HW3 E94104094 游昊蓁

1. $\cos(0.698)=0.7661$
$\cos(0.733)=0.7432$
$\cos(0.768)=0.7193$
$\cos(0.803)=0.6946$ 弧度

- $P_n(x)=\sum_{i=0}^{n} y_i L_i(x)$
- $L_i(x)=\prod_{j=0,j\neq i}^{n}\frac{x-x_j}{x_i-x_j}$
- Error Bound: $|f(x)-P_n(x)| \le \frac{\max|\boxed{f^{(n+1)}(\,)}|}{(n+1)!}\prod_{i=0}^{n}|x-x_i|$ f(x) = cos(x) 的 (n+1) 階導數的最大值
- 本題計算 $P_n(x)$ 並求 Error Bound

(1) 一階：取最接近 0.750 的兩點

$x_1=0.733 \quad f(0.733)=0.7432$

$x_2=0.768 \quad f(0.768)=0.7193$

$P_1(x)=f(x_1)L_1(x)+f(x_2)L_2(x)$

其中 $L_1(x)=\frac{x-x_2}{x_1-x_2}=\frac{0.750-0.768}{0.733-0.768}=\frac{-0.018}{-0.035}\approx 0.5143$

$L_2(x)=\frac{x-x_1}{x_2-x_1}=\frac{0.750-0.733}{0.768-0.733}=\frac{0.017}{0.035}\approx 0.4857$

$P_1(0.750)=0.7432\times 0.5143+0.7193\times 0.4857=0.73159\approx 0.7316$ #

$E_1(x)=\frac{\max|f''(\xi)|}{2!}|(x-x_0)(x-x_1)|$

$f''(x)=-\cos(x)$ 最大值 $\Rightarrow |-\cos(0.733)|=0.74317$

$E_1(0.750)=\frac{0.74317}{2}|(0.750-0.733)(0.750-0.768)|=0.0001137 \Rightarrow E_1(0.750)\le 1.137\times10^{-4}$ #

(2) 二階：$x_0=0.698$，$x_1=0.733$，$x_2=0.768$，$x=0.750$

$P_2(x)=f(x_0)L_0(x)+f(x_1)L_1(x)+f(x_2)L_2(x)$

$L_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(0.750-0.733)(0.750-0.768)}{(0.698-0.733)(0.698-0.768)}=-0.1249$

$L_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\frac{(0.750-0.698)(0.750-0.768)}{(0.733-0.698)(0.733-0.768)}=0.7641$

$L_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{(0.750-0.698)(0.750-0.733)}{(0.768-0.698)(0.768-0.733)}=0.3608$

$P_2(0.750)=0.7661\times(-0.1249)+0.7432\times0.7641+0.7193\times0.3608\approx 0.73171667$ #

$E_2(x)=\frac{\max|f'''(\xi)|}{3!}|(x-x_0)(x-x_1)(x-x_2)|$

$f'''(x)=\sin(x)$ 最大值 $\Rightarrow |\sin(0.768)|=0.694698$

$E_2(0.750)=\frac{0.694698}{6}|(0.750-0.698)(0.750-0.733)(0.750-0.768)|=0.000001842 \Rightarrow E_2(0.750)\le 1.842\times10^{-6}$ #

(3) 三階：$x_0=0.698$，$x_1=0.733$，$x_2=0.768$，$x_3=0.803$

$P_3(x)=f(x_0)L_0(x)+f(x_1)L_1(x)+f(x_2)L_2(x)+f(x_3)L_3(x)$

$L_0(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}=\frac{0.017\times(-0.018)\times(-0.053)}{(-0.035)\times(-0.070)\times(-0.105)}=-0.06304$

$L_1(x)=\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}=\frac{0.052\times(-0.018)\times(-0.053)}{0.035\times(-0.035)\times(-0.070)}=0.5785$

$L_2(x)=\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}=\frac{0.052\times0.017\times(-0.053)}{0.070\times0.035\times(-0.035)}=0.5464$

