HW6

1. Use Gaussian elimination and pivoting technique to solve

$$1.19x_1 + 2.11x_2 - 100x_3 + x_4 = 1.12$$

$$14.2x_1 - 0.112x_2 + 12.2x_3 - x_4 = 3.44$$

$$100x_2 - 99.9x_3 + x_4 = 2.15$$

$$15.3x_1 + 0.110x_2 - 13.1x_3 - x_4 = 4.16$$

Hw6 E94104094 游昊蓁

$$a_{kj} = a_{ij} \times \left(-\frac{a_{k1}}{a_{11}}\right) + a_{kj}$$

$$\Rightarrow \begin{bmatrix} 1.19 & 2.11 & -100 & 1 \\ 14.2 & -0.112 & 12.2 & -1 \\ 0 & 100 & -99.9 & 1 \\ 15.3 & 0.110 & -13.1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.12 \\ 3.44 \\ 2.15 \\ 4.16 \end{bmatrix}$$

$\max\{1.19, 14.2, 15.3\} = 15.3$

1st row $\longleftrightarrow$ 4th row

$$\begin{array}{c} \\ \times\left(-\frac{14.2}{15.3}\right) \\ \\ \times\left(\frac{-1.19}{15.3}\right) \end{array} \begin{bmatrix} 15.3 & 0.110 & -13.1 & -1 \\ 14.2 & -0.112 & 12.2 & -1 \\ 0 & 100 & -99.9 & 1 \\ 1.19 & 2.11 & -100 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4.16 \\ 3.44 \\ 2.15 \\ 1.12 \end{bmatrix} \begin{array}{c} \\ \times\left(-\frac{14.2}{15.3}\right) \\ \\ \times\left(\frac{-1.19}{15.3}\right) \end{array}$$

$$\Rightarrow \begin{bmatrix} 15.3 & 0.110 & -13.1 & -1 \\ 0 & -0.214 & 24.36 & -0.0719 \\ 0 & 100 & -99.9 & 1 \\ 0 & 2.10 & -98.98 & 1.078 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4.16 \\ -0.4209 \\ 2.15 \\ 0.79644 \end{bmatrix}$$

$\max\{100, 2.1, -0.2141\} = 100$

2nd row $\longleftrightarrow$ 3rd row

$$\begin{array}{c} \\ \\ \times\frac{0.214}{100} \\ \times\left(-\frac{2.1}{100}\right) \end{array} \begin{bmatrix} 15.3 & 0.110 & -13.1 & -1 \\ 0 & 100 & -99.9 & 1 \\ 0 & -0.214 & 24.36 & -0.0719 \\ 0 & 2.10 & -98.98 & 1.078 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4.16 \\ 2.15 \\ -0.4209 \\ 0.79644 \end{bmatrix} \begin{array}{c} \\ \\ \times\frac{0.214}{100} \\ \times\left(-\frac{2.1}{100}\right) \end{array}$$

3rd row $\longleftrightarrow$ 4th row

$$\Rightarrow \begin{array}{c} \\ \\ \\ \times\frac{24.15}{96.88} \end{array} \begin{bmatrix} 15.3 & 0.110 & -13.1 & -1 \\ 0 & 100 & -99.9 & 1 \\ 0 & 0 & -96.88 & 1.057 \\ 0 & 0 & 24.15 & -0.0697 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4.16 \\ 2.15 \\ 0.7513 \\ -0.4163 \end{bmatrix} \begin{array}{c} \\ \\ \\ \times\frac{24.15}{96.88} \end{array}$$

$$\Rightarrow \begin{bmatrix} 15.3 & 0.110 & -13.1 & -1 \\ 0 & 100 & -99.9 & 1 \\ 0 & 0 & -96.88 & 1.057 \\ 0 & 0 & 0 & 0.1937 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4.16 \\ 2.15 \\ 0.7513 \\ -0.229 \end{bmatrix}$$

$$x_4 = \frac{-0.229}{0.1937} = -1.1822$$

$$x_3 = \frac{0.7513 - 1.057 \times (-1.1822)}{-96.88} = -0.0207$$

$$x_2 = \frac{2.15 - (-99.9) \times (-0.0207) - 1 \times (-1.1822)}{100} = 0.01264$$

$$x_1 = \frac{4.16 - 0.11 \times 0.01264 + 13.1 \times (-0.0207) + 1 \times (-1.1822)}{15.3} = 0.176$$

#

2. Find the inverse of the matrix A where

$$A=\begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 3 & -1 & 0 \\ -1 & -1 & 6 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

求A^{-1}

$AA^{-1}=I$

$\rightarrow [A|I]$ 整理成 $[I|A^{-1}]$

$$\left[\begin{array}{cccc|cccc} 4 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 6 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \div 4} \left[\begin{array}{cccc|cccc} 1 & 0.25 & -0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 6 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

