HW 12

E94104094 游昱秦

1. Given the problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=xy,\quad 0<x<\pi,\quad 0<y<\pi/2$$

$u(0,y)=\cos y,\ u(\pi,y)=-\cos y,\ 0\le y\le \pi/2,$ ← 左右邊界

$u(x,0)=\cos x,\ u(x,\pi/2)=0,$ ~~$1\le y\le 2$~~ $0<x<\pi$ ← 上下邊界

To calculate $u(x,y)$ by using $h=k=0.1\pi$.

有限差分法：

$h=k=0.1\pi$

區域網格劃分：

$x_i=ih=0.1\pi,\ \pi/h=10,\ i=0\sim10$ ⇒ 共66個網格點

$y_j=jk=0.1\pi,\ \frac{\pi/2}{k}=5,\ j=0\sim5$

(h: 0.1π)

$u_{i,j}\approx u(x_i,y_j)$

$$\frac{\partial^2 u_{i,j}}{\partial x^2}=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2},\quad \frac{\partial^2 u_{i,j}}{\partial y^2}=\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}$$

$$\Rightarrow \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}=x_iy_j$$

由於 $h=k$ ⇒ $$\frac{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}}{h^2}=x_iy_j$$

$$\Rightarrow u_{i,j}=\frac{1}{4}\left(u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-h^2x_iy_j\right)$$

$u(0,y_j)=u_{0,j}=\cos(y_j)$

$u(\pi,y_j)=u_{10,j}=-\cos(y_j)$

$u(x_i,0)=u_{i,0}=\cos(x_i)$

$u(x_i,\frac{\pi}{2})=u_{i,5}=0$

左邊界 $(i=0)$

→ $u_{0,0}=\cos(0)=1$

$u_{0,1}=\cos(0.1\pi)\approx0.9511$

$u_{0,2}=\cos(0.2\pi)\approx0.8090$

$u_{0,3}=\cos(0.3\pi)\approx0.5878$

$u_{0,4}=\cos(0.4\pi)\approx0.3090$

$u_{0,5}=\cos(0.5\pi)=0$

右邊界 $(i=10)$

$u_{10,0}=-\cos(0)=-1$

$u_{10,1}=-\cos(0.1\pi)=-0.9511$

$u_{10,2}=-\cos(0.2\pi)=-0.8090$

$u_{10,3}=-\cos(0.3\pi)=-0.5878$

$u_{10,4}=-\cos(0.4\pi)=-0.3090$

$u_{10,5}=-\cos(0.5\pi)=0$

下邊界 $(j=0)$

$u_{1,0}=\cos(0.1\pi)=0.9511$

$u_{2,0}=\cos(0.2\pi)=0.8090$

$u_{3,0}=\cos(0.3\pi)=0.5878$

$u_{4,0}=\cos(0.4\pi)=0.3090$

$u_{5,0}=\cos(0.5\pi)=0$

$u_{6,0}=\cos(0.6\pi)=-0.3090$

$u_{7,0}=\cos(0.7\pi)=-0.5878$

$u_{8,0}=\cos(0.8\pi)=-0.8090$

$u_{9,0}=\cos(0.9\pi)=-0.9511$

上邊界 $(j=5)$

$u_{i,5}=0$ for $i=1\sim9$

內部點 $i=1\sim9,\ j=1\sim4$，共 $9\times4=36$ 點

已知 $u_{i,j}^{(new)}=\frac{1}{4}\left(u_{i+1,j}^{(old)}+u_{i-1,j}^{(old)}+u_{i,j+1}^{(old)}+u_{i,j-1}^{(old)}-h^2x_iy_j\right)$

$h^2=(0.1\pi)^2=0.01\pi^2\approx0.098696$

$x_i=ih=0.1\pi\cdot i$

$y_j=jk=0.1\pi\cdot j$

算 $u(1,1)=u_{1,1}$

$x_1=0.1\pi=0.314$ ⇒ $h^2x_1y_1=0.098696\times0.314^2=0.00974$

$y_1=0.1\pi=0.314$

$u_{0,1}^{(0)}=\cos(0.1\pi)=0.9511$

$u_{2,1}^{(0)}=0$

$u_{1,0}^{(0)}=\cos(0.1\pi)=0.9511$

$u_{1,2}^{(0)}=0$

代入

⇒ $u_{1,1}^{(1)}=\frac{1}{4}(0.9511+0+0+0.9511)-0.00974)=\frac{1}{4}\times1.89246=0.473115$ 以此類推

