Course Code

CSC302

Course Name

Discrete Structures and Graph Theory

Department of Computer Engineering

AY 2021-2022

Course Objectives

1. Cultivate clear thinking and creative problem solving.
2. Thoroughly train in the construction and understanding of mathematical proofs, exercise common mathematical arguments and proof strategies.
3. To apply graph theory in solving practical problems.
4. Thoroughly prepare for the mathematical aspects of other Computer Engineering courses

Course Outcome

1. Understand the notion of mathematical thinking, mathematical proofs and to apply them in problem solving.
2. Ability to reason logically.
3. Ability to understand relations, functions, Diagraph and Lattice.
4. Ability to understand and apply concepts of graph theory in solving real world problems.
5. Understand use of groups and codes in Encoding-Decoding.
6. Analyze a complex computing problem and apply principles of discrete mathematics to identify solutions.

Syllabus

Module 1 Logic

- Propositional Logic, Predicate Logic, Laws of Logic, Quantifiers, Normal Forms, Inference Theory of Predicate Calculus, Mathematical Induction.

Module 2 Relations and Functions

- Basic concepts of Set Theory
- Relations: Definition, Types of Relations, Representation of Relations, Closures of Relations, Warshall'salgorithm, Equivalence relations and Equivalence Classes
- Functions: Definition, Types of functions, Composition of functions, Identity and Inverse function

Syllabus

Module 3 Posets and Lattice

- Partial Order Relations, Poset, Hasse Diagram, Chain and Anti chains, Lattice, Types of Lattice, Sub lattice

Module 4 Counting

- Basic Counting Principle-Sum Rule, Product Rule, Inclusion-Exclusion Principle, Pigeonhole Principle
- Recurrence relations, Solving recurrence relations

Syllabus

Module 5 Algebraic Structures

- Algebraic structures with one binary operation: Semi group, Monoid, Groups, Subgroups, Abelian Group, Cyclic group, Isomorphism
- Algebraic structures with two binary operations: Ring
- Coding Theory: Coding, binary information and error detection, decoding and error correction

Module 6 Graph Theory

- Types of graphs, Graph Representation, Sub graphs, Operations on Graphs, Walk, Path, Circuit, Connected Graphs, Disconnected Graph, Components, Homomorphism and Isomorphism of Graphs, Euler and Hamiltonian Graphs, Planar Graph, Cut Set, Cut Vertex, Applications.

books

Textbooks

- Bernad Kolman, Robert Busby, Sharon Cutler Ross, Nadeem-ur-Rehman, “Discrete Mathematical Structures”, Pearson Education.
- C. L. Liu “Elements of Discrete Mathematics”, second edition 1985, McGraw-Hill Book Company. Reprinted 2000.
- K. H. Rosen, “Discrete Mathematics and applications”, fifth edition 2003, Tata McGraw Hill Publishing Company

References:

- Y N Singh, “Discrete Mathematical Structures”, Wiley-India.
- J. L. Mott, A. Kandel, T. P. Baker, “Discrete Mathematics for Computer Scientists and Mathematicians”, second edition 1986, Prentice Hall of India.
- J. P. Trembley, R. Manohar “Discrete Mathematical Structures with Applications to Computer Science”, Tata Mcgraw-Hill
- Seymour Lipschutz, Marc Lars Lipson, “Discrete Mathematics” Schaum “sOutline, McGraw Hill Education.
- Narsing Deo,“Graph Theory with applications to engineering and computer science”, PHI Publications.
- P. K. Bisht, H.S. Dhami, “Discrete Mathematics”, Oxford press.

CE– SE–DSGT

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What is Logic?

What is Logic?

- Logic is the basis of all mathematical reasoning, and of all automated reasoning. The rules of logic specify the meaning of mathematical statements.

Importance of Mathematical Logic

- The rules of logic give precise meaning to mathematical statements.
- These rules are used to distinguish between valid and invalid mathematical arguments.
- Apart from its importance in understanding mathematical reasoning, logic has numerous applications in Computer Science, varying from design of digital circuits, to the construction of computer programs and verification of correctness of programs.
- Logic provides rules and techniques for determining whether a given argument is valid or not

Propositional Logic

Sentence – A number of words making complete grammatical structure having sense and meaning is called a sentence. There are two types of sentence.

Declarative sentence

- declarative sentence makes a statement that declared something (or) use reliable information / idea
- **Example**
  - Chennai is capital of Tamil Nadu
  - 1+6 = 7
  - New Delhi is in England

Non-declarative sentence

- statement that does not declare something (or) give idea are called non-declarative sentence
- **Example**
  - Imperative sentence
    command / request / Wishes
  - Exclamatory sentence
    Express strong feeling
  - interrogative sentence
    ?

Propositional Logic

- A **statement** or **proposition** is a declarative sentence that is either true or false but not both.
- <u>**Exercise:**</u>
  a) The Earth is round.
  b) 2 + 3 = 5
  c) Do you speak English?
  d) 3 – x = 5
  e) Take two chocolates.
  f) The temperature on the surface of the planet, Venus is 800F .
  g) The sun will come out tomorrow

Solution:

a) and b) are statements that happen to be true.

c) is a question, so it is not a statement.

d) is declarative sentence, but not a statement, since it is true or false, depending on the value of x.

e) is a command, not a statement.

f) is a declarative sentence as we do not know if its true or false. We can in principle determine if it is true or false.

g) is a declarative sentence as we do not know if its true or false. We will have to wait till tomorrow to find it out.

Logical Connectives & Compound Statements

- **Propositional variables** represent statements in question.
- For Eg:

  p : The sun is shining today.

  q : It is cold.
- A **Connectives statement** is a sentence that consists of two or more **statements** separated by logical connectors.
- For Eg:

  p : It is raining today.

  q : It is cold.

Compound statement: p ∧ q : It is raining ***and*** it is cold.

So here, ***and*** is a connector, connecting statements *p* & *q*

- **However, the truth value of a compound statement depends only on the truth values of the statements being combined and the types of connectives used.**
- Types of Connectives:

NOT: The Negation:

if *p* is a statement, then negation of

p is denoted as '~*p*'

For Eg:

p: it is cold

~*p:* It is not cold.

AND: Conjunction:

The conjunction of two statements *p* & *q* is

denoted by $p \wedge q$.

$p \wedge q$ is true only when both the

statements are true.

Truth Table for Negation

P	~*p*
T	F
F	T

Truth Table for AND

P	Q	P^Q
T	T	T
T	F	F
F	T	F
F	F	F

OR: Disjunction:

The disjunction of two statements *p* & *q* is denoted by *p* ∨ *q.*

Let *p*: 12 + 5 = 17

q: Raj is my neighbor.

So, *p* ∨ *q:* 12 + 5 = 17 or Raj is my neighbor.

Here, statement *p* is always true. Thus, *p* ∨ *q* is true irrespective of *q.*

Truth Table for OR

P	Q	P∨Q
T	T	T
T	F	T
F	T	T
F	F	F

IF *p* THEN *q*: Conditional:

This conditional statement means that *p* implies *q.* It is denoted by: *p*→*q* or *p*⇒q.

Let *p:* I'm hungry.

q: I eat an apple.

So, *p*→*q* : If I'm hungry then I eat an apple.

Truth Table for

P	Q	P→Q
T	T	T
T	F	F
F	T	T
F	F	T

IF AND ONLY IF: Bi-conditional:

The bi-conditional logical connective is true when p and q are same, i.e. both are false or both are true. It is denoted by:

$p \Leftrightarrow q$.

Truth Table for BI-CONDITIONAL

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Excercise:

Using the statements:

F: Tina is fat.

H: Tina is happy.

Write the following statements in symbolic forms, assuming (not fat) is thin.

a) Tina is thin but happy
b) Tina is fat or unhappy
c) Tina is neither fat or happy
d) Tina is thin or she is both fat and unhappy
e) If Tina is fat then she is happy
f) Tina is unhappy implies that Tina is thin
g) It is not true that Tina is not fat and not happy.

- **Solution 2:**

	Statements	Symbolic Forms
a)	Tina is thin but happy	$\sim F \wedge H$
b)	Tina is fat or unhappy	$F \vee \sim H$
c)	Tina is neither fat or happy	$\sim F \wedge \sim H$
d)	Tina is thin or she is both fat and unhappy	$\sim F \vee (F \wedge \sim H)$
e)	If Tina is fat then she is happy	$F \rightarrow H$
f)	Tina is unhappy implies that Tina is thin	$\sim H \rightarrow \sim F$
g)	It is not true that Tina is not fat and not unhappy.	$\sim(\sim F \wedge H)$

Converse, Contrapositive & Inverse

- **Converse** of a statement is the result of reversing its two constituent **statements**.

  For Eg:

  for $P \rightarrow Q$, the **converse** is $Q \rightarrow P$.

For the categorical proposition All S are P, the **converse** is All P are S.

