Assignment 2

1. Evaluate $\oint_C \frac{e^{kz}}{z}\,dz$ over the circle $|z|=1$ and k is real. Hence prove that $\int_0^{\pi} e^{k\cos\theta}\cos(k\sin\theta)\,d\theta = \pi$.

Soln: We obtain integral in two different ways.

(i) $|z|=1$ is a circle with centre at the origin and radius $=1$. The point $z=0$ lies within the circle. Hence, we write $f(z)=e^{kz}$ which is analytic in and on C. Hence, by Cauchy's Integral Formula.

$$\int_C \frac{e^{kz}}{z}\,dz = 2\pi i f(z_0) \text{ where } f(z)=e^{kz} \text{ and } z_0 = 0$$

$= 2\pi i e^{kz_0}$ at $z_0 = 0$

$= 2\pi i e^0 = 2\pi i$

(ii) Now, if we put $z=e^{i\theta}$, $dz = ie^{i\theta}\,d\theta$

$$\int_C \frac{e^{kz}}{z}\,dz = \int_0^{2\pi} \frac{e^{ke^{i\theta}}}{e^{i\theta}}\, ie^{i\theta}\,d\theta = i\int_0^{2\pi} e^{ke^{i\theta}}\,d\theta$$

$$\therefore \int_C \frac{e^{kz}}{z}\,dz = i\int_0^{2\pi} e^{k(\cos\theta + i\sin\theta)}\,d\theta = 0$$

$$i\int_0^{2\pi} e^{k\cos\theta}\cdot e^{ik\sin\theta}\,d\theta$$

$$= i\int_0^{2\pi} e^{k\cos\theta}\{\cos(k\sin\theta) + i\sin(k\sin\theta)\}\,d\theta$$

$$\therefore 2\pi i = i\int_0^{2\pi} e^{k\cos\theta}\{\cos(k\sin\theta) + i\sin(k\sin\theta)\}\,d\theta$$

$$2\pi = 2\int_0^{\pi} e^{k\cos\theta}\cos(k\sin\theta)\,d\theta$$

$\left[\because \int_0^a f(x)\,dx = 2\int_0^{a/2} f(a-x)\,dx \text{ if } f(a-x) = f(x)\right]$

$$\therefore \int_0^{\pi} e^{k\cos\theta}\cos(k\sin\theta)\,d\theta = \pi$$

2. Evaluate $\int_0^{2+i} \bar{z}^2\,dz$ along (i) line $x=2y$ (ii) real axis from 0 to 2 (iii) $2y^2 = x$, the parabola.

Sln: Along the line $x=2y$.

$dx = 2\,dy$

$\therefore dz = dx + i\,dy = 2\,dy + i\,dy = (2+i)\,dy$ & x varies from 0 to 1.

$$\therefore \int_C f(\bar{z})\,dz = \int_0^1 (x-iy)^2 (2+i)\,dy$$

$$= \int_0^1 (x^2 - 2xiy - y^2)(2+i)\,dy$$

$$= \int_0^1 (4y^2 - 4y^2 i - y^2)(2+i)\,dy \quad (\because 4x = 2y)$$

$$= \int_0^1 (3y^2 - 4iy^2)(2+i)\,dy$$

$$= \int_0^1 (3-4i)\,y^2 (2+i)\,dy$$

$$= (3-4i)(2+i)\int_0^1 y^2\,dy$$

$$= (10-5i)\left[\frac{y^3}{3}\right]_0^1 = (10-5i)\frac{1}{3}$$

$$\therefore \int f(\bar{z})\,dz = \frac{10}{3} - \frac{5i}{3}$$

(ii) $\int_0^{2+i} (\bar{z})^2\,dz = \int_0^{2+i} (x+iy)^2(dx + i\,dy)$

$= \int_{OA} (x^2)\,dx + \int_{AB} (2-iy)^2\, i\,dy$ [$\because$ Along OA, $y=0$, $dy=0$, x varies from 0 to 2. Along AB, $x=2$, $dx=0$ ~~x varies~~ and y varies from 0 to 1].

$$= \int_0^2 (x^2)\,dx + \int_0^1 (2+iy)^2\, i\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^2 + i\left[(4y) - \frac{4iy^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{8}{3} + i\left(4 - \frac{4i}{2} \cdot \frac{1}{3}\right) = \frac{1}{3}(14 + 11i) = \frac{14}{3} + \frac{11}{3}i$$

(iii) $\because 2y^2 = x \quad \therefore 4y = dy = dx$

$\therefore dz = dx + i\,dy = 4y\,dy + i\,dy = (4y + i)dy$ & x varies from 0 to 1.