$x\backslash y$	0	0.1π	0.2π	0.3π	0.4π	0.5π
0	1.0000	0.9511	0.8090	0.5878	0.3090	0.0000
0.1π (0.3142)	0.9511	0.4731	0.2897	0.1703	0.0976	0.0000
0.2π (0.6283)	0.8090	0.2897	0.1468	0.0809	0.0418	0.0000
0.3π (0.9425)	0.5878	0.1703	0.0809	0.0386	0.0143	0.0000
0.4π (1.2566)	0.3090	0.0976	0.0418	0.0143	-0.0062	0.0000
0.5π (1.5708)	0.0000	0.0398	0.0124	0.0008	-0.0089	0.0000
0.6π (1.8849)	-0.3090	0.0103	-0.0086	-0.0104	-0.0121	0.0000
0.7π (2.1991)	-0.5878	-0.0226	-0.0284	-0.0288	-0.0291	0.0000
0.8π (2.5133)	-0.8090	-0.0445	-0.0453	-0.0427	-0.0384	0.0000
0.9π (2.8274)	-0.9511	-0.0571	-0.0515	-0.0452	-0.0397	0.0000
π (3.1416)	-1.0000	-0.9511	-0.8090	-0.5878	-0.3090	0.0000

2. Given the problem

$n=1\ j=0\sim5$)

☆ 求 $T(r=0.5\sim1, t=0.5)$ 的值

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}=\frac{1}{4K}\frac{\partial T}{\partial t},\quad \frac{1}{2}\le r\le 1,\quad 0\le t,$$

$T(r,t)$

$$T(1,t)=100+40t,\quad 0\le t\le 10;\qquad \frac{\partial T}{\partial r}+3T=0\quad \text{at}\quad r=\frac{1}{2}$$

$$T(r,0)=200(r-0.5),\quad 0.5\le r\le 1,$$

and use $\Delta t=0.5$, $\Delta r=0.1$, and $K=0.1$ to calculate $T(r,t)$

By (a) the forward-difference method

(b) the backward-difference method

© the Crank-Nicolson algorithm.

✅ 偏微分方程（PDE）：

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial T}{\partial r}=\frac{1}{4K}\cdot\frac{\partial T}{\partial t},\quad \text{for } 0.5\le r\le 1,\ t\ge 0$$

✅ 邊界條件（Boundary Conditions）：

1. 外邊界（右端點 $r=1$）：

$$T(1,t)=100+40t,\quad 0\le t\le 10$$

2. 內邊界（左端點 $r=0.5$）為 Robin 條件（混合邊界）：

$$3T+\frac{\partial T}{\partial r}=0,\quad \text{at } r=0.5$$

✅ 初始條件（Initial Condition）：

$$T(r,0)=200(r-0.5),\quad 0.5\le r\le 1$$

(a) 令 $r_j=0.5+j\cdot\overset{\Delta r}{0.1}$, $j=0,1,\dots,5$

j	0	1	2	3	4	5
r_j	0.5	0.6	0.7	0.8	0.9	1.0

$t_n=0.5n\quad n=0\sim20$

Forward Difference

(時間導數) $\frac{\partial T}{\partial t}\approx\frac{T_j^{n+1}-T_j^n}{\Delta t}$

(空間導數) $\frac{\partial^2 T}{\partial r^2}\approx\frac{T_{j+1}^n-2T_j^n+T_{j-1}^n}{(\Delta r)^2}$, $\frac{\partial T}{\partial r}\approx\frac{T_{j+1}^n-T_{j-1}^n}{2\Delta r}$

$$\frac{T_j^{n+1}-T_j^n}{\Delta t}=4k\left[\frac{T_{j+1}^n-2T_j^n+T_{j-1}^n}{(\Delta r)^2}+\frac{1}{r_j}\times\frac{T_{j+1}^n-T_{j-1}^n}{2\Delta r}\right]$$

$\xrightarrow{\times\Delta t}$ $T_j^{n+1}-T_j^n=4k\Delta t\left[\frac{T_{j+1}^n-2T_j^n+T_{j-1}^n}{(\Delta r)^2}+\frac{1}{r_j}\times\frac{T_{j+1}^n-T_{j-1}^n}{2\Delta r}\right]$