- **Contrapositive** of the statement, $P \rightarrow Q$ *is* $\sim Q \rightarrow \sim P$

  For Eg:

  The **contrapositive** of ***"If it rains, then they cancel school"*** is ***"If they do not cancel school, then it does not rain."***

- The **inverse** of the statement , $P \rightarrow Q$ is $\sim P \rightarrow \sim Q$
- The **inverse** of ***"If it rains, then they cancel school"*** is ***"If it does not rain, then they do not cancel school."***

Example 3: Write the Converse, Contrapositive & Inverse of the following statements.

a) "If Boss then Bad"

b) "If r is rational then r is a real "

Solution 3:

Given Statement	If Boss then Bad
Converse	If Bad then Boss
Contrapositive	If not Bad then not Boss
Inverse	If not Boss then not Bad

a)

Given Statement	If r is rational then r is a real
Converse	If r is real then r is rational
Contrapositive	If r is not real then r is not rational
Inverse	If r is not rational then r is not real

b)

Constructing Truth Tables for a given Statement

1) $(p \wedge q) \vee (\sim p)$

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \vee (\sim p)$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

2) $(p \vee q) \to p$

p	q	$p \vee q$	$p \vee q \to p$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

Exercise: Constructing Truth Tables for the following Statement

Q3. $[P \wedge (P \rightarrow Q)] \rightarrow Q$

Q4. $(P \wedge Q) \vee \sim Q$

Q5. $\sim (\sim P \vee \sim Q)$

Q6. $(P \wedge Q) \rightarrow (Q \vee \sim P)$

Q7. $(P \rightarrow Q) \leftrightarrow \sim (\sim P \vee Q)$

Q8. $\sim [P \leftrightarrow Q] \leftrightarrow [(P \wedge (\sim Q)) \vee (Q \wedge (\sim P))]$

Q9. $[\sim P \wedge (\sim Q \wedge R)] \vee [(Q \wedge R) \vee (P \wedge R)]$

Finding the Truth Values of a given Statement

Example 4: Given that, statements P and Q are true and R and S are false, find the values of

a) **$P \wedge (Q \vee R)$**

Step1: $T \wedge (T \vee F)$

Step2: $T \wedge T$

Step3: T

Thus, the statement is true.

b) **$[P \vee (Q \wedge R)] \vee \sim [(P \wedge Q) \vee (R \vee S)]$**

Step1: $[T \vee (T \wedge F)] \vee \sim [(T \wedge T) \vee (F \vee F)]$

Step2: $[T \vee F] \vee \sim [T \vee F]$ ………….First simplify the brackets

Step3: $T \vee \sim T$

Step4: T

Thus, the statement is true.

c) $\sim [(P \vee Q) \wedge \sim R] \wedge [((\sim P \vee Q) \vee \sim R) \vee S]$

Step1: $\sim[(T \vee T) \wedge \sim F] \wedge [((\sim T \vee T) \vee \sim F) \vee F]$

Step2: $\sim[(T \vee T) \wedge T] \wedge [((F \vee T) \vee T) \vee F]$

Step3: $\sim[T \wedge T] \wedge [(T \vee T) \vee F]$

Step4: $\sim T \wedge [T \vee F]$

Step5: $F \wedge T$

Step6: F

Thus, the statement is false

d) $\sim [(P \wedge Q) \vee \sim R] \vee [((Q \leftrightarrow \sim P) \rightarrow (R \vee \sim S)]$

Step1: $\sim [(T \wedge T) \vee \sim F] \vee [((T \leftrightarrow \sim T) \rightarrow (F \vee \sim F)]$

Step2: $\sim [T \vee T] \vee [((T \leftrightarrow F) \rightarrow (F \vee T)]$

Step3: $\sim [T \vee T] \vee [F \rightarrow T]$

Step4: $\sim T \vee T$

Step5: $F \vee T$

Step6: T

Thus, the statement is true

Exercise: Given that, statements P and Q are true and R and S are false, find the values of

Q1. $\sim(P \rightarrow Q) \rightarrow P$

Q2. $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$

Q3. $(P \wedge \sim Q) \vee (\sim P \wedge Q)$

Q1 $\sim(P \to Q) \to P$

$\sim(T \to T) \to T$

$\sim(T) \to T$

$F \to T$

T //

Q2 $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$

$[(T \to T) \wedge (T \to F)] \to [T \to F]$

$[T \wedge F] \to F$

$F \to F$

T //

Q3 $(P \wedge \sim Q) \vee (\sim P \wedge Q)$

$(T \wedge \sim T) \vee (\sim T \wedge T)$

$(T \wedge F) \vee (F \wedge T)$

$F \vee F$

T //

Constructing Truth Tables for a given Statement

1) $(p \wedge q) \vee (\sim p)$

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \vee (\sim p)$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

2) $(p \vee q) \rightarrow p$

p	q	$p \vee q$	$p \vee q \rightarrow p$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

Exercise: Constructing Truth Tables for the following Statement

Q3. $[P \wedge (P \rightarrow Q)] \rightarrow Q$

Q4. $(P \wedge Q) \vee \sim Q$

Q5. $\sim (\sim P \vee \sim Q)$

Q6. $(P \wedge Q) \rightarrow (Q \vee \sim P)$

Q7. $(P \rightarrow Q) \leftrightarrow \sim (\sim P \vee Q)$

Q8. $\sim [P \leftrightarrow Q] \leftrightarrow [(P \wedge (\sim Q)) \vee (Q \wedge (\sim P))]$

Q9. $[\sim P \wedge (\sim Q \wedge R)] \vee [(Q \wedge R) \vee (P \wedge R)]$

Finding the Truth Values of a given Statement

Example 4: Given that, statements P and Q are true and R and S are false, find the values of

a) **$P \wedge (Q \vee R)$**

Step1: $T \wedge (T \vee F)$

Step2: $T \wedge T$

Step3: T

Thus, the statement is true.

b) **$[P \vee (Q \wedge R)] \vee \sim [(P \wedge Q) \vee (R \vee S)]$**

Step1: $[T \vee (T \wedge F)] \vee \sim [(T \wedge T) \vee (F \vee F)]$

Step2: $[T \vee F] \vee \sim [T \vee F]$ ....First simplify the brackets

Step3: $T \vee \sim T$

Step4: T

Thus, the statement is true.

c) ~ [(P V Q) ∧ ~R] ∧ [((~P V Q) V ~R) V S]

Step1: ~[(T V T) ∧ ~F] ∧ [((~T V T) V ~F) V F]

Step2: ~[(T V T) ∧ T] ∧ [((F V T) V T) V F]

Step3: ~[T ∧ T] ∧ [(T V T) V F]

Step4: ~T ∧ [T V F]

Step5: F ∧ T

Step6: F

Thus, the statement is false

d) ~ [(P ∧ Q) V ~R] V [((Q ↔ ~ P) → (R V~S)]

Step1: ~ [(T ∧ T) V ~F] V [((T ↔ ~ T) → (F V~F)]

Step2: ~ [T V T] V [((T ↔ F) → (F V T)]

Step3: ~ [T V T] V [F → T]

Step4: ~ T V T

Step5: F V T

Step6: T

Thus, the statement is true

Exercise: Given that, statements P and Q are true and R is false, find the values of

Q1. $\sim(P \rightarrow Q) \rightarrow P$

Q2. $[(P \rightarrow Q) \land (Q \rightarrow R)] \rightarrow (P \rightarrow R)$

Q3. $(P \land \sim Q) \lor (\sim P \land Q)$

Q1 $\sim(P \rightarrow Q) \rightarrow P$

$\sim(T \rightarrow T) \rightarrow T$

$\sim(T) \rightarrow T$

$F \rightarrow T$

T //

Q2 $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$

$[(T \rightarrow T) \wedge (T \rightarrow F)] \rightarrow [T \rightarrow F]$

$[T \wedge F] \rightarrow F$

$F \rightarrow F$

T //

Q3 $(P \wedge \sim Q) \vee (\sim P \wedge Q)$

$(T \wedge \sim T) \vee (\sim T \wedge T)$

$(T \wedge F) \vee (F \wedge T)$

$F \vee F$

T //

In Q3, F V F = F, so the statement is False

Tautologies & Contradictions

- *P* is said to be a ***tautology*, if and only if its truth value is T**, for all possible assignments of truth values to P_1, P_2.....$P_{N.}$
- *P* is said to be a ***contradiction*, if and only if its truth value is F**, for all possible assignments of truth values to P_1, P_2, P_3.....$P_{N.}$

Definition of Equivalence : Two statement functions f(P_1, P_2.....$P_{N.}$) and g(P_1, P_2.....$P_{N.}$) are said to be *logically equivalent*, if and only if for every truth values of P_1, P_2.....$P_{N.}$, the corresponding truth values of f(P_1, P_2.....$P_{N.}$) and g(P_1, P_2.....$P_{N.}$) are equal.

Example 1 : Examine whether the following statements are Tautologies, Contradictions or neither.

Q1. $f(P,Q) = [P \wedge (P \rightarrow Q)] \rightarrow Q$

Soln:

P	Q	$P \rightarrow Q$	$P \wedge (P \rightarrow Q)$	$[P \wedge (P \rightarrow Q)] \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Here, all the truth values of f(P,Q) are T. Thus, **$f(P,Q) = [P \wedge (P \rightarrow Q)] \rightarrow Q$ is a Tautology**

Q2. $g(P,Q) = P \wedge [\sim (P \vee Q)]$

Soln:

P	Q	P V Q	~ (P V Q)	P Λ [~ (P V Q)]
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

Here, all the truth values of g(P,Q) are F. Thus **$g(P,Q) = P \wedge [\sim (P \vee Q)]$ is a Contradiction**

Q3. f(P,Q) = (~P V Q) → P ∧ (Q V ~Q)

Soln:

P	Q	~P	~Q	~P V Q	Q V ~Q	P V (Q V ~Q)	(~P V Q) → P ∧ (Q V ~Q)
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	F	F

Here, all the truth values of f(P,Q) are neither all T nor all F. Thus**, f(P,Q) = (~PV Q) → P ∧ (Q V ~Q) is neither a Tautology or a Contradiction**

Negation of P can be written as ¬P or ~P.

