Module 2 - Relations and Functions

CE– SE–DSGT

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Module 2 Relations and Functions

- Basic concepts of **Set Theory**
- **Relations**: Definition, Types of Relations, Representation of Relations, Closures of Relations, Warshall’salgorithm, Equivalence relations and Equivalence Classes
- **Functions**: Definition, Types of functions, Composition of functions, Identity and Inverse function

Sets

Definition

A *set* is an unordered collection of distinct objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Examples

The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$. ◀

The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$. ◀

Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements. For instance, $\{a, 2, \text{Fred}, \text{New Jersey}\}$ is the set containing the four elements a, 2, Fred, and New Jersey. ◀

The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$. ◀

Another way to describe a set is to use **set builder** notation. We characterize all those elements in the set by stating the property or properties they must have to be members. The general form of this notation is $\{x \mid x \text{ has property } P\}$ and is read "the set of all x such that x has property P." For instance, the set O of all odd positive integers less than 10 can be written as

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\},$$

or, specifying the universe as the set of positive integers, as

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}.$$

We often use this type of notation to describe sets when it is impossible to list all the elements of the set. For instance, the set $\mathbf{Q}^+$ of all positive rational numbers can be written as

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = \tfrac{p}{q}, \text{ for some positive integers } p \text{ and } q\}.$$

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of all **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of all **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of all **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{and } q \neq 0\}$, the set of all **rational numbers**
$\mathbf{R}$, the set of all **real numbers**

$\mathbf{R}^+$, the set of all **positive real numbers**
$\mathbf{C}$, the set of all **complex numbers**.

Sets can have other sets as members

The set $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$ is a set containing four elements, each of which is a set. The four elements of this set are $\mathbf{N}$, the set of natural numbers; $\mathbf{Z}$, the set of integers; $\mathbf{Q}$, the set of rational numbers; and $\mathbf{R}$, the set of real numbers. ◀

Types of Sets

1) **Subset:** If every element of a set A is also an element of set B, then we say that A is **subset** of B or A is contained in B. This is denoted by, $A \subseteq B$.

Eg: i) If A = {1,2,3,4,5,6}, B = {2,4,5}, C = {1,2,3,4,5} then $B \subseteq A$, $B \subseteq C$ & $C \subseteq A$

ii) If A = {1,3,6}, B = {-1,1,2,3,4,6}, C = {1,2,3}, then $A \subseteq B$ but $A \not\subset C$.

2) Equal Sets: If A is a subset of B and B is a subset of A, then sets A and B are equal i.e. $A \subseteq B$ and $B \subseteq A$ implies **A = B**.

Eg: i) If A = {1,2,3,4} and B = {3,4,2,1}, then A = B

ii) If A = {1,2,3} and B = {x| x is positive integer $x^2 < 12$}, then A = B

3) Proper SubSet: A set A is called proper subset set of set B if,

- A is subset of B, and
- B is not a subset of A

And is denoted as $A \subset B$

Eg: A = {1,3}, B = {1,2,3}, C = {1,3,2}. Here, $A \subseteq C$ and $B \subseteq C$.

Also, **$A \subseteq C$ but $C \not\subset A$. Thus $A \subset C$.**

But, $B \not\subset C$ as B = C

4) Infinite Sets: A set which is not finite, is said to be infinite set.

Eg: i) Set of natural numbers

ii) Number of points on a line

5) Finite Set: A set containing finite number of elements is a **finite set**.

6) Universal Set: If all sets considered during a 'specific discussion', are subsets of a given set, then this set is called as the '**Universal Set'** or '**Universe of Discourse**' and is denoted by 'U'.

7) Null Set: The set with no elements are called as '**Empty Set**' or '**Null Set**'. It is denoted as '{}' or ϕ.

8) Singleton Set: A set having only one element is known as a **singleton set**.

Eg: i) $A = \{8\}$ ii) $P = \{\phi\}$

9) Super Set: If Set A is a subset of set B, then B is **super set** of A.

Eg: If $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$, then $A \subseteq B$ and thus, B is super set of A

10) Disjoint Set: Two sets are **disjoint** if they have no elements in common.

Eg: If A = {1,2,3} and B = {7,8,9}, then A and B are disjoint sets.

11) Power Set: The set consisting of all the subsets of a given set A as its elements, is called the power set of A and is denoted by P(A) or 2^A. Thus,

P(A) or $2^A = \{X : X \subseteq A\}$. Thus,

(i) $P(\phi) = \{\phi\}$

(ii) if A = {1}, then $P^A = \{\phi, \{1\}\}$

(iii) if A = {1, 2}, then $P^A = \{\phi, \{1\}, \{2\}, A\}$

(iv) if A = {1, 2, 3}, then $P^A = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$

- From these examples we can conclude that if a set A has n elements, then P(A) has 2^n elements.

12) Cardinality of Finite Sets: Cardinality of a finite set is the number of elements in the set. Cardinality of a set A is denoted by |A|.

Eg: If A = {a,b,c,d,e}, then |A| = 5.

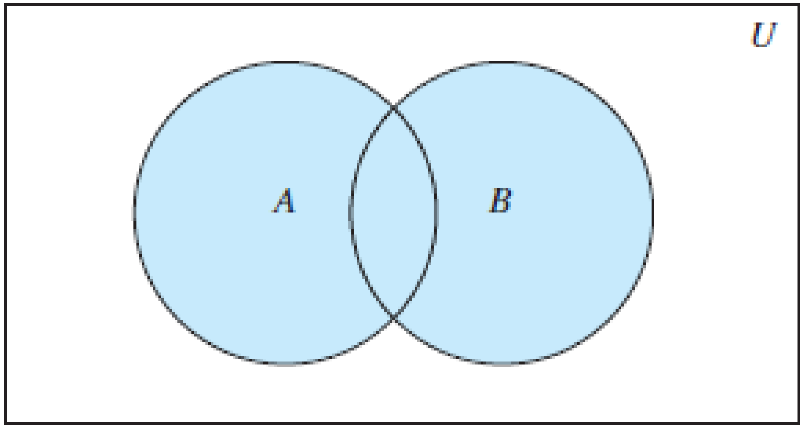
Set Operations

1) Union: The union of two sets A and B is the set consisting of all elements which are in A, or in B or in both the sets i.e. $\mathbf{A \cup B = \{x \mid x \in A \lor x \in B\}}$.

Example1: The union of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 2, 3, 5}; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.

Example2: If $A = \{x \mid x \in N, 4 < x < 12\}$ and $B = \{x \mid x \in N\ 8 < n < 15\}$,

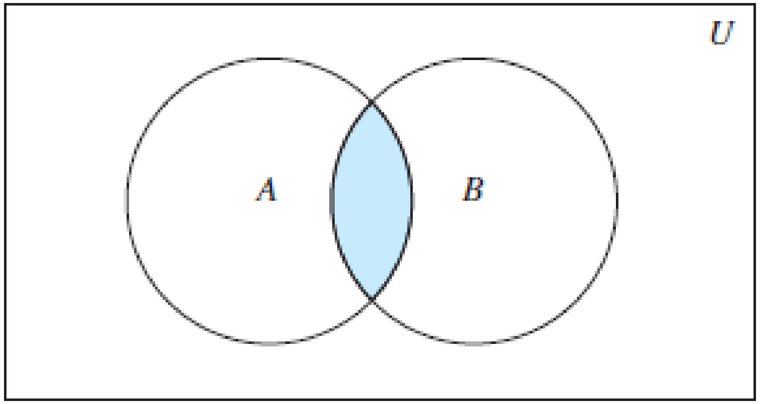
Then, $A \cup B = \{5,6,7,8,9,10,11,12,13,14\}$



2) Intersection: Intersection of two sets A and B (denoted by, $A \cap B$), is the set consisting of elements which are in A as well as in B.

$$\mathbf{A \cap B = \{x \mid x \in A \land x \in B\}}$$

Example1: The intersection of the sets {1,
that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$.



Example2: If $A = \{\phi\}$ and $B = \{a, \phi, \{\phi\}\}$,
then $A \cap B = \{\phi\} = A$

Example3: If $A = \{n \mid n \in N, 4 < n < 12\}$ and $B = \{n \mid n \in N, 5 < n < 10\}$,
then $A \cap B = \{6,7,8,9\} = B$

Example 4: Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint.

3) Complement of a set: If U is a universal set containing set A then U – A is called as **complement** of A and is denoted as $\bar{A}$ or A’

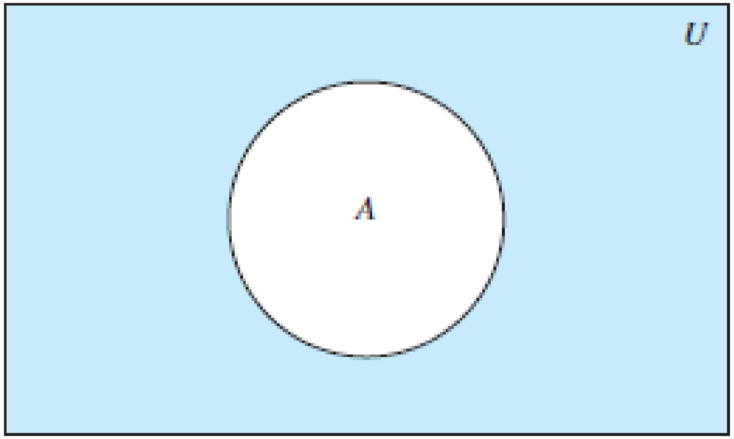
= {x ∈ U | x ∉ A}

Example1: Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then A’ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

Example2: Let A = {a, e, i, o, u} (where the universal set is the set of letters of the English alphabet). Then A = {b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z}.

Example3: If U = {a,b,c,d,e,f} and A = {b,d,e},

Then A’ = U – A = {a,c,f}



<u>4) Difference of sets:</u> the difference of sets A and B (denoted as A – B) is defined as **A – B = {x | x ∈ A ∧ x ∉ B}**. This means set of all elements that belong to set A but not to set B.

Similarly, B – A is the set of elements belonging to set B but not A.

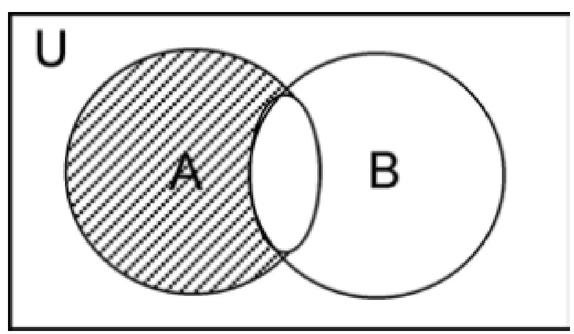
Example1: The difference of {1, 3, 5} and {1, 2, 3} is the set {5}; that is, {1, 3, 5} − {1, 2, 3} = {5}. This is different from the difference of {1, 2, 3} and {1, 3, 5}, which is the set {2}.

Example2: If A = {1,2,3,…,10}

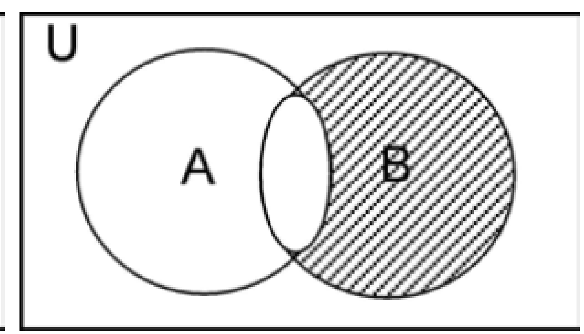
And B = {1,3,5,…,9} then

i) A – B = {2,4,6,8,10}

ii) B – A = ϕ



A – B (Shaded)



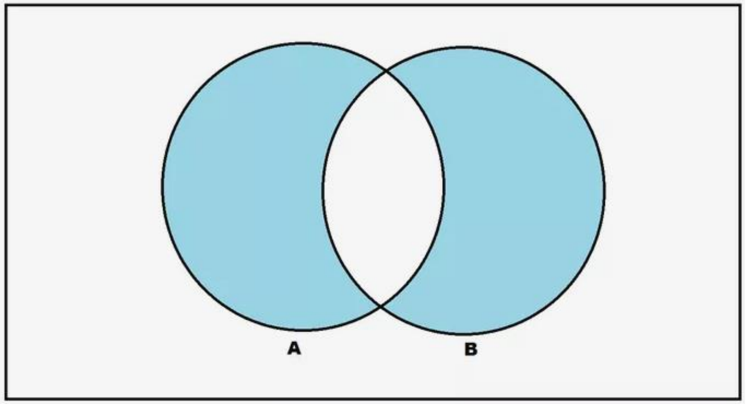
B – A (Shaded)

5) Symmetric Difference: The symmetric difference of sets A and B (denoted as $A \oplus B$) is the set of all elements that belong to A or B but not both i.e.

$\mathbf{A \oplus B = \{x \mid x \in (A - B) \text{ or } x \in (B - A)\}}$ or $\mathbf{A \oplus B = (A - B) \cup (B - A)}$

Example1: If A = {a,b,e,g} and B = {d,e,f,g},

then, $A \oplus B$ = {a,b,d,f}.



Example2: If A = {2,4,5,9} and

$B = \{x \in Z^+ \mid x^2 \leq 16\}$, then $A \oplus B = \{0,1,3,5,9\}$.

Example3: If A = {ϕ} and B = {a, ϕ, {ϕ}}, then $A \oplus B$ = {a, {ϕ}}.

6) Cartesian Product: The **Cartesian product** or **Product set** of two sets A and B (denoted as A X B), is the set of all ordered pairs in which the first element comes from set A and the second element from set B i.e.

A X B = {(x,y)| x ∈ A and y ∈ B }

Example: If A = {a,b} & B = {1,2,3},then, A X B = {(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)}

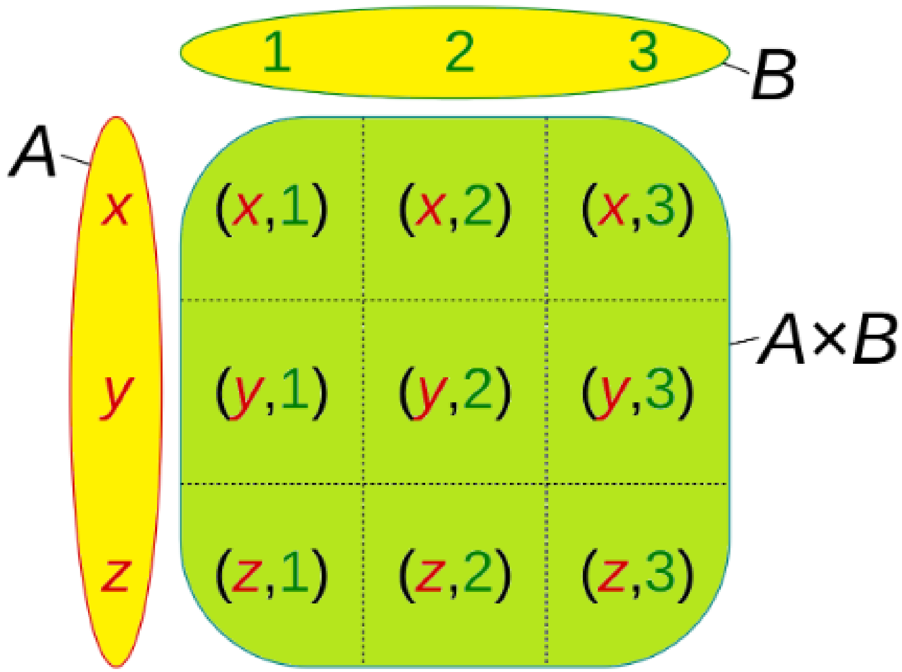


TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Example 1. Let $A = \{a, b, c, e, f\}$ and $B = \{b, d, r, s\}$. Find $A \cup B$.