$\alpha=\frac{4k\Delta t}{(\Delta r)^2}=\frac{4\times0.1\times0.5}{(0.1)^2}=20$

$\beta=\frac{4k\Delta t}{2\Delta r\cdot r_j}=\frac{4\times0.1\times0.5}{2\times0.1\times r_j}=\frac{1}{r_j}$

$\rightarrow T_j^{n+1}=T_j^n+\alpha\left(T_{j+1}^n-2T_j^n+T_{j-1}^n\right)+\beta\left(T_{j+1}^n-T_{j-1}^n\right)$

外邊界 $r=1$ $j=5$ $t_1=0.5\Rightarrow T(1,0.5)=120$

$T_5^n=100+40t_n=100+40\times0.5n$

$=100+20n$

內邊界 $r=0.5$ $j=0$

$3T+\frac{\partial T}{\partial r}=0\Rightarrow\frac{\partial T}{\partial r}=-3T$

$\rightarrow\frac{T_1^n-T_0^n}{\Delta r}=-3T_0^n\Rightarrow T_1^n=T_0^n(1-3\overset{0.1}{\Delta r})\Rightarrow T_0^n=\frac{T_1^n}{0.7}$

初始條件 $T(r,0)=200(r-0.5)$

$n=0$

j	r_j	$T_j^0=200(r_j-0.5)$
0	0.5	~~0~~ 28.5714
1	0.6	20
2	0.7	40
3	0.8	60
4	0.9	80
5	1.0	100

$T_0^0=\frac{T_1^0}{0.7}=\frac{20}{0.7}=28.5714$

⇒ 計算內部節點 ($n=1$, $j=1\sim4$)

$j=1$ ($r_1=0.6$)

$T_0^0=28.5714$

$\beta_1=\frac{1}{r_1}=1.6667$

$T_1^1=T_1^0+\alpha(T_2^0-2T_1^0+T_0^0)+\beta(T_2^0-T_0^0)$

$=20+20(40-2\times20+28.5714)+1.6667(40-28.5714)=610.476$

穩定條件 $\Rightarrow\frac{4k\Delta t}{(\Delta r)^2}\le\frac{1}{2}$

r 0.1 Δt 0.5, $K=0.1$ 代入 $\frac{4k\Delta t}{(\Delta r)^2}=\frac{4\times0.1\times0.5}{(0.1)^2}=20\gg0.5$ (不穩定)

$T(r,t)=T_j^n$ 以 $n=1$, $j=0\sim5$

$T(0.5,0.5)=T_0^1=\frac{T_1^1}{0.7}=872.109$

$T(0.6,0.5)=T_1^1=610.476$

$T(0.7,0.5)=T_2^1=97.143$

$T(0.8,0.5)=T_3^1=110$

$T(0.9,0.5)=T_4^1=124.444$

$T(1,0.5)=T_5^1=120$ (右邊這)

※

(b) Backward Difference

- $\Delta t = 0.5$, $\Delta r = 0.1$, $K = 0.1$
- $t_n = n\Delta t$, $t_0 = 0$, $t_1 = 0.5$ ....

  $r_j = 0.5 + j\Delta r$, $r_0 = 0.5$ $r_1 = 0.6$ ... $r_5 = 1$

  令 $T_j^n \approx T(r_j, t_n)$

PDE: $\dfrac{\partial^2 T}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial T}{\partial r} = \dfrac{1}{4k}\dfrac{\partial T}{\partial t}$

$$\Rightarrow \frac{T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}}{(\Delta r)^2} + \frac{1}{r_j}\cdot\frac{T_{j+1}^{n+1} - T_{j-1}^{n+1}}{2\Delta r} = \frac{1}{4K}\cdot\frac{T_j^{n+1} - T_j^n}{\Delta t}$$

$$\Rightarrow 100\left(T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}\right) + \frac{5}{r_j}\left(T_{j+1}^{n+1} - T_{j-1}^{n+1}\right) = 5\left(T_j^{n+1} - T_j^n\right)$$

$$\Rightarrow 5T_j^n = \left[-100 + \frac{5}{r_j}\right]T_{j-1}^{n+1} + [200+5]\,T_j^{n+1} + \left[-100 - \frac{5}{r_j}\right]T_{j+1}^{n+1}$$

