HW3 E94104094 游旻蓁

1. $\cos(0.698)=0.7661$
$\cos(0.733)=0.7432$
$\cos(0.768)=0.7193$
$\cos(0.803)=0.6946$ 弧度

- $P_n(x)=\sum_{i=0}^{n} y_i L_i(x)$
- $L_i(x)=\prod_{j=0,\, j\neq i}^{n} \frac{x-x_j}{x_i-x_j}$
- Error Bound : $|f(x)-P_n(x)| \le \frac{\max|\boxed{f^{(n+1)}(\ )}|}{(n+1)!}\prod_{i=0}^{n}|x-x_i|$

  $f(x)=\cos(x)$的$(n+1)$階導數的最大值
- 本題計算 $P_n(x)$ 並求 Error Bound

(1) 一階：取最接近 0.750 的兩點

$x_1=0.733 \quad f(0.733)=0.7432$

$x_2=0.768 \quad f(0.768)=0.7193$

$P_1(x)=f(x_1)L_1(x)+f(x_2)L_2(x)$

其中 $L_1(x)=\frac{x-x_2}{x_1-x_2}=\frac{0.750-0.768}{0.733-0.768}=\frac{-0.018}{-0.035}\approx 0.5143$

$L_2(x)=\frac{x-x_1}{x_2-x_1}=\frac{0.750-0.733}{0.768-0.733}=\frac{0.017}{0.035}\approx 0.4857$

$P_1(0.750)=0.7432\times 0.5143+0.7193\times 0.4857=0.73159\approx 0.7316$ #

$E_1(x)=\frac{\max|f''(\xi)|}{2!}|(x-x_0)(x-x_1)|$

$f''(x)=-\cos(x)$ 最大值 $\Rightarrow |-\cos(0.733)|=0.74317$

$E_1(0.750)=\frac{0.74317}{2}|(0.750-0.733)(0.750-0.768)|=0.0001137 \Rightarrow E_1(0.750)\le 1.137\times 10^{-4}$ #

(2) 二階：$x_0=0.698$，$x_1=0.733$，$x_2=0.768$，$x=0.750$

$P_2(x)=f(x_0)L_0(x)+f(x_1)L_1(x)+f(x_2)L_2(x)$

$L_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(0.750-0.733)(0.750-0.768)}{(0.698-0.733)(0.698-0.768)}=-0.1249$

$L_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\frac{(0.750-0.698)(0.750-0.768)}{(0.733-0.698)(0.733-0.768)}=0.7641$

$L_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{(0.750-0.698)(0.750-0.733)}{(0.768-0.698)(0.768-0.733)}=0.3608$

$P_2(0.750)=0.7661\times(-0.1249)+0.7432\times 0.7641+0.7193\times 0.3608\approx 0.73171667$ #

$E_2(x)=\frac{\max|f'''(\xi)|}{3!}|(x-x_0)(x-x_1)(x-x_2)|$

$f'''(x)=\sin(x)$ 最大值 $\Rightarrow |\sin(0.768)|=0.694698$

$E_2(0.750)=\frac{0.694698}{6}|(0.750-0.698)(0.750-0.733)(0.750-0.768)|=0.000001842 \Rightarrow E_2(0.750)\le 1.842\times 10^{-6}$ #

(3) 三階：$x_0=0.698$，$x_1=0.733$，$x_2=0.768$，$x_3=0.803$

$P_3(x)=f(x_0)L_0(x)+f(x_1)L_1(x)+f(x_2)L_2(x)+f(x_3)L_3(x)$

$L_0(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}=\frac{0.017\times(-0.018)\times(-0.053)}{(-0.035)\times(-0.070)\times(-0.105)}=-0.06304$

$L_1(x)=\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}=\frac{0.052\times(-0.018)\times(-0.053)}{0.035\times(-0.035)\times(-0.070)}=0.5785$

$L_2(x)=\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}=\frac{0.052\times 0.017\times(-0.053)}{0.070\times 0.035\times(-0.035)}=0.5464$

$L_3(x)=\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}=\frac{0.052\times 0.017\times(-0.018)}{0.105\times 0.070\times 0.035}=-0.06185$

$P_3(0.750)=0.7661\times(-0.06304)+0.7432\times 0.5785+0.7193\times 0.5464+0.6946\times(-0.06185)\approx 0.7317827$ #

$E_3(x)=\frac{\max|f^{(4)}(\xi)|}{4!}|(x-x_0)(x-x_1)(x-x_2)(x-x_3)|$

$f^{(4)}(x)=\cos(x)$ 最大值 $\Rightarrow |\cos(0.698)|=0.766129$

$E_3(0.750)=\frac{0.766129}{24}|(0.750-0.698)(0.750-0.733)(0.750-0.768)(0.750-0.803)|=2.692\times 10^{-8} \Rightarrow E_3(0.750)\le 2.692\times 10^{-8}$ #

(4) 四階：由於我們只有四個已知點 x_0, x_1, x_2, x_3，
最高只能進行三階插值，
因此四階插值與三階插值相同。

$\Rightarrow E_4(0.750) = E_3(0.750) \leq 3.51 \times 10^{-8}$ ※

2.

