Tutorial 4

Q1) Evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)} dz$ where c is $|z-1|=2$

Solⁿ: C is a closed curve i.e. a circle with centre (1, 0) and radius 2.

Now, for $z=2$

$\therefore C: |z-i| = |2-i| = \sqrt{(2)^2+(-1)^2} = \sqrt{4+1}$

$= \sqrt{5} > 2 = 2.23672$ (inside)

$\rightarrow$ for $z=-1$ $\therefore C: |z-i| = |-1-i| = \sqrt{1+1}$

$= \sqrt{2} = 1.414 < 2$ (outside)

Consider, Residue $(z=2) = \lim_{z \to 2} \left[\frac{(z-2)z-1}{(z+1)^2(z-2)} \right]$

$= \lim_{z \to 2} \left[\frac{z-1}{z^2+2z+1} \right] = \frac{2-1}{4+4+1} = \frac{1}{9}$

$\int_c f(x) = 2\pi i$ [Residue $(z=2)$]

$= 2\pi i \left(\frac{1}{9} \right)$

$= \frac{2\pi i}{9}$ //

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Q2) Evaluate $\int_C \frac{z^2+4}{(z-2)(z+3i)} dz$ where c is (i) $|z+1|=2$
(ii) $|z-2|=2$

Sol: (i) $|z+1|=2$; C is a circle with $(-1,0)$ & radius 2.
Also, $z=2, -3i$ are the simple poles.
→ for $z=2$ ∴ $C: |2+1| = |2+1| = \sqrt{3^2} = \sqrt{9} = 3 > 2$, lies outside the 'C'.
→ for $z=-3i$ ∴ $C: |z+1| = |-3i+1| = \sqrt{(-3)^2+1^2}$
$= \sqrt{10} = 3.16 > 2$, lies outside the 'C'.
$z=2, -3i$ lies outside the 'C'. By Cauchy's Integral theorem, ∴ $\int_C \frac{z^2+4}{(z-2)(z+3i)} = 0$

(ii) $C: |z-2|=2$, C is a circle with $C=(2,0)$ & $R=2$,
also $z=2, -3i$ are simple poles.
→ for $z=2$
∴ $C: |z-2| = |2-2| = 0$, ∴ $z=2$ lies 0 inside C.
→ for $z=-3i$ $C: |z-2| = |-3i-2| = \sqrt{(-3)^2+(-2)^2}$
$= \sqrt{13} = 3.672$ ∴ $z=-3i$ lies outside the c. Lets consider $I = \int_C \frac{z\, dz}{(z-2)(z+3i)}$
$= \int_C \frac{z^2+4 / z+3i}{z-2} dz$ By Cauchy's Integral formula
$I = 2\pi i \left[\frac{z^2+4}{z+3i}\right]_{z=2} = 2\pi i \left[\frac{8}{2+3i}\right]$
∴ $\int_C \frac{z}{(z-2)(z+3i)} = \frac{16\pi i}{2+3i}$

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Q3) If $f(a) = \int_C \frac{4z^2+z+4}{z-a} dz$ where c is the ellipse $4x^2+9y^2=36$.
Find the values of $f(4)$, $f(1)$, $f(i)$, $f'(-1)$, $f''(-i)$

Solution: The circle $|z|=2$ has centre at the origin & radius 2. The point $z=1$ lies inside the circle. $f(z) = 4z^2+z+5$, is analytic in and on C & $z=1$ lies inside it. Hence, by Cauchy's formula
$f(1) = \int_C \frac{4z^2+z+5}{z-1} dz = 2\pi i \phi(z_0)$
where, $\phi(z) = 4z^2+z+5$ and $z_0=1$
∴ $f(1) = 2\pi i(4+1+5) = 20\pi i$
The point $z=i$ also lies inside the circle
∴ $f(i) = \int_C \frac{4z^2+8+5}{z-i} dz = 2\pi i \phi(z_0)$
where, $\phi(z) = 4z^2+z+5$ & $z_0 = i$
∴ $f(i) = 2\pi i(-4+i+5) = 2\pi i(1+i)$
$= 2\pi(i-1)$
∴ $f'(a) = 2\pi i\, \phi'(a) = 2\pi i(8z+1)$
∴ $f'(-1) = -14\pi i$
∴ $f''(a) = 2\pi i(8) = 16\pi i$
∴ $f''(-1) = 16\pi i$

Q4) Evaluate $\int_C \frac{\cos \pi z^2}{z^2-3z+2} dz$ where c is the circle $|z|=3$.

Solution: The circle $|z|=3$ has centre at $(0,0)$ and radius 3. ~~The points~~ Now, $z^2-3z+2=0$ gives $(z-2)(z-1)=0$.

