# Module 6 - Graph Theory

CE– SE–DSGT

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**Module 6 - Graph Theory**

Types of graphs, Graph Representation, Sub graphs, Operations on Graphs, Walk, Path, Circuit, Connected Graphs, Disconnected Graph, Components, Homomorphism and Isomorphism of Graphs, Euler and Hamiltonian Graphs, Planar Graph, Cut Set, Cut Vertex, Applications.

## Definition

A *graph* $G = (V, E)$ consists of $V$, a nonempty set of *vertices* (or *nodes*) and $E$, a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

***Remark:*** The set of vertices $V$ of a graph $G$ may be infinite. A graph with an infinite vertex set or an infinite number of edges is called an **infinite graph**, and in comparison, a graph with a finite vertex set and a finite edge set is called a **finite graph**. In this book we will usually consider only finite graphs.

Now suppose that a network is made up of data centers and communication links between computers. We can represent the location of each data center by a point and each communications link by a line segment, as shown in Figure 1.

This computer network can be modeled using a graph in which the vertices of the graph represent the data centers and the edges represent communication links. In general, we visualize graphs by using points to represent vertices and line segments, possibly curved, to represent edges, where the endpoints of a line segment representing an edge are the points representing he endpoints of the edge.

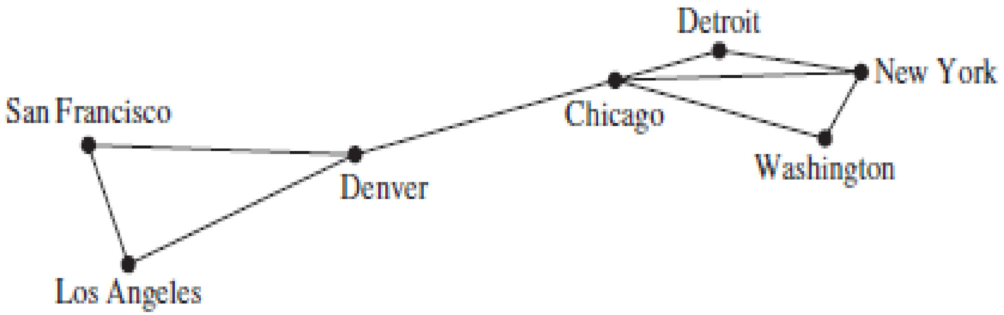


**FIGURE 1** **A computer network.**

# Types of graphs

## Basic Terminology

**Definition 1**

Two vertices $u$ and $v$ in an undirected graph $G$ are called *adjacent* (or *neighbors*) in $G$ if $u$ and $v$ are endpoints of an edge $e$ of $G$. Such an edge $e$ is called *incident with* the vertices $u$ and $v$ and $e$ is said to *connect* $u$ and $v$.

We will also find useful terminology describing the set of vertices adjacent to a particular vertex of a graph.

**Definition 2**

The set of all neighbors of a vertex $v$ of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of $v$. If $A$ is a subset of $V$, we denote by $N(A)$ the set of all vertices in $G$ that are adjacent to at least one vertex in $A$. So, $N(A) = \bigcup_{v \in A} N(v)$.

To keep track of how many edges are incident to a vertex, we make the following definition.

**Definition 3**

The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex $v$ is denoted by $\deg(v)$.

**EXAMPLE 1** What are the degrees and what are the neighborhoods of the vertices in the graphs $G$ and $H$ displayed in Figure 1?

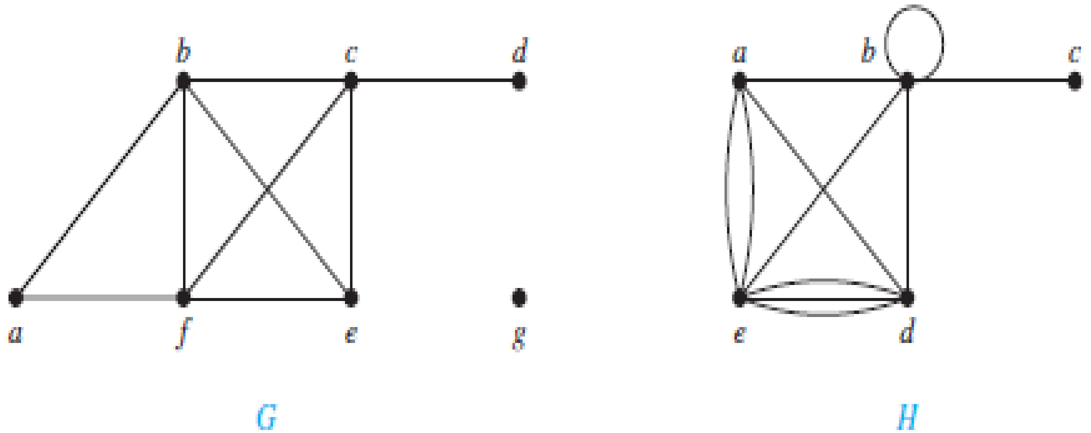


**FIGURE 1** The undirected graphs $G$ and $H$.

*Solution:* In $G$, $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$, $\deg(e) = 3$, ar $\deg(g) = 0$. The neighborhoods of these vertices are $N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c)$ $\{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \emptyset$. In $H$, $\deg(a) =$ $\deg(b) = \deg(e) = 6$, $\deg(c) = 1$, and $\deg(d) = 5$. The neighborhoods of these vertices a $N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$.

A vertex of degree zero is called **isolated**. It follows that an isolated vertex is not adjace to any vertex. Vertex $g$ in graph $G$ in Example 1 is isolated. A vertex is **pendant** if and on if it has degree one. Consequently, a pendant vertex is adjacent to exactly one other verte Vertex $d$ in graph $G$ in Example 1 is pendant.

**THEOREM 1** **THE HANDSHAKING THEOREM** Let $G = (V, E)$ be an undirected graph with $m$ edges. Then

$$2m = \sum_{v \in V} \deg(v).$$

(Note that this applies even if multiple edges and loops are present.)

**EXAMPLE 2:**

How many edges are there in a graph with 10 vertices each of degree six?

**Solution:** Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that $2m = 60$ where m is the number of edges. Therefore, $m = 30$.

**THEOREM 2** An undirected graph has an even number of vertices of odd degree.

*Proof:* Let $V_1$ and $V_2$ be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph $G = (V, E)$ with $m$ edges. Then

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

Because $\deg(v)$ is even for $v \in V_1$, the first term in the right-hand side of the last equality is even. Furthermore, the sum of the two terms on the right-hand side of the last equality is even, because this sum is $2m$. Hence, the second term in the sum is also even. Because all the terms in this sum are odd, there must be an even number of such terms. Thus, there are an even number of vertices of odd degree. ◁

**Definition 4** When $(u, v)$ is an edge of the graph $G$ with directed edges, $u$ is said to be *adjacent to* $v$ and $v$ is said to be *adjacent from* $u$. The vertex $u$ is called the *initial vertex* of $(u, v)$, and $v$ is called the *terminal* or *end vertex* of $(u, v)$. The initial vertex and terminal vertex of a loop are the same.

Because the edges in graphs with directed edges are ordered pairs, the definition of the degree of a vertex can be refined to reflect the number of edges with this vertex as the initial vertex and as the terminal vertex.

**Definition 5** In a graph with directed edges the *in-degree of a vertex* $v$, denoted by $\deg^-(v)$, is the number of edges with $v$ as their terminal vertex. The *out-degree of* $v$, denoted by $\deg^+(v)$, is the number of edges with $v$ as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

**EXAMPLE 4** Find the in-degree and out-degree of each vertex in the graph $G$ with directed edges shown in Figure 2.

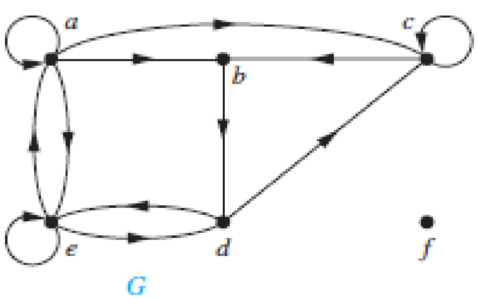


**FIGURE 2** The directed graph $G$.

**EXAMPLE 5** **Complete Graphs** A **complete graph on $n$ vertices**, denoted by $K_n$, is a simple graph that contains exactly one edge between each pair of distinct vertices. The graphs $K_n$, for $n = 1, 2, 3, 4, 5, 6$, are displayed in Figure 3. A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called **noncomplete**. ◀

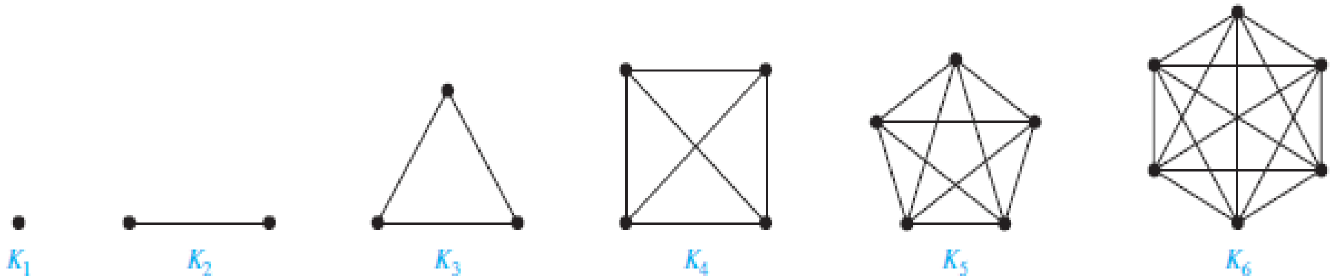


**FIGURE 3** **The graphs $K_n$ for $1 \leq n \leq 6$.**

**EXAMPLE 6** **Cycles** A **cycle $C_n$**, $n \geq 3$, consists of $n$ vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles $C_3$, $C_4$, $C_5$, and $C_6$ are displayed in Figure 4. ◀

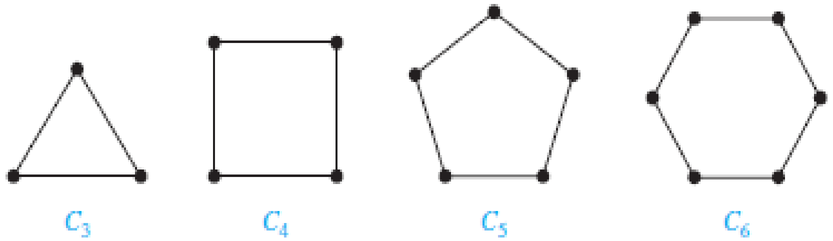


**FIGURE 4** **The cycles $C_3, C_4, C_5$, and $C_6$.**

# Bipartite Graphs

**Definition 6** A simple graph $G$ is called *bipartite* if its vertex set $V$ can be partitioned into two disjoint sets $V_1$ and $V_2$ such that every edge in the graph connects a vertex in $V_1$ and a vertex in $V_2$ (so that no edge in $G$ connects either two vertices in $V_1$ or two vertices in $V_2$). When this condition holds, we call the pair $(V_1, V_2)$ a *bipartition* of the vertex set $V$ of $G$.

**EXAMPLE 9** $C_6$ is bipartite, as shown in Figure 7, because its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of $C_6$ connects a vertex in $V_1$ and a vertex in $V_2$. ◀

**EXAMPLE 10** $K_3$ is not bipartite. To verify this, note that if we divide the vertex set of $K_3$ into two disjoint sets, one of the two sets must contain two vertices. If the graph were bipartite, these two vertices could not be connected by an edge, but in $K_3$ each vertex is connected to every other vertex by an edge. ◀

**EXAMPLE 11** Are the graphs $G$ and $H$ displayed in Figure 8 bipartite?

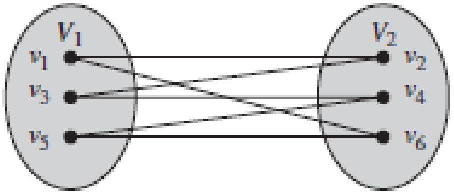


**FIGURE 7** **Showing that $C_6$ is bipartite.**

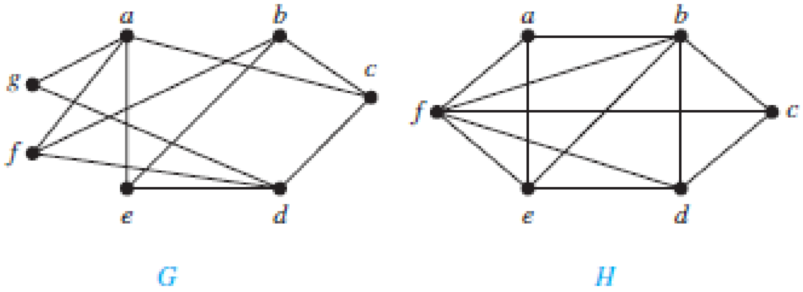


**FIGURE 8** **The undirected graphs $G$ and $H$.**

*Solution:* Graph $G$ is bipartite because its vertex set is the union of two disjoint sets, $\{a, b, d\}$ and $\{c, e, f, g\}$, and each edge connects a vertex in one of these subsets to a vertex in the other subset. (Note that for $G$ to be bipartite it is not necessary that every vertex in $\{a, b, d\}$ be adjacent to every vertex in $\{c, e, f, g\}$. For instance, $b$ and $g$ are not adjacent.)