x	0.3	0.4	0.5	0.6
e^{-x}	0.740818	0.670320	0.606531	0.548812

$f(x) = x - e^{-x} = 0$

$f[x_0] = 0.3 - e^{-0.3} = -0.440818 = a_0$

$f[x_1] = 0.4 - e^{-0.4} = -0.27032$

$f[x_2] = 0.5 - e^{-0.5} = -0.106531$

$f[x_3] = 0.6 - e^{-0.6} = 0.051188$

$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{-0.27032 + 0.440818}{0.4 - 0.3} = 1.70498 = a_1$

$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{-0.106531 + 0.27032}{0.5 - 0.4} = 1.63789$

$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{0.051188 + 0.106531}{0.6 - 0.5} = 1.57719$

$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1.63789 - 1.70498}{0.5 - 0.3} = -0.33545 = a_2$

$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{1.57719 - 1.63789}{0.6 - 0.4} = -0.3035$

$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{-0.3035 + 0.33545}{0.6 - 0.3} = 0.1065 = a_3$

- $P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$

$= -0.440818 + 1.70498(x - 0.3) - 0.33545(x - 0.3)(x - 0.4) + 0.1065(x - 0.3)(x - 0.4)(x - 0.5)$

$= -0.440818 + 1.70498x - 0.511494 - 0.33545x^2 + 0.234815x - 0.040254 + 0.1065x^3 - 0.1278x^2 + 0.050055x - 0.00639$

$= 0.1065x^3 - 0.46325x^2 + 1.98985x - 0.998956$

迭代求解 $\Rightarrow x_0 = 0.5$

Newton's Method

$P_3'(x) = 0.3195x^2 - 0.9265x + 1.98985$

1st iteration

$P_3(x_0) = P_3(0.5) = -0.106531$

$P_3'(x_0) = P_3'(0.5) = 1.606475$

$x_1 = 0.5 - \frac{-0.106531}{1.606475} = 0.56631$

2nd iteration

$P_3(x_1) = P_3(0.56631) = -0.001309$

$P_3'(x_1) = P_3'(0.56631) = 1.56763$

$x_2 = 0.56631 - \frac{-0.001309}{1.56763} = 0.56715$

3rd iteration

$P_3(x_2) = 7.52 \times 10^{-6}$

$P_3'(x_2) = 1.56716$

$x_3 = 0.56715 - \frac{7.52 \times 10^{-6}}{1.56716} = 0.567145$ ※

$\boxed{x = 0.567145 \text{ 為 } x - e^{-x} = 0 \text{ 的近似解}}$ ※

3. A car travelling along a straight road is clocked at a number of points. The data from the observations are given in the following table, where the time ***T*** is in seconds, the distance ***D*** is in feet, and the speed ***V*** is in feet per second.

	T	0 x_0	3 x_1	5 x_2	8 x_3	13 x_4
$f(x)$	D	0	200	375	620	990
$f'(x)$	V	75	77	80	74	72

a. Use a Hermite polynomial to predict the position of the car and its speed when $t = 10$ s.

b. Use the derivative of the Hermite polynomial to determine whether the car ever exceeds a 55 mi/h speed limit on the road. If so, what is the first time the car exceeds this speed?

c. What is the predicted maximum speed for the car ?

一輛沿直線道路行駛的汽車在多個時間點被測速。測量數據如下表，時間 T 單位為秒，距離 D 單位為英尺，速度 V 單位為英尺每秒。

T (s)	0	3	5	8	13
D (ft)	0	200	375	620	990
V (ft/s)	75	77	80	74	72

問題

(a) 使用 Hermite 多項式來預測汽車在 $t = 10$ 秒時的位置和速度。
(b) 使用 Hermite 多項式的導數來確定汽車是否曾超過道路限速 55 英里每小時。如果超過，找出第一次超過該速度的時間點。
(c) 預測汽車的最大速度。

