



Machine Learning

Linear regression with one variable

Model representation

Housing Prices (Portland, OR)

Price
(in 1000s
of dollars)



Supervised Learning

Given the “right answer” for each example in the data.

Regression Problem

Predict real-valued output

Classification: Discrete-valued output

Training set of housing prices (Portland, OR)

| Size in feet ² (x) | Price (\$) in 1000's (y) |
|-----------------------------------|------------------------------|
| → 2104 | 460 |
| 1416 | 232 |
| → 1534 | 315 |
| 852 | 178 |
| ... | ... |

$m = 47$

Notation:

- m = Number of training examples
- x 's = "input" variable / features
- y 's = "output" variable / "target" variable

(x, y) - one training example

$(x^{(i)}, y^{(i)})$ - i^{th} training example

$$\begin{aligned} x^{(1)} &= 2104 \\ x^{(2)} &= 1416 \\ y^{(1)} &= 460 \end{aligned}$$

Training Set

Learning Algorithm

Size of house
x

h

Estimated price
(estimated value of y)

hypothesis

h maps from x's to y's.

How do we represent h ?

$$h_{\theta}(x) = \theta_0 + \theta_1 x$$

Shorthand: $h(x)$



Linear regression with one variable. (x)
Univariate linear regression.
↳ one variable



Machine Learning

Linear regression
with one variable

Cost function

Training Set

| Size in feet ² (x) | Price (\$) in 1000's (y) |
|-------------------------------|--------------------------|
| 2104 | 460 |
| 1416 | 232 |
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| 852 | 178 |
| ... | ... |

} $m = 47$

Hypothesis:
$$h_{\theta}(x) = \theta_0 + \theta_1 x$$

θ_i 's: Parameters

↑ ↑

How to choose θ_i 's ?

$$\underline{h_{\theta}(x)} = \theta_0 + \theta_1 x$$



$$\rightarrow \theta_0 = 1.5$$

$$\rightarrow \theta_1 = 0$$



$$\rightarrow \theta_0 = 0$$

$$\rightarrow \theta_1 = 0.5$$



$$\rightarrow \theta_0 = 1$$

$$\rightarrow \theta_1 = 0.5$$



Idea: Choose $\underline{\theta_0}, \underline{\theta_1}$ so that $\underline{h_\theta(x)}$ is close to \underline{y} for our training examples $\underline{(x, y)}$

$$\begin{array}{l} \boxed{\text{minimize } \underline{\theta_0, \theta_1}} \quad \frac{1}{2m} \sum_{i=1}^m \underbrace{\left(\underbrace{h_\theta(x^{(i)})}_{h_\theta(x^{(i)}) = \underline{\theta_0} + \underline{\theta_1} x^{(i)}} - \underline{y^{(i)}} \right)^2}_{\text{\#training examples}} \end{array}$$

$$J(\underline{\theta_0}, \underline{\theta_1}) = \frac{1}{2m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)})^2$$

$$\begin{array}{l} \text{minimize } J(\underline{\theta_0}, \underline{\theta_1}) \\ \underline{\theta_0, \theta_1} \end{array}$$

Cost function

Squared error function



Machine Learning

Linear regression
with one variable

Cost function
intuition I

Hypothesis:

$$\underline{h_{\theta}(x) = \theta_0 + \theta_1 x}$$

Parameters:

$$\underline{\theta_0, \theta_1}$$



Cost Function:

$$\rightarrow J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$$

Goal: minimize $J(\theta_0, \theta_1)$
 $\nearrow \theta_0, \theta_1$

Simplified

$$h_{\theta}(x) = \underline{\theta_1 x}$$

$$\theta_0 = 0$$

$$\underline{\theta_1}$$



$$J(\theta_1) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$$

minimize $\underline{J(\theta_1)}$
 $\underline{\theta_1}$

$$\theta, x^{(i)}$$

→ $h_{\theta}(x)$

(for fixed θ_1 , this is a function of x)