$L_3(x)=\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}=\frac{0.052\times0.017\times(-0.018)}{0.105\times0.070\times0.035}=-0.06185$

$P_3(0.750)=0.7661\times(-0.06304)+0.7432\times0.5785+0.7193\times0.5464+0.6946\times(-0.06185)\approx 0.7317827$ #

$E_3(x)=\frac{\max|f^{(4)}(\xi)|}{4!}|(x-x_0)(x-x_1)(x-x_2)(x-x_3)|$

$f^{(4)}(x)=\cos(x)$ 最大值 $\Rightarrow |\cos(0.698)|=0.766129$

$E_3(0.750)=\frac{0.766129}{24}|(0.750-0.698)(0.750-0.733)(0.750-0.768)(0.750-0.803)|=2.692\times10^{-8} \Rightarrow E_3(0.750)\le 2.692\times10^{-8}$ #

(4) 四階：由於我們只有四個已知點 x_0, x_1, x_2, x_3,
最高只能進行三階插值,
因此四階插值與三階插值相同。

$\Rightarrow E_4(0.750) = E_3(0.750) \leq 2.692\times10^{-8}$ ※

2.

x	0.3	0.4	0.5	0.6
e^{-x}	0.740818	0.670320	0.606531	0.548812

$f(x) = x - e^{-x} = 0$

$f[x_0] = 0.3 - e^{-0.3} = -0.440818 = a_0$

$f[x_1] = 0.4 - e^{-0.4} = -0.27032$

$f[x_2] = 0.5 - e^{-0.5} = -0.106531$

$f[x_3] = 0.6 - e^{-0.6} = 0.051188$

$f[x_0,x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} = \frac{-0.27032+0.440818}{0.4-0.3} = 1.70498 = a_1$

$f[x_1,x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} = \frac{-0.106531+0.27032}{0.5-0.4} = 1.63789$

$f[x_2,x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2} = \frac{0.051188+0.106531}{0.6-0.5} = 1.57719$

$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{1.63789-1.70498}{0.5-0.3} = -0.33545 = a_2$

$f[x_1,x_2,x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1} = \frac{1.57719-1.63789}{0.6-0.4} = -0.3035$

$f[x_0,x_1,x_2,x_3] = \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0} = \frac{-0.3035+0.33545}{0.6-0.3} = 0.1065 = a_3$

- $P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$

$= -0.440818 + 1.70498(x-0.3) - 0.33545(x-0.3)(x-0.4) + 0.1065(x-0.3)(x-0.4)(x-0.5)$

$= -0.440818 + 1.70498x - 0.511494 - 0.33545x^2 + 0.234815x - 0.040254 + 0.1065x^3 - 0.1278x^2 + 0.050055x - 0.00639$