<u>Example 6:</u> If $f(P,Q) = (P \rightarrow Q)$ and $g(P,Q) = (P \wedge \neg Q)$, then show that $f(P,Q) \equiv g(P,Q)$.

P	Q	¬Q	$f(P,Q) = (P \rightarrow Q)$	$g(P,Q) = (P \wedge \neg Q)$
T	T	F	T	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Since, the columns of $f(P,Q)$ and $g(P,Q)$ are identical, $f(P,Q) \equiv g(P,Q)$

Exercise: Prove that the following statements are equivalent

Q1. $(\neg P) \vee Q$ and $\neg (P \wedge Q) \rightarrow [\neg P \vee (\neg P \vee Q)]$

Q2. $\neg(P \leftrightarrow Q)$ and $(P \wedge \neg Q) \vee (\neg P \wedge Q)$

Soln 1:

P	Q	¬ P	(¬P)V Q	P ∧ Q	¬(P ∧ Q)	¬P V (¬P V Q)	(P ∧ Q) → [¬ P V (¬P V Q)]
T	T	F	T	T	F	F	T
T	F	F	F	F	T	F	F
F	T	T	T	F	T	T	T
F	F	T	T	F	T	T	T

Soln 2:

P	Q	¬P	¬ Q	P ↔ Q	¬(P ↔ Q)	P ∧ ¬Q	¬P ∧ Q	(P ∧ ¬Q) V (¬P ∧ Q)
T	T	F	F	T	F	F	F	F
T	F	F	T	F	T	T	F	T
F	T	T	F	F	T	F	T	T
F	F	T	T	T	F	F	F	F

Laws of Logic

- $\neg(\neg P) \leftrightarrow P$
- **Idempotent Law:**

  $P \vee P \leftrightarrow P$, $P \wedge P \leftrightarrow P$
- **Commutative Law:**

  $P \vee Q \leftrightarrow Q \vee P$, $P \wedge Q \leftrightarrow Q \wedge P$
- **Associate Law:**

  $(P \vee Q) \vee R \leftrightarrow P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \leftrightarrow P \wedge (Q \wedge R)$
- **Distributive Law:**

  $P \vee (Q \wedge R) \leftrightarrow (P \vee Q) \wedge (P \vee R)$

  $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$
- **DeMorgan's Law:**

  $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$

  $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$
- **Conditional is equivalent to its Contrapositive**

  $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$
- **Identity Laws**

  $P \vee \text{False} \leftrightarrow P$ $P \vee \text{True} \leftrightarrow \text{True}$

  $P \wedge \text{False} \leftrightarrow \text{False}$ $P \wedge \text{True} \leftrightarrow P$

Laws of Logic

- **Complement Laws**

  $P \vee \neg P = \text{True}$ $P \wedge \neg P = \text{False}$

- **Implication Law**

  $P \rightarrow Q = \neg P \vee Q$

- **Absorption Law**

  $P \wedge (P \vee Q) = P$ $P \vee (P \wedge Q) = P$

Following are Absorption Laws

Example 8: i) Show that $a \vee (a \wedge b) = a$

Soln: Using Identity Law, $P \wedge True \leftrightarrow P$

$a \vee (a \wedge b) = (a \wedge T) \vee (a \wedge b)$, where T = True

$P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$

Again, $a \vee (a \wedge b) = a \wedge (T \vee b)$............. Distributive Law

$= a \wedge T$...................... Identity Law

$P \vee True \leftrightarrow True$

$= a$............................ Identity Law

$P \wedge True \leftrightarrow P$

ii) Show that $a \wedge (a \vee b) = a$

Using Identity Law,

$a \wedge (a \vee b) = (a \wedge T) \wedge (a \vee b)$, where T = True

Again, $a \wedge (a \vee b) = a \wedge (T \vee b)$............. Distributive Law

$= a \wedge T$...................... Identity Law

$= a$............................ Identity Law

Example 9: i) Show that $P \lor (\neg P \land Q) = P \lor Q$

$P \lor (\neg P \land Q) = (P \lor \neg P) \land (P \lor Q)$ .... Distributive Law

$= T \land (P \lor Q)$ .... Compliment Law

$= P \lor Q$ .... Identity Law

ii) Show that $P \land (\neg P \lor Q) = P \land Q$

$P \land (\neg P \lor Q) = (P \land \neg P) \lor (P \land Q)$ .... Distributive Law

$= F \lor (P \land Q)$ .... Compliment Law

$= P \land Q$ .... Identity Law

Example 10: Simplify the expression: $P \lor \neg(\neg P \to Q)$

$P \lor \neg(\neg P \to Q) = P \lor \neg(\neg\neg P \lor Q)$	-Implication Law
$= P \lor \neg(P \lor Q)$	-Complement Law
$= P \lor (\neg P \land \neg Q)$	-DeMorgan' Law
$= (P \lor \neg P) \land (P \lor \neg Q)$	-Distributive Law
$= T \land (P \lor \neg Q)$	-Complement Law
$= (P \lor \neg Q) \land T$	-Commutative Law
$= P \lor \neg Q$	-Identity Law

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

1 $\neg\neg P$ is equivalent to P.
2 $P \vee P$ is equivalent to P.

3 $(P \wedge \neg P) \vee Q$ is equivalent to Q.
4 $P \vee \neg P$ is equivalent to $Q \vee \neg Q$.

Example 11

Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$.

$$
\begin{aligned}
&(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \\
&\quad \Leftrightarrow (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R) \\
&\quad \Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R) \\
&\quad \Leftrightarrow ((\neg P \wedge \neg Q) \vee (Q \vee P)) \wedge R \\
&\quad \Leftrightarrow (\neg (P \vee Q) \vee (P \vee Q)) \wedge R \\
&\quad \Leftrightarrow \mathbf{T} \wedge R \\
&\quad \Leftrightarrow R
\end{aligned}
$$

Q1 Simplify

a) $((a \wedge \bar{b}) \vee c) \wedge (a \vee \bar{b}) \wedge c$

$\rightarrow$ $(a \vee c) \wedge (\bar{b} \vee c) \wedge (a \vee \bar{b}) \wedge c$ – Distributed law

$(a \vee c) \wedge c \wedge (\bar{b} \vee c) \wedge (a \vee \bar{b})$ – Associative law

$c \wedge (\bar{b} \vee c) \wedge (a \vee \bar{b})$ – Absorption law

$c \wedge (a \vee \bar{b})$ – Absorption law

b) $(a \wedge b) \vee (\bar{a} \wedge b \wedge \bar{c}) \vee (b \wedge c)$

$\rightarrow$ $b \wedge (a \vee (\bar{a} \wedge \bar{c}) \vee c)$ Distributive law

$b \wedge [[(a \vee \bar{a}) \wedge (a \vee \bar{c})] \vee c]$ Distributive law

$b \wedge [[T \wedge (a \vee \bar{c})] \vee c]$ Complement law

$b \wedge [(a \vee \bar{c}) \vee c]$ Identity law

$b \wedge [a \vee (\bar{c} \vee c)]$ Associative law

$b \wedge (a \vee T)$ Complement law

$b \wedge T$ Identity law

$\underline{\underline{b}}$ Identity law

Q2 Prove the following

i) $[(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)] \rightarrow p \wedge q$

$\rightarrow$ LHS: $(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)$

$= \sim(\sim p \vee \sim q) \vee (p \wedge q \wedge r)$ Implication law

$= (p \wedge q) \vee (p \wedge q \wedge r)$ Demorgan's Law

let $p \wedge q$ be x

$= x \vee (x \wedge r)$

$= x$ } Absorption law

$= \underline{\underline{p \wedge q}}$

ii) $[(p \rightarrow q) \wedge [\sim q \wedge (r \vee \sim q)]] \leftrightarrow \sim(q \vee p)$

$(\sim p \vee q) \wedge [\sim q \wedge (r \vee \sim q)]$ Implication law

$(\sim p \vee q) \wedge [\sim q \wedge (\sim q \vee r)]$

$(\sim p \vee q) \wedge [\sim q]$ Absorption law

$(\sim p \wedge \sim q) \vee (q \wedge \sim q)$ Distributive law

$(\sim p \wedge \sim q) \vee F$ Complement law

$\sim(p \vee q) \vee F$ DeMorgan's law

$\sim(q \vee p) \vee F$

$\sim(q \vee p)$

Show that $\sim p \wedge (\sim q \wedge r) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

Solution :

$\sim p \wedge (\sim q \wedge r) \vee (q \wedge r) \vee (p \wedge r)$

$\equiv (\sim p \wedge (\sim q \wedge r)) \vee [(q \vee p) \wedge r]$

$\equiv [(\sim p \wedge \sim q) \wedge r] \vee [(q \vee p) \wedge r]$ Association

$\equiv [(\sim p \wedge \sim q) \vee (q \vee p)] \wedge r$ Distributive

$\equiv [(\sim p \vee \sim q) \vee (p \vee q)] \wedge r$

$\equiv \text{True} \wedge r$ - Identity law

$\equiv r$

Use the laws of logic to show that

$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ is a tautology.

