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Differentiation Under Integral sign

FE-SEM-I(CBCS)

DUIS: If some definite integral satisfies some definite conditions then we can differentiate those functions under integral sign, is called DUIS

- I(α) = $\int_a^b f(x,\alpha)dx$ where x is variable and α is parameter
- $\frac{dI}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha} dx$
- Exa. Using Rule of DUIS prove that $\int_0^\infty \frac{e^{-\alpha x}.sinx}{x} dx = cot^{-1}\alpha$
- Solution : Let , I(α) = $\int_0^\infty \frac{e^{-\alpha x}.sinx}{x} dx$
- By using rule of DUIS $\frac{dI}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha} dx$
- $\frac{dI}{d\alpha} = \int_0^\infty \frac{\partial}{\partial \alpha} \frac{e^{-\alpha x}.sinx}{x} dx = \int_0^\infty (-x) \frac{e^{-\alpha x}.sinx}{x} dx$
- = -$\int_0^\infty \frac{e^{-\alpha x}.sinx}{1} dx$ = - $\frac{e^{-\alpha x}}{\alpha^2+1}[-\alpha sinx - cosx]._{x=0}^{\infty}$
- $\frac{dI}{d\alpha}$ =-[0-($\frac{1}{\alpha^2+1}(0-1)]$

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$\frac{dI}{d\alpha}$=-[0-($\frac{1}{\alpha^2+1}(0-1)$]

$\frac{dI}{d\alpha}$=-($\frac{1}{\alpha^2+1}$]

- Integrating w.r.to α
- I(α) $= -tan^{-1}(\alpha) + c$
- To find c put α=0 in above equation
- I(0) $= \ -tan^{-1}(0) + c$ = c
- But I(α) $= \int_0^\infty \frac{e^{-\alpha x}.sinx}{x}dx$
- I(0)= $\int_0^\infty \frac{.sinx}{x}dx$= $\frac{\pi}{2}$
- I(α) $= \ -tan^{-1}(\alpha) + \frac{\pi}{2}$=$cot^{-1}(\alpha)$

Exa. Using Rule of DUIS prove that $\int_0^{\infty} \frac{\log(1+ax^2)}{x^2} dx = \pi\sqrt{a}(a > 0)$

- Solution : Let I(a) = $\int_0^{\infty} \frac{\log(1+ax^2)}{x^2} dx$
- By using rule of DUIS $\frac{dI}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha} dx$
- $\frac{dI}{da} = \int_0^{\infty} \frac{\partial}{\partial a} \frac{\log(1+ax^2)}{x^2}$ dx
- $= \int_0^{\infty} \frac{1}{1+ax^2} x^2 \frac{1}{x^2}$ dx
- $= \int_0^{\infty} \frac{1}{1+ax^2} dx = \frac{1}{a} \int_0^{\infty} \frac{1}{\sqrt{\frac{1}{a}}^2 + x^2} dx$
- $= \frac{1}{a}\sqrt{a}\ [\tan^{-1} \frac{x}{1/\sqrt{a}}]_{x=0}^{\infty}$
- $\frac{dI}{da} = \frac{1}{\sqrt{a}}(\frac{\pi}{2} - 0)$

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$$\frac{dI}{da} = \frac{1}{\sqrt{a}}\left(\frac{\pi}{2} - 0\right)$$

integrating w.r.to a

- I(a) = $\frac{\pi}{2} 2\sqrt{a}$ +c
- To find c put a =0
- I(0) = $\frac{\pi\sqrt{a}}{1}.o$ +c
- I(a) = $\int_0^\infty \frac{\log(1+ax^2)}{x^2} dx$, I(0) = $\int_0^\infty \frac{\log(1+0)}{x^2} dx$=0
- 0=c
- I(a) = $\pi\sqrt{a}$

DUIS Examples with one variable and two parameters

- I$(\alpha) = \int_a^b f(x,\alpha,\beta)dx$ where x is variable and α, $(\beta)$is parameter
- $\frac{dI}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha,\beta)}{\partial\alpha} dx$
- $\frac{dI}{d\alpha}$ = value
- Integrating w.r.to α
- I(α)= value + c
- To find I put α = (β)
- Have value of c
- I(α)= value

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Exa. Using Rule of DUIS prove that

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x} sinmxdx = \tan^{-1}(\frac{b}{m}) - \tan^{-1}(\frac{a}{m})$$

- Solution : Let I(a) = $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x} sinmxdx$
- $\frac{dI}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha,\beta)}{\partial \alpha} dx$….By using Rule of DUIS
- $\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a} \frac{e^{-ax}-e^{-bx}}{x} sinmxdx$
- $\frac{dI}{da} = \int_0^\infty \frac{e^{-ax}(-x)}{x} sinmxdx$= $\int_0^\infty \frac{e^{-ax}}{-1} sinmxdx$
- $\frac{dI}{da}$= -$[\frac{e^{-ax}}{a^2+m^2}(-asinmx - mcosmx)]_{x=0}^{\infty}$
- $\frac{dI}{da}$= -[0- $\frac{1}{a^2+m^2}(0 - m)$
- $\frac{dI}{da}$= - $\frac{m}{a^2+m^2}$

$\frac{dI}{da}$ = - $\frac{m}{a^2+m^2}$

integrating w.r.to a

- I(a) = - $\frac{m}{m}\tan^{-1}(\frac{a}{m}) + c$
- To find c put a=b in above equation
- I(b) = - $\tan^{-1}(\frac{b}{m}) + c$
- But, I(a) = $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x} sinmxdx$……put a = b to find I(b)
- I(b)=0
- 0= - $\tan^{-1}(\frac{b}{m}) + c$, C = $\tan^{-1}(\frac{b}{m})$
- I(a) = - $\tan^{-1}(\frac{a}{m}) + c$ = $\tan^{-1}(\frac{b}{m})$-$\tan^{-1}(\frac{a}{m})$-----------proved