$= 0.1065x^3 - 0.46325x^2 + 1.98985x - 0.998956$

迭代求解 $\Rightarrow x_0 = 0.5$

Newton's Method

$P_3'(x) = 0.3195x^2 - 0.9265x + 1.98985$

1st iteration

$P_3(x_0) = P_3(0.5) = -0.106531$

$P_3'(x_0) = P_3'(0.5) = 1.606475$

$x_1 = 0.5 - \frac{-0.106531}{1.606475} = 0.56631$

2nd iteration

$P_3(x_1) = P_3(0.56631) = -0.001309$

$P_3'(x_1) = P_3'(0.56631) = 1.56763$

$x_2 = 0.56631 - \frac{-0.001309}{1.56763} = 0.56715$

3rd iteration

$P_3(x_2) = 7.52\times10^{-6}$

$P_3'(x_2) = 1.56716$

$x_3 = 0.56715 - \frac{7.52\times10^{-6}}{1.56716} = 0.567145$ ※

$\boxed{x = 0.567145 \text{ 為 } x - e^{-x} = 0 \text{ 的近似解}}$ ※

3. a. $D \Rightarrow f(x)$ $\quad V \Rightarrow f'(x)$

z	$f(z)$	First divided differences	Second divided differences	3rd	4th
$z_0=x_0=0$	$f[z_0]=f(x_0)=0$				
$z_1=x_0=0$	$f[z_1]=f(x_0)=0$	$f[z_0,z_1]=f'(x_0)=75$			
$z_2=x_1=3$	$f[z_2]=f(x_1)=200$	$f[z_1,z_2]=\frac{f[z_2]-f[z_1]}{z_2-z_1}=\frac{200}{3}=66.\bar{6}$	$f[z_0,z_1,z_2]=-2.\bar{7}=\frac{-25}{9}$		
$z_3=x_1=3$	$f[z_3]=f(x_1)=200$	$f[z_2,z_3]=f'(x_1)=77$	$f[z_1,z_2,z_3]=3.\bar{4}=\frac{31}{9}$	$f[z_0\ldots z_3]=\frac{56}{27}=2.\overline{074}$	
$z_4=x_2=5$	$f[z_4]=f(x_2)=375$	$f[z_3,z_4]=\frac{f[z_4]-f[z_3]}{z_4-z_3}=\frac{175}{2}=87.5$	$f[z_2,z_3,z_4]=5.25$	$f[z_1\ldots z_4]=\frac{13}{36}=0.36\bar{1}$	$f[z_0\ldots z_4]=-\frac{37}{108}=-0.34\overline{259}$
$z_5=x_2=5$	$f[z_5]=f(x_2)=375$	$f[z_4,z_5]=f'(x_2)=80$	$f[z_3,z_4,z_5]=-3.75$	$f[z_2\ldots z_5]=-4.5$	$f[z_1\ldots z_5]=-\frac{35}{36}=-0.97\bar{2}$
$z_6=x_3=8$	$f[z_6]=f(x_3)=620$	$f[z_5,z_6]=\frac{f[z_6]-f[z_5]}{z_6-z_5}=\frac{245}{3}=81.\bar{6}$	$f[z_4,z_5,z_6]=0.\bar{5}=\frac{5}{9}$	$f[z_3\ldots z_6]=\frac{31}{36}=0.86\bar{1}$	$f[z_2\ldots z_6]=\frac{193}{180}=1.07\bar{2}$
$z_7=x_3=8$	$f[z_7]=f(x_3)=620$	$f[z_6,z_7]=f'(x_3)=74$	$f[z_5,z_6,z_7]=-2.\bar{5}=\frac{-23}{9}$	$f[z_4\ldots z_7]=\frac{-28}{27}=-1.\overline{037}$	$f[z_3\ldots z_7]=\frac{-41}{108}=-0.37\overline{962}$
$z_8=x_4=13$	$f[z_8]=f(x_4)=990$	$f[z_7,z_8]=\frac{f[z_8]-f[z_7]}{z_8-z_7}=\frac{370}{5}=74$	$f[z_6,z_7,z_8]=0$	$f[z_5\ldots z_8]=\frac{23}{72}=0.319\bar{4}$	$f[z_4\ldots z_8]=\frac{293}{1728}=0.16956$
$z_9=x_4=13$	$f[z_9]=f(x_4)=990$	$f[z_8,z_9]=f'(x_4)=72$	$f[z_7,z_8,z_9]=-0.4$	$f[z_6\ldots z_9]=-0.08$	$f[z_5\ldots z_9]=\frac{-719}{14400}=-0.049930\bar{5}$

5th	6th	7th	8th	9th
$f[z_0\ldots z_5]=\frac{-17}{135}=-0.1\overline{259}$	$f[z_0\ldots z_6]=\frac{103}{2160}=0.0476\overline{851}$	$f[z_0\ldots z_7]=\frac{-313}{21600}=-0.01449\overline{074}$	$f[z_0\ldots z_8]=0.00172277$	$f[z_0\ldots z_9]=-0.00020460$
$f[z_1\ldots z_6]=\frac{23}{90}=0.2\bar{5}$	$f[z_1\ldots z_7]=\frac{-737}{10800}=-0.0682\overline{407}$	$f[z_1\ldots z_8]=0.00790535$	$f[z_1\ldots z_9]=-0.000937$	
$f[z_2\ldots z_7]=\frac{-196}{675}=-0.29\overline{037}$	$f[z_2\ldots z_8]=0.0345289$	$f[z_2\ldots z_9]=-0.0042764$		
$f[z_3\ldots z_8]=\frac{949}{17280}=0.054919$	$f[z_3\ldots z_9]=-0.008235498$			
$f[z_4\ldots z_9]=-0.027436$				