R_2-R_1 , R_1+R_3

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0.25 & -0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 2.75 & -0.75 & 0 & -0.25 & 1 & 0 & 0 \\ 0 & -0.75 & 5.75 & 2 & 0.25 & 0 & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \div 2.75} \left[\begin{array}{cccc|cccc} 1 & 0.25 & -0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 1 & -0.2727 & 0 & -0.0909 & 0.3636 & 0 & 0 \\ 0 & -0.75 & 5.75 & 2 & 0.25 & 0 & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1-0.25R_2$, $R_3+0.75R_2$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & -0.1818 & 0 & 0.2727 & -0.0909 & 0 & 0 \\ 0 & 1 & -0.2727 & 0 & -0.0909 & 0.3636 & 0 & 0 \\ 0 & 0 & 5.5455 & 2 & 0.1818 & 0.2727 & 1 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \div 5.5455} \left[\begin{array}{cccc|cccc} 1 & 0 & -0.1818 & 0 & 0.2727 & -0.0909 & 0 & 0 \\ 0 & 1 & -0.2727 & 0 & -0.0909 & 0.3636 & 0 & 0 \\ 0 & 0 & 1 & 0.3606 & 0.0328 & 0.0492 & 0.1803 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1+0.1818R_3$, $R_2+0.2727R_3$, R_4-2R_3

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0.0656 & 0.2787 & -0.082 & 0.0328 & 0 \\ 0 & 1 & 0 & 0.0983 & -0.082 & 0.377 & 0.0492 & 0 \\ 0 & 0 & 1 & 0.3606 & 0.0328 & 0.0492 & 0.1803 & 0 \\ 0 & 0 & 0 & 4.2788 & -0.0656 & -0.0984 & -0.3606 & 1 \end{array}\right] \xrightarrow{R_4 \div 4.2788} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0.0656 & 0.2787 & -0.082 & 0.0328 & 0 \\ 0 & 1 & 0 & 0.0983 & -0.082 & 0.377 & 0.0492 & 0 \\ 0 & 0 & 1 & 0.3606 & 0.0328 & 0.0492 & 0.1803 & 0 \\ 0 & 0 & 0 & 1 & -0.0153 & -0.023 & -0.0843 & 0.2337 \end{array}\right]$$

$R_1-0.0656R_4$, $R_2-0.0983R_4$, $R_3-0.3606R_4$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0.2797 & -0.0805 & 0.0383 & -0.0153 \\ 0 & 1 & 0 & 0 & -0.0805 & 0.3793 & 0.0575 & -0.023 \\ 0 & 0 & 1 & 0 & 0.0383 & 0.0575 & 0.2107 & -0.0843 \\ 0 & 0 & 0 & 1 & -0.0153 & -0.023 & -0.0843 & 0.2337 \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} 0.2797 & -0.0805 & 0.0383 & -0.0153 \\ -0.0805 & 0.3793 & 0.0575 & -0.023 \\ 0.0383 & 0.0575 & 0.2107 & -0.0843 \\ -0.0153 & -0.023 & -0.0843 & 0.2337 \end{bmatrix}$$ ＃

3. Use Crout factorization for a tri-diagonal system to solve the problem

$$\underbrace{\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}}_{A}\underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}}_{X}=\underbrace{\begin{Bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{Bmatrix}}_{b}.$$

$A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a:主對角線 → $a = [3,3,3,3]$
b:下對角線 → $b = [-1,-1,-1]$
c:上對角線 → $c = [-1,-1,-1]$

初始 $l_{11} = a_{11} = 3$

$u_{12} = \frac{c_1}{l_{11}} = \frac{-1}{3}$

$i = 2 \sim 4$ $\quad l_{ii} = a_{ii} - \underbrace{l_{i+1,i}}_{b} u_{i,i+1}$

$u_i = \frac{c_i}{l_i}$ (直到 $i=3$, 最後一行沒 c_4)

$$\left[\begin{aligned} a_{21} &= l_{21}u_{11} + l_{22}u_{21} = l_{21} \\ a_{22} &= l_{21}u_{12} + l_{22} \\ a_{12} &= l_{11}u_{12} + l_{12} \end{aligned}\right.$$

$l_{11} = 3$

$l_{22} = 3-(-1)(-\frac{1}{3}) = 3-\frac{1}{3} = 2.667$

$l_{33} = 3-(-1)(-0.375) = 3-0.375 = 2.625$

$l_{44} = 3-(-1)(-0.381) = 3-0.381 = 2.619$

$u_{12} = \frac{-1}{3} \approx -0.333$

$u_{23} = \frac{-1}{2.667} \approx -0.375$

$u_{34} = \frac{-1}{2.625} \approx -0.381$

x

$l_{21} = a_{21} = -1$

$l_{32} = a_{32} = -1$

$l_{43} = a_{43} = -1$

$LY = b$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ -1 & 2.667 & 0 & 0 \\ 0 & -1 & 2.625 & 0 \\ 0 & 0 & -1 & 2.619 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

$y_1 = 2/3 = 0.667$

$y_2 = (3+0.667)/2.667 \approx 1.375$

$y_3 = (4+1.375)/2.625 \approx 2.048$

$y_4 = (1+2.048)/2.619 \approx 1.164$

$UX = Y$

$$\begin{bmatrix} 1 & -0.333 & 0 & 0 \\ 0 & 1 & -0.375 & 0 \\ 0 & 0 & 1 & -0.381 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.667 \\ 1.375 \\ 2.048 \\ 1.164 \end{bmatrix}$$

$x_4 = y_4 = 1.164$

$x_3 = 2.048 - (-0.381)\times 1.164 \approx 2.491$

$x_2 = 1.375 - (-0.375)\times 2.491 \approx 2.309$

$x_1 = 0.667 - (-0.333)\times 2.309 \approx 1.436$

$\Rightarrow \quad X = \begin{bmatrix} 1.436 \\ 2.309 \\ 2.491 \\ 1.164 \end{bmatrix}$ #