Solution :

We have,

$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$

$\equiv \sim [(\sim p \vee q) \wedge \sim q] \vee \sim p$	- Implication law
$\equiv \sim [\sim q \wedge (\sim p \vee q)] \vee \sim p$	- Commutative law
$\equiv \sim [(\sim q \wedge \sim p) \vee (\sim q \wedge q)] \vee \sim p$	- Distributive law
$\equiv \sim [(\sim q \wedge \sim p) \vee (q \wedge \sim q)] \vee \sim p$	- Commutative law
$\equiv \sim [(\sim q \wedge \sim p) \vee F] \vee \sim p$	- Complement law
$\equiv \sim (\sim q \wedge \sim p) \vee \sim p$	- Identity law
$\equiv (\sim \sim q \vee \sim \sim p) \vee \sim p$	- First demorgan's law
$\equiv (q \vee p) \vee \sim p$	- Complement law
$\equiv q \vee (p \vee \sim p)$	- Associative law
$\equiv q \vee T$	- Inverse law
$\equiv T$	- Identity law

$\therefore$ $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ is a tautology.

Example: To prove: $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

Solution:

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ by De Morgan's 2nd law

$\equiv \neg p \wedge (\neg(\neg p) \vee \neg q)$ by De Morgan's first law

$\equiv \neg p \wedge (p \vee \neg q)$ by the double negation law

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ by the 2nd distributive law

$\equiv F \vee (\neg p \wedge \neg q)$ because $\neg p \wedge p \equiv F$

$\equiv (\neg p \wedge \neg q) \vee F$ by commutativity law

$\equiv \neg p \wedge \neg q$ by the identity law for F

Example: Let p, q, r denote primitive statements. Use the laws of logic to show that

$[p \rightarrow (q \vee r)] \Longleftrightarrow [(p \wedge \neg q) \rightarrow r]$

Using the laws of logic we obtain:

$p \rightarrow (q \vee r)$	$\Longleftrightarrow \neg p \vee (q \vee r)$	Logical equivalence
	$\Longleftrightarrow (\neg p \vee q) \vee r$	Associative law
	$\Longleftrightarrow \neg\neg(\neg p \vee q) \vee r$	Double negation law
	$\Longleftrightarrow \neg(\neg\neg p \wedge \neg q) \vee r$	De Morgan's law
	$\Longleftrightarrow \neg(p \wedge \neg q) \vee r$	Double negation law
	$\Longleftrightarrow (p \wedge \neg q) \rightarrow r$	Logical equivalence

Therefore, $[p \rightarrow (q \vee r)] \Longleftrightarrow [(p \wedge \neg q) \rightarrow r]$.

Limitations of Propositional Logic

- Propositional logic cannot adequately express the meaning of a given statement. For Eg: *"Every computer connected to the institute's network, works fine."*

  Here, the propositional logic fails to explain the complete meaning of the statement by not defining the word, '*Every*'.

  – Similarly, in the statement, *"Computer CS-2 is under attack."*, the propositional logic fails to determine which computer is under attack.

Predicates

- A **Predicate** is a sentence containing variable/variables whose values are not specified. For Eg:

  a) $X > 3$ b) $x+y=7$ c) $X^2 + Y^2 = Z^2$

  d) A servant loves to watch a movie. e) x was born in city y in the year z

  Here, *X, Y, Z, servant and movie* are variables.

- The statement *"X is greater than 3"* has two parts:

  i) The variable, X is a subject

  ii) *"is greater than 3"* is a predicate

- As soon as the variables are replaced by the specified values, the predicate becomes a statement having a fixed truth value, T or F.
- The set of values, which upon replacement of a variable, converts a predicate into a statement is called the ***universe of discourse*** or simply the ***universe U*** w.r.t that variable.

- A predicate having n variables $x_1, x_2, \ldots, x_N$ is called ***n-ary predicate*** ***or*** ***n-place predicate*** and is denoted by $P(x_1, x_2, \ldots, x_n)$. For Eg:

  a) $P(x,y)$: $X^2 + Y^2 = 16$

  Here, the universe may be taken as R

  b) Q(s,f): A servant loves a fruit.

  Here, *s* and *f* are the variables representing *servant* and *fruit* respectively. The universe for *s* is the set of all servants and *f* is the set of all the fruits.

  c) R(a,b): a dislikes b

  Here, universe U is the set of all living beings.

Example 12: Let *P(x) denote the statement "x > 3." What are the truth values of P(4)* and *P(2)?*

Soln: a) P(4), putting x = 4, we get 4 > 3, which is true.

b) P(2), putting x = 2, we get 2 > 3, which is false

Similarly,

a) Let, P(x,y): $X^2 + Y^2 = 16$.

For P(3,5), we get $3^2 + 5^2 = 16$, which is a false statement.

b) Let, Q(s,f): Ramu loves an apple.

Here, the truth value depends upon whether this particular servant loves an apple or not.

Quantifiers

Quantifiers

- **Quantification** is a second method to bind individual variables in a predicate.
- It expresses the extent to which a predicate is true over a range of elements.
- In English, the words like *all, some, many, none, and few* are used in *quantifications.*
- There are two types of Quantifiers:
  - Universal Quantifier
  - Existential Quantifier

The Universal Quantifiers

- If P(x) is predicate with a variable x as an argument, then the assertion ***"For all x, P(x)"*** is interpreted as ***"For all values of x, P(x) is true"*** and the variable x is said to be **universally quantified**.
- The phrase, *"for all x" or "for every x"* or *"for each x"* is denoted by ∀ x or by (x).

- **Example**: Let P(x): $X \geq 0$

  If x is any positive integer, then the proposition ∀ x P(x) is true.

  If x is any real number, then the proposition ∀ x P(x) is false.

Exercise: Write in symbols: For all x, $x < 5$ or $x \geq 5$

Soln: Let P(x): $x < 5$ and Q(x): $x \geq 5$

So, the proposition can be written as: $\forall x\, [P(x) \lor Q(x)]$

We can also write: $\forall x\, [P(x) \lor \neg P(x)]$

Remember**

We symbolize a predicate by capital letters and the subject of the sentence is represented by small letters

Consider the following statements:

a) John is a bachelor.

b) Smith is a bachelor.

In these statements, Denote *"is a bachelor"* which is a 'B' John by 'j' and

Smith by 's'. Thus, these statements can be written as: B(j) and B(s).

Consider the following statements:

$M(x)$: x is a man.

$H(y)$: y is a mortal. , then we may write:

$M(x) \wedge H(y)$: x is a man and y is a mortal.

Now,

7 All men are mortal.

8 Every apple is red.

9 Any integer is either positive or negative.

Let us paraphrase these in the following manner.

7a For all x, if x is a man, then x is a mortal.

8a For all x, if x is an apple, then x is red.

9a For all x, if x is a integer, then x is either positive or negative.

The Existential Quantifiers

- Suppose for a predicate P(x), ∀ P(x) is false, **but at least one value of x for P(x) is true**, then here, x is bound by **existential quantifier**.
- The phrase, *"there exists"* or "*some*" is denoted by the symbol ∃.
- **<u>Example</u>**: Let P(x): x + 3 = 5

  Putting x = 2, 2 + 3 = 5. So, the proposition ∃ x P(x) is true.

  But, ∀ x P(x) is false.

<u>Exercise</u> : If M(x): "x is man" and C(x): "x is clever", then translate the statement, ∃ x (M(x) → C(x)) into English.

<u>Soln</u>: *There exits a man who is clever.*

Or it can be written as: *If there exists an x such that if x is a man then x is clever*

Consider the following statements:

11 There exists a man.

12 Some men are clever.

13 Some real numbers are rational.

The first statement can be expressed in various ways, two such ways being

11a There exists an x such that x is a man.

11b There is at least one x such that x is a man.

Similarly, (12) can be written as

12a There exists an x such that x is a man and x is clever.

12b There exists at least one x such that x is a man and x is clever.

and using the existential quantifier, we can write the Statements (11) to (13) as

11c $(\exists x)(M(x))$

12c $(\exists x)(M(x) \wedge C(x))$

13c $(\exists x)(R_1(x) \wedge R_2(x))$

Ex. 11 Anyone who is persistent can learn logic.

Ans. Let $P(x)$: x is persistent

$L(x)$: x can learn logic

The given statement is

$$(x)\,(P(x) \rightarrow L(x))$$

Ex. 12 Some women are beautiful

Ans. Let $W(x)$: x is a woman

$B(x)$: x is beautiful

The given statement is

$$(Ex)\,(W(x) \rightarrow B\,(x))$$

Ex. 13 No woman is intelligent as well as beautiful.

Ans. Let W(X) : x is a woman

I(x) : x is intelligent

B(x) : x is beautiful.

The given statement is

$$(x)\ (W(x) \rightarrow [I(x) \wedge B(x)])$$

Ex. 14 All films except those specially made for children, are liked by children.

Ans. Let F(x) : x is a film

C(x) : x is made for children

L (x) : x is likely by children.

Then, the given statement is

$$(x)\ ([F(x) \rightarrow \sim C(x)] \rightarrow L(x))$$

We now consider examples involving more than one variable.