*Solution:* Graph $G$ is bipartite because its vertex set is the union of two disjoint sets, $\{a, b, d\}$ and $\{c, e, f, g\}$, and each edge connects a vertex in one of these subsets to a vertex in the other subset. (Note that for $G$ to be bipartite it is not necessary that every vertex in $\{a, b, d\}$ be adjacent to every vertex in $\{c, e, f, g\}$. For instance, $b$ and $g$ are not adjacent.)

Graph $H$ is not bipartite because its vertex set cannot be partitioned into two subsets so that edges do not connect two vertices from the same subset. (The reader should verify this by considering the vertices $a$, $b$, and $f$.) ◀

Theorem 4 provides a useful criterion for determining whether a graph is bipartite.

**THEOREM 4** A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

# Graph Representation

One way to represent a graph without multiple edges is to list all the edges of this graph. Another way to represent a graph with no multiple edges is to use **adjacency lists**, which specify the vertices that are adjacent to each vertex of the graph.

**EXAMPLE 1** Use adjacency lists to describe the simple graph given in Figure 1.

*Solution:* Table 1 lists those vertices adjacent to each of the vertices of the graph. ◀

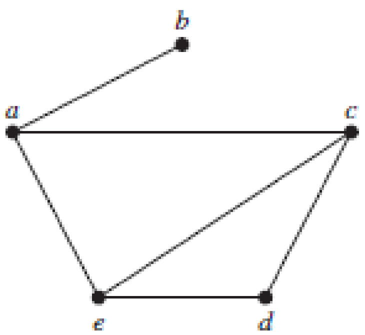


**FIGURE 1** **A simple graph.**

**TABLE 1 An Adjacency List for a Simple Graph.**

| *Vertex* | *Adjacent Vertices* |
|---|---|
| $a$ | $b, c, e$ |
| $b$ | $a$ |
| $c$ | $a, d, e$ |
| $d$ | $c, e$ |
| $e$ | $a, c, d$ |

**EXAMPLE 2** Represent the directed graph shown in Figure 2 by listing all the vertices that are the terminal vertices of edges starting at each vertex of the graph.

*Solution:* Table 2 represents the directed graph shown in Figure 2. ◀

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*Solution:* Table 2 represents the directed graph shown in Figure 2.

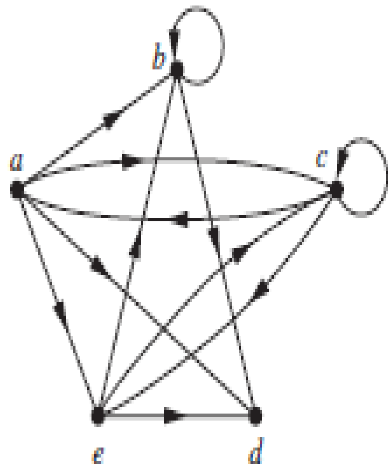



**FIGURE 2** A directed graph.

**TABLE 2** An Adjacency List for a Directed Graph.

| *Initial Vertex* | *Terminal Vertices* |
|---|---|
| $a$ | $b, c, d, e$ |
| $b$ | $b, d$ |
| $c$ | $a, c, e$ |
| $d$ | |
| $e$ | $b, c, d$ |

# Adjacency Matrices

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of $G$ are listed arbitrarily as $v_1, v_2, \ldots, v_n$. The **adjacency matrix A** (or $\mathbf{A}_G$) of $G$, with respect to this listing of the vertices, is the $n$ x $n$ zero–one matrix with 1 as its $(i, j)$th entry when $v_i$ and $v_j$ are adjacent, and 0 as its $(i, j)$th entry when they are not adjacent. In other words, if its adjacency matrix is $\mathbf{A} = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

**EXAMPLE 3** Use an adjacency matrix to represent the graph shown in Figure 3.

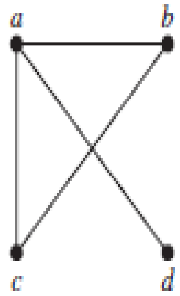



**FIGURE 3**
**A simple graph.**

*Solution:* We order the vertices as $a, b, c, d$. The matrix representing this graph is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

◀

**EXAMPLE 4** Draw a graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices $a, b, c, d$.

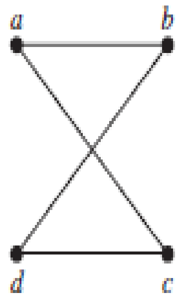



**FIGURE 4**

*Solution:* A graph with this adjacency matrix is shown in Figure 4. ◀

# Incidence Matrices

Another common way to represent graphs is to use **incidence matrices**. Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of $G$. Then the incidence matrix with respect to this ordering of $V$ and $E$ is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

*Solution:* The incidence matrix is

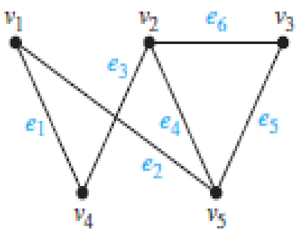



**FIGURE 6 An undirected graph.**

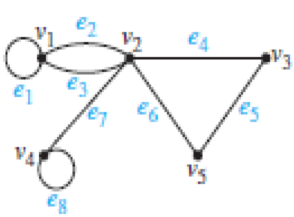
$$
\begin{array}{c}
 \\
v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
\end{array}.
$$

◀

Incidence matrices can also be used to represent multiple edges and loops. Multiple edges are represented in the incidence matrix using columns with identical entries, because these edges are incident with the same pair of vertices. Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop.

**EXAMPLE 7** Represent the pseudograph shown in Figure 7 using an incidence matrix.

*Solution:* The incidence matrix for this graph is




$$
\begin{array}{c}
 \\
v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}.
$$

◀

**FIGURE 7**

# Sub graphs

**Definition**

A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph $H$ of $G$ is a *proper subgraph* of $G$ if $H \neq G$.

Given a set of vertices of a graph, we can form a subgraph of this graph with these vertices and the edges of the graph that connect them.

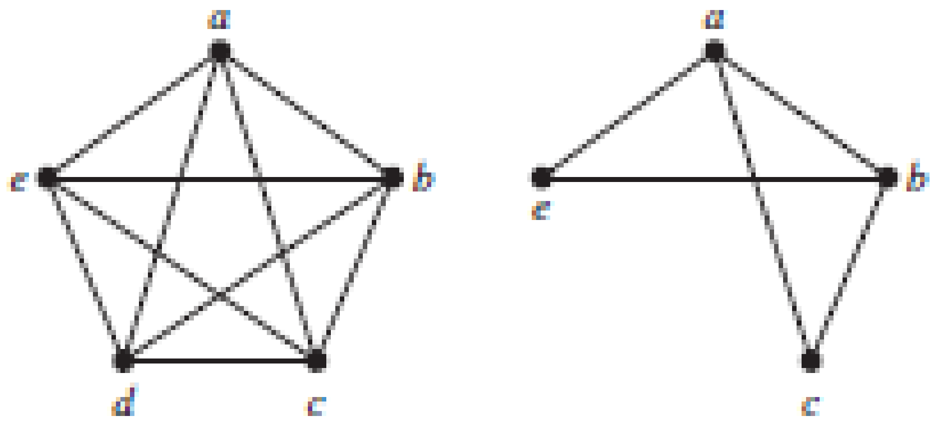



FIGURE 15 A subgraph of $K_5$.

# Operations on Graphs

**REMOVING VERTICES FROM A GRAPH** When we remove a vertex $v$ and all edges incident to it from $G = (V, E)$, we produce a subgraph, denoted by $G - v$. Observe that $G - v = (V - \{v\}, E')$, where $E'$ is the set of edges of $G$ not incident to $v$. Similarly, if $V'$ is a subset of $V$, then the graph $G - V'$ is the subgraph $(V - V', E')$, where $E'$ is the set of edges of $G$ not incident to a vertex in $V'$. (See Example 19 for an example of the removal of a vertex from a graph.)

Figure 16 displays an undirected graph $G$ with four different graphs that are the result of different operations on $G$. These are:

**(a)** $G - \{b, c\}$, constructed from $G$ by removing the edge $\{b, c\}$
**(b)** $G + \{e, d\}$, constructed from $G$ by adding the edge $\{e, d\}$
**(c)** the contraction of $G$, constructed from $G$ by replacing the edge $\{b, c\}$ with a new vertex $f$, and replacing the edges $\{c, d\}$, $\{a, b\}$, $\{b, e\}$, and $\{c, e\}$ with the new edges $\{a, f\}$, $\{f, d\}$, and $\{f, e\}$
**(d)** $G - c$, constructed from $G$ by removing the vertex $c$ and the edges $\{b, c\}$, $\{c, d\}$ and $\{c, e\}$

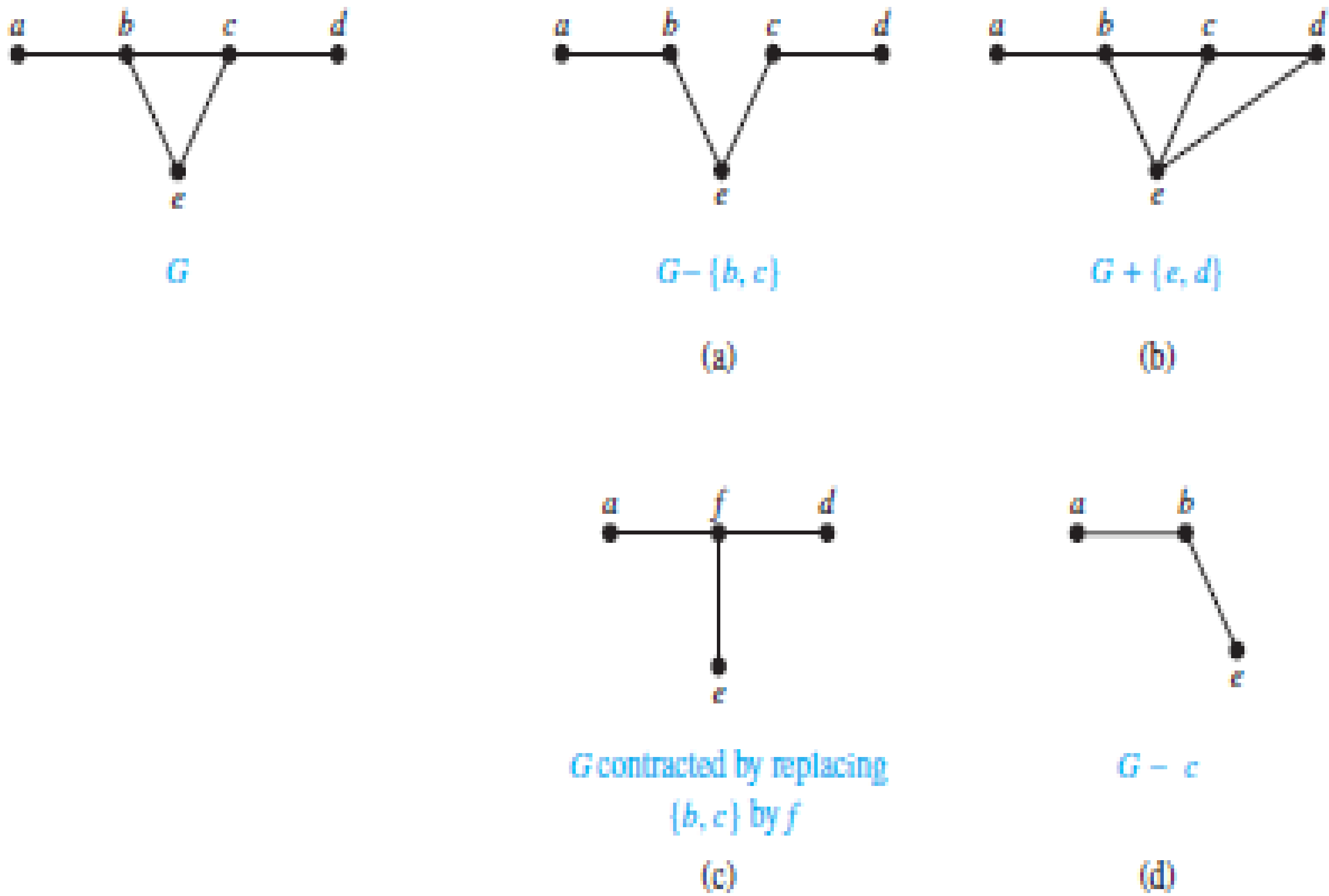



**FIGURE 16** The graph *G* and four graphs resulting from different operations on *G*.

**GRAPH UNIONS** Two or more graphs can be combined in various ways. The new graph that contains all the vertices and edges of these graphs is called the **union** of the graphs. We will give a more formal definition for the union of two simple graphs.

