17.02.2022

Seminar 1

alg

1) $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$

$J = \{i_1, \dots, i_m\} \subset [p]$; $A \in M_p(\mathbb{R})$

$\det(A) : \sum M \cdot M'$

M minor de ordin m pe liniile I și coloanele J $= \{j_1, \dots, j_m\}$

$M' = (-1)^{i_1 + \dots + i_m + j_1 + \dots + j_m} \cdot \det(A_{I,J})$

$I = \{1\}$

$J : \{1\}, \{2\}, \{3\} \Rightarrow C_3^1 = 3$

C_p^m

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2) $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = \begin{vmatrix} 2b+a & 2b+a & 2b+a \\ b & a & b \\ b & b & a \end{vmatrix}$

$= (2b+a) \begin{vmatrix} 1 & 1 & 1 \\ b & a & b \\ b & b & a \end{vmatrix} = (2b+a) \begin{vmatrix} 1 & 0 & 0 \\ b & a-b & 0 \\ b & 0 & a-b \end{vmatrix}$

$= -(2b+a)(a-b)(b-a) = +(2b+a)(a-b)^2$

3) $\begin{vmatrix} a^2 & ab & b^2 \\ b^2 & a^2 & ab \\ b & b & a \end{vmatrix} = a^5 + b^5 + \cancel{a^2b^3} - \cancel{a^2b^3} - a^3b^2 - a^2b^3$

$= a^3(a^2-b^2) + b^3(b^2-a^2)$

$= (a-b)(a+b)(a^3-b^3)$

$= (a-b)(a+b)(a-b)(a^2+ab+b^2)$

$= (a-b)^2(a+b)(a^2+ab+b^2)$

4) $\begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix} = (a+b)^2 \begin{vmatrix} 1 & 1 & 1 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix}$

$= (a+b)^2 \begin{vmatrix} 0 & 1 & 0 \\ b^2-a^2 & a^2 & 2ab-a^2 \\ 2ab-b^2 & b^2 & a^2-b^2 \end{vmatrix}$

$= (a+b)^2(-1)\left[(b^2-a^2)(a^2-b^2) - (2ab-b^2)(2ab-a^2)\right]$

$= -(a+b)^2\left[-(a^2-b^2)^2 - ab(2a-b)(2b-a)\right]$

$= (a+b)^2\left[(a^2-b^2)^2 + ab(2a-b)(2b-a)\right]$

$= (a+b)^2(a^4+b^4-2a^2b^2+4a^2b^2-2a^3b-2b^3a+a^2b^2)$

$= (a+b)^2(a^4+b^4+3a^2b^2-2a^3b-2b^3a)$

$= (a^4-2ab^3)(-1)(a^2b^2-2a^2b^2)(b^4-2a^3b)$

$= +\,a(a^3-2b^3)(b^3-2a^3)(a^2b^2)b$

$= a^3b^3(a^3-2b^3)(b^3-2a^3)$

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7 $(a+b)^2\left[(a-b)^4 + 2ab(a-b)^2\right] =$

$= (a+b)^2(a^4 - C_4^1 a^3b + C_4^2 a^2b^2 - C_4^3 \cdot ab^3 + b^4 + 2a^3b - 4a^2b^2 +$
$+ 2ab^3) =$

$= (a+b)^2(a-b)^2(a^2+b^2)$

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5) $\begin{vmatrix} a+b & b+c & c+a \\ a^2+b^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix} =$

sau cu sumă de 2 det

$= 2 \begin{vmatrix} a+b+c & b+c & c+a \\ a^2+b^2+c^2 & b^2+c^2 & c^2+a^2 \\ a^3+b^3+c^3 & b^3+c^3 & c^3+a^3 \end{vmatrix} = \begin{vmatrix} a & \dots \\ a^2 & \dots \\ a^3 & \dots \end{vmatrix} + \begin{vmatrix} b & \dots \\ b^2 & \dots \\ b^3 & \dots \end{vmatrix}$

$= 2 \begin{vmatrix} a+b+c & -a & -b \\ a^2+b^2+c^2 & -a^2 & -b^2 \\ a^3+b^3+c^3 & -a^3 & -b^3 \end{vmatrix} =$

$= 2 \begin{vmatrix} c & +a & b \\ c^2 & +a^2 & b^2 \\ c^3 & a^3 & b^3 \end{vmatrix} =$

$= 2abc \begin{vmatrix} 1 & 1 & 1 \\ c & a & b \\ c^2 & a^2 & b^2 \end{vmatrix} =$

$= 2abc \begin{vmatrix} 0 & 0 & 1 \\ c-b & a-b & b \\ c^2-b^2 & a^2-b^2 & b^2 \end{vmatrix} = 2abc[(a^2-b^2)(c-b)-(a-b)(c^2-b^2)]$

$= 2abc(a-b)(c-b)(a+\cancel{b}-c-\cancel{b})$

~~$= 2abc(c-a)(c-b)(a-b)$~~

$= 2abc(a-c)(b-a)(b-c)$

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$$= \begin{vmatrix} a & b+c & c+a \\ a^2 & b^2+c^2 & c^2+a^2 \\ a^3 & b^3+c^3 & c^3+a^3 \end{vmatrix} + \begin{vmatrix} b & b+c & c+a \\ b^2 & b^2+c^2 & c^2+a^2 \\ b^3 & b^3+c^3 & c^3+a^3 \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} + \begin{vmatrix} b & c & a \\ b^2 & c^2 & a^2 \\ b^3 & c^3 & a^3 \end{vmatrix}$$

$$= abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ b^2 & c^2 & a^2 \end{vmatrix}$$

$$= abc(b-a)(c-b)(c-a) + abc(c-b)(a-c)(a-b)$$

$$= 2abc(b-a)(c-b)(c-a)$$

$$\begin{vmatrix} a & b & a+b \\ a^2 & b^2 & a^2+b^2 \\ a^3 & b^3 & a^3+b^3 \end{vmatrix} \underset{}{=} 0$$

$$\det(A+B) \neq \det(A)+\det(B) \quad !$$

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

$$\begin{vmatrix} L_1 \\ L_2 \\ L_3 \end{vmatrix} = \begin{vmatrix} L_1+L_2+L_3 \\ L_2 \\ L_3 \end{vmatrix}$$

$$= \begin{vmatrix} L_1 \\ L_2 \\ L_3 \end{vmatrix} + \underbrace{\begin{vmatrix} L_2 \\ L_2 \\ L_3 \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} L_3 \\ L_2 \\ L_3 \end{vmatrix}}_{=0} \quad !$$

7) $\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 5 & 5 & 5 & 5 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$

$= 5 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = 5 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$

$= 5 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 5$

(*) $\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{matrix} \qquad \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{matrix}$

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8) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 10 & 10 & 10 & 10 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$

$= 10 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = 10 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & +2 & -1 \\ 3 & 1 & -2 & -1 \\ 4 & -3 & -2 & -1 \end{vmatrix}$

$= 10 \begin{vmatrix} 1 & +2 & -1 \\ 1 & -2 & -1 \\ -3 & -2 & -1 \end{vmatrix} = 10 \begin{vmatrix} 0 & -2 & -1 \\ 0 & -2 & -1 \\ -4 & -2 & -1 \end{vmatrix}$ ‡

$= 10 \cdot 0 = 10 \overset{''}{0}$

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8) $A = \begin{pmatrix} M & N \\ O_{p,m} & P \end{pmatrix} \in M_{m+p,m+p}$

$M \in M_m(\mathbb{R})$;
$P \in M_p(\mathbb{R})$;
$N \in M_{m,p}(\mathbb{R})$

$\det(A) \overset{?}{=} \det(M) \cdot \det(P)$

$$A = \left(\begin{array}{cc|ccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hline 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & a_{53} & a_{54} & a_{55} \end{array}\right)$$

$m = 2$
$p = 3$

$I = \{1,2\}$ FIXAT! $\quad |I| = |J| = 2$

$J: \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}$
$\{3,4\}, \{3,5\}, \{4,5\}$.

$$\det(A) \xrightarrow[\text{Laplace}]{\text{formula}} \sum_J \overbrace{\det(A_{I,J})}^{M} \cdot (-1)^{i_1+i_2+j_1+j_2} \cdot \det(A_{\bar{I},\bar{J}}) = \ldots$$

$\bar{I} = [5] \setminus I$; $\bar{J} = [5] \setminus J$.
$\{1,2,3,4,5\}$ $\quad M'$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot (-1)^{1+2+1+2} \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix} +$$

$J\{1,2\}$

$$+ \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot (-1)^{1+2+1+3} \cdot \underbrace{\begin{vmatrix} 0 & a_{34} & a_{35} \\ 0 & a_{44} & a_{45} \\ 0 & a_{54} & a_{55} \end{vmatrix}}_{=0} +$$

$J=\{1,3\}$

$$+ \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \cdot (-1)^{1+2+1+4} \cdot \underbrace{\begin{vmatrix} 0 & a_{33} & a_{35} \\ 0 & a_{43} & a_{45} \\ 0 & a_{53} & a_{55} \end{vmatrix}}_{=0} +$$

$J=\{1,4\}$

$$+ \begin{vmatrix} a_{11} & a_{15} \\ a_{21} & a_{25} \end{vmatrix} \cdot (-1)^{1+2+1+5} \cdot \underbrace{\begin{vmatrix} 0 & a_{33} & a_{34} \\ 0 & a_{43} & a_{44} \\ 0 & a_{53} & a_{54} \end{vmatrix}}_{=0} +$$

$J=\{1,5\}$

$$+ \begin{vmatrix} \; & \; \end{vmatrix} \cdot (-1)^{1+2+2+3} \cdot \underbrace{\begin{vmatrix} 0 & \cdots \\ 0 & \\ 0 & \end{vmatrix}}_{=0} + \begin{vmatrix} \; & \; \end{vmatrix} \cdot (-1)^{1+2+2+4} \underbrace{\begin{vmatrix} 0 & \\ 0 & \\ 0 & \end{vmatrix}}_{=0} +$$

$J=\{2,3\}$ $\qquad$ $J=\{2,4\}$

$$+ \begin{vmatrix} \; & \; \end{vmatrix} \cdot (-1)^{1+2+2+5} \underbrace{\begin{vmatrix} 0 & \cdots \\ 0 & \cdots \\ 0 & \end{vmatrix}}_{=0} + \begin{vmatrix} \; & \; \end{vmatrix} \cdot (-1)^{1+2+3+4} \cdot \underbrace{\begin{vmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \end{vmatrix}}_{=0}$$

$J=\{2,5\}$ $\qquad$ $J=\{3,4\}$

$$+ \begin{vmatrix} \; & \; \end{vmatrix} \cdot (-1)^{1+2+3+5} \cdot \underbrace{\begin{vmatrix} 0 & 0 & \cdots \\ 0 & 0 & \\ 0 & 0 & \end{vmatrix}}_{=0} + \begin{vmatrix} \; & \; \end{vmatrix} \cdot (-1)^{1+2+4+5} \cdot \underbrace{\begin{vmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \end{vmatrix}}_{=0}$$

$J=\{3,5\}$ $\qquad$ $J=\{4,5\}$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{53} & a_{55} \end{vmatrix}$$

$= \det(M) \cdot \det(P) \cdot (-1)^{mp}$

!?

24.02.2022.

