FE-SEM-II(CBCS)-C-SCHEME

DOUBLE INTEGRATION

Module at a glance: A double integration can be evaluated by two successive integration in the following way

- The integration in the rectangular bracket will be evaluated first then the other integration with respect to x is evaluated.
  Important steps to follow while doing double integration:
- Draw the rough diagram of all the curves which are the boundary of the region.
- Find the point of intersection of all the curves for finding the region of intersection.
- Decide the strip (Horizontal or Vertical) this decision is based on the shape of region or sometimes on the functions whose integration is to be evaluated.
- If the strip is taken then limit is always written down to upward where as for horizontal strip the limit is always left to right.
- The inside integral is always on the strip so its limit is generally written using the equation of the curve from where the strip start and where it ends.
- The outer integral's limit is always constant because once the strip is bounded between two curves then we can move it freely from one point to other point to cover the entire region of integration.

Double Integrals

What is a double integral?

Recall that a single integral is something of the form

$$\int_a^b f(x)\,dx$$

A double integral is something of the form

$$\iint_R f(x,y)\,dxdy$$

where R is called the region of integration and is a region in the XY plane.

We can evaluate double integrals in two steps:

First evaluate the inner integral, and then plug this solution into the outer integral and solve that.

Outer Integral   Inner Integral

$$\int \int f(x,y)\,dxdy$$

Note:

1. **Limits of Outer Integral are always constant**
2. **Limits of Inner Integral are constant or function of only one variable (either *x* or *y*)**

- $$\int_{x=a}^{x=b}\left[\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dy\right]dx$$

EXAMPLE Evaluating a Double Integral

Calculate $\iint_R f(x,y)\,dA$ for

$$f(x,y) = 1 - 6x^2y \qquad \text{and} \qquad R:\quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution By Fubini's Theorem,

$$\iint_R f(x,y)\,dA = \int_{-1}^{1}\int_0^2 (1 - 6x^2y)\,dx\,dy = \int_{-1}^{1}\left[x - 2x^3y\right]_{x=0}^{x=2} dy$$

Reversing the order of integration gives the same answer:

$$\int_0^2\int_{-1}^{1}(1 - 6x^2y)\,dy\,dx = \int_0^2\left[y - 3x^2y^2\right]_{y=-1}^{y=1} dx$$

$$= \int_0^2 [(1 - 3x^2) - (-1 - 3x^2)]\,dx$$

$$= \int_0^2 2\,dx = 4.$$

3)EVALUATE

- $\int_{-1}^{1}\int_{-2}^{2}(x^2 - y^2)\,dy\,dx$

$= \int_{x=-1}^{1}[x^2y - \frac{y^3}{3}]_{y=-2}^{2}\,dx$

$=\int_{x=-1}^{1}[x^2y - \frac{y^3}{3}]_{y=-2}^{2}\,dx$

$= \int_{x=-1}^{1}[4x^2 - \frac{16}{3}]\,dx$

$= [4\frac{x^3}{3} - \frac{16}{3}x]_{x=-1}^{1}$

$= [4\frac{(1-(-1))}{3} - \frac{16}{3}(1-(-1))]$

$= \frac{-24}{3} = -8$

Exa:- Evaluate $\int_0^1 dx \int_0^x e^{y/x}\, dy$

Sol^n:- Let $I = \int_{x=0}^{1} dx \left\{ \int_{y=0}^{x} e^{y/x}\, dy \right\}$

$$I = \int_{x=0}^{1} dx \left[\frac{e^{y/x}}{\frac{1}{x}(1)} \right]_{y=0}^{x}$$

$$= \int_{x=0}^{1} x \cdot dx \left[e^{\cancel{x}/\cancel{x}} - e^{0} \right]$$

$$= \int_{x=0}^{1} x \cdot dx \, (e-1)$$

$$= (e-1)\left[\frac{x^2}{2}\right]_{x=0}^{1}$$

$$= (e-1)\left[\frac{1}{2} - 0\right] = \boxed{\frac{(e-1)}{2}}$$

$$\int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\,dxdy$$

Step (i): First integrate $f(x,y)$ with respect to y (keeping x constant)

Step (ii) : Integrate the result obtained in step (i) with respect to x

Here order of integration is first w.r.t y and then w.r.t. x (YX order)

Note that order of integration is decided by the limits of inner integral and not by $dxdy$ or $dydx$

For Example: Evaluate $\int_0^2 \int_0^{x^2} xydxdy$

Solution: Note that upper limit of inner integral is function of x. Therefore, order of integration is first w.r.t y and then w.r.t. x.

$$I = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} xydxdy$$

Place the inner integral in parentheses so you can better see what you're working with:

$$I = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=x^2} xydy \right) dx$$

Now focus on what's inside the parentheses. For the moment, you can ignore the rest. Your integration variable is y, so treat the variable x as a constant, moving it outside the inner integral:

$$I = \int_{x=0}^{x=2} x \left(\int_{y=0}^{y=x^2} ydy \right) dx$$

$$= \int_{x=0}^{x=2} x \left(\frac{y^2}{2} \right)_0^{x^2} dx$$

$$= \frac{1}{2} \int_{x=0}^{x=2} x\left(x^4 - 0\right) dx \;\; = \frac{1}{2} \int_{x=0}^{x=2} x^5 dx$$

Case II: Limits of Inner Integral are function of y. That is,

$$\int_{c}^{d}\int_{k_1(y)}^{k_2(y)} f(x,y)\,dxdy$$

Step (i): First integrate $f(x,y)$ with respect to x (keeping y constant)

Step (ii) : Integrate the result obtained in step (i) with respect to y

Here order of integration is first w.r.t x and then w.r.t y(XY order)

For Example: Evaluate $\int_{0}^{1}\int_{0}^{\sqrt{1+y^2}} \frac{1}{1+x^2+y^2}dxdy$

Solution: Note that upper limit of inner integral is function of y. Therefore, order of integration is first w.r.t x and then w.r.t. y.

$$I=\int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{1+y^2}} \frac{1}{1+x^2+y^2}dxdy$$

Place the inner integral in parentheses so you can better see what you're working with:

$$I=\int_{y=0}^{y=1}\left(\int_{x=0}^{x=\sqrt{1+y^2}} \frac{1}{1+x^2+y^2}dx\right)dy$$

Now focus on what's inside the parentheses. For the moment, you can ignore the rest. Your integration variable is x, so treat the variable y as a constant:

$$I=\int_{y=0}^{y=1}\left(\int_{x=0}^{x=\sqrt{1+y^2}} \frac{1}{(1+y^2)+x^2}dx\right)dy$$

$$=\int_{y=0}^{y=1}\left[\frac{1}{\sqrt{1+y^2}}\tan^{-1}\left(\frac{x}{\sqrt{1+y^2}}\right)\right]_0^{\sqrt{1+y^2}}dy \qquad \left\{\text{Using } \int\frac{1}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right\}$$

$$=\int_{y=0}^{y=1}\frac{1}{\sqrt{1+y^2}}\left[\tan^{-1}\left(\frac{\sqrt{1+y^2}}{\sqrt{1+y^2}}\right)-\tan^{-1}(0)\right]dy$$

$$=\int_{y=0}^{y=1}\frac{1}{\sqrt{1+y^2}}\left[\tan^{-1}(1)-0\right]dy=\frac{\pi}{4}\int_{y=0}^{y=1}\frac{1}{\sqrt{1+y^2}}dy$$

$$=\frac{\pi}{4}\left[\log\left(y+\sqrt{1+y^2}\right)\right]_0^1=\frac{\pi}{4}\left[\log\left(1+\sqrt{1+1^2}\right)-\log(1)\right]=\frac{\pi}{4}\log(1+\sqrt{2})$$

Chapter-9	Double Integrals
Topic-9.2-A	

Type 2: Evaluation of Double Integral over given region

Working Rule to Evaluate Double Integrals when limits of integral are not given

1. Draw all the curves, find points of intersection and locate region of integration.

2. Decide the order of integration i.e. XY or YX

3. Function $f(x, y)$ will decide the order of integration. If integration of $f(x, y)$ w.r.t x (keeping y constant) is easy order is XY, otherwise order is YX.

Case 1: If order of integration is YX, then consider a strip parallel to Y axis.

(a) This strip is an imaginary strip of variable length which moves from left to right end of the region so that the entire region of integration is swept.

(b) During its journey from to left to right end, lower and upper end of the strip should touch single curve. Otherwise partition the region of integration.