## Definition

The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of $G_1$ and $G_2$ is denoted by $G_1 \cup G_2$.

## EXAMPLE

Find the union of the graphs $G_1$ and $G_2$ shown in Figure 17(a). ◀

*Solution:* The vertex set of the union $G_1 \cup G_2$ is the union of the two vertex sets, namely, $\{a, b, c, d, e, f\}$. The edge set of the union is the union of the two edge sets. The union is displayed in Figure 17(b).

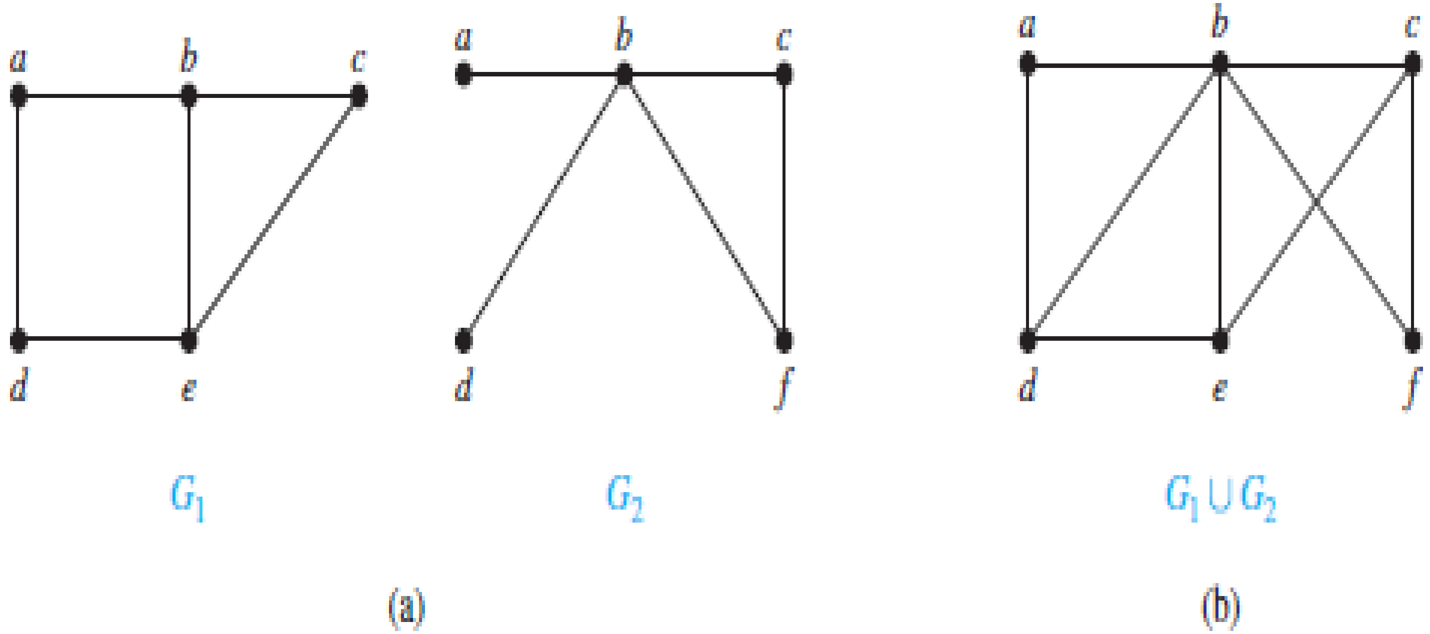



**FIGURE 17** (a) The simple graphs $G_1$ and $G_2$. (b) Their union $G_1 \cup G_2$.

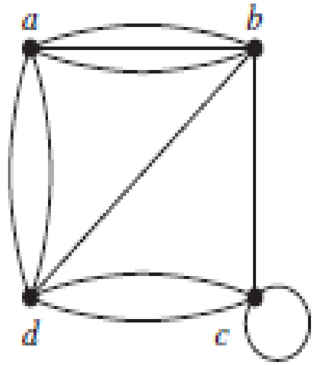
# Connectivity

## Paths

Let $n$ be a nonnegative integer and $G$ an undirected graph. A *path* of *length* $n$ from $u$ to $v$ in $G$ is a sequence of $n$ edges $e_1, \ldots, e_n$ of $G$ for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that $e_i$ has, for $i = 1, \ldots, n$, the endpoints $x_{i-1}$ and $x_i$. When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$ (because listing these vertices uniquely determines the path). The path is a *circuit* if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero. The path or circuit is said to *pass through* the vertices $x_1, x_2, \ldots, x_{n-1}$ or *traverse* the edges $e_1, e_2, \ldots, e_n$. A path or circuit is *simple* if it does not contain the same edge more than once.

When it is not necessary to distinguish between multiple edges, we will denote a path $e_1, e_2, \ldots, e_n$, where $e_i$ is associated with $\{x_{i-1}, x_i\}$ for $i = 1, 2, \ldots, n$ by its vertex sequence $x_0, x_1, \ldots, x_n$. This notation identifies a path only as far as which vertices it passes through. Consequently, it does not specify a unique path when there is more than one path that passes through this sequence of vertices, which will happen if and only if there are multiple edges between some successive vertices in the list. Note that a path of length zero consists of a single vertex.

**EXAMPLE 5** Use an adjacency matrix to represent the pseudograph shown in Figure 5.




*Solution:* The adjacency matrix using the ordering of vertices $a, b, c, d$ is

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

## Connectedness in Undirected Graphs

An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph. An undirected graph that is not *connected* is called *disconnected*. We say that we *disconnect* a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

**EXAMPLE**

The graph $G_1$ in Figure 2 is connected, because for every pair of distinct vertices there is a path between them (the reader should verify this). However, the graph $G_2$ in Figure 2 is not connected. For instance, there is no path in $G_2$ between vertices $a$ and $d$.

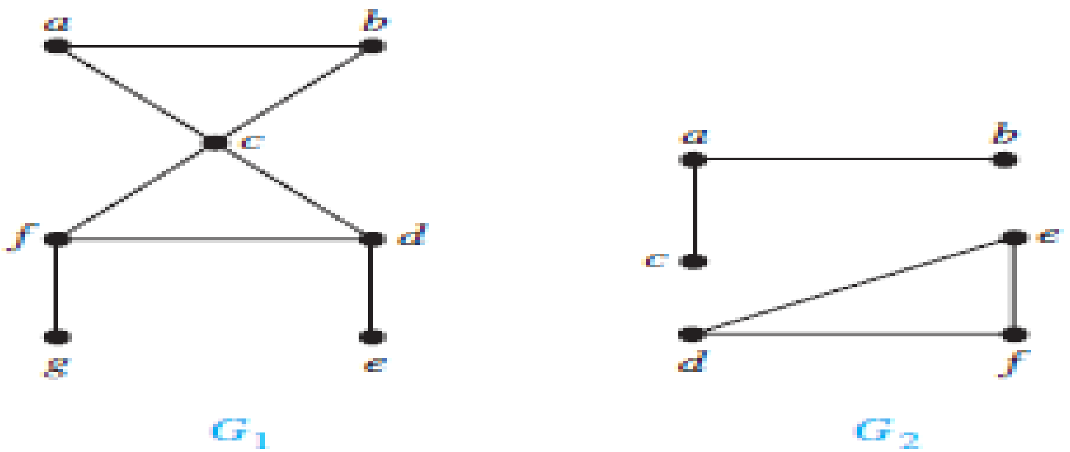



**FIGURE 2** **The graphs $G_1$ and $G_2$.**

**CONNECTED COMPONENTS** A **connected component** of a graph $G$ is a connected subgraph of $G$ that is not a proper subgraph of another connected subgraph of $G$. That is, a connected component of a graph $G$ is a maximal connected subgraph of $G$. A graph $G$ that is not connected has two or more connected components that are disjoint and have $G$ as their union.

What are the connected components of the graph $H$ shown in Figure 3?

*Solution:* The graph $H$ is the union of three disjoint connected subgraphs $H_1$, $H_2$, and $H_3$, shown in Figure 3. These three subgraphs are the connected components of $H$.

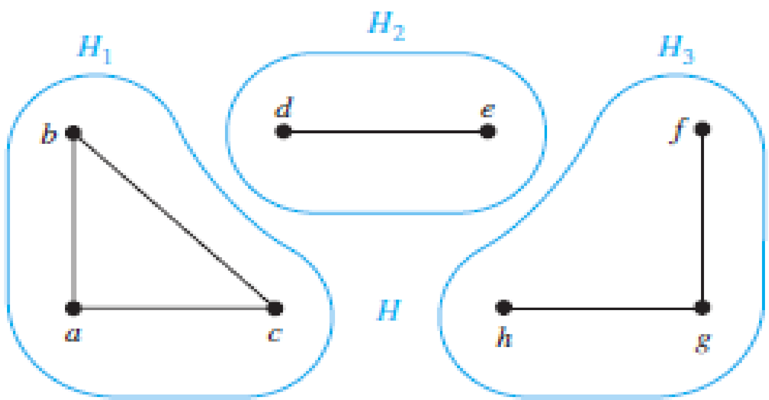



**FIGURE 3** **The graph $H$ and its connected components $H_1$, $H_2$, and $H_3$.**

# Isomorphism of Graphs

**Definition 1** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a one-to-one and onto function $f$ from $V_1$ to $V_2$ with the property that $a$ and $b$ are adjacent in $G_1$ if and only if $f(a)$ and $f(b)$ are adjacent in $G_2$, for all $a$ and $b$ in $V_1$. Such a function $f$ is called an *isomorphism*.* Two simple graphs that are not isomorphic are called *nonisomorphic*.

**EXAMPLE: 1**

Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure 8, are isomorphic.

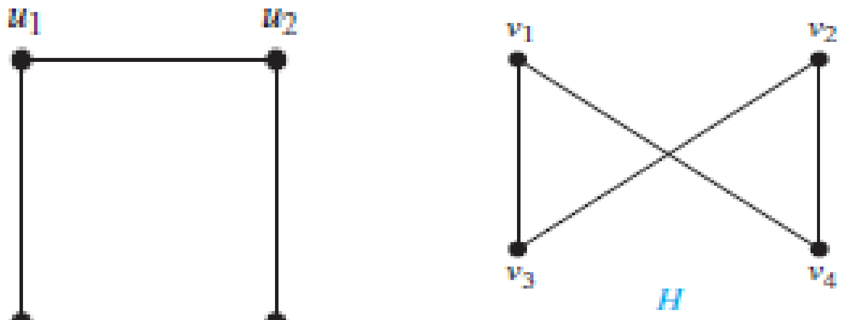



**FIGURE 8** **The graphs $G$ and $H$.**

*Solution:* The function $f$ with $f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between $V$ and $W$. To see that this correspondence preserves adjacency, note that adjacent vertices in $G$ are $u_1$ and $u_2$, $u_1$ and $u_3$, $u_2$ and $u_4$, and $u_3$ and $u_4$, and each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, and $f(u_3) = v_3$ and $f(u_4) = v_2$ consists of two adjacent vertices in $H$. ◀

# Determining whether Two Simple Graphs are Isomorphic

Sometimes it is not hard to show that two graphs are not isomorphic. In particular, we can show that two graphs are not isomorphic if we can find a property only one of the two graphs has, but that is preserved by isomorphism. A property preserved by isomorphism of graphs is called a **graph invariant**. For instance, isomorphic simple graphs must have the same number of vertices, because there is a one-to-one correspondence between the sets of vertices of the graphs.