$A \cdot T = b$

初始值 $n=0$, $t=0$

$r_j = 0.5 + 0.1j$ 代入 $T(r,0) = 200(r-0.5)$

j	r_j	T_j^0
0	0.5	0
1	0.6	20
2	0.7	40
3	0.8	60
4	0.9	80
5	1.0	100

邊界條件離散化

- $T_5^{n+1} = 100 + 40t_{n+1}$
- $3T_0^{n+1} + \dfrac{T_1^{n+1} - T_0^{n+1}}{\Delta r} = 0$

$\Rightarrow T_1^{n+1} = 0.7\,T_0^{n+1}$

求 $T^1 = [T_0^1, T_1^1, T_2^1, T_3^1, T_4^1]^T$, $T_5^1 = 100 + 40\times 0.5 = 120$

令 $A\cdot T^1 = b$, $b_j = 5\cdot T_j^0$, $T_1^1 = 0.7T_0^1$

$T^0 = [0, 20, 40, 60, 80, 100]$

$\Rightarrow$ 方程①, $j=1$, $r_1 = 0.6$

$a_0 = -100 + \frac{5}{r_1} = -100 + 8.333 = -91.667$

$a_1 = 205$

$a_2 = -100 - 8.333 = -108.333$

$5\cdot T_1^0 = 100$

$-91.667T_0^1 + 205T_1^1 - 108.333T_2^1 = 100$

$\xrightarrow{T_1^1 = 0.7T_0^1 \text{代入}} -91.667T_0^1 + 205\times 0.7T_0^1 - 108.333T_2^1 = 100$

$\Rightarrow 51.833T_0^1 - 108.333T_2^1 = 100$ —— ①

方程② , $j=2$, $r_2 = 0.7$

$a_1 = -100 + \frac{5}{0.7} = -92.857$ $\quad 5\cdot T_2^0 = 200$

$a_2 = 205$

$a_3 = -100 - \frac{5}{0.7} = -107.143$

$\Rightarrow -92.857T_1^1 + 205T_2^1 - 107.143T_3^1 = 200$

$\xrightarrow{T_1^1 = 0.7T_0^1 \text{代入}} -64.9999T_0^1 + 205T_2^1 - 107.143T_3^1 = 200$ —— ②

方程③ $j=3$, $r_3 = 0.8$

$a_2 = -100 + \frac{5}{0.8} = -93.75$ $\quad 5\cdot T_3^0 = 300$

$a_3 = 205$

$a_4 = -100 + \frac{5}{0.8} = -106.25$

$\Rightarrow -93.75T_2^1 + 205T_3^1 - 106.25T_4^1 = 300$ —— ③

方程④ $j=4$, $r_4 = 0.9$

$a_3 = -100 + \frac{5}{0.9} = -94.444$ $\quad 5\cdot T_4^0 = 400$

$a_4 = 205$

$a_5 = -100 - \frac{5}{0.9} = -105.556$

$\Rightarrow -94.444T_3^1 + 205T_4^1 - 105.556T_5^1 = 400$

$\xrightarrow{T_5^1 = 120 \text{代入}} -94.444T_3^1 + 205T_4^1 = 13066.72$ —— ④

$$\begin{cases} 51.833T_0^1 - 108.333T_2^1 = 100 \\ -64.9999T_0^1 + 205T_2^1 - 107.143T_3^1 = 200 \\ -93.75T_2^1 + 205T_3^1 - 106.25T_4^1 = 300 \\ -94.444T_3^1 + 205T_4^1 = 13066.72 \end{cases}$$

python 解出 →

```python
import numpy as np

A = np.array([
    [51.833, -108.333, 0, 0],
    [-65.0, 205, -107.143, 0],
    [0, -93.75, 205, -106.25],
    [0, 0, -94.444, 205]
])

b = np.array([100, 200, 300, 13066.72])

x = np.linalg.solve(A, b)
T0, T2, T3, T4 = x
T1 = 0.7 * T0
T5 = 120

T_values = [T0, T1, T2, T3, T4, T5]
T_values_rounded = [round(v, 4) for v in T_values]
print("T^1 =", T_values_rounded)
```

```
T^1 = [2267.691, 1587.3837, 1084.0762, 696.599, 384.665, 120]
```