(c) If lower end of the strip touches the curve $y = h(x)$, upper end touches the curve $y = k(x)$, X co-ordinate of a point on extreme left of the region is a and X coordinate of a point on extreme right of the region is b then

$$I = \int_{x=a}^{x=b} \int_{y=h(x)}^{y=k(x)} f(x, y) dxdy$$

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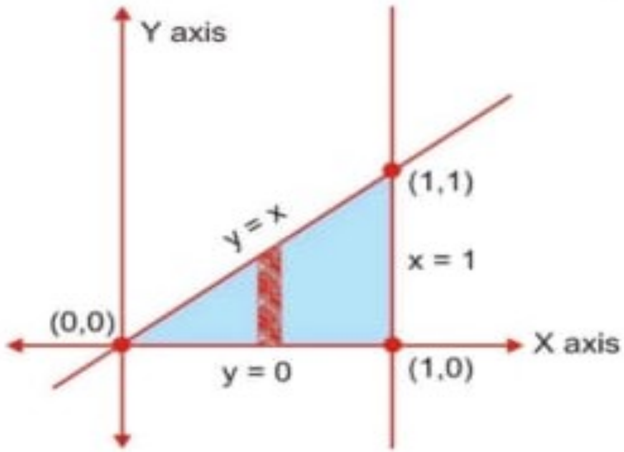
$$I = \int_{x=a}^{x=b} \int_{y=h(x)}^{y=k(x)} f(x,y)dxdy$$

by the straight lines $y=0$, $x=1$ and $x=y$

Solution: $y=0$ is X axis, $x=1$ is a straight line parallel Y axis and passing through $(1,0)$ and $x=y$ is a straight line passing through $(0,0)$ and $(1,1)$



Integration is easy w.r.t y. So, let's evaluate using order YX. That is first integrate w.r.t y (keeping x constant) and then integrate w.r.t x.

To write limits, consider a strip **parallel to Y axis**, which moves from left to right end of the region of integration without changing the curves.

Lower end of the strip touches X axis ($y=0$). Therefore, lower limit of inner integral is $y=0$

Upper end of the strip touches the straight line $y=x$. Therefore, upper limit of inner integral is $y=x$.

Value of x at left end of the region is 0. Therefore, lower limit of outer integral is $x=0$

Value of x at right end of the region is 1 . Therefore, upper limit of outer integral is $x=1$.

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=x} e^{y/x}\,dy\,dx = \int_{x=0}^{x=1} \left[\int_{y=0}^{y=x} e^{y/x}\,dy \right] dx$$

$$= \int_{x=0}^{x=1} \left[x e^{y/x} \right]_{y=0}^{y=x} dx \qquad \left\{ \because \int e^{ay}\,dy = \frac{e^{ay}}{a}, \text{ where } a = \frac{1}{x} \right\}$$

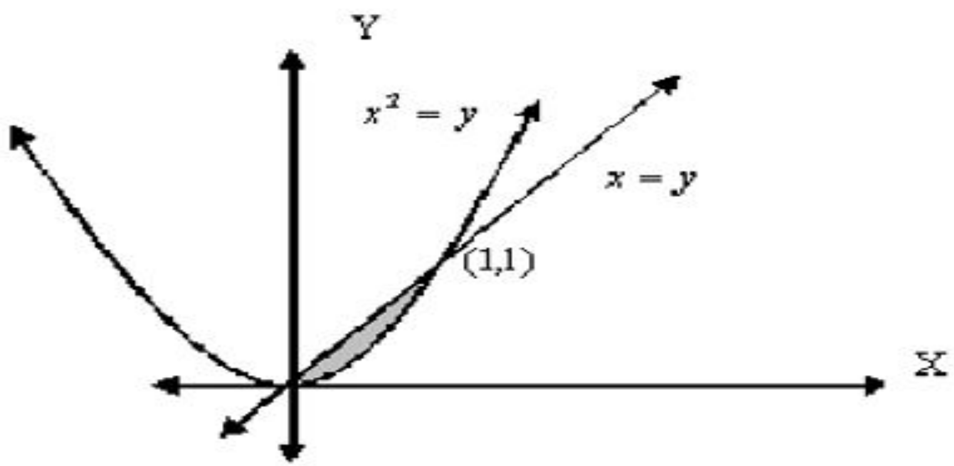
$$= \int_{x=0}^{x=1} [xe - x]\,dx$$

$$= \left[e\frac{x^2}{2} - \frac{x^2}{2} \right]_{x=0}^{x=1} = \frac{e}{2} - \frac{1}{2} = \frac{e-1}{2}$$

$$\therefore \boxed{\iint_R e^{y/x}\,dx\,dy = \frac{e-1}{2}}$$

1.Evaluate $\iint xy(x+y)\,dx\,dy$ over the region bounded by $x^2 = y, y = x$.

Solution:



Solution:

$$\iint xy(x+y)\,dx\,dy = \int_0^1 x\left(x\frac{y^2}{2} + \frac{y^3}{3}\right)_{x^2}^{x} dx$$

$$= \int_0^1 x\left(\frac{x^3}{2} + \frac{x^3}{3} - \frac{x^5}{2} - \frac{x^6}{3}\right)dx$$

$$= \int_0^1 x\left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right)dx$$

$$= \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}$$

Change the order of integration

$$\iint f(x,y)\,dxdy \neq \iint f(x,y)\,dydx \text{ .... always}$$

Change of the order of Int.

$$\iint_R f(x,y)\,dydx \Longleftrightarrow \iint_R f(x,y)\,dxdy$$

$$I = \iint_R f(x,y)\,dy\,dx = \int_{x=a}^{b} \int_{y=y_1}^{y_2} f(x,y)\,dy\,dx$$

Change of the order of Int.

$$\iint_R f(x,y)\,dy\,dx \Longleftrightarrow \iint_R f(x,y)\,dx\,dy$$

$$I = \iint_R f(x,y)\,dy\,dx = \int_{x=a}^{b}\int_{y=y_1}^{y_2} f(x,y)\,dy\,dx$$

Step I :- Curves are

$x=a,\ x=b,\ y=y_1,\ y=y_2$

Step II :- Draw curves & fix Region

Step III :- Strip parallel to x-axis.

Step IV : Find limits of x and y

$$\iint_R f(x,y)\,dy\,dx = \int_{y=c}^{d}\int_{x=x_1}^{x_2} f(x,y)\,dx\,dy$$

Exa:- Change the order of Int. and evaluate $\int_1^2 \int_1^{x^2} \frac{x^2}{y}\,dy\,dx$

