Lambda Calcul

Lambda termeni

lambda termen = variabila
| aplicare
| abstracticare

$M, N ::= x|(MN)|(\lambda x.M)$

Conventii:

- Se elimina parantezele exterioare (M N) P inseamna M N P
- Aplicarea este asociativa la stanga
  - M N P inseamna (M N) P
  - f x y z inseamna ((f x) y) z
- Corpul abstractizarii se extinde la dreapta $\lambda x.MN$ inseamna $\lambda x.(MN)$
- In termenul $M \equiv (\lambda x.xy)(\lambda x.yz)$
  - λ se numeste operator de legare
  - x este variabila legata
  - z este variabila libera
  - y are o aparitie legata, si una libera
  - multlimea variabilelor libere ale lui M este y, z
- N din $\lambda x.N$ se numeste **domeniul** de legare a lui x si toate aparitiile lui x in N sunt **legate**
- un termen fara variabile libere se numeste **inchis**(closed) sau **combinator**

$\beta - reductii$ reguli

Un pas de $\beta - reductie$ $\rightarrow_\beta$ este cea mai mica relatie pe lamba termeni care satisface regulile:

$$\overline{(\lambda.M)N \rightarrow_\beta M[N/x]}$$

$$\frac{M \rightarrow_\beta M'}{MN \rightarrow_\beta M'N}$$

$$\frac{N \rightarrow_\beta N'}{MN \rightarrow_\beta MN'}$$

$$\frac{M \rightarrow_\beta N'}{\lambda x.M \rightarrow_\beta \lambda x.M'}$$

Redenumire de variabile

M $\langle y/x \rangle$ - rezultatul obtinut dupa redenumirea lui x cu y in M

- $x\langle y/x\rangle \equiv y$
- $z\langle y/x\rangle \equiv z$, daca $x \neq z$
- $(MN)\langle y/x\rangle \equiv (M\langle y/x\rangle)(N\langle y/x\rangle)$
- $(\lambda x.M)\langle y/x\rangle \equiv \lambda y.(M\langle y/x\rangle)$
- $(\lambda z.M)\langle y/z\rangle \equiv \lambda z.(M\langle y/x\rangle)$, daca $x \neq z$

$\alpha - echivalenta$

$\alpha - echivalenta$ - cea mai mica relatie de congruenta pe multimea lambda termenilor, astfel incat pentru orice termen M si orice variabila y care nu apare in M, avem: $\lambda.M =_\alpha \lambda y.(M\langle y/x\rangle)$

Substitutii

$M[N/x]$ este rezultatul obtinut dupa inlocuirea lui x cu N in M.
1. Vrem sa inlocuim doar variabile libere: $x(\lambda xy.x)[N/x]$ este $N(\lambda xy.x)$ (NU $N(\lambda xy.N)$ sau $N(\lambda Ny.N)$
2. Nu vream sa legam variabile libere neintentionat asa ca redenumim variabilele legate inainte de substitutie.

$M[N/x]$

$x[N/x] \equiv N$
$y[N/x] \equiv y$, daca $x \neq y$
$(MP)[N/x] \equiv (M[N/x])(P[N/x])$
$(\lambda x.M)[N/x] \equiv \lambda x.M$
$(\lambda y.M)[N/x] \equiv \lambda y.(M[N/x])$, $x \neq y, y \notin FV(N)$
$(\lambda y.M)[N/x] \equiv \lambda y'.(M\langle y'/y\rangle[N/x])$, daca $x \neq y$, $y \in FV(N)$, y' variabila noua
$FV(N)$ - multimea variabilelor libere ale lui N

$\beta - reductii$

$\beta - reductie$ = procesul de a evalua lambda termeni prin "pasarea de argumente"
$\beta - redex$ = un termen de forma $(\lambda x.M)N$
redusul unui redex $(\lambda x.M)N$ este $M[N/x]$
forma normala = un lambda termen fara redex-uri

$\beta - forma$ normala

$M \twoheadrightarrow_{beta}$ - M poate fi $\beta - redus$ pana la M' in 0 sau mai multi pasi
M este:
slab normalizabil daca exista N in forma normala a.i. $M \twoheadrightarrow_\beta N$.
puternic normalizabil daca nu exista reduceri infinite care incep din M.
$(\lambda xy.y)((\lambda x.xx)(\lambda x.xx))(\lambda z.z)$ este **slab normalizabil**, dar NU puternic normalizabil.