a. $D \Rightarrow f(x)$ $\quad V \Rightarrow f'(x)$

z	$f(z)$	First divided differences	Second divided differences
$z_0 = x_0 = 0$	$f[z_0] = f(x_0) = 0$	$f[z_0, z_1] = f'(x_0) = 75$	$f[z_0, z_1, z_2] = -2.\overline{7} = \frac{-25}{9}$
$z_1 = x_0 = 0$	$f[z_1] = f(x_0) = 0$	$f[z_1, z_2] = \frac{f[z_2]-f[z_1]}{z_2 - z_1} = \frac{200}{3} = 66.\overline{6}$	$f[z_1, z_2, z_3] = 3.\overline{4} = \frac{31}{9}$
$z_2 = x_1 = 3$	$f[z_2] = f(x_1) = 200$	$f[z_2, z_3] = f'(x_1) = 77$	$f[z_2, z_3, z_4] = 5.25$
$z_3 = x_1 = 3$	$f[z_3] = f(x_1) = 200$	$f[z_3, z_4] = \frac{f[z_4]-f[z_3]}{z_4 - z_3} = \frac{175}{2} = 87.5$	$f[z_3, z_4, z_5] = -3.75$
$z_4 = x_2 = 5$	$f[z_4] = f(x_2) = 375$	$f[z_4, z_5] = f'(x_2) = 80$	$f[z_4, z_5, z_6] = 0.\overline{5} = \frac{5}{9}$
$z_5 = x_2 = 5$	$f[z_5] = f(x_2) = 375$	$f[z_5, z_6] = \frac{f[z_6]-f[z_5]}{z_6 - z_5} = \frac{245}{3} = 81.\overline{6}$	$f[z_5, z_6, z_7] = -2.\overline{5} = \frac{-23}{9}$
$z_6 = x_3 = 8$	$f[z_6] = f(x_3) = 620$	$f[z_6, z_7] = f'(x_3) = 74$	$f[z_6, z_7, z_8] = 0$
$z_7 = x_3 = 8$	$f[z_7] = f(x_3) = 620$	$f[z_7, z_8] = \frac{f[z_8]-f[z_7]}{z_8 - z_7} = \frac{370}{5} = 74$	$f[z_7, z_8, z_9] = -0.4$
$z_8 = x_4 = 13$	$f[z_8] = f(x_4) = 990$	$f[z_8, z_9] = f'(x_4) = 72$	
$z_9 = x_4 = 13$	$f[z_9] = f(x_4) = 990$		

3rd

$f[z_0 \ldots z_3] = \frac{56}{27} = 2.\overline{074}$

$f[z_1 \ldots z_4] = \frac{13}{36} = 0.36\overline{1}$

$f[z_2 \ldots z_5] = -4.5$

$f[z_3 \ldots z_6] = \frac{31}{36} = 0.86\overline{1}$

$f[z_4 \ldots z_7] = \frac{-28}{27} = -1.\overline{037}$

$f[z_5 \ldots z_8] = \frac{23}{72} = 0.319\overline{4}$

$f[z_6 \ldots z_9] = -0.08$

4th

$f[z_0 \ldots z_4] = \frac{-37}{108} = -0.34\overline{259}$

$f[z_1 \ldots z_5] = \frac{-35}{36} = -0.97\overline{2}$

$f[z_2 \ldots z_6] = \frac{193}{180} = 1.07\overline{2}$

$f[z_3 \ldots z_7] = \frac{-41}{108} = -0.3\overline{796}2$

$f[z_4 \ldots z_8] = \frac{293}{1728} = 0.16956$

$f[z_5 \ldots z_9] = \frac{-719}{14400} = -0.049930\overline{5}$

5th

$f[z_0 \ldots z_5] = \frac{-17}{135} = -0.1\overline{259}$

$f[z_1 \ldots z_6] = \frac{23}{90} = 0.2\overline{5}$

$f[z_2 \ldots z_7] = \frac{-196}{675} = -0.2\overline{9037}$

$f[z_3 \ldots z_8] = \frac{949}{17280} = 0.054919$

$f[z_4 \ldots z_9] = -0.027436$

6th

$f[z_0 \ldots z_6] = \frac{103}{2160} = 0.047\overline{685}1$

$f[z_1 \ldots z_7] = \frac{-737}{10800} = -0.068\overline{2407}$

$f[z_2 \ldots z_8] = 0.0345289$

$f[z_3 \ldots z_9] = -0.008235498$

7th

$f[z_0 \ldots z_7] = \frac{-313}{21600} = -0.01449\overline{074}$

$f[z_1 \ldots z_8] = 0.00790535$

$f[z_2 \ldots z_9] = -0.0042764$

8th

$f[z_0 \ldots z_8] = 0.00172277$

$f[z_1 \ldots z_9] = -0.000937$

9th

$f[z_0 \ldots z_9] = -0.00020460$

a.

$$H_9(x) = f[z_0] + \sum_{k=1}^{9} f[z_0,\ldots,z_k](x-z_0)(x-z_1)\cdots(x-z_{k-1})$$ $\quad x^3-3x^2$

$$= 0 + 75(x-0) + \left(-\frac{25}{9}\right)(x-0)^2 + \frac{56}{27}(x-0)^2(x-3) + \left(\frac{-37}{108}\right)(x-0)^2(x-3)^2 + \left(\frac{-17}{135}\right)(x-0)^2(x-3)^2(x-5) + \frac{103}{2160}(x-0)^2(x-3)^2(x-5)^2$$

$$+ \left(-\frac{313}{21600}\right)(x-0)^2(x-3)^2(x-5)^2(x-8) + (0.00172277)(x-0)^2(x-3)^2(x-5)^2(x-8)^2 + (-0.00020460)(x-0)^2(x-3)^2(x-5)^2(x-8)^2(x-13)$$

$x=10$ 代入

$H_9(10) = 768.96$ ft

$V_9(10) = H_9'(10) = 74.64$ ft/s

b. 55 mph

$\Rightarrow 55 \times \frac{5280}{3600} = 80.67$ ft/s

$V(t) > 80.67$

$t \approx 3.15$ s #

c. $V_{max} \approx 92.04$ ft/s #