Ans. $j = 0\sim5$ $n=1$, $T(r,t)$

$T_0^1 = T(0.5, 0.5) = 2267.691$

$T_1^1 = T(0.6, 0.5) = 1587.3837$

$T_2^1 = T(0.7, 0.5) = 1084.0762$

$T_3^1 = T(0.8, 0.5) = 696.599$

$T_4^1 = T(0.9, 0.5) = 384.665$

$T_5^1 = T(1, 0.5) = 120$

#

→ 整体是穩定

(c) Crank-Nicolson algorithm

PDE : $\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{4k}\frac{\partial T}{\partial t}$ $\quad 0.5 \le r \le 1,\ t \ge 0$

邊界條件 :

$T(1,t) = 100 + 40t$

$3T + \frac{\partial T}{\partial t} = 0$ at $r = 0.5$

$T(r,0) = 200(r - 0.5)$

網格條件

$\Delta r = 0.1 \rightarrow r_0 = 0.5,\ r_1 = 0.6 \ \dots\ r_5 = 1$

$\Delta t = 0.5,\ k = 0.1$

$\gamma = \frac{1}{4k\Delta t} = \frac{1}{4\times 0.1 \times 0.5} = 5$

Crank-Nicolson 離散公式

$$\rightarrow \left(\frac{T_j^{n+1} - T_j^n}{\Delta t}\right) = \frac{4k}{2}\left[\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)^n + \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)^{n+1}\right] \rightarrow \gamma T_j^{n+1} - \frac{1}{2}(D_j^{n+1}) = \gamma T_j^n + \frac{1}{2}(D_j^n)$$

其中 $D_j^n = \frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\Delta r^2} + \frac{1}{r_j}\frac{T_{j+1}^n - T_{j-1}^n}{2\Delta r}$

$$\xrightarrow{\text{代入}} \gamma T_j^{n+1} - \frac{1}{2}\left(\frac{T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}}{\Delta r^2} + \frac{1}{r_j}\frac{T_{j+1}^{n+1} - T_{j-1}^{n+1}}{2\Delta r}\right) = \gamma T_j^n + \frac{1}{2}\left(\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\Delta r^2} + \frac{1}{r_j}\frac{T_{j+1}^n - T_{j-1}^n}{2\Delta r}\right)$$

令 $h = \Delta r = 0.1$

$\gamma = 5$

$\frac{1}{h^2} = 100$

$\frac{1}{2h^2} = 50$

$\frac{1}{4h} = 2.5$

$a_j = -\left(\frac{1}{2h^2} - \frac{1}{4hr_j}\right)$ $\qquad x_j = \frac{1}{2h^2} - \frac{1}{4hr_j}$

$b_j = \gamma + \frac{1}{h^2}$ $\qquad y_j = \gamma - \frac{1}{h^2}$

$c_j = -\left(\frac{1}{2h^2} + \frac{1}{4hr_j}\right)$ $\qquad z_j = \frac{1}{2h^2} + \frac{1}{4hr_j}$

$$a_j T_{j-1}^{n+1} + b_j T_j^{n+1} + c_j T_{j+1}^{n+1} = RHS_j = x_j T_{j-1}^n + y_j T_j^n + z_j T_{j+1}^n$$

初始 $T^0 = [0, 20, 40, 60, 80, 100]$, $T_1^{n+1} = 0.7\,T_0^{n+1}$, $n = 0$

方程① $j=1$

$r_1 = 0.6 \rightarrow \frac{1}{4hr_1} = \frac{1}{4\times 0.1\times 0.6} = 4.1667$

$a_1 = -(50 - 4.1667) = -45.8333$

$b_1 = 5 + 100 = 105$

$c_1 = -(50 + 4.1667) = -54.1667$

$x_1 = 50 - 4.1667 = 45.8333$

$y_1 = -95$

$z_1 = 50 + 4.1667 = 54.1667$

$RHS_1 = 45.8333\cdot T_0^0 + (-95)\times T_1^0 + 54.1667\,T_2^0$

$= 0 - 1900 + 2166.668 = 266.668$

$-45.8333\,T_0^1 + 105\,T_1^1 - 54.1667\,T_2^1 = 266.668$

$\xrightarrow{T_1 = 0.7T_0 \text{代入}} 27.6667\,T_0^1 - 54.1667\,T_2^1 = 266.668$ —— ①

