Tutorial (NLPP)

1. Obtain the relative maxima or minima (if any) of the functions $z = x_1 + 2x_3 + x_2 x_3 - x_1^2 - x_2^2 - x_3^2$

Sol^n: We have $f(x_1, x_2, x_3) = x_1 + 2x_3 + x_2 x_3 - x_1^2 - x_2^2 - x_3^2$

The stationary points are given by

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \quad \frac{\partial f}{\partial x_3} = 0$$

Now, $\dfrac{\partial f}{\partial x_1} = 1 - 2x_1 \quad \therefore \ 1 - 2x_1 = 0 \quad \therefore \ x_1 = \dfrac{1}{2}$

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$$\frac{\partial f}{\partial x_2} = x_3 - 2x_2 \quad \therefore x_3 - 2x_2 = 0$$

$$\frac{\partial f}{\partial x_3} = 2 + x_2 - 2x_3 \quad \therefore 2 + x_2 - 2x_3 = 0$$

Solving the last two simultaneous eq^n, we get $x_2 = 2/3$, $x_3 = 4/3$.

$\therefore$ Thus, X_0 $(1/2, 2/3, 4/3)$ is the stationary point. To check whether the point is a point of minima or a point of maxima, we apply the sufficiency test.

Clearly, $\frac{\partial^2 f}{\partial x_1^2} = -2$, $\frac{\partial^2 f}{\partial x_2^2} = -2$, $\frac{\partial^2 f}{\partial x_3^2} = -2$,

$\frac{\partial^2 f}{\partial x_3 \partial x_2} = 1$ & the remainging second order derivatives are zero.

Now, consider the Hessian matrix at X_0 $(1/2, 2/3, 4/3)$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

The principal minors of this matrix are

$$[-2], \quad \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

The values of their determinants are $-2, 4, -6$.

$\therefore$ the values of the determinants are alternatively $-, +$ & $-$, X_0 $(1/2, 2/3, 4/3)$ is a maxima.

$$\therefore Z_{max} = \frac{1}{2} + 2 \cdot \frac{4}{3} + \frac{2}{3} \cdot \frac{4}{3} - \left(\frac{1}{2}\right)^2 - \left(\frac{2}{3}\right)^2 -$$

$\left(\frac{4}{3}\right)^2 = \frac{19}{12}$

2. Using the method of Lagrange's multipliers solve the foll. NLPP.

optimise $z = 4x_1 + 8x_2 - x_1^2 - x_2^2$

Subs to $x_1 + x_2 = 4$

$x_1, x_2 \geq 0$

Sol^n We have the Lagrangian function

$L(x_1, x_2, \lambda) = (4x_1 + 8x_2 - x_1^2 - x_2^2) - \lambda(x_1 + x_2 - 4)$

We, now, obtain the following partial derivatives

$\frac{\partial L}{\partial x_1} = 4 - 2x_1 - \lambda$, $\frac{\partial L}{\partial x_2} = 8 - 2x_2 - \lambda$,

$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 4)$

Solving the eq^ns $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial \lambda} = 0$, we get

$\therefore 4 - 2x_1 - \lambda = 0$, $8 - 2x_2 - \lambda = 0$, $x_1 + x_2 = 4$

Adding the first two, we get

$\cancel{4 - 2x_1}$ $12 - 2(x_1 + x_2) - 2\lambda = 0$ $\quad \therefore 12 - 8 = 2\lambda$

$\therefore \lambda = 2$.

Hence, from the 1st eq^n, we get

$4 - 2x_1 - 2 = 0 \quad \therefore 2x_1 = 2 \quad \therefore x_1 = 1$

And from the 2nd eq^n, we get

$8 - 2x_2 - 2 = 0 \quad \therefore 2x_2 = 6 \quad \therefore x_2 = 3$.

Hence, X_0 is $(1, 3)$

Now, $h(x_1, x_2) = x_1 + x_2 - 4 = 0$

$\therefore \frac{\partial h}{\partial x_1} = 1$, $\frac{\partial h}{\partial x_2} = 1$ & all other partial derivatives are zero.

