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Assignment 3

1: Using Cauchy's Residue theorem evaluate $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx$

Sol^n i) Considering the contour as above and nothing that $z f(z) \to 0$ as $|z| \to \infty$, we find the poles,

(ii) $z^6 + 1 = 0$ gives $z^6 = e^{(2n+1)\pi i}$

$\therefore z = e^{(2n+1) i\pi/6}$ $n = 0, 1, 2, 3, 4, 5.$

$\therefore z = e^{(i\pi)/6}, e^{(i\pi)/2}, e^{(5i\pi)/6}, e^{(7i\pi)6}, e^{(3i\pi)/2}, e^{(11 i\pi)/6}$

(iii) Of these 1st three poles lie in the upper half plane. Let α lie one of these poles.

(iv) Residue (at $z = \alpha$) $= \lim_{z \to \alpha} \frac{(z-\alpha)(z^2)}{z^6+1}$

[form $\frac{0}{0}$]

$= \lim_{z \to \alpha} \frac{(z-\alpha) 2z + z^2}{6z^5}$ [L Hospital's Rule]

$= \frac{\alpha^2}{6\alpha^5} = \frac{\alpha^3}{6\alpha} = \frac{-\alpha^3}{6}$ [$\because \alpha^6 = -1$]

$\therefore$ Sum of the residues $= \frac{-1}{6} [\alpha_1^3 + \alpha_2^3 + \alpha_3^3]$

$$= -\frac{1}{6}\left[e^{(i\pi)/2} + e^{(3i\pi)/2} + e^{(5i\pi)/2}\right]$$

$$= -\frac{1}{6}\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) + \cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right)\right]$$

$$= -\frac{1}{6}\left[i - i + i\right] = -\frac{i}{6}$$

2. Use Taylor's or Laurent's series to expand $f(z) = \frac{z-1}{z^2+2z-3}\,dz$ indicating the regions of convergence.

Ans: Let $f(z) = \frac{z-1}{(z+3)(z-1)} = \frac{a}{z+3} + \frac{b}{z-1}$

$\therefore z-1 = a(z-1) + b(z+3)$

Putting $z = -3$, $-4 = -4a$ $\therefore a = 1$

Putting $z = 1$, $0 = 4b$ $\therefore b = 0$

$\therefore \frac{z-1}{(z+3)(z-1)} = \frac{1}{z+3}$

Sol: let $f(z) = \frac{z-1}{(z+1)(z-3)} = \frac{a}{z+1} + \frac{b}{z-3}$

$\therefore z-1 = a(z-3) + b(z+1)$

Putting $z = -1$, $-2 = -4a$ $\therefore a = 1/2$

Putting $z = 3$, $2 = 4b$ $\therefore b = \frac{1}{2}$

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$$\therefore \frac{z-1}{(z+1)(z-3)} = \frac{1/2}{z+1} + \frac{1/2}{z-3}$$

Hence, $f(z)$ is not analytic at $z=-1$ and $z=3$.

$\therefore f(z)$ is analytic in (i) $|z|<1$, (ii) $1<|z|<3$ (iii) $|z|>3$

case (i) when $|z|<1$, we also get $|z|<3$

$$\therefore f(z) = \frac{1/2}{z+1} + \frac{1/2}{z-3} = \frac{1}{2}\frac{1}{1+z} + \frac{1}{2}\frac{1}{(-3)}\cdot\frac{1}{1-(z/3)}$$

$$= \frac{1}{2}[1+z]^{-1} - \frac{1}{6}\left(1-\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}[1-z+z^2-z^3+\cdots] - \frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right]$$

$$= \frac{1}{3} - \frac{5z}{9} + \frac{13z^2}{27} + \cdots$$

case (ii) when $1<|z|<3$, we get $|1/z|<1$ and $|z/3|<1$

$$\therefore f(z) = \frac{1}{2}\frac{1}{1+z} + \frac{1}{2}\frac{1}{z-3}$$

$$= \frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1} - \frac{1}{6}\left[1-\frac{z}{3}\right]^{-1}$$

$$= \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right] - \frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\cdots\right]$$

$$= \frac{1}{2z}\left[\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots\right] - \frac{1}{6}$$