Seminar 2

12) $\begin{vmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \\ x_3 & x_1 & x_2 \end{vmatrix} = 3x_1x_2x_3 - (x_1^3 + x_2^3 + x_3^3) = 3\cdot 17 + 55 = 4$

$f = x^3 - 2x^2 + 2x + 17 = 0$

$S = x_1 + x_2 + x_3 = -\frac{b}{a} = 2$

$P = x_1x_2 + x_2x_3 + x_3x_1 = 2 \left(\frac{c}{a}\right)$

$x_1x_2x_3 = -\frac{d}{a} = -17$

$(x_1+x_2+x_3)^3 = x_1^3 + x_2^3 + x_3^3 + 3 \ldots$

$x_1^3 + x_2^3 + x_3^3 = (x_1+x_2+x_3)^3 - 3SQ + 3P =$

$= S^3 - 3SQ + 3P = 8 - 12 + 51 = 55$

$\boxed{x_1^3 + x_2^3 + x_3^3 = S^3 - 3SQ + 3P}$!

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13) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 4a+4 \\ b^2 & 2b+1 & 4b+4 \\ c^2 & 2c+1 & 4c+4 \end{vmatrix}$

$= \begin{vmatrix} a^2 & 2a+1 & 2 \\ b^2 & 2b+1 & 2 \\ c^2 & 2c+1 & 2 \end{vmatrix} = 2 \begin{vmatrix} a^2 & 2a+1 & 1 \\ b^2-a^2 & 2(b-a) & 0 \\ c^2-a^2 & 2(c-a) & 0 \end{vmatrix}$

$= 2\,[\,2(c-a)(b-a)(b+a) - 2(b-a)(c-a)(c+a)\,]$

$= 4(c-a)(b-a)(b+\cancel{a}-c-\cancel{a})$

$= 4(c-a)(b-a)(b-c)$

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14) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (x-y)(z-y)(x-z)(xy+zx+yz)$

14) $\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & & & \\ & & & \end{vmatrix}$

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15) $\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} \Rightarrow \begin{vmatrix} \cancel{a-b} & \cancel{b-a} & \cancel{0} & \cancel{0} \\ a-b & 0 & 0 & b-a \\ 0 & a-b & 0 & b-a \\ 0 & 0 & a-b & b-a \\ b & b & b & a \end{vmatrix}$

$= (a-b)^3 \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ b & b & b & a \end{vmatrix} = (a-b)^3 \begin{vmatrix} 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & -1 \\ a+b & b & b & a \end{vmatrix}$

$= (a-b)^3 \cdot (-1) \Big|$

$\times$

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15) $\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} = (a+3b) \begin{vmatrix} 1 & b & b & b \\ 1 & a & b & b \\ 1 & b & a & b \\ 1 & b & b & a \end{vmatrix}$ $L_1 - L_4$ $L_2 - L_4$ $L_3 - L_4$ ⋮

$= (a+3b) \begin{vmatrix} 1 & b & b & b \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix} = (a+3b)(a-b)^3.$

$a^2+3a+a\ldots-3$

$\cancel{a(a+3)}$

14) $\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} a^3-1 & 3(a^2-1) & 3(a-1) & 0 \\ a^2-1 & a^2+2a-3 & 2(a-1) & 0 \\ a-1 & 2(a-1) & a-1 & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix}$

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Temă dom.

$$a^2+2a-3 = a^2+3a-a-3 = a(a-1)+3(a-1) = (a-1)(a+3)$$

$(a-1)(a^2+\cancel{2}a+1)$

$a^3+\cancel{2}a^2+a$

$-a^2-2a$

-1

$(a-1)(a^2+a+1)$

$a^3+\cancel{a^2}+\cancel{a}-\cancel{a^2}-\cancel{a}-1$

$$= (a-1)^3 \begin{array}{c} \\ \begin{vmatrix} \overset{(a+1)^2}{a^2+2a+1} & 3(a+1) & 3 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{vmatrix}\end{array}$$

$$= (a-1)^3 \begin{vmatrix} (a+1)^2-3 & 3(a+1)-6 & 3 \\ a-1 & a-1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-1)^3\left[(a+1)^2(a-1) - 3(a-1) - 3(a+1)(a-1) + 6(a-1)\right]$$

$$= (a-1)^3(a-1)(a^2+2a+1+\cancel{3a}-3-\cancel{3a}-3)$$

$$= (a-1)^3(a-1)(a^2+2a-5)$$

$\cancel{a^2+5a+3a}$

16)
$$\begin{vmatrix} 1+x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & 1+x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & 1+x_3 & \dots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \dots & 1+x_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \dots & -1 \\ 0 & 1 & 0 & \dots & -1 \\ 0 & 0 & 1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cancel{0}\, x_1 & \cancel{0}\, x_2 & \cancel{0}\, x_3 & \dots & 1+x_n \end{vmatrix}$$

$\cancel{1+x_n}$

$$= \begin{vmatrix} 0 \\ -1 \\ -1 \\ \vdots \\ 1+x_1+\dots+x_n \end{vmatrix} \qquad \phi$$

17)
$$\begin{vmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & n-1 \end{vmatrix} =$$

$$= \begin{vmatrix} n & 0 & 0 & \dots & -n \\ 0 & n & 0 & \dots & -n \\ 0 & 0 & n & \dots & -n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & n-1 \end{vmatrix} =$$

$$\cancel{= (n-1)(n)^{n-1} = n^{n-2} - n^{n-1}}$$

$$\begin{vmatrix} 1+x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & 1+x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & 1+x_3 & \dots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \dots & x_n+1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \dots & -1 \\ 0 & 1 & 0 & \dots & -1 \\ 0 & 0 & 1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \dots & x_n+1 \end{vmatrix}$$

Adunăm toate coloanele la ultima.

$$= \cancel{\begin{vmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_2 & x_3 & \dots & & x_n+1 \end{vmatrix}} \quad \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & -1 \\ 0 & 0 & 1 & \dots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1 & x_2 & x_3 & \dots & x_1+x_n+1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & \dots & -1 \\ 0 & 1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & \dots & x_1+x_n+1 \end{vmatrix} = \dots = \begin{vmatrix} 1 & -1 \\ x_{n-1} & x_1+x_2+\dots x_n+1 \end{vmatrix}$$

$$= x_1+x_2+\dots+x_n+1$$

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17) $\begin{vmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & n-1 \\ -1 & -1 & -1 & \dots & \end{vmatrix} =$

$=$

$\frac{2n-n-1+1}{(n)}$

$\frac{(n+1)n}{2}$

18) $\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ n+1 & n+2 & n+3 & \dots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \dots & 3n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & \dots & n^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2 & 3 & \dots & n \\ n & n & n & \dots & n \\ 2n & 2n & 2n & \dots & 2n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & \dots & n^2 \end{vmatrix} = 0$

19) x_1 x_2

20) $\Delta_n = \begin{vmatrix} 3 & 2 & 0 & \dots & 0 \\ 1 & 3 & 2 & \dots & 0 \\ 0 & 1 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 3 \end{vmatrix} =$

n $\Delta_2 = \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = 7$

$\Delta_3 = \begin{vmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{vmatrix} = 15$

$$\Delta_4 = \begin{vmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 \end{vmatrix} =$$

$$= 3 \begin{vmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= 3\Delta_3 - 2\Delta_2 = 45 - 14 = \underline{31}$$

$$\boxed{\Delta_n = 3\Delta_{n-1} - 2\Delta_{n-2}}$$

$$\Leftrightarrow \Delta_n - 3\Delta_{n-1} + 2\Delta_{n-2} = 0$$

$$\Leftrightarrow \alpha^2 - 3\alpha + 2 = 0$$

$$\Leftrightarrow (\alpha - 2)(\alpha - 1) = 0 \Rightarrow \begin{cases} \alpha_1 = 2 \\ \alpha_2 = +1 \end{cases}$$

Pt. $\alpha_2 \neq \alpha_1$

$$\Rightarrow \Delta_n = \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} = \frac{\alpha_2^{n+1} - \alpha_1^{n+1}}{\alpha_2 - \alpha_1}$$

!

$$\frac{\alpha_2^{n+1} - \alpha_1^{n+1}}{\alpha_2 - \alpha_1} = \frac{(\alpha_2 - \alpha_1)(\alpha_2^n + \alpha_2^{n-1}\cdot\alpha_1 + \alpha_2^{n-2}\cdot\alpha_1^2 + \ldots + \alpha_2\cdot\alpha_1^{n-1} + \alpha_1^n)}{\alpha_2 - \alpha_1}$$

$$= \underbrace{(\alpha_2^n + \alpha_2^{n-1}\cdot\alpha_1 + \alpha_2^{n-2}\cdot\alpha_1^2 + \ldots + \alpha_2\cdot\alpha_1^{n-1} + \alpha_1^n)}_{(n+1)\text{ termeni}}$$

Pt. cazul $\alpha_1 = \alpha_2 = \alpha$

$$\Downarrow$$

$$\underline{\Delta_n = (n+1)\alpha^n}$$

$$\Delta_n - \alpha\Delta_{n-1} + \beta\gamma\Delta_{n-2} = 0$$

Dc. $\begin{cases} \alpha_1 \neq \alpha_2 \Rightarrow \Delta_n = \dfrac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} \\ \alpha_1 = \alpha_2 = \alpha \Rightarrow \Delta_n = (n+1)\alpha^n \end{cases}$

!

Scanned with CamScanner

Temă
...

~~$\alpha = 2\lambda$~~ ; $\beta\gamma = \lambda^2$

$\alpha = 2\lambda$; $\beta\gamma = \lambda^2$

$D_n - 2\lambda D_{n-1} - \lambda^2 \cdot D_{n-2} = 0$

$\Rightarrow D_n - \lambda D_{n-1} = \lambda D_{n-1} - \lambda^2 \cdot D_{n-2}$

$= \lambda (D_{n-1} - \lambda D_{n-2})$

$= \lambda^{n-3} (D_3 - \lambda D_2) = \lambda^n$

~~$D_3 = \alpha^3 - 2\beta\gamma = \ldots$~~

$D_3 = \alpha^3 - 2\alpha\beta\gamma = \cancel{8\gamma^3 - 4\lambda}$

$8\lambda^3 - 4\lambda^3$

$= 4\lambda^3$

$D_2 = \alpha^2 - \beta\gamma = 4\lambda^2 - \lambda^2 = 3\lambda^2$

pt. $\lambda_1 = \lambda_2 = \lambda$;

$\Rightarrow D_n = (n+1)\lambda^n$

Temă:

$$\begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{vmatrix}$$

$= \ldots (-1)^{1+2+1+2+3} \ldots \begin{vmatrix} 1 & 1 \\ x_3 & x_4 \\ x_3^2 & x_4^2 \end{vmatrix}$

$= \Big|$ $I = 1,2$

$J = 1,2$; $1,3$; ~~$1,4$~~ ; ~~$1,5$~~ ; $2,3$; ~~$2,4$~~ ;

~~$2,5$~~ ; ~~$3,4$~~ ; ~~$3,5$~~ ; ~~$4,5$~~

al II-lea dă 0

$$\begin{vmatrix} M & N \\ P & 0 \end{vmatrix} = -\det(P)\cdot\det(N)$$

01.03.2022.

Curs 3

Spațiu vectorial peste un corp comutativ $\underline{\underline{K}}$

ex: (matrice)

= grup abelian cu o multiplicare externă cu scalari din K $(\alpha, v) \to \alpha v \in V$; $\alpha \in K$, $v \in V$

Exemple: 0 !

+ subsp. afini

04.03.2022.

