

Hw3 E94104094 游昊羣

$$\begin{aligned} 1. \cos(0.698) &= 0.7661 \\ \cos(0.733) &= 0.7432 \\ \cos(0.768) &= 0.7193 \\ \cos(0.803) &= 0.6946 \end{aligned}$$

$$\begin{aligned} P_n(x) &= \sum_{i=0}^n y_i L_i(x) \\ L_i(x) &= \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j} \end{aligned}$$

$$\bullet \text{Error Bound: } |f(x) - P_n(x)| \leq \frac{\max |f^{(n+1)}(\xi)|}{(n+1)!} \prod_{i=0}^n |x - x_i|$$

• 本題計算 $P_n(x)$ 並求 Error Bound

$f(x) = \cos(x)$ 的 $(n+1)$ 階導數的最大值

(1) 一階：取最接近 0.750 的兩點

$$x_1 = 0.733 \quad f(0.733) = 0.7432$$

$$x_2 = 0.768 \quad f(0.768) = 0.7193$$

$$P_1(x) = f(x_1)L_1(x) + f(x_2)L_2(x)$$

$$\text{其中 } L_1(x) = \frac{x - x_2}{x_1 - x_2} = \frac{0.750 - 0.768}{0.733 - 0.768} = \frac{-0.018}{-0.035} \approx 0.5143$$

$$L_2(x) = \frac{x - x_1}{x_2 - x_1} = \frac{0.750 - 0.733}{0.768 - 0.733} = \frac{0.017}{0.035} \approx 0.4857$$

$$P_1(0.750) = 0.7432 \times 0.5143 + 0.7193 \times 0.4857 = 0.73159 \approx 0.7316 *$$

$$E_1(x) = \frac{\max |f''(\xi)|}{2!} |(x - x_0)(x - x_1)|$$

$$f''(x) = -\cos(x) \text{ 最大值} \Rightarrow |\cos(0.733)| = 0.74319$$

$$E_1(0.750) = \frac{0.74319}{2} |(0.750 - 0.733)(0.750 - 0.768)| = 0.0001137 \Rightarrow E_1(0.750) \leq 1.137 \times 10^{-4} *$$

(2) 二階： $x_0 = 0.698, x_1 = 0.733, x_2 = 0.768, x = 0.750$

$$P_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)$$

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(0.750 - 0.733)(0.750 - 0.768)}{(0.698 - 0.733)(0.698 - 0.768)} = -0.1249$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(0.750 - 0.698)(0.750 - 0.768)}{(0.733 - 0.698)(0.733 - 0.768)} = 0.7641$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(0.750 - 0.698)(0.750 - 0.733)}{(0.768 - 0.698)(0.768 - 0.733)} = 0.3608$$

$$P_2(0.750) = 0.7661 \times (-0.1249) + 0.7432 \times 0.7641 + 0.7193 \times 0.3608 \approx 0.73171667 *$$

$$E_2(x) = \frac{\max |f'''(\xi)|}{3!} |(x - x_0)(x - x_1)(x - x_2)|$$

$$f'''(x) = \sin(x) \text{ 最大值} \Rightarrow |\sin(0.768)| = 0.694698$$

$$E_2(0.750) = \frac{0.694698}{6} |(0.750 - 0.698)(0.750 - 0.733)(0.750 - 0.768)| = 0.0000001842 \Rightarrow E_2(0.750) \leq 1.842 \times 10^{-6} *$$

(3) 三階： $x_0 = 0.698, x_1 = 0.733, x_2 = 0.768, x_3 = 0.803$

$$P_3(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x) + f(x_3)L_3(x)$$

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = \frac{0.017 \times (-0.018) \times (-0.053)}{(-0.035) \times (-0.070) \times (-0.105)} = -0.06304$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} = \frac{0.052 \times (-0.018) \times (-0.053)}{0.035 \times (-0.035) \times (-0.070)} = 0.5785$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} = \frac{0.052 \times 0.017 \times (-0.053)}{0.070 \times 0.035 \times (-0.035)} = 0.5464$$

$$L_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = \frac{0.052 \times 0.017 \times (-0.018)}{0.105 \times 0.070 \times 0.035} = -0.06185$$

$$P_3(0.750) = 0.7661 \times (-0.06304) + 0.7432 \times 0.5785 + 0.7193 \times 0.5464 + 0.6946 \times (-0.06185) \approx 0.7317827 *$$

$$E_3(x) = \frac{\max |f^{(4)}(\xi)|}{4!} |(x - x_0)(x - x_1)(x - x_2)(x - x_3)|$$

$$f^{(4)}(x) = \cos(x) \text{ 最大值} \Rightarrow |\cos(0.698)| = 0.766129$$

$$E_3(0.750) = \frac{0.766129}{24} |(0.750 - 0.698)(0.750 - 0.733)(0.750 - 0.768)(0.750 - 0.803)| = 2.692 \times 10^{-8} \Rightarrow E_3(0.750) \leq 2.692 \times 10^{-8} *$$

