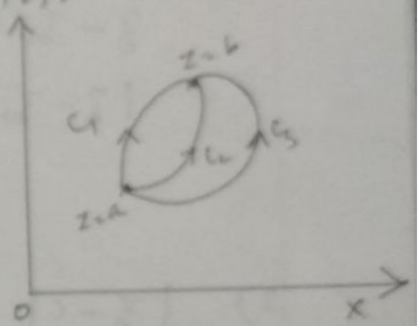
Sem IV A.M. IV Leture : 1

Complex Integration

Path of Integration :-

The definite integral of a complex variable may depend upon the path of integration in the complex plane. The path of definite integral $\int_a^b f(z)\,dz$ is a curve joining the pts $z=a$, $z=b$. The value of the integration depends upon the path.



Let $f(z)$ be a continuous fn of the complex variable $z=x+iy$ defined at every point of a curve 'C' with end points A & B. Divide 'C' into 'n' parts.

$A = P_0(z_0),\ P_1(z_1),\ P_2(z_2) \dots P_n(z_n) = B.$

Let $\delta z_i = z_i - z_{i-1}$ and let ξ_i be a point on the arc $P_{i-1}P_i$. Then the limit of the sum $\sum_{i=1}^{n} f(\xi_i)\cdot\delta z_i$ as $n\to\infty$ such that $\delta z_i \to 0$. If its exists is called the line integral of $f(z)$ along C denoted by $\int_C f(z)\,dz$

If C is a closed ~~[illegible]~~ curve ie, P_0 & P_n coincide, the integral is called the contour integral and defined by $\oint f(z)\,dz$.

Evaluation of line Integral

Let $z = x+iy \qquad f(z) = u+iv$

$dz = dx + i\,dy$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int (v\,dx + u\,dy)$$

2. When the Contour is a Circle

Note :- When the contour is a circle, it is better to use polar form $z = re^{i\theta}$

(1) $|z| = r$

$z = x + iy \quad |z| = \sqrt{x^2+y^2}$

$x^2 + y^2 = z = r$, is a circle with centre $(0,0)$, radius $= r$

(2) $|z - z_0| = r$

$z - z_0 = (x+iy) - (x_0 + iy_0)$

$|z - z_0| = r = (x-x_0)^2 + (y-y_0)^2$ represents a circle with centre (x_0, y_0) and radius r.

3) $|z - c| + |z + c| = k$

This is an ellipse with focii $(c,0)$ & $(-c,0)$ and major axis equal to k.

$\{$ In general, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse with focii $(\pm ae, 0)$ and semi major axis $b^2 = a^2(1-e^2)$ $\}$.

$\therefore b^2 = \left(\frac{k}{2}\right)^2 - c^2$.

Problems:-

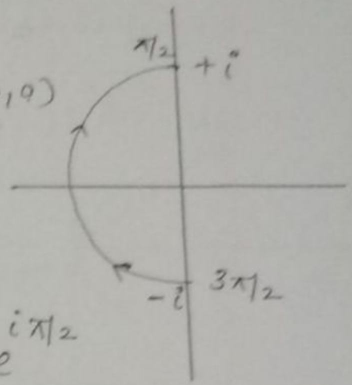
1. Evaluate $\int_C |z|\,dz$, where C is the left half of unit circle $|z|=1$ from $z=-i$ to i

Let $z = re^{i\theta}$ where $r=1$

$z = e^{i\theta}$, $dz = ie^{i\theta}\,d\theta$

$|z|=1$

$r=1$, $(0,0)$



$$\int_C |z|\,dz = \int_{3\pi/2}^{\pi/2} 1.\,i e^{i\theta}\,d\theta$$

$$= \left[i\frac{e^{i\theta}}{i} \right]_{3\pi/2}^{\pi/2} = e^{i3\pi/2} - e^{i\pi/2}$$

$$= \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) - \left(\cos\pi/2 + i\sin\pi/2\right)$$

$$= 2i$$

2. Evaluate $\int \frac{2z+3}{z}\,dz$ where C is the
1) upper half of the circle $|z|=2$
2) lower half of the circle $|z|=2$
3) whole circle in anti-clockwise direction