2^{nd} $\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\cdots\right.$

case (iii) when $|z|>3$, clearly, $|z|>1$

$\therefore \frac{|z|}{3}>1$ and $\frac{|z|}{1}>1$ $\therefore \frac{3}{|z|}<1$ and

$\frac{1}{|z|}<1$

$$\therefore f(z)=\frac{1}{2}\frac{1}{z+1}+\frac{1}{2}\frac{1}{z-3}$$

$$=\frac{1}{2z}\frac{1}{1+(1/z)}+\frac{1}{2z}\frac{1}{1-(3/z)}$$

$$=\frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1}+\frac{1}{2z}\left[1-\frac{3}{z}\right]^{-1}$$

$$=\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right]+\frac{1}{2z}\left[1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+\cdots\right]$$

$$= \frac{1}{2z}\left\{2 + \frac{2}{z} + \frac{10}{z^2} + \frac{26}{z^3} + \cdots\right\}$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{5}{z^3} + \frac{13}{z^4} + \cdots$$

This is the required laurent's series.

3. Evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$

Ans: $z = e^{i\theta}$ $\therefore dz = i \cdot z \cdot d\theta$

$\therefore dz = i \cdot z \cdot d\theta$ $\therefore d\theta = dz/iz$

$\therefore \cos = \frac{z^2+1}{2z}$

$$\therefore I = \int_C \frac{1}{5+3\cdot\frac{(z^2+1)}{2z}} \frac{dz}{iz}$$

$$= \int_C \frac{2\,dz}{(3z^2+10z+3)i}$$

where C is the circle $|z|=1$.

Now, the poles of $f(z)$ are given by $z = \cancel{-3}\ \frac{-10 \pm \sqrt{100-36}}{2\times3}$

$$= \frac{-10\pm 8}{6}$$

$$= \frac{-10+8}{6} \text{ or } \frac{-10-8}{6}$$

$$= \frac{-2}{6} \text{ or } -\frac{18}{6}$$

$$\alpha = \frac{-1}{3} \text{ or } \beta = -3$$

$\because a > b > 0$, α lies inside & β lies outside the circle $|z|=1$

$\therefore$ Residue of $f(z)$ (at $z=\alpha$)

$$= \lim_{z\to\alpha} (z-\alpha)\left\{\frac{2}{3(z-\alpha)(z-\beta)i}\right.$$

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$$= \frac{2}{3(\alpha-\beta)i} = \frac{2}{3\left(\frac{-1}{3}+3\right)} = \frac{2}{3 \times \frac{8}{3}}$$

$$= \frac{1}{4}$$

Now, $\alpha - \beta = \sqrt{[(\alpha-\beta)^2 - 4\alpha\beta}$

But $\alpha + \beta = \frac{-2a}{\beta}$ and $\alpha\beta = \frac{b}{b} = 1$

$\cancel{5+3 = -2a}$

$$-\frac{1}{3} - 3 = \frac{-2a}{-3}$$

$$-\frac{10}{3} = \frac{2a}{3}$$

$$a = -5$$

$\therefore \alpha - \beta = \sqrt{\cancel{[100}} \quad \sqrt{\left[\frac{4a^2}{b^2} - 4\right]}$

$$= \frac{2\sqrt{a^2-b^2}}{b} = \frac{2\sqrt{25-9}}{3} = \frac{2 \times 4}{3}$$

$$= \frac{8}{3}$$

Residue of $f(z)$ (at $z = \alpha$)

$$= \frac{2}{bi \cdot \frac{2\sqrt{a^2-b^2}}{b}} = \frac{1}{i\sqrt{a^2-b^2}}$$

$$\therefore I = 2\pi i \left(\frac{1}{i\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$= \frac{2\pi}{\sqrt{5^2 - 3^2}} = \frac{2\pi}{4} = \frac{\pi}{2}$$

