Cauchy's Residue theorem.

If $f(z)$ is analytic inside and on a simple closed curve C, except, at a finite number of isolated singular points $z_1, z_2, \cdots z_n$. inside C then

$$\oint_C f(z)\,dz = 2\pi i \text{ (Sum of residues of } z_1, z_2 \cdots z_n).$$

① Evaluate $\int_C \frac{z^2}{(z-1)^2(z-2)}\,dz$, where C is the circle $|z| = 2.5$

$f(z)$ has a simple pole at $z = 2$ and at $z=1$. is a pole of order 2.

Both poles lies inside the curve.

$$\text{Res of } f(z) \text{ at } (z=2) = \lim_{z\to 2} (z-2)\frac{z^2}{(z-1)^2(z-2)}$$

$$= \underline{\underline{4}}$$

$$\text{Res of } f(z) \text{ at } z=1 = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2\frac{z^2}{(z-1)^2(z-2)}$$

$$= -3.$$

$$\therefore \int_C f(z)\,dz = 2\pi i \text{ (Sum of Residues)}$$

$$= 2\pi i\,[4-3]$$

$$= \underline{\underline{2\pi i}}$$

2. Evaluate $\int_C \frac{dz}{z \sin z}$, $C: x^2+y^2=1$

$z \sin z = 0$

$z = 0$, $\sin z = 0$

$z = n\pi$, $n = 0$, $n = \pm 1, \ldots$

~~Here.~~ $C: x^2+y^2=1$

$(0,0)$, $r=1$

Here $z=0$ lies inside the circle.

Res. $f(z)$ at $z=0$ =

③. Evaluate $\int_C \frac{dz}{\cos z}$, C is $|z|=2$.

$\cos z = \pm \pi/2, \pm 3\pi/2, \ldots$

Here $z = \pm \pi/2$ lie inside the circle with centre $(0,0)$ and $r=2$.

Res. at $z = +\pi/2$

Res at $z = -\pi/2$.

$\oint_c \frac{e^z}{\cos \pi z} dz, \quad |z|=1$

$\cos \pi z = 0, \quad z = \underline{\underline{\pm 1/2}}, \pm 3/2 \ldots$

6). $\int_c \tan z \, dz$. ① c is $|z|=2$ (pt inside).

2) $|z|=1$ (pt outside).

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$\cos z = 0, \quad z = 0 \pm \pi/2, \pm 3\pi/2, \ldots$

$z = \pm \pi/2$.

7). $\int \frac{\cos \pi z}{z^2-1} dz$ c is the rectangle whose vertices are $2 \pm i, -2 \pm i$

8) $\oint \frac{e^z}{(z^2+\pi^2)^2} dz$.

Singular pole

$$\text{Res. } f(z) \text{ at } (z = z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

pole of order m

$$\text{Rs } f(z) \text{ at } z = z_0 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m$$

1. Evaluate $\int_C \frac{dz}{\cos z}$ where C is $|z|=2$

$\cos z = 0$

$z = \pm \pi/2, \pm 3\pi/2, \ldots$

$z = \pm \pi/2$ lies inside the circle with centre $(0,0)$ and radius 2.

Res of $f(z)$ at $z=\pi/2 = \lim_{z\to\pi/2} (z-\frac{\pi}{2})\frac{1}{\cos z}$ $\left(\frac{0}{0}\right)$, (L'Hopital's Rule)

$= \lim_{z\to\pi/2} 1 \cdot \frac{1}{-\sin z} =$

$= \underline{\underline{-1}}$

Res of $f(z)$ at $z=-\pi/2 = \lim_{z\to-\pi/2} (z+\pi/2)\frac{1}{\cos z}$ $(0/0)$

$= \lim_{z\to-\pi/2} \frac{1}{-\sin z} = \underline{\underline{1}}$

$\therefore \int_C f(z)\,dz = 2\pi i(-1+1) = \underline{\underline{0}}.$

2. Evaluate $\oint_C \frac{e^z}{\cos \pi z}\,dz$, where C is $|z|=1$

$\cos \pi z = 0$

$\pi z = \pm \pi/2, \pm 3\pi/2,$

$z = \pm 1/2, \pm 3/2,$

Here $z = \pm 1/2$ lie inside the circle with centre $(0,0)$ and radius 1.

Res. of $f(z)$ at $z=1/2 = \lim_{z\to 1/2} (z-1/2)\frac{e^z}{\cos \pi z}$ $\left(\frac{0}{0}\right)$

$= \lim_{z\to 1/2} 1 \cdot \frac{[e^z] + e^z(z-1/2)}{-\pi \sin \pi z}$

$= -\frac{1}{\pi}\cdot e^{1/2}$ //

Residue of $f(z)$ at $(z=-1/2) = \frac{e^{-1/2}}{\pi}$

$\therefore \int_C f(z)\,dz = 2\pi i\left[-\frac{e^{1/2}}{\pi}+\frac{e^{-1/2}}{\pi}\right]$

$= \frac{-2\pi i}{\pi}\left(e^{1/2}-e^{-1/2}\right)$

$= -2i \times 2\left(\frac{e^{1/2}-e^{-1/2}}{2}\right)$

$= -4i\sinh 1/2$

(3) Evaluate $\int_C \tan z\,dz$ where C is the circle (1) $|z|=2$ (2) $|z|=1$

$\tan z = \frac{\sin z}{\cos z}$ $\qquad \cos z = 0$

$\qquad z = \pm\pi/2$

(1) $z = \pm\pi/2$ lie inside the circle with centre $(0,0)$ and radius 2.