Let $z = re^{i\theta}$ $\quad |z|=2$

$z = 2e^{i\theta}$

$dz = 2ie^{i\theta}\,d\theta$

$$\frac{2z+3}{z} = 2 + \frac{3}{z} = 2 + \frac{3e^{-i\theta}}{2}$$

(1). $$\int_C f(z)\,dz = \int_C \frac{2z+3}{z}\,dz = 2\int_0^{\pi}\left(2 + \frac{3}{2}e^{-i\theta}\right) i\,d\theta . e^{+i\theta}$$

$$= 2i\int (2e^{+i\theta} + \tfrac{3}{2})\,d\theta$$

$$= 2i\left(\frac{2e^{i\theta}}{i} + \frac{3}{2}\theta\right)_0^{\pi}$$

$$= 2i\left\{ \cancel{2}\theta + \cancel{\tfrac{3}{2}}\cancel{\frac{e^{-i\theta}}{-i}} \right\}_0^{\pi}$$

$$= \cancel{2i\left\{\left(2\pi + \frac{3}{-2i}e^{-i\pi}\right) - \left(0 + \frac{3}{-2i}e^{0}\right)\right\}}$$

$$= \cancel{4\pi i - 3e^{-i\pi} + \frac{3}{2}}$$

$$= \cancel{4\pi i + \frac{3}{2} - 3(\cos\pi - i\sin\pi)} = 4\pi i + \frac{3}{2} + 3$$

$$2i\left\{\frac{2}{i}e^{i\pi} + \frac{3}{2}\pi - \frac{2}{i}\right\}$$

$$4(-1) + 3i\pi - 2$$

$$= -6 + 3i\pi$$

(2) $\int_C f(z)\,dz = \underline{\underline{\quad}}$

3) $\int_C f(z)\,dz = 2i \int_0^{2\pi} (2 + \frac{3}{2} e^{-i\theta}) e^{i\theta} d\theta$

$= 2i \int_0^{2\pi} (2e^{i\theta} + \frac{3}{2})\,d\theta$

$= 2i \left[\frac{2e^{i\theta}}{i} + \frac{3}{2}\theta \right]_0^{2\pi}$

$= \left[4e^{i\theta} + 3i\theta \right]_0^{2\pi}$

$= (4e^{2\pi i} + 3i\,2\pi) - (4+0).$

$= 4 + 6\pi i - 4$

$= \underline{\underline{6\pi i}}.$

3. Show that $\int_C \log z\,dz = 2\pi i$ where C is the unit circle in the z-plane.

$z = re^{i\theta} \quad |z| = 1$

$z = e^{i\theta}$

$dz = ie^{i\theta} d\theta$

$\int_C \log z\,dz = \int_0^{2\pi} \log e^{i\theta} \cdot ie^{i\theta} d\theta$

$= i\left\{ \int_0^{2\pi} i\theta \cdot e^{i\theta} d\theta \right\} = -\int_0^{2\pi} \theta e^{i\theta} d\theta$

$= -\left\{ \left(\theta \cdot \frac{e^{i\theta}}{i}\right)_0^{2\pi} - \int_0^{2\pi} \frac{e^{i\theta}}{i} \cdot 1\,d\theta \right\}.$

$= -\left\{ \frac{2\pi}{i}(1) - \frac{1}{i}\left[\frac{e^{i\theta}}{i}\right]_0^{2\pi} \right.$

$= -\frac{2\pi}{i} + e^{2\pi i} - e^0$

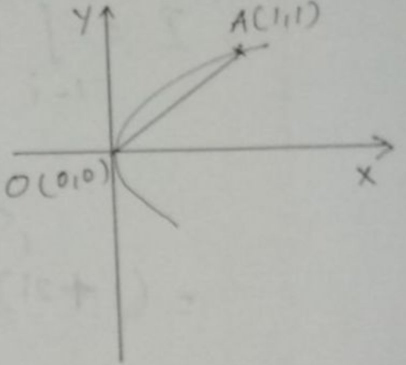
$= -\frac{2\pi}{i} + 1 - 1 = \frac{-2\pi}{i} \times \frac{i}{i} = \underline{\underline{2\pi i}}.$

HW: Evaluate $\int_C z^2\,dz$, where C is the circle $x = r\cos\theta$, $y = r\sin\theta$ from $\theta = 0$ to $\pi/3$. $\left\{ \text{Ans}: -\frac{2}{3}r^3 \right\}$

Lecture : 02

When the contour is a straight line or Parabola.

1. Evaluate $\int_0^{1+i} z^2\, dz$ along (1) the line $y=x$ (2) the parabola $x=y^2$. Is the integral independent of the path?



(1) On the line OA, $y=x$

$dy = dx$

$z = x+iy$

$dz = dx + i\,dy$

$= dx + i\,dx = (1+i)\,dx$

$x \to 0$ to 1

$\cancel{I = \int_0^{1+i} (x+iy)^2\, dz}$

$$I = \int_0^{1+i} z^2\, dz = \int_0^{1+i} (x+iy)^2\, dz$$

$$= \int_0^1 (x+ix)^2 (1+i)\,dx$$

$$= (1+i)\int_0^1 x^2 (1+i)^2\, dx$$

$$= (1+i)^3 \int_0^1 x^2\, dx = (1+3i-1+3i-i)\left(\frac{x^3}{3}\right)_0^1$$

$$= 2(-1+i)\frac{1}{3} = \frac{2}{3}(i-1)$$

(I (a) Next page)

(2). Evaluate $\int_{1-i}^{2+i} (2x+iy+1)\,dz$ along

(1) the straight line joining $(1-i)$ to $(2+i)$

(2) $x=t+1$, $y=2t^2-1$, a parabola.