$\therefore z=2, -1$. Both these points lie inside the circle. Hence, using partial fractions, $\frac{1}{(z-2)(z-1)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$

$f(z) = \cos \pi z^2$ which is analytic in C.

$$\therefore \int_C \frac{\cos \pi z^2}{(z-2)(z-1)} dz = \int_C \frac{\cos \pi z^2}{(z-2)} dz - \int_C \frac{\cos \pi z^2}{(z-1)} dz$$

By Cauchy's Integral theorem,

$= 2\pi i f(2) - 2\pi i f(1)$, we have $f(z) = \cos \pi z^2$

$= 2\pi i + 2\pi i$

$= 4\pi i$

Q5) Evaluate $\int_C \frac{\sin 3z}{z+\pi/2} dz$ where c is $|z|=5$.

Sol^n: The circle $|z|=5$ has centre $(0,0)$ and radius 5. Now $z+\pi/2=0$ gives $z=-\pi/2$. Here, the value of z lies inside the circle.

By Cauchy's Integral theorem,

$$\int_C \frac{f(z)}{z-(-\pi/2)} dz = 2\pi i f(-\pi/2)$$

$f(z) = \sin 3z$

$f(-\pi/2) = -\sin\left(\frac{-3\pi}{2}\right) = -(-1) = 1$

$= 2\pi i (1)$

$$\therefore \int_C \frac{f(z)}{z+\pi/2} = 2\pi i$$

Q6) Evaluate $\int_C z-z^2 \, dz$ along the upper half of the circle $|z|=1$

Sol^n Let us put $z = e^{i\theta}$ $\therefore dz = e^{i\theta} d\theta$. And θ varies from 0 to π. $\therefore \int_C (z-z^2)dz = \int_0^\pi (e^{i\theta} - e^{2i\theta})$ $e^{i\theta}\cdot i\,d\theta = i\int_0^\pi (e^{2i\theta} - e^{2i\theta})\, d\theta$

$= i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_0^\pi = \left[\frac{e^{2i\pi}}{2} - \frac{e^{3i\pi}}{3} - \frac{1}{2} + \frac{1}{3}\right]$

$= \left[\frac{1}{2}(\cos 2\pi + i\sin 2\pi) - \frac{1}{3}(\cos 3\pi + i \sin 3\pi) - \frac{1}{2} + \frac{1}{3}\right]$

$= \left[\frac12 + \frac13 - \frac12 + \frac13\right] = \frac23$. The value of the integral for the lower half of the same circle in the same positive direction i.e. when θ varies from π to 2π.

$= \int_C (z-z^2)\, dz = i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_\pi^{2\pi} = i\left[\frac{e^{4i\pi}}{2i} - \frac{e^{6i\pi}}{3i} - \frac{e^{2i\pi}}{2i} + \frac{e^{3i\pi}}{3i}\right] = \left[\frac{\cos 4\pi + i\sin 4\pi}{2} - \frac{\cos 6\pi + i\sin 6\pi}{3} - \frac{\cos 2\pi + i\sin 2\pi}{2} + \frac{\cos 3\pi + i\sin 3\pi}{2}\right] = \left[\frac12 - \frac13 - \frac12 - \frac13\right]$

$= -\frac{2}{3}$

Q7) Evaluate $\int_0^{2+i} z^{-2}\, dz$ along (i) line $x=2y$ (ii) real axis from 0 to 2 (iii) $2y^2 = x$, the parabola.

Sol^n i) Along the line $x=2y$, $dx = 2dy$

$\therefore dz = dx + i\,dy = 2dy + i\,dy$

$= (2+i)dy$

And x varies from 0 to 1.

$\therefore \int_C f(\bar z)\, dz = \int_0^1 (x-iy)^2 (2+i)\, dy$

$= \int_0^1 (x^2 - 2xiy - y^2)(2+i)\,dy$

$= \int_0^1 (4y^2 - 4y^2 i - y^2)(2+i)\,dy \; \ldots \; (\because x = 2y)$

$= \int_0^1 (3-4i)y^2(2+i)\,dy = (10-5i)\left[\frac{y^3}{3}\right]_0^1$

$= (10-5i)\left(\frac13\right) = \frac{10}{3} - \frac{5i}{3}$

(ii) $\int_0^{2+i} (\bar z)^2\, dz = \int_0^{2+i} (x-iy)^2 (dx + i\,dy)$

$= \int_{OA} (x^2)\, dx + \int_{AB} (2-iy^2)\, i\,dy$ ∴ Since

[Along OA, $y=0$, $dy=0$, x varies 0 to 2,
Along AB, $x = 2$, $dx=0$ & y varies 0 to 1