Seminar 3

21.
$$\begin{vmatrix} 1 & x & x & x \\ 1 & a & 0 & 0 \\ 1 & 0 & b & 0 \\ 1 & 0 & 0 & c \end{vmatrix} = -x\begin{vmatrix} 1 & a & 0 \\ 1 & 0 & b \\ 1 & 0 & 0 \end{vmatrix} + c\begin{vmatrix} 1 & x & x \\ 1 & a & 0 \\ 1 & 0 & b \end{vmatrix}$$

$$= +xa(-b) + c\left[(-x)\cdot b + a(b-x)\right] =$$

$$= -abx - bxc + abc - acx$$

22)
$$\left|\begin{array}{ccc|cc} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 1 \\ 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{array}\right| = (-1)^{1+2+1+2}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\cdot\begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ x_2^2 & x_3^2 & x_4^2 \end{vmatrix}$$

$$+ (-1)^{1+2+1+3}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_3 & x_4 \\ x_1^2 & x_3^2 & x_4^2 \end{vmatrix} + (-1)^{1+2+2+3}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}\begin{vmatrix} 0 & 1 & 1 \\ 0 & x_3 & x_4 \\ 0 & x_3^2 & x_4^2 \end{vmatrix}$$

Scanned with CamScanner

Temă
acasă.

$= (x_3-x_2)(x_4-x_3)(x_4-x_2) - 2(x_3-x_1)(x_4-x_3)(x_4-x_1)$

24. $D_n = \begin{vmatrix} 1+x^2 & x & 0 & \cdots & 0 \\ x & 1+x^2 & x & \cdots & 0 \\ 0 & x & 1+x^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1+x^2 \end{vmatrix}$

$= (1+x^2) \underbrace{\begin{vmatrix} 1+x^2 & x & \cdots & 0 \\ x & 1+x^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1+x^2 \end{vmatrix}}_{D_{n-1}} - x \begin{vmatrix} x & x & \cdots & 0 \\ 0 & 1+x^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1+x^2 \end{vmatrix}$

$= (1+x^2) D_{n-1} - x^2 \cdot D_{n-2}$

$\Rightarrow \cancel{D_n + x^2 \cdot D_{n-2} - (1+x^2) \cdot D_{n-1} = 0}$

$\Leftrightarrow \cancel{r^2 + x^2 \cdot r - (1+x^2) = 0}$

$\cancel{\Delta = x^4 - 4x^2 + 4 = (x^2-2)^2}$

$\cancel{\Rightarrow r_{1,2} = \frac{-x^2 \pm (x^2-2)^2}{2}} \quad \frac{x^4-5x^2+4}{2}, \quad \frac{x^4+4x^2+4}{2}$

$\cancel{D_n = \frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}} = \cancel{\frac{x^4-5x^2}{2^{n+1}} (x^4-5x^2+4)(x^4+4x^2+4)}$

$=$

$$D_2 = \begin{vmatrix} 1+x^2 & x \\ x & 1+x^2 \end{vmatrix} = (1+x^2)^2 - x^2$$
$$= (1+x^2-x)(1+x^2+x)$$
$$= x^4 + x^2 + 1$$

$$D_n - (1+x^2)\cdot D_{n-1} + x^2 \cdot D_{n-2} = 0$$

$$\rho^2 - (1+x^2)\rho + x^2 = 0$$

$$\Delta = x^4 + 2x + 1 - 4x^2 = x^2 - 2x + 1 = (x-1)^2$$

$$\rho_{1,2} = \frac{1+x^2 \pm (x^2 - 2x + 1)}{2} \quad \begin{matrix} x \\ x^2 - x + 1 \end{matrix} \quad /$$

$$D_n = \frac{\rho_1^{n+1} - \rho_2^{n+1}}{\rho_1 - \rho_2} = \frac{x^{n+1} - (x^2 - x + 1)^{n+1}}{-x^2 - 1}$$

$(n+1)\rho^n$

$$\boxed{\rho_{1,2} = 1 \text{ sau } x^2}$$

$\Leftrightarrow$ Dacă $\rho_1 = \rho_2 = \rho \Rightarrow D_n = (n+1)\rho^n$

Dacă $\rho_1 \neq \rho_2 \Rightarrow D_n = \dfrac{\rho_2^{n+1} - \rho_1^{n+1}}{\rho_2 - \rho_1} = \dfrac{x^{2n+2} - 1}{x^2 - 1}$ $\quad 2(n+1)$

$$= \frac{(x^2-1)(x^{n+1} + x^n + \dots)}{x^2-1}$$

$\Leftrightarrow$ Dacă $\rho_2 = 1 \Rightarrow x = \pm 1 \Rightarrow D_n = n+1$

Dacă $\rho_2 \neq 1 \Leftrightarrow x \neq \pm 1 \Rightarrow D_n = \dfrac{(x^2)^{n+1} - 1}{x^2 - 1}$

Spații vectoriale

$(V, +)$ sp. vectorial?

29. $V = \{ f \in \mathbb{R}[x] \mid grad(f) = 2k+1 \}$ NU! pt. că $0 \notin V$, trb. ca V să fie grup abelian, deci să conțină elementul neutru

28). $V = \{ f \in \mathbb{R}[X] \mid f \text{ monic} \}$

$f(X) = X^n + a_{n-1} \cdot X^{n-1} + a_{n-2} \cdot X^{n-2} + \ldots + a_1 \cdot X + a_0, a_i \in \mathbb{R}$

șir funcții $f : \mathbb{N} \to \mathbb{R}$; $f(n) \in \mathbb{R}$

$\{ f(n) \}$

$X = (0^{-f(0)}, 1^{-f(1)}, 0, 0 \ldots)$

---

monic = coeficientul ~~gr~~ elem. cu gradul cel mai mare $= 1$.

f monic $\Leftrightarrow a_n = 1$

$f(X) = X^2 + 3x + 5 \in V$

$g(X) = X^2 + 7x + 2 \in V$

$f(x) + g(x) = 2x^2 + 10x + 7 \notin V \Rightarrow$ nu e parte stabilă,

deci nu poate fi spațiu vectorial.

---

27). $V = \{ f \in \mathbb{R}[x] \mid f(a) = f(-a), \forall a \in \mathbb{R} \}$

fct. pară

e.n.: $0 \in V$

~~fie $f(X), g(X) \in V$, dem. că $f(x)+g(x) = f(-$~~

fie $f, g \in V$, dem. că $(f+g)(a) = (f+g)(-a)$

$(f+g)(a) = f(a) + g(a) \overset{def.}{=} f(-a) + g(-a) = (f+g)(-a) \in V$

$\Rightarrow$ parte stabilă

$(f+g)+h = f+(g+h) \to$ asociativitatea unui polinom.

$(-f)(a) = -f(a) = -f(-a) = (-f)(-a) \Rightarrow$
$\forall a \in \mathbb{R}$

$\Rightarrow -f \in V$ ~~[illegible]~~ invers.

$f+g=g+f$ (din comutativitatea adunării nr. reale

$\Rightarrow (V,+) \rightarrow$ grup abelian

Axiome de dem.:

I $1\cdot f=f$ ✓

$(1\cdot f)(a)=1\cdot f(a)=f(a),\ \forall a\in\mathbb{R} \Rightarrow 1\cdot f=f$

. . . . .

(V $\alpha\cdot(\beta\cdot f)=(\alpha\beta)\cdot f$) ?

II $(\alpha\cdot f)(a)=\alpha\cdot f(a)=\alpha\cdot f(-a)=(\alpha\cdot f)(-a) \Rightarrow \alpha\cdot f\in V$
$\forall a\in\mathbb{R}$

din toate (gr. abelian + axiome)

$\Rightarrow V$ este spațiu vectorial $/\mathbb{R}$

spațiu vectorial peste $\mathbb{R}$

3) $V\subset\mathbb{R}^2,\ V=\left\{\begin{pmatrix}x\\y\end{pmatrix} \middle| \ ax+by=0\right\}$

V subspațiu ?

$\boxed{V \text{ subspațiu} \Leftrightarrow \forall \alpha,\beta\in\mathbb{R} \text{ și } \forall v,w_2\in V \Rightarrow \alpha v+\beta w_2\in V}$

$v=\begin{pmatrix}x_1\\y_1\end{pmatrix}\in V \Rightarrow ax_1+by_1=0$

$\alpha v_1+\beta v_2=\begin{pmatrix}\alpha x_1+\beta x_2\\ \alpha y_1+\beta y_2\end{pmatrix}$

$w_2=\begin{pmatrix}x_2\\y_2\end{pmatrix}\in V \Rightarrow ax_2+by_2=0$

$\Rightarrow a(\alpha x_1+\beta x_2)+b(\alpha y_1+\beta y_2)=0$

$\Leftrightarrow \underbrace{a\alpha x_1+a\beta x_2+b\alpha y_1}+b\beta y_2=0$

$\Rightarrow \underbrace{\alpha\underbrace{(ax_1+by_1)}_{0}+\beta\underbrace{(ax_2+by_2)}_{0}=0}_{\text{Adev.}}$

$\Rightarrow \alpha v_1+\beta v_2\in V,\ \forall \alpha,\beta\in\mathbb{R}$ și $\forall v_1,v_2\in V$
$\Rightarrow V$ subspațiu vectorial.

ūmă

m.

$ax + by = 0$

ec. omogenă (are „= 0")

$W = \{ (x, y) \in \mathbb{R}^2 \mid x + y - 1 = 0 \}$

Subspațiu?

$0 + 0 - 1 = -1 \neq 0 \Rightarrow \underset{\underset{0_{\mathbb{R}^2}}{\parallel}}{(0,0)} \notin W \Rightarrow W$ nu e subspațiu vectorial.

vectorul nul e din spațiul acela vectorial

$\forall v_1, v_2 \in W; \alpha, \beta \in \mathbb{R} \Rightarrow \alpha \cdot v_1 + \beta v_2 \in W$

$\Rightarrow \begin{cases} v_1 = (x_1, y_1) \text{ cu } x_1 + y_1 - 1 = 0 \\ v_2 = (x_2, y_2) \text{ cu } x_2 + y_2 - 1 = 0 \end{cases}$

$\Rightarrow \alpha v_1 + \beta v_2 = (\alpha x_1 + \beta x_2, \alpha y_2 + \beta y_2)$