(1) $1-i \Rightarrow (1,-1)$ & $2+i \Rightarrow (2,1)$

The eqn of the line joining $(1,-1)$ & $(2,1)$

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2} \Rightarrow \frac{y+1}{-1-1} = \frac{x-1}{1-2}$$

i.e, $y+1 = 2x-2 \Rightarrow y = 2x-3$.

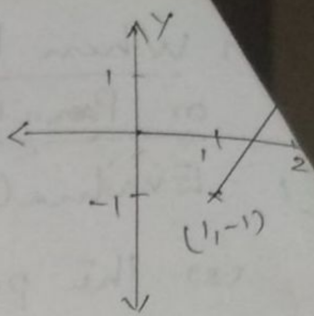
$y = 2x - 3$

$dy = 2 \cdot dx$

$z = x + iy$

$dz = dx + i\,dy = dx + i\,2dx$

$= (1+2i)\,dx$



$$I = \int_{1-i}^{2+i} (2x + iy + 1)\, dz = \int$$

$$= \int_1^2 \left[2x + i(2x-3) + 1\right](1+2i)\,dx$$

$$= (1+2i)\int_1^2 (2x + i\,2x - 3i + 1)\, dx$$

$$= (1+2i)\left(x^2 + ix^2 - 3ix + x\right)\Big|_1^2$$

$$= (1+2i)\left[(4+4i-6i+2) - (1+i-3i+1)\right]$$

$$= (1+2i)\{(6-2i) - (2-2i)]$$

$$= (1+2i)(4 + 0i)$$

$$= \underline{\underline{4(1+2i)}}$$

(2) $x = t+1, \quad y = 2t^2 - 1$

$z = x + iy = (t+1) + i(2t^2 - 1)$

$dz = dt + i\,4t \cdot dt = (1 + i\,4t)\,dt$

When $z = 1-i, \Rightarrow t = 0$

When $z = 2+i, \Rightarrow t = 1$

$$I = \int_{1-i}^{2+i} (2x + iy + 1)\, dz$$

$$= \int_0^1 \left[2(t+1) + i(2t^2 - 1) + 1\right](1 + i\,4t)\,dt$$

$$= \int_0^1 (1 + i\,4t) \int (2t + 2 + i\,2t^2 - i + 1)\, dt$$

$$= (1 + i\,4t)\left[t^2 + 2t + i\frac{2}{3}t^3 - it + t\right]_0^1$$

$$= (1 + i\,4t)\left[1 + 2 + i\frac{2}{3} - i + 1\right]$$

$$= (1 + i\,4t)\left(4 - \frac{1}{3}i\right) \;//$$

$= \int_0^{\;} \quad 4 + \frac{25}{3} i$

Q. I (b) On the arc OA, $x = y^2$

$dx = 2y\,dy.$

$z = x + iy$

$dz = dx + i\,dy$

$$I = \int_0^{1+i} z^2\,dz = \int_0^{1+i} (x+iy)^2\,dz$$

$$= \int_0^{1+i} (x^2 + 2xiy - y^2)(dx + i\,dy)$$

$$= \int_0^1 (y^4 + 2y^2 \cdot iy - y^2)(2y\,dy + i\,dy)$$

$$= \int_0^1 (y^4 + i\,2y^3 - y^2)(2y + i)\,dy$$

$$= \int_0^1 (2y^5 + i\,4y^4 - 2y^3 + iy^4 - 2y^3 - iy^2)\,dy$$

$$= \int_0^1 (2y^5 - 4y^3)\,dy + i\int_0^1 (5y^4 - y^2)\,dy$$

$$= \left[\frac{2y^6}{6} - 4\frac{y^4}{4}\right]_0^1 + i\left[\frac{5y^5}{5} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{2}{3}(i-1)$$

Since the two integrals are equal, the integral is independent of path.

Practice Problems.

HW 1. Evaluate $\int_0^{1+i}(x^2+iy)\,dz$ along the path (i) $y=x$ (ii) $y=x^2$. Is the line integral independent of the path?

HW 2. Evaluate $\int_0^{2+i}(\bar{z})^2\,dz$ along

1) the line $x=2y$
2) the real axis from 0 to 2 and then vertically to $2+i$
3) the parabola $2y^2=x$.

Hint: $z^2=(x-iy)^2$.

3. Evaluate the integral $\int_0^{1+i}(x-y+ix^2)\,dz$

1) along the line from $z=0$, $z=1+i$
2) along the real axis from $z=0$ to $z=1$ and then along the line parallel to the imaginary axis from $z=1$ to $z=1+i$
3) along the imaginary axis from $z=0$ to $z=i$ and then along the line parallel to the real axis from $z=i$ to $z=i+1$
4) along the parabola $y^2=x$.