$J(\theta_1)$ = $\frac{1}{2m} \sum_{i=1}^3 (h_{\theta}(x^{(i)}) - y^{(i)})^2$

= $\frac{1}{2m} \sum_{i=1}^3 (\theta_1 x^{(i)} - y^{(i)})^2 = \frac{1}{2m} (0^2 + 0^2 + 0^2) = 0^2$

→ $J(\theta_1)$

(function of the parameter θ_1)



$\theta_1 = 0.5?$

$J(1) = 0$

$$h_{\theta}(x)$$

(for fixed θ_1 , this is a function of x)



$$J(0.5) = \frac{1}{2m} [(0.5-1)^2 + (1-2)^2 + (1.5-3)^2]$$

$$= \frac{1}{2 \times 3} (3.5) = \frac{3.5}{6} \approx \underline{0.58}$$

$$J(\theta_1)$$

(function of the parameter θ_1)



$$\theta_1 = 0?$$

$$J(0) = ?$$

$$h_{\theta}(x)$$

(for fixed θ_1 , this is a function of x)



$$J(0) = \frac{1}{2m} (1^2 + 2^2 + 3^2)$$

$$= \frac{1}{6} \cdot 14 \approx 2.3$$



$$h(x) = -0.5x$$

minimize $J(\theta_1)$

θ_1



Machine Learning

Linear regression
with one variable

Cost function
intuition II

Hypothesis: $h_{\theta}(x) = \theta_0 + \theta_1 x$

Parameters: θ_0, θ_1

Cost Function: $J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$

Goal: minimize $J(\theta_0, \theta_1)$
 θ_0, θ_1

$$\underline{h_{\theta}(x)}$$

(for fixed θ_0, θ_1 , this is a function of x)



$$h_{\theta}(x) = 50 + 0.06x$$

$$\underline{\underline{J(\theta_0, \theta_1)}}$$

(function of the parameters θ_0, θ_1)



Contour plots
Contour figures -



$$h_{\theta}(x)$$

$$J(\theta_0, \theta_1)$$

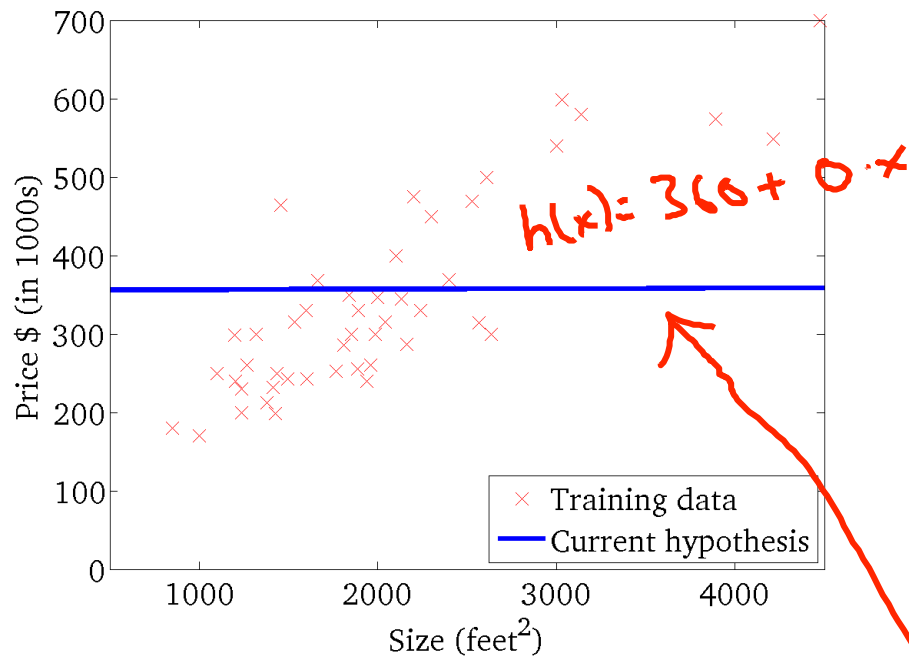
(for fixed θ_0, θ_1 , this is a function of x)

