Linear Differential Equation with constant coefficient

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The n^{th} order linear differential equation with constant coefficient

The Differential Equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_{n-1} \frac{dy}{dx} + a_n y = Q$$

Example

$$\frac{d^3 y}{dx^3} + 3\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 2y = \sin 5x$$

$$If \quad \frac{d}{dx} = D$$

$$F(D)y = Q$$

$$Where \quad F(D) = a_o D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_{n-1} D + a_n$$

$$Example \quad \frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 2y = \sin 5x$$

$$\Rightarrow (D^3y + 3D^2y - 6Dy + 2y) = Sin5x$$

$$\Rightarrow (D^3 + 3D^2 - 6D + 2)y = Sin5x$$

$$\Rightarrow F(D)y = Sin5x$$

$$\therefore F(D) = (D^3 + 3D^2 - 6D + 2)$$

Auxiliary Equation(A.E.)

$$Suppose\ L.D.E.\ \ is\ \ F(D)y = Q$$

$$A.E.\ is\ \ F(m) = 0$$

$$OR\quad a_o m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_{n-1} m + a_n = 0$$

$$Example\quad \frac{d^3 y}{dx^3} + 3\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 2y = \sin 5x$$

$$\Rightarrow (D^3 + 3D^2 - 6D + 2)y = Sin5x\quad \Rightarrow F(D)y = Sin5x$$

$$\therefore F(D) = (D^3 + 3D^2 - 6D + 2)$$

$$Hence\ \ A.E. is\ \ F(m) = 0 \Rightarrow m^3 + 3m^2 - 6m + 2 = 0$$

Complementary Function (C.F.) of L.D.E.

A function of *'x'* which satisfies the L.D.E $F(D)y = 0$ is known as complementary function of L.D.E . .

Particular Integral (P.I.) of L.D.E.

A function of *'x'* which satisfies the L.D.E. $F(D)y = Q$ is known as particular integral of L.D.E .

General Solution of L.D.E.

The general solution of L.D.E $F(D)y = Q$ is given by

$$y = C.F. + P.I$$

General Solution of L.D.E.

$$Suppose\ L.D.E.\ is\ \ F(D)y = Q$$

Complete Solution :

$$y = C.F + P.I$$

Where $C.F \longrightarrow Complementary\ Function$

$P.I \longrightarrow Particular\ Integral$

Complementary Function

A function of 'x' which satisfies the L.D.E

$$F(D)y = 0$$

is known as complementary function of L.D.E . .

Determination of C.F.

- *Consider the L.D.E . $F(D)y = 0$*

- *Write A.E. of L.D.E. $F(m) = 0$*

$$\Rightarrow \quad a_o m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_{n-1} m + a_n = 0$$

- Solve A.E.

- Suppose $m_1, m_2, m_3, \ldots\ldots\ldots, m_n$

are the ‘n’ roots of the auxiliary equation.

Case I: (Roots are real)

- $If \quad m_1, m_2, m_3, \dots\dots\dots, m_n \quad are \quad distinct$

  $then \quad C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots\dots + c_n e^{m_n x}$

Determination of C.F.

Consider the L.D.E . $F(D)y = Q$

Write A.E. of L.D.E. $F(m) = 0$

i.e. $a_o m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_{n-1} m + a_n = 0$

.

Solve A.E.

Suppose $m_1, m_2, m_3, \ldots\ldots\ldots, m_n$ are the ‘n’ roots of the auxiliary equation.