Solution: Since $A \cup B$ consists of all the elements that belong to either A or B, $A \cup B = \{a, b, c, d, e, f, r, s\}$. ◆

Example 2. Let $A = \{a, b, c, e, f\}$, $B = \{b, e, f, r, s\}$, and $C = \{a, t, u, v\}$. Find $A \cap B$, $A \cap C$, and $B \cap C$.

Solution: The elements b, e, and f are the only ones that belong to both A and B, so $A \cap B = \{b, e, f\}$. Similarly, $A \cap C = \{a\}$. There are no elements that belong to both B and C, so $B \cap C = \{\ \}$. ◆

Example 3. Let $A = \{1, 2, 3, 4, 5, 7\}$, $B = \{1, 3, 8, 9\}$, and $C = \{1, 3, 6, 8\}$. Then $A \cap B \cap C$ is the set of elements that belong to A, B, and C. Thus $A \cap B \cap C = \{1, 3\}$. ◆

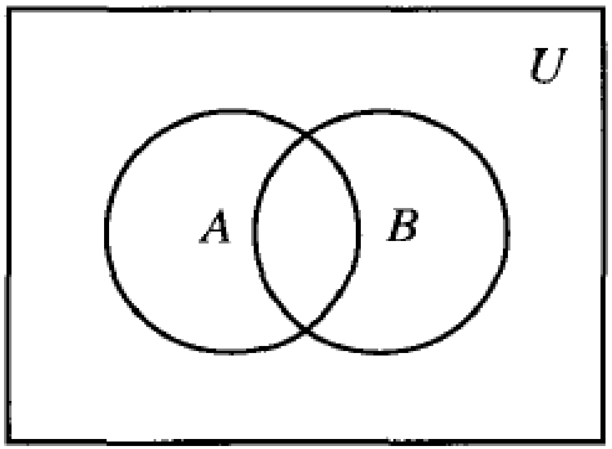
Example 4. Let $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$. Then $A - B = \{a\}$ and $B - A = \{d, e\}$. ◆

Example 5. Let $A = \{x \mid x \text{ is an integer and } x \leq 4\}$ and $U = Z$. Then $A = \{x \mid x \text{ is an integer and } x > 4\}$. ◆

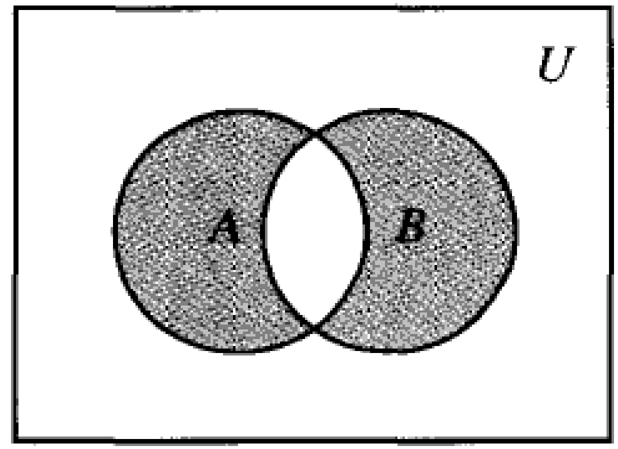
Example 6. Let $A = \{a, b, c, d\}$ and $B = \{a, c, e, f, g\}$. Then $A \oplus B = \{b, d, e, f, g\}$. ◆

If A and B are as indicated in Figure 1.10(a), their symmetric difference is the shaded region shown in Figure 1.10(b). It is easy to see that

$$A \oplus B = (A - B) \cup (B - A).$$



(a)



(b) $A \oplus B$

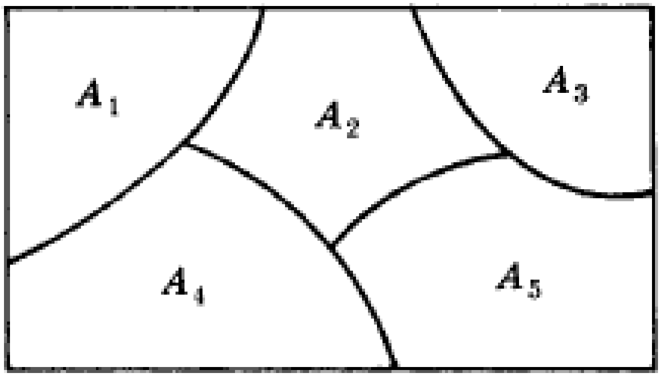
Partitions

Let S be a nonempty set. A *partition* of S is a subdivision of S into nonoverlapping, nonempty subsets. Precisely, a *partition* of S is a collection $\{A_i\}$ of nonempty subsets of S such that:

(i) Each a in S belongs to one of the A_i.
(ii) The sets of $\{A_i\}$ are mutually disjoint; that is, if

$$A_j \neq A_k \quad \text{then} \quad A_j \cap A_k = \emptyset$$

The subsets in a partition are called *cells*. Figure 1-6 is a Venn diagram of a partition of the rectangular set S of points into five cells, A_1, A_2, A_3, A_4, A_5.



EXAMPLE 1.11 Consider the following collections of subsets of $S = \{1, 2, \ldots, 8, 9\}$:

(i) $[\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}]$

(ii) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}]$

(iii) $[\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$

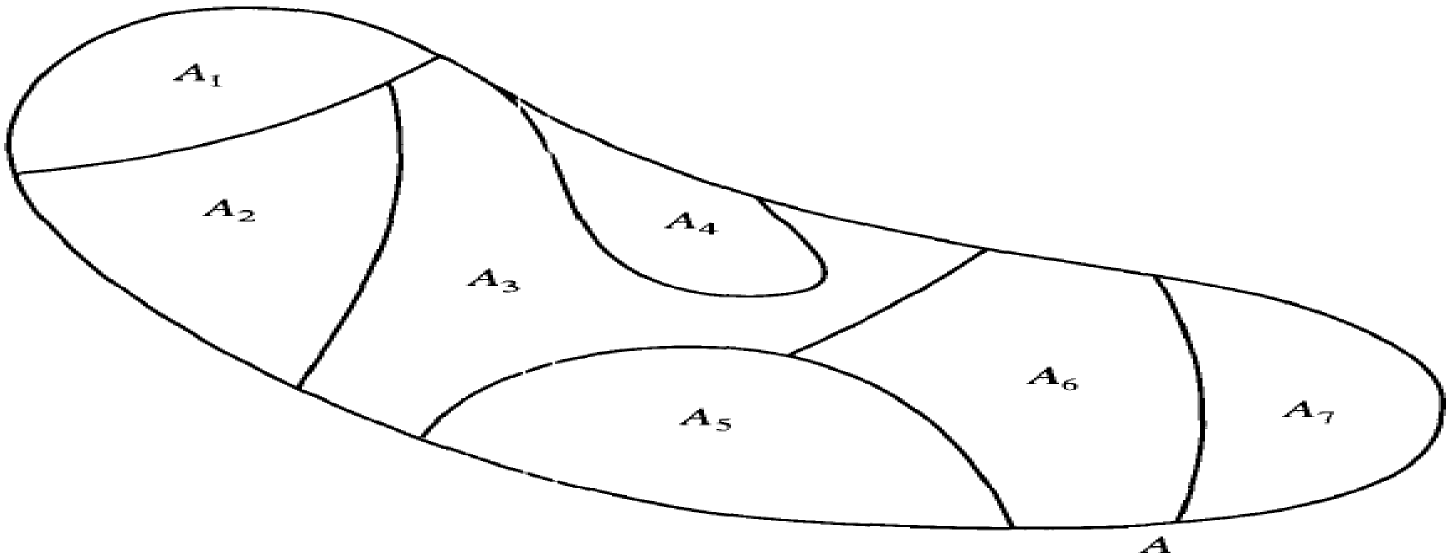
Then (i) is not a partition of S since 7 in S does not belong to any of the subsets. Furthermore, (ii) is not a partition of S since $\{1, 3, 5\}$ and $\{5, 7, 9\}$ are not disjoint. On the other hand, (iii) is a partition of S.

Partitions

A **partition** or **quotient set** of a nonempty set A is a collection $\mathcal{P}$ of nonempty subsets of A such that

1. Each element of A belongs to one of the sets in $\mathcal{P}$.
2. If A_1 and A_2 are distinct elements of $\mathcal{P}$, then $A_1 \cap A_2 = \varnothing$.

The sets in $\mathcal{P}$ are called the **blocks** or **cells** of the partition. Figure 4.2 shows a partition $\mathcal{P} = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ of A into seven blocks.



Example 6. Let

$$A = \{a, b, c, d, e, f, g, h\}.$$

Consider the following subsets of A:

$$A_1 = \{a, b, c, d\}, \quad A_2 = \{a, c, e, f, g, h\}, \quad A_3 = \{a, c, e, g\},$$

$$A_4 = \{b, d\}, \quad A_5 = \{f, h\}.$$

Then $\{A_1, A_2\}$ is not a partition since $A_1 \cap A_2 \neq \emptyset$. Also, $\{A_1, A_5\}$ is not a partition since $e \notin A_1$ and $e \notin A_5$. The collection $\mathcal{P} = \{A_3, A_4, A_5\}$ is a partition of A. ♦

Example 7. Consider the set A of all employees of General Motors. If we form subsets of A by grouping in a subset all employees who make exactly the same salary, we obtain a partition of A. Each employee will belong to exactly one subset. ♦

Example 8. Let

$$Z = \text{set of all integers,}$$
$$A_1 = \text{set of all even integers, and}$$
$$A_2 = \text{set of all odd integers.}$$

Then $\{A_1, A_2\}$ is a partition of Z. ♦

Since the members of a partition of a set A are subsets of A, we see that the partition is a subset of $P(A)$, the power set of A. That is, partitions can be considered as particular kinds of subsets of $P(A)$.

Product Sets

An **ordered pair** (a, b) is a listing of the objects a and b in a prescribed order, with a appearing first and b appearing second. Thus an ordered pair is merely a **Sequence of length 2.**

it follows that the ordered pairs (a_1, b_1) and (a_2, b_2) are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.

If A and B are two nonempty sets, we define the **product set** or **Cartesian product** $A \times B$ as the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. Thus

$$A \times B = \{(a, b) \mid a \in A \quad \text{and} \quad b \in B\}.$$

Example 1. Let

3 2 6

$$A = \{1, 2, 3\} \quad \text{and} \quad B = \{r, s\},$$

then

$$A \times B = \{(1, r), (1, s), (2, r), (2, s), (3, r), (3, s)\}.$$

Observe that the elements of $A \times B$ can be arranged in a convenient tabular array as shown in Figure 4.1. ◆

A \ B	r	s
1	$(1, r)$	$(1, s)$
2	$(2, r)$	$(2, s)$
3	$(3, r)$	$(3, s)$

Example 2. If A and B are as in Example 1, then

$$B \times A = \{(r, 1), (s, 1), (r, 2), (s, 2), (r, 3), (s, 3)\}. \quad \blacklozenge$$

From Examples 1 and 2, we see that $A \times B$ need not equal $B \times A$.

$\neq$

Example 3. If $A = B = \mathbb{R}$, the set of all real numbers, then $\mathbb{R} \times \mathbb{R}$, also denoted by $\mathbb{R}^2$, is the set of all points in the plane. The ordered pair (a, b) gives the coordinates of a point in the plane. ◆

Example 4. A marketing research firm classifies a person according to the following two criteria:

Gender: male (m); female (f)
Highest level of education completed: elementary school (e); high school (h); college (c); graduate school (g)

Relations

Let $S = \{m, f\}$ and $L = \{e, h, c, g\}$. Then the product set $S \times L$ contains all the categories into which the population is classified. Thus the classification (f, g) represents a female who has completed graduate school. There are eight categories in this classification scheme. ◆

Let A and B be nonempty sets. A **relation R from A to B** is a subset of $A \times B$. If $R \subseteq A \times B$ and $(a, b) \in R$, we say that a **is related to b by R**, and we also write $a\ R\ b$. If a is not related to b by R, we write $a\ \not{R}\ b$. Frequently, A and B are equal. In this case, we often say that $R \subseteq A \times A$ **is a relation on A**, instead of a relation from A to A.