$\Rightarrow \alpha x_1 + \beta x_2 + \alpha y_2 + \beta y_1 - 1 = 0$

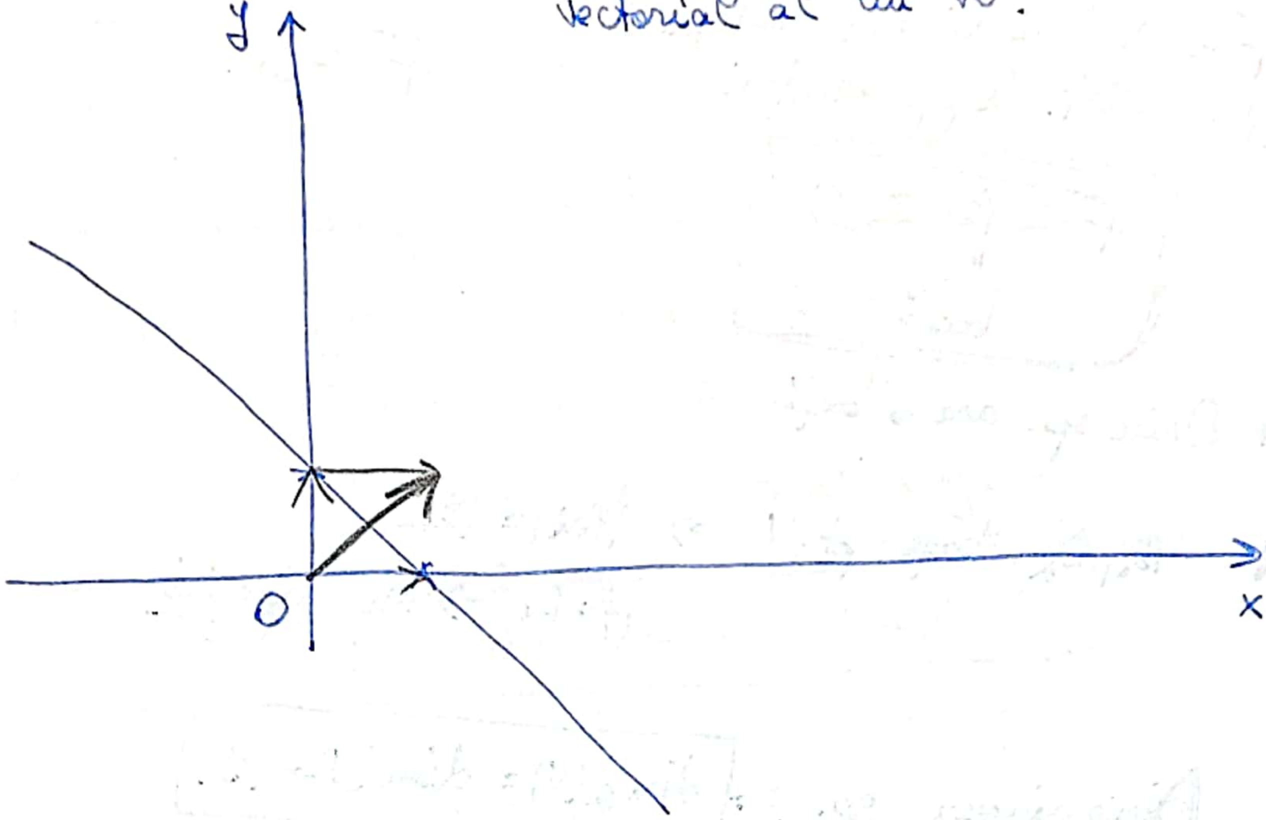
$\Leftrightarrow \underbrace{\alpha(x_1 + y_1 - 1)}_{a_1} + \alpha - \alpha y_1 + \underbrace{\beta(x_2 + y_2 - 1)}_{a_2} - \beta y_2 + \beta = 0$

$\Leftrightarrow \underbrace{a_1 + a_2}_{\overset{\shortparallel}{0}} + \alpha(1 - y_1) + \beta(1 - y_2) = 0$

$\Rightarrow y_1 = y_2 = 1$

$\alpha(x_1+y_1)+\beta(x_2+y_2) = 0$

$\Leftrightarrow \alpha+\beta = 1 \neq 0$ pt. $\alpha \neq 1-\beta \Rightarrow$ nu e subspațiu vectorial al lui W.



$y_1 = -x_1$
$y_2 = -x_2$

$y_1+y_2 = -(x_1+x_2)$

$y_1 = -x_1+1$
$y_2 = -x_2+1$

$y_1+y_2 = -(x_1+x_2)+2$

dă o dr. ∥ nu aceeași

$\Downarrow$

~~...~~ spațiu afin

$\boxed{\text{Bază}}$ = sist. de vectori liniar independent + sist. de generatori pt. V.

$\boxed{\text{Sistem de generatori:}}$

Fie $v \in \mathbb{R}^2$. Dacă $\{v_1, v_2\}$ sist. de gen. în $\mathbb{R}^2 \Rightarrow$

$\Rightarrow \exists\, \alpha, \beta \in \mathbb{R}$ a.î. $\boxed{v = \alpha v_1 + \beta v_2}$

$\boxed{\text{Liniar independenți:}}$ $\Leftrightarrow$ orice combinație liniară nulă de acești vectori este ~~lin~~ trivială.

Temă
...

Teorema 9

F = vectori liniar independenți, $F \subset S$
S = sist. de generatori

$$\boxed{F \subset \underset{\substack{\shortparallel \\ \text{bază}}}{B} \subset S}$$

- Orice sp. are o bază.

! B_1, B_2 baze pt. V $\Rightarrow |B_1| = |B_2|$
$f: B_1 \to B_2$ bijecție

Dimensiunea sp. $= \boxed{\dim_{\mathbb{R}}(V) = \dim(V) = |B|}$

! $\dim_{\mathbb{R}} 0 = 0$
$\dim(M_{m,n}(\mathbb{R})) = m \cdot n$
$\dim(\mathbb{R}[X]_2) = 3$

Teorema Grassmann: PiE

$$\boxed{\dim(V_1+V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)}$$

Morfism de spații vectoriale: = o fct.

- aditivă: $\cancel{f(x+y) = f(x) + f(y)}$
$f(V_1+V_2) = f(V_1) + f(V_2)$

- omogenă: $f(\alpha V) = \alpha \cdot f(V)$

$$\boxed{!\ \forall \alpha, \beta \in \mathbb{R} \text{ și } \forall v_1, v_2 \in V \Rightarrow f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)}$$

$f(0_V) = 0_W$ $\quad$ $f: V \longrightarrow W$ morfism

- $U \subset V$ subsp $\Rightarrow f(U)$ subsp. în W

! $Ker(f) = \{ v \in V \mid f(v) = 0_W \} = f^{-1}(0_W) =$
$=$ nucleul morfismului f = subsp. în V

T. Rang-Defect $\quad f: V \longrightarrow W$ morf. de sp.
FINIT DIMENSIONALE

$$\boxed{\dim(V) = \underbrace{\dim(Im(f))}_{\text{rang}} + \underbrace{\dim(Ker(f))}_{\text{defect}}}$$

Spații vectoriale factor:
$V/X = \{ \hat{v} \mid v \in V \}$; $\hat{v}_1 = \hat{v}_2 \Leftrightarrow v_1 - v_2 \in X$

T. fund. de izomorfism: $V / Ker(f) \cong Im(f)$

V-sp ; X-ssp $\Rightarrow \boxed{\dim(V/X) = \dim(V) - \dim(X)}$

$$\boxed{\text{Subspații} \Leftrightarrow \forall \alpha, \beta \in \mathbb{R} \text{ și } \forall v_1, v_2 \in V \Rightarrow \alpha v_1 + \beta v_2 = V}$$

$S = \{ A = (a_{ij}) \in M_n(\mathbb{R}) \mid a_{ij} = a_{ji} \}$ mulț. mat. sim. față de diag. pr.

Fie $A =$ ~~$a_{11}\ a_{12} \ldots a$~~
$a_{11} \quad a_{12} \ldots a_{1n}$
$a_{21} \quad a_{22}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

pt. $A = 0_n \in S$
$\downarrow$
el. neutru ✓

27) $\{f \in \mathbb{R}[X] \mid f(a) = f(-a), \forall a \in \mathbb{R}\}$

$\to \forall f, g \in \mathbb{R}[X] \Rightarrow (f+g)(a) = (f+g)(-a)$

$\Rightarrow f(a) + g(a) = f(-a) + g(-a) = (f+g)(-a) \in V \Rightarrow$ p.s. ✓

$\to \forall f, g, h \in \mathbb{R}[X] \Rightarrow (f+g)+h = f+(g+h)$ asoc. pol. ~~[illegible]~~ ✓

$\to \forall f, g \in \mathbb{R}[X] \Rightarrow f+g = g+f$ ✓

$\to (-f)(a) = -f(a) = -f(-a) = (-f)(-a) \in V \Rightarrow$ invers ✓

$\to 0 \in V$ e.n ✓

gr. ab.

Axiome: $1 \cdot f = f$; $\alpha(\beta f) = (\alpha\beta) \cdot f$

$(1 \cdot f)(a) = 1 \cdot f(a) = 1 \cdot f(-a) = (1 \cdot f)(-a)$

$(\alpha f)(a) = \alpha \cdot f(a) = \alpha \cdot f(-a) = (\alpha f)(-a)$

$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$; $v_2 = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix}$

$\alpha \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} \alpha + 3\beta \\ 2\alpha + 6\beta \\ 3\alpha + 7\beta \end{pmatrix} = 0$

$\Rightarrow \begin{cases} \alpha + 3\beta = 0 \Rightarrow \alpha = -3\beta \\ \alpha + 3\beta = 0 \\ 3\alpha + 7\beta = 0 \end{cases}$

$-2\beta = 0 \Rightarrow \beta = \alpha = 0$

$\Rightarrow$ DA e. ind.

$\alpha + 3\beta$

$\alpha + 3\beta$

$\alpha + 3\beta$

$\alpha + 3\beta = 0$

$\boxed{\alpha = -3\beta} \Rightarrow$ NU e l. ind.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $x=0$; $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $y=0$

$x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (z_1+z_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

10.03.2022.

Seminar 4

32) $V \subset \mathbb{R}^2$; $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, ax+by=0 \right\}$

$\forall \alpha, \beta \in \mathbb{R}$; $v_1, v_2 \in V$, $\alpha v_1 + \beta v_2 \in V$

$\Leftrightarrow \alpha \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \beta \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \end{pmatrix}$

unde $a(\alpha x_1 + \beta x_2) + b(\alpha y_1 + y_2 \cdot \beta) = 0$

$\Leftrightarrow a\alpha x_1 + a\beta x_2 + \alpha b y_1 + b\beta \cdot y_2 = 0$

$\Leftrightarrow \alpha(\underbrace{ax_1 + by_1}_{0}) + \beta(\underbrace{ax_2 + by_2}_{0}) = 0$

$\Rightarrow$ Adev. : V subspațiu vect.

---

33) $U \subset \mathbb{R}^3$, $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, ax+by+cz=0 \right\}$

$\forall \alpha, \beta \in \mathbb{R}$; $v_1, v_2 \in V$: $\alpha v_1 + \beta v_2 \in V$

$\Leftrightarrow a(\alpha x_1 + \beta y_1) + b(\alpha x_2 + \beta y_2) + c(\alpha z_1 + \beta y_2) =$

$= a\alpha x_1 + a\beta y_1 + b\alpha x_2 + b\beta y_2 + c\alpha x_3 + c\beta y_3 =$

$= \alpha(\underbrace{ax_1 + by_1 + cz_1}_{=0}) + \beta(\underbrace{ax_2 + by_2 + cz_2}_{=0}) =$

$= 0$

$\Rightarrow$ QED.

temă
un.

34) $S = \{A=(a_{ij}) \in M_n(\mathbb{R}) \mid a_{ij}=a_{ji}\}$ subsp.

Fie $A, B \in S$; $\alpha,\beta\in\mathbb{R}$: $\alpha\cdot A+\beta\cdot B\in S$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$A=(a_{ij})\in S$

$B=(b_{ij})\in S$

$\alpha\cdot A = \alpha(a_{ij}) = (\alpha a_{ij})$

$= (\alpha a_{ji}) = \alpha(a_{ji})$

$\alpha A+\beta\cdot B = (\alpha a_{ij}+\beta b_{ij}) = \alpha a_{ij}+\beta b_{ij} = \alpha a_{ji}+\beta b_{ji}$

$(\alpha A+\beta B)_{ij} = (\alpha A+\beta B)_{ji} \Rightarrow \alpha A+\beta B\in S \Rightarrow S$ subsp

$a_{ij}=a_{ji} \leftarrow A^t \quad = \{A\in M_n(\mathbb{R}) \mid A=A^t\}$

35) $D=\{B=(b_{ij})\in M_n(\mathbb{R}) \mid b_{ij}=0, i\neq j\}$ sbsp.