Solution:- Let, $I = \int_1^2 \int_1^{x^2} \frac{x^2}{y}\,dy\,dx$

$$I = \int_{x=1}^{2} \left\{ \int_{y=1}^{x^2} \frac{x^2}{y}\,dy \right\} dx$$

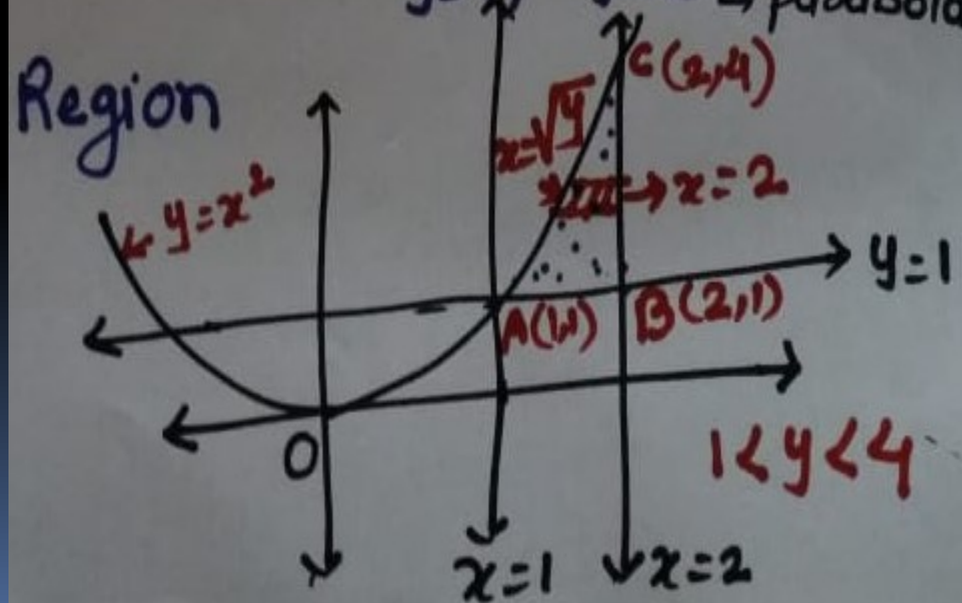
Curve are $x=1$, $x=2$

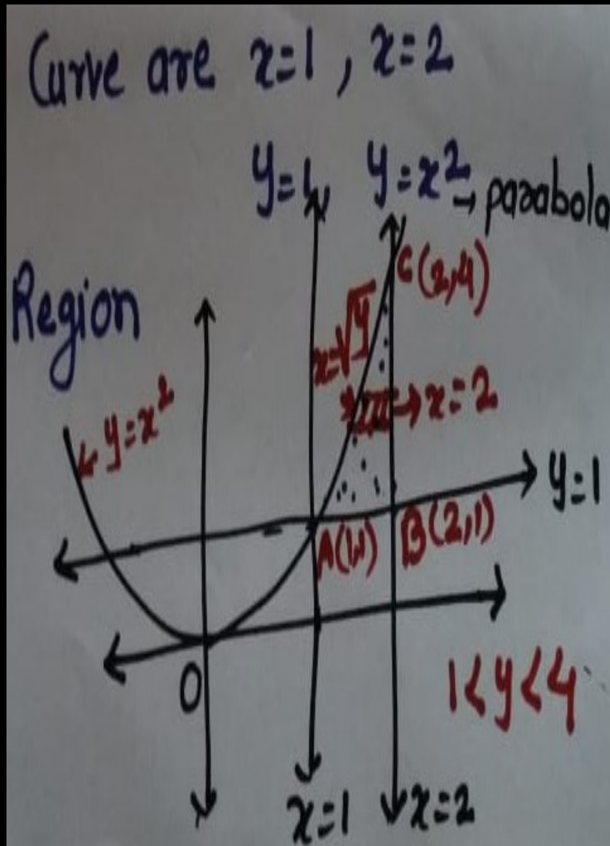
$y=1$, $y=x^2 \rightarrow$ parabola

Curve are $x=1$, $x=2$

$y=1$, $y=x^2 \rightarrow$ parabola

Region





Limits are

$$\sqrt{y} < x < 2$$

$$1 < y < 4$$

$$I = \int_{x=1}^{2}\int_{y=1}^{x^2} \left(\frac{x^2}{y}\right) dy\, dx$$

$$= \int_{y=1}^{4}\left\{\int_{x=\sqrt{y}}^{2} \frac{1}{y}\, x^2 . dx\right\} dy$$

$$I = \int_{y=1}^{4} \frac{1}{y}\left[\frac{x^3}{3}\right]_{x=\sqrt{y}}^{2} dy = \frac{1}{3}\int_{y=1}^{4}\frac{1}{y}\left[8 - y^{3/2}\right]$$

$$I = \frac{1}{3} \int_{y=1}^{4} \left[\frac{8}{y} - y^{1/2}\right] dy$$

$$I = \frac{1}{3}\left[8\log y - \frac{y^{3/2}}{3/2}\right]_{y=1}^{4}$$

$$= \frac{8}{3} \log 4 - 0 - \frac{2}{9}\left[4^{3/2} - 1\right]$$

$$= \frac{8}{3}\, 2\log 2 - \frac{2}{9}[7]$$

$$= \boxed{\frac{2}{9}\left[24\log 2 - 7\right]}$$

Evaluate $\int_0^1 dx \int_1^\infty e^{-y} . y^x . \log y \, dy$

Sol^n: - Let, $I = \int_{x=0}^{1} dx \int_{y=1}^{\infty} e^{-y} . \log y \; y^x . dy$

$$I = \int_{y=1}^{\infty} dy \int_{x=0}^{1} e^{-y} . \log y \; y^x \, dx$$

$$I = \int_{y=1}^{\infty} e^{-y} . \log y \, dy \int_{x=0}^{1} y^x . dx$$

$$I = \int_{y=1}^{\infty} e^{-y} . \log y \, dy \left[\frac{y^x}{\log y}\right]_{x=0}^{1}$$

$$I = \int_{y=1}^{\infty} e^{-y} . \log y . dy . \frac{1}{\log y} [y - 1]$$

$$I = \int_{y=1}^{\infty} (y-1) e^{-y} \, dy$$

$$I = \left[(y-1)\frac{e^{-y}}{(-1)} - (1)\frac{e^{-y}}{(-1)^2}\right]_1^{\infty}$$

$$= \left[0 - 0 + \frac{e^{-1}}{(1)}\right] = \boxed{\frac{1}{e}}$$

NOTE: To Evaluate Integral of the form

$$\iint \frac{f(x)}{\sqrt{(a-y)(y-x)}}dxdy \quad \text{Or} \quad \iint \frac{g(y)}{\sqrt{(a-x)(x-y)}}dxdy$$

Case 1: If $I = \iint \frac{f(x)}{\sqrt{(a-y)(y-x)}}dxdy$, integral can be evaluated easily using order YX. That is first integrating w.r.t y and then integrating w.r.t x.

Write $I = \int f(x)\left[\int \frac{1}{\sqrt{(a-y)(y-x)}}dy\right]dx$

Evaluate inner integral $I_1 = \int \frac{1}{\sqrt{(a-y)(y-x)}}dy$ using substitution $y - x = t^2$

Case 2: If $\iint \frac{g(y)}{\sqrt{(a-x)(x-y)}}dxdy$, integral can be evaluated easily using order XY. That is first integrating w.r.t x and then integrating w.r.t y.

Write $I = \int g(y)\left[\int \frac{1}{\sqrt{(a-x)(x-y)}}dx\right]dy$

Evaluate inner integral $I_1 = \int \frac{1}{\sqrt{(a-x)(x-y)}}dy$ using substitution $x - y = t^2$

Ex- Evaluate $\int_0^{\pi}\int_0^{x} \frac{\sin y}{\sqrt{(\pi-x)(x-y)}}\,dx\,dy$ by changing order of integration.

Solution- Let,

$$I = \int_0^{\pi}\int_0^{x} \frac{\sin y\,dx\,dy}{\sqrt{(\pi-x)(x-y)}}$$

$$I = \int_{x=0}^{\pi}\int_{y=0}^{x} \frac{\sin y}{\sqrt{(\pi-x)(x-y)}}\,dy\,dx$$

Curves are $x=0 \rightarrow$ y-axis, $x=\pi$

$y=0 \rightarrow$ x-axis, $y=x \rightarrow$ line

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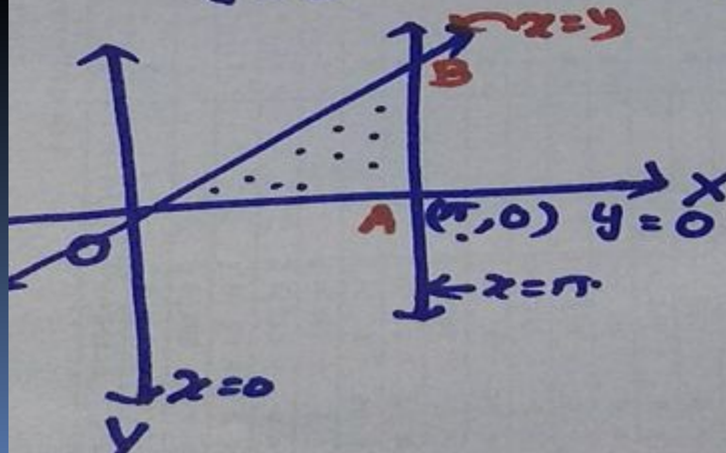
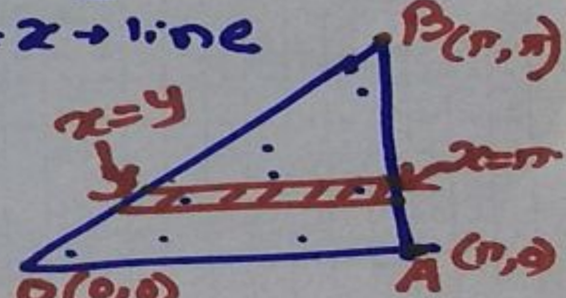
$y=0 \rightarrow$ x-axis, $y=x \rightarrow$ line