And $f(x_1, x_2) = 4x_1 + 8x_2 - x_1^2 - x_2^2$

$\therefore \frac{\partial f}{\partial x_1} = 4 - 2x_1, \quad \frac{\partial^2 f}{\partial x_1 x_2} = 0, \quad \frac{\partial^2 f}{\partial x_1^2} = -2,$

$\frac{\partial f}{\partial x_2} = 8 - 2x_2, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0, \quad \frac{\partial^2 f}{\partial x_2^2} = -2,$

$$\therefore \Delta = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = -1 \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix}$$

$= 2 + 2 = 4$

$\therefore$, Δ is positive, X_0 is a maxima.

Hence, $x_1 = 1$, $x_2 = 3$, $z_{max} = 18$.

3. Using the method of Lagrange's multipliers, solve the following

optimise $z = x_1^2 + x_2^2 + x_3^2$

sub. to $x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

$x_1, x_2 \geq 0$

Sol^n: We have $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$

$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 2;$

$h_2(x_1, x_2, x_3) = 5x_1 + 2x_2 + x_3 - 5$

Now, the Lagrangian f^n is,

$L(x_1, x_2, x_3 \lambda_1, \lambda_2) = f(x_1, x_2, x_3) - \lambda_1 h_1$

$(x_1, x_2, x_3) - \lambda_2 h_2(x_1, x_2, x_3)$

$= x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$

We, now, obtain the following partial derivatives

$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2, \quad \frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2$

$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2, \quad \frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_2 - 2)$

$\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5)$

We, then, solve the following eq^{n}s.

$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial x_3} = 0, \quad \frac{\partial L}{\partial \lambda_1} = 0,$

$\frac{\partial L}{\partial \lambda_2} = 0.$

$\therefore 2x_1 - \lambda_1 - 5\lambda_2 = 0$ — (1)

$2x_2 - \lambda_1 - 2\lambda_2 = 0$ — (2)

$2x_3 - 3\lambda_1 - \lambda_2 = 0$ — (3)

$x_1 + x_2 + x_3 - 2 = 0$ — (4)

$5x_1 + 2x_2 + x_3 - 5 = 0$ — (5)

We have to solve these eq^{n}s to find out the value of $x_1, x_2, x_3, \lambda_1, \lambda_2$.

Add 3 times (3) to the sum of (1) & (2).

$\therefore 2x_1 - \lambda_1 - 5\lambda_2 + 2x_2 - \lambda_1 - 2\lambda_2 + 6x_3 - 9\lambda_1 - 3\lambda_2 = 0$

$\therefore 2(x_1 + x_2 + 3x_3) - 11\lambda_1 - 10\lambda_2 = 0$

By (4), we get $11\lambda_1 + 10\lambda_2 = 4$ — (6)

Now, multiply ① by 5, ② by 2 & ③ by 1 and add.

$\therefore\ 10x_1 - 5\lambda_1 - 25\lambda_2 + 4x_2 - 2\lambda_1 - 4\lambda_2 + 2x_3 - 3\lambda_1 - \lambda_2 = 0$

$2(5x_1 + 2x_2 + x_3) - 10\lambda_1 - 30\lambda_2 = 0$

By ⑤, we get $10\lambda_1 + 30\lambda_2 = 10$ i.e. $\lambda_1 + 3\lambda_2 = 1$ — ⑦

Solving ⑥ & ⑦, we get $\lambda_1 = \frac{2}{23}$, $\lambda_2 = \frac{7}{23}$

Adding ①, ② & ③, we get

$x_1 = \frac{37}{46} = 0.804$, $x_2 = \frac{16}{46} = 0.348$, $x_3 = \frac{13}{46} = 0.283$

Now, $\frac{\partial h_1}{\partial x_1} = 1$, $\frac{\partial h_1}{\partial x_2} = 1$, $\frac{\partial h_1}{\partial x_3} = 3$

$\frac{\partial h_2}{\partial x_1} = 5$, $\frac{\partial h_2}{\partial x_2} = 2$, $\frac{\partial h_2}{\partial x_3} = 1$