Relations

Example 1. Let

$$A = \{1, 2, 3\} \quad \text{and} \quad B = \{r, s\}.$$

Then

$$R = \{(1, r), (2, s), (3, r)\}$$

is a relation from A to B. ◆

Example 2. Let A and B be sets of real numbers. We define the following relation R (equals) from A to B:

$$a\ R\ b \quad \text{if and only if} \quad a = b.$$ ◆

Example 3. Let

$$A = \{1, 2, 3, 4, 5\}.$$

Define the following relation R (less than) on A:

$$a\ R\ b \quad \text{if and only if} \quad a < b.$$

Then

$$R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}.$$ ◆

Example 4. Let $A = Z^+$, the set of all positive integers. Define the following relation R on A:

$$a \; R \; b \quad \text{if and only if} \quad a \text{ divides } b.$$

Then $4 \; R \; 12$, but $5 \not R \; 7$. ◆

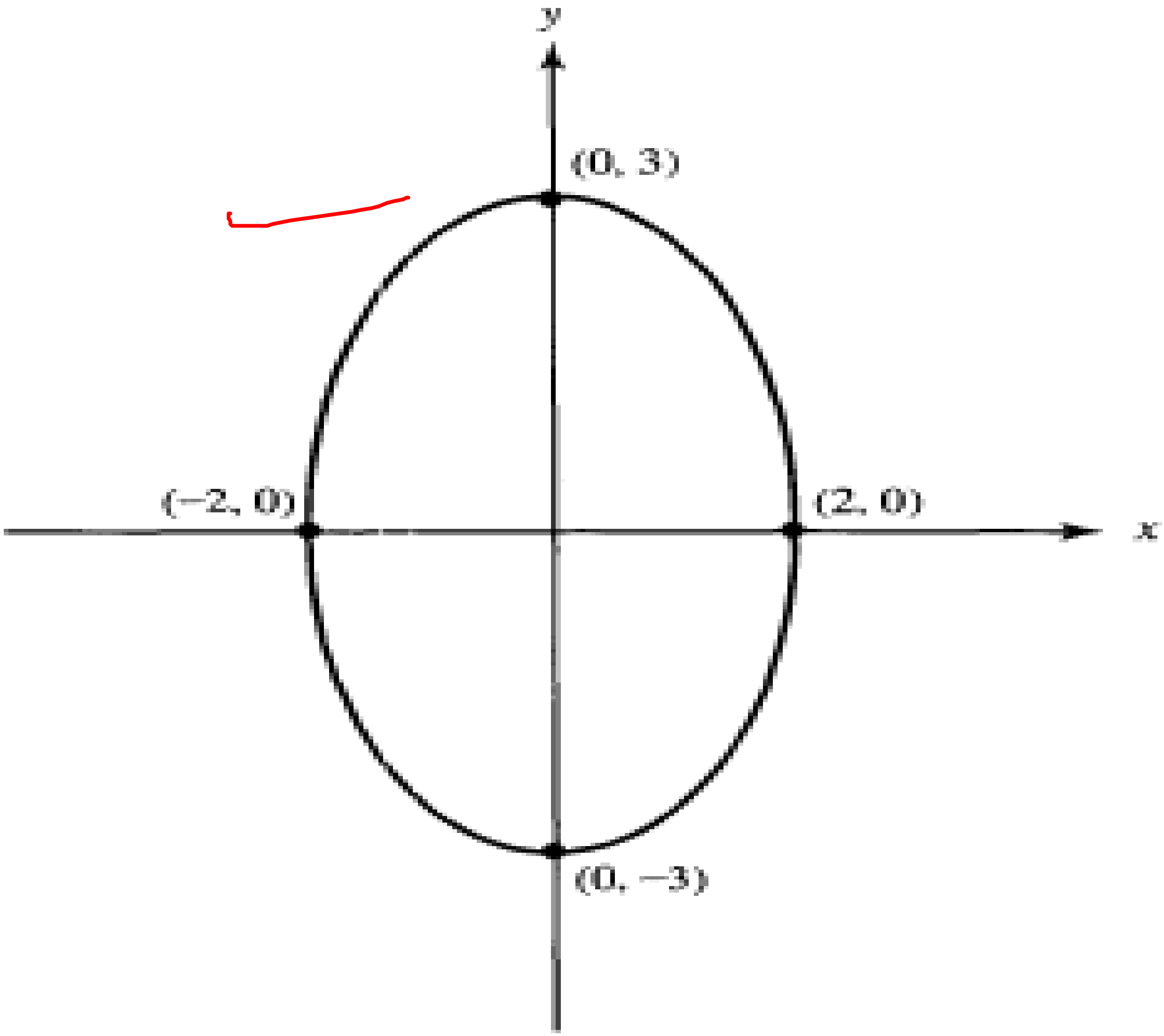
Example 5. Let A be the set of all people in the world. We define the following relation R on A: $a \; R \; b$ if and only if there is a sequence $a_0, a_1, \ldots, a_n$ of people such that $a_0 = a$, $a_n = b$ and a_{i-1} knows a_i, $i = 1, 2, \ldots, n$ (n will depend on a and b). ◆

Example 6. Let $A = \mathbb{R}$, the set of all real numbers. We define the following relation R on A:

$$x \; R \; y \quad \text{if and only if} \quad x \text{ and } y \text{ satisfy the equation } \frac{x^2}{4} + \frac{y^2}{9} = 1.$$

The set R consists of all points on the ellipse shown in Figure 4.3. ◆

Relations



Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$

How many relations are there on a set with n elements?

Solution: A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements. For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a, b, c\}$. ◀

Domain & Ranges

- **<u>Domain:</u>** Let R be the relation from X to Y. The domain of R (abbreviated as, *dom R*) is the set of all elements x ∈ X, that occurs in at least in one pair (x,y) ∈ R i.e.

  dom R = {x|∃y ((x,y) ∈ R)}

- **<u>Range:</u>** The range of R (abbreviated as, *ran R*) is the set of all y ∈ Y that occurs in at least one pair (x,y) ∈ R i.e.

  ran R = {y|∃x ((x,y) ∈ R)}

<u>Example:</u> Find the domain and range of relation R from the set {1,2,3,4} to the set {a,b,c} given by R = {(2,c),(1,d),(3,d),(2,a)}

<u>Soln:</u> **dom R = {1,2,3}** and **ran R = {a,c,d}**

Properties of Relations

1) **Reflexive Relations:** A relation R on a set A is called *reflexive* if (a,a) ∈ R for every element a∈A

i.e. $\forall a(aRa)$

Example1: Consider the following relations on {1,2,3,4}:

R_1 = {(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}

R_2 = {(1,1),(1,2),(2,1)}

R_3 = {(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}

R_4 = {(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}

R_5 = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}

R_6 = {(3,4)}

Soln: The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a,a), namely, (1,1),(2,2),(3,3) and (4,4). The other relations do not contain all such pairs.

Example2: Consider the following relations on the set of integers:

$R_1 = \{(a,b) | a \leq b\}$

$R_2 = \{(a,b) | a > b\}$

$R_3 = \{(a,b) | a = b \text{ or } a = (-b)\}$

$R_4 = \{(a,b) | a = b\}$

$R_5 = \{(a,b) | a = b+1\}$

$R_6 = \{(a,b) | a + b \leq 3\}$

Soln = R_1, R_3 and R_4 are reflexive relations.

Example3: Is the *'divides'* relation on the set of positive integers reflexive?

Soln: As $x|x$ whenever x is a positive integer, the *'divides'* relation is reflexive. However, if the set of positive integers is replaced with the set of all integers, then the relation is not reflexive because 0 does not divide 0.

2) Irreflexive Relations: A relation R on set A is irreflexive if for every a ∈ A, (x,x) ∉ R. This means there is **no x ∈ X such thar xRx.**

Example1: Let A ={1,2} and let R ={(1,2),(2,1)}, then R is irreflexive as (1,1),(2,2) ∉ R

3) Symmetric Relations: A relation R on a set A is called symmetric if (b,a) ∈ R whenever (a,b) ∈ R, for all a,b ∈ A i.e.

$$\forall a\ \forall b(aRb \rightarrow bRa)$$

Example1: let A be set of people. Let aRb if a is a friend of b. Then obviously, b is related to a. Thus, relation 'friend' is symmetric.

Example2: Let A = {1,2} and let R = {(1,1),{2,2)}. This is an example of symmetric relation which is also reflexive.

Example3: Let A = {1,2,3,4} and R = {(1,2),(2,2),(3,4),(4,1)}. Here, R is not symmetric as (1,2) ∈ R but (2,1) ∉ R.

Example4: Let R = {(1, 1),(1, 2),(2, 1),(2, 2),(3, 3),(4, 4)} on set A = {1,2,3,4}. Here R is symmetric as for every (a,b) we have a (b,a)

Example5: Is the *'divides'* relation on the set of positive integers symmetric?

Soln: The relation is not symmetric because, ½ is not equal to 2/1.

<u>4) Asymmetric Relations:</u> A relation R on a set A is asymmetric if whenever $(a,b) \in R$, we have $(b,a) \notin R$.

Example1: If A is set of R(set of real numbers) and R is the relation '<'. So, if $a < b$, then $b \nless a$. So, '<' is asymmetric.

<u>5) Antisymmetric Relations:</u> A relation R on a set A such that for all $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$, then a=b is called antisymmetric i.e.

$$\forall a\, \forall b (aRb \wedge bRa \rightarrow a = b)$$

Example1: The relation *'mother of'* is antisymmetric because "x is mother y" precludes "y is mother of x".

Example2: i) $R_1 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

ii) $R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

iii) $R_3 = \{(3,4)\}$

iv) $R_4 = \{(a,b) | a \leq b\}$ is antisymmetric because the inequalities $a \leq b$ and $b \leq a$ implies that $a = b$.

v) $R_5 = \{(a,b) | a > b\}$ is antisymmetric because it is impossible for $a > b$ and $b > a$.

vi) $R_6 = \{(a,b) | a = b\}$ is antisymmetric.

Vii) $R_7 = \{(a,b) | a = b+1\}$ is antisymmetric because it is impossible for $a = b + 1$ and $b = a + 1$.

Example3: Is the *'divides'* relation on the set of positive integers antisymmetric?

Soln: The relation is antisymmetric as for a and b are positive integers with a/b and b/a, then a = b.

6) Transitive Relations: A relation R on a set A is transitive when $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ for all $a,b,c \in A$, i.e. $\forall a\, \forall b\, \forall c(aRb \wedge bRc \rightarrow aRc)$

Example1: i) $R_1 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$ is transitive because there are (3,2),(2,1),(3,1); (4,2),(2,1),(4,1); (2,1),(4,3),(3,2).

ii) $R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$ is transitive

iii) $R_3 = \{(a,b) | a \leq b\}$ is transitive because $a \leq b$ and $b \leq c$ implies that $a \leq c$.

iv) $R_4 = \{(a,b) | a > b\}$ is transitive because $a > b$ and $b > c$ implies that $a > c$.

v) $R_5 = \{(a,b) | a = b \text{ or } a = (-b)\}$ is transitive because $a = \pm b$ and $b = \pm c$ implies that $a = \pm c$.

vi) $R_6 = \{(a,b) | a = b\}$ is transitive because $a = b$ and $b = c$ implies that $a = c$.

vii) $R_7 = \{(a,b) | a = b+1\}$ is not transitive because, (2,1) and (1,0) belong to R_7 but (2,0) does not.

viii) $R_8 = \{(a,b) | a + b \leq 3\}$ is not transitive because (2,1) and (1,2) belong to R_8 but not (2,2)

Example2: Is the *'divides'* relation on the set of positive integers transitive?

Soln: Suppose that a divides b and b divides c. Then there are positive integers k and l such that b=ak and c=bl. Hence c = a(kl) so a divides c. It follows that this relation is transitive.

Exercise

Q1. For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) {(2, 2),(2, 3),(2, 4),(3, 2),(3, 3),(3, 4)}

Not reflexive because we do not have (1, 1), (3, 3), and (4, 4).

Not symmetric because while we have (3, 4), we do not have (4, 3).

Not antisymmetric because we have both (2, 3) and (3, 2).

Transitive because whenever we have both (a, b) and (b, c), then we have (a, c) which makes this relation transitive.

b) {(1, 1),(1, 2),(2, 1),(2, 2),(3, 3),(4, 4)}

Reflexive because (a, a) is in the relation for all a = 1, 2, 3, 4.

Symmetric because for every (a, b), we have a (b, a).

Not antisymmetric because we have (1, 2) and (2, 1).

Transitive because while we have (1, 2) and (2, 1), we also have (1, 1) and (2, 2) in the relation.

c) {(2, 4),(4, 2)}

Not reflexive because we do not have (a, a) for all a = 1, 2, 3, 4.

Symmetric because for every (a, b), we have a (b, a).

Not antisymmetric because we have both (2, 4) and (4, 2).

Not transitive because we are missing (2, 2) and (4, 4).

d) {(1, 2),(2, 3),(3, 4)}

Not reflexive because we do not have (a, a) for all a = 1, 2, 3, 4.

Not symmetric because we do not have (2, 1),(3, 2), and (4, 3).

Antisymmetric because for every (a, b), we do not have a (b, a).

Not transitive because we do not have (1, 3) for (1, 2) and (2, 3).

e) {(1, 1),(2, 2),(3, 3),(4, 4)}

Reflexive because we have (a, a) for every a = 1, 2, 3, 4.

Symmetric because we do not have a case where (a, b) and a 6= b.

Antisymmetric because we do not have a case where (a, b) and a 6= b.

Transitive because we can satisfy (a, b) and (b, c) when a = b = c.

f) {(1, 3),(1, 4),(2, 3),(2, 4),(3, 1),(3, 4)}

Not reflexive because we do not have (a, a) for all a = 1, 2, 3, 4.

Not symmetric because the relation does not contain (4, 1), (3, 2), (4, 2), and (4, 3).

Not antisymmetric because we have (1, 3) and (3, 1).

Not transitive because we do not have (2, 1) for (2, 3) and (3, 1).

Q2. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where (x, y) ∈ R if and only if

a) $x \neq y$.

Not reflexive because it's not the case $1 \neq 1$.

Is symmetric because $x \neq y$ and $y \neq x$.

Not antisymmetric because we have $x \neq y$ and $y \neq x$.

Not transitive because we can have $1 \neq 2$ and $2 \neq 1$ but not $1 \neq 1$.

b) $xy \geq 1$.

Not reflexive because we can't have (0, 0).

Is symmetric because we have $xy = yx$.

Not antisymmetric because we have $xy = yx$.

Is transitive because if we have (a, b) ∈ R and that (b, c) ∈ R, it follows that (a, c) ∈ R.

Note that in order for the relation to be true, a, b, and c will have to be all positive or all negative

c) $x = y + 1$ or $x = y - 1$.

Not reflexive because we can't have (1, 1)

Is symmetric because we have $x = y + 1$ and $y = x - 1$.

Not antisymmetric because of the same reason above.

Not transitive because if we have (1, 2) and (2, 1) in the relation, (1, 1) is not in relation.

d) $x = y^2$.

Not reflexive because (2, 2) does not satisfy.

Not symmetric because although we can have (9, 3), we can't have (3, 9).

Is antisymmetric because each integer will map to another integer but not in reverse (besides 0 and 1).

Not transitive because if we have (16, 4) and (4, 2), it's not the case that 16 = 22 .

e) $x \geq y^2$.

Not reflexive because we can't have (2, 2).

Not symmetric because if we have (9, 3), we can't have (3, 9).

Is antisymmetric, because each integer will map to another integer but not in reverse (besides 0 and 1).

Is transitive because if $x \geq y^2$ and $y \geq z^2$, then $x \geq z^2$.

Equivalence Relations

Equivalence Relation on set A is a relation which is reflexive, symmetric and transitive.

Example1:

Given $A = \{1, 2, 3, 4\}$

(i) Let $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

R is reflexive since $(1, 1), (2, 2), (3, 3), (4, 4) \in R$

R is symmetric since $(1, 2) \in R$ and $(2, 1) \in R$

R is Transitive since if $(1\ R\ 2)$ and $(2\ R\ 1)$ then $(1\ R\ 1)$

Hence R is reflexive, symmetric, Transitive.

Hence, R is an Equivalence Relation.

Example 2 :

Let A = {a, b, c} and let

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow R = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Determine whether R is an equivalence relation.

Solution : R = {(a, a), (b, b), (b, c), (c, b), (c, c)}

R is reflexive since (a, a), (b, b), (c, c) ∈ R

R is symmetric since (b, c) ∈ R → (c, b) ∈ R

R is transitive since,

(b,b)	and	(b, c) ∈ R	implies	(b, c) ∈ R.
(b, c)	and	(c, b) ∈ R	implies	(b, b) ∈ R,
(c, c)	and	(c, b) ∈ R	implies	(c, b) ∈ R,
(c, b)	and	(b, b) ∈ R	implies	(c, b) ∈ R,
(c, b)	and	(b, c) ∈ R	implies	(c, c) ∈ R,
(b, c)	and	(c, c) ∈ R	implies	(b, c) ∈ R,

Hence R is an equivalence relation.