Fie $A,B\in D$ și $\alpha,\beta\in\mathbb{R}$

$(\alpha A+\beta\cdot B)_{ij} = \alpha\cdot a_{ij}+\beta\cdot b_{ij} = \alpha\cdot 0+\beta\cdot 0 = 0$

$\Rightarrow \alpha A+\beta\cdot B\in D \Rightarrow$ sbsp.

$\dim(\mathbb{R})=1$; bază $\forall x\neq 0$

baza canonică $= 1 \in\mathbb{R}$

$\dim(\mathbb{R}^2)=2$, baza canonică $=\{e_1, e_2\}$

$\mathbb{R}^2 = \left\{\begin{pmatrix}x\\y\end{pmatrix} \mid x,y\in\mathbb{R}^2\right\}$

$\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}0\\1\end{pmatrix} = x\cdot e_1+y\cdot e_2$

$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \middle| \; x_i \in \mathbb{R} \right\}$

$\dim(\mathbb{R}^n) = n$, baza canonică $= \{ e_1, e_2, \dots e_n \}$

Fie $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = x_1 \cdot e_1 +$

$+ \dots + x_n \cdot e_n$.

---

38) Fie $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix}$; $v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_4 = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$

$\alpha v_1 + \beta v_2 = 0_{\mathbb{R}^3}$

$\begin{pmatrix} \alpha + 3\beta \\ 2\alpha + 6\beta \\ 3\alpha + 7\beta \end{pmatrix} = 0_{\mathbb{R}^3} \Rightarrow \begin{cases} \alpha + 3\beta = 0 \\ 2\alpha + 6\beta = 0 \\ 3\alpha + 7\beta = 0 \end{cases}$

$\Rightarrow \begin{matrix} \alpha + 3\beta = 0 \\ 3\alpha + 7\beta = 0 \end{matrix} \Rightarrow \alpha = \beta = 0$

$\Rightarrow v_1, v_2 =$ liniar independenți

$\alpha v_3 + \beta v_4 = 0_{\mathbb{R}^3}$; $\begin{cases} \alpha + 3\beta = 0 \\ 2\alpha + 6\beta = 0 \\ 3\alpha + 9\beta = 0 \end{cases}$

$\Rightarrow \alpha = -3\beta \Rightarrow$ NU lin-ind.

$v_4 = 3 \cdot v_3 \Leftrightarrow v_4 - 3v_3 = 0_{\mathbb{R}^3}$

$\Rightarrow -3v_3 + v_4 = 0_{\mathbb{R}^3} \Rightarrow v_3, v_4$ nu sunt liniar ind.

$\begin{vmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{vmatrix} \Rightarrow$ rang $= 1$

39) $H_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3,\ x = 0 \right\} \subset \mathbb{R}^3$

$H_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3,\ y = 0 \right\} \subset \mathbb{R}^3$

$H_1 \cap H_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3,\ x = 0 \text{ și } y = 0 \right\} =$

$= \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \in \mathbb{R}^3,\ z \in \mathbb{R} \right\}$

$H_1 = \left\{ \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \middle|\ y, z \in \mathbb{R} \right\}$; $H_2 = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \middle|\ x, z \in \mathbb{R} \right\}$

$\begin{pmatrix} 0 \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $\mathcal{B} = \left\{ \underset{e_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{e_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$

$\dim(H_1) = 2$

$\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $\mathcal{B} = \left\{ \underset{e_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{e_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$

$\dim(H_2) = 2$

$\mathcal{B}(H_1 \cap H_2) = \left\{ \underset{e_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\} \Rightarrow \dim(H_1 \cap H_2) = 1$

Prin Teorema Grassmann.

$\underset{3}{\dim(H_1 + H_2)} = \underset{2}{\dim(H_1)} + \underset{2}{\dim(H_2)} - \underset{1}{\dim(H_1 \cap H_2)}$

Vr $H_1 + H_2 = \langle H_1 \cup H_2 \rangle = \{ v_1 + v_2 \mid v_1 \in H_1, v_2 \in H_2 \}$

$= \underbrace{\alpha e_2 + \beta e_3}_{v_1 \in H_1} + \underbrace{\gamma e_1 + \delta e_3}_{v_2 \in H_2} = \gamma e_1 + \alpha e_2 + (\beta + \delta) e_3$

$\cup$ $H_1 + H_2 \underset{2}{\subseteq} \mathbb{R}^3$

$$\mathbb{R}^3 \ni \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ae_1 + be_2 + ce_3 = \underbrace{be_2 + ce_3}_{\in H_1} + \underbrace{ae_1 + 0 \cdot e_3}_{\in H_2}$$

T: 40

11) $tr : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$

$Ker(tr) = ?$, $Im(tr) = ?$

$$tr(\alpha \cdot A + \beta \cdot B) = \alpha \cdot tr(A) + \beta \cdot tr(B)$$

$\Rightarrow tr$ - morfism

$Ker(tr) = \{ A \in M_n(\mathbb{R}) \mid tr(A) = 0 \} = sl_n(\mathbb{R})$

tr e surjectivă $\quad tr\left(\begin{pmatrix} a & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \right) = a$, $a \in \mathbb{R}$, a arb.

$Im(tr) = \mathbb{R}$

Dimensiunea lui $M_n(\mathbb{R})$ este n^2.

Teorema rang-defect: $\dim(sl_n(\mathbb{R})) = n^2 - 1$

$$\dim(M_n(\mathbb{R})) = \dim(Im(tr)) + \dim(Ker(tr))$$

Curs 5

1) $rang(A \cdot B) \leq \min \{ rang(A), rang(B) \}$

2) $rang(U \cdot A) = rang(A) = rang(V \cdot A)$

$U, V \rightarrow$ mat. inversabile oarecare

$\boxed{\text{Teorema Kronecker}}$: $rang(A) = \dim \langle C_1(A), C_2(A), \dots, C_n(A) \rangle$

(la fel la linii)

pt. că $rang(A) = rang({}^tA)$

~~02~~ 17. 02. 2022.

Seminar 5

Fie $A \in \mathcal{M}_{m,n}(\mathbb{R})$:

1°) $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$; Este f aplicație liniară?

$f(v) = A \cdot v \quad \in \boxed{\mathcal{M}_{m,1}(\mathbb{R}) = \mathbb{R}^m}$

f aplicație liniară: $\forall\, v_1, v_2 \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R} \Rightarrow$

// $\alpha v_1 + \beta v_2 \in V$

$\Leftrightarrow f(\alpha v_1 + \beta v_2) = \alpha \cdot f(v_1) + \beta \cdot f(v_2)$

$f(\alpha v_1 + \beta v_2) = A(\alpha v_1 + \beta v_2) = \alpha \cdot A v_1 + \beta \cdot A + v_2 = \alpha \cdot f(v_1) + \beta \cdot f(v_2)$

$\mathrm{Ker}\, f = \{ [f: V \xrightarrow{\text{morfism}} W;\ V, W$ spații vectoriale reale$]$

$u \in V \mid f(v) = 0_W \}$ (*)

$f: V \longrightarrow W$ morfism

$\underbrace{V \supset \mathcal{B} \quad W \supset \mathcal{C}}_{\text{baze}}$

V, W sp. vectoriale reale

$A = A_{\mathcal{B},\mathcal{C}}(f) \in \mathcal{M}_{m,n}(\mathbb{R})$

$\mathrm{Ker}\, f = \{u \in V \mid f(v) = 0_W\}$

$\{u \in V \mid \underbrace{A \cdot V = 0_W}\} \longrightarrow$ sistem omogen

cu m ec. și n necunoscute.

$Av = w \Rightarrow$ sol. sist. cu m ec. și n nec.

$A^e = (A \mid w)$

$$\begin{cases} a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n = w_1 \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mn}v_n = w_m \end{cases}$$

A rezolva ec. sist. $\Leftrightarrow$ a vedea dc. $w \in \mathrm{Im}(f) = \{f(v) \mid v \in V\}$

$\{w \in W \mid \exists\, v \in V$ a.î. $f(v) = w\}$

$Av = w$

$\hookrightarrow v_1 C_1(A) + v_2 C_2(A) + \dots + v_n \cdot C_n(A) = w$

w e în $\mathrm{Im}(f) \Longleftrightarrow \exists\, v$ a.î. $\uparrow$

$$\boxed{\begin{gathered} \det(X \cdot A \cdot X^{-1}) = \det(A) \\ \mathrm{tr}(X \cdot A \cdot X^{-1}) = \mathrm{tr}(A) \end{gathered}}$$

4.) $\det : M_n(\mathbb{R}) \to \mathbb{R}$; Este determinantul o aplicație liniară? NU de exemplu:

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{B} = I_2;\ \det(I_2) = 1$$

$$\left.\begin{aligned} \det(A) = 0 \\ \det(B) = 0 \end{aligned}\right\} \Rightarrow \det(A) + \det(B) \neq \det(A+B)$$

49) $A \in M_n(\mathbb{R})$ fixată ; $f_A(X) = AX - XA$ aplicație liniară.

$f_A : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

$n=2$

$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \{E_{ij} \mid 1 \le i \le n ; 1 \le j \le n\}$ bază

$E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$f_A(\alpha X + \beta Y) = \alpha f_A(X) + \beta f_A(Y)$?

$f_A(\alpha X + \beta Y) = f_A(\alpha X) + f_A(\beta Y)$

$= \alpha f_A$

$\operatorname{tr}(AX - XA) = 0$

$f_A(\alpha X + \beta Y) = A(\alpha X + \beta Y) - (\alpha X + \beta Y)A$

$= A\alpha X + A\beta Y - \alpha XA - \beta YA =$

$= \alpha(AX - XA) + \beta(AY - YA)$

$= \alpha \cdot f_A(X) + \beta \cdot f_A(Y)$ ∎

$E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = E$

$F = E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$;

$H = E_{11} - E_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$f_E(E) = EE - EE = O_2 = 0E + 0F + 0H$

$f_E(F) = \underbrace{EF}_{= E_{12}E_{21} = E_{11}} - \underbrace{FE}_{= E_{21}E_{12} = E_{22}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H$

$= 0E + 0 \cdot F + 1 \cdot H$

$E_{ij} \cdot E_{ke} = \delta_{jk} \cdot E_{ie}$

$\delta_{jk} = \begin{cases} 1, & j=k \\ 0, & j \neq k \end{cases}$

sl (special liniare)

$f_E(H) = EH - HE = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = -2E + 0F + 0H$

$\{E, F, H\} \subset sl_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \;\middle|\; \begin{matrix} tr(A) = 0 \\ a + d = 0 \Rightarrow a = -d \end{matrix} \right\}$

↳ bază

$= \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\};$

$\parallel \; aH + bE + cF$

$\alpha H + \beta E + \gamma F = O_2 \Leftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow \alpha = \beta = \gamma = 0$

$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$f_E : sl_2(\mathbb{R}) \to sl_2(\mathbb{R})$

$\Downarrow$

$\dim(sl_2(\mathbb{R})) = 3 \Leftrightarrow \{E, F, H\}$

$\alpha + 2\beta = 0$

$2\alpha = 0$

$\alpha = 0$

$\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \alpha + 2\beta \\ 2\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{matrix} \alpha + 2\beta = 0 \Rightarrow \beta = 0 \\ 2\alpha = 0 \\ \alpha = 0 \end{matrix}$

$\Rightarrow$ lin. ind.

