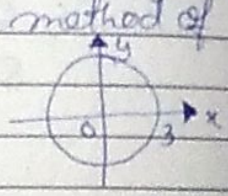
Tutorial No: 2

1. Evaluate $\int_c \frac{e^z}{(z-1)(z-4)} dz$, where c is the circle $|z|=2$ by Cauchy's Integral formula.

Sol^n: The circle $|z|=3$ has centre at $(0,0)$ and radius 3. The points $z=1$ & $z=2$ lie inside the circle. Hence, $f(z)$ is not analytic in C. Hence we use the method of partial fractions & write



$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

and write $f(z) = e^{2z}$ which is analytic in C.

$$\therefore \int_C \frac{e^{2z}}{(z-i)(z-2)} dz = \int_C \frac{e^{2z}}{z-2} dz - \int_C \frac{e^{2z}}{z-1} dz$$

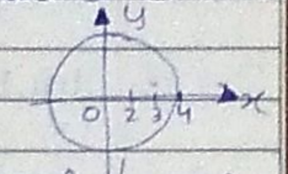
$= 2\pi i f(2) - 2\pi i f(1)$ where $f(z) = e^{2z}$

$= 2\pi i e^4 - 2\pi i \, e^2 = 2\pi i e^2 (e^2 - 1)$

2. Evaluate $\int_c \frac{\cos \pi z^2 + \sin \pi z^2}{(z-2)(z-3)}$, where C is the circle $|z|=4$.

Sol^n: $|z|=4$ is a circle with centre at the origin and radius $=4$. Hence $z=2, z=3$ lie inside the circle. Hence, the given function $f(z)$ is not analytic in C. Hence we use the method of partial fractions & write

$$\frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2}$$



& write $f(z) = \sin \pi z^2 + \cos \pi z^2$ which is analytic in C. By Cauchy's integral formula,

$$\therefore \int_C \frac{e^{2z}}{(z-\ldots)} \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz$$

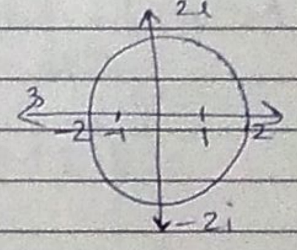
$$= \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-3} dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

$= 2\pi i f(3) + 2\pi i f(2)$ where $f(z) = \sin \pi z^2 + \cos \pi z^2$

$= 2\pi i (\sin 9\pi + \cos 9\pi) - 2\pi i (\sin 4\pi + \cos 4\pi)$

$= -2\pi i - 2\pi i = -4\pi i$

3) Evaluate $\int_C \frac{3z^2+z+1}{(z^2-1)(z+3)} dz$, where C is the circle $|z|=2$.



Ans. $I = \int_C \frac{3z^2+z+1}{(z^2-1)(z+3)}$, $(|z|=2)$

$\because z^2-1=0$ gives $(z+1)(z-1)=0$

$\therefore z = -1, 1, 3$ are the simple poles

Also 'c' is $|z|=2$ i.e. circle with $C=(0,0)$ & $R=2$, As $-1, 1$ lies inside circle. By partial fraction

$$\frac{1}{(z+1)(z-1)} = \frac{A}{(z+1)} + \frac{B}{(z-1)} = \frac{2}{(z+1)} - \frac{2}{(z-1)}$$

$$= 2\int_C \frac{(3z^2+z+1)/(z+3)}{z+1} - 2\int_C \frac{(3z^2+z+1)/(z+3)}{z-1}$$

By Cauchy's ~~theorem~~ Integral formula,

$$I = 2\left[2\pi i\left(\frac{3}{2}\right) - 4\pi i\left(\frac{5}{4}\right)\right]$$

$$= 4\pi i \times \frac{3}{2} - 4\pi i \times \frac{5}{4}$$

$$= 6\pi i - 5\pi i$$

$$\therefore I = \pi i$$

4) If $f(\xi) = \int \frac{4z^2+z+5}{z-\xi}\, dz$, where c is the ellipse, find the values of $f(i)$, $f'(-1)$, $f''(-i)$ & $f(3)$.