Example3:

Let $A = \{1, 2, 3, 4\}$ and let

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$$

Is R is an equivalence relation.

Solution :

R is reflexive since $(1, 1), (2, 2), (3, 3)$ and $(4, 4) \in R$.

R is symmetric since

$(1, 2) \in R$ and $(2, 1) \in R$,

$(3, 4) \in R$ and $(4, 3) \in R$,

R is transitive since.

$(1, 2)$,	$(2, 1) \in R$	implies	$(1, 1) \in R$
$(3, 4)$,	$(4, 3) \in R$	implies	$(3, 3) \in R$
$(1, 1)$,	$(2, 1) \in R$	implies	$(1, 1) \in R$
$(1, 2)$,	$(2, 2) \in R$	implies	$(1, 2) \in R$
$(2, 1)$,	$(1, 2) \in R$	implies	$(2, 2) \in R$
$(4, 3)$,	$(4, 4) \in R$	implies	$(4, 4) \in R$
$(3, 3)$,	$(3, 4) \in R$	implies	$(3, 4) \in R$
$(4, 4)$,	$(4, 3) \in R$	implies	$(4, 3) \in R$
$(2, 2)$,	$(2, 1) \in R$	implies	$(2, 1) \in R$

Hence R is an equivalence relation.

Example4: Let $A = Z$, the set of integers, and let R be defined by $a \, R \, b$ if and only if $a \leq b$. Is R an equivalence relation?

Solution :

Since $a \leq a$, R is reflexive.

If $a \leq b$, it need not follow that $b \leq a$, so R is not symmetric.

Incidentally, R is transitive, since $a \leq b$ and $b \leq c$ imply that $a \leq c$.

We see that R is not an equivalence relation.

Example 4 :

Let $A = \{a, b, c, d\}$, $R = \{(a, a), (b, a), (b, b), (c, c), (d, d), (d, c)\}$.

Determine whether R is an equivalence relation.

Solution :

R is reflexive since $(a, a), (b, b), (c, c), (d, d) \in R$.

But R is not symmetric since $(b, a) \in R$ but $(a, b) \notin R$.

R is transitive since.

(b, a) ,	$(a, a) \in R$	implies	$(b, a) \in R$
(b, b) ,	$(b, a) \in R$	implies	$(b, a) \in R$
(d, c) ,	$(c, c) \in R$	implies	$(d, c) \in R$
(d, d) ,	$(d, c) \in R$	implies	$(d, c) \in R$

Hence R is not an equivalence relation.

Example 6:

Let $A = Z$. Let R be defined as

aRb iff $a \le b$. is R equivalence Relation?

sol^n:-

As $a \le a$ $\therefore$ R is reflexive

if $a \le b$ then $b \le a$ not true

$\therefore$ not symmetric.

if $a \le b$ and $b \le c$ then $a \le c$

$\therefore$ R is transitive

$\therefore$ R is not equivalence Relation.

Example 7

One of the most widely used equivalance Relation is congruence modulo m.

In this, m is possitive integer > 1.

Ex. Let m be the possitive integer >1. show that, the relation

$$R = \{ (a, b) \mid a \equiv b \pmod{m} \}$$

is equivalance relation on set of Integers.

Solution:- (i) $a \equiv b \pmod{m}$ means if and only if m divides $(a-b)$.

now $(a-a) = 0$, divisible by m

As $0 = 0 \times m$.

Therefore

$a \equiv a \pmod{m}$ is true.

$\therefore$ congruence modulo m is Reflexive.

Example 7 continue…

(ii) suppose $a \equiv b \pmod{m}$

It shows that $(a-b)$ is divisible by m.

$\therefore$ $a-b = km$ whenever k is integer

$\therefore$ $(b-a) = (-k).m$

so that $b \equiv a \pmod{m}$.

hence Congruence modulo m is symetric.

Example 7 continue…

(iii) Assume

$(a \equiv b) (\mathrm{mod}\ m)$ and $(b \equiv c) (\mathrm{mod}\ m)$

It shows m divides both $(a-b)$ and $(b-c)$.

Let integers k and l such that

$$\frac{(a-b)}{m} = k \quad \text{and} \quad \frac{(b-c)}{m} = l$$

$$\Rightarrow \quad (a-b) = mk \quad \text{and} \quad (b-c) = ml.$$

Example 7 continue…

now $(a-b)+(b-c) = a-c$

$\therefore (a-c) = (a-b)+(b-c)$

$= mk + ml$

$= m\cdot(k+l)$

hence $(a-c)$ is divisible by m.

$\therefore \quad a \equiv c \pmod{m}$

$\therefore$ Congruence modulo m is transitive

$\therefore$ given relation is equivalence relation.

Example 7 continue…

now $(a-b)+(b-c) = a-c$

$$\therefore (a-c) = (a-b)+(b-c)$$
$$= mk + ml$$
$$= m\cdot(k+l)$$

hence $(a-c)$ is divisible by m.

$$\therefore a \equiv c \pmod{m}$$

$\therefore$ Congruence modulo m is transitive

$\therefore$ given relation is equivalence relation.

Representation of Relations – Using Matrix

- A relation between finite sets can be represented using a zero–one matrix. Suppose that *R is a* relation from $A = \{a_1, a_2, \ldots, a_m\}$ *to* $B = \{b_1, b_2, \ldots, b_n\}$. The relation *R can be represented by the matrix* $\mathbf{M}_R = [m_{ij}]$, ***where***

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

EXAMPLE 1 Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The 1s in $\mathbf{M}_R$ show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R. ◀

Example 2:

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

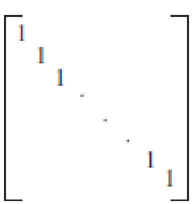
Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$ ◀

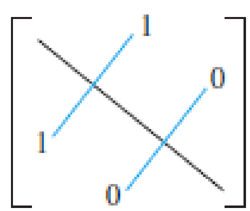
1) **<u>Matrix for Reflexive Relations:</u>** A relation R on A is reflexive if (a, a) ∈ R whenever a ∈ A. Thus, R is reflexive if and only if $m_{ii} = 1$, for i = 1, 2,…, n. In other words, R is reflexive if all the elements on the main diagonal of M_R are equal to 1. Note that, the diagonal elements can be either 0 or 1.

2) **<u>Matrix for Symmetric Relations:</u>** The relation R is symmetric if (a, b) ∈ R implies that (b, a) ∈ R. Thus, R is symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means, $m_{ji} = 0$ whenever $m_{ij} = 0$. Moreover, R is symmetric if and only if $m_{ij} = m_{ji}$. We also see that, we see that R is symmetric if and only if $M_R = (M_R)^t$

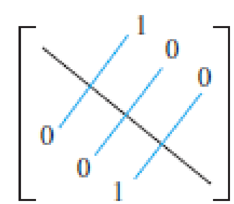
3) **<u>Matrix for Antisymmetric Relations:</u>** The relation R is antisymmetric if and only if (a, b) ∈ R and (b, a) ∈ R imply that a = b. Consequently, the matrix of an antisymmetric relation has the property that if $m_{ij} = 1$with i ≠ j, then $m_{ji} = 0$.



Matrix for Reflexive Relations



Matrix for Symmetric Relations



Matrix for Antisymmetric Relations

Example 1:

Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because $\mathbf{M}_R$ is symmetric, it follows that R is symmetric. It is also easy to see that R is not antisymmetric. ◀

Representation of Relations – Using Digraphs

- There is another important way of representing a relation using a pictorial representation. Each element of the set is represented by a point, and each ordered pair is represented using an arc with its direction indicated by an arrow. We use such pictorial representations when we think of relations on a finite set as **directed graphs, or digraphs.**
- In other words, the relation R on a set A is represented by the **directed graph that has the elements of A as its vertices and the ordered pairs (a, b), where (a, b) ∈ R, as edges.**

Example 1: The directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b) and (d, b) is displayed in the figure (a).

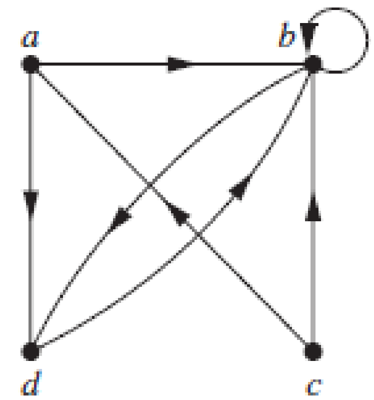


Figure (a)

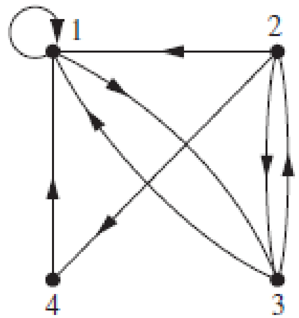


Figure (b)

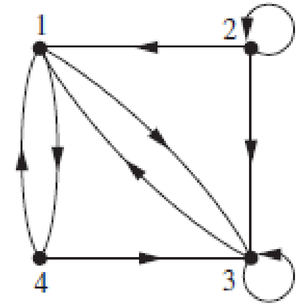


Figure (c)

Example 2: The directed graph of the relation R1 = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)} on the set {1, 2, 3, 4} is shown in Figure (b).

Example 3: The ordered pairs (x, y) in the relation are R2 = {(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)}. Each of these pairs corresponds to an edge of the directed graph, with (2, 2) and (3, 3) corresponding to loops in figure (c)

Example 4: **Determine whether the relations for the directed graphs shown in Figure (d) are reflexive, symmetric, antisymmetric, and/or transitive.**

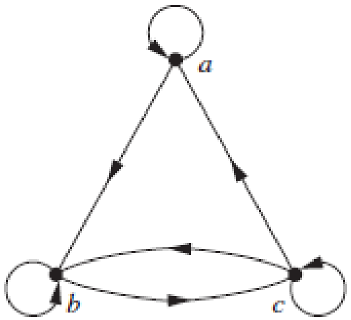


Figure (d)

1) Because there are loops at every vertex of the directed graph of *S1, it is reflexive.*
2) The relation *S1 is neither symmetric nor antisymmetric because there is an edge from a to b but not* one from *b to a, but there are edges in both directions connecting b and c.*
3) *S1 is not* transitive because there is an edge from *a to b and an edge from b to c, but no edge from a to c.*

Example 5: **Determine whether the relations for the directed graphs shown in Figure (d) are reflexive, symmetric, antisymmetric, and/or transitive.1**

1) Because loops are not present at all the vertices of the directed graph of *S2, this relation* is not reflexive.
2) It is symmetric and not antisymmetric, because every edge between distinct vertices is accompanied by an edge in the opposite direction.
3) The graph that *S2 is not transitive, because (c, a) and (a, b) belong to S2, but (c, b) does not* belong to *S2.*

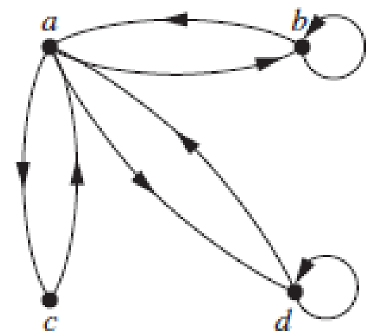


Figure (e)

Example 6: Let A = {1,4,5} and let R be the relation given by the diagraph (fig: f). Find M_R

Solution:

$$M_R = \begin{array}{c} \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$R = \{(1,4), (1,5), (4,1), (4,4), (5,4), (5,5)\}$

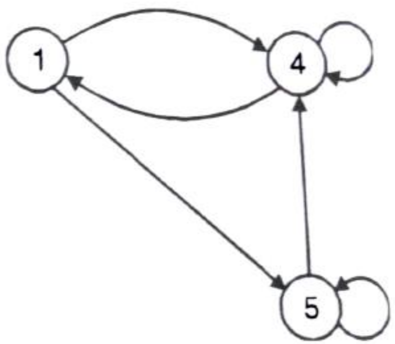


Figure (f)

Example 7: Let A = {2,3,4,5} and R = {(2,3),(3,2),(3,4),(3,5),(4,3),(4,4),(4,5)}. Draw its digraph.

Solution:

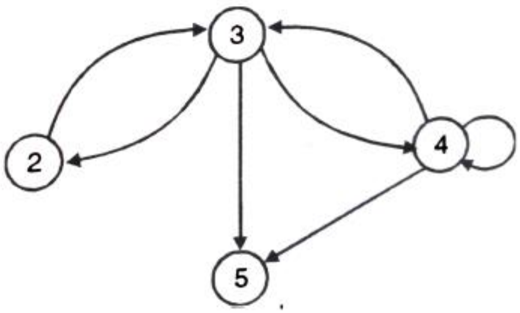


Figure (g)

Example 8: Find the relation from the given digraph (fig: h) and give its matrix.

Solution :

$A = \{1,2,3,4,5\}$

$R = \{(1,2), (2,2), (2,3), (3,4), (4,4), (5,1), (5,4)\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

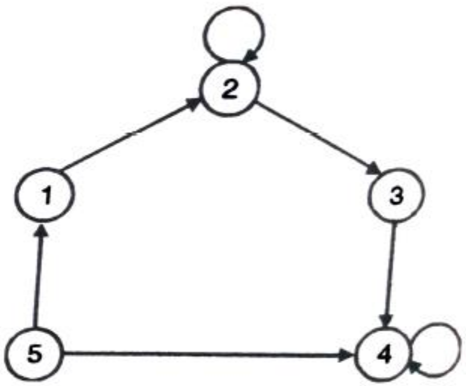


Figure (h)

Example 9: Draw the digraph for relation R = {(1,2),(2,2),(2,4),(3,2),(3,4),(4,1),(4,3)} where A = {1,2,3,4}

Solution: See fig: (i)

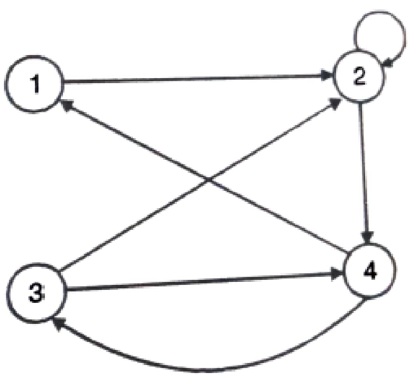


Figure (i)

Example 10: Draw the graphical representation of the relation ‘less than’ on {1,2,3,4} and the matrix for it.

Solution: $R = \{(1,2), (1,4), (1,3), (2,3), (2,4), (3,4)\}$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

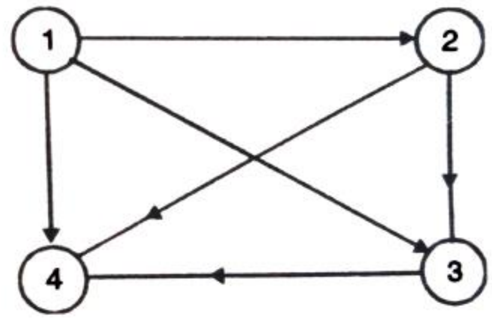


Figure (j)

Example 11: Let A = {1,2,3,4,6} and let R be the relation defined as ‘x divides y’ on A. Find the relation and draw the digraph of R.