$\therefore \int_0^{2+i} (\bar z)^2\, dz = \int_0^2 (x^2)\,dx + \int_0^1 (2-iy)^2\, i\,d$

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$$= \left[\frac{x^3}{3}\right]_0^2 + \left[\left(4y - \frac{4iy^2}{2} - \frac{y^3}{3}\right)\right]_0^1$$

$$= \frac{8}{3} + i\left[4 - 4i\cdot\frac{1}{2} - \frac{1}{3}\right) = \frac{14}{3} - \frac{11i}{3}$$ //

iii) $\because 2y^2 = x \quad \therefore 4y\,dy = dx$

$\therefore dz = dx + i\,dy = 4y\,dy + i\,dy = (4y+i)\,dy$

find x varies 0 to 1.

$$\therefore \int_C f(z)dz = \int_0^1 (x+iy)^2 (4y+i)dy$$

$$= \int_0^1 (4y^4 - 4y^3 - 1y^2)(4y+i)dy \ldots$$

$[\because 2y^2 = x]$

$$= \left[\frac{16}{6} - \frac{16i}{5} - 1 + \frac{4i}{5} + 2 - \frac{1}{3}\right]$$

$$= \left[\frac{8}{3} - \frac{4li}{15}\right]$$

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Q8) Evaluate $\int_C z^2 dz$ where C is the arc of the circle $x = r\cos\theta$, $y = r\sin\theta$, from $\theta = 0$ to $\pi/3$

Sol^n As above $I = \int_C z^2 dz$ ~~$= \frac{z^3}{3}$~~

$$= \int_0^{\pi/3} r^2 e^{2i\theta} \cdot re^{i\theta} \cdot i\,d\theta$$

$$= r^3 i \int_0^{\pi/3} e^{3i\theta}\,d\theta$$

$$= r^3 i \left[\frac{e^{3i\theta}}{3i}\right]_0^{\pi/3}$$

$$= \frac{r^3}{3}\left[e^{i\pi} - 1\right] = \frac{r^3}{3}\left[\cos\pi + i\sin\pi - 1\right]$$

$$= \frac{-2r^3}{3}.$$

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Q9) Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z-1| = 3$

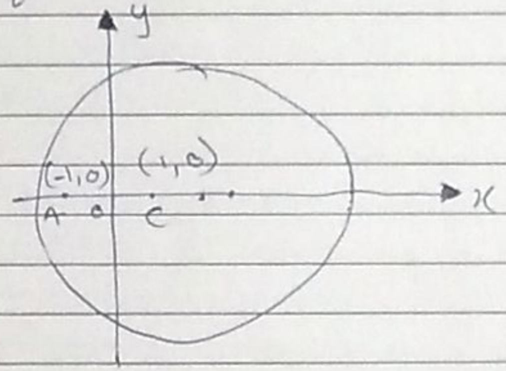
Sol^n: The circle $|z-1| = 3$ has centre at $C(1,0)$ and radius 3. Further, $z+1=0$ gives A. $z=-1$. The point A lies inside the circle. Hence, $e^{2z}/(z+1)^4$ is not analytic in C. We take $f(z) = e^{2z}$ which is analytic in C. By corollary of Cauchy's Formula, $\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$

$\therefore \int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f^3(z_0)$

$= \frac{2\pi i}{3!} \cdot \frac{8}{e^2}$

$[\because f(z) = e^{2z}, f^3(z) = 8e^{2z} \text{ and } z_0 = -1]$

$= \frac{8\pi i}{3e^2}$



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Q10) Evaluate $\int_C \frac{dz}{e^z z^4}$ where C is the circle $|z|=1$.

Sol^n $|z|=1$ is a circle ~~$|z|=1$~~ with centre at origin and radius 1. $\therefore \int_C \frac{e^{-z}}{z^4} dz$

$\therefore z=0$ lies inside the circle. By corollary of Cauchy's formula, $\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)} f^{(n-1)}(z_0)$ and $z_0 = 0$

$\therefore \int_C \frac{f(z)}{(z-z_0)^4} dz = \frac{2\pi i}{3!} f^3(z_0)$

$\therefore f'(z) = e^{-z} = e^{-z};$

$f''(z) = e^{-z};$

$f'''(z) = e^{-z}$

$\therefore \int_C \frac{-e^{-z}}{z^4} dz = \frac{2\pi i}{3!} + (-2)$

$\therefore \int_C \frac{-e^{-z}}{z^4} = \frac{-\pi i}{3}$