$0 < y < \pi$

$y < x < \pi$

$$\int_0^{\pi}\int_0^{x}\frac{\sin y}{\sqrt{(\pi-x)(x-y)}}\,dy\,dx=\int_{y=0}^{\pi}\int_{x=y}^{\pi}\frac{\sin y}{\sqrt{(\pi-x)(x-y)}}\,dx\,dy$$

$$=\int_{y=0}^{\pi}\sin y\int_{x=y}^{\pi}\frac{dx}{\sqrt{(\pi-x)(x-y)}}\,dy$$

Put, $(x-y)=t^2$

$$x=y+t^2\Rightarrow dx=2t\,dt$$

when, $x=y,\ t=0$

$$x=\pi,\ t^2=\pi-y,\ t=\sqrt{\pi-y}$$

$$I=\int_{y=0}^{\pi}\sin y\int_{t=0}^{t=\sqrt{\pi-y}}\frac{2t\cdot dt}{\sqrt{(\pi-y-t^2)\,t^2}}$$

$$I=2\int_{y=0}^{\pi}\sin y\int_{t=0}^{\sqrt{\pi-y}}\frac{1}{\sqrt{(\sqrt{\pi-y})^2-t^2}}\,dt$$

$$I=2\int_{y=0}^{\pi}\sin y\left[\sin^{-1}\left(\frac{t}{\sqrt{\pi-y}}\right)\right]_{t=0}^{\sqrt{\pi-y}}dt$$

$$I = 2\int_{y=0}^{\pi} \sin y \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy$$

$$I = 2\int_{y=0}^{\pi} \sin y \left[\frac{\pi}{2} - 0\right] dy$$

$$I = \pi\left[-\cos y\right]_{y=0}^{\pi}$$

$$I = -\pi\left[-1-1\right]$$

$$\boxed{I = 2\pi}$$

Exa: Change the order of integrati

$$\int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y)\,dx\,dy$$

Solution - Let, $I = \int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y)\,dx$

$$I = \int_{x=0}^{a} \left\{ \int_{y=\sqrt{a^2-x^2}}^{x+3a} f(x,y)\,dy \right\} dx$$

Curves are, $x=0 \to$ y-axis $x=a \to$ li

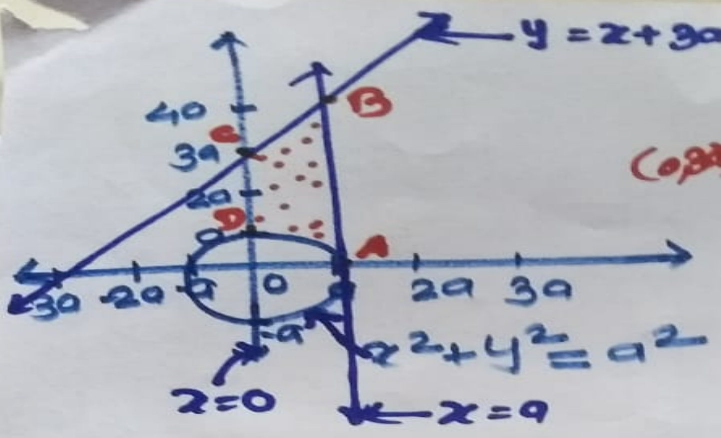
$y=\sqrt{a^2-x^2}$ and $y=x+3a$

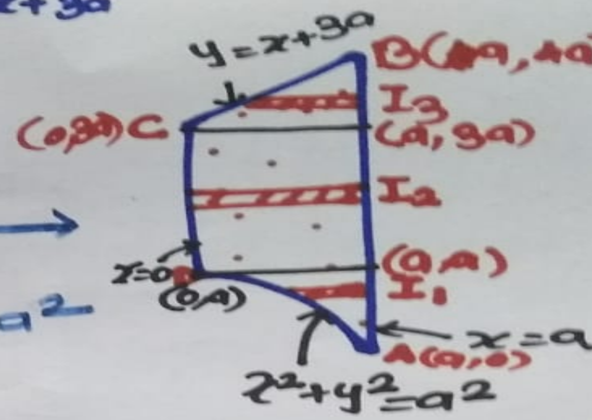
$y^2=a^2-x^2$ $x-y=3a$

$x^2+y^2=a^2$

$\uparrow$ $\uparrow$

circle line





For I_1 $\sqrt{a^2-y^2} < x < a, \quad 0 < y < a$

For I_2 $0 < x < a, \quad a < y < 3a$

For I_3 $y-3a < x < a, \quad 3a < y < 4a$

$$I = \int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y)\,dx\,dy$$

$$I = \int_{y=0}^{a} \int_{x=\sqrt{a^2-y^2}}^{a} f(x,y)\,dx\,dy + \int_{y=a}^{3a} \int_{x=0}^{a} f(x,y)\,dx\,dy + \int_{y=3a}^{4a} \int_{x=y-3a}^{a} f(x,y)\,dx\,dy$$

Polar form of Double Integration

Double integral in Polar Co-ordinate

We can evaluate double integrals in two steps:

First evaluate the inner integral, and then plug this solution into the outer integral and solve that.

$$\int \int f(r,\theta)\, dr d\theta$$

Note:

1. **Limits of Outer Integral are always constant**
2. **Limits of Inner Integral are constant or function of only one variable θ**

Polar form of Double Integration

Working Rule to Evaluate Double Integrals in polar

To Evaluate $\int_{\theta_1}^{\theta_2} \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\,dr\,d\theta$

Step (i): First integrate $f(r,\theta)$ with respect to r (keeping θ constant)

Step (ii) : Integrate the result obtained in step (i) with respect to θ

Here order of integration is always first w.r.t r and then w.r.t θ

Example 1.5.1: Evaluate $\int_{0}^{\pi/4} \int_{0}^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta$

Solution: $I = \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta$

$$I = \int_{\theta=0}^{\theta=\pi/4} \left(\int_{r=0}^{r=\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr \right) d\theta \text{-----------}(1)$$

Exa:- Evaluate $\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr \cdot d\theta$.

Solution - Let, $I = \int_{\theta=0}^{\pi/4}\int_{r=0}^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta$

$$I = \int_{\theta=0}^{\pi/4}\left\{\frac{1}{2}\int_{r=0}^{\sqrt{\cos 2\theta}} \frac{2r}{(1+r^2)^2}\, dr\right\} d\theta$$

Put $1+r^2 = t$ | when $r=0$, $t=1$

$2r \cdot dr = dt$ | $r = \sqrt{\cos 2\theta}$

$t = 1+r^2 = 1+\cos 2\theta$

$= 2\cdot\cos^2\theta$

$$I = \frac{1}{2}\int_{\theta=0}^{\pi/4}\left\{\int_{t=1}^{2\cos^2\theta} \frac{dt}{t^2}\right\} d\theta$$

$$I = \frac{1}{2}\int_{\theta=0}^{\pi/4}\left[\frac{t^{-1}}{-1}\right]_{t=1}^{2\cos^2\theta} d\theta$$

$$I = \left(-\frac{1}{2}\right) \int_{\theta=0}^{\pi/4} \left[\frac{1}{2\cos^2\theta} - 1\right] d\theta$$

$$I = \left(-\frac{1}{2}\right) \int_{\theta=0}^{\pi/4} \left[\frac{1}{2}\sec^2\theta - 1\right] d\theta$$

$$I = \left(-\frac{1}{2}\right) \left[\frac{1}{2}\tan\theta - \theta\right]_{\theta=0}^{\pi/4}$$