4. Determine the residue of $f(z) = \frac{z^3}{(z-1)^4 (z-2)(z-3)}$ at its simple poles.

Sol^n: we have $f(z) = \frac{z^3}{(z-1)^4 (z-2)(z-3)}$

$\therefore$ $z = 1$ is a pole of order 4 and $z=2$ and $z=3$ is a simple poles.

$$\text{Residue (at } z=2) = \lim_{z \to 2} \left[(z-2) \frac{z^3}{(z-1)^4 (z-2)(z-3)} \right]$$

$$= \lim_{z \to} \left[\frac{z^3}{(z-1)^4 (z-3)} \right]$$

$$= \left[\frac{z^3}{(z-1)^4 (z-3)} \right] = \frac{8}{-1} = -8$$

$$\therefore \text{Residue (at } z=3) = \lim_{z \to 3} \left[(z-3) . \frac{z^3}{(z-1)^4 (z-2)(z-3)} \right]$$

$$= \lim_{z \to 3} \left[\frac{z^3}{(z-1)^4 (z-2)} \right] = \left[\frac{(3)^3}{(3-1)^4 (3-2)} \right]$$

$$= \frac{27}{16 \times 1} = \frac{27}{16}$$

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5. Evaluate the following integral using Residue Theorem

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$$

Sol^n The pole of integrand is given by,
$z(z-1)(z-2)=0 \Rightarrow z=0,1,2.$
Now, $z=0 \Rightarrow |z|=|0|=0<\frac{3}{2}$ [lies within C]
$z=1 \Rightarrow |z|=|1|=1<\frac{3}{2}$ [lies within C]
and $z=2 \Rightarrow |z|=|2|=2>\frac{3}{2}$ [outside of C]
By Cauchy integral formula,

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_{C_1} \frac{\frac{4-3z}{(z-1)(z-2)}}{z}\,dz + \int_{C_2} \frac{\frac{4-3z}{z(z-2)}}{z-1} = 2\pi i\left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} + 2\pi i\left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

$$= 2\pi i\left[\frac{4-0}{(0-1)(0-2)}\right] + 2\pi i\left[\frac{4-3}{1(1-2)}\right]$$

$$= 4\pi i - 2\pi i = 2\pi i$$

Thus, $\boxed{\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i}$

6. Evaluate $\int_C \frac{z^2}{(z-1)^2(z+1)}\, dz$ where C is $|z| = 2$ using Residue theorem.

Ans: If C is $|z| = 2$, both poles lie inside C.

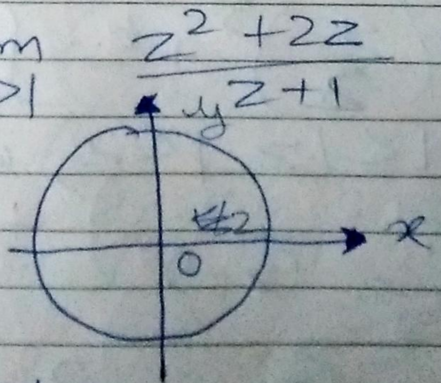
Now, Residue (at $z = -1$) = $\lim_{z \to -1} (z+1) f(z)$

$$= \lim_{z \to -1} \frac{z^2}{(z-1)^2} = \frac{1}{4}.$$

Residue (at $z = 1$) = $\lim_{z \to 1} \frac{d}{dz} \left[(z-1)^2 f(z) \right]$

$$= \lim_{z \to 1} \frac{d}{dz} \left[\frac{z^2}{z+1} \right] = \lim_{z \to 1} \frac{(z+1)2z - z^2}{(z+1)^2}$$

$$= \lim_{z \to 1} \frac{z^2 + 2z}{z+1} = \frac{3}{2}$$



$$\therefore \int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$= 2\pi i \left[\frac{1}{4} + \frac{3}{2} \right] = \frac{7\pi i}{2}$$

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7. Find Laurent's series of $f(z) = \frac{2}{(z-2)(z-1)}$, indicating the regions of convergence.