Res. of $f(z)$ at $(z=\pi/2) = \lim_{z\to\pi/2}(z-\pi/2)\cdot\frac{\sin z}{\cos z}\left(\frac{0}{0}\right)$

$= \lim_{z\to\pi/2}\frac{(z-\pi/2)\cdot\cos z+\sin z}{-\sin z}$

$= -1$

Res of $f(z)$ at $(z=-\pi/2) = \lim_{z\to-\pi/2}\left(z+\frac{\pi}{2}\right)\cdot\frac{\sin z}{\cos z}$

$= \lim_{z\to-\pi/2}\cdot\frac{(z+\pi/2)\cdot\cos z+\sin z}{\cancel{\cos}-\sin z}$

$= \frac{-1}{+1} = -1$

$\therefore \int f(z)\,dz = 2\pi i(-1-1)$

$= -4\pi i$

$z = \pm \pi/2, \pm 3\pi/2, \ldots$, lie outside the circle, $|z|$ centre $(0,0)$ and $r = 1$

By Cauchy's integral theorem,

$$\oint_c \tan z \, dz = 0$$

Application of Residues:

Integral of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$.

$z = e^{i\theta}$ $\quad \theta \to (0, 2\pi) \Rightarrow$ a unit circle $|z| = 1$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{z^2-1}{2iz}$$

$$d\theta = \frac{dz}{iz}$$

1. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$

Let $e^{i\theta} = z$, $e^{i\theta} i\, d\theta = dz$; $d\theta = \dfrac{dz}{iz}$

$$\sin\theta = \frac{z^2-1}{2iz}$$

$$I = \int_c \frac{1}{5+3\left(\frac{z^2-1}{2iz}\right)} \cdot \frac{dz}{iz}$$

$$= \int_c \frac{1}{\frac{10iz+3z^2-3}{2iz}} \cdot \frac{dz}{iz} = \int_c \frac{2}{3z^2+10iz-3}\, dz$$

$I = \int_c \frac{2}{(3z+i)(z+3i)} dz$ where c is $|z|$

$(3z+i)(z+3i) = 0$

$3z+i=0 \qquad z+3i=0$

$3z = -i \qquad z = -3i$

$z = -i/3$

Here $z = -i/3$ lies inside the circle $|z|=1$ and $z=-3i$ lies outside the circle $|z|=1$

$\therefore$ Res. of $f(z)$ at $z = -i/3$

$= \lim_{z \to -i/3} (z - -i/3) \frac{2}{(3z+i)(z+3i)}$

$= \lim_{z \to -i/3} (z+i/3) \frac{2}{(3z+i)(z+3i)}$

$= \lim_{z \to -i/3} \left(\frac{3z+i}{3}\right) \frac{2}{(3z+i)(z+3i)}$

$= \lim_{z \to -i/3} \frac{2}{3} \cdot \frac{1}{z+3i}$

$= \underline{\underline{1/4i}}$

$\therefore \quad I = 2\pi i \times \frac{1}{4i} = \underline{\underline{\pi/2}}$

2. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$

Note: $\cos 2\theta$ is RP of $e^{2i\theta}$

Let $I = \int_0^{2\pi} \frac{e^{i2\theta}}{5+4\cos\theta}\, d\theta$

$z = e^{i\theta}$, $dz = iz\,d\theta$ $\quad \cos\theta = \frac{z^2+1}{2}$

$d\theta = dz/iz$,

$\therefore I = \int_C \frac{z^2}{5+4\left(\frac{z^2+1}{2z}\right)} dz/iz$

$= \int_C \frac{z^2}{i\left[\frac{10z+4z^2+4}{2}\right)} dz$

$I = \int_C \frac{z^2}{i\,(2z^2+5z+2)} dz$

$2z^2+5z+2=0$

$(2z+1)(z+2)=0$

$2z=-1 \qquad \underline{\underline{z=-2}}$

$\underline{\underline{z=-1/2}}$

Here $z=-1/2$ lies inside the unit circle $|z|=1$

and $z=-2$ lies outside the same circle.