Isomorphic simple graphs also must have the same number of edges, because the one-to-one correspondence between vertices establishes a one-to-one correspondence between edges. In addition, the degrees of the vertices in isomorphic simple graphs must be the same. That is, a vertex $v$ of degree $d$ in $G$ must correspond to a vertex $f(v)$ of degree $d$ in $H$, because a vertex $w$ in $G$ is adjacent to $v$ if and only if $f(v)$ and $f(w)$ are adjacent in $H$.

**EXAMPLE :** Show that the graphs displayed in Figure 9 are not isomorphic.

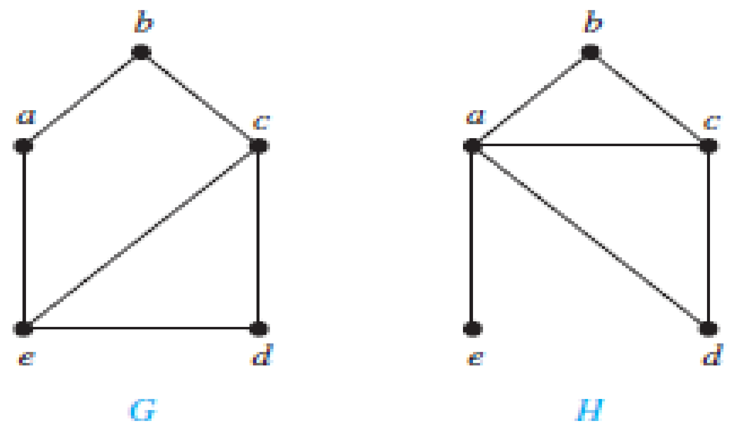



**FIGURE 9** **The graphs $G$ and $H$.**

*Solution:* Both $G$ and $H$ have five vertices and six edges. However, $H$ has a vertex of degree one, namely, $e$, whereas $G$ has no vertices of degree one. It follows that $G$ and $H$ are not isomorphic.

The number of vertices, the number of edges, and the number of vertices of each degree are all invariants under isomorphism. If any of these quantities differ in two simple graphs, these graphs cannot be isomorphic. However, when these invariants are the same, it does not necessarily mean that the two graphs are isomorphic. There are no useful sets of invariants currently known that can be used to determine whether simple graphs are isomorphic.

**EXAMPLE:** Determine whether the graphs shown in Figure 10 are isomorphic.

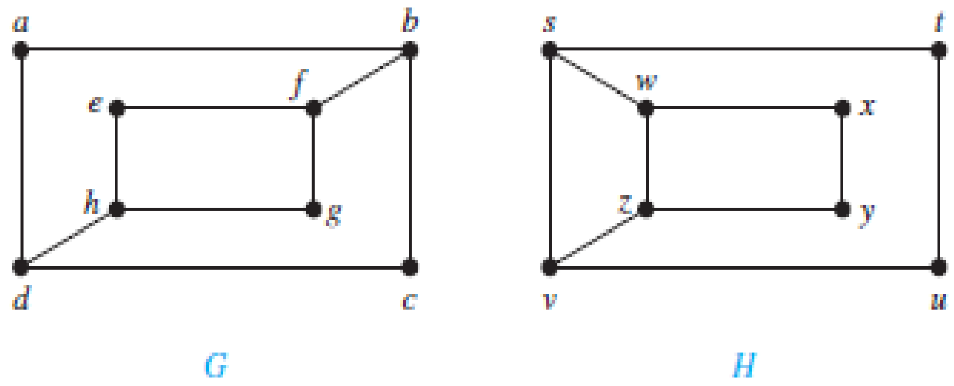



**FIGURE 10** **The graphs $G$ and $H$.**

*Solution:* The graphs $G$ and $H$ both have eight vertices and 10 edges. They also both have four vertices of degree two and four of degree three. Because these invariants all agree, it is still conceivable that these graphs are isomorphic.

However, $G$ and $H$ are not isomorphic. To see this, note that because $\deg(a) = 2$ in $G$, $a$ must correspond to either $t$, $u$, $x$, or $y$ in $H$, because these are the vertices of degree two in $H$. However, each of these four vertices in $H$ is adjacent to another vertex of degree two in $H$, which is not true for $a$ in $G$.

Another way to see that $G$ and $H$ are not isomorphic is to note that the subgraphs of $G$ and $H$ made up of vertices of degree three and the edges connecting them must be isomorphic if these two graphs are isomorphic (the reader should verify this). However, these subgraphs, shown in Figure 11, are not isomorphic. ◀

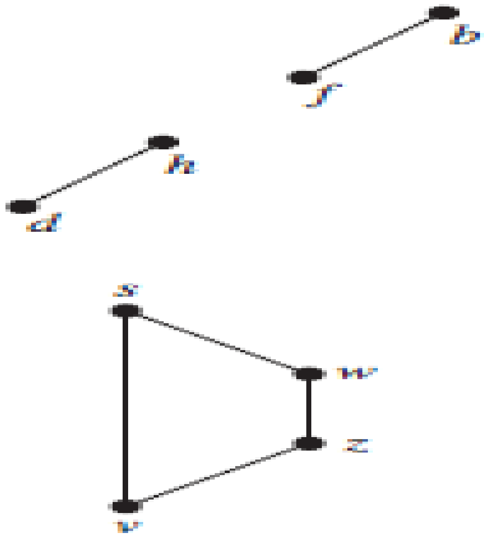



**FIGURE 11** The subgraphs of $G$ and $H$ made up of vertices of degree three and the edges connecting them.

To show that a function $f$ from the vertex set of a graph $G$ to the vertex set of a graph $H$ is an isomorphism, we need to show that $f$ preserves the presence and absence of edges. One helpful way to do this is to use adjacency matrices. In particular, to show that $f$ is an isomorphism, we can show that the adjacency matrix of $G$ is the same as the adjacency matrix of $H$, when rows and columns are labeled to correspond to the images under $f$ of the vertices in $G$ that are the labels of these rows and columns in the adjacency matrix of $G$. We illustrate how this is done in Example 11.

Determine whether the graphs $G$ and $H$ displayed in Figure 12 are isomorphic.

*Solution:* Both $G$ and $H$ have six vertices and seven edges. Both have four vertices of degree two and two vertices of degree three. It is also easy to see that the subgraphs of $G$ and $H$ consisting of all vertices of degree two and the edges connecting them are isomorphic (as the reader should verify). Because $G$ and $H$ agree with respect to these invariants, it is reasonable to try to find an isomorphism $f$.

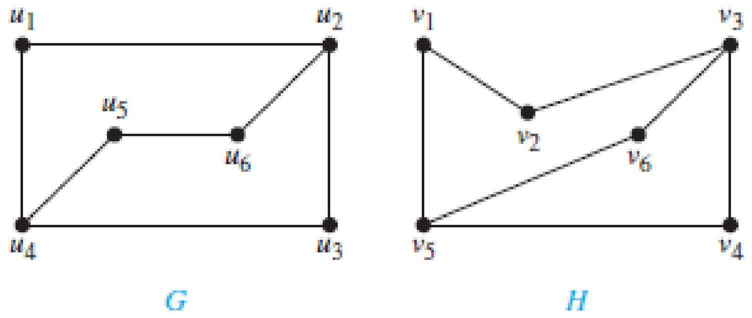



**FIGURE 12** **Graphs $G$ and $H$.**

We now will define a function $f$ and then determine whether it is an isomorphism. Because $\deg(u_1) = 2$ and because $u_1$ is not adjacent to any other vertex of degree two, the image of $u_1$ must be either $v_4$ or $v_6$, the only vertices of degree two in $H$ not adjacent to a vertex of degree two. We arbitrarily set $f(u_1) = v_6$. [If we found that this choice did not lead to isomorphism, we would then try $f(u_1) = v_4$.] Because $u_2$ is adjacent to $u_1$, the possible images of $u_2$ are $v_3$ and $v_5$. We arbitrarily set $f(u_2) = v_3$. Continuing in this way, using adjacency of vertices and degrees as a guide, we set $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5) = v_1$, and $f(u_6) = v_2$. We now have a one-to-one correspondence between the vertex set of $G$ and the vertex set of $H$, namely, $f(u_1) = v_6$, $f(u_2) = v_3$, $f(u_3) = v_4$, $f(u_4) = v_5$, $f(u_5) = v_1$, $f(u_6) = v_2$. To see whether $f$ preserves edges, we examine the adjacency matrix of $G$,

# Path Example

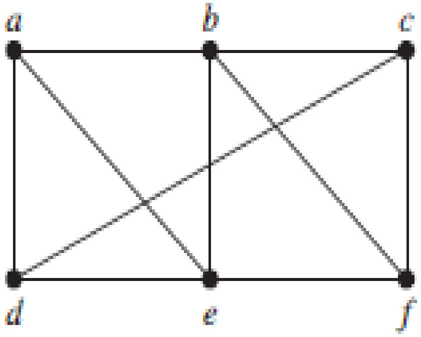



**FIGURE 1** **A simple graph.**

particular book or article when you read about traversing edges of a graph. The text [GrYe06] is a good reference for the alternative terminology described in this remark.

In the simple graph shown in Figure 1, $a, d, c, f, e$ is a simple path of length 4, because $\{a, d\}$, $\{d, c\}$, $\{c, f\}$, and $\{f, e\}$ are all edges. However, $d, e, c, a$ is not a path, because $\{e, c\}$ is not an edge. Note that $b, c, f, e, b$ is a circuit of length 4 because $\{b, c\}$, $\{c, f\}$, $\{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at $b$. The path $a, b, e, d, a, b$, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice. ◀

**Definition 2** Let $n$ be a nonnegative integer and $G$ a directed graph. A *path* of length $n$ from $u$ to $v$ in $G$ is a sequence of edges $e_1, e_2, \ldots, e_n$ of $G$ such that $e_1$ is associated with $(x_0, x_1)$, $e_2$ is associated with $(x_1, x_2)$, and so on, with $e_n$ associated with $(x_{n-1}, x_n)$, where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, x_2, \ldots, x_n$. A path of length greater than zero that begins and ends at the same vertex is called a *circuit* or *cycle*. A path or circuit is called *simple* if it does not contain the same edge more than once.

# How Connected is a Graph?

Sometimes the removal from a graph of a vertex and all incident edges produces a subgraph with more connected components. Such vertices are called **cut vertices** (or **articulation points**). The removal of a cut vertex from a connected graph produces a subgraph that is not connected. Analogously, an edge whose removal produces a graph with more connected components than in the original graph is called a **cut edge** or **bridge**. Note that in a graph representing a computer network, a cut vertex and a cut edge represent an essential router and an essential link that cannot fail for all computers to be able to communicate.

Find the cut vertices and cut edges in the graph $G_1$ shown in Figure 4.

*Solution:* The cut vertices of $G_1$ are $b$, $c$, and $e$. The removal of one of these vertices (and its adjacent edges) disconnects the graph. The cut edges are $\{a, b\}$ and $\{c, e\}$. Removing either one of these edges disconnects $G_1$. ◀

