CURS 2

11.10.2018

Ω ↗ evenimentul sigur

$\mathcal{F} \subseteq \mathcal{P}(\Omega)$ $\begin{cases} \emptyset \in \mathcal{F} \\ A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \end{cases}$

$(\Omega, \mathcal{F})$

$A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

$A \setminus B \in \mathcal{F}$

$A \Delta B \in \mathcal{F}$

Repetăm un experiment arbitrar de N ori

Abordarea frecventionistă: A și $N(A)$

e.g. : $N = 1000$

$A = \{$nr. de puncte de pe fața superioară $> 3\}$

$2, 3, 3, \underline{6, 5}, 1\} \ldots$
2

$\frac{N(A)}{N} \xrightarrow[N \to \infty]{} \mathbb{P}(A) \quad \left(\mathbb{P}(A) = \lim_{N \to \infty} \frac{N(A)}{N}\right)$

$\mathbb{P}(A) \in [0,1]$

$N(\Omega) = N \Rightarrow \frac{N(\Omega)}{N} = \frac{N}{N} = 1 \Rightarrow P(\Omega) = 1$

$\emptyset$ - evenimentul imposibil

$\frac{N(\emptyset)}{N} = 0 \Rightarrow P(\emptyset) = 0$

Presupunem că avem 2 evenimente A, B, $A \cap B = \emptyset$

$N(A \cup B) = N(A) + N(B)$

$\frac{N(A \cup B)}{N} = \frac{N(A)}{N} + \frac{N(B)}{N} \Rightarrow P(A \cup B) = P(A) + P(B)$ (finit aditivitate)

Def: Fie $(\Omega, \mathcal{F})$ un spațiu probabilizabil, o funcție $P: \mathcal{F} \to [0,1]$

a) $P(\Omega) = 1$

b) $\forall (A_n)_n \subseteq \mathcal{F}$, $A_i \cap A_j = \emptyset \Rightarrow P(\bigcup_n A_n) = \sum_n P(A_n)$

(σ-aditivitate)

Def: Se numește câmp de probabilitate asociat unui experiment aleator, un triplet $(\Omega, \mathcal{F}, P)$

Propoziție:

Experimentul 1 (aruncarea cu banul)

$\Omega = \{H, T\}$ // $\{H\} \cup \{T\}$

$\mathcal{F} = \{\{H, T\}, \{H\}, \{T\}, \emptyset\} = \mathcal{P}(\Omega)$

$P: \mathcal{F} \to [0,1]$

$P(\{H\}) = p$, $p \in [0,1]$

Experimentul 2 (aruncarea cu zarul)

$\Omega = \{1, 2, 3, 3, 4, 5, 6\}$

$\mathcal{F} = \mathcal{P}(\Omega)$ (2^6 elemente)