Res. of $f(z)$ at $(z=-1/2) = \lim_{z\to -1/2} (z+1/2) \cdot \frac{z^2}{i(2z^2+5z+2)}$

$= \lim_{z\to -1/2} (z+1/2) \cdot \frac{z^2}{(2z+1)(z+2)}$

$= \lim_{z \to -1/2} \left(\frac{2z+1}{2} \right) \cdot \frac{z^2}{i(2z+1)(z+2)}$

$= \lim_{z \to -1/2} \frac{z^2}{i\,2(z+2)}$

$= \frac{(-1/2)^2}{i\,2(-\frac{1}{2}+2)} = \frac{1/4}{i\,2 \times 3/2} = 1/12i$

$\therefore \int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta}\, d\theta = 2\pi i \times \frac{1}{12i} = \frac{\pi}{6}$

$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = R.P \text{ of } \int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta}\, d\theta$

$= \underline{\underline{\pi/6}}$

Q-3. Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ where $a > b > 0$

$z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$

$I = \int_C \frac{1}{a+b\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz}$

$= \int_C \frac{1}{\frac{bz^2+2az+b}{2z}} \cdot \frac{dz}{iz} = \int_C \frac{2}{(bz^2+2az+b)i}\, dz$

$$bz^2 + 2az + b = 0.$$

$$z = \frac{-2a \pm \sqrt{4a^2 - 4\times b^2}}{2b}$$

$$= \frac{-2a \pm 2\sqrt{a^2-b^2}}{2b} = \frac{-a \pm \sqrt{a^2-b^2}}{b}.$$

Let $\alpha = \frac{-a + \sqrt{a^2-b^2}}{b}$, $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}.$

$a > b > 0$

α lies inside the circle and

β lies ~~inside~~ outside the circle $|z| = 1$

$$\text{Res. of } f(z) \text{ at } (z=\alpha) = \lim_{z\to\alpha} (z-\alpha)\frac{2}{b(z-\alpha)(z-\beta)i}$$

$$= \lim_{z\to\alpha} \frac{2}{bi(z-\beta)}$$

$$= \frac{2}{bi(\alpha-\beta)}.$$

$$\alpha - \beta = \frac{-a+\sqrt{a^2-b^2}}{b} - \left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)$$

$$= \frac{1}{b}\left[2\sqrt{a^2-b^2}\right]$$

$$\text{Res of } f(z) \text{ at } (z=\alpha) = \frac{2}{bi\left[\frac{1}{b}2\sqrt{a^2-b^2}\right]} = \frac{1}{i\sqrt{a^2-b^2}}$$

$$\therefore\ I = 2\pi i \frac{1}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Integral of the form $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}$

where $P(x)$ & $Q(x)$ are polynomials and degree of $Q(x)$ is greater than that of $P(x)$.

Q.1. Evaluate $\int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4-10x^2+9}\,dx$ using contour integration.

Consider a contour consisting of a large semi circle with centre $(0,0)$, in the upper half of the plane and its diameter on the real axis.

$$z f(z) \to \frac{z^3+z^2+2}{z^4-10z^2+9} \to 0 \text{ as } |z| \to \infty$$

$$z^4 - 10z^2 + 9 = 0 \Rightarrow (z^2+1)(z^2+9) = 0.$$

$$z = +i, -i, +3i, -3i$$

The poles lying in the upper half of the semi circle is $+i, +3i$

$$\text{Res of } f(z) \text{ at } (z=i) = \lim_{z\to i} \frac{(z-i)(z^2+z+2)}{(z+i)(z-i)(z^2+9)}$$

$$= \lim_{z\to i} \frac{z^2+z+2}{(z+i)(z^2+9)}$$

$$= \frac{1+i}{16i}$$

$$\text{Res of } f(z) \text{ at } (z=3i) = \lim_{z\to 3i} \frac{(z-3i)(z^2+z+2)}{(z^2+1)(z+3i)(z-3i)}$$

$$= \frac{7-3i}{48i}$$

$I = 2\pi i \left[\frac{\cdots}{16i} + \frac{\cdots}{48i} \right] = 5\pi/12$

Q.2. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^3}$, $a>0$.

Consider the contour consisting of a semicircle and diameter on the real axis with centre at the origin

$z\, f(z) = \frac{z}{(z^2+a^2)^3} \to 0, \quad |z| \to \infty$

$(z^2+a^2)^3 = 0$

$z^2 + a^2 = 0 \qquad z = \pm ai$

Let $z = ~~a~~ + ai$ [upper half of the plane only positive values to be considered].

~~It lies inside is~~ a pole of order 3.

$z = +ai$ lies inside the region

Res of $f(z)$ at $z = ai = \lim_{z \to ai} \frac{1}{2!} \frac{d^2}{dz^2} (z-ai)^3 \frac{1}{(z+ai)^3 (z-ai)^3}$

$= \lim_{z \to ai} \frac{1}{2} \cdot \frac{d^2}{dz} \frac{1}{(z+ai)^3} = 3/16a^5 i$

$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} = 2\pi i \frac{3}{16a^5 i} = \frac{3\pi}{8a^5}$

$\therefore \int_0^\infty \frac{dx}{(x^2+a^2)^3} = \frac{1}{2} \frac{3\pi}{8a^5} = \frac{3\pi}{16a^5}$