方程② $j=2$

$r_2 = 0.7 \rightarrow \frac{1}{4hr_2} = 3.5714$

$a_2 = -46.4286$

$b_2 = 105$

$c_2 = -53.5714$

$x_1 = 46.4286$

$y_2 = -95$

$z_2 = 53.5714$

$RHS_2 = 46.4286\times 20 + (-95)\times 40 + 53.5714\times 60 = 342.857$

$-46.4286\,T_1^1 + 105\,T_2^1 - 53.5714\,T_3^1 = 342.857$

$\xrightarrow{T_1 = 0.7T_0 \text{代入}} -32.5\,T_0^1 + 105\,T_2^1 - 53.5714\,T_3^1 = 342.857$ —— ②

方程③ $j=3$

$r_3 = 0.8 \rightarrow \frac{1}{4hr_3} = 3.125$

$a_3 = -46.875$

$b_3 = 105$

$c_3 = -53.125$

$x_3 = 46.875$

$y_3 = -95$

$z_3 = 53.125$

$RHS_3 = 46.875\times 40 + (-95)\times 60 + 53.125\times 80 = 425$

$-46.875\,T_2^1 + 105\,T_3^1 - 53.125\,T_4^1 = 425$ —— ③

方程④ $j=4$

$r_4 = 0.9 \rightarrow \frac{1}{4hr_4} = 2.7778$

$a_4 = -47.2222$

$b_4 = 105$

$c_4 = -52.7778$

$x_4 = 47.2222$

$y_4 = -95$

$z_4 = 52.7778$

$RHS_4 = 511.112$

$-47.222\,T_3^1 + 105\,T_4^1 = 511.112 + 52.7778\times 120 = 6844.448$ —— ④

$$\begin{cases} 27.6667\,T_0^1 - 54.1667\,T_2^1 = 266.668 \\ -32.5\,T_0^1 + 105\,T_2^1 - 53.5714\,T_3^1 = 342.857 \\ -46.875\,T_2^1 + 105\,T_3^1 - 53.125\,T_4^1 = 425 \\ -47.222\,T_3^1 + 105\,T_4^1 = 6844.448 \end{cases} \longrightarrow$$

$\boxed{\begin{aligned} &\text{Ans. } j = 0\sim 5,\ n = 1,\ T(r,t) \\ &T_0^1 = T(0.5, 0.5) = 616.0121 \\ &T_1^1 = T(0.6, 0.5) = 431.2085 \\ &T_2^1 = T(0.7, 0.5) = 309.7171 \\ &T_3^1 = T(0.8, 0.5) = 226.9317 \\ &T_4^1 = T(0.9, 0.5) = 167.244 \\ &T_5^1 = T(1, 0.5) = 120 \end{aligned}}$

#

3. Given the problem

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}=0,\quad \frac{1}{2}\le r\le 1,\quad 0\le t\le \pi/3,$$

$$T(r,0)=0,\quad T(r,\pi/3)=0,\quad T(1/2,\theta)=50,\quad T(1,\theta)=100.$$

用有限差分法(FDM)

1. 將區域離散化

$M+1$ 個點在 $r\in[\frac{1}{2},1]$, $\Delta r=\frac{\frac{1}{2}}{M}=\frac{1}{2M}$

$N+1$ 個點在 $\theta\in[0,\frac{\pi}{3}]$, $\Delta\theta=\frac{\pi/3}{N}=\frac{\pi}{3N}$

令 $r_i=\frac{1}{2}+i\cdot\Delta r$, $i=0,1,\dots,M$

$\theta_j=j\cdot\Delta\theta$, $j=0,1\dots N$

用 $T_{i,j}$ 表 $T(r_i,\theta_j)$

2. PDE 可離散為

$$\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{(\Delta r)^2}+\frac{1}{r_i}\cdot\frac{T_{i+1,j}-T_{i-1,j}}{2\Delta r}+\frac{1}{r_i^2}\cdot\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{(\Delta\theta)^2}=0$$

3. 設定 $M=2$, $N=3$

$\Delta r=\frac{1}{4}\Rightarrow r_i=\frac{1}{2}+i\cdot\frac{1}{4}$ $\quad r_0=0.5$, $r_1=0.75$, $r_2=1$