Ans: Let $\frac{2}{(z-1)(z-2)} = \frac{a}{z-1} + \frac{b}{z-2}$

$\therefore 2 = a(z-2) + b(z-1)$

when $z=1$, $2 = -a$ $\therefore a=-2$

when $z=2$, $2 = b$

$\therefore \frac{2}{(z-1)(z-2)} = \frac{-2}{z-1} + \frac{2}{z-2}$

Case (i): when $|z|<1$, clearly $|z|<2$

$\therefore f(z) = \frac{2}{1-z} - \frac{2}{2[1-(z/2)]}$

$= 2[1-z]^{-1} - [1-(z/2)]^{-1}$

$= 2[1+z+z^2+z^3+\ldots] - \left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\ldots\right]$

Case (ii): when $1<|z|<2$, we write

$\frac{2}{(z-1)(z-2)} = -\frac{2}{(z-1)} + \frac{2}{(z-2)}$

as

$= -\frac{2}{z[1-(1/z)]} - \frac{2}{2[1-(z/2)]}$

$= -\frac{2}{z}\{1-(1/z)\}^{-1} - [1-(z/2)]^{-1}$

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$$= -\frac{2}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] - \left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right]$$

case (iii) when $|z| > 2$, $\frac{|z|}{2} > 1$ i.e. $\frac{2}{|z|} < 1$.

Also when $|z| > 2$, $|z| > 1$

$\therefore \frac{1}{|z|} < 1$. we write

$$f(z) = -\frac{2}{z} \cdot \frac{1}{[1-(1/z)]} + \frac{2}{z} \cdot \frac{1}{[1-(2/z)]}$$

$$= -\frac{2}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{-2}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) + \frac{2}{z}\left(\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right)$$

$$= -2\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots\right) + 4\left(\frac{1}{z^2}+\frac{2}{z^3}+\frac{4}{z^4}+\cdots\right)$$

8. Expand $f(z) = \frac{z^2-1}{z^2+5z+6}$ along $z=1$

Ans: $f(z) = 1 - \frac{8}{z+3} + \frac{3}{z+2}$

$\because$, we want expansion around $z=1$, we have to obtain Laurent's series in powers of $(z-1)$.

$$\therefore \frac{z^2-1}{z^2+5z+6} = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3}$$

There now arise 3 cases

Case (i): when $|z-1| < 3$, we write (when $|z-1| < 3$ clearly $|z-1| < 4$)

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3} \text{ as}$$

$$= 1 - \frac{8}{4[1+(z-1)/4]} + \frac{3}{3[1+(z-1)/3]}$$

$$\therefore f(z) = 1 - \frac{8}{4}\left[1+\left(\frac{z-1}{4}\right)\right]^{-1} + \frac{3}{3}\left[1+\left(\frac{z-1}{3}\right)\right]^{-1}.$$

$$= 1 - 2\left[1 - \left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \cdots\right] + \left[1 - \left(\frac{z-1}{3}\right) + \left(\frac{z-1}{3}\right)^2 - \left(\frac{z-1}{3}\right)^3 + \cdots\right]$$

Case ii) When $3 < |z-1| < 4$, we write

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3}$$ as

$$= 1 - \frac{8}{4[1+(z-1)/4]} + \frac{3}{(z-1)[1+3/(z-1)]^{-1}}$$

$$= 1 - 2\left[1+\left(\frac{z-1}{4}\right)\right]^{-1} + \frac{3}{(z-1)}\left[1+\left(\frac{3}{z-1}\right)\right]$$

$$= 1 - 2\left[1 - \left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \cdots\right]$$

$$+ \frac{3}{(z-1)}\left[1 - \left(\frac{3}{z-1}\right) + \left(\frac{3}{z-1}\right)^2 - \left(\frac{3}{z-1}\right)^3 + \cdots\right]$$

Case (iii) When $|z-1| > 4$, we write

(When $|z-1| > 4$, clearly $|z-1| > 3$)