$$I = \left(-\frac{1}{2}\right) \left[\frac{1}{2} - \frac{\pi}{4}\right]$$

$$\boxed{I = \frac{\pi - 2}{8}}$$

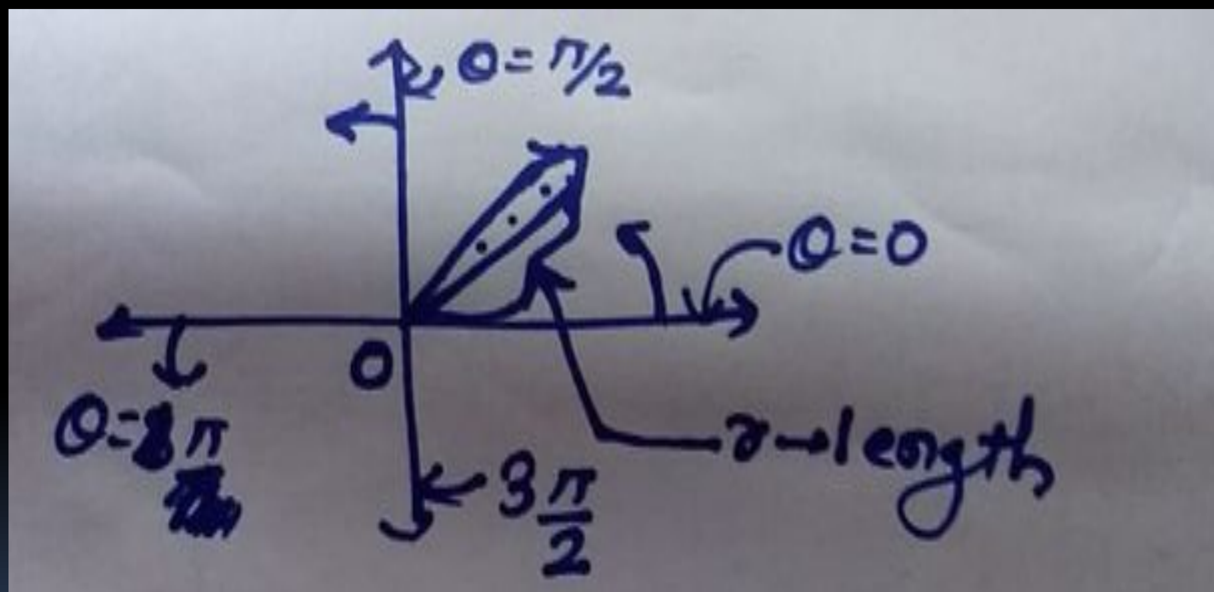
Cartecian to Polar Form

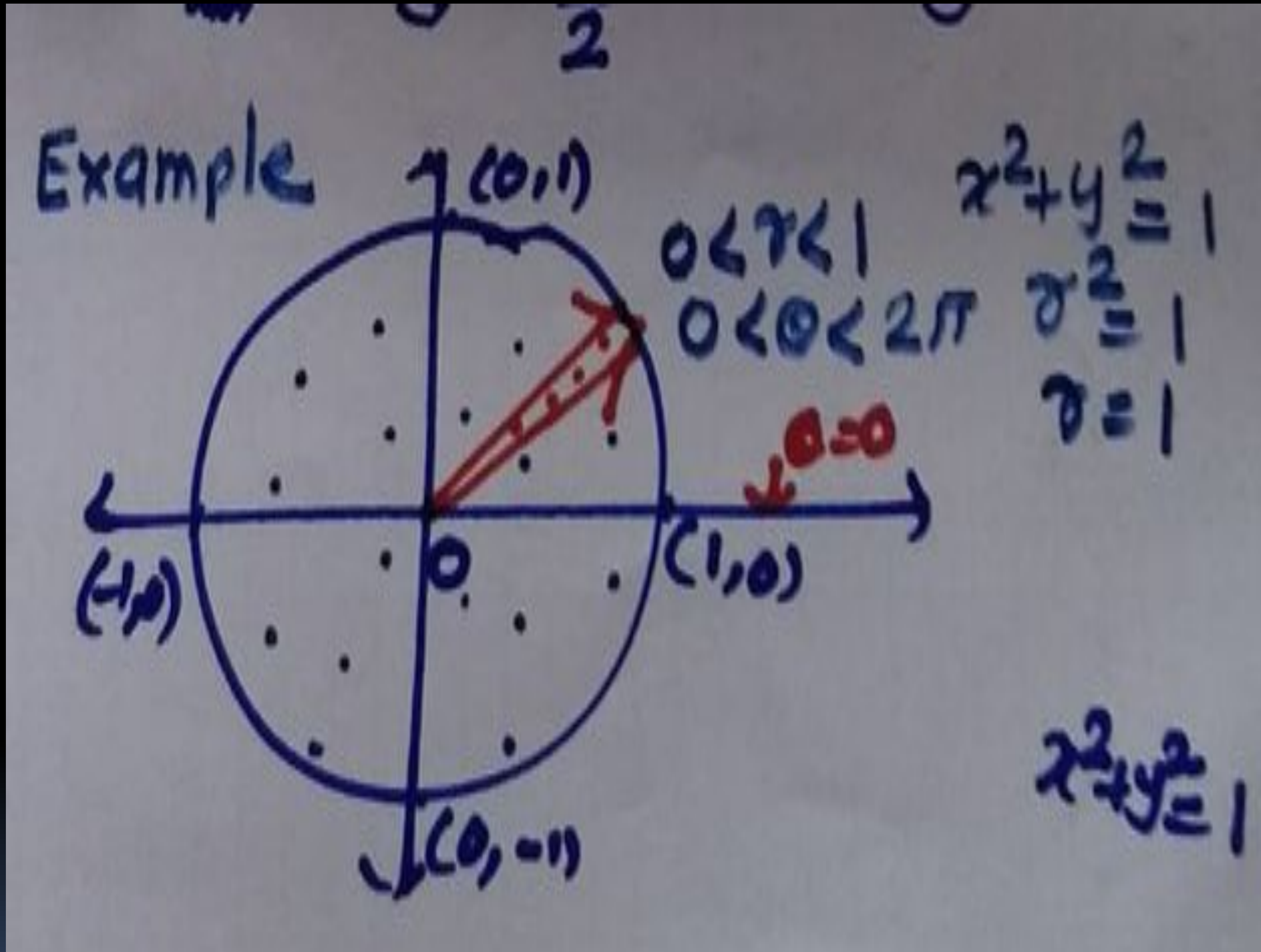
Change to polar co-ordinate system

$$\iint f(x,y)\,dx\,dy \rightarrow \iint f(r,\theta)\,dr\,d\theta$$

Put $x = r\cdot\cos\theta$ & $y = r\sin\theta$

$$dx\,dy = r\,dr\,d\theta$$




Example
(0,1)
$0<r<1$
$0<\theta<2\pi$
$x^2+y^2=1$
$r^2=1$
$r=1$
$\theta=0$
(1,0)
(-1,0)
O
(0,-1)
$x^2+y^2=1$

Q. Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$ by changing to 'polar co-ordinates'.

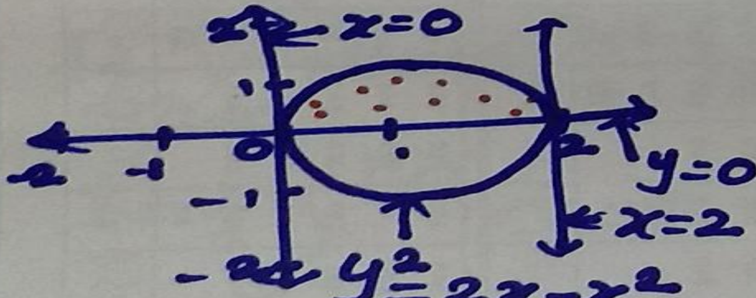
Sol. Let, $I = \int_{x=0}^{2} \int_{y=0}^{y=\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$

Curves are

$x=0$, $x=2$, $y=0 \longrightarrow$ lines

$y=\sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 - 2x + 1 + y^2 = 1$

$\Rightarrow (x-1)^2 + (y-0)^2 = 1^2 \rightarrow$ Circle $C(1,0)$, $r=1$



← Cartecian form

For polar form $\rightarrow$ $x = r\cos\theta$, $y = r.\sin\theta$

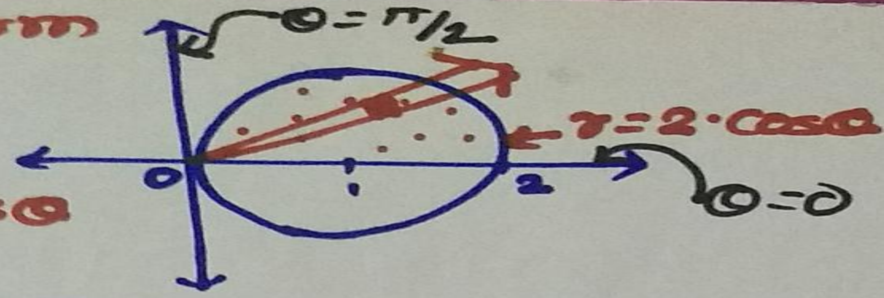
$dx\,dy = r.dr.d\theta$

$y^2 = 2x - x^2$

$x^2 + y^2 = 2x \Rightarrow r^2 = 2r.\cos\theta$

$r = 2.\cos\theta$

Polar form



$0 < r < 2\cos\theta$

$0 < \theta < \frac{\pi}{2}$

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cdot\cos\theta} \frac{r\cdot\cos\theta}{\sqrt{r^2}}\, r\cdot dr\cdot d\theta$$

$$I = \int_{\theta=0}^{\pi/2} \cos\theta \cdot \left[\frac{r^2}{2}\right]_{r=0}^{2\cdot\cos\theta} d\theta$$

$$I = \frac{1}{2}\int_{\theta=0}^{\pi/2} \cos\theta \cdot 4\cdot\cos^2\theta\, d\theta$$

$$I = 2\int_{\theta=0}^{\pi/2} \sin^0\theta\cdot\cos^3\theta\, d\theta$$

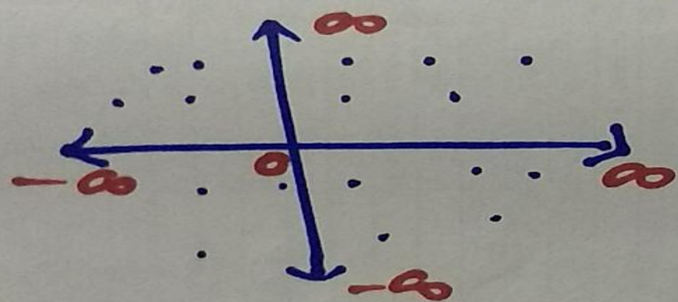
$$I = 2\cdot\frac{1}{2}\,\frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{3+1}{2}\right)}{\Gamma\left(\frac{3+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)(1)}{\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)} = \boxed{\frac{4}{3}}$$

Q. Evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(a^2+x^2+y^2)^{3/2}}\,dx\,dy$ by changing to polar co-ordinates.