$\Delta\theta=\frac{\pi}{9}\Rightarrow\theta_j=\frac{\pi}{9}j$ $\quad \theta_0=0$, $\theta_1=\frac{\pi}{9}$, $\theta_2=\frac{2\pi}{9}$, $\theta_3=\frac{\pi}{3}$

$\Delta r^2=0.0625$ $\quad \Delta\theta^2=(\frac{\pi}{9})^2\approx 0.1228$

4. ① 套入 $T_{1,1}$ 的公式 ($i=1$, $j=1$, $r_1=0.75$)

$$\frac{T_{2,1}-2T_{1,1}+T_{0,1}}{0.0625}+\frac{1}{0.75}\cdot\frac{T_{2,1}-T_{0,1}}{2\times 0.25}+\frac{1}{0.75^2}\cdot\frac{T_{1,2}-2T_{1,1}+T_{1,0}}{0.1228}=0$$

已知 $T_{2,1}=100$ ($r=1$), $T_{0,1}=50$ ($r=0.5$), $T_{1,0}=0$ ($\theta=0$) 代入

$\Rightarrow 2400-32T_{1,1}+133.33+14.48(T_{1,2}-2T_{1,1})=0$

$-60.96T_{1,1}+14.48T_{1,2}=-2533.33$ —— ①

② 套入 $T_{1,2}$ 的公式 ($i=1$, $j=2$, $r_1=0.75$)

$$\frac{T_{2,2}-2T_{1,2}+T_{0,2}}{0.0625}+\frac{1}{0.75}\times\frac{T_{2,2}-T_{0,2}}{2\times 0.25}+\frac{1}{0.75^2}\times\frac{T_{1,3}-2T_{1,2}+T_{1,1}}{0.1228}=0$$

已知 $T_{2,2}=100$ ($r=1$), $T_{0,2}=50$ ($r=0.5$), $T_{1,3}=0$ ($\theta=0$) 代入

$\Rightarrow 2400-32T_{1,2}+133.33+14.48(T_{1,1}-2T_{1,2})=0$

$14.48T_{1,1}-60.96T_{1,2}=-2533.33$ —— ②

5. 解聯立

$$\begin{cases}-60.96T_{1,1}+14.48T_{1,2}=-2533.33\\ 14.48T_{1,1}-60.96T_{1,2}=-2533.33\end{cases}$$

$T_{1,1}=T_{1,2}=54.504$

$T_{0,0}=T(\frac{1}{2},0)=0$	$T_{1,0}=T(\frac{3}{4},0)=0$	$T_{2,0}=T(1,0)=0$
$T_{0,1}=T(\frac{1}{2},\frac{\pi}{9})=50$	$T_{1,1}=T(\frac{3}{4},\frac{\pi}{9})=54.504$	$T_{2,1}=T(1,\frac{\pi}{9})=100$
$T_{0,2}=T(\frac{1}{2},\frac{2\pi}{9})=50$	$T_{1,2}=T(\frac{3}{4},\frac{2\pi}{9})=54.504$	$T_{2,2}=T(1,\frac{2\pi}{9})=100$
$T_{0,3}=T(\frac{1}{2},\frac{\pi}{3})=0$	$T_{1,3}=T(\frac{3}{4},\frac{\pi}{3})=0$	$T_{2,3}=T(1,\frac{\pi}{3})=0$

#

4. Given the problem

$$\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2}, \quad 0 \le x \le 1, \quad 0 \le t$$

$$p(0,t)=1, \quad p(1,t)=2, \quad p(x,0)=\cos(2\pi x), \quad \frac{\partial p}{\partial t}(x,0) = 2\pi\sin(2\pi x), \quad 0 \le x \le 1$$

To calculate p by using $\Delta x = \Delta t = 0.1$.