Example 6 :

Write the following two propositions in symbols.

(i) 'For every number x there is a number y such that $y = x + 1$.'

(ii) 'There is a number y such that, for every number x, $y = x + 1$.'

Solution :

Let $P(x,y)$ denote the predicate '$y = x + 1$'.

(i) The first proposition is :

$\forall x \exists y\, P(x, y)$

(ii) The second proposition is :

$\exists y \forall x\, P(x,y)$

Example 13 :

Write English sentenses corresponding to following :

(i) $\forall x \exists y\, R(x, y)$

(ii) $\exists x \forall y\, R(x, y)$

(iii) $\forall x (\sim Q(x))$

(iv) $\exists y (\sim P(y))$

(v) $\forall x\, P(x)$

where $P(x)$: x is even

$Q(x)$: x is prime no.

$R(x, y)$: $x + y$ is even

Solution :

(i) $\forall x \exists y\, R(x, y)$

For all x there exists y, such that x + y is even integer.

(ii) $\exists x \forall y\, R(x, y)$

For all y there exists x, such that x + y is even integer.

(iii) $\forall x (\sim Q(x))$

For all x, x is not prime numbers.

(iv) $\exists y (\sim P(y))$

There exists y, such that y is odd.

(v) $\forall x\, P(x)$

For all x, x is even.

Normal Forms

Normal Forms

- A product of the variables and their negations in a formula is called an **elementary product.**

Eg: $\neg p \wedge q$, $q \wedge r \wedge \neg s$, q

P^Q^~R

- A sum of variables and their negations in a formula is called an **elementary sum.**

Eg: $\neg p \vee q$, $q \vee p \vee s$, p

P v ~P v Q v ~R

A necessary and sufficient condition for an elementary product to be identically false is that it contain at least one pair of factors in which one is the negation of the other.

P ^ Q ^ ~R ^ ~ Q .....^ 0 = 0

A necessary and sufficient condition for an elementary sum to be identically true is that it contain at least one pair of factors in which one is the negation of the other.

P v `~P v Q v ~R .... v 1 =1

Normal Forms

- Well formed formula(wff) of propositional logic is string that contains propositional variables, connectives and parenthesis used in a proper manner.

Eg: $((p \lor q) \land (\neg p \lor q \lor r)) \land (\neg s \lor q)$ is wff with the variables *p, q, r, s.*

- An expression of the form ***'p v q v ¬s'*** **is a disjunction**
- An expression of the form ***'p ∧ ¬q ∧ r'*** **is a conjunction**
- It is convenient to use word 'product' for conjunction and 'sum' for disjunction.

Logical Equivalences using Conditional and Biconditionals

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(q \rightarrow \neg p)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

DISJUNCTIVE NORMAL FORMS

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula.

$$(p \wedge q) \vee (\neg p \wedge \neg q) \vee (\cdot \wedge \cdots \wedge \cdots)$$

$$\left(P \Rightarrow (Q \wedge R)\right) \wedge \left(\neg P \Rightarrow (\neg Q \wedge \neg R)\right)$$

$$\left(\neg P \vee (Q \wedge R)\right) \wedge \left(\neg(\neg P) \vee (\neg Q \wedge \neg R)\right)$$

$$\left(\neg P \vee (Q \wedge R)\right) \wedge \left(P \vee (\neg Q \wedge \neg R)\right)$$

$$\left[\left(\neg P \vee (Q \wedge R)\right) \wedge P\right] \vee \left[\left(\neg P \vee (Q \wedge R)\right) \wedge (\neg Q \wedge \neg R)\right]$$

$$\left[(\neg P \wedge P) \vee (Q \wedge R \wedge P)\right] \vee \left[\left(\neg P \wedge (\neg Q \wedge \neg R)\right) \vee \left((Q \wedge R) \wedge (\neg Q \wedge \neg R)\right)\right]$$

$$(\neg P \wedge P) \vee (Q \wedge R \wedge P) \vee (\neg P \wedge \neg Q \wedge \neg R) \vee (Q \wedge R \wedge \neg Q \wedge \neg R)$$

CONJUNCTIVE NORMAL FORMS

A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a conjunctive normal form of the given formula.

$$(P \vee Q) \wedge (\cdots \vee \cdots \vee \cdots) \wedge (\cdots \vee \cdots \vee \cdots)$$

$$(\neg P \vee \neg Q) \Rightarrow (P \Leftrightarrow \neg Q)$$

$$\neg(\neg P \vee \neg Q) \vee [(P \Rightarrow \neg Q) \wedge (\neg Q \Rightarrow P)]$$

$$(P \wedge Q) \vee [(\neg P \vee \neg Q) \wedge (\neg\neg Q \vee P)]$$

$$(P \wedge Q) \vee [(\neg P \vee \neg Q) \wedge (P \vee Q)]$$

$$P \vee [(\neg P \vee \neg Q) \wedge (P \vee Q)] \quad \wedge$$

$$Q \vee [(\neg P \vee \neg Q) \wedge (P \vee Q)]$$

$$(P \vee \neg P \vee \neg Q) \wedge (P \vee P \vee Q)$$

$$\wedge\ (Q \vee \neg P \vee \neg Q) \wedge (Q \vee P \vee Q)$$

Ex. 1 obtain disjuctive normal form of

$(P \to q) \land \neg q$

Solution: $(P \to q) \land \neg q$

$\Leftrightarrow (\neg P \lor q) \land \neg q$ Distributive Law

$\Leftrightarrow (\neg P \land \neg q) \lor (q \land \neg q)$

Ex. 2: $\neg(p \wedge q) \leftrightarrow (p \vee q)$

Sol^n: Apply law – $(p \leftrightarrow q) \leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

It gives

$\leftrightarrow [\neg(p \wedge q) \wedge (p \vee q)] \vee [\neg\neg(p \wedge q) \wedge \neg(p \vee q)]$

$\leftrightarrow [(\neg p \vee \neg q) \wedge (p \vee q)] \vee [(p \wedge q) \wedge (\neg p \wedge \neg q)]$

$\leftrightarrow [(\neg p \wedge p) \vee (\neg q \wedge p) \vee (\neg q \wedge p) \vee (\neg q \wedge q)] \vee$
$(p \wedge q \wedge \neg p \wedge \neg q)$

$\leftrightarrow (\neg p \wedge p) \vee (\neg q \wedge p) \vee (\neg q \wedge p) \vee (\neg q \wedge q) \vee$
$(p \wedge q \wedge \neg p \wedge \neg q)$

obtain conjunctive normal form

Ex 1 $(p \to q) \land \neg q$

Sol^n: $\Leftrightarrow (\neg p \lor q) \land q$

Above is in CNF.

Ex :2

$$(\neg(p \wedge q)) \leftrightarrow ((p \vee q))$$

Solution:- Apply law $(p \leftrightarrow q) \leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

$$\leftrightarrow [\neg(p \wedge q) \rightarrow (p \vee q)] \wedge [(p \vee q) \rightarrow \neg(p \wedge q)]$$

Apply law $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

$$\leftrightarrow [\neg\neg(p \wedge q) \vee (p \vee q)] \wedge [\neg(p \vee q) \vee \neg(p \wedge q)]$$

$$\leftrightarrow [(p \wedge q) \vee (p \vee q)] \wedge [\neg(p \vee q) \vee \neg(p \wedge q)]$$

$$\leftrightarrow [(p \vee (p \vee q)) \wedge (q \vee (p \vee q))] \wedge [\neg(p \vee q) \vee (\neg p \vee \neg q)]$$

$$\Leftrightarrow [(p \lor p \lor q) \land (q \lor p \lor q)] \land [(\neg p \land \neg q) \lor (\neg p \lor \neg q)]$$

$$\Leftrightarrow [(p \lor p \lor q) \land (q \lor p \lor q)] \land [(\neg p \lor \neg p \lor \neg q) \land (\neg q \lor \neg p \lor \neg q)]$$

$$\Leftrightarrow (p \lor p \lor q) \land (q \lor p \lor q) \land (\neg p \lor \neg p \lor \neg q) \land (\neg q \lor \neg p \lor \neg q)$$

EXAMPLE 1 Obtain disjunctive normal forms of (a) $P \wedge (P \rightarrow Q)$; (b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$.

SOLUTION

(a) $P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q) \Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

(b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q))$$

[using $R \rightleftarrows S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S)$]

$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$$

$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg P)$$

$$\vee ((P \vee Q) \wedge \neg Q)$$

$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P)$$

$$\vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$$

For above same problems the conjunctive normal forms are

(a) $P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q)$. Hence $P \wedge (\neg P \vee Q)$ is a required form.