(function of the parameters θ_0, θ_1)



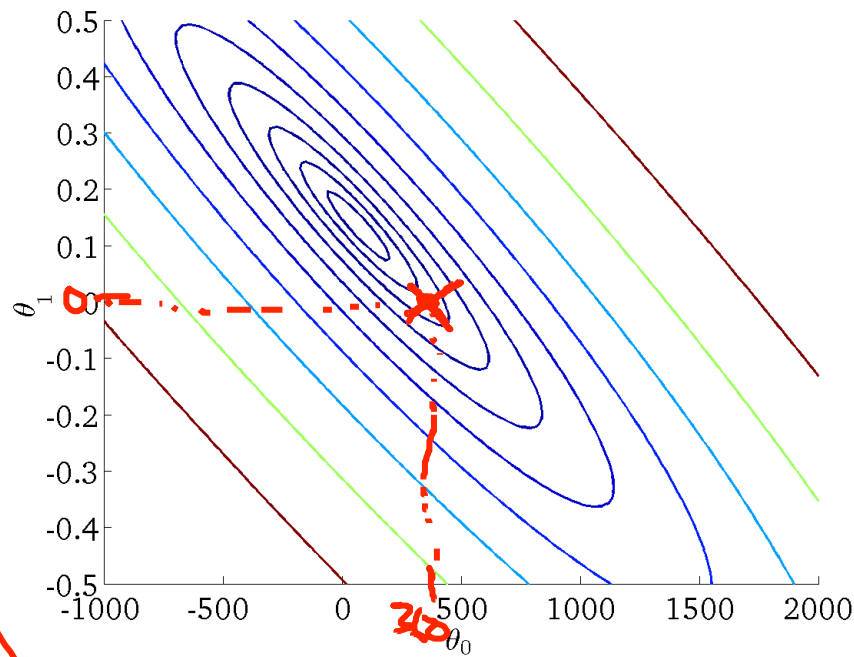
$$h_{\theta}(x)$$

(for fixed θ_0, θ_1 , this is a function of x)



$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)



$$\begin{cases} \theta_0 = 360 \\ \theta_1 = 0 \end{cases}$$

$$h_{\theta}(x)$$

(for fixed θ_0, θ_1 , this is a function of x)



$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)



$$h_{\theta}(x)$$

(for fixed θ_0, θ_1 , this is a function of x)



$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)





Machine Learning

Linear regression
with one variable

Gradient
descent

Have some function $J(\theta_0, \theta_1)$ $J(\theta_0, \theta_1, \theta_2, \dots, \theta_n)$

Want $\min_{\theta_0, \theta_1} J(\theta_0, \theta_1)$ $\min_{\theta_0, \dots, \theta_n} J(\theta_0, \dots, \theta_n)$

Outline:

- Start with some θ_0, θ_1 (say $\theta_0 = 0, \theta_1 = 0$)
- Keep changing θ_0, θ_1 to reduce $J(\theta_0, \theta_1)$
until we hopefully end up at a minimum





Gradient descent algorithm

θ_0, θ_1

repeat until convergence {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1)$$

learning rate

(for $j = 0$ and $j = 1$)

Simultaneously update
 θ_0 and θ_1

Assignment

$$a := b$$

$$a := a + 1$$

Truth assertion

$$a = b$$

$$a = a + 1 \times$$

Correct: Simultaneous update

$$\rightarrow \text{temp0} := \theta_0 - \alpha \frac{\partial}{\partial \theta_0} J(\theta_0, \theta_1)$$

$$\rightarrow \text{temp1} := \theta_1 - \alpha \frac{\partial}{\partial \theta_1} J(\theta_0, \theta_1)$$

$$\rightarrow \theta_0 := \text{temp0}$$

$$\rightarrow \theta_1 := \text{temp1}$$

Incorrect:

$$\rightarrow \text{temp0} := \theta_0 - \alpha \frac{\partial}{\partial \theta_0} J(\theta_0, \theta_1)$$

$$\rightarrow \theta_0 := \text{temp0}$$

$$\rightarrow \text{temp1} := \theta_1 - \alpha \frac{\partial}{\partial \theta_1} J(\theta_0, \theta_1)$$

$$\rightarrow \theta_1 := \text{temp1}$$



Machine Learning

Linear regression
with one variable

Gradient descent
intuition

Gradient descent algorithm

repeat until convergence {

→ $\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1)$

}

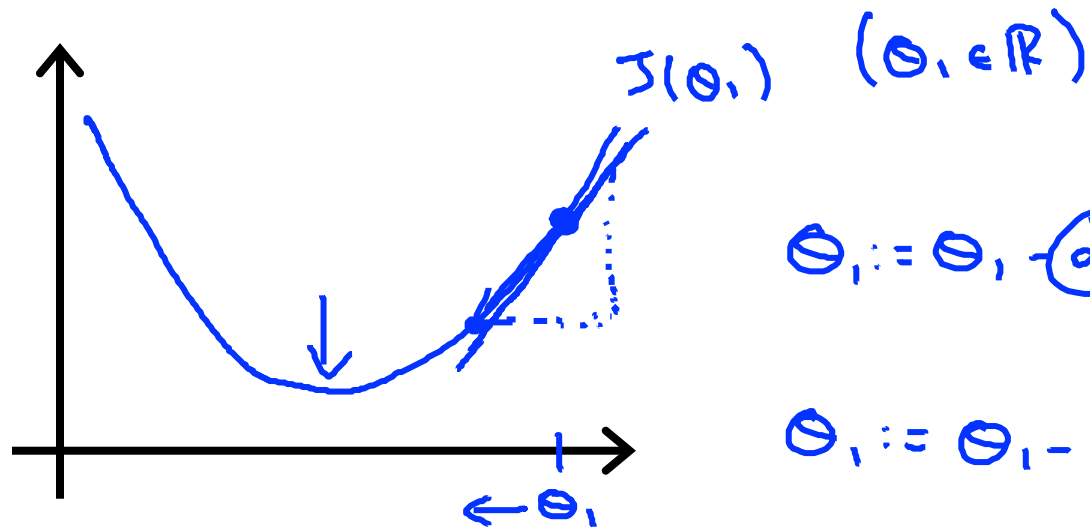
learning
rate

derivative

(simultaneously update
 $j = 0$ and $j = 1$)

$$\min_{\theta_1} J(\theta_1)$$

$$\theta_1 \in \mathbb{R}.$$



$$\theta_1 := \theta_1 - \alpha \left(\frac{\partial}{\partial \theta_1} J(\theta_1) \right) \geq 0$$

$\frac{\partial}{\partial \theta_1} \leftarrow$

$$\theta_1 := \theta_1 - \alpha \cdot (\text{positive number})$$



$$\frac{\partial}{\partial \theta_1} J(\theta_1) \leq 0$$

$$\theta_1 := \theta_1 - \alpha \cdot (\text{negative number})$$

$\uparrow \qquad \qquad \uparrow$

$$\theta_1 := \theta_1 - \alpha \frac{\partial}{\partial \theta_1} J(\theta_1)$$

If α is too small, gradient descent can be slow.

If α is too large, gradient descent can overshoot the minimum. It may fail to converge, or even diverge.





Current value of θ_1

$$\theta_1 := \theta_1 - \alpha \frac{d}{d\theta_1} J(\theta_1)$$

Gradient descent can converge to a local minimum, even with the learning rate α fixed.

$$\theta_1 := \theta_1 - \alpha \frac{d}{d\theta_1} J(\theta_1)$$

As we approach a local minimum, gradient descent will automatically take smaller steps. So, no need to decrease α over time.