求 $p(x,t)$， $t=0.1, 0.2$

$x=0, 0.1 \sim 1$

有限差分法（FDM）

時間格點 $t_n = n\Delta t$

空間格點 $x_i = i\Delta x$，其中 $i=0,1,2,\ldots,10$

$P_i^n \approx p(x_i, t_n)$

$\frac{\partial^2 p}{\partial t^2} \approx \frac{P_i^{n+1} - 2P_i^n + P_i^{n-1}}{(\Delta t)^2}$ ， $\frac{\partial^2 p}{\partial x^2} \approx \frac{P_{i+1}^n - 2P_i^n + P_{i-1}^n}{(\Delta x)^2}$

DE $\rightarrow \frac{P_i^{n+1} - 2P_i^n + P_i^{n-1}}{(\Delta t)^2} = \frac{P_{i+1}^n - 2P_i^n + P_{i-1}^n}{(\Delta x)^2}$

$\xrightarrow{\Delta x 及 \Delta t} P_i^{n+1} = 2P_i^n - P_i^{n-1} + P_{i+1}^n - 2P_i^n + P_{i-1}^n = P_{i+1}^n + P_{i-1}^n - P_i^{n-1}$

初始已知

$P_i^0 = \cos(2\pi x)$

$\left.\frac{\partial p}{\partial t}\right|_{t=0} = 2\pi\sin(2\pi x_i)$

$\frac{\partial p}{\partial t}(x,0) \Rightarrow P_i^1 = P_i^0 + \Delta t \left.\frac{\partial p}{\partial t}\right|_{t=0}$

$\Rightarrow P_i^1 = P_i^0 + \Delta t \cdot \underline{\underline{v_i}} \leftarrow 2\pi\sin(2\pi x)$

實際計算 $\Delta x = 0.1$ $x_i = i \cdot 0.1$， $i = 0,1,\ldots,10$

Step 1. 初始化 $P_i^0 = \cos(2\pi x_i)$

$P_0^0 = \cos(0) = 1$ （左边界）

$P_{10}^0 = \cos(2\pi) = 1$ （右边界）

$P_1^0 = \cos(0.2\pi) \approx 0.809$

$P_2^0 = \cos(0.4\pi) \approx 0.309$

$P_5^0 = \cos(\pi) = -1$

step 2、初始化 $P_i^1 = P_i^0 + \Delta t \cdot 2\pi\sin(2\pi x_i)$ $n=1$， $t=0.1$， $i=1\sim 9$

$P(x_i, t_n) = P_i^n$

$x = 0.1 \sim 0.9$

$n=1 \rightarrow t=0.1$

$P_0^1 = 1$

$P_{10}^1 = 2$

$P_1^1 = \cos(0.2\pi) + 0.1 \cdot 2\pi\sin(0.2\pi) \approx 0.809 + 0.2\pi \times 0.5878 = 1.178$

$P_2^1 = \cos(0.4\pi) + 0.1 \cdot 2\pi\sin(0.4\pi) \approx 0.309 + 0.2\pi \times 0.95105 = 0.9066$

$P_3^1 = \cos(0.6\pi) + 0.2\pi\sin(0.6\pi) \approx -0.309 + 0.2\pi \times 0.95105 = 0.2885$

$P_4^1 = -0.4397$

$P_5^1 = -1$

$P_6^1 = -1.1783$

$P_7^1 = -0.9066$

$P_8^1 = -2.885$

$P_9^1 = 0.4397$

#

step 3、時間層使用 $P_i^{n+1} = P_{i+1}^n + P_{i-1}^n - P_i^{n-1}$

$t=0.2$， $n=1\rightarrow 2$ $P_i^2 = P_{i+1}^1 + P_{i-1}^1 - P_i^0$

當 $i=1 \Rightarrow P_1^2 = P_2^1 + P_0^1 - P_1^0$ （由前一步算出 P_2^1， $P_0^1 = 1$）

$= 0.9066 + 1 - 0.809 = 1.0976$

當 $i=2 \Rightarrow P_2^2 = P_3^1 + P_1^1 - P_2^0$

$= 0.2885 + 1.178 - 0.309 = 1.1575$

以此類推 … $\Rightarrow$ 求 P_j^2 $j=3\sim 9$

$P(x,t)$

$x = 0 \sim 1$

$t = 0.2$

$P_0^2 = 1$

$P_1^2 = 1.0976$

$P_2^2 = 1.1575$

$P_3^2 = 0.7759$

$P_4^2 = 0.0976$

$P_5^2 = -0.6180$

$P_6^2 = -1.0976$

$P_7^2 = -1.1579$

$P_8^2 = -0.7759$

$P_9^2 = 0.9024$

$P_{10}^2 = 2$

*