(b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q) \Leftrightarrow (\neg(P \vee Q) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q)$

$$\rightarrow \neg(P \vee Q))$$

$$[\text{using } R \rightleftarrows S \Leftrightarrow (R \rightarrow S) \wedge (S \rightarrow R)]$$

$$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q)$$

$$\vee (\neg P \wedge \neg Q))$$

$$\Leftrightarrow ((P \vee Q \vee P) \wedge (P \vee Q \vee Q))$$

$$\wedge ((\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q))$$

$$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q)$$

$$\wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)$$

Q. Convert the following expression into normal form.

$$\neg((P \vee \neg Q) \wedge \neg R)$$

Solution The conjunctive normal form can be found by the following derivation:

$$\begin{aligned}
\neg((P \vee \neg Q) \wedge \neg R) &\equiv \neg(P \vee \neg Q) \vee \neg\neg R && \text{De Morgan} \\
&\equiv \neg(P \vee \neg Q) \vee R && \text{Double negation} \\
&\equiv (\neg P \wedge \neg\neg Q) \vee R && \text{De Morgan} \\
&\equiv (\neg P \wedge Q) \vee R && \text{Double negation} \\
&\equiv (\neg P \vee R) \wedge (Q \vee R) && \text{By (1.19)}
\end{aligned}$$

Q. Convert the following expression into conjunctive normal form.

$$(P_1 \wedge P_2) \vee (P_3 \wedge (P_4 \vee P_5))$$

Solution

$$
\begin{aligned}
&(P_1 \wedge P_2) \vee (P_3 \wedge (P_4 \vee P_5)) \\
&\quad \equiv (P_1 \vee (P_3 \wedge (P_4 \vee P_5))) \\
&\quad \wedge (P_2 \vee (P_3 \wedge (P_4 \vee P_5))) && \text{By (1.19)} \\
&\quad \equiv (P_1 \vee P_3) \wedge (P_1 \vee P_4 \vee P_5) && \text{By (1.18)} \\
&\quad \wedge (P_2 \vee P_3) \wedge (P_2 \vee P_4 \vee P_5) && \text{By (1.18)}
\end{aligned}
$$

Inference Theory of Predicate Calculus

Inference Theory

- Every Theorem in Mathematics, or any subject for that matter, is supported by underlying proofs.
- These proofs are nothing but a set of arguments that are conclusive evidence of the validity of the theory.
- The arguments are chained together using Rules of Inferences to deduce new statements and ultimately prove that the theorem is valid.

Inference Theory of Predicate Calculus

- A proof is a way to derive statements from other statements.
- It starts with axioms (statements that are assumed in the current context always to be true), theorems or lemmas (statements that were proved already; the difference between a theorem and a lemma is whether it is intended as a final result or an intermediate tool), and premises P (assumptions we are making for the purpose of seeing what consequences they have), and uses inference rules to derive Q.

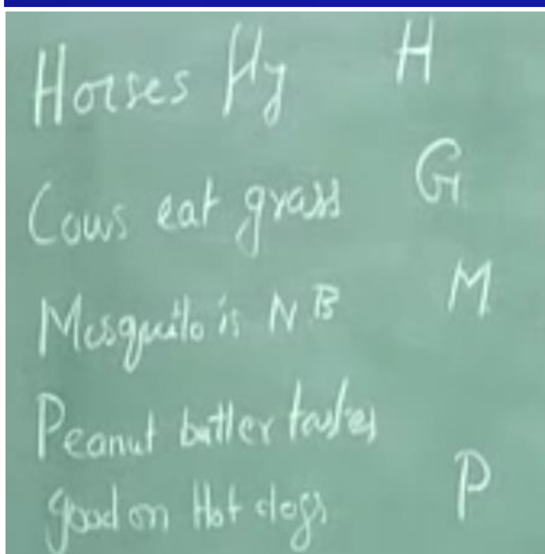
- The axioms, theorems, and premises are in a sense the starting position of a game whose rules are given by the inference rules
- The goal of the game is to apply the inference rules until Q pops out.
- We refer to anything that isn't proved in the proof itself (i.e., an axiom, theorem, lemma, or premise) as a hypothesis and the result Q is the conclusion.

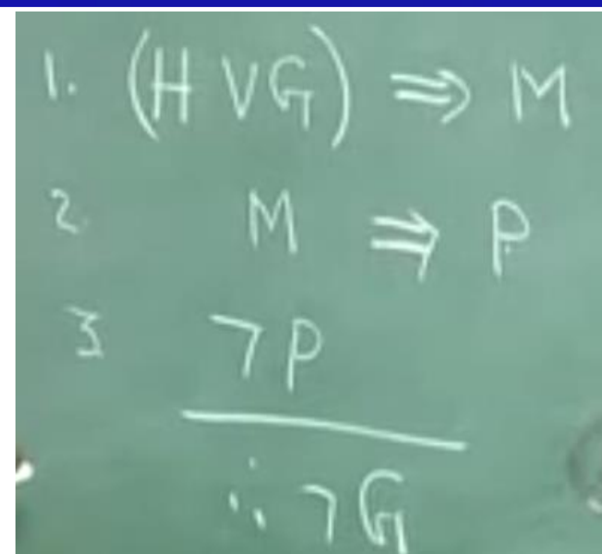
Rules of inference related to the language of propositions

Rule of inference	Tautological Form	Name
$\dfrac{P}{\therefore P \vee Q}$	$P \Rightarrow (P \vee Q)$	Addition
$\dfrac{P \wedge Q}{\therefore P}$	$(P \wedge Q) \Rightarrow P$	Simplification
$\dfrac{\begin{matrix} P \\ P \Rightarrow Q \end{matrix}}{\therefore Q}$	$[P \wedge (P \Rightarrow Q)] \Rightarrow Q$	*Modus ponens*
$\dfrac{\begin{matrix} \neg Q \\ P \Rightarrow Q \end{matrix}}{\therefore \neg P}$	$[\neg Q \wedge (P \Rightarrow Q)] \Rightarrow \neg P$	*Modus tollens*

$P \vee Q$ $\neg P$ $\therefore Q$	$[(P \vee Q) \wedge \neg P] \Rightarrow Q$	**Disjunctive syllogism**
$P \Rightarrow Q$ $Q \Rightarrow R$ $\therefore P \Rightarrow R$	$[P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow [P \Rightarrow R]$	**Hypothetical syllogism**
P Q $\therefore P \wedge Q$		**Conjunction**

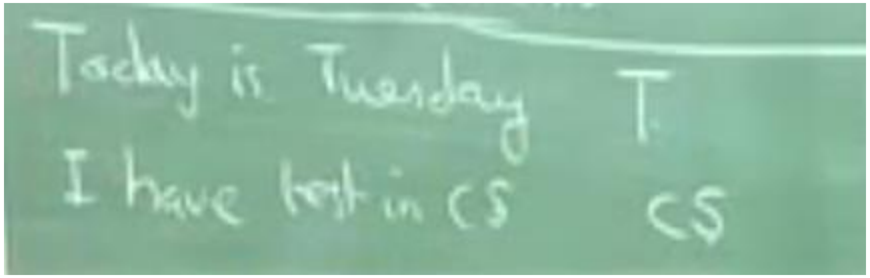
- If horses fly or cows eat grass , then the mosquito is the national bird. If the mosquito is the national bird, then peanut butter tastes good on hot dogs. But peanut tastes terrible on hot dogs. Therefore, cows don't eat grass.

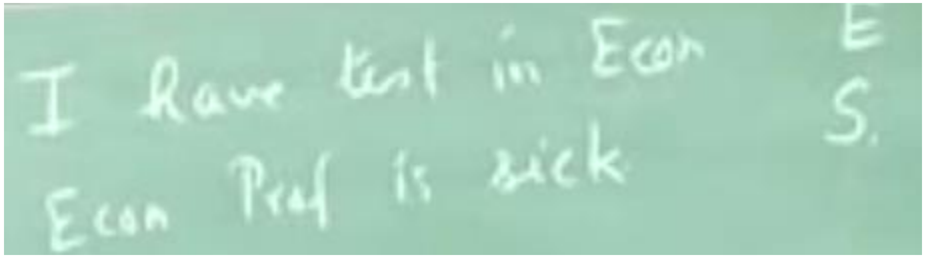




1 & 2 Hypothetical Syllogism	4.	$(H \vee G) \Rightarrow P$
4 & 3 modus tollens	5.	$\neg(H \vee G)$
5. De Morgan's law	6.	$\neg H \wedge \neg G$
6 .. simplification		$\neg G$.

- If today is Tuesday, then I have a test in computer science or a test in Econ. If my econ professor is sick, then I will not have a test in econ. Today is Tuesday and my econ professor is sick. Therefore, I have a test in computer science





1. $T \Rightarrow (CS \vee E)$
2. $S \Rightarrow \neg E$
3. $T \wedge S$

$\therefore CS$

From (3) by simplification (4) T

(5) S

From (1) and (4) modus ponens (6) $CS \vee E$

From (2) and (5) . . . . . . (7) $\neg E$

. . . (6) . . . (7) disjunctive syllogism CS

- We will now describe some important rules of inference for statements involving quantifiers.

1. **Universal instantiation** is the rule of inference used to conclude that *P(c) is true,* where *c* is a particular member of the domain, given the premise $\forall x P(x)$.

For Eg: Consider following sentences:

a) “All women are wise”

b) “Lisa is wise”

Here, we conclude statement b) from statement a) where Lisa is a member of the domain of all women.

2. **Universal generalization** is the rule of inference that states that ∀*xP(x) is true, given the* premise that *P(c) is true for all elements c in the domain.*

   Universal generalization is used when we show that ∀xP(x) is true by taking an arbitrary element c from the domain and showing that P(c) is true.

3. **Existential instantiation** is the rule that allows us to conclude that there is an element *c in* the domain for which *P(c) is true if we know that ∃xP(x) is true.*

   We cannot select an arbitrary value of c here, but rather it must be a c for which P(c) is true.