Machine Learning

Linear regression with one variable

Gradient descent for linear regression

Gradient descent algorithm

repeat until convergence {
 $\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1)$
 (for $j = 1$ and $j = 0$)
}

Linear Regression Model

$$h_{\theta}(x) = \theta_0 + \theta_1 x$$

$$J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$$

$$\frac{\partial}{\partial \theta_j} \underline{J(\theta_0, \theta_1)} = \frac{2}{2\theta_j} \frac{1}{2m} \sum_{i=1}^m \underline{(h_0(x^{(i)}) - y^{(i)})^2}$$

$$= \frac{2}{2\theta_j} \frac{1}{2m} \sum_{i=1}^m \underline{(\theta_0 + \theta_1 x^{(i)} - y^{(i)})^2}$$

$$j = 0 : \underline{\frac{\partial}{\partial \theta_0} J(\theta_0, \theta_1)} = \frac{1}{m} \sum_{i=1}^m (h_0(x^{(i)}) - y^{(i)})$$

$$j = 1 : \underline{\frac{\partial}{\partial \theta_1} J(\theta_0, \theta_1)} = \frac{1}{m} \sum_{i=1}^m (h_0(x^{(i)}) - y^{(i)}) \cdot x^{(i)}$$

Gradient descent algorithm

repeat until convergence {

$$\theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})$$

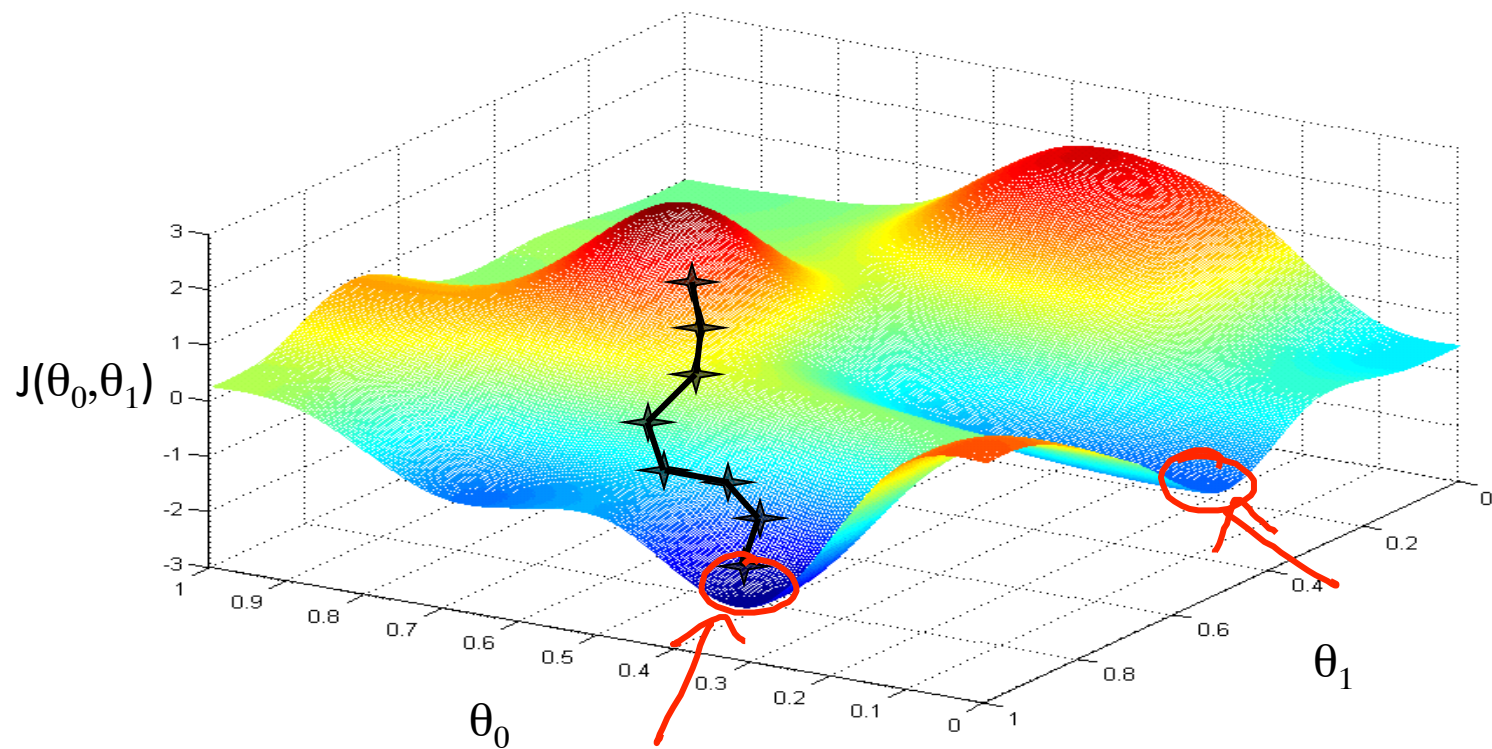
$$\theta_1 := \theta_1 - \alpha \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) \cdot x^{(i)}$$