Sol. Let, $I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(a^2+x^2+y^2)^{3/2}}\,dx\,dy$

$$I = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} \frac{1}{(a^2+x^2+y^2)^{3/2}}\,dx\,dy$$

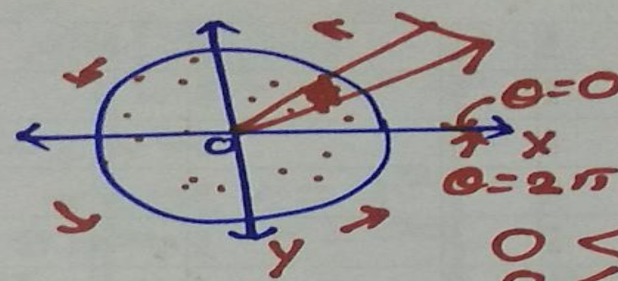
$$-\infty < x < \infty \text{ and } -\infty < y < \infty$$



Put, $x = r\cdot\cos\theta,\ y = r\cdot\sin\theta$

$$x^2+y^2 = r^2$$

$$dx\,dy = r\cdot dr\,d\theta$$



$0 < \theta < 2\pi$

$0 < r < \infty$

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}\frac{1}{(a^2+x^2+y^2)^{3/2}}\,dx\,dy$$

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty}\frac{r\cdot dr\cdot d\theta}{(a^2+r^2)^{3/2}} = \int_{\theta=0}^{2\pi} d\theta\left\{\int_{r=0}^{\infty}\frac{r\cdot dr}{(a^2+r^2)^{3/2}}\right.$$

$$\int\frac{r\,dr}{(a^2+r^2)^{3/2}} = \frac{1}{2}\int\frac{2r\cdot dr}{(a^2+r^2)^{3/2}}$$

Put $a^2+r^2=t$

$2r\,dr = dt$

$$\int\frac{r\cdot dr}{(a^2+r^2)^{3/2}} = \frac{1}{2}\int\frac{dt}{t^{3/2}} = \frac{1}{2}\left[\frac{t^{-1/2}}{-1/2}\right]$$

$$=(-1)\frac{1}{\sqrt{t}}$$

$$I = \int_{\theta=0}^{2\pi} d\theta \left[\frac{(-1)}{\sqrt{t}} \right]_{r=0}^{\infty}$$

$t = a^2 + r^2$

when $r = 0,\ t = a^2$

$r = \infty,\ t = \infty$

$$I = \int_{\theta=0}^{2\pi} d\theta \left[\frac{(-1)}{\sqrt{t}} \right]_{t=a^2}^{t=\infty}$$

$$= (-1) \int_{\theta=0}^{2\pi} d\theta \left[0 - \frac{1}{a} \right]$$

$$= \frac{1}{a} \left[\theta \right]_{\theta=0}^{2\pi} = \boxed{\frac{2\pi}{a}}$$

Q. By changing into polar co-ordinate, evaluate $\iint \frac{4xy\, e^{-(x^2+y^2)}}{x^2+y^2}\, dx\, dy$ over the region bounded by the circle $x^2+y^2-x=0$ in the I^{st} quadrant.

Solution :- Let, $I = \iint \frac{4xy\, e^{-(x^2+y^2)}}{x^2+y^2}\, dx\, dy$

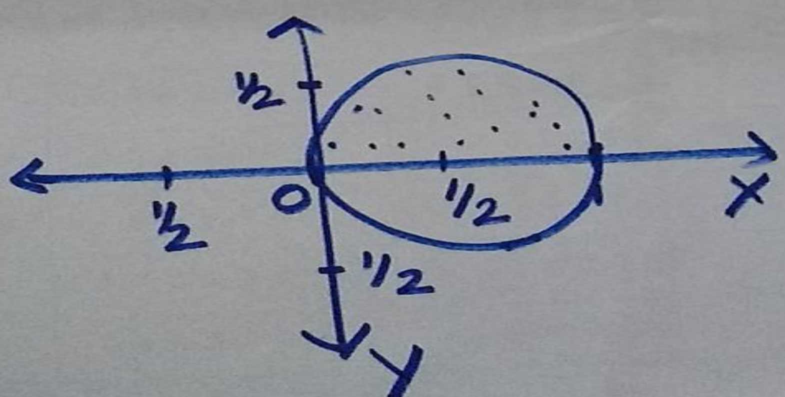
Region bounded by circle,

$$x^2 - x + y^2 = 0$$

$$\left(x^2 - 2\frac{x}{2} + \frac{1}{4}\right) + (y-0)^2 = \frac{1}{4}$$

$$\left(x - \frac{1}{2}\right)^2 + (y-0)^2 = \left(\frac{1}{2}\right)^2$$

Circle $c\left(\frac{1}{2}, 0\right)$, $r = \frac{1}{2}$



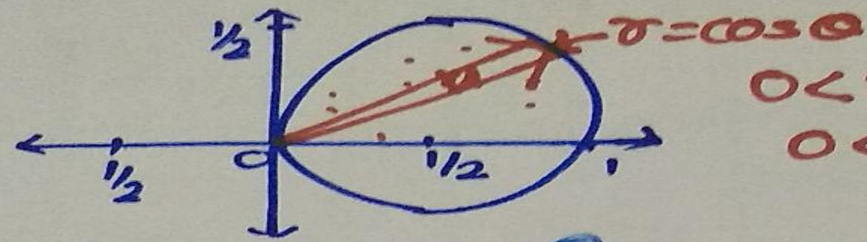
$$x^2 + y^2 = x$$

$$r^2 = r \cdot \cos\theta$$

$$r = \cos\theta$$

circle

For Polar for



$0 < r < \cos\theta$

$0 < \theta < \frac{\pi}{2}$

$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\cos\theta} \frac{4xy\, e^{-(x^2+y^2)}}{x^2+y^2}\, dx\, dy$$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\cos\theta} \frac{4\cancel{r}\cdot\cos\theta\cdot\cancel{r}\sin\theta\, e^{-r^2}}{\cancel{r^2}}\, r\cdot dr\, d\theta$$

$$I = 2\int_{\theta=0}^{\pi/2} 2\sin\theta\cos\theta \int_{r=0}^{\cos\theta} e^{-r^2}\cdot r\cdot dr$$

$\int r\cdot e^{-r^2} = \frac{1}{2}\int 2r\cdot e^{-r^2}\, dr$ Put $r^2 = t$, $2r\cdot dr = dt$

$= \frac{1}{2}\int e^{-t}\, dt$

$= -\frac{1}{2} e^{-t} = -\frac{1}{2} e^{-r^2}$

$$I = 2\int_{\theta=0}^{\pi/2} 2\sin\theta\cos\theta\left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{\cos\theta} d\theta$$

$$I = (-1)\int_{\theta=0}^{\pi/2} 2\cdot\sin\theta\cos\theta\left[e^{-\cos^2\theta}-1\right]d\theta$$

$$I = \int_{\theta=0}^{\pi/2}\sin 2\theta\, d\theta - \int_{\theta=0}^{\pi/2} 2\sin\theta\cdot\cos\theta\, e^{-\cos^2\theta}d\theta$$

$$= \left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} + \int_{\theta=0}^{\pi/2}(-2\sin\theta\cos\theta)e^{-\cos^2\theta}d\theta$$

$$= -\frac{1}{2}\left[-1-1\right]$$

$$= 1 + \int_{t=1}^{0} e^{-t}dt$$

Put $\cos^2\theta = t$

$-2\cos\theta\sin\theta\, d\theta = dt$

when $\theta = 0,\ t = 1$

$\theta = \frac{\pi}{2},\ t = 0$

$$= 1 + \left[\frac{e^{-t}}{-1}\right]_{t=1}^{0}$$

$$= 1 - \left[1 - e^{-1}\right]$$

$$= \boxed{\frac{1}{e}}$$

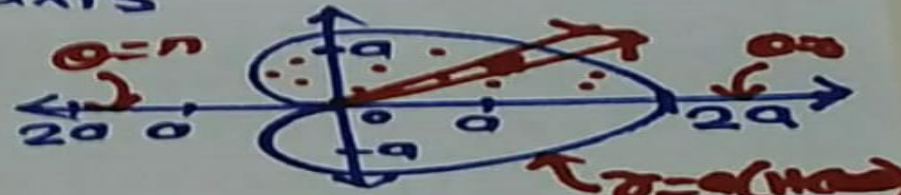
Q. Evaluate $\iint r \cdot \sin\theta \, dA$ over the cardiode $r = a(1+\cos\theta)$ above initial line.