Solution: R = {(1,1),(1,2),(1,3),(1,4),(1,6),(2,2),(2,4),(2,6),(3,3),(3,6), (4,4) (6,6)}

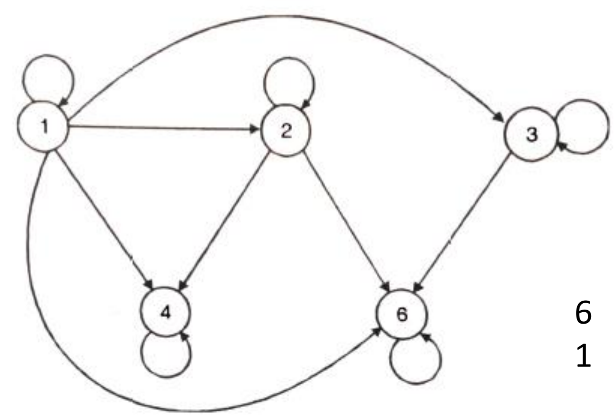


Figure (k)

Example 12: Let A = {2,3,4,6,9} and let R be the relation defined as 'x is relatively prime to y' i.e. the only positive divisor of x and y is 1. Find the relation and draw the digraph of R.

Solution: R = {(2,3),(2,9),(1,3),(3,2),(3,4),(4,3),(4,9),(9,2),(9,4)}

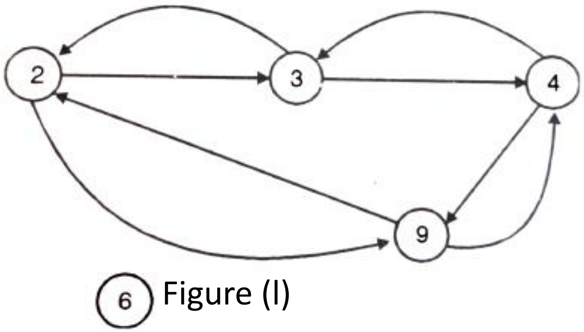


Figure (l)

Example 13: Let A = {1,2,3,4,6} where a R b iff a is a multiple of b. Find R, M_R and the draw the digraph.

Solution: R = {(1,1),(2,1),(2,2),(3,1),(3,3),(4,1),(4,2),(4,4),(6,1),(6,2),(6,3),(6,6)}

$$M_R = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

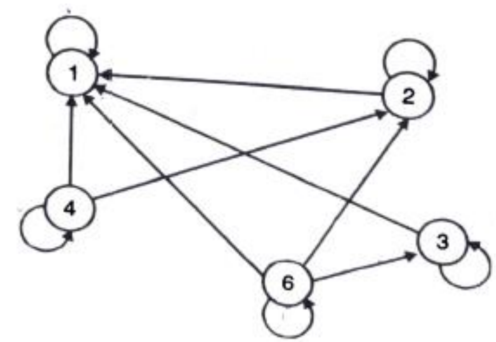


Figure (m)

Degree of Vertex

1) **<u>In-degree:</u>** In-degree of a vertex in a digraph of R is the <u>number of edges terminating at the vertex.</u>
2) **<u>Out-degree</u>**: In-degree of a vertex in a digraph of R is the <u>number of edges leaving the vertex</u>.

Example1: List the in-degrees and out-degrees of the

Solution:

	a	b	c	d
In-degree	2	3	1	1
Out-degree	1	1	3	2

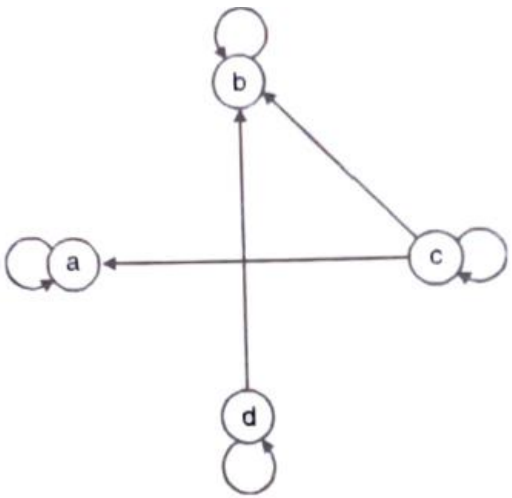


Figure (n)

Example2: List the in-degrees and out-degrees of the digraph (fig: o)

Solution:

	1	2	3	4
In-degree	2	2	2	2
Out-degree	3	2	2	1

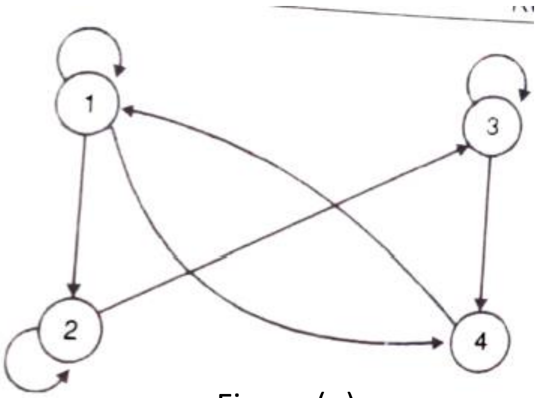


Figure (o)

Example3: List the in-degrees and out-degrees of the digraph (fig: p)

Solution:

	1	2	3	4
In-degree	2	2	2	2
Out-degree	3	2	2	1

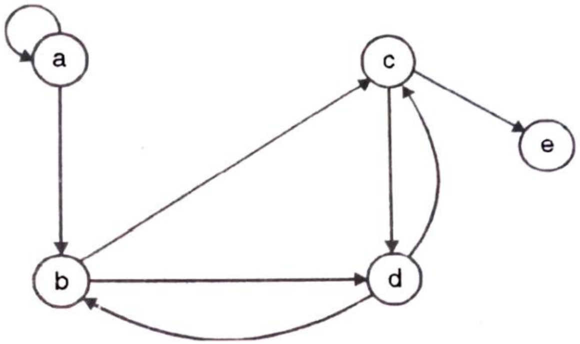


Figure (p)

Combining Relations, Closures of Relation

Example 1:

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

$$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\},$$

$$R_1 \cap R_2 = \{(1, 1)\},$$

$$R_1 - R_2 = \{(2, 2), (3, 3)\},$$

$$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}.$$

◀

Example 2:

Let R_1 be the "less than" relation on the set of real numbers and let R_2 be the "greater than" relation on the set of real numbers, that is, $R_1 = \{(x, y) \mid x < y\}$ and $R_2 = \{(x, y) \mid x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R_1 \oplus R_2$?

Solution: We note that $(x, y) \in R_1 \cup R_2$ if and only if $(x, y) \in R_1$ or $(x, y) \in R_2$. Hence $(x, y) \in R_1 \cup R_2$ if and only if $x < y$ or $x > y$. Because the condition $x < y$ or $x > y$ is the same as the condition $x \neq y$, it follows that $R_1 \cup R_2 = \{(x, y) \mid x \neq y\}$. In other words, the union of the "less than" relation and the "greater than" relation is the "not equals" relation.

Next, note that it is impossible for a pair (x, y) to belong to both R_1 and R_2 since it is impossible for $x < y$ and $x > y$. It follows that $R_1 \cap R_2 = \emptyset$. We also see that $R_1 - R_2 = R_1$, $R_2 - R_1 = R_2$, and $R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 = \{(x, y) \mid x \neq y\}$. ◀

Definition of Composite of Relation

Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A, c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Example 1:

What is the composite of the relations R and S where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution: $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs $(2, 3)$ in R and $(3, 1)$ in S produce the ordered pair $(2, 1)$ in $S \circ R$. Computing all the ordered pairs in the composite, we find

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}.$$

◀

Example 2:

Composing the Parent Relation with Itself Let R be the relation on the set of all people such that $(a, b) \in R$ if person a is a parent of person b. Then $(a, c) \in R \circ R$ if and only if there is a person b such that $(a, b) \in R$ and $(b, c) \in R$, that is, if and only if there is a person b such that a is a parent of b and b is a parent of c. In other words, $(a, c) \in R \circ R$ if and only if a is a grandparent of c. ◀

The powers of a relation R can be recursively defined from the definition of a composite of two relations.

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by

$$R^1 = R \qquad \text{and} \qquad R^{n+1} = R^n \circ R.$$

The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

Example 3:

Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots$.

Solution: Since $R^2 = R \circ R$, we find that $R^2 = \{(1,1), (2,1), (3,1), (4,2)\}$. Furthermore, since $R^3 = R^2 \circ R$, $R^3 = \{(1,1), (2,1), (3,1), (4,1)\}$. Additional computation shows that R^4 is the same as R^3, so $R^4 = \{(1,1), (2,1), (3,1), (4,1)\}$. It also follows that $R^n = R^3$ for $n = 5, 6, 7, \ldots$. The reader should verify this. ◀

Example 4:

Example 10. Let $A = \{1, 2, 3, 4\}$, $R = \{(1,2), (1,1), (1,3), (2,4), (3,2)\}$, and $S = \{(1,4), (1,3), (2,3), (3,1), (4,1)\}$. Since $(1,2) \in R$ and $(2,3) \in S$, we must have $(1,3) \in S \circ R$. Similarly, since $(1,1) \in R$ and $(1,4) \in S$, we see that $(1,4) \in S \circ R$. Proceeding in this way, we find that $S \circ R = \{(1,4), (1,3), (1,1), (2,1), (3,3)\}$. ◆

Example :

Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Proof: We first prove the "if" part of the theorem. We suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$. In particular, $R^2 \subseteq R$. To see that this implies R is transitive, note that if $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, $(a, c) \in R^2$. Since $R^2 \subseteq R$, this means that $(a, c) \in R$. Hence R is transitive.

We will use mathematical induction to prove the only if part of the theorem. Note that this part of the theorem is trivially true for $n = 1$.

Assume that $R^n \subseteq R$ where n is a positive integer. This is the inductive hypothesis. To complete the inductive step we must show that this implies that R^{n+1} is also a subset of R. To show this, assume that $(a, b) \in R^{n+1}$. Then, since $R^{n+1} = R^n \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$. The inductive hypothesis, namely, that $R^n \subseteq R$, implies that $(x, b) \in R$. Furthermore, since R is transitive, and $(a, x) \in R$ and $(x, b) \in R$, it follows that $(a, b) \in R$. This shows that $R^{n+1} \subseteq R$, completing the proof. $\lhd$

Paths in Relations and Digraphs

Suppose that R is a relation on a set A. A **path of length n** in R from a to b is a finite sequence $\pi : a, x_1, x_2, \ldots, x_{n-1}, b$, beginning with a and ending with b, such that

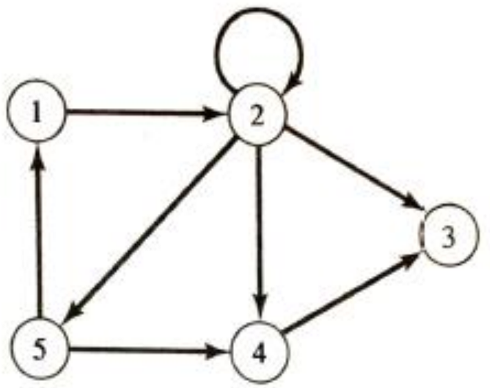
$$a \; R \; x_1, x_1 \; R \; x_2, \ldots, x_{n-1} \; R \; b.$$

Note that a path of length n involves $n + 1$ elements of A, although they are not necessarily distinct.

A path is most easily visualized with the aid of the digraph of the relation. It appears as a geometric *path* or succession of edges in such a digraph, where the indicated directions of the edges are followed, and in fact a path derives its name from this representation. Thus the length of a path is the number of edges in the path, where the vertices need not all be distinct.

Example 1:

Consider the digraph given.
Then Path1:1,2,5, 4,3 is a path of length 4 from vertex 1 to vertex 3;
Path2: 1,2, 5, 1 is a path of length 3 from vertex 1 to itself;
Path3 :2,2 is a path of length 1 from vertex 2 to itself.



A path that begins and ends at the same vertex is called a **cycle**. In Example 1, π_2 and π_3 are cycles of length 3 and 1, respectively. It is clear that the paths of length 1 can be identified with the ordered pairs (x, y) that belong to R. Paths in a relation R can be used to define new relations that are quite useful. If n is a fixed positive integer, we define a relation R^n on A as follows: $x\ R^n\ y$ means that there is a path of length n from x to y in R. We may also define a relation R^∞ on A, by letting $x\ R^\infty\ y$ mean that there is some path in R from x to y. The length of such a path will depend, in general, on x and y. The relation R^∞ is sometimes called the **connectivity relation** for R.

Note that $R^n(x)$ consists of all vertices that can be reached from x by means of a path in R of length n. The set $R^\infty(x)$ consists of all vertices that can be reached from x by some path in R.

Closures of Relations

If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R w.r.t P.

1) **Reflexive Closure:**

As this example illustrates, given a relation R on a set A, the reflexive closure of R can be formed by adding to R all pairs of the form (a, a) with $a \in A$, not already in R. The addition of these pairs produces a new relation that is reflexive, contains R, and is contained within any reflexive relation containing R. We see that the reflexive closure of R equals $R \cup \Delta$, where $\Delta = \{(a, a) \mid a \in A\}$ is the **diagonal relation on A.**

Example: What is the reflexive closure of the relation $R = \{(a,b) \mid a < b\}$ on the set of integers?

Solution: The reflexive closure of R is $R \cup \Delta = \{(a,b) \mid (a < b)\} \cup \{(a,a) \mid a \in Z\} = \{(a,b) \mid a \le b\}$

2) Symmetric Closure:

Consider relation R = {(1,1),(1,2),(2,1),(3,2)} on set A = {1,2,3}. It is clearly not a symmetric relation. So, how can we produce the symmetric relation containing R that is small as possible?

This is done by adding all pairs of the form (b,a), where (a,b) is in the relation that are not already in R to R. Adding such pairs will produces a symmetric relation that contains R and any other symmetric relation that contains R.

So, here we see that the **symmetric closure of R equals to R U R^{-1}**

Here, $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

Example: What is the symmetric closure of the relation $R = \{(a,b) \mid a > b\}$ on the set of positive integers?

Solution: The symmetric closure of R is R U $R^{-1} = \{(a,b) \mid (a > b)\} \cup \{(b,a) \mid a > b\}$ $= \{(a,b) \mid a \neq b\}$

Example 8. Suppose that R is a relation on a set A, and R is not reflexive. This can only occur because some pairs of the diagonal relation Δ are not in R. Thus $R_1 = R \cup \Delta$ is the smallest reflexive relation on A containing R; that is, the **reflexive closure** of R is $R \cup \Delta$. ◆

Example 9. Suppose now that R is a relation on A that is not symmetric. Then there must exist pairs (x, y) in R such that (y, x) is not in R. Of course, $(y, x) \in R^{-1}$, so if R is to be symmetric we must add all pairs from R^{-1}; that is, we must enlarge R to $R \cup R^{-1}$. Clearly, $(R \cup R^{-1})^{-1} = R \cup R^{-1}$, so $R \cup R^{-1}$ is the smallest symmetric relation containing R; that is, $R \cup R^{-1}$ is the **symmetric closure** of R.

