HW7

Solve the problem

$4x_1 - x_2 - x_4 = 0$

$-x_1 + 4x_2 - x_3 - x_5 = -1$

$-x_2 + 4x_3 + x_5 - x_6 = 9$

$-x_1 + 4x_4 - x_5 - x_6 = 4$

$-x_2 - x_4 + 4x_5 - x_6 = 8$

$-x_3 - x_5 + 4x_6 = 6$

by (a) Jacobi method, (b) Gauss-Seidel method, (c) SOR method, and (d) the conjugate gradient method.

$A\vec{x} = \vec{b}$

$$A = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 1 & -1 \\ -1 & 0 & 0 & 4 & -1 & -1 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ -1 \\ 9 \\ 4 \\ 8 \\ 6 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

(a) Jacobi Iteration

$A = L + D + U$

$(L+D+U)\vec{x} = \vec{b}$

$D\vec{x} = -(L+U)\vec{x} + \vec{b}$

$\Rightarrow \vec{x} = \underline{-D^{-1}(L+U)}\vec{x} + \underline{D^{-1}\vec{b}} \triangleq \underline{T}\vec{x} + \underline{\vec{c}}$

$\begin{cases} T = -D^{-1}(L+U) \\ \vec{c} = D^{-1}\vec{b} \end{cases}$

$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)}\right)$

初始猜測 $x^{(0)} = [0\ 0\ 0\ 0\ 0\ 0]^T$

1st

$x_1^{(1)} = \frac{1}{4}(0 + x_2^{(0)} + x_4^{(0)}) = \frac{1}{4}(0) = 0$

$x_2^{(1)} = \frac{1}{4}(-1 + x_1^{(0)} + x_3^{(0)} + x_5^{(0)}) = \frac{1}{4}(-1) = -0.25$

$x_3^{(1)} = \frac{1}{4}(9 + x_2^{(0)} - x_5^{(0)} + x_6^{(0)}) = \frac{1}{4}(9) = 2.25$

$x_4^{(1)} = \frac{1}{4}(4 + x_1^{(0)} + x_5^{(0)} + x_6^{(0)}) = \frac{1}{4}(4) = 1$

$x_5^{(1)} = \frac{1}{4}(8 + x_2^{(0)} + x_4^{(0)} + x_6^{(0)}) = \frac{1}{4}(8) = 2$

$x_6^{(1)} = \frac{1}{4}(6 + x_3^{(0)} + x_5^{(0)}) = \frac{1}{4}(6) = 1.5$

2nd

$x_1^{(2)} = \frac{1}{4}(0 + (-0.25) + 1) = 0.1875$

$x_2^{(2)} = \frac{1}{4}(-1 + 0 + 2.25 + 2) = 0.5625$

$x_3^{(2)} = \frac{1}{4}(9 + 0.25 - 2 + 1.5) = 2.1875$

$x_4^{(2)} = \frac{1}{4}(4 + 0 - 2 - 1.5) = 0.125$

$x_5^{(2)} = \frac{1}{4}(8 + 0.25 - 1 + 1.5) = 2.1875$

$x_6^{(2)} = \frac{1}{4}(6 - 2.25 - 2) = 0.4375$

…… 多次迭代

→ 迭代 25 次 $\underline{x^{(25)} = [1.174786,\ 1.643168,\ 2.448249,\ 3.055976,\ 3.949655,\ 3.099473]}$ 接近 python 的結果 ↓

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Jacobi: [1.1747883  1.64317298 2.44824825 3.05598016 3.94965738 3.09947604]
Gauss-Seidel: [1.17478836 1.64317351 2.44824812 3.05598056 3.94965762 3.09947644]
SOR (ω=1.25): [1.17478873 1.6431735  2.44824802 3.05598063 3.94965756 3.0994764 ]
Conjugate Gradient: [1.17656665 1.64269366 2.44433267 3.06002082 3.95260785 3.09922059]
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(b) Gauss - Seidel method

Gauss-Seidel method 和 Jacobi 相似，但每一步更新都立即用於下一個變數計算，收斂速度較快。

$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}\right)$

$(D+L)\vec{x} = -U\vec{x} + \vec{b}$

$\rightarrow \vec{x} = -(D+L)^{-1}U\vec{x} + (D+L)^{-1}\vec{b}$

1st

$x_1^{(1)} = \frac{1}{4}(0 + x_2^{(0)} + x_4^{(0)}) = \frac{1}{4}(0) = 0$

$x_2^{(1)} = \frac{1}{4}(-1 + x_1^{(1)} + x_3^{(0)} + x_5^{(0)}) = \frac{1}{4}(-1) = -0.25$