Case I: (Roots are real)

$If \quad m_1, m_2, m_3, \ldots\ldots\ldots, m_n \quad are \ \ distinct$

If are distinct then

$then \quad C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$

If $m_1 = m_2 = k(say)$ and $m_3, m_4, \ldots\ldots\ldots, m_n$ are distinct then

$$C.F = (c_1 + c_2 x)e^{kx} + c_3 e^{m_3 x} + c_4 e^{m_4 x} \ldots\ldots + c_n e^{m_n x}$$

If $m_1 = m_2 = m_3 = k(say)$ and $m_4, m_5, \ldots\ldots\ldots, m_n$ are distinct then

$$C.F = (c_1 + c_2 x + c_3 x^2)e^{kx} + c_4 e^{m_4 x} + c_5 e^{m_5 x} \ldots\ldots + c_n e^{m_n x}$$

If $m_1 = \alpha + \sqrt{\beta}$ $m_2 = \alpha - \sqrt{\beta}$ and $m_3, m_4, \ldots\ldots\ldots, m_n$ are distinct then

$$C.F = e^{\alpha x}(c_1 \cosh \beta x + c_2 \sinh \beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} \ldots\ldots + c_n e^{m_n x}$$

If $m_1 = m_2 = \alpha + \sqrt{\beta}$, $m_3 = m_4 = \alpha - \sqrt{\beta}$, and $m_5, \ldots\ldots, m_n$ are distinct then

$$C.F = e^{\alpha x}[(c_1 + c_2 x)\cosh \beta x + (c_3 + c_4 x)\sinh \beta x)] + c_5 e^{m_5 x} + \ldots\ldots + c_n e^{m_n x}$$

Case II: (Roots are comlex)

If $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta \; and \; m_3, m_4......, m_n$ are real and distinct then

$$C.F = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x 0 + c_3 e^{m_3 x} + c_4 e^{m_4 x} .... + c_n e^{m_n x}$$

If $m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta \; and \; m_5, ..., m_n$ are real and distinct then

$$C.F = e^{\alpha x}[(c_1 + xc_2)\cos \beta x + (c_3 + xc_4)\sin \beta x] + c_5 e^{m_3 x} + ... + c_n e^{m_n x}$$

Determination of P.I.

P.I. of L.D.E. $F(D)y = Q$ is given by $\dfrac{1}{F(D)}Q$

Thus P.I. = $\dfrac{1}{F(D)}Q$

Case I: when *CASE I:* $when \quad Q = e^{ax}$

$$P.I = \frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}, \quad F(a) \neq 0$$

\# If $F(a) = 0$ then

$$P.I = \frac{1}{F(D)}e^{ax} = x\frac{1}{F'(a)}e^{ax}, \quad F'(a) \neq 0$$

if $F^{'}(a)=0$ then

$$then \quad P.I.=\frac{1}{F(D)}e^{ax}, \qquad F(a)=0$$

$$=x\frac{1}{F^{'}(D)}e^{ax}, \quad F^{'}(a)=0$$

$$=x^{2}\frac{1}{F^{''}(a)}e^{ax}, \quad F^{''}(a)\neq 0$$

Case II: when $Q=\sin ax \quad or \quad \cos(ax+b)$

$$P.I=\frac{1}{[F(D)]}Sin(ax+b)$$

$$=\frac{1}{[F(D)]_{D^{2}=-a^{2}}}Sin(ax+b), \quad [F(D)]_{D^{2}=-a^{2}}\neq 0$$

if $If\ \ [F'(D)]_{D^2=-a^2} = 0$

$$P.I = \frac{1}{[F(D)]} Sin(ax+b), \quad [F(D)]_{D^2=-a^2} = 0$$

$$= x\frac{1}{[F'(D)]_{D^2=-a^2}} Sin(ax+b), \quad [F'(D)]_{D^2=-a^2} \neq 0$$

if $[F'(D)]_{D^2=-a^2} = 0$

$$P.I = \frac{1}{[F(D)]} Sin(ax+b), \quad [F(D)]_{D^2=-a^2} = 0$$

$$= x\frac{1}{[F'(D)]} Sin(ax+b), \quad [F'(D)]_{D^2=-a^2} = 0$$

$$= x^2\frac{1}{[F''(D)]_{D^2=-a^2}} Sin(ax+b), \quad [F''(D)]_{D^2=-a^2} \neq 0$$