If $A = \{a, b, c, d\}$ and $R = \{(a, b), (b, c), (a, c), (c, d)\}$, then $R^{-1} = \{(b, a), (c, b), (c, a), (d, c)\}$, so the symmetric closure of R is

$$R \cup R^{-1} = \{(a, b), (b, a), (b, c), (c, b), (a, c), (c, a), (c, d), (d, c)\}.$$ ◆

2) Transitive Closure:

Theorem 1. *Let R be a relation on a set A. Then R^∞ is the transitive closure of R.*

Proof: We recall that if a and b are in the set A, then $a\ R^\infty\ b$ if and only if there is a path in R from a to b. Now R^∞ is certainly transitive since, if $a\ R^\infty\ b$ and $b\ R^\infty\ c$, the composition of the paths from a to b and from b to c form a path from a to c in R, and so $a\ R^\infty\ c$. To show that R^∞ is the smallest transitive relation containing R, we must show that if S is any transitive relation on A and $R \subseteq S$, then $R^\infty \subseteq S$. Theorem 1 of Section 4.4 tells us that if S is transitive, then $S^n \subseteq S$ for all n; that is, if a and b are connected by a path of length n, then $a\ S\ b$. It follows that $S^\infty = \bigcup_{n=1}^{\infty} S^n \subseteq S$. It is also true that if $R \subseteq S$, then $R^\infty \subseteq S^\infty$, since any path in R is also a path in S. Putting these facts together, we see that if $R \subseteq S$ and S is transitive on A, then $R^\infty \subseteq S^\infty \subseteq S$. This means that R^∞ is the smallest of all transitive relations on A that contain R. ◆

Matrics of Composite Relations

Let's see determining the matrix for the composite of relations. This matrix can be found using the Boolean product of the matrices

for these relations. In particular, suppose that R is a relation from A to B and S is a relation from B to C. Suppose that A, B, and C have m, n, and p elements, respectively. Let the zero–one matrices for $S \circ R$, R, and S be $\mathbf{M}_{S \circ R} = [t_{ij}]$, $\mathbf{M}_R = [r_{ij}]$, and $\mathbf{M}_S = [s_{ij}]$, respectively (these matrices have sizes $m \times p$, $m \times n$, and $n \times p$, respectively). The ordered pair (a_i, c_j) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S. It follows that $t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for some k. From the definition of the Boolean product, this means that

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S.$$

Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero–one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero–one matrix. Then the *Boolean product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ matrix with (i, j)th entry c_{ij} where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Note that the Boolean product of $\mathbf{A}$ and $\mathbf{B}$ is obtained in an analogous way to the ordinary product of these matrices, but with addition replaced with the operation $\vee$ and with multiplication replaced with the operation $\wedge$. We give an example of the Boolean products of matrices.

Example 1:

Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution: The matrix for $S \circ R$ is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix representing the composite of two relations can be used to find the matrix for $\mathbf{M}_{R^n}$. In particular,

$$\mathbf{M}_{R^n} = \mathbf{M}_R^{[n]},$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$c_{11} = (1 \wedge 0) \vee (0 \wedge 0) \vee (1 \wedge 1) = 1$

$c_{12} = (1 \wedge 1) \vee (0 \wedge 0) \vee (0 \wedge 0) = 1$

$c_{13} = (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) = 1$

$c_{21} = (1 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) = 0$

$c_{22} = (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) = 1$

$c_{23} = (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) = 1$

$c_{31} = (0 \wedge 0) \vee (0 \wedge 0) \vee (0 \wedge 1) = 0$

$c_{32} = (0 \wedge 1) \vee (0 \wedge 0) \vee (0 \wedge 0) = 0$

$c_{33} = (0 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 1) = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 2:

Find the matrix representing the relation R^2, where the matrix representing R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution: The matrix for R^2 is

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

Find the zero–one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: By Theorem 3, it follows that the zero–one matrix of R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}.$$

Because

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

it follows that

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

◀

Example 10. Let $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (1, 1), (1, 3), (2, 4), (3, 2)\}$, and $S = \{(1,4), (1,3), (2,3), (3,1), (4,1)\}$. Since $(1,2) \in R$ and $(2,3) \in S$, we must have $(1,3) \in S \circ R$. Similarly, since $(1,1) \in R$ and $(1,4) \in S$, we see that $(1,4) \in S \circ R$. Proceeding in this way, we find that $S \circ R = \{(1,4), (1,3), (1,1), (2,1), (3,3)\}$. ◆

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so

$$S \circ R = \{(1,1), (1,3), (1,4), (2,1), (3,3)\}$$

Theorem

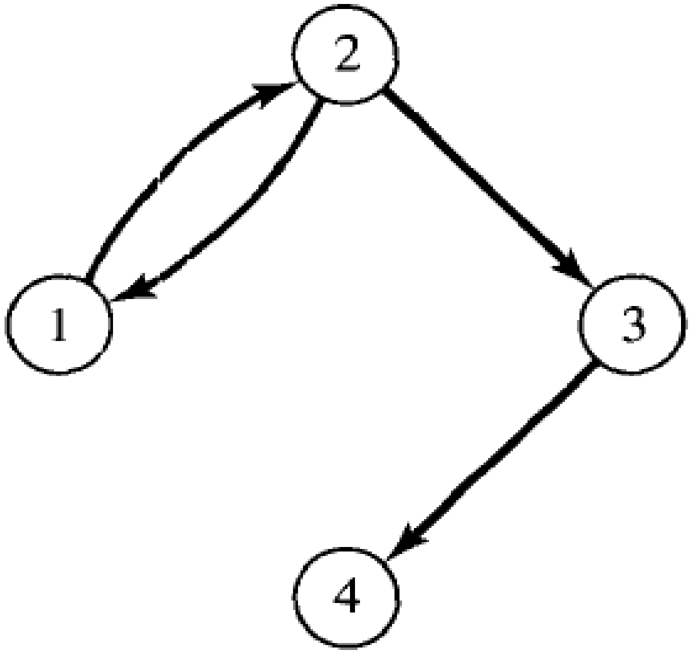
Let R be a relation on a set A. Then R^{∞} is the transitive closure of R.

Example 1. Let $A = \{1, 2, 3, 4\}$, and let $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$. Find the transitive closure of R.

Solution

METHOD 1. The digraph of R is shown in Figure 4.42. Since R^{∞} is the transitive closure, we can proceed geometrically by computing all paths. We see that from vertex 1 we have paths to vertices 2, 3, 4, and 1. Note that the path from 1 to 1 proceeds from 1 to 2 to 1. Thus we see that the ordered pairs (1, 1), (1, 2), (1, 3), and (1, 4) are in R^{∞}. Starting from vertex 2, we have paths to vertices 2, 1, 3, and 4, so the ordered pairs (2, 1), (2, 2), (2, 3), and (2, 4) are in R^∞. The only other path is from vertex 3 to vertex 4, so we have

$$R^\infty = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)\}.$$



METHOD 2. The matrix of R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We may proceed algebraically and compute the powers of $\mathbf{M}_R$. Thus

$$(\mathbf{M}_R)^2_\odot = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (\mathbf{M}_R)^3_\odot = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(\mathbf{M}_R)^4_\odot = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Continuing in this way, we can see that $(\mathbf{M}_R)^n_\odot$ equals $(\mathbf{M}_R)^2_\odot$ if n is even and equals $(\mathbf{M}_R)^3_\odot$ if n is odd and greater than 1. Thus

$$\mathbf{M}_{R^\infty} = \mathbf{M}_R \vee (\mathbf{M}_R)^2_\odot \vee (\mathbf{M}_R)^3_\odot = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and this gives the same relation as Method 1. ◆

Suppose that R is a relation on a set with n elements. Let $v_1, v_2, \ldots, v_n$ be an arbitrary listing of these n elements. The concept of the **interior vertices** of a path is used in Warshall's algorithm. If $a, x_1, x_2, \ldots, x_{m-1}, b$ is a path, its interior vertices are $x_1, x_2, \ldots, x_{m-1}$, that is, all the vertices of the path that occur somewhere other than as the first and last vertices in the path. For instance, the interior vertices of a path a, c, d, f, g, h, b, j in a directed graph are c, d, f, g, h, and b. The interior vertices of a, c, d, a, f, b are c, d, a, and f. (Note that the first vertex in the path is not an interior vertex unless it is visited again by the path, except as the last vertex. Similarly, the last vertex in the path is not an interior vertex unless it was visited previously by the path, except as the first vertex.)

Warshall's algorithm is based on the construction of a sequence of zero–one matrices. These matrices are $\mathbf{W}_0, \mathbf{W}_1, \ldots, \mathbf{W}_n$, where $\mathbf{W}_0 = \mathbf{M}_R$ is the zero–one matrix of this relation, and $\mathbf{W}_k = [w_{ij}^{(k)}]$, where $w_{ij}^{(k)} = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, \ldots, v_k\}$ (the first k vertices in the list) and is 0 otherwise. (The first and last vertices in the path may be outside the set of the first k vertices in the list.) Note that $\mathbf{W}_n = \mathbf{M}_{R^*}$, because the (i, j)th entry of $\mathbf{M}_{R^*}$ is 1 if and only if there is a path from v_i to v_j, with all interior vertices in the set $\{v_1, v_2, \ldots, v_n\}$ (but these are the only vertices in the directed graph).

Example :

Let R be the relation with directed graph shown in Figure 3. Let a, b, c, d be a listing of the elements of the set. Find the matrices $\mathbf{W}_0$, $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, and $\mathbf{W}_4$. The matrix $\mathbf{W}_4$ is the transitive closure of R.

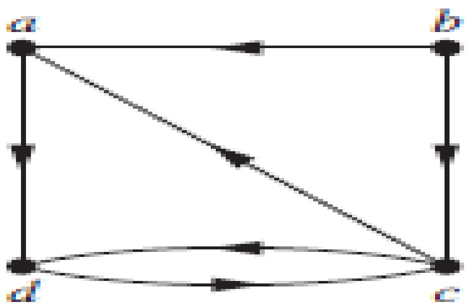


FIGURE 3
The directed graph of the relation R.

Solution: Let $v_1 = a$, $v_2 = b$, $v_3 = c$, and $v_4 = d$. $\mathbf{W}_0$ is the matrix of the relation. Hence,

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$\mathbf{W}_1$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has only $v_1 = a$ as an interior vertex. Note that all paths of length one can still be used because they have no interior vertices. Also, there is now an allowable path from b to d, namely, b, a, d. Hence,

$$\mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$\mathbf{W}_2$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has only $v_1 = a$ and/or $v_2 = b$ as its interior vertices, if any. Because there are no edges that have b as a terminal vertex, no new paths are obtained when we permit b to be an interior vertex. Hence, $\mathbf{W}_2 = \mathbf{W}_1$.

$\mathbf{W}_3$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has only $v_1 = a$, $v_2 = b$, and/or $v_3 = c$ as its interior vertices, if any. We now have paths from d to a, namely, d, c, a, and from d to d, namely, d, c, d. Hence,

$$\mathbf{W}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Finally, $\mathbf{W}_4$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has $v_1 = a$, $v_2 = b$, $v_3 = c$, and/or $v_4 = d$ as interior vertices, if any. Because these are all the vertices of the graph, this entry is 1 if and only if there is a path from v_i to v_j. Hence,

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

This last matrix, $\mathbf{W}_4$, is the matrix of the transitive closure. ◀

Let $\mathbf{W}_k = [w_{ij}^{[k]}]$ be the zero–one matrix that has a 1 in its (i, j)th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}),$$

whenever i, j, and k are positive integers not exceeding n.

ALGORITHM 2 Warshall Algorithm.

procedure *Warshall* ($\mathbf{M}_R$: $n \times n$ zero–one matrix)
$\mathbf{W} := \mathbf{M}_R$
for $k := 1$ **to** n
  for $i := 1$ **to** n
    for $j := 1$ **to** n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return $\mathbf{W}$ $\{\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_{R^*}\}$

Functions

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Functions are specified in many different ways.

1) Sometimes we explicitly state the assignments, as in Figure 1.

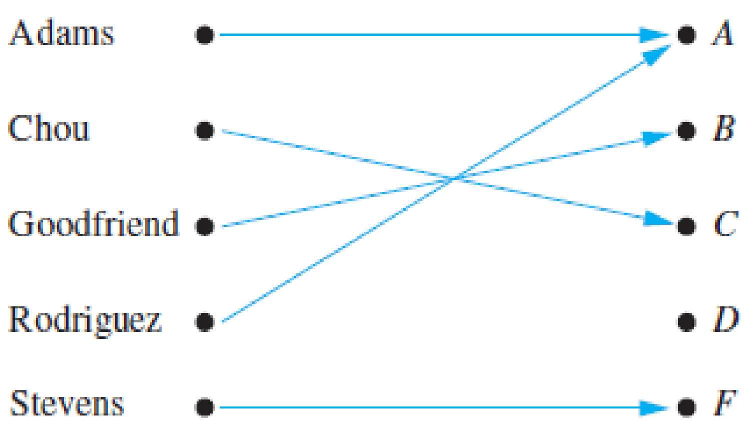


FIGURE 1 **Assignment of grades in a discrete mathematics class.**

2) Often we give a formula, such as $f(x) = x + 1$, to define a function.

3) A function $f : A \to B$ can also be defined in terms of a relation from A to B. A relation from A to B is just a subset of $A \times B$. A relation from A to B that contains one, and only one, ordered pair (a, b) for every element $a \in A$, defines a function f from A to B. This function is defined by the assignment $f(a) = b$, where (a, b) is the unique ordered pair in the relation that has a as its first element.

Also, if f is a function from A to B, we say that f *maps* A to B.

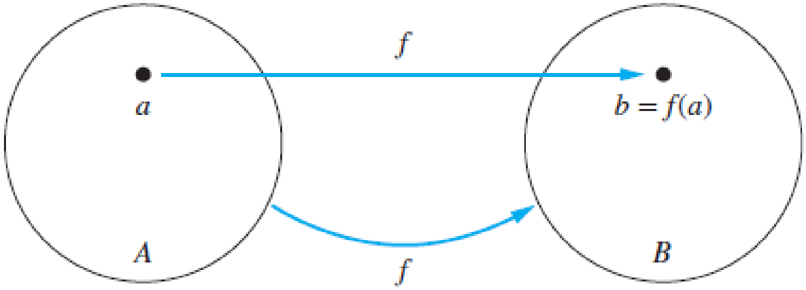


FIGURE 2 **The function f maps A to B.**

Parameters of Functions:

1) If f is a function from A to B, we say that A is the domain of f and B is the codomain of function f.
2) If f (a) = b, we say that b is the image of a and a is a preimage of b.

3) The range, or image, of f is the set of all images of elements of A.