$x_3^{(1)} = \frac{1}{4}(9 + x_2^{(1)} - x_5^{(0)} + x_6^{(0)}) = \frac{1}{4}(9 - 0.25 + 0) = 2.1875$

$x_4^{(1)} = \frac{1}{4}(4 + x_1^{(1)} + x_5^{(0)} + x_6^{(0)}) = \frac{1}{4}(4) = 1$

$x_5^{(1)} = \frac{1}{4}(8 + x_2^{(1)} + x_4^{(1)} + x_6^{(0)}) = \frac{1}{4}(8 - 0.25 + 1 + 0) = 2.1875$

$x_6^{(1)} = \frac{1}{4}(6 + x_3^{(1)} + x_5^{(1)}) = \frac{1}{4}(6 - 2.1875 - 2.1875) = 0.40625$

2nd

$x_1^{(2)} = \frac{1}{4}(x_2^{(1)} + x_4^{(1)}) = \frac{1}{4}(-0.25 + 1) = 0.1875$

$x_2^{(2)} = \frac{1}{4}(-1 + x_1^{(2)} + x_3^{(1)} + x_5^{(1)}) = \frac{1}{4}(-1 + 0.1875 + 2.1875 + 2.1875) = 0.890625$

$x_3^{(2)} = \frac{1}{4}(9 + x_2^{(2)} - x_5^{(1)} + x_6^{(1)}) = \frac{1}{4}(9 + 0.890625 - 2.1875 + 0.40625) = 2.02734375$

$x_4^{(2)} = \frac{1}{4}(4 + x_1^{(2)} + x_5^{(1)} + x_6^{(1)}) = \frac{1}{4}(4 + 0.1875 + 2.1875 + 0.40625) = 1.6953125$

$x_5^{(2)} = \frac{1}{4}(8 + x_2^{(2)} + x_4^{(2)} + x_6^{(1)}) = \frac{1}{4}(8 + 0.890625 + 1.6953125 + 0.40625) = 2.748046875$

$x_6^{(2)} = \frac{1}{4}(6 + x_3^{(2)} + x_5^{(2)}) = \frac{1}{4}(6 - 2.02734375 - 2.748046875) = 0.306152334375$

…… 多次迭代

→ 迭代 14 次 $\underline{x^{(14)} = [1.174788,\ 1.643173,\ 2.448248,\ 3.055980,\ 3.949657,\ 3.099476]}$ 接近 python 結果 ↑

(c) SOR method

$$D\vec{x} + wA\vec{x} = D\vec{x} + w\vec{b} \quad , \; 0 < w < 2$$

$$\Rightarrow D\vec{x} + w(L+D+U)\vec{x} = D\vec{x} + w\vec{b}$$

$$D\vec{x} + w(D+L)\vec{x} = D\vec{x} - wU\vec{x} + w\vec{b}$$

$$\rightarrow (D+wL)\vec{x} = [(1-w)D - wU]\vec{x} + w\vec{b}$$

$$\vec{x} = (D+wL)^{-1}[(1-w)D - wU]\vec{x} + w(D+wL)^{-1}\vec{b} \equiv T_w\vec{x} + \vec{c}$$

$$x_i^{(k+1)} = (1-w)x_i^{(k)} + \frac{w}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right)$$

set w: 鬆弛參數 $(0 < w < 2)$

假設 $w = 1.25$. 初始 $x^{(0)} = [0\ 0\ 0\ 0\ 0\ 0]^T$

1st
$x_1^{(1)} = (1-w)\cdot 0 + \frac{w}{4}(0+0) = 0$
$x_2^{(1)} = (1-w)\cdot 0 + \frac{1.25}{4}(-1+0+0+0) = -0.3125$
$x_3^{(1)} = (1-w)\cdot 0 + \frac{1.25}{4}(9+(-0.3125)-0+0) = 2.7148$
$x_4^{(1)} = (1-w)\cdot 0 + \frac{1.25}{4}(4+0+0+0) = 1.25$
$x_5^{(1)} = (1-w)\cdot 0 + \frac{1.25}{4}(8+(-0.3125)+1.25+0) = 2.79297$
$x_6^{(1)} = (1-w)\cdot 0 + \frac{1.25}{4}(6-2.7148-2.79297) = 0.1538$