Teorema Church-Rosser

Daca $M \twoheadrightarrow_\beta M_1$ si $M \twoheadrightarrow_\beta M_2$ atunci exista M' a.i. $M_1 \twoheadrightarrow_\beta M'$ si $M_2 \twoheadrightarrow_\beta M'$.
Consecinta. Un lambda termen are cel mult o $\beta - forma$ normala (modulo $\alpha - echivalenta$).

Strategii de evaluare

Strategia normala(leftmost-outermost)

Daca M_1 si M_2 sunt redex-uri si M_1 este un subtermen al lui M_2, atunci M_1 NU va fi urmatorul redex ales.
$\underline{(\lambda xy.y)((\lambda x.xx)(\lambda x.xx))}(\lambda z.z) \rightarrow_\beta \underline{(\lambda y.y)(\lambda x.x)}$
$\rightarrow_\beta \lambda x.x$

Strategia aplicativa(leftmost-innermost)

Daca M_1 si M_2 sunt redex-uri si M_1 este un subtermen al lui M_2, atunci M_2 NU va fi urmatorul redex ales.
$(\lambda xy.y)\underline{((\lambda x.xx)(\lambda x.xx))}(\lambda z.z)$
$\rightarrow_\beta (\lambda xy.y)((\lambda x.xx)(\lambda x.xx))(\lambda z.z)$

CBN si CBV

Strategia call-by-name(CBN) = strategia normala fara a face reducrei in corpul unei $\lambda - abstractizari$
Strategia call-by-value(CBV) = strategia aplicativa fara a face reduceri in corpui unei $\lambda - abstractizari$

Expresivitatea λ - calculului

Booleeni

T $\triangleq \lambda xy.x$
F $\triangleq \lambda xy.y$
if $\triangleq \lambda btf$.b t f
and $\triangleq \lambda b_1 \lambda b_2$ **if** $b_1 b_2$ **F**
or $\triangleq \lambda b_1 \lambda b_2$ **if** b_1 **T** b_2
not $\triangleq \lambda b_1$ **if** b_1 **FT**

Numere naturale

Numeralii Church: $\lambda fx.f^n x$
$\bar{0} \triangleq \lambda fx.f^0 x = \lambda fx.x$
$\bar{1} \triangleq \lambda fx.f^1 x = \lambda fx.fx$
$\bar{2} \triangleq \lambda fx.f^1 x = \lambda fx.f(fx)$
Succesor: **Succ** $\triangleq \lambda nfx.f(nfx)$
Adunare:
add $\triangleq \lambda mnfx.mf(nfx)$
add $\triangleq \lambda mn.m$ **Succ** n
Inmultirea: **mul** $\triangleq \lambda mn.m(\mathbf{add}\, n)\bar{0}$
Ridicarea la putere: **exp** $\triangleq \lambda mn.m(\mathbf{mul}\, n)\bar{1}$
Verifica numar = 0: **isZero** $\triangleq \lambda nxy.n(\lambda z.y)x$

Puncte fixe

Spunem ca x este un punct fix al functiei f daca $f(x) = x$
THM. In lambda calcul fara tipuri, orice termen are un punct fix.
Daca F si M sunt λ-termeni, spunem ca M este un **punct fix** al lui F daca $FM =_\beta M$

Combinatorii de puncte fixe:
Curry: $\mathbf{Y} \triangleq y.(\lambda x.y(xx))(\lambda x.y(xx))$
Turing: $\mathbf{\Theta} \triangleq (\lambda xy.y(xxy))(\lambda xy.y(xxy))$

Factorial

fact $n \;=\;$ if(**isZero** n)($\bar{1}$)(**mul** n(**fact**(**pred** n)))
fact $\triangleq$ **Y**(fn.if (**isZero** n)($\bar{1}$)(**mul** n(f(**pred** n)))

Lambda calcul cu tipuri

Tipuri simple

Fie $\mathbb{V} = \{\alpha, \beta, \gamma, ...\}$ o multime infinita de tipuri variabila. Multimea tuturor tipurilor simple $\mathbb{T}$ este definita prin:

$$\mathbb{T} = \mathbb{V} | \mathbb{T} \to \mathbb{T}$$

(Tipul variabila) Daca $\alpha \in \mathbb{V}$, atunci $\alpha \in \mathbb{T}$
(Tipul sageata) Daca $\sigma, \tau \in \mathbb{T}$, atunci $(\sigma \to \tau) \in \mathbb{T}$