Ans: i) The point $z = i$ i.e., $(0,1)$ lies inside the ellipse, $f(z) = 4z^2+z+5$ is analytic in & on C. Hence, by Cauchy's formula

$$f(i) = \int_C \frac{4z^2+z+5}{z-i}\, dz = 2\pi i\, \phi(z_0)$$

where, $\phi(z) = 4z^2+z+5$ & $z_0 = i$

$$\therefore f(i) = 2\pi i\,(4i^2+i+5)$$

$$= 2\pi i\,(-4+i+5) = 2\pi i\,(1+i)$$

(ii) The point $(-1,0)$ also lies inside the ellipse. Hence, we take $\phi(z) = 4z^2+z+5$, which is analytic in & on C.

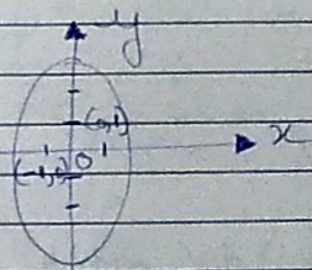
$$\therefore f(\xi) = \int \frac{4z^2+z+5}{z-\xi}\, dz = 2\pi i\,(4\xi^2+\xi+5)$$

$$\therefore f'(\xi) = 2\pi i\,(8\xi+1) \;\&\; f''(\xi) = 2\pi i \cdot 8$$

$$\therefore f'(-1) = 2\pi i\,[8(-1)+1] = -14\pi i$$

(iii) $f''(\xi) = 16\pi i \quad \therefore f''(-1) = 16\pi i$

(iv) The point $(3,0)$ lies outside the ellipse. Hence, $\int_C \frac{4z^2+z+5}{z-\xi}\, dz = 0$



5) Evaluate $\int_C \frac{e^{3iz}}{(z+\pi)^3}\, dz$, where c is a circle $|z-\pi| = 3.2$, with centre $(\pi, 0)$, and radius $= 3.2$.

Sol^n $I = \int_C \frac{e^{3iz}}{(z+\pi)^3}$, $C : |z-\pi| = 3.2$

$\therefore z = -\pi$ is a pole of order $n = 3$

Also 'c' is $|z-\pi| = 3.2$ i.e. circle with $C = (\pi, 0)$

& $r = 3.2$, $|-\pi-\pi| = |-2\pi| = 2\pi$

$= 2 \times 3.14 = 6.28 > 3.2$

$\therefore z = -\pi$ lies outside 'C'

By Cauchy's Integral formula

$$\int_C \frac{z\, e^{3iz}}{(z+\pi)^3} = 6$$

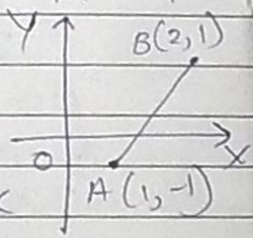
6) Evaluate $\int_{1-i}^{2+i} (ix+iy+1)\,dz$ along
i) straight line joining $(1-i)$ to $(2+i)$
ii) the curve $x = t+1,\ y = 2t^2-1$.

Ans: i) We first find the eqⁿ of the line through the given points $(1,-1)$ & $(2,1)$. The eqⁿ is $\frac{y+1}{-1-1} = \frac{x-1}{1-2}$

i.e. $\frac{y+1}{2} = \frac{x-1}{1}$

$\therefore y+1 = 2x-2 \quad \therefore y = 2x-3$

$\therefore dy = 2dx \quad \therefore dz = dx + i\,dy$
$= (1+2i)\,dx$



Hence, the integral becomes

$$\int_{1-i}^{2+i} (ix+iy+1)\,dz = \int_1^2 [ix + i(2x-3)+1](1+2i)\,dx$$

$$= (1+2i)\left[\frac{ix^2}{2} + i(x^2-3x) + x\right]_1^2$$

$$= (1+2i)\left\{\left[\frac{i4}{2} + i(4-6) + 2\right] - \left[\frac{i}{2} + i[1-3] + 1\right]\right\}$$

$$= (1+2i)\left(1+\frac{3i}{2}\right) = 1 + i + \frac{3i}{2} - \frac{3}{2} = -\frac{1}{2} + \frac{5i}{2}$$

$$= \frac{1+5i}{2}$$

(ii) $I = \int$ $x = t+1,\ y = 2t^2-1,\ z = (t+1) + i(2t^2-1)$
$\therefore dz = (1+4it)\,dt$ when $z = 1-i$,
$t = 0$ & when $z = 2+i$, $t = 1$