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3}$$

as

$$= 1 - \frac{8}{(z-1)[1+4/(z-1)]} + \frac{3}{(z-1)[1+3/(z-1)]}$$

When $|z-1| > 4$, clearly $|z-1| > 3$

$$\therefore f(z) = 1 - \frac{8}{z-1}\left[1+\left(\frac{4}{(z-1)}\right)\right]^{-1} + \frac{3}{z-1}\left[1+\left(\frac{3}{z-\cancel{1}}\right)\right]^{-1}$$

9. Expand $\cos z$ as Taylor's series at $z = \pi/2$

Ans: By Taylor's series

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

Here $f(z) = \cos z$ and $a = \pi/2$

$\therefore f(\pi/2) = \cos(\pi/2) = 0$

$\therefore f''(z) = -\cos z \quad \therefore f''(\pi/2) = -\cos(\pi/2) = 0$

$f'''(z) = \sin z \quad \therefore f'''(\pi/2) = \sin(\pi/2) = 1$

$$\therefore f(z) = 0 + [z-(\pi/2)](-1) + \frac{[z-(\pi/2)]^3}{3!}(1) + \frac{[z-(\pi/2)]^5}{5!}(-1) + \frac{[z-(\pi/2)]^7}{7!}(1) + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{[z-(\pi/2)]^{2n+1}}{(2n+1)!}$$

Alter: Put $z = u + (\pi/2)$

$\therefore \cos z = \cos[u + (\pi/2)] = \cos u \cos(\pi/2) - \sin u \sin(\pi/2) = -\sin u$

$$= -\left[u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots\right]$$

$$= -u + \frac{u^3}{3!} - \frac{u^5}{5!} + \frac{u^7}{7!} + \cdots$$

$$= -[z-(\pi/2)] + \frac{[z-(\pi/2)]^3}{3!} - \frac{[z-(\pi/2)]^5}{5!}$$

$+ \cdots$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{[z - (\pi/2)]^{2n+1}}{(2n+1)!}$$

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10 Determine the nature of poles & find the residue at each pole

(i) $\dfrac{z e^{x}}{(z-a)^3}$

Sol

$z = a$ is a pole of order 3.

$\therefore$ Residue of $f(z)$ (at $z=a$)

$$= \lim_{z \to a} \frac{1}{2!} \frac{d^2}{dz^2} \left\{ (z-a)^3 \frac{z e^{z}}{(z-a)^3} \right\}$$

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$$= \frac{1}{2!} \lim_{z \to a} \frac{d^2}{dz^2} (z e^z) = \frac{1}{2!}$$

$$\lim_{z \to a} \frac{d}{dz} (z e^z + e^z)$$

$$= \frac{1}{2!} \lim_{z \to a} [z e^z + e^z + e^z]$$

$$= \frac{1}{2} (a e^a + 2e^a) = \frac{1}{2} (a+2) e^a.$$

(ii) $\frac{1 - e^{2z}}{z^3}$

$z = 0$ is a pole of order 3.

$\therefore$ Residue of $f(z)$ (at $z=0$) $=$

$$\lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2} [z^3 f(z)]$$

$$= \lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2} \left[\frac{z^3 \cdot (1 - e^{2z})}{z^3} \right]$$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} (1 - e^{2z}) = \frac{1}{2!}$$

$$\lim_{z \to 0} \frac{d}{dz} (-2e^{2z})$$

$$= \frac{1}{2!} \lim_{z \to 0} (-4 e^{2z}) = \frac{1}{2} (-4)$$

$$= -2$$

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Alt.

$$f(z) = \frac{1-e^{2z}}{z^3} = \frac{1-\left\{1+2z+\frac{4z^2}{2!}+\frac{8z^3}{3!}+\cdots\right\}}{z^3}$$

$$= -\frac{2}{z^2} - \frac{2}{z} - \frac{4}{3} - \cdots$$

$\therefore$ Residue (at $z=0$) $= b_1 =$ coefficient of $\frac{1}{z} = -2$.