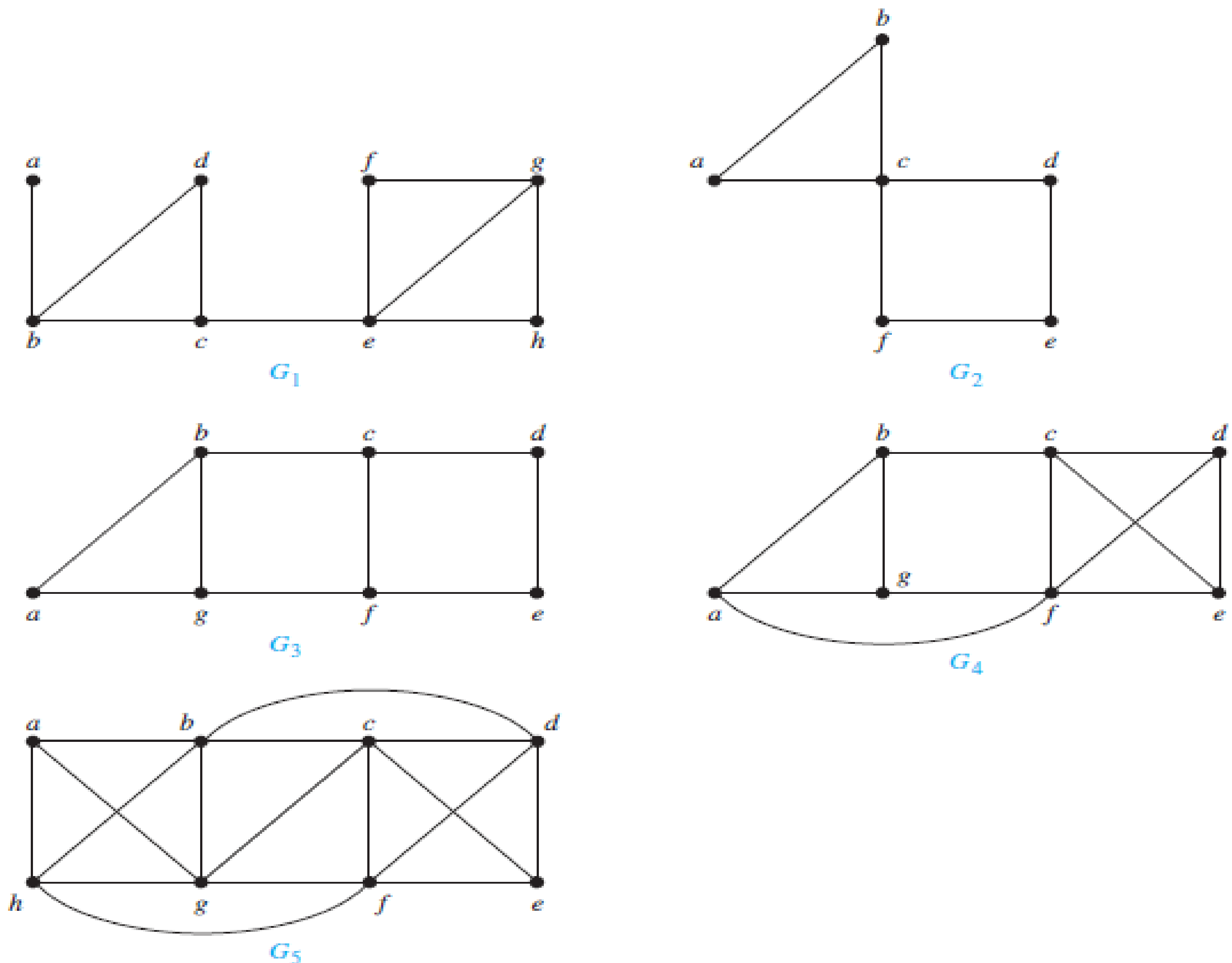



**FIGURE 4** Some connected graphs.

# Planar Graphs

Consider the problem of joining three houses to each of three separate utilities, as shown in Figure 1. Is it possible to join these houses and utilities so that none of the connections cross? This problem can be modeled using the complete bipartite graph $K_{3,3}$. The original question can be rephrased as: Can $K_{3,3}$ be drawn in the plane so that no two of its edges cross?

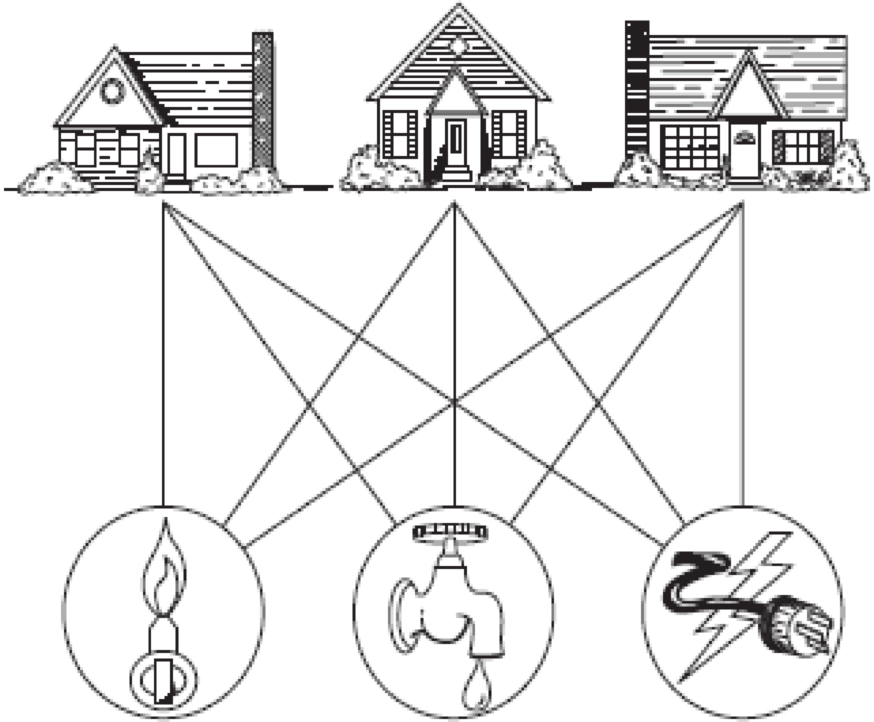


**FIGURE 1** **Three houses and three utilities.**

In this section we will study the question of whether a graph can be drawn in the plane without edges crossing. In particular, we will answer the houses-and-utilities problem.

There are always many ways to represent a graph. When is it possible to find at least one way to represent this graph in a plane without any edges crossing?

A graph is called *planar* if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a *planar representation* of the graph.

A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

Is $K_4$ (shown in Figure 2 with two edges crossing) planar?

*Solution:* $K_4$ is planar because it can be drawn without crossings, as shown in Figure 3. ◀

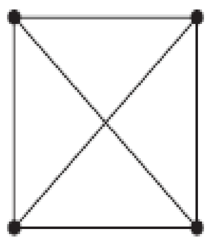


**FIGURE 2** The graph $K_4$.

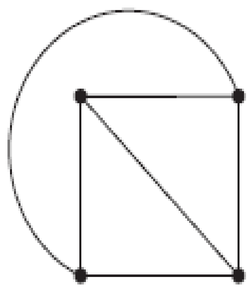


**FIGURE 3** $K_4$ drawn with no crossings.

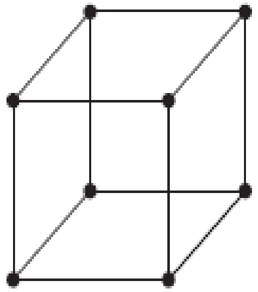


**FIGURE 4** The graph $Q_3$.

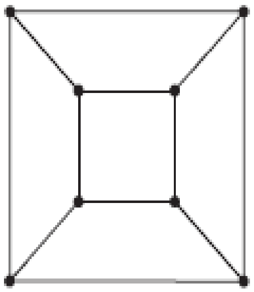


**FIGURE 5** A planar representation of $Q_3$.

**EXAMPLE 1** Is $K_4$ (shown in Figure 2 with two edges crossing) planar?

*Solution:* $K_4$ is planar because it can be drawn without crossings, as shown in Figure 3. ◀

**EXAMPLE 2** Is $Q_3$, shown in Figure 4, planar?

*Solution:* $Q_3$ is planar, because it can be drawn without any edges crossing, as shown in Figure 5. ◀

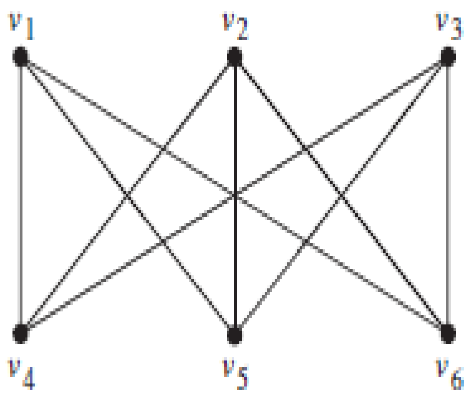



**FIGURE 6** **The graph $K_{3,3}$.**

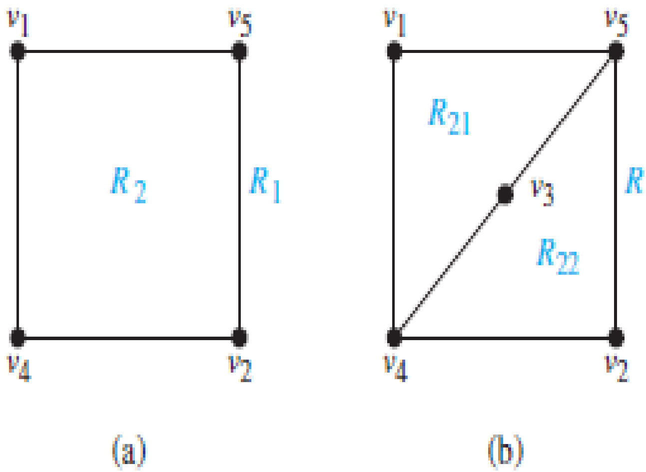



**FIGURE 7** **Showing that $K_{3,3}$ is nonplanar.**

A similar argument can be used when $v_3$ is in $R_1$. The completion of this argument is left for the reader (see Exercise 10). It follows that $K_{3,3}$ is not planar. ◀

**EXAMPLE 3** Is $K_{3,3}$, shown in Figure 6, planar?

*Solution:* Any attempt to draw $K_{3,3}$ in the plane with no edges crossing is doomed. We now show why. In any planar representation of $K_{3,3}$, the vertices $v_1$ and $v_2$ must be connected to both $v_4$ and $v_5$. These four edges form a closed curve that splits the plane into two regions, $R_1$ and $R_2$, as shown in Figure 7(a). The vertex $v_3$ is in either $R_1$ or $R_2$. When $v_3$ is in $R_2$, the inside of the closed curve, the edges between $v_3$ and $v_4$ and between $v_3$ and $v_5$ separate $R_2$ into two subregions, $R_{21}$ and $R_{22}$, as shown in Figure 7(b).

Next, note that there is no way to place the final vertex $v_6$ without forcing a crossing. For if $v_6$ is in $R_1$, then the edge between $v_6$ and $v_3$ cannot be drawn without a crossing. If $v_6$ is in $R_{21}$, then the edge between $v_2$ and $v_6$ cannot be drawn without a crossing. If $v_6$ is in $R_{22}$, then the edge between $v_1$ and $v_6$ cannot be drawn without a crossing.

**APPLICATIONS OF PLANAR GRAPHS** Planarity of graphs plays an important role in the design of electronic circuits. We can model a circuit with a graph by representing components of the circuit by vertices and connections between them by edges. We can print a circuit on a single board with no connections crossing if the graph representing the circuit is planar. When this graph is not planar, we must turn to more expensive options. For example, we can partition the vertices in the graph representing the circuit into planar subgraphs. We then construct the circuit using multiple layers. (See the preamble to Exercise 30 to learn about the thickness of a graph.) We can construct the circuit using insulated wires whenever connections cross. In this case, drawing the graph with the fewest possible crossings is important. (See the preamble to Exercise 26 to learn about the crossing number of a graph.)

The planarity of graphs is also useful in the design of road networks. Suppose we want to connect a group of cities by roads. We can model a road network connecting these cities using a simple graph with vertices representing the cities and edges representing the highways connecting them. We can built this road network without using underpasses or overpasses if the resulting graph is planar.

# Euler Paths and Circuits

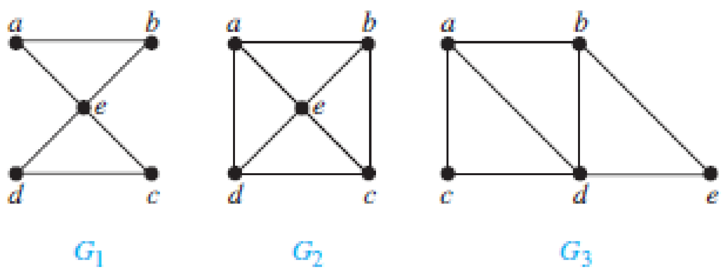
**Definition 1** An *Euler circuit* in a graph $G$ is a simple circuit containing every edge of $G$. An *Euler path* in $G$ is a simple path containing every edge of $G$.

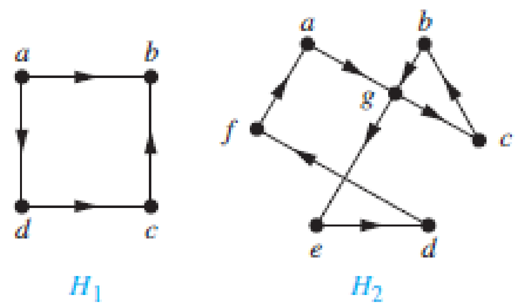
Examples 1 and 2 illustrate the concept of Euler circuits and paths.

**EXAMPLE 1** Which of the undirected graphs in Figure 3 have an Euler circuit? Of those that do not, which have an Euler path?

*Solution:* The graph $G_1$ has an Euler circuit, for example, $a, e, c, d, e, b, a$. Neither of the graphs $G_2$ or $G_3$ has an Euler circuit (the reader should verify this). However, $G_3$ has an Euler path, namely, $a, c, d, e, b, d, a, b$. $G_2$ does not have an Euler path (as the reader should verify). ◀

**EXAMPLE 2** Which of the directed graphs in Figure 4 have an Euler circuit? Of those that do not, which have an Euler path?