$< v_1, v_2 > = \{\alpha v_1 + \beta v_2 \mid \alpha, \beta \in \mathbb{R}\}$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right) \sim$$

Sist. omogene eşalonăm matricea sistemului.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 1 & 0 \end{pmatrix} \xrightarrow[L_3' = L_3 - L_1]{L_2' = L_2 - 2L_1} \begin{pmatrix} 1 & 2 \\ 0 & -4 \\ 0 & -2 \end{pmatrix} \xrightarrow{L_2' = -\frac{1}{4}L_2} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -2 \end{pmatrix}$$

$$\xrightarrow[L_3 = L_3 + 2L_2]{L_1' = L_1 - 2L_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} \alpha + 0\beta = 0 \\ 0 \cdot \alpha + \beta = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}$$

Nucleul este nul.

$\text{rang}(A) = 2$

$$A\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$A \cdot : \mathbb{R}^2 \to \mathbb{R}^3$

$A \cdot$ (înmulţ. la st. cu mat)

Teorema rang - defect:

$\dim(\mathbb{R}^2) = 2 = \dim(\text{Ker}(A \cdot)) + \dim(\text{Im}(A \cdot))$

Ac. rang e maxim $\Rightarrow$ vect. lin. ind. $\Rightarrow$ T.K

$\text{rang}(A) = \dim < C_1, \dots >$

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Esalonați:

45)
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \\ 2 & 3 & 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -5 & -4 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \\ 0 & \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -5 & -4 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -7 & -4 \\ 0 & -1 & -5 & -4 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{L_2=-L_2} \begin{pmatrix} 1 & 0 & -7 & -4 \\ 0 & 1 & +5 & +4 \\ 0 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & +9 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

46)
$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & -3 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -4 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -5 \end{pmatrix} \longrightarrow$$

$$\xrightarrow{L_4=\frac{L_4}{-5}} \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4$$

Forma eşalon a unei mat. inv. e identitatea!

Curs 6

Sisteme de ecuații lin

$$(1)\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$\rightsquigarrow$ forma matriceală $AX = B$ unde

$$\begin{cases} A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \in \mathcal{M}_{m,n}(\mathbb{R}) \\ X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}_{n,1}(\mathbb{R}), \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathcal{M}_{m,1}(\mathbb{R}) \end{cases}$$

Matricea extinsă: $A^e = (A|B) \in \mathcal{M}_{m,n+1}(\mathbb{R})$

Def.: Sist. (1) s.n. compatibil cu $\exists X \in \mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$ a.î. toate ec. din (1) să fie satisfăcute.

Un astfel de $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ s.n. soluție a sist. (1)

T. K-C: Sistemul (1) e comp. dc. și numai dc.

$$\text{rang}(A) = \text{rang}(A^e)$$

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Seminar 6

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 2 & 5 & 5 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 6 & 4 & 4 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x - 2y + z = 0 \\ t = 1 \end{cases}$$

x, t – nec. pr.

y, z – nec. sec.

$\Rightarrow x = 2y - z$

$t = 1$

$y = \alpha$

$z = \beta$

$$\Rightarrow S = \left\{ \begin{pmatrix} 2\alpha - \beta \\ \alpha \\ \beta \\ 1 \end{pmatrix} \middle| \ \alpha, \beta \in \mathbb{R} \right\}$$

sist. de gen.

$v_1 = (2, 1, 0, 1)$

$v_2 = (-1, 0, 1, 1)$

46)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & -1 & -3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & +2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & -4 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -6 \end{array}\right)$$

---

47)

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 1 & -1 & -1 \\ -1 & -4 & -3 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -1 \\ -1 & -4 \end{vmatrix}$$

$$= 3$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -3 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -3 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -4 & -3 & 0 & 1 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & 3 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -5 & -4 & 1 & 1 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & 3 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & +\frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & +\frac{4}{5} \\ 0 & 1 & 0 & 0 & -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 1 & \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \end{array}\right)$$

$$\text{Jw} = \frac{1}{(-5)} \begin{pmatrix} +1 & 1 & +1 & -4 \\ 1 & 1 & -4 & 1 \\ 1 & -4 & 1 & 1 \\ -4 & 1 & 1 & 1 \end{pmatrix}$$

A^{-1} //

$$\left(\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \Bigg($$

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$$\begin{cases} x + t = 1 \\ y + z + t = -1 \\ x + y + t = -1 \\ x + t = 1 \end{cases}$$

$$A^e = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x + t = 1 \\ y = -2 \\ z + t = 1 \end{cases}$$

$x, y, z \rightarrow$ nec. pr.

$t \rightarrow$ nec. sec. $= \alpha$

$\Rightarrow x = 1 - \alpha;\ y = -2;\ z = 1 - \alpha$

c) $\mathcal{L} = \left\{ \begin{pmatrix} 1-\alpha \\ -2 \\ 1-\alpha \\ \alpha \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 1 & 3 & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \\ 2 & 1 & 0 & 2 & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \\ 1 & 0 & 0 & 1 & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 1 & 3 & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \\ 0 & 1 & 0 & 0 & \cancel{-2} & \cancel{0} & \cancel{1} & \cancel{0} \\ 0 & 0 & 0 & 0 & \cancel{-1} & \cancel{0} & \cancel{0} & \cancel{1} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 1 & 3 & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \\ 0 & 0 & -1 & -3 & \cancel{-2} & \cancel{-1} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 0 & \cancel{-1} & \cancel{0} & \cancel{0} & \cancel{1} \end{array}\right)$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \nexists\, X$$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\cancel{\frac{1}{3}} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right)$$

$-\frac{1}{3} + \frac{1}{12}$

$\frac{2}{3} - \frac{1}{6}$

$-\frac{1}{3} - \frac{2}{3} \cdot \frac{1}{4}$

$\frac{2}{3} + \frac{1}{2} \cdot \frac{2}{3}$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{6} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right)$$

pb. 66

Matricea unui morfism
Vectori și valori proprii

$\dim(V) = n$, $\dim(W) = m$

$V \xrightarrow{f} W$; f morfism

$\{v_1, \dots v_n\} = \mathcal{B} \subset V$, $\mathcal{C} = \{w_1, \dots, w_m\} \subset W$

$f(v_j) = \sum_{i=1}^{m} a_{ij} w_i$ – combinație liniară unică

$f \in Hom(V, W) \xrightarrow{A} A \in M_{m,n}(\mathbb{R})$

$f \mapsto A = M_{\mathcal{B},\mathcal{C}}(f) \in M_{m,n}(\mathbb{R})$ izomorfism de spații vectoriale

$A = (a_{ij})_{i=1,m;\ j=1,n}$

1) **OBS**: $v \in V$; $(\exists)!\ \alpha_1, \dots, \alpha_n$ a.î. $v = \alpha_1 v_1 + \dots + \alpha_n v_n$

$$f(v) = f\left(\sum_{j=1}^{n} \alpha_j v_j\right) \overset{f \text{ morfism}}{=} \sum_{j=1}^{n} \alpha_j f(v_j)$$

$$= \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{m} a_{ij} w_i\right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \alpha_j\right) w_i =$$

$$= \beta_1 w_1 + \dots + \beta_m \cdot w_m$$

$$\cancel{\beta_i = \sum_{j=1}^{n} a_{ij} \alpha_j}$$

$$\beta_i = \sum_{j=1}^{n} a_{ij} \alpha_j = \underset{\substack{\cap \\ \mathcal{M}_{1,n}(\mathbb{R})}}{L_i(A)} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \iff$$

$$\iff A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

coord. vect. v în $\mathcal{B}$ → coord. lui $f(v)$ în $\mathcal{C}$

---

2) Schimbarea mat. unui morfism f la schimbarea bazelor.

$$\begin{tikzcd} {}_{\mathcal{B}}V \arrow[r, "f"] & W_{\mathcal{C}} \arrow[d, "id_W"] \\ {}_{\mathcal{B}'}V \arrow[u, "id_V"] \arrow[r, "f"] & W_{\mathcal{C}'} \end{tikzcd}$$

* $M_{\mathcal{B}',\mathcal{C}'}(f)\ ?\ M_{\mathcal{B},\mathcal{C}}(f) \iff id_W \circ f \circ id_V = f$

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(I) <u>Obs</u>: Fie $f \in \mathrm{Hom}(V,W)$; $g \in \mathrm{Hom}(W,U)$

$\dim(U)=p$; $\underset{\text{bază}}{\mathcal{D}} \subset U$

$$\begin{tikzcd} V_{\mathcal{B}} \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & W_{\mathcal{C}} \arrow[r, "g"] & U_{\mathcal{D}} \end{tikzcd}$$

$$\underset{\underset{M_{p,n}(\mathbb{R})}{\cap}}{M_{\mathcal{B},\mathcal{D}}(g\circ f)} = \underset{\underset{M_{p,m}}{\cap}}{M_{\mathcal{C},\mathcal{D}}(g)} \cdot \underset{\underset{M_{m,n}}{\cap}}{M_{\mathcal{B},\mathcal{C}}(f)}$$

~~Obs.: (COR.1)~~ $\cancel{\longrightarrow}$

(II) <u>Obs.</u>: $V_{\mathcal{B}} \xrightarrow{id} V_{\mathcal{B}}$ $\quad id(v_j)=v_j$

$$I_n = M_{\mathcal{B}}(id) = M_{\mathcal{B},\mathcal{B}}(id)$$

<u>COR1</u>: $\begin{tikzcd} V_{\mathcal{B}} \arrow[r, "id"] \arrow[rr, bend right, "id"'] & V_{\mathcal{B}'} \arrow[r, "id"] & V_{\mathcal{B}} \end{tikzcd}$

$\mathcal{B}, \mathcal{B}'$ baze pt. V.

$I_n = M_{\mathcal{B}}(id) = M_{\mathcal{B}'\mathcal{B}}(id) \cdot M_{\mathcal{B}\mathcal{B}'}(id)$ $\quad$ sunt inverse. și inverse una alteia

$\Rightarrow M_{\mathcal{B}'\mathcal{B}}(id_V) = M_{\mathcal{B},\mathcal{B}'}(id_V)^{-1}$ și viceversa

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$\star$ <u>COR2</u>: $\mathcal{B}, \mathcal{B}'$ baze în V; $\mathcal{C}, \mathcal{C}'$ baze în W

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$$\underbrace{M_{\mathcal{B}',\mathcal{C}'}(f)} = \underbrace{M_{\mathcal{C},\mathcal{C}'}(id_W)}_{\shortparallel \; M_{\mathcal{C}',\mathcal{C}}(id_W)^{-1}} \cdot \underbrace{M_{\mathcal{B},\mathcal{C}}(f)} \cdot M_{\mathcal{B}',\mathcal{B}}(id_V)$$

au aceeași rang (se păstrează)

$\boxed{\text{Ex.:}}$ Fie $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = f(x,y,z) = \begin{pmatrix} x+y+z \\ x-y+z \end{pmatrix}$$

$\mathcal{B} = \{e_1, e_2, e_3\} \subset \mathbb{R}^3 \quad ; \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ldots$

$\mathcal{C} = \{\overline{e_1}, \overline{e_2}\} \subset \mathbb{R}^2 \quad ; \quad \overline{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \overline{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{cases} f(e_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \overline{e_1} + 1 \cdot \overline{e_2} \rightsquigarrow f(e_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ f(e_2) = \begin{pmatrix} 0+1+0 \\ 0-1+0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \overline{e_1} - 1 \cdot \overline{e_2} \Longrightarrow f(e_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ f(e_3) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \overline{e_1} + 1 \cdot \overline{e_2} \Rightarrow f(e_3) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{cases}$$

$$\to M_{\mathcal{B},\mathcal{C}}(f) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = A$$

$$\Rightarrow f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

NUCLEUL

OBS.: $\ker(f) = \{v \in V \mid f(v) = 0_W\} = \{v \in V \mid Av = 0_W\}$

$f: V_{\mathcal{B}} \to W_{\mathcal{C}}$ $\qquad A = M_{\mathcal{B},\mathcal{C}}(f)$

Ex.: Fie $D: \mathbb{R}[X]_2 \longrightarrow \mathbb{R}[X]_2$

$D(P(X)) = P'(X)$

↳ derivarea este liniară!