4. **Existential generalization** is the rule of inference that is used to conclude that ∃*xP(x) is* true when a particular element *c with P(c) true is known.*

   That is, if we know one element c in the domain for which P(c) is true, then we know that ∃xP(x) is true

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

EXAMPLE 12: Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Solution: *Let $D(x)$ denote "x is in this discrete mathematics class," and let $C(x)$ denote "x has* taken a course in computer science." Then the premises are $\forall x(D(x) \rightarrow C(x))$ *and* $D(Marla)$.

The conclusion is $C(Marla)$.

The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(D(x) \rightarrow C(x))$	*Premise*
2. $D(Marla) \rightarrow C(Marla)$	*Universal instantiation from (1)*
3. $D(Marla)$	*Premise*
4. $C(Marla)$	*Modus ponens from (2) and (3)*

EXAMPLE 13 Show that the premises “A student in this class has not read the book,” and “Everyone in this class passed the first exam” imply the conclusion “Someone who passed the first exam has not read the book.”

Solution**: Let* ***C(x) *be “x is in this class,”* ***B(x)*** *be “x has read the book,” and* ***P(x)*** *be “x passed* the first exam.” The premises are $\exists x(C(x) \land B(x))$ *and* $\forall x(C(x) \to P(x))$. *The conclusion is* $\exists x(P(x) \land B(x))$. *These steps can be used to establish the conclusion from the premises.*

Step	**Reason**
1. $\exists x(C(x) \land B(x))$	*Premise*
2. $C(a) \land B(a)$	*Existential instantiation from (1)*
3. $C(a)$	*Simplification from (2)*
4. $\forall x(C(x) \to P(x))$	*Premise*
5. $C(a) \to P(a)$	*Universal instantiation from (4)*
6. $P(a)$	*Modus ponens from (3) and (5)*
7. $.B(a)$	*Simplification from (2)*
8. $P(a) \land .B(a)$	*Conjunction from (6) and (7)*
9. $\exists x(P(x) \land B(x))$	*Existential generalization from (8)*

Mathematical Induction

Mathematical Induction

- **Mathematical Induction** is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.
- Let P(n) be a statement involving a natural number n.

1. If P(n) is true for n = n_o, and
2. Assuming P(k) is true, ($k \geq n_o$) we prove P(k + 1) is also true,

   then P(n) is true for all natural numbers $n \geq n_o$

Step (1) is called as the **Basis of Induction**

Step (2) is called as the **Induction step.**

Example 1. Show by mathematical induction that, for all $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Solution: Let P(n) be the predicate $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. In this example, $n_0 = 1$.

BASIS STEP. We must first show that P(1) is true. P(1) is the statement $1 = \frac{1(1+1)}{2}$, which is clearly true.

INDUCTION STEP. We must now show that for $k \geq 1$, if P(k) is true, then P($k + 1$) must also be true. We assume that for some fixed $k \geq 1$,

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}. \tag{1}$$

We now wish to show the truth of P($k + 1$):

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)((k+1)+1)}{2}.$$

The left-hand side of P($k + 1$) can be written as $1 + 2 + 3 + \cdots + k + (k + 1)$, and we have

$$
\begin{aligned}
(1+2+3+\cdots+k)+(k+1) &= \frac{k(k+1)}{2}+(k+1) && \text{Using (1) to replace } 1+2+\cdots+k \\
&= (k+1)\left[\frac{k}{2}+1\right] && \text{Factoring} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}. && \text{The right-hand side of } \mathrm{P}(k+1)
\end{aligned}
$$

Thus we have shown that the left-hand side of $\mathrm{P}(k+1)$ equals the right-hand side of $\mathrm{P}(k+1)$. By the principle of mathematical induction, it follows that $\mathrm{P}(n)$ is true for all $n \geq 1$. ◆

Example 3

Consider the following function given in pseudocode.

FUNCTION SQ(A)
1. $C \leftarrow 0$
2. $D \leftarrow 0$
3. **WHILE** ($D \neq A$)
   a. $C \leftarrow C + A$
   b. $D \leftarrow D + 1$
4. **RETURN** (C)

END OF FUNCTION SQ

The name of the function, SQ, suggests that it computes the square of A. Step 3b shows that A must be a positive integer if the looping is to end. A few trials with particular values of A will provide evidence that the function does carry out this task. However, suppose that we now want to prove that SQ always computes the square of the positive integer A, no matter how large A might be. We shall give a proof by mathematical induction. For each integer $n \geq 0$, let C_n and D_n be the values of the variables C and D, respectively, after passing through the **WHILE** loop n times. In particular, C_0 and D_0 represent the values of the variables before looping starts. Let P(n) be the predicate $C_n = A \times D_n$. We shall prove by induction that $\forall n \geq 0$ P(n) is true. Here n_0 is 0.

BASIS STEP. P(0) is the statement $C_0 = A \times D_0$, which is true since the value of both C and D is zero "after" zero passes through the **WHILE** loop.

INDUCTION STEP. We must now use

$$\mathrm{P}(k)\colon C_k = A \times D_k \tag{2}$$

to show that $\mathrm{P}(k+1)\colon C_{k+1} = A \times D_{k+1}$. After a pass through the loop, C is increased by A, and D is increased by 1, so $C_{k+1} = C_k + A$ and $D_{k+1} = D_k + 1$.

left-hand side of P($k+1$): $C_{k+1} = C_k + A$

$= A \times D_k + A$ Using (2) to replace C_k

$= A \times (D_k + 1)$ Factoring

$= A \times D_{k+1}$, Right-hand side of P($k+1$)

By the principle of mathematical induction, it follows that as long as looping occurs $C_n = A \times D_n$. The loop must terminate. (Why?) When the loop terminates, $D = A$, so $C = A \times A$, or A^2, and this is the value returned by the function SQ. ◆

Example 4

Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Solution: The sums of the first n positive odd integers for $n = 1, 2, 3, 4, 5$ are

$$1 = 1, \quad 1 + 3 = 4, \quad 1 + 3 + 5 = 9,$$
$$1 + 3 + 5 + 7 = 16, \quad 1 + 3 + 5 + 7 + 9 = 25.$$

From these values it is reasonable to conjecture that the sum of the first n positive odd integers is n^2, that is, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. We need a method to *prove* that this *conjecture* is correct. if in fact it is.

BASIS STEP: $P(1)$ states that the sum of the first one odd positive integer is 1^2. This is true because the sum of the first odd positive integer is 1. The basis step is complete.

INDUCTIVE STEP: To complete the inductive step we must show that the proposition $P(k) \to P(k + 1)$ is true for every positive integer k. To do this, we first assume the inductive hypothesis. The inductive hypothesis is the statement that $P(k)$ is true for an arbitrary positive integer k, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

To show that $\forall k(P(k) \to P(k+1))$ is true, we must show that if $P(k)$ is true (the inductive hypothesis), then $P(k+1)$ is true. Note that $P(k+1)$ is the statement that

$$1 + 3 + 5 + \cdots + (2k-1) + (2k+1) = (k+1)^2.$$

Before we complete the inductive step, we will take a time out to figure out a strategy. At this stage of a mathematical induction proof it is time to look for a way to use the inductive hypothesis to show that $P(k+1)$ is true. Here we note that $1 + 3 + 5 + \cdots + (2k-1) + (2k+1)$ is the sum of its first k terms $1 + 3 + 5 + \cdots + (2k-1)$ and its last term $2k-1$. So, we can use our inductive hypothesis to replace $1 + 3 + 5 + \cdots + (2k-1)$ by k^2.

We now return to our proof. We find that

$$\begin{aligned}
1 + 3 + 5 + \cdots + (2k-1) + (2k+1) &= [1 + 3 + \cdots + (2k-1)] + (2k+1) \\
&\overset{\text{IH}}{=} k^2 + (2k+1) \\
&= k^2 + 2k + 1 \\
&= (k+1)^2.
\end{aligned}$$

Example 5

Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Solution: Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

BASIS STEP: $P(0)$ is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis, we assume that $P(k)$ is true for an arbitrary nonnegative integer k. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k + 1)$ is also true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned} 1+2+2^2+\cdots+2^k+2^{k+1} &= (1+2+2^2+\cdots+2^k)+2^{k+1} \\ &\overset{\text{IH}}{=} (2^{k+1}-1)+2^{k+1} \\ &= 2\cdot 2^{k+1}-1 \\ &= 2^{k+2}-1. \end{aligned}$$

Note that we used the inductive hypothesis in the second equation in this string of equalities to replace $1+2+2^2+\cdots+2^k$ by $2^{k+1}-1$. We have completed the inductive step.

Because we have completed the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true for all nonnegative integers n. That is, $1+2+\cdots+2^n = 2^{n+1}-1$ for all nonnegative integers n.

Example 1

Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

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$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2.$$

Before we complete the inductive step, we will take a time out to figure out a strategy. At this stage of a mathematical induction proof it is time to look for a way to use the inductive hypothesis to show that $P(k+1)$ is true. Here we note that $1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1)$ is the sum of its first k terms $1 + 3 + 5 + \cdots + (2k - 1)$ and its last term $2k - 1$. So, we can use our inductive hypothesis to replace $1 + 3 + 5 + \cdots + (2k - 1)$ by k^2.