*Solution:* The graph $H_2$ has an Euler circuit, for example, $a, g, c, b, g, e, d, f, a$. Neither $H_1$ nor $H_3$ has an Euler circuit (as the reader should verify). $H_3$ has an Euler path, namely, $c, a, b, c, d, b$, but $H_1$ does not (as the reader should verify). ◀

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

**Theorem 1**

(a) *If a graph G has a vertex of odd degree, there can be no Euler circuit in G.*

(b) *If G is a connected graph and every vertex has even degree, then there is an Euler circuit in G.*

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

**Theorem 2**

(a) *If a graph G has more than two vertices of odd degree, then there can be no Euler path in G.*

(b) *If G is connected and has exactly two vertices of odd degree, there is an Euler path in G. Any Euler path in G must begin at one vertex of odd degree and end at the other.*

Example 4. Which of the graphs in Figures 6.29, 6.30, and 6.31 have an Euler circuit, an Euler path but not an Euler circuit, or neither?

*Solution:* (a) In Figure 6.29, each of the four vertices has degree 3; thus, by Theorems 1 and 2, there is neither an Euler circuit nor an Euler path.

(b) The graph in Figure 6.30 has exactly two vertices of odd degree. There is no Euler circuit, but there must be an Euler path.

(c) In Figure 6.31, every vertex has even degree; thus the graph must have an Euler circuit. ♦

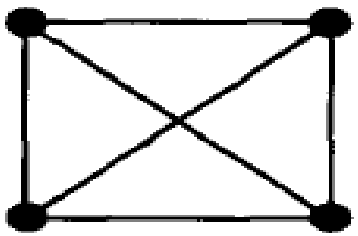


Figure 6.29

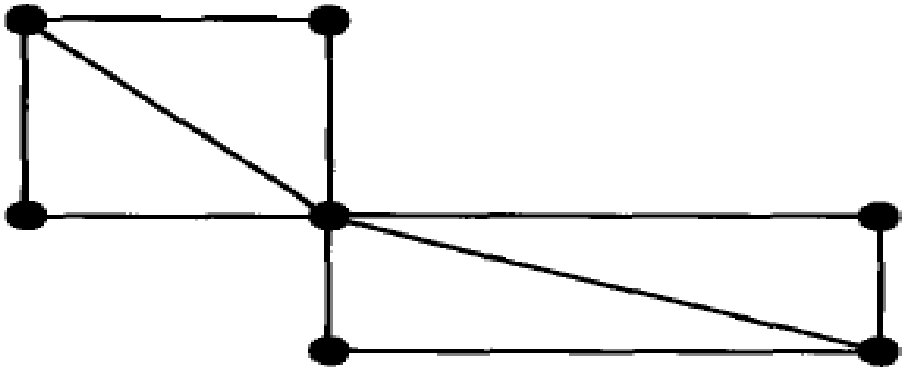


Figure 6.30

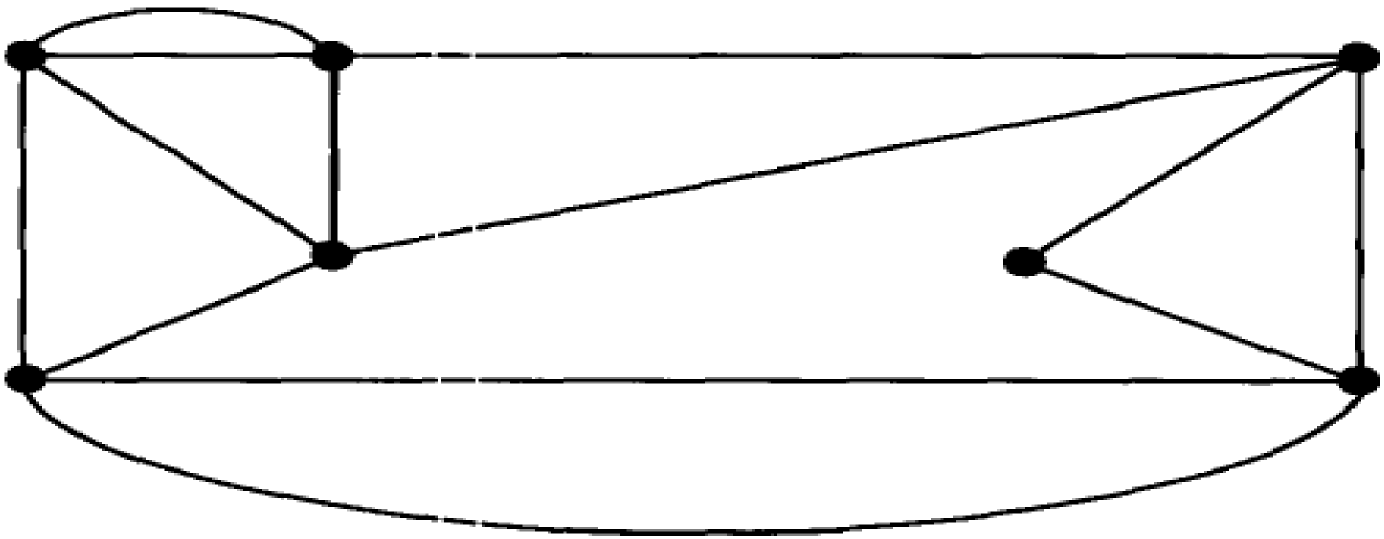


Figure 6.31

Which graphs shown in Figure 7 have an Euler path?

*Solution:* $G_1$ contains exactly two vertices of odd degree, namely, $b$ and $d$. Hence, it has an Euler path that must have $b$ and $d$ as its endpoints. One such Euler path is $d, a, b, c, d, b$. Similarly, $G_2$ has exactly two vertices of odd degree, namely, $b$ and $d$. So it has an Euler path that must have $b$ and $d$ as endpoints. One such Euler path is $b, a, g, f, e, d, c, g, b, c, f, d$. $G_3$ has no Euler path because it has six vertices of odd degree. ◀

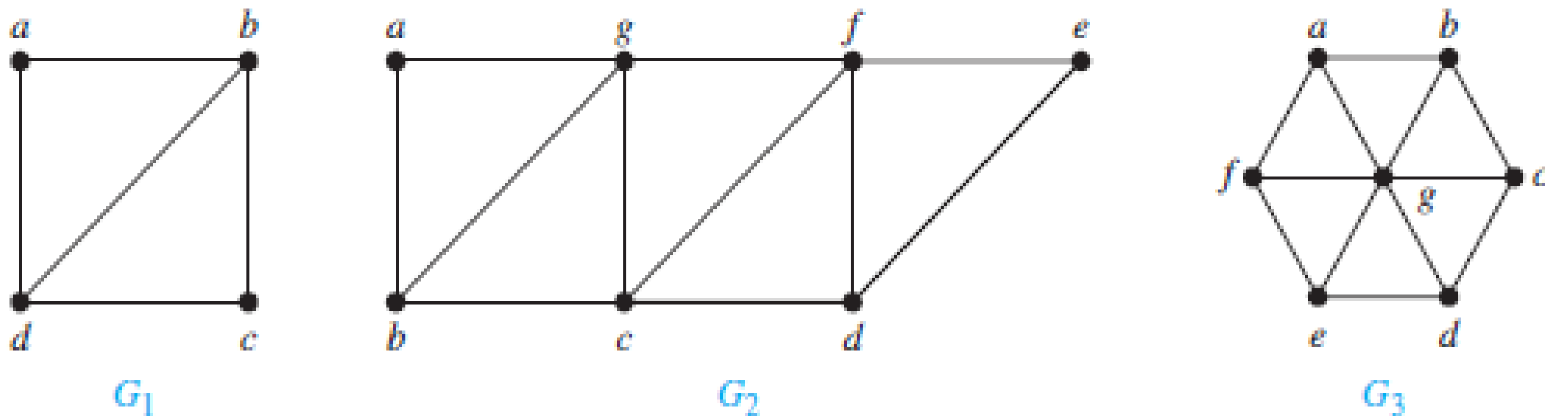



**FIGURE 7** **Three undirected graphs.**

# Hamiltonian Paths and Circuits

A simple path in a graph $G$ that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph $G$ that passes through every vertex exactly once is called a *Hamilton circuit*. That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

**EXAMPLE 5** Which of the simple graphs in Figure 10 have a Hamilton circuit or, if not, a Hamilton path?

Extra Examples

*Solution:* $G_1$ has a Hamilton circuit: $a, b, c, d, e, a$. There is no Hamilton circuit in $G_2$ (this can be seen by noting that any circuit containing every vertex must contain the edge $\{a, b\}$ twice), but $G_2$ does have a Hamilton path, namely, $a, b, c, d$. $G_3$ has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once.

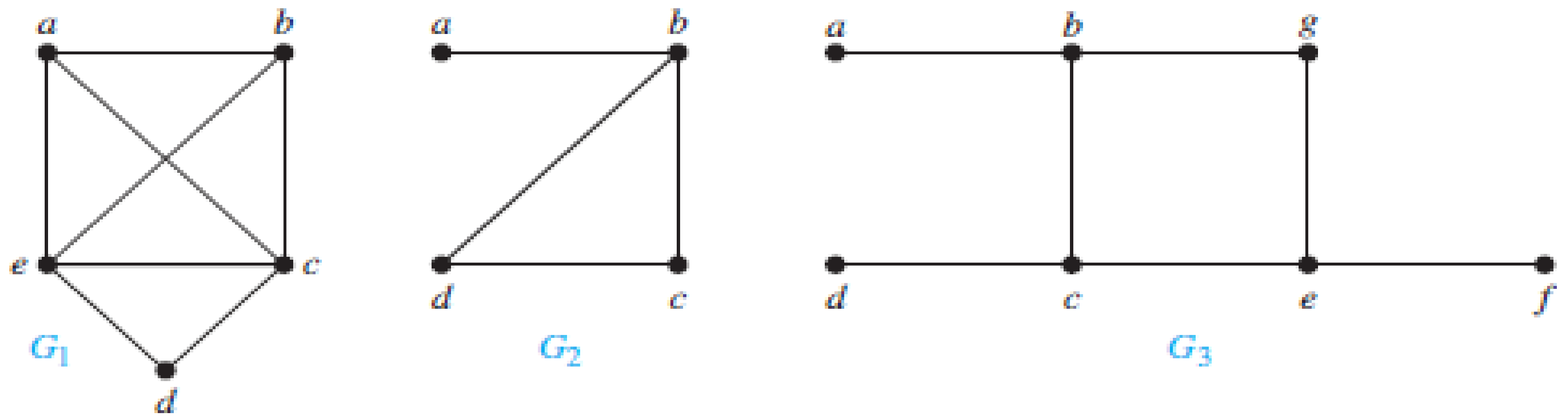



**FIGURE 10** **Three simple graphs.**

Example 1. Consider the graph in Figure 6.53. The path $a, b, c, d, e$ is a Hamiltonian path because it contains each vertex exactly once. It is not hard to see, however, that there is no Hamiltonian circuit for this graph. For the graph shown in Figure 6.54, the path $A, D, C, B, A$ (choosing either edge from $B$ to $A$) is a Hamiltonian circuit. In Figures 6.55 and 6.56, no Hamiltonian path is possible. (Verify this.) ◆