Example 1: For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set {*A, B, C, D, F*}. And suppose that the grades are *A* for Adams, *C* for Chou, *B* for Goodfriend, *A* for Rodriguez, and *F* for Stevens. What are the domain, codomain, and range of the function that assigns grades to students?

Solution: Let *G* be the function that assigns a grade to a student in our discrete mathematics class. Note that *G*(Adams) = *A*, for instance. The domain of *G* is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}, and the codomain is the set {*A, B, C, D, F*}. The range of *G* is the set {*A, B,C, F*}, because each grade except *D* is assigned to some student.

EXAMPLE 2 Let R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.

Solution: If f is a function specified by R, then $f(\text{Abdul}) = 22$, $f(\text{Brenda}) = 24$, $f(\text{Carla}) = 21$, $f(\text{Desire}) = 22$, $f(\text{Eddie}) = 24$, and $f(\text{Felicia}) = 22$. [Here, $f(x)$ is the age of x, where x is a student.] For the domain, we take the set {Abdul, Brenda, Carla, Desire, Eddie, Felicia}. We also need to specify a codomain, which needs to contain all possible ages of students. Because it is highly likely that all students are less than 100 years old, we can take the set of positive integers less than 100 as the codomain. (Note that we could choose a different codomain, such as the set of all positive integers or the set of positive integers between 10 and 90, but that would change the function. Using this codomain will also allow us to extend the function by adding the names and ages of more students later.) The range of the function we have specified is the set of different ages of these students, which is the set {21, 22, 24}. ◀

EXAMPLE 3 *Extra Examples* Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, $f(11010) = 10$. Then, the domain of f is the set of all bit strings of length 2 or greater, and both the codomain and range are the set {00, 01, 10, 11}. ◀

EXAMPLE 4 Let $f: \mathbf{Z} \rightarrow \mathbf{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \dots\}$. ◀

Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$
$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Note that the functions $f_1 + f_2$ and $f_1 f_2$ have been defined by specifying their values at x in terms of the values of f_1 and f_2 at x.

Let f_1 and f_2 be functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution: From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$ ◀

When f is a function from A to B, the image of a subset of A can also be defined.

Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$
$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Note that the functions $f_1 + f_2$ and $f_1 f_2$ have been defined by specifying their values at x in terms of the values of f_1 and f_2 at x.

Let f_1 and f_2 be functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution: From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$

◀

When f is a function from A to B, the image of a subset of A can also be defined.

Types of Functions

1) One to One Function(Injection):

A function f is said to be *one-to-one*, or an *injection*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one.

We can express that f is one-to-one using quantifiers as $\forall a\, \forall b\, (f(a) = f(b) \to a = b)$ or equivalently $\forall a\, \forall b(a \neq b \to f(a) \neq f(b))$, where the universe of discourse is the domain of the function

EXAMPLE 9 Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution: The function $f(x) = x^2$ is not one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 \neq -1$. ◀

NOTE: The function $f(x) = x2$ with domain **Z**+ is one-to-one.

EXAMPLE 8 *Extra Examples*

Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Solution: The function f is one-to-one because f takes on different values at the four elements of its domain. This is illustrated in Figure 3. ◀

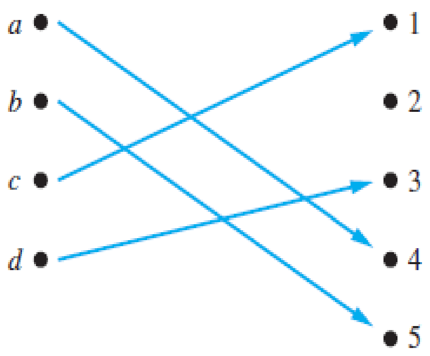


FIGURE 3 **A one-to-one function.**

Example 10:

Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

Solution: Suppose that x and y are real numbers with $f(x) = f(y)$, so that $x + 1 = y + 1$. This means that $x = y$. Hence, $f(x) = x + 1$ is a one-to-one function from $\mathbf{R}$ to $\mathbf{R}$. ◀

2) Onto Function(Surjection):

A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

A function f is onto if $\forall y \exists x(f(x) = y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function

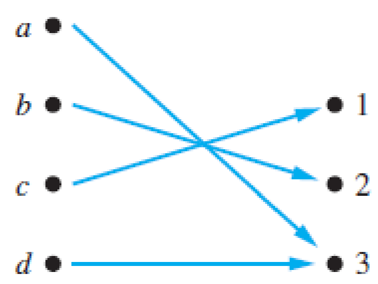


FIGURE 4 **An onto function.**

EXAMPLE 13 Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Extra Examples

Solution: Because all three elements of the codomain are images of elements in the domain, we see that f is onto. This is illustrated in Figure 4. Note that if the codomain were $\{1, 2, 3, 4\}$, then f would not be onto. ◀

EXAMPLE 14 Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: The function f is not onto because there is no integer x with $x^2 = -1$, for instance. ◀

EXAMPLE 15 Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

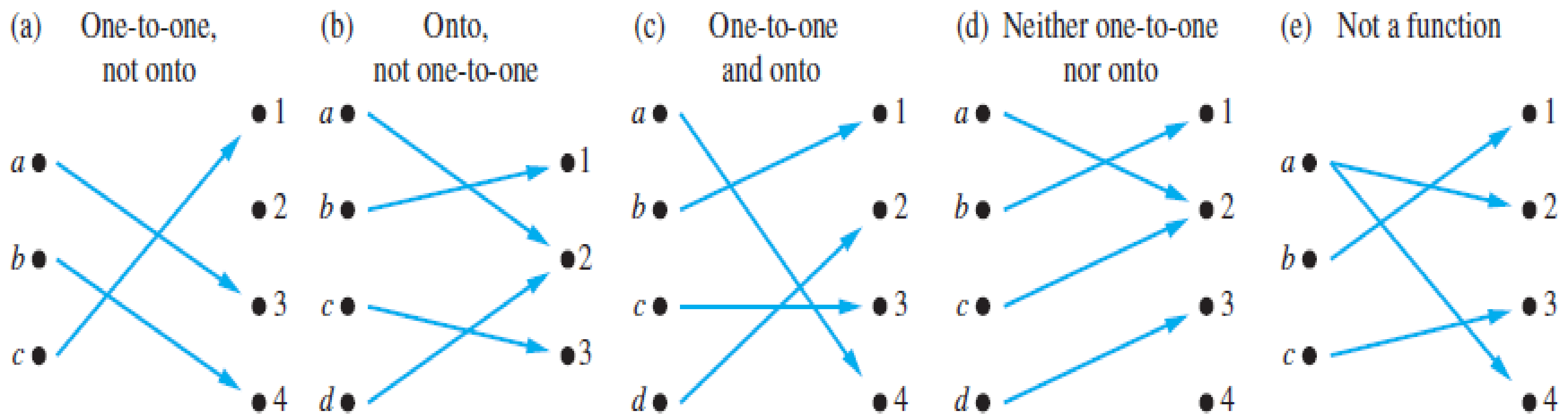
Solution: This function is onto, because for every integer y there is an integer x such that $f(x) = y$. To see this, note that $f(x) = y$ if and only if $x + 1 = y$, which holds if and only if $x = y - 1$. (Note that $y - 1$ is also an integer, and so, is in the domain of f.) ◀

3) One to One Correspondence Function(Bijection):

The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.

EXAMPLE 17 Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4, f(b) = 2, f(c) = 1$, and $f(d) = 3$. Is f a bijection?

Solution: The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection. ◀



Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Exercise 1: Let $A = B = Z$ and let $f : A \to B$ be defined by

$$f(a) = a + 1, \quad \text{for } a \in A.$$

Determine whether the given function is one to one, onto and one to one correspondence.

Solution: Since the formula defining f makes sense for all integers, $\mathrm{Dom}(f) = Z = A$, and so f is everywhere defined.

Suppose that

$$f(a) = f(a')$$

for a and a' in A. Then

$$a + 1 = a' + 1$$

so

$$a = a'.$$

Hence f is one to one.

To see if f is onto, let b be an arbitrary element of B. Can we find an element $a \in A$ such that $f(a) = b$?

Since

$$f(a) = a + 1,$$

we need an element a in A such that

$$a + 1 = b.$$

Of course,

$$a = b - 1$$

will satisfy the desired equation since $b - 1$ is in A. Hence $\mathrm{Ran}(f) = B$; therefore, f is onto. ◆

As, the given function is one to one and onto**, it is one to one correspondence as well.**

Exercise 2:

Example 12. Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2\}$, and $D = \{d_1, d_2, d_3, d_4\}$. Consider the following four functions, from A to B, A to D, B to C, and D to B, respectively.

(a) $f_1 = \{(a_1, b_2), (a_2, b_3), (a_3, b_1)\}$.
(b) $f_2 = \{(a_1, d_2), (a_2, d_1), (a_3, d_4)\}$.
(c) $f_3 = \{(b_1, c_2), (b_2, c_2), (b_3, c_1)\}$.
(d) $f_4 = \{(d_1, b_1), (d_2, b_2), (d_3, b\ )\}$.

Determine whether or not each function is one to one, whether each function is onto, and whether each function is everywhere defined.

Solution

(a) f_1 is everywhere defined, one to one, and onto.
(b) f_2 is everywhere defined and one to one, but not onto.
(c) f_3 is everywhere defined and onto, but is not one to one.
(d) f_4 is not everywhere defined, not one to one, and not onto. ◆

Exercise 3:

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. Determine whether the relation R from A to B is a function. If it is a function, give its range.

(a) $R = \{(a, 1), (b, 2), (c, 1), (d, 2)\}$
(b) $R = \{(a, 1), (b, 2), (a, 2), (c, 1), (d, 2)\}$
(c) $R = \{(a, 3), (b, 2), (c, 1)\}$
(d) $R = \{(a, 1), (b, 1), (c, 1), (d, 1)\}$

Solution

(a) Yes, $\mathrm{Ran}(R) = \{1, 2\}$. (b) No.
(c) Yes, $\mathrm{Ran}(R) = \{1, 2, 3\}$.
(d) Yes, $\mathrm{Ran}(R) = \{1\}$.

Inverse Functions

Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

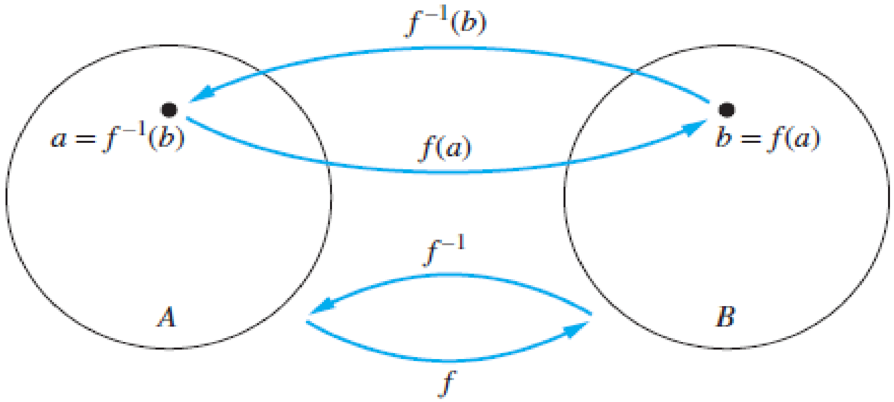


FIGURE 6 **The function f^{-1} is the inverse of function f.**

A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Example 1

Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$. ◀

Example 2

Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it is a one-to-one correspondence, as follows from Examples 10 and 15. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbf{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$. ◀

Example 3

Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ with $f(x) = x^2$. Is f invertible?

Solution: Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible. (Note we can also show that f is not invertible because it is not onto.) ◀

Theorem 1. *Let $f : A \to B$ be a function.*

(a) *Then f^{-1} is a function from B to A if and only if f is one to one.*

(b) *If f^{-1} is a function, then the function f^{-1} is also one to one.*

(c) *f^{-1} is everywhere defined if and only if f is onto.*

(d) *f^{-1} is onto if and only if f is everywhere defined.*

Proof: (a) We prove the following equivalent statement.

f^{-1} is not a function if and only if f is not one to one.

Suppose first that f^{-1} is not a function. Then, for some b in B, $f^{-1}(b)$ must contain at least two distinct elements, a_1 and a_2. Then $f(a_1) = b = f(a_2)$, so f is not one to one.

Conversely, suppose that f is not one to one. Then $f(a_1) = f(a_2) = b$ for two distinct elements a_1 and a_2 of A. Thus $f^{-1}(b)$ contains both a_1 and a_2, so f^{-1} cannot be a function.

(b) Since $(f^{-1})^{-1}$ is the function f, part (a) shows that f^{-1} is one to one.

(c) Recall that $\text{Dom}(f^{-1}) = \text{Ran}(f)$. Thus $B = \text{Dom}(f^{-1})$ if and only if $B = \text{Ran}(f)$. In other words, f^{-1} is everywhere defined if and only if f is onto.

(d) Since $\text{Ran}(f^{-1}) = \text{Dom}(f)$, $A = \text{Dom}(f)$ if and only if $A = \text{Ran}(f^{-1})$. That is, f is everywhere defined if and only if f^{-1} is onto. ◆

Composition of Functions

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is the function from A to C defined by

$$(f \circ g)(a) = f(g(a)).$$

In other words, $f \circ g$ is the function that assigns to the element a of A the element assigned by f to $g(a)$. The domain of $f \circ g$ is the domain of g. The range of $f \circ g$ is the image of the range of g with respect to the function f. That is, to find $(f \circ g)(a)$ we first apply the function g to a to obtain $g(a)$ and then we apply the function f to the result $g(a)$ to obtain $(f \circ g)(a) = f(g(a))$. Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f. In Figure 7 the composition of functions is shown.