$(x_1, \dots, x_6)$

$\downarrow$

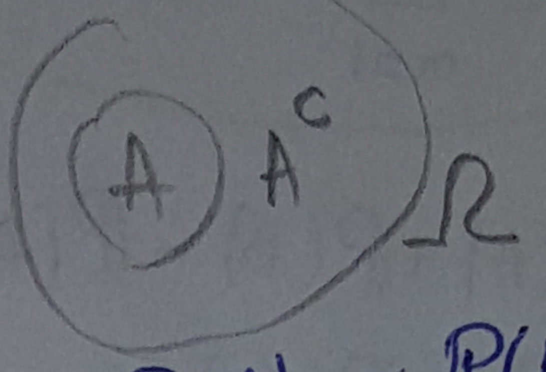
$0, 1$ 2^6

$\mathbb{P}: \mathcal{F} \to [0,1]$

$\mathbb{P}(i) = p_i \in [0,1], \; \sum_{i=1}^{6} p_i = 1$

Proprietăți: Fie $(\Omega, \mathcal{F}, \mathbb{P})$

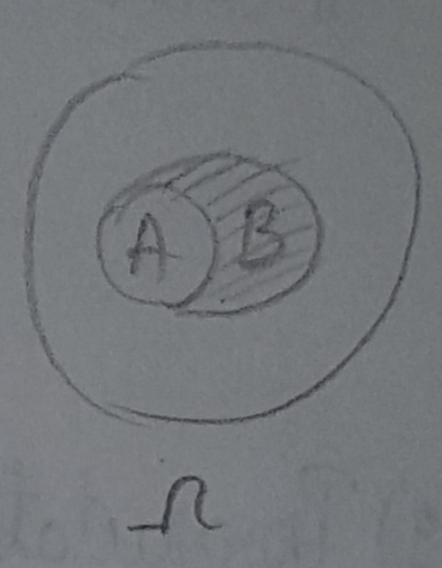
a) $\mathbb{P}(\emptyset) = 0$



b) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

c) Fie A, B, $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$ (monotonie)

d) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$



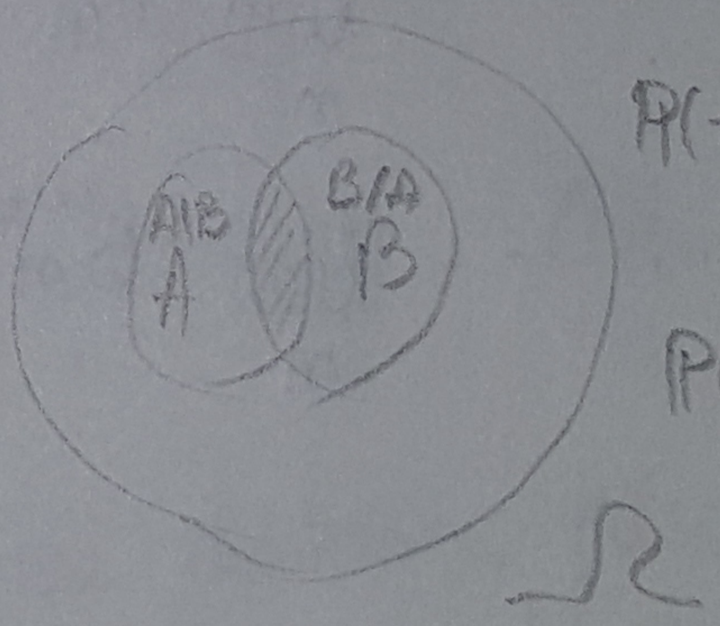
$A \cap B = \emptyset$

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

Presupunem $\mathbb{P}(\emptyset) > 0$, $A_n = \emptyset$

$\bigcup A_n = \emptyset$

$\sum_{n}^{n} \mathbb{P}(\emptyset) = \infty$ (contradicție)



$\mathbb{P}(A) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B)$

$\mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$

$\mathbb{P}(A \cup B) + \mathbb{P}(A \cap B)$

e) (Formula lui Poincaré)

$\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) \ldots (-1)^{n+1} \cdot \mathbb{P}(A_1 \cap \ldots \cap A_n)$

Dem. inducție

f) (Ineg. Boole)

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} P(A_n) = \sum_n P(B_n)$$

$B_1 = A_1$

$B_2 = A_2 \setminus A_1$

$B_3 = A_3 \setminus (A_1 \cup A_2)$

$\ldots$

$B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) \le P(A) + P(B)$

$P(B_n) \le P(A_n)$

$\bigcup_n B_n = \bigcup_n A_n$

g) Proprietatea de continuitate

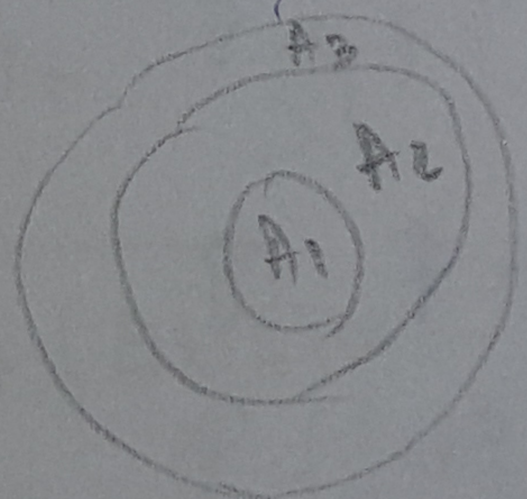
$(x_n) \nearrow a \quad f(a_n) \to f(a)$