}

$$\frac{\partial}{\partial \theta_0} J(\theta_0, \theta_1)$$

update
 θ_0 and θ_1
simultaneously

$$\frac{\partial}{\partial \theta_1} J(\theta_0, \theta_1)$$





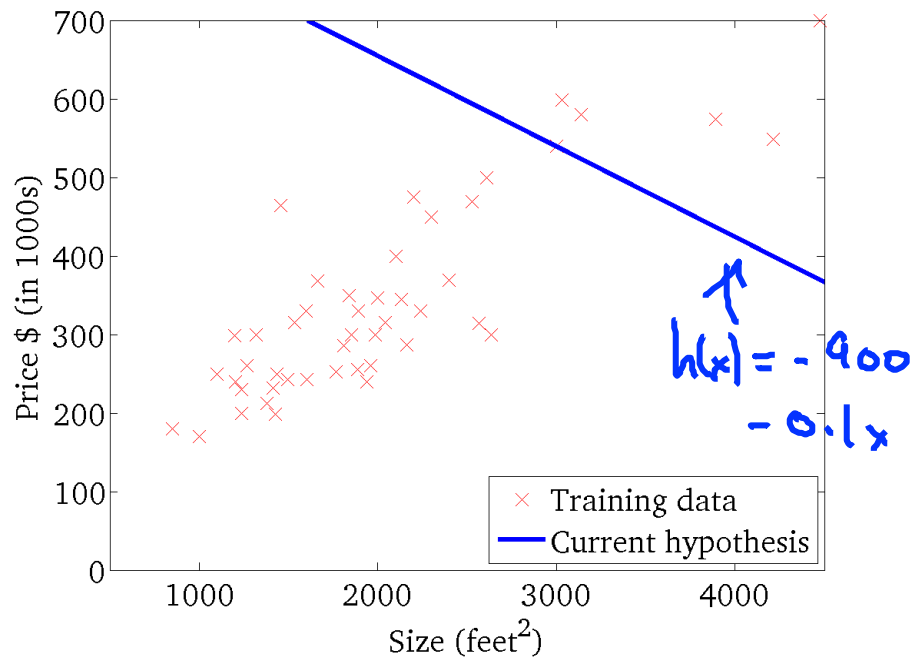


"Convex function"

Bowl-shaped

$$\underline{h_{\theta}(x)}$$

(for fixed θ_0, θ_1 , this is a function of x)



$$\underline{J(\theta_0, \theta_1)}$$

(function of the parameters θ_0, θ_1)



$$h_{\theta}(x)$$

(for fixed θ_0, θ_1 , this is a function of x)



$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)



$$h_{\theta}(x)$$

(for fixed θ_0, θ_1 , this is a function of x)