$\frac{\partial^2 L}{\partial x_1^2} = 2$, $\frac{\partial^2 L}{\partial x_1 \partial x_2} = 0$, $\frac{\partial^2 L}{\partial x \partial x_3} = 0$

$\frac{\partial^2 L}{\partial x_2 \partial x_1} = 0$, $\frac{\partial^2 L}{\partial x_2^2} = 2$, $\frac{\partial^2 L}{\partial x_2 \partial x_3} = 0$

$\frac{\partial^2 L}{\partial x_3 \partial x_1} = 0$, $\frac{\partial^2 L}{\partial x_3 \partial x_2} = 0$, $\frac{\partial^2 L}{\partial x_3^2} = 2$

Hence, $U = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_3} & \frac{\partial h_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{bmatrix}$

$V = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Hence, the bordered Hessian matrix at X_0 $(0.804, 0.348, 0.283)$ & $\lambda_1 = \frac{2}{23}$, $\lambda_2 = \frac{7}{23}$ is

$$H_0^B = \begin{bmatrix} O & \vdots & U \\ \cdots & & \cdots \\ U' & \vdots & V \end{bmatrix} = \begin{bmatrix} 0 & 0 & \vdots & 1 & 1 & 3 \\ 0 & 0 & \vdots & 5 & 2 & 1 \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ 1 & 5 & \vdots & 2 & 0 & 0 \\ 1 & 2 & \vdots & 0 & 2 & 0 \\ 3 & 1 & \vdots & 0 & 0 & 2 \end{bmatrix}$$

$\because n = 3$, $m = 2$, $n - m = 1$ & $2m + 1 = 5$, we have to check whether the principal minor of order 5 has the sign of $(-1)^2$.

By Laplace method the determinant

$$H_0^B = (-1)^{3+4+1} \begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} \begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} + (-1)^{3+5+1} \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} \begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix} + (-1)^{4+5+1} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 5 & 2 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix}$$

$= (2-5)(+2)(2-5) + (-1)(1-15)(-2)(1-15) - (1-6)(+2)(1-6) = -460.$

$\therefore$, the determinant is +ve X_0 (0.804

0.348, 0.283) is a minima.

$\therefore Z_{min} = (0.804)^2 + (0.348)^2 + (0.283)^2$

$= 0.8476$

4. Solve the NLPP

Minimize $z = x_1^3 - 4x_1 - 2x_2$

Sub. to $x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$

Sol^n we write the problem as $f(x_1, x_2) = x_1^3 - 4x_1 - 2x_2$ & $h(x_1, x_2) = x_1 + x_2 - 1$

The Kuhn-Tucker conditions for minima are $\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$, $\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$

$\partial h(x_1, x_2) = 0$, $h(x_1, x_2) \leq 0$, $\lambda \leq 0$

$\therefore$ we ge get

$3x_1^2 - 4 - \lambda = 0$ — (1) $\quad -2 - \lambda = 0$

$\lambda(x_1 + x_2 - 1) = 0$ — (3) $\quad x_1 + x_2 - 1 \leq 0$

$\lambda \leq 0$ — (5)

from (2), we get $\lambda = -2$ & from (3), we get,

$\therefore \lambda \neq 0$, $x_1 + x_2 - 1 = 0$

from (1), we get $3x_1^2 = 4 + \lambda = 4 - 2 = 2$

$\therefore x_1 = \sqrt{2/3}$

$\because x_1 + x_2 = 1$, $x_2 = 1 - x_1 = 1 - \sqrt{2/3}$

These values satisfy all the above conditions (1) to (6),

$\therefore$ The optimal sol^{n}s are $x_1 = \sqrt{2/3}$, $x_2 = 1 - \sqrt{2/3}$, $\therefore Z_{min} = \frac{2}{3}\sqrt{\frac{2}{3}} - 4\sqrt{\frac{2}{3}}$

$$-2\left(1-\sqrt{\frac{2}{3}}\right)$$

$$=\frac{2}{3}(0.82)-4(0.81)-2(1-0.82)$$

$$=\underline{\underline{-3.093}}$$