Beta & Gamma functions

Nirav B. Vyas

Department of Mathematics
Atmiya Institute of Technology and Science
Yogidham, Kalavad road
Rajkot - 360005 . Gujarat

Introduction

- The Gamma function and Beta functions belong to the category of the special transcendental functions and are defined in terms of improper definite integrals.

- The Gamma function and Beta functions belong to the category of the special transcendental functions and are defined in terms of improper definite integrals.
- These functions are very useful in many areas like asymptotic series, Riemann-zeta function, number theory, etc. and also have many applications in engineering and physics.

Introduction

- The Gamma function and Beta functions belong to the category of the special transcendental functions and are defined in terms of improper definite integrals.
- These functions are very useful in many areas like asymptotic series, Riemann-zeta function, number theory, etc. and also have many applications in engineering and physics.
- The Gamma function was first introduced by Swiss mathematician **Leonhard Euler**(1707-1783).

- Definition:

  Let n be any positive number. Then the definite integral $\displaystyle\int_0^\infty e^{-x}x^{n-1}dx$ is called gamma function of n which is denoted by Γn and it is defined as

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx, n > 0$$

Properties of Gamma function

(1) $\Gamma(n+1) = n\Gamma n$

Properties of Gamma function

(1) $\Gamma(n+1) = n\Gamma n$

(2) $\Gamma(n+1) = n!$, where n is a positive integer

Properties of Gamma function

(1) $\Gamma(n+1) = n\Gamma n$

(2) $\Gamma(n+1) = n!$, where n is a positive integer

(3) $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$

Properties of Gamma function

(1) $\Gamma(n+1) = n\Gamma n$

(2) $\Gamma(n+1) = n!$, where n is a positive integer

(3) $\Gamma(n) = 2\displaystyle\int_0^\infty e^{-x^2} x^{2n-1} dx$

(4) $\dfrac{\Gamma n}{t^n} = \displaystyle\int_0^\infty e^{-tx} x^{n-1} dx$

Properties of Gamma function

(1) $\Gamma(n+1) = n\Gamma n$

(2) $\Gamma(n+1) = n!$, where n is a positive integer

(3) $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$

(4) $\dfrac{\Gamma n}{t^n} = \int_0^\infty e^{-tx} x^{n-1} dx$

(5) $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$

Exercise

(1) $\int_{-\infty}^{\infty} e^{-k^2x^2}\,dx$

Exercise

(1) $\int_{-\infty}^{\infty} e^{-k^2x^2}\,dx$

(2) $\int_{0}^{\infty} e^{-x^3}\,dx$

Exercise

(1) $\displaystyle\int_{-\infty}^{\infty} e^{-k^2x^2}dx$

(2) $\displaystyle\int_{0}^{\infty} e^{-x^3}dx$

(3) $\displaystyle\int_{0}^{1} x^m \left(log\frac{1}{x}\right)^n dx$

Exercise

(1) $\displaystyle\int_{-\infty}^{\infty} e^{-k^2x^2}\,dx$

(2) $\displaystyle\int_{0}^{\infty} e^{-x^3}\,dx$

(3) $\displaystyle\int_{0}^{1} x^m \left(log\frac{1}{x}\right)^n dx$

Beta Function

- Definition:

  The Beta function denoted by $\beta(m, n)$ or $B(m, n)$ is defined as

  $$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, (m > 0, n > 0)$$

Properties of Beta Function

(1) $B(m,n) = B(n,m)$

Properties of Beta Function

(1) $B(m, n) = B(n, m)$

(2) $B(m, n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta \, cos^{2n-1}\theta \, d\theta$

Properties of Beta Function

(1) $B(m,n) = B(n,m)$

(2) $B(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\, cos^{2n-1}\theta\, d\theta$

(3) $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Properties of Beta Function

(1) $B(m,n) = B(n,m)$

(2) $B(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta \, cos^{2n-1}\theta \, d\theta$

(3) $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

(4) $B(m,n) = \int_0^{1} \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx$

Exercise

Ex. Prove that $\displaystyle\int_0^{\frac{\pi}{2}} sin^p\theta\, cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Exercise

Ex. Prove that $\displaystyle\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}dx = \frac{\beta(m,n)}{a^n b^m}$

Relation between Beta and Gamma functions

- $\beta(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Exercise

Ex. Prove that $\int_0^{\frac{\pi}{2}} sin^p\theta\, cos^q\theta\, d\theta = \frac{1}{2}\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q+2}{2})}$

Ex. Prove that $\displaystyle\int_0^{\frac{\pi}{2}} sin^p\theta\, cos^q\theta\, d\theta = \frac{1}{2}\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q+2}{2})}$

Ex. Prove that: $B(m,n) = B(m,n+1) + B(m+1,n)$