Problem Set 1

6.S091: Causality

IAP 2023

Due: Thursday, January 19th at 1pm EST

- Problem sets **must** be done in LaTeX.
- Printed problem sets must be turned in at the beginning of lecture. If you cannot attend, please find a classmate to turn the problem set in for you.
- You may use any programming language for your solutions, and you are not required to turn in your code.

Problem 1: Interventions and Adjustment [4 points]

Mickey and Minnie are deciding whether to begin a new venture training other mice to solves mazes. They know that humans reward mice with cheese based on how long they take to run a maze. Before beginning their venture, they want to figure out if their training program actually helps mice earn more cheese. They don't have the funding for a trial, but they already have some data: for the past few years, they have been offering this training to their friends. However, they've noticed the gender is a possible confounding factor. Their male mouse friends are more likely to sign up for the training program, and they cannot rule out sexism on the part of the humans in terms of how much cheese is rewarded.

To answer their questions, they have developed a structural causal model over the following exogenous variables:

- $U = 1$ for male, $U = 0$ for female
- $A = 1$ if training is received, $A = 0$ otherwise
- $M = 1$ if the maze is solved "quickly" (in less than one minute), $M = 0$ otherwise
- $Y = 1$ if cheese is rewarded, $Y = 0$ otherwise

The structural causal model M is as follows:

$$
\begin{aligned}
U &= \varepsilon_u & \varepsilon_u &\sim \mathsf{Ber}(0.5) \\
A &= \mathbb{1}_{\{\varepsilon_a + U/4 \geq 1\}} & \varepsilon_a &\sim \mathsf{Unif}(0,1) \\
M &= \mathbb{1}_{\{\varepsilon_m + 10(1-A) \leq 60\}} & \varepsilon_m &\sim \mathsf{Unif}(0,100) \\
Y &= \mathbb{1}_{\{\varepsilon_y + M/2 + U/4 \geq 1\}} & \varepsilon_y &\sim \mathsf{Unif}(0,1)
\end{aligned}
$$

Preliminaries [1 point]

(a) What is the entailed distribution $\mathbb{P}_{\mathcal{X}}$ of the SCM M?
(b) Assume that $\mathbb{P}$ is a distribution over binary-valued variables. Write a (`Python/Julia/other`) method which takes in two disjoint sets $\mathbf{A}, \mathbf{B} \subseteq \mathcal{X}$ and returns the function $f(\mathbf{a}, \mathbf{b}) = \mathbb{P}(\mathbf{A} = \mathbf{a} \mid \mathbf{B} = \mathbf{b})$. Report the following probabilities for the entailed distribution $\mathbb{P}_{\mathcal{X}}$ from (a):

- $\mathbb{P}_{\mathcal{X}}(Y = 1)$
- $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid M = 0, A = 0)$
- $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid M = 0, A = 1)$

Interventional [1 point]

(c) What is the interventional distribution $\mathbb{P}_{\mathcal{X}}(U, A, M, Y \mid \mathtt{do}(A = 1))$?
(d) Use the method you wrote for **(b)** to report the following probabilities:

- $\mathbb{P}_{\mathcal{X}}(Y \mid \mathtt{do}(A = 1))$
- $\mathbb{P}_{\mathcal{X}}(Y \mid \mathtt{do}(A = 0))$

Backdoor Adjustment [1 point]

(e) Since U is an adjustment set for $\mathbb{P}_{\mathcal{X}}(Y \mid \mathtt{do}(A = a))$, we have

$$\mathbb{P}_{\mathcal{X}}(Y \mid \mathtt{do}(A = a)) = \sum_{u \in \{0,1\}} \mathbb{P}_{\mathcal{X}}(Y \mid A = a, U = u)\mathbb{P}_{\mathcal{X}}(U = u)$$

Report the probabilities $\mathbb{P}_{\mathcal{X}}(Y \mid A = a, U = u)$ for $(a, u) \in \{0, 1\} \times \{0, 1\}$ and confirm numerically that the right-hand side of the above equation matches your answers from **(d)**.