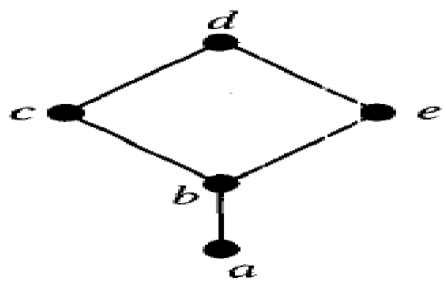


Figure 6.53

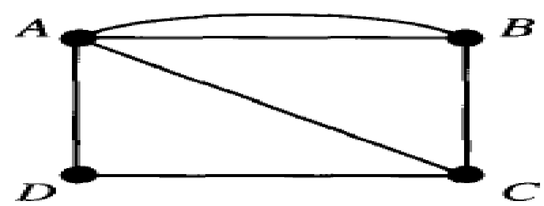


Figure 6.54

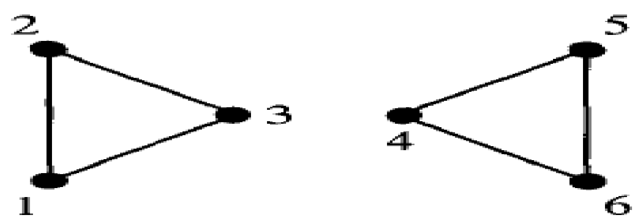


Figure 6.55

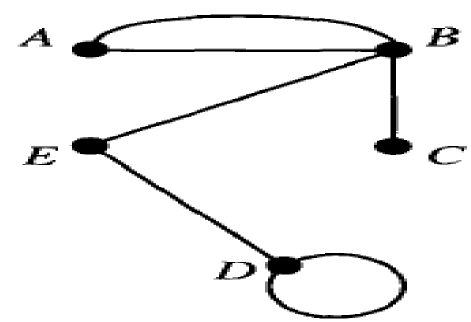


Figure 6.56

**EXAMPLE 6** Show that neither graph displayed in Figure 11 has a Hamilton circuit.

*Solution:* There is no Hamilton circuit in $G$ because $G$ has a vertex of degree one, namely, $e$. Now consider $H$. Because the degrees of the vertices $a$, $b$, $d$, and $e$ are all two, every edge incident with these vertices must be part of any Hamilton circuit. It is now easy to see that no Hamilton circuit can exist in $H$, for any Hamilton circuit would have to contain four edges incident with $c$, which is impossible. ◀

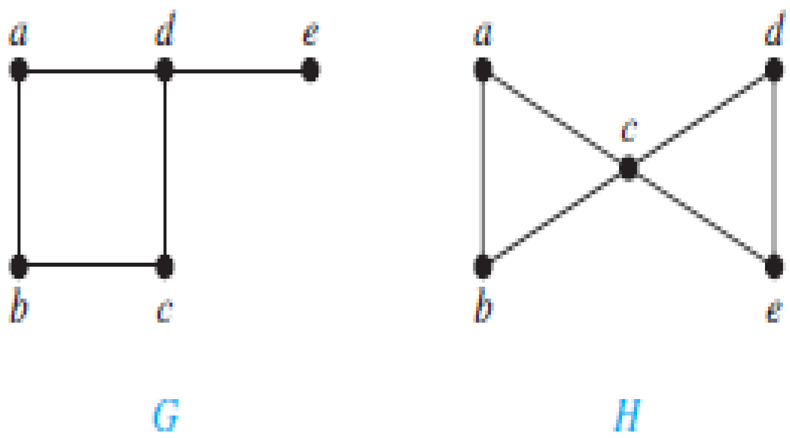



**FIGURE 11** **Two graphs that do not have a Hamilton circuit.**

# Applications of Hamilton Circuits

Hamilton paths and circuits can be used to solve practical problems. For example, many applications ask for a path or circuit that visits each road intersection in a city, each place pipelines intersect in a utility grid, or each node in a communications network exactly once. Finding a Hamilton path or circuit in the appropriate graph model can solve such problems. The famous **traveling salesperson problem** or **TSP** (also known in older literature as the **traveling salesman problem**) asks for the shortest route a traveling salesperson should take to visit a set of cities. This problem reduces to finding a Hamilton circuit in a complete graph such that the total weight of its edges is as small as possible. We will return to this question in Section 10.6.

# Cut Set, Cut Vertex

Whether it is possible to traverse a graph from one vertex to another is determined by how a graph is connected. Connectivity is a basic concept in Graph Theory. Connectivity defines whether a graph is connected or disconnected.

## Connectivity

A graph is said to be **connected if there is a path between every pair of vertex**. From every vertex to any other vertex, there should be some path to traverse. That is called the connectivity of a graph. A graph with multiple disconnected vertices and edges is said to be disconnected.

## Cut Vertex

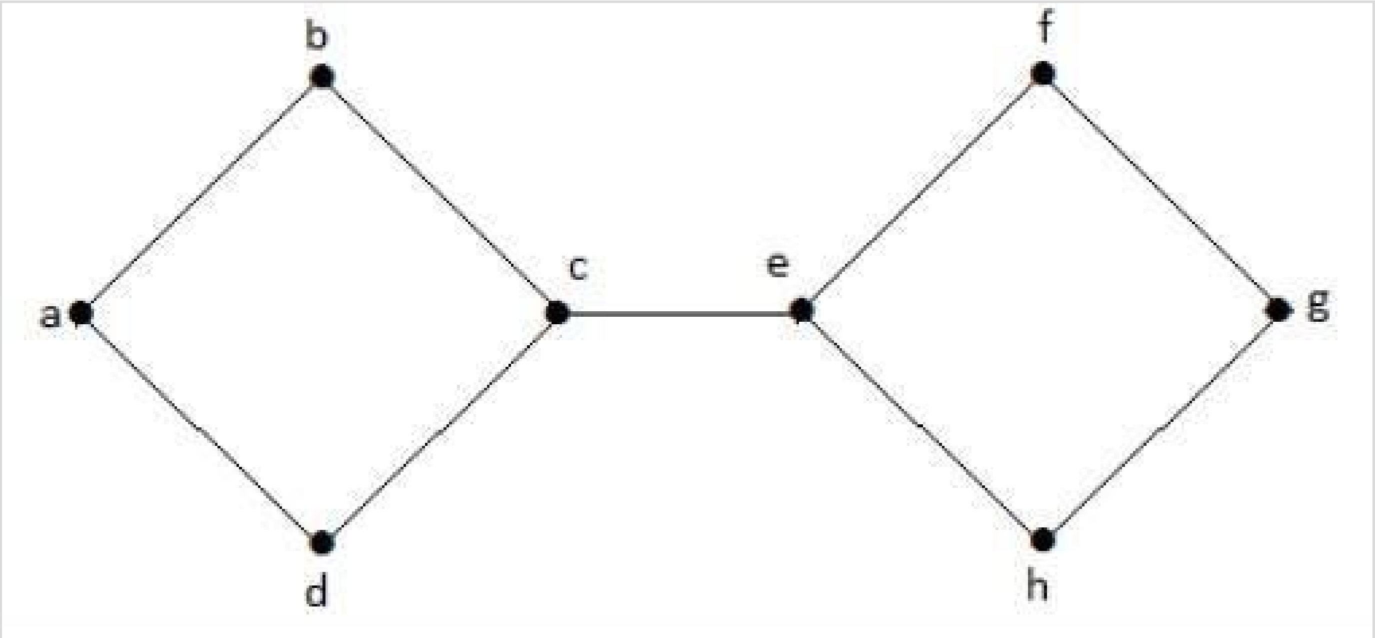
Let 'G' be a connected graph. A vertex V ∈ G is called a cut vertex of 'G', if 'G-V' (Delete 'V' from 'G') results in a disconnected graph. Removing a cut vertex from a graph breaks it in to two or more graphs.

**Note** − Removing a cut vertex may render a graph disconnected.

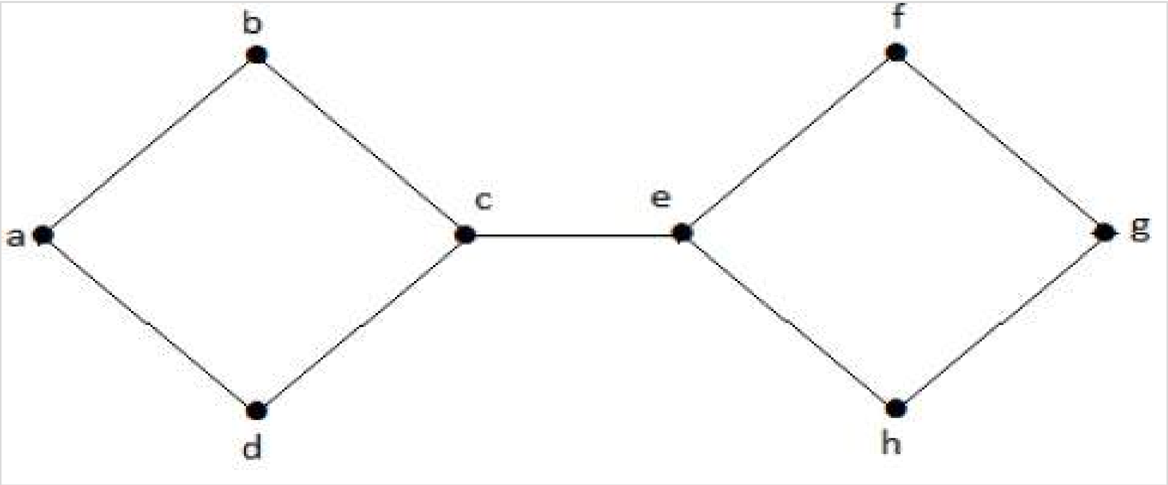
A connected graph 'G' may have at most (n–2) cut vertices.

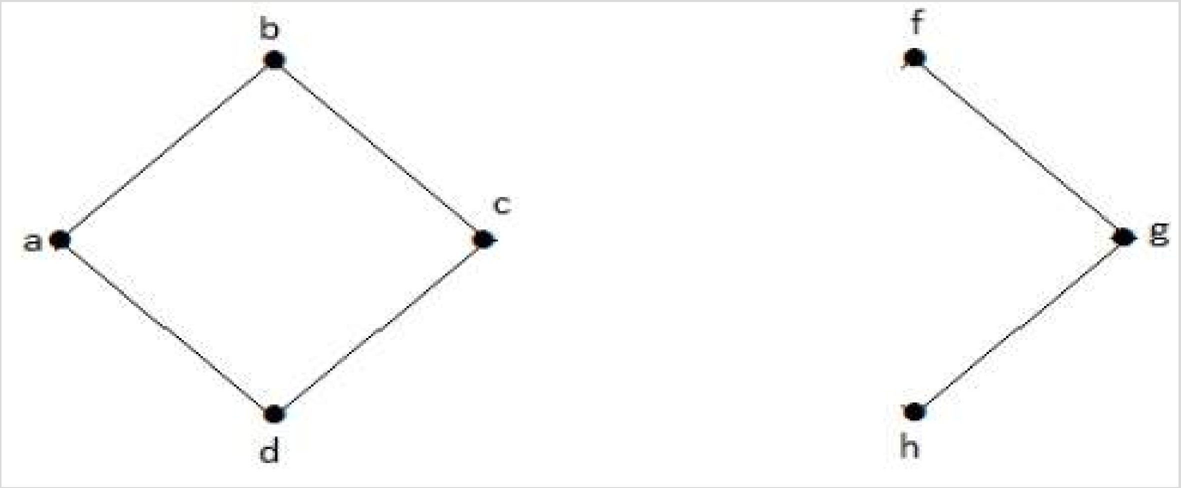
## Example

In the following graph, vertices 'e' and 'c' are the cut vertices.



By removing 'e' or 'c', the graph will become a disconnected graph.




Without 'g', there is no path between vertex 'c' and vertex 'h' and many other. Hence it is a disconnected graph with cut vertex as 'e'. Similarly, 'c' is also a cut vertex for the above graph.

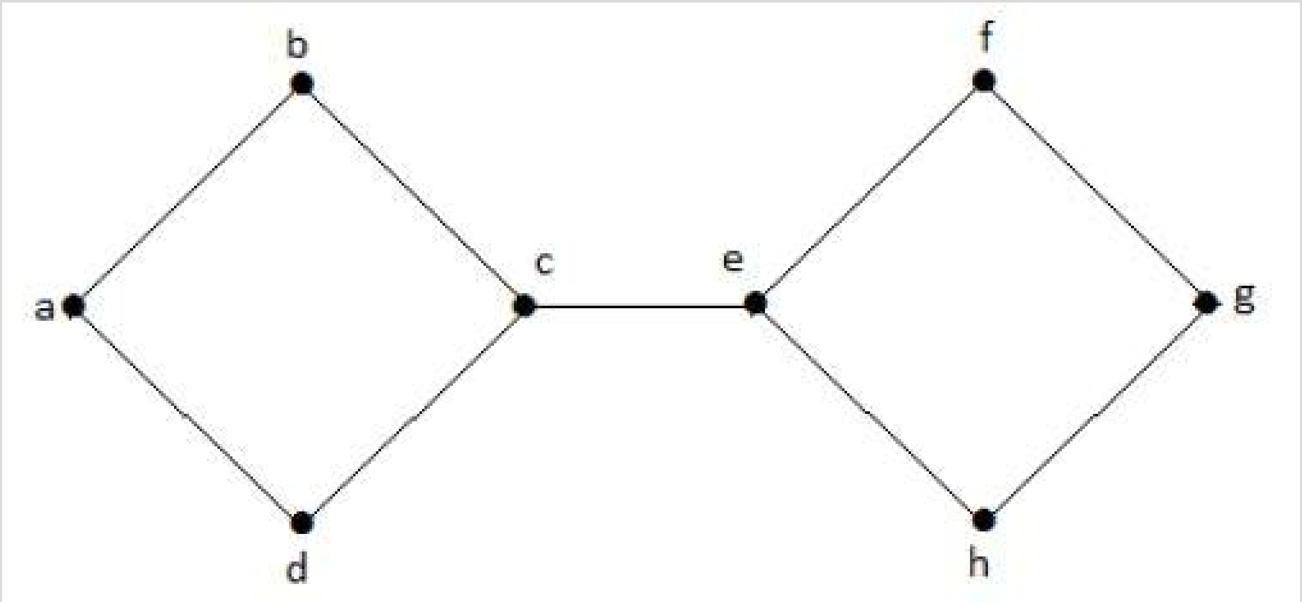
## Cut Edge (Bridge)

Let 'G' be a connected graph. An edge 'e' ∈ G is called a cut edge if 'G-e' results in a disconnected graph.