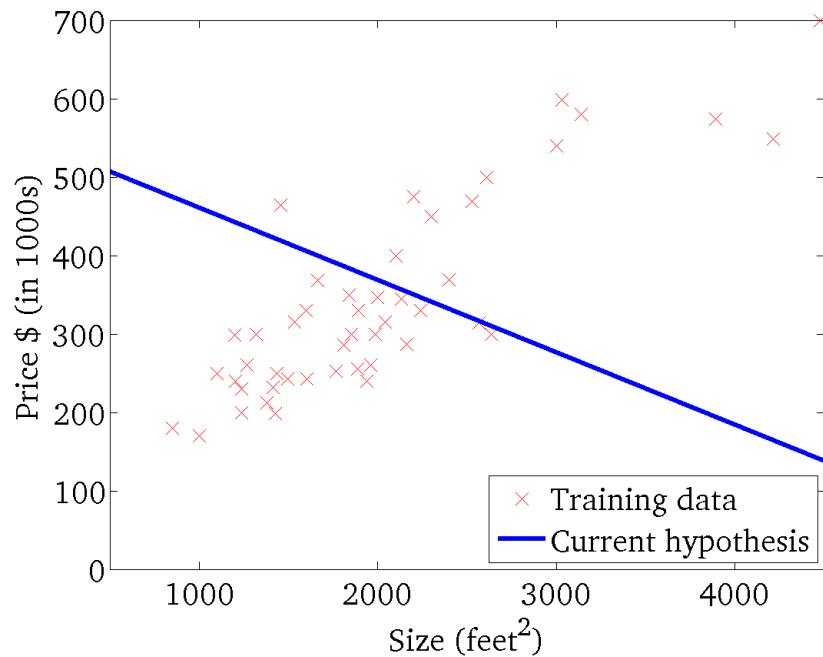
$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)



$$h_{\theta}(x)$$

(for fixed θ_0, θ_1 , this is a function of x)



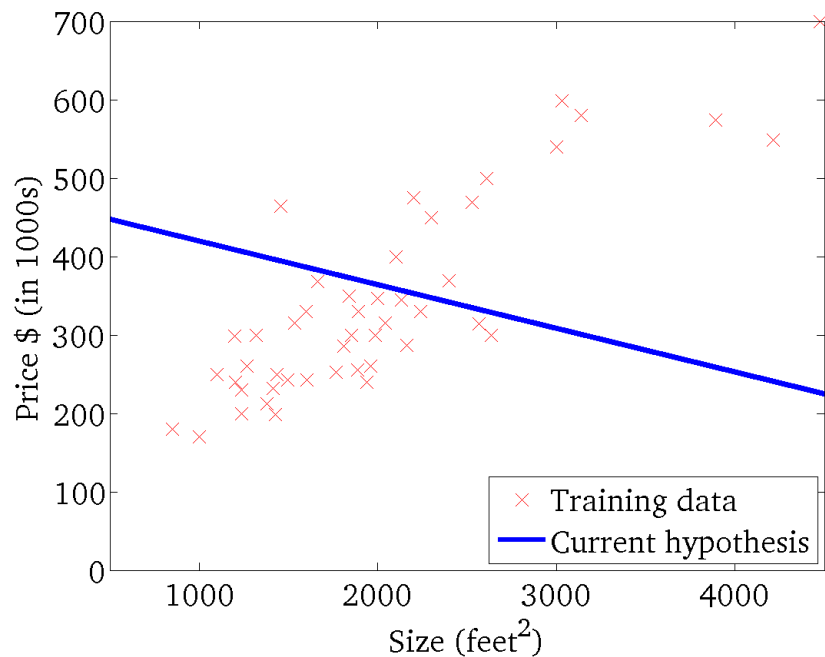
$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)