$A_1 \subseteq A_2 \subseteq \ldots \qquad B_1 \supseteq B_2 \supseteq \ldots$

$(\nearrow)$: $\qquad (\searrow)$

$\lim_n A_n = \bigcup_n A_n \qquad P(\lim_n A_n) = \lim_n P(A_n)$

$\lim_n B_n = \bigcap_n B_n \qquad P(\lim_n B_n) = \lim_n P(B_n)$



Modelul clasic de probabilitate (câmpul de probabilitate al lui Laplace)

$\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$

$\mathcal{F} = \mathcal{P}(\Omega) \quad (2^n)$

$\mathbb{P} : \mathcal{F} \to [0,1]$ echirepartiția

$\mathbb{P}(\{\omega_i\}) = \frac{1}{n}$

$$\mathbb{P}(A) = \mathbb{P}\Big(\bigcup_{\{i \mid \omega_i \subset A\}} \{\omega_i\}\Big) = \sum_{\{i \mid \omega_i \subset A\}} \mathbb{P}(\omega_i) = \frac{1}{n} \sum_{\{i \mid \omega_i \in A_i\}} 1 = \frac{\# A}{n} = \frac{\# A}{\# \Omega}$$

$\# A$ – nr. caz. fav.

$\# \Omega$ – nr. caz. total posibile

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

Formula lui Poincaré, în mod Laplace, devine formula includerii - excluderii:

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n|$$

Experiment: $A = \{$capul apare mai devreme sau mai târziu$\}$

$$\Omega = \{H, T\}^{\mathbb{N}} = \{(\omega_k)_{k \in \mathbb{N}} \mid \omega_k \in \{H, T\}\}$$

$$B^A = \{f : A \to B\}$$

$$x_n = x(n), \quad x : \mathbb{N} \to \mathbb{R}$$

$$\mathbb{P}(A) = \lim_n \mathbb{P}(A_n) = \lim_n \left(1 - \frac{1}{2^n}\right) = 1$$

evenimentul aproape sigur

$\mathbb{P}(B) = 0 \Rightarrow$
$\Rightarrow B$ - even. nul

Prob. ca în n aruncări să pice numai T

$A_n = \{(w_k)_k \mid \exists\, i \in \{1, \dots, n\}$ a.î. $w_i = H\}$

$\Updownarrow$

În primele n aruncări am obținut k

$A = \bigcup_n A_n \qquad A_1 \subseteq A_2 \subseteq A_3 \subseteq$

A, B

$|A \times B| = |A| \cdot |B|$

$$\begin{array}{cc} a & b \\ \downarrow & \downarrow \\ n & p \text{ moduri} \end{array}$$

$(a, b) = np$ moduri

$|B| = p \quad (x_1, \dots, x_n),\ x_i \in B$

$\quad \overset{\shortparallel}{p^n}$

$n = p \Rightarrow p!$

$n < p \Rightarrow \dfrac{p!}{(p-n)!}$

<u>Experiment</u> (Problema aniversărilor)

Printre n persoane prezente, care este probabilitatea ca 2 persoane să aibă aceeași zi de naștere?

$\Omega = \{(w_1, \dots, w_n) \mid w_i \in \{1, \dots, 365\}\}$

$|\Omega| = 365^n$

A - evenimentul în care avem 2 persoane cu aceeași zi de naștere

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \frac{\# A^c}{\# \Omega} = 1 - \frac{365 \cdot 364 \cdot \ldots \cdot (365 - n + 1)}{365^n}$$

$n \in \mathbb{N}^*$

(*) $\varphi(n)$ funcția lui Euler

$$\frac{\varphi(n)}{n} = \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$$

Plecând de la formula lui Poincaré, ar. (*)