Case III: when $Q = x^m$, m non negative integer

$$P.I = \frac{1}{F(D)} x^m$$

$$= \frac{1}{Lowest\ \ \deg ree\ \ term[1 \pm \phi(D)]} x^m$$

$$= \frac{1}{LDT} [1 \pm \phi(D)]^{-1} (x^m)$$

Expending $[1 \pm \phi(D)]^{-1}$ by Binomial theorem *P.I.* can be evaluated

Case IV: when $Q = e^{ax} V$

$$P.I = \frac{1}{F(D)} e^{ax} V = e^{ax} \frac{1}{F(D+a)} V$$

Case V: (General Method), Q is any function of *'x'*

$$P.I = \frac{1}{F(D)} Q = \frac{1}{\phi(D)(D-\alpha)} Q$$

$$= \frac{1}{\phi(D)} \left[\frac{1}{(D-\alpha)} Q \right]$$

$$= \frac{1}{\phi(D)} e^{\alpha x} \int e^{-\alpha x} Q \, dx$$

1. Solve

Solution: The d.e. is

The A.E. is

Factorizing

The roots are

$$P.I. = \frac{1}{(D^3 - 3D^2 + 4)} e^{2x} = x \frac{1}{3D^2 - 6D} e^{2x}$$

$$= x^2 \frac{1}{(6D - 6)} e^{2x} = \frac{x^2 e^{2x}}{6}.$$

The complete solution is

2. **Solve**

Solution: The d.e. is

The a.e. is

Factorizing

The roots are

The complete solution is

3. Solve

Solution: The d.e. is

The a.e. is Factorizing

The roots are

And

The complete solution is

4. Solve

Solution: The d.e. is

The a.e. is

The complete solution is

5. Solve

Solution: The d.e. is

The a.e. is

Factorizing

The roots of A.E. are

$\therefore$

The complete solution is

$\therefore$

6. Solve

Solution:

Here

But ,

and

The complete solution is

Legendre’s Linear Equations

A Legendre’s linear differential equation is of the form

where are constants and

This differential equation can be converted into L.D.E with constant coefficient by subsitution

and so on

Note: If then Legendre's equation is known as Cauchy- Euler's equation

7. Solve

Put Then

The C.S. is

Simultaneous Linear Differential Equations

The most general form a system of simultaneous linear differential equations containing two dependent variable *x*, *y* and the only independent variable *t* is

…………………(1),

where are constants and and are functions of *t* only.

8. Solve :

Solution: The system is

Eleminating 'y' between Equations (1) and (2), we get

It is L.D.E. with constant coefficient.

Solution of eqn (3) *is given by*

$$x = e^t(C_1\cos t + C_2\sin t) - \frac{1}{2}\cos 2t. ----------(4)$$

From (1) and (2),

$$(1)+(2) \Rightarrow 2x' - 2x + 2y = \sin 2t + \cos 2t$$

$$\Rightarrow 2y = \sin 2t + \cos 2t + 2x - 2x'$$

$$= \sin 2t + \cos 2t + 2\left[e^t(C_1\cos t + C_2\sin t) - \frac{1}{2}\cos 2t\right]$$

$$-2\left[e^t(C_1\cos t + C_2\sin t) + e^t(-C_1\sin t + C_2\cos t) + \sin 2t\right] \text{ by using (3)}$$

$$= 2e^t[C_1\cos t + C_2\sin t - C_1\cos t - C_2\sin t + C_1\sin t - C_2\cos t]$$

$$+\sin 2t + \cos 2t - \cos 2t - 2\sin 2t$$

$$= 2e^t(C_1\sin t - C_2\cos t) - \sin 2t$$

$$\therefore y = e^t(C_1\sin t - C_2\cos t) - \frac{1}{2}\sin 2t..................(5)$$

Equations (5) *and* (6) *give complete solution of given simultaneous equations.*