Solution – Let, $I = \iint r \cdot \sin\theta \, dA$

Region → $r = a(1+\cos\theta)$ is symmetrical about initial axis

$0 < \theta < \pi$

$0 < r < a(1+\cos\theta)$



$$I = \iint r \cdot \sin\theta \, dA = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r \cdot \sin\theta \, dA$$

$dA = dx\,dy = r \cdot dr\,d\theta$

$$I = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r \cdot \sin\theta \, r \cdot dr \cdot d\theta$$

$$I = \int_{\theta=0}^{\pi} \sin\theta \int_{r=0}^{a(1+\cos\theta)} r^2 \, dr \cdot d\theta$$

$$I = \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_{r=0}^{a(1+\cos\theta)} d\theta$$

$$I = \frac{1}{3}\int_{\theta=0}^{\pi} \sin\theta \; a^3(1+\cos\theta)^3 \, d\theta$$

$$I = \frac{a^3}{3}\int_{\theta=0}^{\pi} 2\cdot\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}\right)^3 d\theta$$

$$I = \frac{16a^3}{3}\int_{\theta=0}^{\pi} \sin\frac{\theta}{2}\cos^7\frac{\theta}{2}\, d\theta$$

Put $\frac{\theta}{2} = t \Rightarrow d\theta = 2dt$

when $\theta = 0,\ t = 0$

$\theta = \pi,\ t = \frac{\pi}{2}$

$$I = \frac{16}{3}a^3\int_{t=0}^{\pi/2} \sin^1 t\cdot\cos^7 t\cdot 2dt$$

$$I = \frac{32}{3}a^3\cdot\frac{1}{2}\,\frac{\Gamma\left(\frac{1+1}{2}\right)\Gamma\left(\frac{7+1}{2}\right)}{\Gamma\left(\frac{8+2}{2}\right)}$$

$$I = \frac{16}{3} a^3 \frac{\lceil 2 \lceil 4}{\lceil 5}$$

$$I = \frac{16}{3} a^3 \frac{3!}{4!}$$

$$I = \frac{16}{3} a^3 \frac{1}{4}$$

$$\boxed{I = \frac{4}{3} a^3}$$

Application of Double Integral to Find Area for region 'R'

Cartesian co-ordinates is,

$$A = \iint_R dx\,dy$$

Polar co-ordinates is,

$$A = \iint_R r\,dr\,d\theta$$

Q. Find the area included between the curves $y^2 = 4ax$ and $x^2 = 4by$

Solution– We have, Area for the region R is,

$$A = \iint_R dx\,dy$$

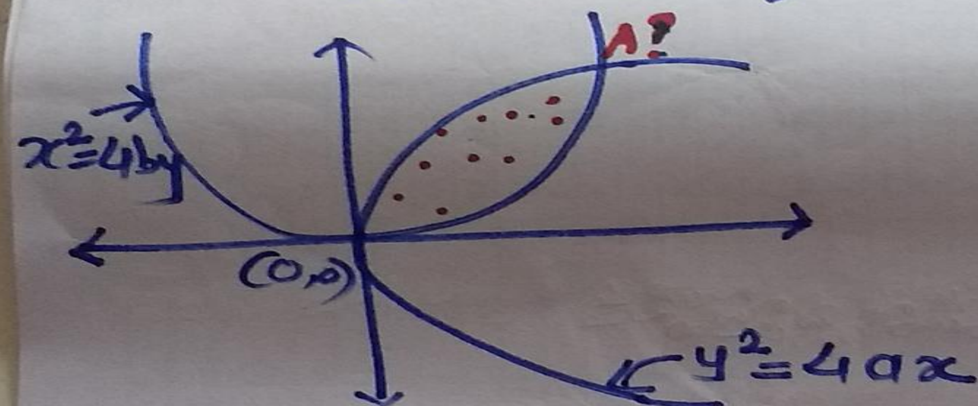
$R = ?$

R → area included between

$y^2 = 4ax$ } parabola
$x^2 = 4by$

symmetrical about x-axis & vertex (0,0)

Symmetrical about y-axis & vertex (0,0)

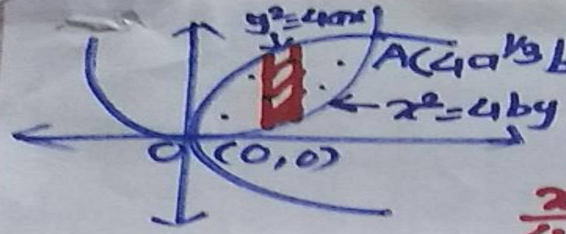


A=? $y^2=4ax$, $x^2=4by$

Solve simultaneously

$\left(\frac{x^2}{4by}\right)^2 = 4ax$

$x = 0,\ 4a^{1/3}b^{2/3}$ & $y = 0,\ 4a^{2/3}b^{1/3}$



$$\frac{x^2}{4b} < y < 2\sqrt{ax}$$

$$0 < x < 4a^{1/3}b^{2/3}$$

$$\text{Area, } A = \iint_R dx\,dy = \int_{x=0}^{4a^{1/3}b^{2/3}} \int_{y=\frac{x^2}{4b}}^{2\sqrt{ax}} dy\,dx$$

$$= \int_{x=0}^{4a^{1/3}b^{2/3}} dx \Big[\, y \,\Big]_{y=x^2/4b}^{2\sqrt{ax}}$$

$$A = \int_{x=0}^{4a^{1/3}b^{2/3}} \left[2\sqrt{a}\, x^{1/2} - \frac{1}{4b} x^2\right] dx$$

$$A = \left[2\sqrt{a}\, \frac{x^{3/2}}{3/2} - \frac{1}{4b} \frac{x^3}{3}\right]_{x=0}^{4a^{1/3}b^{2/3}}$$

$$A = \frac{4}{3}\sqrt{a}\,(4a^{1/3}b^{2/3})^{3/2} - \frac{1}{12b}(4a^{1/3}b^{2/3})^3 - 0$$

$$\boxed{A = \frac{16}{3}ab}$$

Q: Find the area included between the parabola $y=x^2-6x+3$ & the straight line $y=2x-9$