(4) 四階：由於我們只有四個已知點 x_0, x_1, x_2, x_3 ，
最高只能進行三階插值，
因此四階插值與三階插值相同。

$$\Rightarrow E_4(0.750) = E_3(0.750) \leq 2.692 \times 10^{-8} \text{ **}$$

X	0.3	0.4	0.5	0.6
e^{-x}	0.740818	0.670320	0.606531	0.548812

$$f(x) = x - e^{-x} = 0$$

$$f[x_0] = 0.3 - e^{-0.3} = -0.440818 = a_0$$

$$f[x_1] = 0.4 - e^{-0.4} = -0.27032$$

$$f[x_2] = 0.5 - e^{-0.5} = -0.106531$$

$$f[x_3] = 0.6 - e^{-0.6} = 0.051188$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{-0.27032 + 0.440818}{0.4 - 0.3} = 1.70498 = a_1$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{-0.106531 + 0.27032}{0.5 - 0.4} = 1.63789$$

$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{0.051188 + 0.106531}{0.6 - 0.5} = 1.59919$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1.63789 - 1.70498}{0.5 - 0.3} = -0.33545 = a_2$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{1.59919 - 1.63789}{0.6 - 0.4} = -0.3035$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{-0.3035 + 0.33545}{0.6 - 0.3} = 0.1065 = a_3$$

$$\bullet P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$= -0.440818 + 1.70498(x - 0.3) - 0.33545(x - 0.3)(x - 0.4) + 0.1065(x - 0.3)(x - 0.4)(x - 0.5)$$

$$= -0.440818 + 1.70498x - 0.511494 - 0.33545x^2 + 0.234815x^3 - 0.040254x^4 + 0.1065x^5 - 0.1278x^6 + 0.050055x^7 - 0.00639$$

$$= 0.1065x^3 - 0.46325x^2 + 1.98985x - 0.998956$$

迭代求解 $\Rightarrow x_0 = 0.5$

Newton's Method

$$P_3'(x) = 0.3195x^2 - 0.9265x + 1.98985$$

1st iteration

$$P_3(x_0) = P_3(0.5) = -0.106531$$

$$P_3'(x_0) = P_3'(0.5) = 1.606495$$

$$x_1 = 0.5 - \frac{-0.106531}{1.606495} = 0.56631$$

2nd iteration

$$P_3(x_1) = P_3(0.56631) = -0.001309$$

$$P_3'(x_1) = P_3'(0.56631) = 1.56763$$

$$x_2 = 0.56631 - \frac{-0.001309}{1.56763} = 0.56715$$

3rd iteration

$$P_3(x_2) = 1.52 \times 10^{-6}$$

$$P_3'(x_2) = 1.56716$$

$$x_3 = 0.56715 - \frac{1.52 \times 10^{-6}}{1.56716} = 0.567145 \text{ **}$$

$x = 0.567145$ 為 $x - e^{-x} = 0$ 的近似解 **

3. A car travelling along a straight road is clocked at a number of points.

The data from the observations are given in the following table, where the time T is in seconds, the distance D is in feet, and the speed V is in feet per second.

T	0	x_0	3	x_1	5	x_2	8	x_3	13	x_4
$f(x)$	D	0	200	375	620	990				
$f'(x)$	V	75	77	80	74	72				

- Use a Hermite polynomial to predict the position of the car and its speed when $t = 10$ s.
- Use the derivative of the Hermite polynomial to determine whether the car ever exceeds a 55 mi/h speed limit on the road. If so, what is the first time the car exceeds this speed?
- What is the predicted maximum speed for the car?

一輛沿直線道路行駛的汽車在多個時間點被測速。測量數據如下表，時間 T 單位為秒，距離 D 單位為英尺，速度 V 單位為英尺每秒。

$T(s)$	0	3	5	8	13
$D(\text{ft})$	0	200	375	620	990
$V(\text{ft/s})$	75	77	80	74	72

問題

- 使用 Hermite 多項式來預測汽車在 $t = 10$ 秒時的位置和速度。
- 使用 Hermite 多項式的導數來確定汽車是否曾超過道路限速 55 英里每小時。如果超過，找出第一次超過該速度的時間點。
- 預測汽車的最大速度。

$$a. D \Rightarrow f(x) \quad V \Rightarrow f'(x)$$

$$z \quad f(z)$$

First divided
differences

Second divided
differences

3rd 4th

$$z_0 = x_0 = 0 \quad f[z_0] = f(x_0) = 0$$

$$z_1 = x_0 = 0 \quad f[z_1] = f(x_0) = 0$$

$$z_2 = x_1 = 3 \quad f[z_2] = f(x_1) = 200$$

$$z_3 = x_1 = 3 \quad f[z_3] = f(x_1) = 200$$

$$z_4 = x_2 = 5 \quad f[z_4] = f(x_2) = 375$$

$$z_5 = x_2 = 5 \quad f[z_5] = f(x_2) = 375$$

$$z_6 = x_3 = 8 \quad f[z_6] = f(x_3) = 620$$

$$z_7 = x_3 = 8 \quad f[z_7] = f(x_3) = 620$$

$$z_8 = x_4 = 13 \quad f[z_8] = f(x_4) = 990$$

$$z_9 = x_4 = 13 \quad f[z_9] = f(x_4) = 990$$

$$f[z_0, z_1] = f(x_0) = 75$$

$$f[z_1, z_2] = \frac{f[z_2] - f[z_1]}{z_2 - z_1} = \frac{200 - 75}{3 - 0} = 66.6$$

