Taylor's and Laurent's Series.

If $f(z)$ is analytic inside a circle c with centre at z_0, then for all z inside c $f(z)$ can be expanded as

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \dots \infty.$$

This series is convergent at every point inside c and is known as Taylor's series.

Laurent's Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

$$a_n = \frac{1}{2\pi i} \int_{c_1} \frac{f(w)}{(w-z_0)^{n+1}} dw \, ; \quad b_n = \frac{1}{2\pi i} \int_{c} \frac{f(w)}{(w-z_0)^{-n+1}} dw.$$

Q.1. Expand $\cos z$ as Taylor's series at $z = \pi/2$

By Taylor's Series

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots$$

$a = \pi/2$, $f(z) = \cos z$; $f(\pi/2) = \cos \pi/2 = 0$.

$f'(z) = -\sin z$; $f'(\pi/2) = -\sin \pi/2 = -1$

$f''(z) = -\cos z$. ; $f''(\pi/2) = -\cos z = 0$

$f'''(z) = \sin z$; $f'''(\pi/2) = \sin \pi/2 = 1$.

$$f(z) = 0 + (z-\pi/2) \times -1 + \frac{(z-\pi/2)^2}{2!} \times 0 + \frac{(z-\pi/2)^3}{3!} \times 1 + \dots$$

Q.2. Find Laurents' Series for

$$f(z) = \frac{e^{3z}}{(z-1)^3}, \text{ about } z=1$$

$z = 1$

$z-1 = u.$

$\boxed{z = u+1}$

$$f(u) = \frac{e^{3(u+1)}}{(u+1-1)^3} = \frac{e^{3u} \cdot e^{+3}}{u^3}.$$

$$= \frac{e^3 \cdot e^{3u}}{u^3}$$

$$= \frac{e^3}{u^3}\left\{1 + \frac{3u}{1!} + \frac{(3u)^2}{2!} + \frac{(3u)^3}{3!} + \cdots\right\}$$

$$= \frac{e^3}{u^3}\left\{1 + 3u + 9\frac{u^2}{2} + 27\frac{u^3}{6} + \cdots\right\}.$$

$$\therefore f(z) = \frac{e^3}{(z-1)^3}\left\{1 + 3(z-1) + \frac{9}{2}(z-1)^2 + \frac{9}{2}(z-1)^3 + \cdots\right\}.$$

Q.4. Find the Laurents' series which represents the function

$$f(z) = \frac{2}{(z-1)(z-2)} \quad \text{when (1) } |z|<1$$

(2) $1<|z|<2$, (3) $|z|>2$.

$$f(z) = \frac{2}{(z-1)(z-2)}$$

$$f(z) = \frac{-2}{z-1} + \frac{2}{z-2} \quad \text{(By partial fractions)}$$

Case I : $|z| < 1$

$|z| < 1$ means $|z| < 2$

$\frac{|z|}{2} < 1$

$$f(z) = \frac{-2}{z-1} + \frac{2}{z-2}$$

$$= \frac{2}{1-z} - \frac{2}{2-z}$$

$$= \frac{2}{1-z} - \frac{2}{2(1-z/2)}$$

$$= 2(1-z)^{-1} - (1-z/2)^{-1}$$

$$= 2[1+z+z^2+\cdots] - [1+\frac{z}{2}+(\frac{z}{2})^2+\cdots]$$

Case II : $1 < |z| < 2$

$1 < |z|$ $\qquad$ $|z| < 2$

$\frac{1}{|z|} < 1$ $\qquad$ $\frac{|z|}{2} < 1$

$$f(z) = \frac{-2}{z-1} + \frac{2}{z-2}$$

$$= \frac{-2}{z(1-1/z)} + \frac{2}{2(\frac{z}{2}-1)}$$

$$= \frac{-2}{z(1-z)} - \frac{1}{(1-z/2)}$$

$$= -\frac{2}{z}(1-z)^{-1} - (1-z/2)^{-1}$$

$$= -\frac{2}{z}[1+z+z^2+\cdots] - [1+z/2+(z/2)^2+\cdots]$$

Case III : $|z| > 2$

$|z| > 2$ ~~mean~~ $|z| > 1$

$\frac{1}{|z|} < \frac{1}{2}$ $\qquad$ $\frac{1}{|z|} < 1$

$\frac{2}{|z|} < 1$

$$f(z) = \frac{-2}{z-1} + \frac{2}{z-2}$$

$$= \frac{-2}{z(1-1/z)} + \frac{2}{z(1-2/z)}$$

$$= \frac{-2}{z}(1-1/z)^{-1} + \frac{2}{z}(1-2/z)^{-1}$$

$$= -\frac{2}{z}\left[1 + 1/z + 1/z^2 + \cdots\right] + \frac{2}{z}\left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots\right]$$

Q.5 Obtain Taylor's & Laurent's series of $f(z) = \frac{z-1}{z^2-2z-3}$ indicating region of convergence

Let $f(z) = \frac{z-1}{z^2-2z-3} = \frac{z-1}{(z-3)(z+1)}$

$$= \frac{1/2}{z+1} + \frac{1/2}{z-3} \quad \text{(By partial fraction)}$$

Case I : $|z| < 1$

Case II : $1 < |z| < 3$

Case III : $|z| > 3$.

Case I : $|z| < 1$, $|z| < 3$

$\frac{|z|}{3} < 1$

$$f(z) = \frac{1/2}{z+1} - \frac{1/2}{z-3}$$

$$= \frac{1}{2} \cdot \frac{1}{1+z} + \frac{1}{2} \frac{1}{3-z}$$

$$= \frac{1}{2} \frac{1}{1+z} + \frac{1}{2} \frac{1}{3(1-z/3)}$$

$$= \frac{1}{2}(1+z)^{-1} + \frac{1}{6}(1-z/3)^{-1}$$

Case II : $1 < |z| < 3$.

$1 < |z|$ $\qquad$ $|z| < 3$

$\frac{1}{|z|} < 1$ $\qquad$ $\frac{|z|}{3} < 1$

$$f(z) = \frac{1}{2} \cdot \frac{1}{1+z} + \frac{1}{2} \cdot \frac{1}{3-z}$$

$$= \frac{1}{2} \frac{1}{z(1+1/z)} + \frac{1}{2} \cdot \frac{1}{3(1-z/3)}$$

$$= \frac{1}{2z}(1+1/z)^{-1} + \frac{1}{6}(1-z/3)^{-1}$$

Case III $|z| > 3$ $\Rightarrow$ $|z| > 1$

$\frac{1}{|z|} < \frac{1}{3}$ $\qquad$ $\frac{1}{|z|} < 1$

$\frac{3}{|z|} < 1$

$$f(z) = \frac{1}{2} \cdot \frac{1}{1+z} + \frac{1}{2} \frac{1}{3-z}$$

$$= \frac{1}{2} \frac{1}{z(1+1/z)} + \frac{1}{2} \cdot \frac{1}{z(\frac{3}{z}-1)}$$

$$= \frac{1}{2} \frac{1}{z(1+1/z)} - \frac{1}{2z(1-3/z)} = \frac{1}{2z}(1+1/z)^{-1} - \frac{1}{2z}\left(1-\frac{3}{z}\right)^{-1}$$