Termeni si tipuri

Variabila. $x : \sigma$
Aplicare. Daca $M : \sigma \to \tau$ si $N : \sigma$, atunci $MN : \tau$.
Abstractizare. Daca $x : \sigma$ si $M : \tau$, atunci $\lambda x.M : \sigma \to \tau$

Deductie pentru Church $\lambda \to$

Multimea λ-termenilor cu pre-tipuri $\Lambda_\mathbb{T}$ este

$$\Lambda_\mathbb{T} = x | \Lambda_\mathbb{T} \Lambda_\mathbb{T} | \lambda x : \mathbb{T}.\Lambda_\mathbb{T}$$

O afirmatie este o expresie de form $M : \sigma$, unde $M \in \Lambda_\mathbb{T}$ si $\sigma \in \mathbb{T}$ (M- subiect, σ- tip)
O declaratie este o afirmatie in care subiectul este o variabila ($x : \sigma$)
Un context este o lista de declaratii cu subiecti diferiti.
O judecata este o expresie de forma $\Gamma \vdash M : \sigma$, unde $\vdash$ este context si $M : \sigma$ este o afirmatie.

Reguli

$$\frac{}{\Gamma \vdash x : \sigma}, \text{ daca } x : \in \Gamma \ (\text{var})$$

$$\frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} \ (\text{app})$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x : \sigma.M) : \sigma \to \tau} \ (\text{abs})$$

Strategii de evaluare

Multimea tipurilor:

$$\mathbb{T} = \mathbb{V} \mid \mathbb{T} \to \mathbb{T} \mid \text{Unit} \mid \text{Void} \mid \mathbb{T} \times \mathbb{T} \mid \mathbb{T} + \mathbb{T}$$

Multimea λ-termenilor cu pre-tipuri $\Lambda_\mathbb{T}$ este:

$$\Lambda_\mathbb{T} = x | \Lambda_\mathbb{T} \Lambda_\mathbb{T} | \lambda x : \mathbb{T}.\Lambda_\mathbb{T} \mid \text{unit} | \langle \Lambda_\mathbb{T}, \Lambda_\mathbb{T} \rangle$$

$|\text{fst } \Lambda_\mathbb{T} \ |\text{snd } \Lambda_\mathbb{T} \ |\text{Left } \Lambda_\mathbb{T} \ |\text{Right } \Lambda_\mathbb{T} \ |\text{case } \Lambda_\mathbb{T} \text{of } \Lambda_\mathbb{T}; \Lambda_\mathbb{T}$

Corespondenta Curry-Howard

λ - calcul cu tipuri	Deductie naturala
$\dfrac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash \langle M, N \rangle : \sigma \times \tau} (\times_I)$	$\dfrac{\Gamma \vdash \sigma \quad \Gamma \vdash \tau}{\Gamma \vdash \sigma \wedge \tau} (\wedge_I)$
$\dfrac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash fst M : \sigma} (\times_{E_1})$	$\dfrac{\Gamma \vdash \sigma \wedge \tau}{\Gamma \vdash \sigma} (\wedge_{E_1})$
$\dfrac{\Gamma \vdash p : \sigma \times \tau}{\Gamma \vdash snd\, p : \tau} (\times_{E_2})$	$\dfrac{\Gamma \vdash \sigma \wedge \tau}{\Gamma \vdash \tau} (\wedge_{E_2})$
$\dfrac{\Gamma \cup \{x : \sigma\} \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau} (\to_I)$	$\dfrac{\Gamma \cup \{\sigma\} \vdash \tau}{\Gamma \vdash \sigma \supset \tau} (\supset_I)$
$\dfrac{\Gamma \vdash \lambda x.M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} (\to_E)$	$\dfrac{\Gamma \vdash \sigma \supset \tau \quad \Gamma \vdash \sigma}{\Gamma \vdash \tau} (\supset_E)$

Teoria tipurilor $\to$ *Logica*
tipuri $\to$ *formule*
termeni $\to$ *demonstratii*
inhabitation a tipului $\sigma \to$ demonstratie a lui σ
tip produs $\to$ conjunctie
tip functie $\to$ implicatie
tip suma $\to$ disjunctie
tipul void $\to$ false
tipul unit $\to$ true
Propositions are types!
Proofs are terms!

Template folosit: Nelson Elias Rincon Quiñones