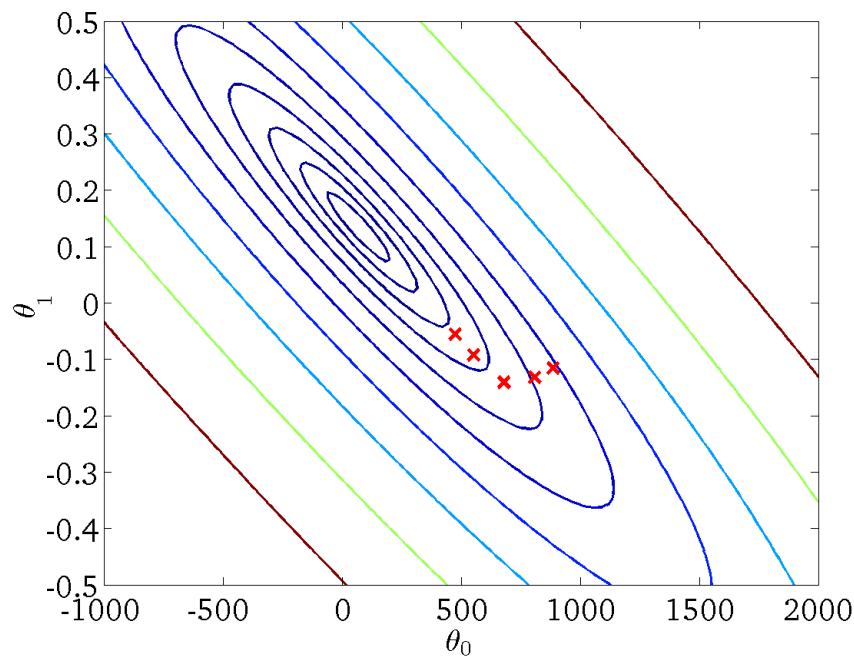
$$h_{\theta}(x)$$

(for fixed θ_0, θ_1 , this is a function of x)



$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)



$$h_{\theta}(x)$$

(for fixed θ_0, θ_1 , this is a function of x)



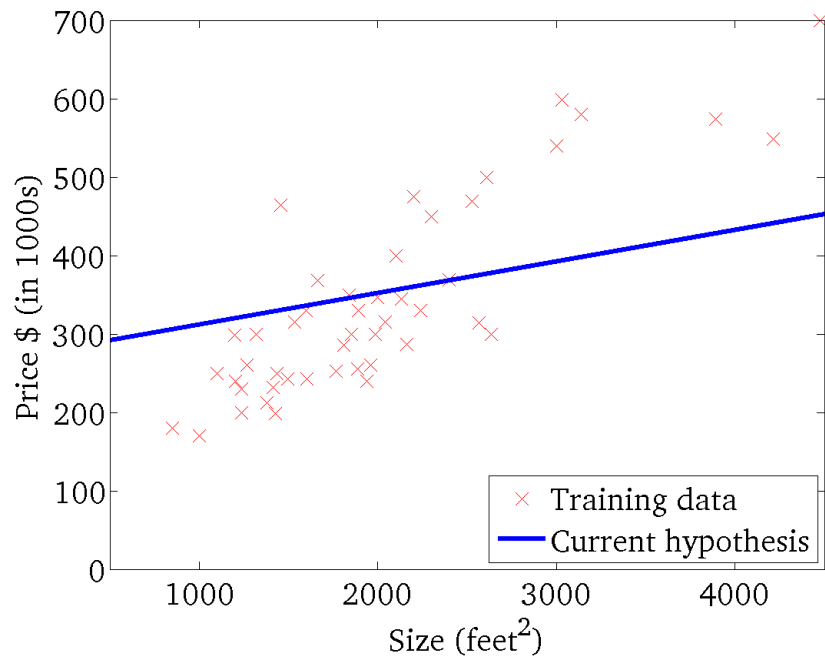
$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)



$$h_{\theta}(x)$$

(for fixed θ_0, θ_1 , this is a function of x)



$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)



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$$h_{\theta}(x)$$

(for fixed θ_0, θ_1 , this is a function of x)



$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)



“Batch” Gradient Descent

“Batch”: Each step of gradient descent uses all the training examples.

$$\rightarrow \sum_{i=1}^n (h_{\theta}(x^{(i)}) - y^{(i)})$$