$$\therefore \int_c f(z)\,dz = \int_0^1 (x - iy)^2 (4y + i)\,dy$$

$$= \int_0^1 (x^2 - 2xiy - y^2)(4y + i)\,dy$$

$$= \int_0^1 (4y^4 - 4y^3 i - y^2)(4y + i)\,dy \quad [\because 2y^2 = x]$$

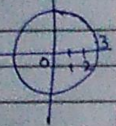
$$= \int_0^1 (16y^5 - 16iy^4 - 4y^3 + 4y^4 i + 4y^3 - iy^2]$$

$$= \left[16 \cdot \frac{1}{6} - \frac{16i}{\cancel{25}\,5} - 1 + \frac{4i}{5} + 1 - \frac{i}{3}\right]$$

$$= \left[\frac{8}{3} - \frac{41i}{15}\right]$$

3. Evaluate $\int_c \frac{\cos \pi z^2}{z^2 - 3z + 2}\,dz$ where c is the circle $|z| = 3$.

Solution: The circle $|z| = 3$ has centre at $(0, 0)$ and radius 3. Now $z^2 - 3z + 2 = 0$ gives $(z-2)(z-1) = 0$. $\therefore z = 2, 1$. Both these points lie inside the circle. Hence, we use partial fractions and write $\frac{1}{(z-2)(z-1)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$



and write $f(z) = \cos \pi z^2$ which is analytic in C.

$$\therefore \int_c \frac{\cos \pi z^2}{(z-2)(z-1)}\,dz = \int_c \frac{\cos \pi z^2}{(z-2)}\,dz - \int_c \frac{\cos \pi z^2}{(z-1)}\,dz$$

$= 2\pi i f(2) - 2\pi i f(1)$ where $f(z) = \cos \pi z^2$

$= 2\pi i + 2\pi i = \underline{4\pi i}$

4. Evaluate $\int_c \frac{z^2 + 4}{(z-2)(z+3i)}\,dz$ where c is (i) $|z+1| = 2$ (ii) $|z-2| = 2$.

Sol^n (i) $|z+1| = 2$

c is a circle with centre $(-1, 0)$ and radius 2. $f(z)$ is not analytic at $z = 4i$

For ~~$|z^2+4$~~ $|z+1| = |-3i + 1| = \sqrt{16+1} = \sqrt{10} > 1$

$z = -3i$ lies outside the curve c. So $f(z)$ is analytic on and inside c. $f'(z)$ is continuous on and inside c. $\therefore$, by Cauchy Integral theorem,

$$\int_c \frac{z^2 + 4}{(z-2)(z+3i)}\,dz = 0$$

(ii) $|z - 2| = 2$ centre $(2, 0)$ radius $= 2$.

$f(z) = \frac{z^2 + 4}{z + 3i}$ ($z = 2$ lies inside C)

$f(z)$ is analytic everywhere inside & on C. Now, by Cauchy's Integral formula

$$\int_c \frac{z^2 + 4}{(z-2)(z+3i)}\,dz = 2\pi i f(z)$$

$$= 2\pi i f(2)$$

$$= 2\pi i \left(\frac{2^2 + 4}{2 + 3i}\right)$$

$$\boxed{\int_c \frac{z^2 + 4}{(z-2)(z+3i)}\,dz = \frac{16\pi i}{(2 + 3i)}}$$

6. Evaluate $\int_c \frac{e^{2z}}{(z+1)^4}\,dz$ where c is the circle $|z-1| = 3$.

Sol^n. The circle $|z-1| = 3$ has centre at $C(1, 0)$ and

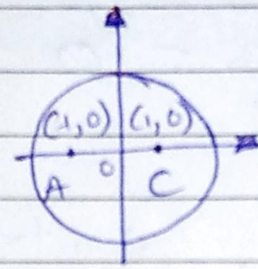
radius 3. Further, $z+1=0$ gives A. $z=-1$. The point A lies inside the circle. Hence, $\frac{e^{2z}}{(z+1)^4}$ is not analytic in C. We take $f(z)=e^{2z}$ which is analytic in C.
By corollary of Cauchy's Formula.

$$\int_c \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!} f^3(z_0)$$

$$= \frac{2\pi i}{3!} \cdot \frac{8}{e^2} \quad [\because f(z)=e^{2z} \text{ \& } f^3(z)=8e^{2z} \text{ \& } z_0=-1]$$

$$= \frac{8\pi i}{3e^2}$$



7. Evaluate $\int_c \frac{dz}{e^z z^4}$ where c is the circle $|z|=1$.

Soln: $\because$ the point $z=0$ lies inside the circle $|z|=1$
$f(z)=e^{-z}$ is analytic in and on c. Hence, by corollary of Cauchy's integral formula $\int_c \frac{f(z)}{(z-a)^n} = \frac{2\pi i}{(n-1)!} f^{n-1}(a)$

Now, $f(z)=e^{-z}$ $\therefore f'(z)=-e^{-z}$

$f''(z)=e^{-z}$, $f'''(z)=-e^{-z}$, $f''''(z)=e^{-z}$

$$\therefore \int \frac{dz}{e^z z^4} = \frac{2\pi i}{3!} e^{-z} = \frac{2\pi i}{3\times 2} \cdot e^0 = \frac{\pi i}{3}$$