Frontdoor Adjustment [1 point]

(f) Suppose Mickey and Minnie forgot to record which data point U: they have only samples of (A, M, Y). Luckily, they recall the frontdoor adjustment formula for their problem setup:

$$\mathbb{P}_{\mathcal{X}}(Y \mid \texttt{do}(A = a)) = \sum_{m \in \{0,1\}} \mathbb{P}_{\mathcal{X}}(M = m \mid A = a) \sum_{a' \in \{0,1\}} (\mathbb{P}_{\mathcal{X}}(Y \mid M = m, A = a')\mathbb{P}_{\mathcal{X}}(A = a'))$$

Confirm numerically that the right-hand side of the above equation matches your answers from **(d)** by writing the eight values in the summation, e.g. $1 = 0.125 + 0.125 + \ldots + 0.125$. You are encouraged to write a (`Python/Julia/other`) function which return computes $\mathbb{P}_{\mathcal{X}}(Y \mid \texttt{do}(A = a))$ using frontdoor adjustment.

Problem 2: Moral separation implies d-separation [2 points]

In this problem, we prove the converse of Theorem 2.2 from Lecture 2. In particular, we wish to show that $\mathcal{I}^m_{\perp\!\!\!\perp}(\mathcal{G}) \subseteq \mathcal{I}_{\perp\!\!\!\perp}(\mathcal{G})$. Suppose there is a d-connecting path γ from $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{G}$ given $\mathbf{S}$.

(a) Show that all nodes in γ are in $\mathbf{V} = \mathrm{an}_{\mathcal{G}}(\mathbf{A} \cup \mathbf{B} \cup \mathbf{S})$.
(b) Show that there is a path in $\overline{\mathcal{G}[\mathbf{V}]}$ from $\mathbf{A}$ to $\mathbf{B}$ which does not pass through $\mathbf{S}$.

Problem 3: Simpson's paradox [2 points]

Let $S \in \{L, R\}$ denote the size of a patient's kidney stone, where $S = L$ denotes that the patient has large kidney stones and $S = R$ denotes that they have a regular kidney stones. Let $A \in \{0, 1\}$ denote the patient's treatment, where $A = N$ denotes that the patient receives the new treatment and $A = O$ denotes that the patient receives the old treatment. Finally, let $Y \in \{0, 1\}$ denote whether the kidney stones are eliminated after one month, where $Y = 1$ denotes that they are eliminated and $Y = 0$ denotes that they are not.

Suppose that $\mathbb{P}_{\mathcal{X}}(S, A, Y)$ factorizes according to the DAG in Figure 2.1(b) of the lecture notes, with the following conditional distributions:

- $\mathbb{P}_{\mathcal{X}}(S = L) = 0.49$
- $\mathbb{P}_{\mathcal{X}}(A = N \mid S = L) = 0.77$
- $\mathbb{P}_{\mathcal{X}}(A = N \mid S = R) = 0.24$
- $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid S = L, A = N) = 0.73$
- $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid S = L, A = O) = 0.69$
- $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid S = R, A = N) = 0.93$
- $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid S = R, A = O) = 0.87$

(a) What is $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid A = N)$? What is $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid A = O)$?
(b) What is $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid \mathtt{do}(A = N))$? What is $\mathbb{P}_{\mathcal{X}}(Y = 1 \mid \mathtt{do}(A = O))$?

Problem 4: Instrumental Variables [2 points]

You will receive 1000 samples of (W, A, Y) generated from the following SCM, for $\beta_{wa} \in \{0.05, 0.5, 5\}$:

$$
\begin{aligned}
U &= \varepsilon_u & \varepsilon_u &\sim \mathcal{N}(0,1) \\
W &= \varepsilon_w & \varepsilon_w &\sim \mathcal{N}(0,1) \\
A &= 10U + \beta_{wa} W + \varepsilon_a & \varepsilon_a &\sim \mathcal{N}(0,1) \\
Y &= 2.5U + 7.5A + \varepsilon_y & \varepsilon_y &\sim \mathcal{N}(0,1)
\end{aligned}
$$

The samples for $\beta_{wa} = x$ are in the file `samples_x.csv` at

`https://github.com/csquires/6.S091-causality/tree/main/psets/pset1`,

with W in the first column, A in the second column, and Y in the third column. For each value of β_{wa}, report the following estimates:

- $\widehat{\beta}_{aw}$, the coefficient of a linear regression (with intercept) of A onto W.
- $\widehat{\beta}_{yw}$, the coefficient of a linear regression (with intercept) of Y onto W.
- The ratio $\widehat{\beta}_{yw}/\widehat{\beta}_{aw}$