$$H_9(x)=f[z_0]+\sum_{k=1}^{9} f[z_0,\ldots,z_k](x-z_0)(x-z_1)\cdots(x-z_{k-1})$$

x^3-3x^2

$$=0+75(x-0)+\left(-\frac{25}{9}\right)(x-0)^2+\frac{56}{27}(x-0)^2(x-3)+\left(\frac{-37}{108}\right)(x-0)^2(x-3)^2+\left(\frac{-17}{135}\right)(x-0)^2(x-3)^2(x-5)+\frac{103}{2160}(x-0)^2(x-3)^2(x-5)^2$$
$$+\left(-\frac{313}{21600}\right)(x-0)^2(x-3)^2(x-5)^2(x-8)+(0.00172277)(x-0)^2(x-3)^2(x-5)^2(x-8)^2+(-0.00020460)(x-0)^2(x-3)^2(x-5)^2(x-8)^2(x-13)$$

$x=10$ 代入

$H_9(10)=768.96$ ft

$V_9(10)=H_9'(10)=74.64$ ft/s

b. 55 mph

$\Rightarrow 55\times\frac{5280}{3600}=80.67$ ft/s

$\Rightarrow$ car never exceeds 55 mph

c. $V_{max}\approx 92.04$ ft/s at $t=4.06$ s

3、

```python
import numpy as np
import scipy.interpolate as spi

# Given data
T = np.array([0, 3, 5, 8, 13])  # Time in seconds
D = np.array([0, 200, 375, 620, 990])  # Distance in feet
V = np.array([75, 77, 80, 74, 72])  # Speed in feet per second

# Construct Hermite interpolating polynomial
hermite_interp = spi.CubicHermiteSpline(T, D, V)

# Part (a) Predict position and speed at t = 10 sec
t_target = 10
distance_at_10 = hermite_interp(t_target)  # Position
speed_at_10 = hermite_interp.derivative()(t_target)  # Speed
print(f"Predicted position at t=10s: {distance_at_10:.2f} feet")
print(f"Predicted speed at t=10s: {speed_at_10:.2f} ft/s")

# Part (b) Find the first time exceeding 55 mph (convert to ft/s)
limit_speed = 55 * 5280 / 3600  # Convert mph to ft/s
speed_func = hermite_interp.derivative()

# Define a function to find root where speed exceeds limit
def speed_exceeds_limit(t):
    return speed_func(t) - limit_speed

from scipy.optimize import brentq

# Find the first time speed exceeds limit within given range
t_exceed = None
for i in range(len(T) - 1):
    if speed_exceeds_limit(T[i]) * speed_exceeds_limit(T[i+1]) < 0:
        t_exceed = brentq(speed_exceeds_limit, T[i], T[i+1])
        break

if t_exceed:
    print(f"Car first exceeds 55 mph at t={t_exceed:.2f} seconds")
else:
    print("Car never exceeds 55 mph")

# Part (c) Find the maximum speed
from scipy.optimize import minimize_scalar

res = minimize_scalar(lambda t: -speed_func(t), bounds=(T[0], T[-1]), method='bounded')
max_speed = -res.fun
max_speed_time = res.x

print(f"Predicted maximum speed: {max_speed:.2f} ft/s at t={max_speed_time:.2f} seconds")
```

```
Predicted position at t=10s: 768.96 feet
Predicted speed at t=10s: 74.64 ft/s
Car never exceeds 55 mph
Predicted maximum speed: 92.04 ft/s at t=4.06 seconds
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