Solution- Area, $A=\iint\limits_R dx\,dy$

$R = ?$

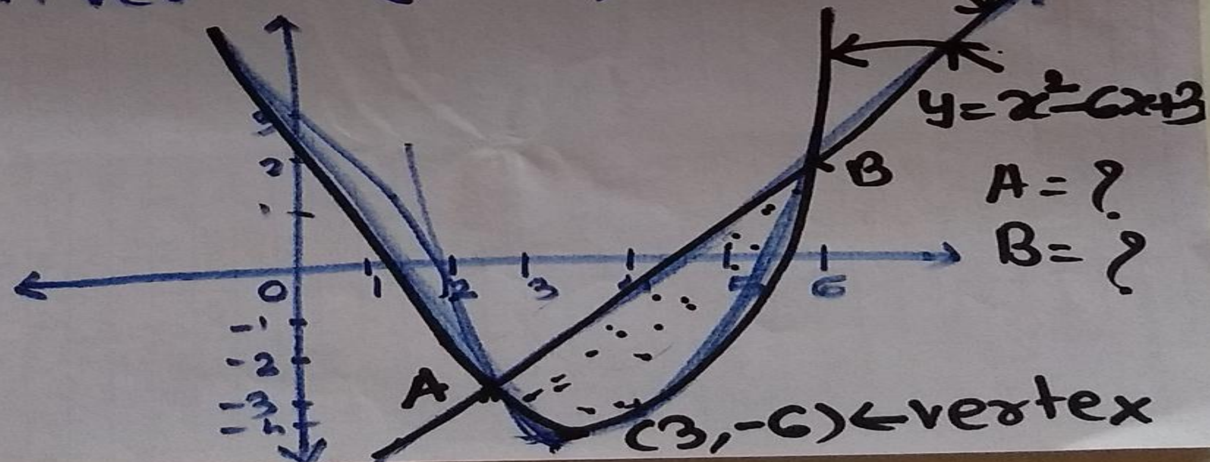
parabola $y=x^2-6x+3$

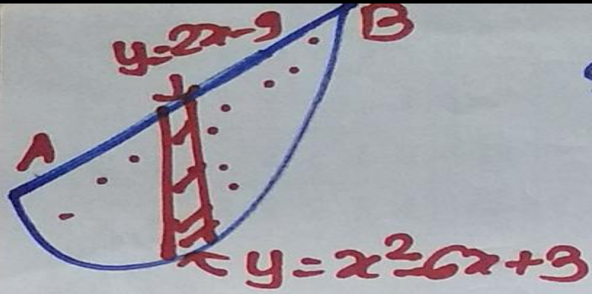
$$y=(x^2-6x+9)+3-9$$

$$(y+6)=(x-3)^2 \Rightarrow (x-3)^2=(y-(-6))$$

parabola symmetrical about x-axis with vertex $(3,-6)$

$y=2x-9$





To Find A, B

Solve, $y=2x-9$

$y=x^2-6x+3$

$\Rightarrow x^2-6x+3=2x-9$

$x^2-8x+12=0$

$x=6,\ 2$

$\&\ y=2,\ -5$

$\therefore A=A(2,-5)$

$B=B(6,2)$

$\therefore\ x^2-6x+3<y<2x-9$

$2<x<6$

$$A=\iint dx\,dy=\int_{x=2}^{6}\int_{y=x^2-6x+3}^{2x-9} dy\,dx$$

$$A=\int_{x=2}^{6}\left[y\right]_{x^2-6x+3}^{2x-9} dx$$

$$A=\int_{x=2}^{6}\left[-x^2+8x-12\right] dx$$

$$A=\left[-\frac{x^3}{3}+8\frac{x^2}{2}-12x\right]_{x=2}^{6}=\boxed{\frac{32}{3}}$$

Q. Find by double integration the area between the circles $r = \sin\theta$ & $r = 2\sin\theta$

Solution – We have, $A = \iint_R r\,dr\,d\theta$

R is region bounded between

$r = \sin\theta$ $\qquad$ $r = 2\sin\theta$

$r^2 = r\sin\theta$ $\qquad$ $r^2 = 2r\sin\theta$

$x^2+y^2 = y$ $\qquad$ $x^2+y^2 = 2y$

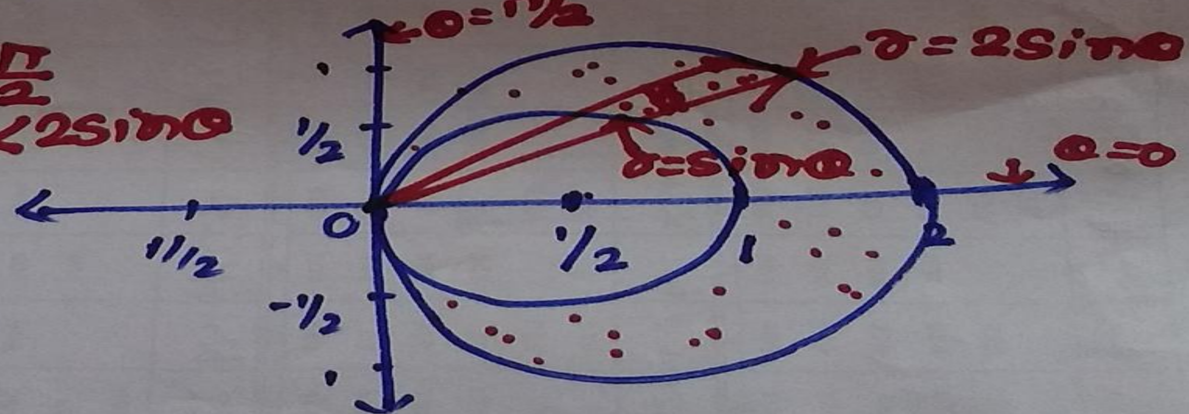
$x^2+y^2 - 2\frac{y}{2} + \frac{1}{4} = \frac{1}{4}$ $\qquad$ $x^2+y^2-2y+1 = 1$

$(x-0)^2+(y-\frac{1}{2})^2 = (\frac{1}{2})^2$ $\qquad$ $(x-0)^2+(y-1)^2 = 1^2$

$C(0,\frac{1}{2}),\ r = \frac{1}{2}$ $\qquad$ $C(0,1),\ r = 1$

$0 < \theta < \frac{\pi}{2}$

$\sin\theta < r < 2\sin\theta$



$$\text{Area} = 2\int_{\theta=0}^{\pi/2}\int_{r=\sin\theta}^{r=2\sin\theta} r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2}\left[\frac{r^2}{2}\right]_{r=\sin\theta}^{2\sin\theta} d\theta$$

$$A = \frac{2}{2}\int_{\theta=0}^{\pi/2}\left[4\sin^2\theta - \sin^2\theta\right] d\theta$$

$$A = 3\int_{0}^{\pi/2}\sin^2\theta\cdot\cos^0\theta\,d\theta$$

$$A = 3\cdot\frac{1}{2}\,\beta\left(\frac{2+1}{2},\frac{0+1}{2}\right)$$

$$A = \frac{3}{2}\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{4}{2}} = \frac{3}{2}\,\frac{\frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{(1)}$$

$$= \boxed{\frac{3\pi}{4}}$$

Q. Find the area inside the circle $r = 2\sin\theta$ & outside the cardiode $r = 2(1-\cos\theta)$

Solution. Area $= \iint_R r\,dr\,d\theta$

Region is

$r = 2\sin\theta \rightarrow$ circle

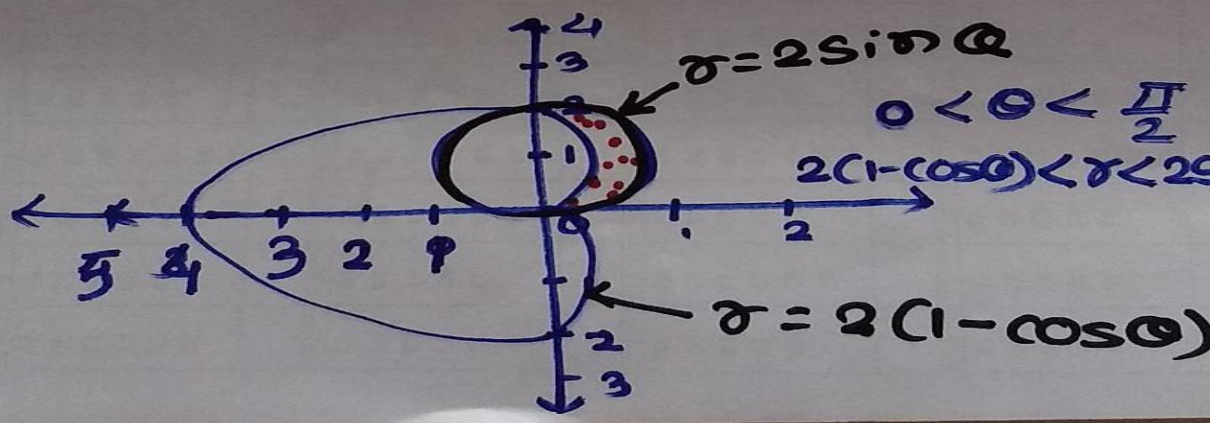
$r^2 = 2r\sin\theta$

$x^2 + y^2 = 2y$

$x^2 + (y-1)^2 = 1 \qquad C(0,1),\ r = 1$

$r = 2(1-\cos\theta) \rightarrow$ cardiode

symmetrical about initial axis



$0 < \theta < \dfrac{\pi}{2}$

$2(1-\cos\theta) < r < 2\sin\theta$

$$A = \int_{\theta=0}^{\pi/2} \int_{r=2(1-\cos\theta)}^{r=2\sin\theta} r \cdot dr \cdot d\theta$$

$$A = \int_{0}^{\pi/2} \left[\frac{r^2}{2}\right]_{r=2(1-\cos\theta)}^{2\sin\theta} d\theta$$

$$A = \frac{1}{2} \int_{\theta=0}^{\pi/2} [4\sin^2\theta - 4(1-2\cos\theta+\cos^2\theta)] d\theta$$

$$= \frac{4}{2} \int_{\theta=0}^{\pi/2} [-\cos 2\theta - 1 + 2\cos\theta] d\theta$$

$$= 2 \left[-\frac{\sin 2\theta}{2} - \theta + 2\sin\theta\right]_{\theta=0}^{\pi/2}$$

$$= 2 \left[-0 - \frac{\pi}{2} + 2 - 0\right]$$

$$= \boxed{4-\pi}$$