$$\therefore I = \int_0^1 [i(t+1) + i(2t^2-1) + 1](1+4it)\,dt$$

$$= \int_0^1 \left\{i\left(\frac{t^2}{2}+t\right) + i\left(\frac{2t^3}{3} - t\right) + t\right\}_0^1 + \left\{-4\left(\frac{t^3}{3} + \frac{t^2}{2}\right) - 4\left(\frac{t^4}{2} - \frac{t^2}{2}\right) + 2it^2\right\}_0^1$$

$$= \left[\left\{i\left(\frac{1}{2}+1\right) + i\left(\frac{2}{3}-1\right) + 1\right\}\right] + \left[\left\{-4\left(\frac{1}{3}+\frac{1}{2}\right) - 4\left(\frac{1}{2}-\frac{1}{2}\right) + 2i(1)^2\right\}\right]$$

$$= \left[i\left(\frac{3}{2}\right) + i\left(\frac{-1}{3}\right) + 1\right] + \left[\left\{-4\left(\frac{5}{6}\right) - 4(0) + 2i\right\}\right]$$

$$= \frac{3i}{2} - \frac{i}{3} + 2i + 1 - \frac{20}{6} = \frac{19i}{6} - \frac{7}{3}$$

7) Obtain the Taylor's or Laurent's series which represents the function

$$f(z) = \frac{1}{(1+z^2)(z+2)}$$

Ans. Let $f(z) = \frac{a}{z+2} + \frac{bz+c}{z^2+1}$

$\therefore \frac{1}{(z^2+1)(z+2)} = \frac{a(z^2+1)+(bz+c)(z+2)}{(z^2+1)(z+2)}$

$\therefore 1 = (a+b)z^2 + (2b+c)z + (a+2c)$

Equating the coefficients of like powers of z, on both sides

$a+b=0, \quad 2b+c=0 \quad \therefore a+2c=1$

$\therefore 2c-b=1$ i.e. $4c-2b=2$ and $2b+c=0$

$\therefore 5c=2$ i.e. $c=2/5$, $\therefore a=1-2c=1-(4/5)$

$= 1/5$

$\therefore b=-a=-1/5$

$\therefore f(z) = \frac{1}{5}\frac{1}{z+2} - \frac{z-2}{5(z^2+1)}$

$= \frac{1}{5}\left[\frac{1}{z+2} - \frac{z-2}{z^2+1}\right]$

Case(i): when $1<|z|<2$, ~~$|z|<2$ i.e. $\frac{|z|}{2}<1$ & $|z^2|>1$ i.e. $\frac{1}{|z^2|}<1$~~

~~Hence we~~

8) Find the Laurent's expansion for $f(z) = \frac{7z-2}{z^3-z^2-2z}$ about $z=1$

Ans: $\because f(z) = \frac{7z-2}{z^3-z^2-2z} = \frac{7z-2}{z(z-2)(z+1)}$

Let $\frac{7z-2}{z(z-2)(z+1)} = \frac{a}{z} + \frac{b}{(z-2)} + \frac{c}{(z+1)}$

$\therefore 7z-2 = a(z-2)(z+1) + bz(z+1) + cz(z-2$

when $z=0$, $-2=-2a$ $\therefore a=1$

when $z=-1$, $-9=3c$ $\therefore c=-3$

when $z=2$, $12=6b$ $\therefore b=2$

$\therefore \frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$

$\therefore f(z) = \frac{1}{(z-1)-1} + \frac{2}{(z-1)-3} - \frac{3}{z-1}$

case(i): when $|z-1|<1$, we write $f(z) = \frac{-3}{z-1} - \frac{1}{1-(z-}$

$- \frac{2}{3-(z-1)}$

when $|z-1|<$, clearly $|z-1|<3$

$\therefore f(z) = \frac{-3}{z-1} - [1-(z-1)]^{-1} - \frac{2}{3}\left\{1-\left(\frac{z-1}{3}\right.\right.$

$= \frac{-3}{z-1} - [1+(z-1)+(z-1)^2+(z-1)^3+\cdots] - \frac{2}{3}\left[1+\frac{(z-1)}{3}+\frac{(z-1)^2}{3^2}+\frac{(z-1)^3}{3^3}+\cdots\right]$