$$f[z_2, z_3] = f(x_1) = 77$$

$$f[z_3, z_4] = \frac{f[z_4] - f[z_3]}{z_4 - z_3} = \frac{375 - 200}{8 - 5} = 87.5$$

$$f[z_4, z_5] = f'(x_2) = 80$$

$$f[z_5, z_6] = \frac{f[z_6] - f[z_5]}{z_6 - z_5} = \frac{620 - 375}{8 - 5} = 81.6$$

$$f[z_6, z_7] = f'(x_3) = 74$$

$$f[z_7, z_8] = \frac{f[z_8] - f[z_7]}{z_8 - z_7} = \frac{990 - 620}{13 - 8} = 74$$

$$f[z_8, z_9] = f'(x_4) = 72$$

$$f[z_0, z_1, z_2] = -2.7 = -\frac{25}{9}$$

$$f[z_1, z_2, z_3] = 3.4 = \frac{31}{9}$$

$$f[z_2, z_3, z_4] = 5.25$$

$$f[z_3, z_4, z_5] = -3.75$$

$$f[z_4, z_5, z_6] = 0.5 = \frac{5}{9}$$

$$f[z_5, z_6, z_7] = -2.5 = -\frac{23}{9}$$

$$f[z_6, z_7, z_8] = 0$$

$$f[z_7, z_8, z_9] = -0.4$$

$$f[z_0, z_1, z_2, z_3] = \frac{56}{27} = 2.074$$

$$f[z_1, z_2, z_3, z_4] = -\frac{37}{108} = -0.34259$$

$$= \frac{13}{36} = 0.361$$

$$f[z_2, z_3, z_4, z_5] > \frac{35}{36} = -0.972$$

$$f[z_3, z_4, z_5, z_6] > \frac{193}{180} = 1.072$$

$$f[z_4, z_5, z_6, z_7] > \frac{41}{108} = -0.37962$$

$$f[z_5, z_6, z_7, z_8] > \frac{91}{1728} = 0.054956$$

$$f[z_6, z_7, z_8, z_9] > \frac{91}{14400} = -0.0499305$$

5th

6th

7th

8th

9th

$$f[z_0, z_1, z_2]$$

$$= \frac{-12}{735} = -0.1259$$

$$f[z_1, z_2, z_3]$$

$$= \frac{103}{2100} = 0.0476851$$

$$f[z_2, z_3, z_4]$$

$$= \frac{103}{2100} = 0.0476851$$

$$f[z_3, z_4, z_5]$$

$$= \frac{-313}{21000} = -0.01449074$$

$$f[z_4, z_5, z_6]$$

$$= 0.00172277$$

$$f[z_5, z_6, z_7]$$

$$= -0.00020460$$

$$f[z_6, z_7, z_8]$$

$$= -0.00020460$$

$$f[z_7, z_8, z_9]$$

$$= -0.00020460$$

$$a. H_9(x) = f(z_0) + \sum_{k=1}^9 f(z_0, \dots, z_k)(x-z_0)(x-z_1)\dots(x-z_{k-1}) \quad x^3 - 3x^2$$

$$= 0 + 95(x-0) + \left(\frac{25}{9}\right)(x-0)^2 + \left(\frac{56}{27}\right)(x-0)^3(x-3) + \left(\frac{37}{108}\right)(x-0)^2(x-3)^2 + \left(\frac{-7}{33}\right)(x-0)^2(x-3)(x-5) + \left(\frac{103}{360}\right)(x-0)^2(x-3)^2(x-5)^2$$

$$+ \left(-\frac{313}{21600}\right)(x-0)^2(x-3)^2(x-5)(x-8) + (0.00172277)(x-0)^2(x-3)^2(x-5)^2(x-8)^2 + (-0.00020460)(x-0)^2(x-3)^2(x-5)^2(x-8)^2(x-13)^2$$

$x=10$ ft

$$H_9(10) = 768.96 \text{ ft}$$

$$V_9(10) = H'_9(10) = 74.64 \text{ ft/s}$$

b. 55 mph

$$\Rightarrow 55 \times \frac{5280}{3600} = 80.67 \text{ ft/s}$$

$$V(t) > 80.67$$

$$t \approx 3.15 \text{ s}$$

c. $V_{\max} \approx 92.04 \text{ ft/s}$