$\mathcal{B} = \{X^2, X, 1\}$ baza canonică în $\mathbb{R}[X]_2$

$\mathcal{B}' = \{(X-1)^2, X, 1\}$. Temă: $\mathcal{B}'$ = este bază pt. $\mathbb{R}[X]_2$

$$M_{\mathcal{B}}(D) = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$D(X^2) = 2X = 0X^2 + 2X + 0 \cdot 1$

$D(X) = 1$

$D(1) = 0$

$$M_{\mathcal{B}',\mathcal{B}}(id_{\mathbb{R}[X]_2}) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \neq M_{\mathcal{B},\mathcal{B}'}(id) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = M_{\mathcal{B}',\mathcal{B}}(id)^{-1}$$

$id((X-1)^2) = (X-1)^2 = X^2 - 2X + 1$

$id(X) = X$

$id(1) = 1$

$$M_{\mathcal{B}'}(D) = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix}$$

$D((X-1)^2) = 2(X-1) = 2X - 2$

$D(X) = 1$

$D(1) = 0$

$$\cancel{M_{\mathcal{B}'}(D) = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix} \overset{?}{=} M_{\mathcal{B}',\mathcal{B}}(id)^{-1} \cdot M_{\mathcal{B}}(D) \cdot M_{\mathcal{B},\mathcal{B}'}(id)}$$

$$\Leftrightarrow \cancel{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$$

verif. a rel.

$$M_{B'}(D) = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix} \overset{?}{=} M_{B,B'}(id)^{-1} \cdot M_B(D) \cdot M_{B,B'}(id)$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

✓

Vectori și valori proprii

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

3 2
2 1

Fibonacci

Dorim să lucrăm cu matrice diagonale.

<u>Exemplu:</u>
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \dots$$

$$\to \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$\dots \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

Def.: Fie V un sp. vect $/\mathbb{R}$; $\dim(V) = n$; $f : V \to W$ morfism

$\lambda \in \mathbb{R}$ s.n. valoare proprie pt. f dc. și numai dc.
$(\exists)\ v \in V \setminus \{0\}$ a.î. $f(v) = \lambda \cdot v$

Def. : $\bullet\, v \in V\setminus\{0_V\}$ s.n. vector propriu dc. $(\exists)\, \lambda \in \mathbb{R}$

$\bullet\, v \in V\setminus\{0_V\}$ —— // —— pt. f dc. $(\exists)\, \lambda \in \mathbb{R}$ a.î. $f(v) = \lambda v$

(f endomorfism)

Fie $\mathcal{B} \subset V$ și $A = M_{\mathcal{B}}(f) = M_{\mathcal{B},\mathcal{B}}(f)$

PROP.: Fie $f: V \to V$ morfism; $\mathcal{B} \subset V$ bază. Atunci λ e val. prop. pt. f dc. și num. dc. $\boxed{\det(\lambda I_n - A) = 0}$

Dem.: " $\Rightarrow$ " $\lambda \in \mathbb{R}$ val. prop. $\Leftrightarrow (\exists)\, v \in V\setminus\{0_V\}$ a.î. $f(v) = \lambda \cdot \underset{id(v)}{v}$

$\Leftrightarrow (\exists)\, v \in V\setminus\{0_V\}$ a.î. $Av = \lambda \cdot I v \Leftrightarrow$

$\Leftrightarrow \underset{(\exists) v \in V\setminus\{0_V\}}{0_V = \lambda I_n v - Av} \Leftrightarrow \underset{(\exists) v \in V\setminus\{0_V\}}{0_V = (\lambda I_n - A)\cdot v} = 0_V \longrightarrow$

~~$\Leftrightarrow (\lambda I_n - A)\, v =$~~

$\rightarrow$ sist. omogen de ecuații liniare cu mat. $\lambda I_n - A \in M_n(\mathbb{R})$ cu soluția nenulă v

$\Updownarrow$

$\det(\lambda I_n - A) = 0 \Leftrightarrow \text{rang}(\lambda I_n - A) < n$

Def.: S.n. polinomul caracteristic al morfismului f

$\det(\lambda I_n - A)$ unde $A = M_{\mathcal{B}}(f)$ pt. $\underset{\text{bază}}{\mathcal{B}} \subset V$

not. $P_f(X)$

PROP.: $P_f(X)$ NU depinde de baza $\mathcal{B}'$ aleasă.

Dem.: Fie $\mathcal{B}' \subset V$ o altă bază $A' = M_{\mathcal{B}'}(f) = R^{-1} \underbrace{M_{\mathcal{B}}(f)}_{A} \cdot R$, $M_{\mathcal{B}',\mathcal{B}}(id)$

$P_{f,\beta'}(X) = \det(XI_n - A') = \det(X \cdot \underset{R^{-1}R}{\overset{\shortparallel}{I_n}} - R^{-1}AR)$

$= \det(R^{-1}(XI_n - A)R) = \det(R^{-1}) \cdot \det(XI_n - A) \cdot \det(R)$

$= \underset{P_{f,\beta}(X)}{\overset{\shortparallel}{\det(XI_n - A)}} \cdot \underbrace{\det(R^{-1}) \cdot \det(R)}_{=1}$

<u>Def.</u> : Pt. o matrice $A \in M_n(\mathbb{R})$:

$P_A(X) = \det(XI_n - A)$ pt. polinomul caracteristic (mat. pătratică)

Polinomul caract. pt. $A \in M_2(\mathbb{R})$:

→ Fie $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$P_A(X) = \det(XI_2 - A) = \begin{vmatrix} X-a & -b \\ -c & X-d \end{vmatrix} = (X-a)(X-d) - bc$

$= X^2 - (a+d)X + ad - bc$

$\Rightarrow \boxed{P_A(X) = X^2 - tr(A) \cdot X + \det(A)}$

@ $\Rightarrow$ Pt. $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$; $P_A(X) = X^2 - X - 1$

<u>Def.</u> : $f : V \to V$ morfism s.n. $\boxed{\text{diagonalizabil}}$ dc. și numai dc. ($\exists$) $v_1, v_2 \ldots v_n$ vect. prop. (prin def. nenuli) <u>liniar independenți</u> pt. f. ; <u>$n = \dim(V)$</u>

<u>Obs.</u>: $f(v_i) = \lambda_i \cdot v_i$

dc. $v_1, \ldots v_n$ liniar independenți $\Rightarrow$ formează baza $\mathcal{B} = \{v_1, \ldots, v_n\}$

$\Rightarrow M_B(f) = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}$

$\boxed{\text{PROP.}}$ Pt. $\boxed{\lambda \neq \mu}$ val. prop. distincte pt. $f: V \to V$, vect. prop. asociați val. prop. λ, μ sunt $\boxed{\text{liniar independenți.}}$

$\boxed{\text{COR}}$: Ac. $f: V \to V$ are n val. prop. distincte $\Rightarrow$
$\dim(V) = n$
$\Rightarrow$ avem n vect. asoc. lin. ind. $\Rightarrow f$ este diagonalizabilă
$\left(\xrightarrow{\text{PROP}}\right)$

$\boxed{\text{Obs.}}$: Fie $B' = \{v_1, \dots v_n\}$ bază de vect. prop. lin. indep. pt. f (pp. că există)

$M_{B'}(f) = D$, și $\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} = M_{B'}(f)$

Fie B o bază pt. V; $A = M_B(f)$ mat. lui f în baza B

$\to f(v_i) = \lambda_i v_i \Leftrightarrow A \cdot v_i = \lambda_i \cdot v_i$

Fie $Q = (v_1 \dots v_n)$; $C_j(Q) = v_j$

$\Leftrightarrow AQ = QD$ $\quad$ $|\cdot Q^{-1}$

$\Leftrightarrow Q^{-1} A Q = D$

$\Leftrightarrow A = Q D Q^{-1}$

Dem. că $AQ = QD$ demonstrând că $C_j(AQ) = C_j(QD)$

$C_j(AQ) = A \cdot C_j(Q) = A \cdot v_j$

$C_j(QD) = Q \cdot C_j(D) = Q \begin{pmatrix} 0 \\ 0 \\ \lambda_j \\ 0 \end{pmatrix} \to$ linia j $=$

$= 0 \cdot C_1(Q) + 0 \cdot C_2(Q) + \dots + \lambda_j C_j(Q) + \dots 0 \cdot C_n(Q)$

$= \lambda_j v_j$

Dar $A v_j = \lambda_j v_j$

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Seminar 7

$$\mathbb{R}^3_{\mathcal{B}} \xrightarrow{id} \mathbb{R}^3_{\mathcal{C}} \xrightarrow{id} \mathbb{R}^3_{\mathcal{B}}$$

1) $M_{\mathcal{B},\mathcal{C}}(id_{\mathbb{R}^3})$ = mat. de trecere din baza $\mathcal{C}$ in baza canonică

$\mathcal{B} = \{e_1, e_2, e_3\}$

$\mathcal{C} = \left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

$id(e_1) = e_1 = a_{11} \cdot v_1 + a_{21} \cdot v_2 + a_{31} \cdot v_3$

$M_{\mathcal{C},\mathcal{B}}(id_{\mathbb{R}^3})^{-1}$

$id(v_1) = v_1 = 1 \cdot e_1 + 1 \cdot e_2 + 1 \cdot e_3$

$id(v_2) = v_2 = e_1 + e_2$

$id(v_3) = v_3 = e_1 - e_3$

$a \cdot v_1 + b \cdot v_2 + c \cdot v_3 = 0_{\mathbb{R}^3}$

$\Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -2 + 1 = -1 \neq 0 \Rightarrow$ mat. invers.; singura soluție este cea nulă $\Rightarrow$ sist. liniar ind.

$\left.\begin{array}{l} \langle v_1, v_2, v_3 \rangle \subset \mathbb{R}^3 \\ \dim \langle v_1, v_2, v_3 \rangle = 3 \end{array}\right\} \Rightarrow \langle v_1, v_2, v_3 \rangle = \mathbb{R}^3 \Rightarrow \mathcal{C}$ este bază pt. $\mathbb{R}^3$

$$M_{\mathcal{C},\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$M_{\mathcal{C},\mathcal{B}}(id_{\mathbb{R}})^{-1} = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & -2 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 & -2 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) = M_{\mathcal{E},\mathcal{B}}^{-1} = M_{\mathcal{B},\mathcal{E}}$$

$id(e_1) = e_1 = a_{11} v_1 + a_{2,1} v_2 + a_{3,1} v_3$

$id(e_2) = e_2 = a_{1,2} v_1 + a_{2,2} v_2 + a_{3,2} v_3$

---

2) $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \{ a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \mid a,b,c,d \in \mathbb{Q} \} = V$

$\mathcal{B} = \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ formează bază $/\mathbb{Q}$

S.L.I – temă ; S.G = sisteme generatori $\to$ prin definiția lui V

$T : V \longrightarrow V$; $\dim_{\mathbb{Q}}(V) = 4$

$T(x) = (\sqrt{2}+\sqrt{3})x$. Să se determine $M_{\mathcal{B},\mathcal{B}}(T)$.