We now return to our proof. We find that

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + \cdots + (2k - 1)] + (2k + 1) \\ &\overset{\text{IH}}{=} k^2 + (2k + 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2. \end{aligned}$$

Example 2

Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Solution: Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

BASIS STEP: $P(0)$ is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis, we assume that $P(k)$ is true for an arbitrary nonnegative integer k. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k + 1)$ is also true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned}
1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\
&\overset{\text{IH}}{=} (2^{k+1} - 1) + 2^{k+1} \\
&= 2 \cdot 2^{k+1} - 1 \\
&= 2^{k+2} - 1.
\end{aligned}$$

Note that we used the inductive hypothesis in the second equation in this string of equalities to replace $1 + 2 + 2^2 + \cdots + 2^k$ by $2^{k+1} - 1$. We have completed the inductive step.

Because we have completed the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true for all nonnegative integers n. That is, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n. ◀

Example 4

An Inequality for Harmonic Numbers The **harmonic numbers** H_j, $j = 1, 2, 3, \ldots,$ are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}.$$

For instance,

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2},$$

whenever n is a nonnegative integer.

Solution: To carry out the proof, let $P(n)$ be the proposition that $H_{2^n} \geq 1 + \frac{n}{2}$.

BASIS STEP: $P(0)$ is true, because $H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $H_{2^k} \geq 1 + \frac{k}{2}$, where k is an arbitrary nonnegative integer. We must show that if $P(k)$ is true, then $P(k+1)$, which states that $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$, is also true. So, assuming the inductive hypothesis, it follows that

$$
\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of harmonic number} \\
&= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of } 2^k\text{th harmonic number} \\
&\overset{\text{IH}}{\geq} \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the inductive hypothesis} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} && \text{because there are } 2^k \text{ terms each} \geq 1/2^{k+1} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} && \text{canceling a common factor of } 2^k \text{ in second term} \\
&= 1 + \frac{k+1}{2}.
\end{aligned}
$$

This establishes the inductive step of the proof.

We have completed the basis step and the inductive step. Thus, by mathematical induction $P(n)$ is true for all nonnegative integers n. That is, the inequality $H_{2^n} \geq 1 + \frac{n}{2}$ for the harmonic numbers holds for all nonnegative integers n. ◀

Remark: The inequality established here shows that the **harmonic series**

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is a divergent infinite series. This is an important example in the study of infinite series.

Example 5

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer. (Note that this is the statement with $p = 3$ of Fermat's little theorem, which is Theorem 3 of Section 4.4.)

Solution: To construct the proof, let $P(n)$ denote the proposition: "$n^3 - n$ is divisible by 3."

BASIS STEP: The statement $P(1)$ is true because $1^3 - 1 = 0$ is divisible by 3. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true; that is, we assume that $k^3 - k$ is divisible by 3 for an arbitrary positive integer k. To complete the inductive step, we must show that when we assume the inductive hypothesis, it follows that $P(k + 1)$, the statement that $(k + 1)^3 - (k + 1)$ is divisible by 3, is also true. That is, we must show that $(k + 1)^3 - (k + 1)$ is divisible by 3. Note that

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$$
$$= (k^3 - k) + 3(k^2 + k).$$

Using the inductive hypothesis, we conclude that the first term $k^3 - k$ is divisible by 3. The second term is divisible by 3 because it is 3 times an integer. So, by part (i) of Theorem 1 in Section 4.1, we know that $(k+1)^3 - (k+1)$ is also divisible by 3. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $n^3 - n$ is divisible by 3 whenever n is a positive integer. ◀

Example 6

Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Solution: To construct the proof, let $P(n)$ denote the proposition: "$7^{n+2} + 8^{2n+1}$ is divisible by 57."

BASIS STEP: To complete the basis step, we must show that $P(0)$ is true, because we want to prove that $P(n)$ is true for every nonnegative integer n. We see that $P(0)$ is true because $7^{0+2} + 8^{2\cdot 0+1} = 7^2 + 8^1 = 57$ is divisible by 57. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k; that is, we assume that $7^{k+2} + 8^{2k+1}$ is divisible by 57. To complete the inductive step, we must show that when we assume that the inductive hypothesis $P(k)$ is true, then $P(k+1)$, the statement that $7^{(k+1)+2} + 8^{2(k+1)+1}$ is divisible by 57, is also true.

The difficult part of the proof is to see how to use the inductive hypothesis. To take advantage of the inductive hypothesis, we use these steps:

$$\begin{aligned} 7^{(k+1)+2} + 8^{2(k+1)+1} &= 7^{k+3} + 8^{2k+3} \\ &= 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1} \\ &= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1} \\ &= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}. \end{aligned}$$

We can now use the inductive hypothesis, which states that $7^{k+2} + 8^{2k+1}$ is divisible by 57. We will use parts (i) and (ii) of Theorem 1 in Section 4.1. By part (ii) of this theorem, and the inductive hypothesis, we conclude that the first term in this last sum, $7(7^{k+2} + 8^{2k+1})$, is divisible by 57. By part (ii) of this theorem, the second term in this sum, $57 \cdot 8^{2k+1}$, is divisible by 57. Hence, by part (i) of this theorem, we conclude that $7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1} = 7^{k+3} + 8^{2k+3}$ is divisible by 57. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n. ◀

2. P.T. by induction $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}\ \forall n \in N$.**[D-98, 00, M-99]**

Soln. :

For n = 1

L.H.S. = $1^2 = 1$

R.H.S. = $\frac{1.(2+1)(1)}{3} = 1$

L.H.S. = R.H.S.

The result is true for n = 1

Let the result is true for n = k

$1^2 + 3^2 + \ldots. + (2k-1)^2 = \frac{k(2k+1)(2k-1)}{3}$ [i..e P(k)]

Now we have to prove that P(k) $\Rightarrow$ P(k + 1)

Hence P(k + 1) $\Rightarrow 1^2 + 3^2 + \ldots.. + (2k+1)^2 = \frac{(k+1)(2k+3)(2k+1)}{3}$

L.H.S. = $1^2 + 3^2 + \ldots.. + (2k-1)^2 + (2k+1)^2$

$$= \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2 \quad \text{by } P(k)$$

$$= (2k+1)\left[\frac{k(2k-1)+3(2k+1)}{3}\right]$$

$$= \frac{(2k+1)}{3}[2k^2 - k + 6k + 3]$$

$$= \frac{2k+1}{3}[2k^2 + 5k + 3]$$

$$= \frac{2k+1}{3}[2k^2 + 2k + 3k + 3]$$

$$= \frac{2k+1}{3}[2k(k+1) + 3(k+1)]$$

$$= \frac{(2k+1)}{3}(k+1)(2k+3)$$

$$= \frac{(k+1)(2k+3)(2k+1)}{3}$$

= R.H.S.

$\therefore$ $P(k) \Rightarrow P(k+1)$

$\therefore$ the result is true $n = k + 1$

Hence the result is true for all n.

5. Use induction to show that, $1 + 2 + 2^2 + 2^3 + \ldots\ldots\ldots + 2^n = 2^{n+1} - 1$. **[M-04, N-04]**

Soln.:

We put n = 1,

L.H.S. = 3, R.H.S. = $2^{1+1} - 1 = 3$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ The result is true for n = 1

Let the result is true for n = k

i.e. $1 + 2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 1$

Now we have to prove that

$P(k) \Rightarrow P(k+1)$

i.e. $1 + 2 + 2^2 + 2^3 + \ldots + 2^{k+1} = 2^{k+2} - 1$

$$\begin{aligned}
\text{L.H.S.} &= 1 + 2^1 + 2^2 + \ldots + 2^k + 2^{k+1} \\
&= 2^{k+1} - 1 + 2^{k+1} \quad [\text{by } P(k)] \\
&= 2 \cdot 2^{k+1} - 1 \\
&= 2^{(k+1)+1} - 1 \\
&= 2^{k+2} - 1 \\
&= \text{R.H.S.}
\end{aligned}$$

$\therefore P(k) \Rightarrow P(k+1)$

Hence result is true for n = k + 1

$\therefore$ The result is true for all n.

8. Use induction to show that, $1 + a + a^2 + \ldots\ldots + a^{n-1} = \dfrac{a^n - 1}{a - 1}$, $a \neq 1$

Soln.:

For n = 1,

L.H.S. = 1 R.H.S. = $\dfrac{a^1 - 1}{a - 1} = 1$

$\therefore$ the result is true for n = 1

Let the result is true for n = k

$$1 + a + a^2 + \ldots\ldots + a^{k-1} = \frac{a^k - 1}{a - 1} \quad \text{[i.e. P(k)]}$$

Now we have to prove that $P(k) \Rightarrow P(k+1)$

i.e. $1 + a + a^2 + \ldots\ldots + a^k = \dfrac{a^{k+1} - 1}{a - 1}$

L.H.S. = $1 + a + a^2 + \ldots\ldots + a^{k-1} + a^k$

$= \dfrac{a^k - 1}{a - 1} + a^k$ [by P(k)]

$= \dfrac{a^k - 1 + a^{k+1} + a^k}{a - 1}$

$= \dfrac{a^{k+1} - 1}{a - 1}$

= R.H.S.

$\therefore P(k) \Rightarrow P(k+1)$

Hence the result is true for n = k + 1

$\therefore$ The result is true for all n.