Exa. Using Rule of DUIS prove that

$$\int_0^\infty cos\beta x \frac{e^{-ax}-e^{-bx}}{x}dx = \frac{1}{2}\log(\frac{b^2+\beta^2}{a^2+\beta^2})$$

- Solution : Let I(a) = $\int_0^\infty cos\beta x \frac{e^{-ax}-e^{-bx}}{x}dx$
- $\frac{dI}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha,\beta)}{\partial \alpha}dx$....By using Rule of DUIS
- $\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a} cos\beta x \frac{e^{-ax}-e^{-bx}}{x}dx$
- $\frac{dI}{da} = \int_0^\infty \frac{e^{-ax}(-x)}{x} cos\beta x dx = \int_0^\infty \frac{e^{-ax}}{-1} cos\beta x dx$
- $\frac{dI}{da}$=- $[\frac{e^{-ax}}{a^2+\beta^2}(-acos\beta x + \beta sin\beta x)]_{x=0}^\infty$
- $\frac{dI}{da}$= -[0- $\frac{1}{a^2+\beta^2}(-a)$]
- $\frac{dI}{da} = \frac{-a}{a^2+\beta^2}$

$\frac{dI}{da} = \frac{-a}{a^2+\beta^2}$ integrating w.r.to a

- I(a) =- $\frac{1}{2}\log(a^2 + \beta^2)$+c
- To find c, put a =b
- I(b) =- $\frac{1}{2}\log(b^2 + \beta^2)$+c
- But, I(a) = $\int_0^\infty cos\beta x \frac{e^{-ax} - e^{-bx}}{x} dx$…put a =b
- I(b) = 0
- C= $\frac{1}{2}\log(b^2 + \beta^2)$
- I(a) = $\frac{1}{2}\log(a^2 + \beta^2)$+c = - $\frac{1}{2}\log(a^2 + \beta^2)$+ $\frac{1}{2}\log(b^2 + \beta^2)$
- I(a) = = $\frac{1}{2}\log(\frac{b^2+\beta^2}{a^2+\beta^2})$

Exa: Evaluate $\int_0^{\pi} \frac{dx}{a+bcosx}, a, b > 0$

Hence,deduce that $\int_0^{\pi} \frac{cosx\,.dx}{(a+bcosx)^2}, a, b > 0$

- Solution : I = $\int_0^{\pi} \frac{dx}{a+bcosx}, a, b > 0$
- Put $\tan\left(\frac{x}{2}\right) = t$, dx =$\frac{dt}{1+t^2}$, cosx = $\frac{1-t^2}{1+t^2}$
- When x= 0,t=0,and x= π,t= ∞
- I= $\int_0^{\infty} \frac{\frac{dt}{1+t^2}}{a+b[\frac{1-t^2}{1+t^2}]}, a, b > 0$
- I = $\int_0^{\infty} \frac{dt}{a(1+t^2)+b[1-t^2]]}, = \int_0^{\infty} \frac{dt}{(a+b)+[a-b]t^2}, = \frac{1}{(a-b)} \int_0^{\infty} \frac{dt}{(a+b)/(a-b)+t^2},$
- I = $\frac{1}{\sqrt{\frac{a+b}{a-b}}(a-b)} [\tan^{-1} \frac{x}{\sqrt{\frac{a+b}{a-b}}}]._0^{\infty} = \frac{\pi}{2\sqrt{a^2-b^2}}$

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$$I = \int_0^{\pi} \frac{dx}{a+bcosx}, a, b > 0$$

$$I = \frac{\pi}{2\sqrt{a^2-b^2}}$$

- $\int_0^{\pi} \frac{dx}{a+bcosx} = \frac{\pi}{2\sqrt{a^2-b^2}}$
- Differentiating w.r.to b
- $\int_0^{\pi} \frac{\partial}{\partial b} \frac{dx}{a+bcosx} = \frac{d}{db}\left[\frac{\pi}{2\sqrt{a^2-b^2}}\right]$
- $\int_0^{\pi} \frac{\partial}{\partial b} \frac{dx}{a+bcosx} = \int_0^{\pi} \frac{(-1)dx}{(a+bcosx)^2} . cosx$
- $\frac{d}{db}\left[\frac{\pi}{2\sqrt{a^2-b^2}}\right] = \frac{\pi}{2}\left(\frac{-1}{2}\right)(a^2 - b^2)^{-3/2}$(-2b)
- $= \frac{\pi b}{2(a^2-b^2)^{3/2}}$
- $\int_0^{\pi} \frac{(-1)dx}{(a+bcosx)^2} . cosx = \frac{\pi b}{2(a^2-b^2)^{3/2}}$
- $\int_0^{\pi} \frac{cosx .dx}{(a+bcosx)^2} =$ - $\frac{\pi b}{2(a^2-b^2)^{3/2}}$

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Exa: Evaluate $\int_0^{\pi} \frac{dx}{a+bcosx}, a, b > 0$

Hence,deduce that $\int_0^{\pi} \frac{cosx\ .dx}{(a+bcosx)^2}, a, b > 0$

Similar pattern of examples

- Exa.Evaluate $\int_0^{\pi} \frac{dx}{a-cosx}, a > 0$

  Hence, deduce $\int_0^{\pi} \frac{dx}{(a-cosx)^2}, a > 0$

- Exa. Evaluate $\int_0^{\pi/2} \frac{dx}{a^2sin^2x+b^2cos^2x},$

  Hence, deduce $\int_0^{\pi/2} \frac{cosx\ .dx}{(a^2sin^2x+b^2cos^2x)^2}$