Fie $x_1, x_2 \in V$; $x_1 = a_1 + b_1\sqrt{2} + c_1\sqrt{3} + d_1\sqrt{6}$

$x_2 = a_2 + b_2\sqrt{2} + c_2\sqrt{3} + d_2\sqrt{6}$

$\alpha, \beta \in \mathbb{Q}$; $a_i, b_i \in \mathbb{Q}$
$c_i, d_i \in \mathbb{Q}$

$T(\alpha x_1 + \beta x_2) = \cancel{T(\alpha a} \overset{*}{=} (\sqrt{2}+\sqrt{3})(\alpha_1(a_1+\dots) + \beta(a_2+\dots))$

$= T = (\sqrt{2}+\sqrt{3})((\alpha a_1+\beta a_2) + (\alpha b_1+\beta b_2)\sqrt{2} + (\alpha c_1+\beta c_2)\sqrt{3} +$
$+(\alpha d_1+\beta d_2)\sqrt{6}) \Rightarrow (\alpha a_1+\beta a_2)\sqrt{2} + (\alpha b_1+\beta b_2)2 +$
$+ (\alpha c_1+\beta c_2)\sqrt{6} + (\alpha d_1+\beta d_2)2\sqrt{3} + (\alpha a_1+\beta a_2)\sqrt{3} +$
$+ (\alpha b_1+\beta b_2)\sqrt{6} + (\alpha c_1+\beta c_2)3 + (\alpha d_1+\beta d_2)3\sqrt{2}.$

$= (\sqrt{2}+\sqrt{3})((\alpha a_1 + \alpha b_1\sqrt{2} + \alpha c_1\sqrt{3} + \alpha d_1\sqrt{6}) + (\beta a_2 + \beta b_2\sqrt{2} +$
$+ \beta c_2\sqrt{3} + \beta d_2\sqrt{6}))$

$\overset{*}{=} (\sqrt{2}+\sqrt{3})\,\alpha \cdot x_1 + (\sqrt{2}+\sqrt{3})\,\beta \cdot x_2$
$= \alpha \underset{T}{\cancel{f}}(x_1) + \beta \underset{T}{\cancel{f}}(x_2)$ apl. lin.

$T(1) = (\sqrt{2}+\sqrt{3})\cdot 1 = 0\cdot 1 + 1\cdot\sqrt{2} + 1\cdot\sqrt{3} + 0\cdot\sqrt{6}$
$T(\sqrt{2}) = (\sqrt{2}+\sqrt{3})\cdot\sqrt{2} = 2+\sqrt{6}$
$T(\sqrt{3}) = (\sqrt{2}+\sqrt{3})\sqrt{3} = \sqrt{6}+3$
$T(\sqrt{6}) = 2\sqrt{3}+3\sqrt{2}$

$$\Rightarrow M_{BB}(T) = \begin{pmatrix} 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

65) $(\exists)$ morfisme injectiv $f: \mathbb{R}^3 \to \mathbb{R}^2$?

Pp. că da. Fie $f: \mathbb{R}^3 \to \mathbb{R}^2$ morfism injectiv.

$\Leftrightarrow \mathrm{Ker}(f) \underset{\text{"}\subseteq\text{"}}{=} 0_{\mathbb{R}^3}$

Teor. rang-defect: $\dim(\mathbb{R}^3) = \dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f))$.

$Im(f) \leq \mathbb{R}^2$
$\Rightarrow$ subspațiu

$\Rightarrow \dim(Im(f)) \leq 2$

$\dim(\mathbb{R}^3) = \dim(Ker(f)) + \dim(Im(f))$

$\quad\; \shortparallel \qquad\qquad\quad \shortparallel$

$\quad\; 3 \qquad\qquad\quad 0$

$\Rightarrow \dim(Im(f)) = 3$

Contradicție
$\rightarrow$ nu putem avea un morfism injectiv $\mathbb{R}^3 \rightarrow \mathbb{R}^2$

baza pt $0_{\mathbb{R}^3} = \emptyset$

Morfism injectiv:

$f(v_1) = f(v_2) \Rightarrow v_1 = v_2$

$\Downarrow f(v_1) - f(v_2) = 0_V$

$f(v_1 - v_2) = 0_V \Rightarrow v_1 - v_2 = 0_V$

$f : \mathbb{R}^2_B \longrightarrow \mathbb{R}^3_{B'}$; $f(x,y) = f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+2y \\ -x+y \end{pmatrix}$

$B = \{e_1, e_2\}$; $B' = \{\overline{e_1}, \overline{e_2}, \overline{e_3}\} \Rightarrow M_{B,B'}(f) = \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}$

$f(e_1) = f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 1\overline{e_1} + \overline{e_2} - \overline{e_3}$

$f(e_2) = f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -\overline{e_1} + 2\overline{e_2} + \overline{e_3}$

$f: \mathbb{R}^2 \to \mathbb{R}^3$; $M_{B,\mathcal{E}_0}(f) = \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix} = A$

$f\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ este aplic. liniară pt-că înmulțirea la stânga cu o mat este o aplicație liniară.

$Ker(f)$, $Im(f)$ + dim.

$\{v \in \mathbb{R}^2 \mid Av = 0_{\mathbb{R}^3}\}$

$\Updownarrow$

$A\begin{pmatrix} x \\ y \end{pmatrix} = 0_{\mathbb{R}^3} \Leftrightarrow x\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 2 \\ +1 \end{pmatrix} = 0_{\mathbb{R}^3}$

$\Rightarrow \begin{cases} x - y = 0 \\ x + 2y = 0 \\ -x + y = 0 \end{cases}$

$\begin{pmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix} \xrightarrow[L_3' = L_3 + L_1]{L_2' = L_2 - L_1} \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \xrightarrow{L_2' = L_2/3} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \Rightarrow Ker(f) = 0_{\mathbb{R}^2}$

<u>Teorema rang-defect:</u>

$$\dim(\mathbb{R}^2) = \underset{\substack{\shortparallel \\ 0}}{\dim(Ker(f))} + \underbrace{\dim(Im(f))}_{2}$$

$Im(f) = \{w \in \mathbb{R}^3 \mid (\exists)\, v \in \mathbb{R}^2,\ f(v) = w\} = \{f(v) \mid v \in \mathbb{R}^2\} =$

$= \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\rangle = \left\{ x\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$

$\begin{pmatrix}1\\2\\3\end{pmatrix} \in \mathrm{Im}(f) \subset \mathbb{R}^3$

$(\exists)\ x, y$ a.î. $A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$?

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{array}\right)$$

pivot pe ult. col.

$\Downarrow$

Sist. incomp.

$$\boxed{\begin{cases} x - y = 1 \\ x + 2y = 2 \\ -x + y = 3 \end{cases}} \Leftrightarrow \nexists$$

$\Leftrightarrow$

$\begin{pmatrix}1\\2\\3\end{pmatrix} \notin \mathrm{Im}(f)$

Vectori și valori prop.

$Av = \lambda v \ ;\ v \neq 0_V$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

$$P_A(X) = \det(XI_3 - A) = \begin{vmatrix} X & -1 & 0 \\ 4 & X-4 & 0 \\ 2 & -1 & X-2 \end{vmatrix}$$

$$= (X-2)\begin{vmatrix} X & -1 \\ 4 & X-4 \end{vmatrix} = (X-2)((X-4)X + 4)$$

$$= (X-2)(X^2 - 4X + 4)$$

$$= (X-2)^3 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 2$$

$Av = 2v \Leftrightarrow Av = 2I_3\, v \Leftrightarrow (A - 2I_3)v = 0$

$$\underbrace{(A-2I_3)}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad -2x + y = 0 \Rightarrow x = \frac{y}{2}$$

$y, z \in \mathbb{R}$

rang = 1 (o var. princ.)

$$\mathcal{S} = \left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

v_1: $y = 1$, $z = 0$ $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $y = 2$, $z = 0$

v_2: $y = 0$, $z = 1$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}; \det(xI_4 - A) = \begin{vmatrix} x-1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ -1 & 0 & 0 & x-1 \end{vmatrix}$$

$= x^2(x-1)^2$

$\lambda_1, \lambda_2 = 0$; $\lambda_3, \lambda_4 = 1$

Vectorii prop. asoc. val. prop. 0

$$\underbrace{(A - 0 \cdot I_4)}_{A}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x = 0 \\ t = 0 \end{cases} \Rightarrow \quad v = \begin{pmatrix} 0 \\ y \\ z \\ 0 \end{pmatrix}$$

$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

* vect. din baza canonică

Scanned with CamScanner

Obs.: $\lambda = 0$ este val. ~~pr~~ proprie pt. $A \Leftrightarrow \det(A) = 0$

Vectorii prop. asoc. val. prop $\lambda_3 = \lambda_4 = 1$

$$(A - I_4)\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} y = 0 \\ z = 0 \\ x = 0 \end{cases} \Rightarrow V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Aceleași exerciții pt. $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

simetrică, diagonalizabilă

$$A = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 5 & 5 & -3 \\ 4 & -1 & 3 & -1 \end{pmatrix}$$

$$\det(xI_4 - A) = \begin{vmatrix} x-3 & 1 & 0 & 0 \\ -1 & x-1 & 0 & 0 \\ -3 & -5 & x-5 & 3 \\ -4 & 1 & -3 & x+1 \end{vmatrix}$$

$= ((x-3)(x-1)+1)((x-5)(x+1)+9)$

$= (x-3)(x-5)(x^2-1) + (x-5)(x+1)$

$= (x^2 - 4x + 3 + 1)(x^2 - 4x - 5 + 9)$

$= (x-2)^2 (x-2)^2 = (x-2)^4 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$

multiplicitate 4

$$(A-2I_4)\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Leftrightarrow \begin{pmatrix}1&-1&0&0\\1&-1&0&0\\3&5&3&-3\\4&-1&3&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\rightarrow \begin{pmatrix}1&-1&0&0\\0&0&0&0\\0&8&3&-3\\0&5&3&-3\end{pmatrix} \rightarrow \dots$$

Geom.

1) 2 pct. det. o dr. şi numai una
2) o dr. are cel puţin 2 pct.
3) $\exists$ 3 pct. nesituate pe ac. dr.

<u>Paralelism</u>:

2 dr. par. sunt 2 dr. coplanare care nu au pct comun.

3 pct. necol. $\longrightarrow$ 1 plan şi numai unul

G. euclidiană

Plan euclidian $(\mathcal{P}, \mathcal{D})$

$\mathcal{P} \rightarrow$ mulţime ; mulţimea punctelor

$\mathcal{D} \rightarrow$ mulţime ; mulţimea dreptelor

→ Verifică prop. exprimate prin:

I ax. incidenţă
II ax. paralelism
III ax. ordonare
IV ax. continuitate (Cantor, Dedekind)
V ax. congruenţă

K. Teleman, Stăfescu, Moraru, Florescu..
IX, X "Geom. şi trig."

carte