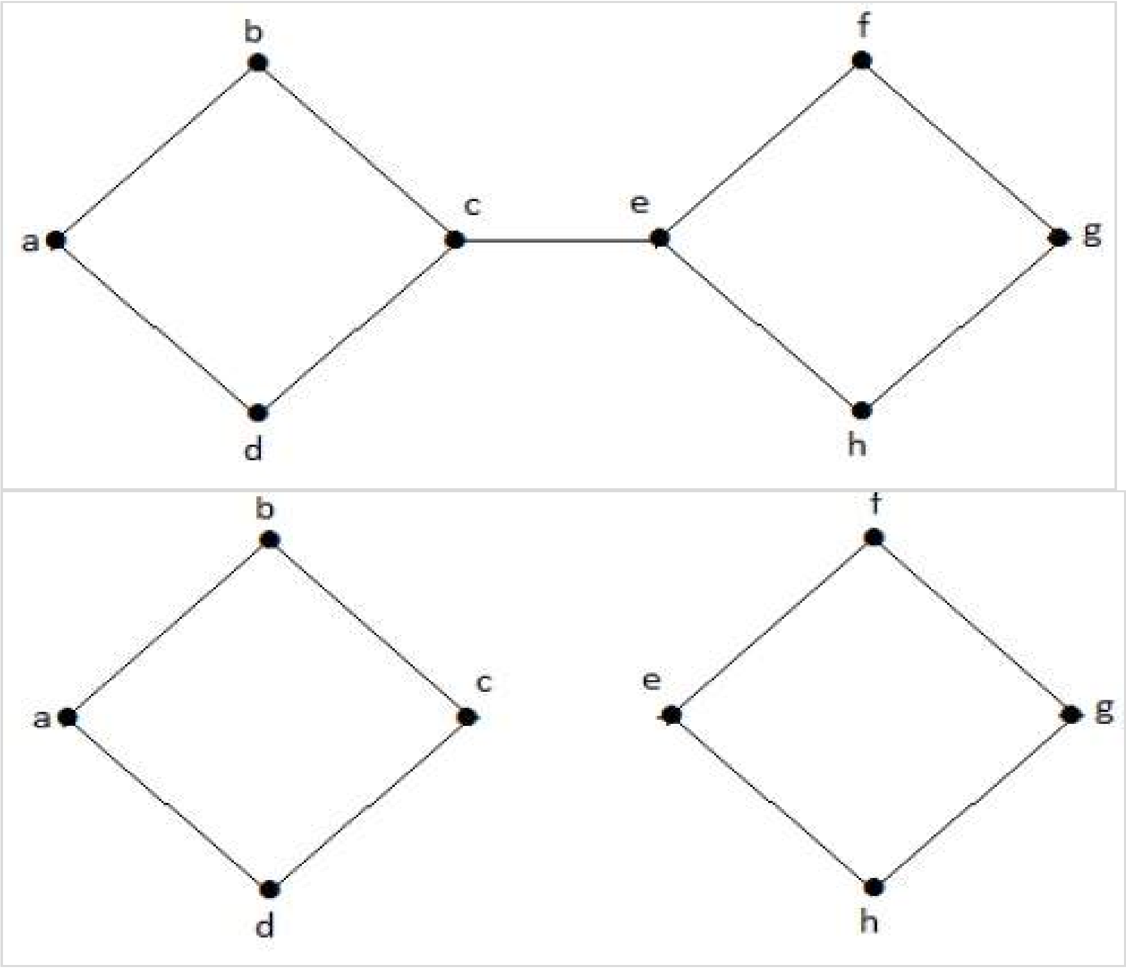
If removing an edge in a graph results in to two or more graphs, then that edge is called a Cut Edge.

## Example

In the following graph, the cut edge is [(c, e)]



By removing the edge (c, e) from the graph, it becomes a disconnected graph.




In the above graph, removing the edge (c, e) breaks the graph into two which is nothing but a disconnected graph. Hence, the edge (c, e) is a cut edge of the graph.

**Note** – Let 'G' be a connected graph with 'n' vertices, then

- a cut edge e ∊ G if and only if the edge 'e' is not a part of any cycle in G.
- the maximum number of cut edges possible is 'n-1'.
- whenever cut edges exist, cut vertices also exist because at least one vertex of a cut edge is a cut vertex.
- if a cut vertex exists, then a cut edge may or may not exist.

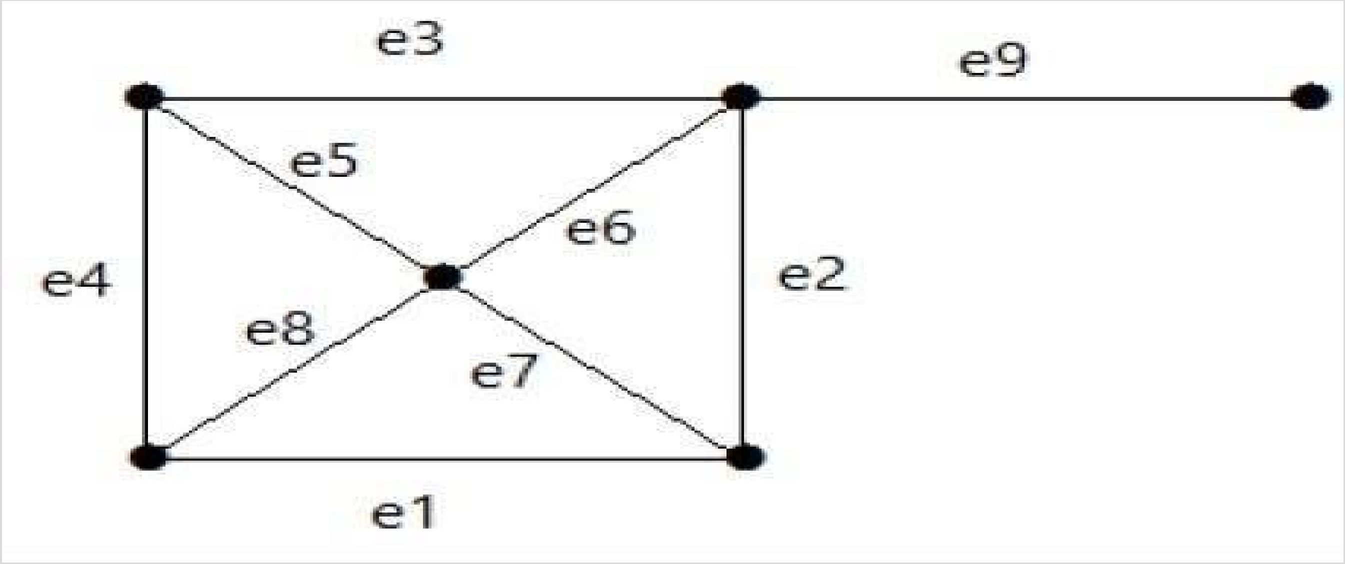
## Cut Set of a Graph

Let 'G'= (V, E) be a connected graph. A subset E' of E is called a cut set of G if deletion of all the edges of E' from G makes G disconnect.

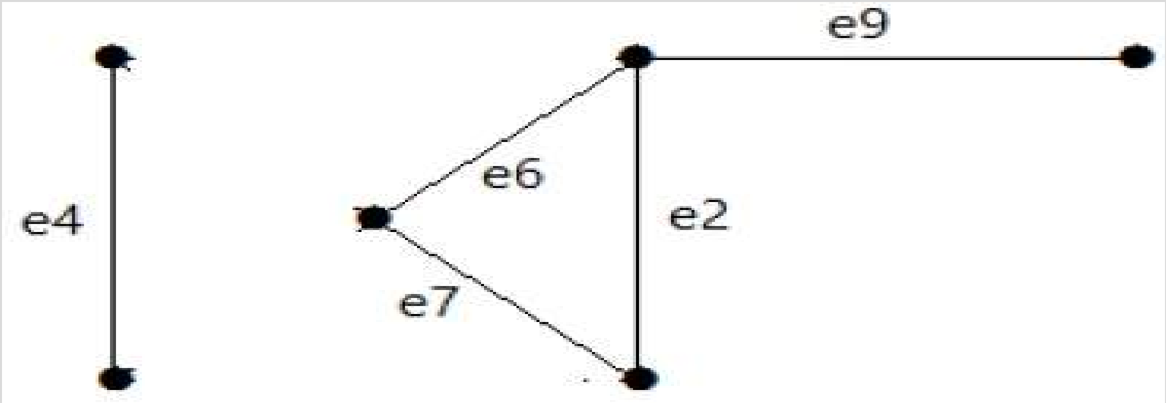
If deleting a certain number of edges from a graph makes it disconnected, then those deleted edges are called the cut set of the graph.

## Example

Take a look at the following graph. Its cut set is E1 = {e1, e3, e5, e8}.




After removing the cut set E1 from the graph, it would appear as follows −



We next turn to a discussion of important subsets of a group. Let $H$ be a subset of a group $G$ such that

(a) The identity $e$ of $G$ belongs to $H$.

(b) If $a$ and $b$ belong to $H$, then $ab \in H$.

(c) If $a \in H$, then $a^{-1} \in H$.

Then $H$ is called a **subgroup** of $G$. Part (b) says that $H$ is a subsemigroup of $G$. Thus a subgroup of $G$ can be viewed as a subsemigroup having properties (a) and (c).

Observe that if $G$ is a group and $H$ is a subgroup of $G$, then $H$ is also a group with respect to the operation in $G$, since the associative property in $G$ also holds in $H$.

Congruence relations on groups have a very special form, which we will now develop. Let $H$ be a subgroup of a group $G$, and let $a \in G$. The **left coset** of $H$ in $G$ determined by $a$ is the set $aH = \{ah \mid h \in H\}$. The **right coset** of $H$ in $G$ determined by $a$ is the set $Ha = \{ha \mid h \in H\}$. Finally, we will say that a subgroup $H$ of $G$ is **normal** if $aH = Ha$ for all $a$ in $G$.

WARNING. If $Ha = aH$, it does *not* follow that, for $h \in H$ and $a \in G$, $ha = ah$. It does follow that $ha = ah'$, where $h'$ is some element in $H$.

If $H$ is a subgroup of $G$, we shall need to compute all the left cosets of $H$ in $G$. First, suppose that $a \in H$. Then $aH \subseteq H$, since $H$ is a subgroup of $G$; moreover, if $h \in H$, then $h = ah'$, where $h' = a^{-1}h \in H$, so that $H \subseteq aH$. Thus, if $a \in H$, then $aH = H$. This means that, when finding all the cosets of $H$, we need not compute $aH$ for $a \in H$, since it will always be $H$.

Example 3. Let $G$ be the symmetric group $S_3$ discussed in Example 6 of Section 9.4. The subset $H = \{f_1, g_2\}$ is a subgroup of $G$. Compute all the distinct left cosets of $H$ in $G$.

*Solution:* If $a \in H$, then $aH = H$. Thus

$$f_1H = g_2H = H.$$

Also,

$$\begin{aligned} f_2H &= \{f_2, g_1\} \\ f_3H &= \{f_3, g_3\} \\ g_1H &= \{g_1, f_2\} = f_2H \\ g_3H &= \{g_3, f_3\} = f_3H. \end{aligned}$$

The distinct left cosets of $H$ in $G$ are $H$, $f_2H$, and $f_3H$. ◆

Example 4. Let $G$ and $H$ be as in Example 3. Then the right coset $Hf_2 = \{f_2, g_3\}$. In Example 3 we saw that $f_2H = \{f_2, g_1\}$. It follows that $H$ is not a normal subgroup of $G$. ◆