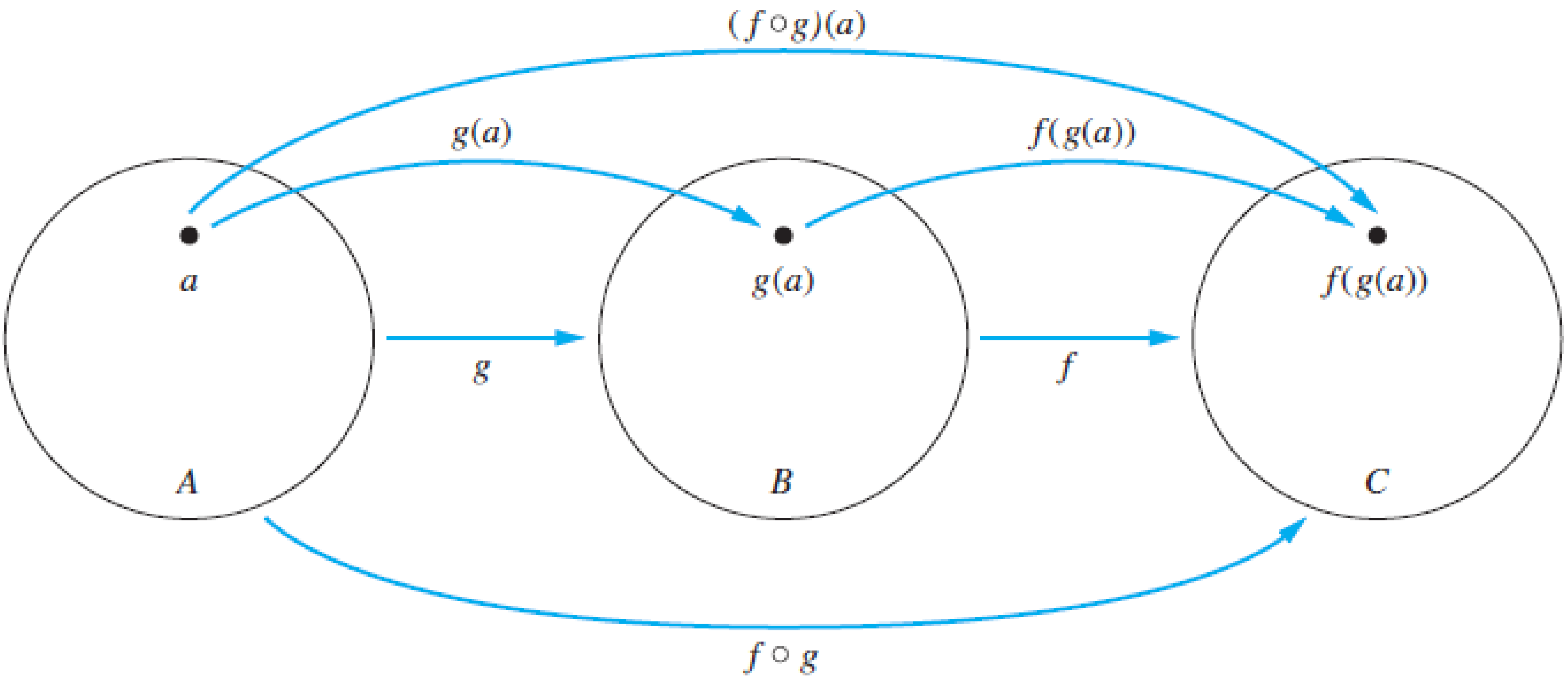


FIGURE 7 **The composition of the functions *f* and *g*.**

Example 1

Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Solution: The composition $f \circ g$ is defined by $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$.

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Example 2

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

Example 3

Let f and g be the functions defined by $f : \mathbf{R} \to \mathbf{R}^+ \cup \{0\}$ with $f(x) = x^2$ and $g : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}$ with $g(x) = \sqrt{x}$ (where $\sqrt{x}$ is the nonnegative square root of x). What is the function $(f \circ g)(x)$?

Solution: The domain of $(f \circ g)(x) = f(g(x))$ is the domain of g, which is $\mathbf{R}^+ \cup \{0\}$, the set of nonnegative real numbers. If x is a nonnegative real number, we have $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \left(\sqrt{x}\right)^2 = x$. The range of $f \circ g$ is the image of the range of g with respect to the function f. This is the set $\mathbf{R}^+ \cup \{0\}$, the set of nonnegative real numbers. Summarizing, $f : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}^+ \cup \{0\}$ and $f(g(x)) = x$ for all x. ◀

Example 4

Example 9. Let $A = B = Z$, and C be the set of even integers. Let $f: A \to B$ and $g: B \to C$ be defined by

$$f(a) = a + 1$$
$$g(b) = 2b.$$

Find $g \circ f$.

Solution: We have

$$\begin{aligned}(g \circ f)(a) &= g(f(a)) \\ &= g(a + 1) \\ &= 2(a + 1).\end{aligned}$$

Thus, if f and g are functions specified by giving formulas, then so is $g \circ f$, and the formula for $g \circ f$ is produced by substituting the formula for f into the formula for g. ◆

Identity Functions

- The identity function on a set A is the function LA: A → A,

  where LA(x) = x, and x ∈ A.

- In other words, identity function is the function that assigns each element to itself.
- This function is one to one, onto and thus, bijective.

When the composition of a function and its inverse is formed, in either order, an identity function is obtained. To see this, suppose that f is a one-to-one correspondence from the set A to the set B. Then the inverse function f^{-1} exists and is a one-to-one correspondence from B to A. The inverse function reverses the correspondence of the original function, so $f^{-1}(b) = a$ when $f(a) = b$, and $f(a) = b$ when $f^{-1}(b) = a$. Hence,

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a,$$

and

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b.$$

Consequently $f^{-1} \circ f = \iota_A$ and $f \circ f^{-1} = \iota_B$, where ι_A and ι_B are the identity functions on the sets A and B, respectively. That is, $(f^{-1})^{-1} = f$.

Theorem 2. *Let R and S be relations from A to B. If $R(a) = S(a)$ for all a in A, then $R = S$.*

Proof: If $a\,R\,b$, then $b \in R(a)$. Therefore, $b \in S(a)$ and $a\,S\,b$. A completely similar argument shows that, if $a\,S\,b$, then $a\,R\,b$. Thus $R = S$. ◆

Theorem 2. *Let $f : A \to B$ be any function. Then*

(a) $1_B \circ f = f$.

(b) $f \circ 1_A = f$.

If f is a one-to-one correspondence between A and B, then

(c) $f^{-1} \circ f = 1_A$.

(d) $f \circ f^{-1} = 1_B$.

Proof: (a) $(1_B \circ f)(a) = 1_B(f(a)) = f(a)$, for all a in $\mathrm{Dom}(f)$. Thus, by Theorem 2 of Section 4.2, $1_B \circ f = f$.

(b) $(f \circ 1_A)(a) = f(1_A(a)) = f(a)$, for all a in $\mathrm{Dom}(f)$, so $f \circ 1_A = f$.

Suppose now that f is a one-to-one correspondence between A and B. As we pointed out above, the equation $b = f(a)$ is equivalent to the equation $a = f^{-1}(b)$. Since f and f^{-1} are both everywhere defined and onto, this means that, for all a in A and b in B, $f(f^{-1}(b)) = b$ and $f^{-1}(f(a)) = a$. Then

(c) For all a in A, $1_A(a) = a = f^{-1}(f(a)) = (f^{-1} \circ f)(a)$. Thus $1_A = f^{-1} \circ f$.

(d) For all b in B, $1_B(b) = b = f(f^{-1}(b)) = (f \circ f^{-1})(b)$. Thus $1_B = f \circ f^{-1}$.

◆

Check whether the following functions are bijective.

Ex. 9 Let $f : R \to R$ such that for all $x \in R$, $f(x) = 7x + 1$.

Solution :

For all $x_1, x_2 \in R$, $f(x_1) = f(x_2)$

$$\Rightarrow 7x_1 + 1 = 7x_2 + 1$$
$$\Rightarrow 7x_1 = 7x_2$$
$$\Rightarrow x_1 = x_2$$

$\therefore$ f is an injective function.

We now check whether f is a surjective function.

Let $y \in R$, y has a pre-image x if $\exists\, x \in R$ such that $f(x) = y$

i.e. if $7x + 1 = y$ i.e. if $7x = y - 1$ i.e. if $x = \dfrac{y-1}{7}$

Thus every $y \in R$ has pre-image $\dfrac{y-1}{7} \in R$.

$\therefore$ f is surjective.

$\therefore$ f is a bijective function.

Ex. 10 Let $f : R - \left\{\frac{3}{2}\right\} \to R - \{0\}$ such that for all $x \in R - \left\{\frac{3}{2}\right\}$, $f(x) = \frac{1}{2x-3}$

Solution :

For all $x_1, x_2 \in R - \left\{\frac{3}{2}\right\}$

$$f(x_1) = f(x_2) \Rightarrow \frac{1}{2x_1 - 3} = \frac{1}{2x_2 - 3}$$

$$\Rightarrow 2x_1 - 3 = 2x_2 - 3$$

$$\Rightarrow x_1 = x_2$$

$\therefore$ f is an injective function.

We now check whether f is a surjective function.

Let $y \in R - \{0\}$ so that $y \in R$ but $y \neq 0$.

y has a pre-image x if $\exists\, x \in R - \left\{\frac{3}{2}\right\}$ such that $f(x) = y$.

Continue…

i.e. if $\dfrac{1}{2x-3} = y$

i.e. if $2x - 3 = \dfrac{1}{y}$ $\qquad \because \quad y \neq 0$

i.e. if $x = \dfrac{1}{2}\left(3 + \dfrac{1}{y}\right)$

Thus every $y \in R - \{0\}$ has pre-image $\dfrac{1}{2}\left(3 + \dfrac{1}{y}\right) \in R - \left\{\dfrac{3}{2}\right\}$

$\therefore$ f is surjective

$\therefore$ f is a bijective function.

Ex. 11 $f : R - \left\{\frac{2}{5}\right\} \to R - \left\{\frac{4}{5}\right\}$ such that for all x $R - \left\{\frac{2}{5}\right\}$, $f(x) = \frac{4x+3}{5x-2}$

Solution :

For all $x_1, x_2 \in R - \left\{\frac{2}{5}\right\}$

Sem-V

$$f(x_1) = f(x_2) \Rightarrow \frac{4x_1+3}{5x_1-2} = \frac{4x_2+3}{5x_2-2}$$

$$\Rightarrow (4x_1 + 3(5x_2 - 2) = (4x_2 + 3)(5x_1 - 2)$$

$$\Rightarrow 23x_1 = 23x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore$ f is injective.

We now check whether f is surjective.

Let $y \in R - \left\{\frac{4}{5}\right\}$ so that $y \in R$ but $y \neq \frac{4}{5}$. y has a pre-image x if $\exists\, x \in R - \left\{\frac{2}{5}\right\}$ such that $f(x) = y$.

Continue…

i.e. if $\dfrac{4x+3}{5x-2} = y$

i.e. if $4x + 3 = y(5x - 2)$

i.e. if $x = \dfrac{2y+3}{5y-4}, y \neq \dfrac{4}{5}$

Thus every $y \in R - \left\{\dfrac{4}{5}\right\}$ has a pre–image $\dfrac{2y+3}{5y-4}$ in $R - \left\{\dfrac{2}{5}\right\}$

$\therefore$ f is surjective.

$\therefore$ f is bijective function.

Ex. 12 Let $f : R \to R$ such that for all $x \in R$, $f(x) = 3x^2 + 2$.

Solution :

Let $x_1, x_2 \in R$.

$f(x_1) = f(x_2)$

$\Rightarrow \quad 3x_1^2 + 2 = 3x_2^2 + 2$

$\Rightarrow \quad 3x_1^2 = 3x_2^2$

$\Rightarrow \quad x_1^2 = x_2^2$

$\Rightarrow \quad x_1 = \pm x_2 \qquad \therefore$ f is not an injective function.

$\therefore \quad$ f is not a bijective function.

Ex. 13 $f : R \to R$ is defined as $f(x) = x^3$.

$g : R \to R$ is defined as $g(x) = 4x^2 + 1$

$h : R \to R$ is defined as $h(x) = 7x - 2$.

Find the rule of defining (hog)of, go(hof).

Solution :

$$
\begin{aligned}
((hog)of)\,(x) &= (hog)\,(f(x)) \\
&= (hog)\,(x^3) = h(g(x^3)) \\
&= h(4(x^3)^2 + 1) = h(4x^6 + 1) \\
&= 7(4x^6 + 1) - 2 = 28x^6 + 5
\end{aligned}
$$

$$
\begin{aligned}
(go(hof))\,(x) &= go(hof)\,(x) \\
&= g(h(f(x)) = g(h(x^3)) \\
&= g(7(x^3) - 2) = g(7x^3 - 2) \\
&= 4(7x^3 - 2)^2 = 4(49x^6 - 28^3 + 4) + 1 \\
&= 196x^6 - 112x^3 + 17
\end{aligned}
$$

Ex. 14 $f : R \to R$ is defined as $f(x) = 3x - 1$. Find the rule of finding $f^3 : R \to R$ where $f^3 = fo(fof)$.

Solution :

$$f^3(x) = [fo(fof)]\,(x)$$

Now, $(fof)\,(x) = f(f(x))$

$$= 3f(x) - 1 = 3(3x - 1) - 1$$

$$= 9x - 4$$

$$f^3(x) = [fo(fof)]\,9x)$$

$$= f(9x - 4) = 3(9x - 4) - 1$$

$$= 27x - 13$$

Ex. 16 $f : R \to R$ is bijective function given by $f(x) = 4x - 5$. Find f^{-1}.

Solution :

$f : R \to R$ is bijective, $f^{-1} : R \to R$ is defined.

Let y be a point of domain R of f^{-1}.

fof^{-1} is identity on $B = R$.

$\Rightarrow \quad (fof^{-1})(y) = y$

$\Rightarrow \quad f(f^{-1}(y)) = y$

$\Rightarrow \quad 4\,f^{-1}(y) - 5 = y,$ by definition of f

$\Rightarrow \quad f^{-1}(y) = \dfrac{y+5}{4}.$

This is the required inverse function f^{-1}.

Ex. 17 Find f^{-1} if $f : R \to R$ is such that $f(x) - x^3$.

Solution :

fof^{-1} is the identity in co–domain R.

$\Rightarrow \quad (fof^{-1})(x) = x$

$\Rightarrow \quad f(f^{-1}(x)) = x$

$\Rightarrow \quad (f^{-1}(x))^3 = x$, by definition of f.

$\Rightarrow \quad f^{-1}(x) = (x)^{1/3}$

Ex. 18 Let $f : R - \left\{\frac{3}{2}\right\} \to R - \{0\}$ be such that for all $x \in R - \left\{\frac{3}{2}\right\}$, $f(x) = \frac{1}{2x-3}$.

Find f^{-1}.

Solution :

We have seen that f is bijective.

$\therefore \quad f^{-1} : R - \{0\} \to R - \left\{\frac{3}{2}\right\}$ is defined.

By definition of inverse function $(fof^{-1})(x) = x$.

$$\Rightarrow \quad f(f^{-1}(x)) = x$$

$$\Rightarrow \quad \frac{1}{2f^{-1}(x)-3} = x, \qquad \text{by definition of f.}$$

$$\Rightarrow \quad 2f^{-1}(x) = \frac{1}{x} + 3$$

$$\Rightarrow \quad f^{-1}(x) = \frac{1}{2}\left(\frac{1}{x} + 3\right)$$

Ex. 19 Let $f : R - \left\{\frac{2}{5}\right\} \to R - \left\{\frac{4}{5}\right\}$ be defined as $f(x) = \frac{4x+3}{5x-2}$. Find f^{-1}.

Solution :

We have seen that f is bijective.

$$\therefore \quad f^{-1} : R - \left\{\frac{4}{5}\right\} \to R - \left\{\frac{2}{5}\right\} \text{ is defined.}$$

By definition of inverse function $(f o f^{-1})(x) = x$.

$$\Rightarrow \quad \frac{4f^{-1}(x)+3}{5f^{-1}(x)-2} = x \qquad \text{by definition of f.}$$

$$\Rightarrow \quad 4f^{-1}(x)+3 = x\left(5f^{-1}(x)-2\right)$$

$$\Rightarrow \quad f^{-1}(x) = \frac{-3-2x}{4-5x} = \frac{2x+3}{5x-4}.$$