{1}{X}$ dx
- = - logx
- Complete Solution = C.F. +P.I.=$(C_1e^{2x}+C_2e^{-x})$-logx

Method of $Variation\ of\ Parameters$:For second order D.E.
$(D^2 + aD + b)y = X$
Find it's C.F. = $C_1 y_1(x) + C_2 y_2$(x)

- Method of $Variation\ of\ Parameters$ $\text{P.I.} = u(x)y_1(x) + v(x)y_2(x)$

where $u = -\int \frac{y_2 X}{W} dx$, $v = \int \frac{y_1 X}{W} dx$

- $W = wronskian = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Method of $Variation\ of\ Parameters\quad y = uy_1 + vy_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$u = -\int \frac{y_2 X}{W} dx\ , \qquad v = \int \frac{y_1 X}{W} dx$$

- 1) Use the method of variation of parameters to solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$
- Solution: Given D.E. is $D^2 + 3D + 2 = 0$
- The auxiliary equation is $m^2 + 3m + 2 = 0$
- $\therefore (m+1)(m+2) = 0\ , m = -1, -2.$
- $\therefore The\ C.F.is\ \ y = C_1 e^{-x} + C_2 e^{-2x}$
- $y_1 = e^{-x}, y_2 = e^{-2x}\ , X = e^{e^x}$

Let, P.I. be $y = uy_1 + vy_2$

Now $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$

$$\therefore u = -\int \frac{y_2 X}{W} dx = -\int \frac{e^{-2x} e^{e^x}}{-e^{-3x}} dx = \int e^{e^x} e^x dx = e^{e^x} \qquad [Put\ e^x = t]$$

- $v = \int \frac{y_1 X}{W} dx = \int \frac{e^{-x} e^{e^x}}{-e^{-3x}} dx = \int e^{e^x} e^{2x} dx$
- Putting $e^x = t,\ v = \int e^t \cdot t dt = te^t - e^t \qquad \therefore v = e^x e^{e^x} - e^{e^x}$
- $\therefore P.I. = e^{e^x} \cdot e^{-x} - \left(e^x e^{e^x} - e^{e^x}\right) \cdot e^{-2x} = e^{-2x} \cdot e^{e^x}$
- $\therefore$ *The complete solution is*
- $y = C_1 e^{-x} + C_2 e^{-2x} + e^{-2x} \cdot e^{e^x}$

Use the method of variation of parameters to solve $\frac{d^2y}{dx^2} + a^2y = \text{tanax}$

- Solution: Given D.E. is $(D^2+a^2)y = tanax$
- The auxiliary equation is $m^2 + a^2 = 0$
- $\therefore\ m = ai, -ai.$
- $\therefore The\ C.F.is\ \ y = (C_1cosax + C_2\text{sinax})$
- $y_1 = cosax, y_2 = sinax\ , X = \text{tanax}$
- $W = wronskian = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} cosax & sinax \\ -asinax & acosax \end{vmatrix} = \text{a}$
- $u = -\int \frac{y_2X}{W} dx = -\int \frac{sinax.tanaX}{a} dx$

$$u = -\int \frac{y_2 X}{W} dx = -\int \frac{sinax.tanaX}{a} dx$$
$$= -\int \frac{secax - cosaX}{a} dx$$
$$= \frac{-secax.tanax + sinaX}{a^2}$$

- $v = \int \frac{y_1 X}{W} dx = \int \frac{cosax.tanaX}{a} dx$
- $= \int \frac{sinax}{a} dx$ = - cosax.$\frac{1}{a^2}$
- $\mathrm{P.I.} = u(x)y_1(x) + v(x)y_2(x)$

$= [\frac{-secax.tanax+sinaX}{a^2}]\ cosax$ - cosax.$\frac{1}{a^2}.\ sinax$

$= \frac{-tanax}{a^2}$

Complete Solution = C.F. +P.I.

=($C_1 cosax + C_2$sinax)- $\frac{tanax}{a^2}$

Apply method of variation of parameters to solve

$$(D^3-6D^2+12D-8)Y=\frac{e^{2x}}{x}$$

Solution: The A.E.

$(D^3-6D^2+12D-8)=0$

$$(D-2)^3=0$$

- $\therefore D=2,2,2$
- The C.F. is y = $(C_1+C_2x+C_3x^2)e^{2x}$
- $y_1=e^{2x}, y_2=xe^{2x}, y_3=x^2e^{2x}$
- W = $\begin{vmatrix} e^{2x} & xe^{2x} & x^2e^{2x} \\ 2e^{2x} & (2x+1)e^{2x} & (2x^2+2x)e^{2x} \\ 4e^{2x} & 4(x+1)e^{2x} & (4x^2+8x+2)e^{2x} \end{vmatrix} = 2e^{6x}\begin{vmatrix} 1 & x & x^2 \\ 2 & (2x+1) & (2x^2+2x) \\ 2 & 2(x+1) & (2x^2+4x+1) \end{vmatrix}$
- By R_2-2R_1 , $R_3-\ 2R_2$
- W = 2 $e^{6x}\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 1 & 2x+1 \end{vmatrix} = 2\ e^{6x}$
- P.I. = uy_1 +Vy_2 $+w\ y_3$

u = $\int \frac{(y_2 y_3' - y_3 y_2')X}{W} \; dx$

$= \int \frac{[xe^{2x}((2x^2+2x)e^{2x} - x^2 e^{2x(2x+1)e^{2x}}]}{2\, e^{6x}} \; \frac{e^{2x}}{x} \; dx$

$= \int \frac{x}{2} \; dx \; = \; \frac{x^2}{2}$

- V = $\int \frac{(Y_3 Y_1{\prime} - Y_1 \, Y_3{\prime})X}{W} \, dx$ = - x
- W = $\int \frac{(y_1 y_2' - y_2 y_1')X}{W} d\,x$
- = $\frac{1}{2} \log x$
- ∴P.I. = uy_1 +Vy_2 $+ w\, y_3$
- $= \frac{x^2}{2} e^{2x} \; + \; -x \; xe^{2x} + \frac{1}{2} \log x \; x^2 e^{2x}$
- ∴ $\mathit{The\ complete\ solution\ is}$
- $y \; = (C_1 + C_2 + C_3 x^2) e^{2x} + \frac{x^2}{2} \; e^{2x} \; + \; - x \; xe^{2x} + \frac{1}{2} \log x \; x^2 e^{2x}$