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$$= \frac{-3}{z-1} - \sum\left(1+\frac{2}{3^{n+1}}\right)(z-1)^n, \quad 0<|z-1|<1$$

case(ii): when $1<|z-1|<3$

$$f(z) = \frac{-3}{z-1} + \frac{1}{(z-1)\left(1-\frac{1}{z-1}\right)} - \frac{2}{3\left[1-\frac{(z-1)}{3}\right]}$$

$$= \frac{-3}{z-1} + \frac{1}{z-1}\left(1-\frac{1}{z-1}\right)^{-1} - \frac{2}{3}\left(1-\frac{\cancel{z}-1}{3}\right)^{-2}$$

$$= \frac{-3}{z-1} + \frac{1}{z-1}\left[1+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^2}+\cdots\right] - \frac{2}{3}\left[1+\frac{(z-1)}{3} + \frac{(z-1)^2}{3^2}+\frac{(z-1)^3}{3^3}+\cdots\right]$$

$$= \frac{-3}{z-1} + \sum\frac{1}{(z-1)^{n+1}} - 2\sum\frac{(z-1)^n}{3^{n+1}}; \quad 1<|z-1|<3$$

case(iii): when $|z-1|>3$, we write

$$f(z) = \frac{-3}{z-1} + \frac{1}{(z-1)\left(1-\frac{1}{z-1}\right)} + \frac{2}{(z-1)\left(1-\frac{3}{z-1}\right)}$$

when $|z-1|>3$, $|z-1|>1$

$$\therefore f(z) = \frac{-3}{z-1} + \frac{1}{z-1}\left(1-\frac{1}{z-1}\right)^{-1} + \frac{2}{z-1}\left(1-\frac{3}{z-1}\right)^{-1}$$

$$= \frac{-3}{z-1} + \frac{1}{z-1}\left[1+\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+\cdots\right] + \frac{2}{z-1}\left[1+\frac{3}{z-1}+\frac{3^2}{(z-1)^2}+\frac{3^3}{(z-1)^3}+\cdots\right]$$

$$\therefore f(z) = \frac{-3}{z-1} + \sum\frac{1+2\cdot 3^{n-1}}{(z-1)^n}; \quad |z-1|>3.$$

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10) Find the Taylor's & Laurents series which represent the function $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ about $z=0$.

Ans: ∵ the degree of the numerator is equal to the degree of the denominator we first divide the numerator by the denominator.

$$\therefore f(z) = \frac{z^2-1}{(z^2+5z+6)} = 1 - \frac{5z+7}{z^2+5z+6}$$

Let $\dfrac{-5z-7}{z^2+5z+6} = \dfrac{a}{z+3} + \dfrac{b}{z+2}$

$\therefore -5z-7 = a(z+2)+b(z+3)$

when $z=-2$; $b=3$;

when $z=-3$, $a=-8$

$$\therefore f(z) = \frac{z^2-1}{z^2+5z+6} = 1 - \frac{8}{z+3} + \frac{3}{z+2} \quad \text{—①}$$

Case (i): when $|z|<2$, we write

$$f(z) = 1 - \frac{8}{3[1+(z/3)]} + \frac{3}{2[1+(z/2)]}$$

when $|z|<2$, clearly $|z|<3$

$$\therefore f(z) = 1 - \frac{8}{3}[1+(z/3)]^{-1} + \frac{3}{2}[1+(z/2)]^{-1}$$

$$= 1 - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \cdots\right] + \frac{3}{2}$$

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$$\left[1-\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right]$$

Case (ii): when $2<|z|<3$, we write

$$f(z)=1-\frac{8}{3[1+(z/3)]}+\frac{3}{z[1+(2/z)]}$$

$$=1-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}$$

$$=1-\frac{8}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\cdots\right]+\frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\cdots\right]$$

Case (iii): when $|z|>3$, we write

$$f(z)=1-\frac{8}{z[1+(3/z)]}+\frac{3}{z[1+(2/z)]}$$

when $|z|>3$, clearly $|z|>2$

$$\therefore f(z)=1-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}$$

$$=1-\frac{8}{z}\left[1-\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\cdots\right]$$

$$+\frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]$$