2nd
$x_1^{(2)} = (1-w)\cdot 0 + \frac{1.25}{4}(-0.3125+1.25) = 0.29297$
$x_2^{(2)} = (1-w)(-0.3125) + \frac{1.25}{4}(-1+0.29297+2.7148+2.7930) = 1.8909$
$x_3^{(2)} = (1-w)\cdot 2.7148 + \frac{1.25}{4}(9-1.8909-2.7930+0.1538) = -0.5029$
$x_4^{(2)} = (1-w)\cdot 1.25 + \frac{1.25}{4}(4+0.29297+2.7930+0.1538) = 2.4965$
$x_5^{(2)} = (1-w)\cdot 2.7930 + \frac{1.25}{4}(8-1.8909-2.4965+0.1538) = 0.2584$
$x_6^{(2)} = (1-w)\cdot 0.1538 + \frac{1.25}{4}(6-(-0.5029)-0.2584) = 1.8997$

…… 多次迭代

→ 迭代17次 $x^{(17)} = [1.174789, 1.643174, 2.448248, 3.055981, 3.949658, 3.099476]$ 接近 python 結果

```
Jacobi: [1.1747883  1.64317298 2.44824825 3.05598016 3.94965738 3.09947604]
Gauss-Seidel: [1.17478836 1.64317351 2.44824812 3.05598056 3.94965762 3.09947644]
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(d) the conjugate gradient method

$$\vec{v}^{(k)} = b - A\vec{x}^{(k)} \qquad \vec{v}^{(k+1)} = \vec{v}^{(k)} - t_k A\vec{p}^{(k)}$$

$$t_k = \frac{\vec{v}^{(k)T}(\vec{b} - A\vec{x}^{(k)})}{\vec{p}^{(k)T}A\vec{p}^{(k)}} \qquad \beta_k = \frac{\vec{v}^{(k+1)T}\vec{v}^{(k+1)}}{\vec{v}^{(k)T}\vec{v}^{(k)}}$$

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} + t_k\vec{p}^{(k)} \qquad \vec{p}^{(k+1)} = \vec{v}^{(k+1)} + \beta_k\vec{p}^{(k)}$$

初始化
$x^{(0)} = 0$
$\vec{v}^{(0)} = b - A\vec{x}^{(0)}$
$\vec{p}^{(0)} = \vec{v}^{(0)}$

1st $x^{(0)} = [0, 0, 0, 0, 0, 0]$

$\vec{v}^{(0)} = b - Ax^{(0)} = b = [0, -1, 9, 4, 8, 6]$

$\vec{p}^{(0)} = \vec{v}^{(0)} = [0, -1, 9, 4, 8, 6]$

$$t_0 = \frac{\vec{v}^{(0)T}\vec{v}^{(0)}}{\vec{p}^{(0)T}A\vec{p}^{(0)}} = \frac{198}{606} = \frac{33}{101} \approx 0.3267$$

$\vec{x}^{(1)} = \vec{x}^{(0)} + t_0\vec{p}^{(0)} = [0, -0.3267, 2.9406, 1.3069, 2.6139, 1.9604]$

$\vec{v}^{(1)} = \vec{v}^{(0)} - t_0 A\vec{p}^{(0)} = [0, -1, 9, 4, 8, 6] - 0.3267[-3, -21, 39, 2, 23, 7]$
$= [0.9802, 5.8614, -3.7426, 3.3465, 0.4851, 3.7129]$

$$\beta_0 = \frac{\vec{v}^{(1)T}\vec{v}^{(1)}}{\vec{v}^{(0)T}\vec{v}^{(0)}} = \frac{74.518}{198} \approx 0.3764$$

$\vec{p}^{(1)} = \vec{v}^{(1)} + \beta_0\vec{p}^{(0)} = [0.9802, 5.485, -0.355, 4.8521, 3.4963, 5.9713]$

$$t_1 = \frac{\vec{v}^{(1)T}\vec{v}^{(1)}}{\vec{p}^{(1)T}A\vec{p}^{(1)}} = \frac{74.518}{253.998} \approx 0.2934$$

$\vec{x}^{(2)} = \vec{x}^{(1)} + t_1\vec{p}^{(1)} = [0.2876, 1.2826, 2.8364, 2.7305, 3.6397, 3.7124]$

…… 多次迭代, 結果接近 python (如上)